At various points in the text, special notations and conventions are defined. Some general notations are, however, made use of throughout. $i:j;k$ indicates that the integer i should take the values $i=j,\ldots,k$. If the lower limit j is unity and the upper limit is bounded, the former and the subsequent semi-colon are omitted; thus $i:k$ is an abbreviation for $i=1,\ldots,k$. If the upper limit is unbounded, it is omitted; thus $i:1;$ is an abbreviation for $i=1,2,\ldots$. If the lower and upper limits take unbounded negative and positive values respectively, both, together with the intervening semi-colon, are omitted; thus $i;$ is an abbreviation for $i=\ldots,-1,0,1,\ldots$. Duplicated descriptions are indicated simply by separating the variables concerned by a comma; thus $i,j:1;$ is an abbreviation for $i=1,2,\ldots;\ j=1,2,\ldots$. Conjoint descriptions are separated by a vertical bar; thus $i:n\,|\,j:i;n$ is an abbreviation for $i=1,\ldots,n;\ j=i,\ldots,n$.

Angular brackets are used to condense two statements into one in a way made obvious by the context: 1⟨2⟩ is odd ⟨even⟩; braces {...} are used for the same purpose.

Definition. A stratified commutative ring, or S-ring, is a system S of numbers arranged into strata $S\{j\}$

$(j:)$ such that

(i) the members of $S\{j\}$ form an additive Abelian group $(j:)$;

(ii) commutative and associative multiplication is defined between elements of the various $S\{j\}$: with $a \in S\{j\}$, $b \in S\{k\}$, $ab = ba \in S\{j+k\}$ $(j,k:)$ and $(ab)c = a(bc)$ for such products;

(iii) multiplication is distributive with respect to addition within appropriate strata: with $a, b \in S\{j\}$, $c \in S\{k\}$, $(a+b)c = ac + bc$, addition on the left being within $S\{j\}$, that on the right within $S\{j+k\}$ $(j,k:)$.

A number belonging to the above system is an S-number and the $S\{j\}$ to which it belongs is its stratum.

$\mathbb{S}$ is the class of S-rings.

It is not assumed in the above definition that addition and subtraction between disparate strata should be possible, although in special cases this may be so. It is so, for example, when all $S\{j\}$ are the same additive Abelian group; S is then simply a commutative ring. More generally, it is also possible between congruent strata if $S\{j\} \equiv S\{k\}$ when $j = k \bmod m$, m being a prescribed integer.

By way of motivation, two special S-rings are described.

a) With K a commutative ring, the strata $S\{j\}$ with $j < 0$

each consist of the number $0 \in \mathbb{K}$. With $n \geq 1$ and $j \geq 0$, $S\{j\}$ is the aggregate of jth degree quintic forms in n variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$; each quintic is expressible in the form

$$\sum_{k(1)+\ldots+k(n)=j} c\{k(1), \ldots, k(n)\} x_1^{k(1)} \ldots x_n^{k(n)} =$$

$$\sum_{j(1)=0}^{j} \sum_{j(2)=0}^{m(j,1)} \ldots \sum_{j(n)=0}^{m(j,n-1)} c\{j(1), j(2), \ldots, j(n-1), m(j,n)\} x_1^{j(1)} \ldots x_{n-1}^{j(n-1)} x_n^{m(j,n)}$$

where $m(j,h) = j - j(1) - \ldots - j(h)$. Both sums are of $\binom{j+n-1}{j}$ terms, the ranges of the successive integer arguments of the coefficients c being as indicated by the limits of summation in the second expression. The free variables $x_1, \ldots, x_n$ are introduced merely to motivate the arithmetic operations upon the coefficients c resulting from addition, subtraction and multiplication of quintic forms in which the variables are treated as members of $S\{0\} \equiv \mathbb{K}$.

b) Let Δ be a domain and $\alpha^{(r)}$ $(r:m)$ a set of points (possibly not belonging to Δ) and let a nonnegative real distance $d(\alpha^{(r)}, z)$ between each $\alpha^{(r)}$ and all $z \in \Delta$ be defined; assume further that Δ contains m sequences of points $\{z_k^{(r)}\}$ such that $\lim d(\alpha^{(r)}, z_k^{(r)}) = 0$ as k

tends to infinity $(r:m)$. Let $\mathbb{F}$ be a class of mappings f over Δ such that for each $z \in \Delta$, $f(z) \in R$, where R is a normed commutative ring, with addition, subtraction and multiplication over $\mathbb{F}$ being defined by use of these operations over R, and let $\mathbb{F}$ also contain the subclass $S\{j\}$ of mappings f for which $\| f(z_k) \| / \{d(\alpha^{(r)}, z_k^{(r)})\}^j$ remains bounded as k tends to infinity $(j:/r:m)$. The mappings of all $S\{j\}$ form a normed S-ring.

The above examples serve to draw attention to a salient feature of theory to follow. It will be shown that the end results of certain computations involving S-numbers are members of fixed strata — for example that certain bilinear forms with S-number coefficients are expressible as linear sums of squares with coefficients that are S-numbers. Application of this theory to the system of the first example then reveals that the end product of certain computations is a pure quintic and not, as might be supposed, a mixed expression involving quintic forms of varying degrees; application to the second example reveals that a

constructed function possesses the asymptotic properties appropriate to its class.

Definition. A i) $O_{(j)}$ is the zero element in the stratum $S\{j\}$ $(j:)$.

(ii) $S[0]$ is the complete system of numbers $O_{(j)}$ $(j:)$.

(iii) An S-stratum containing a zero element alone is a null stratum.

(iv) An S-ring consisting exclusively of null strata is a null S-ring.

B) A void system of S-numbers contains no members.

C i) A, B being two systems of S-numbers, the notation $A \subseteq B$ means that either A is void or that all numbers belonging to A also belong to B.

(ii) The notation $A \equiv B$ $\langle A \subset B \rangle$ is synonymous with $A \subseteq B$, $B \subseteq A$ $\langle A \subseteq B, B \nsubseteq A \rangle$.

D) Let $A, B \subseteq \mathfrak{S} \in \mathbb{S}$.

(i) $A \cap B$ is void if either A or B is void and is the system of numbers common to both A and B otherwise.

(ii) $A \cup B$ is $A\langle B \rangle$ if $B\langle A \rangle$ is void and is otherwise the system of numbers belonging to either A or B.

Ei) With $I, J, K \subseteq R \subseteq S \in \mathbb{S}$ and $K \equiv I \cap J$, I and J are said to be disjoint with respect to K in R; if K is void I and J are simply said to be disjoint in R.

(ii) With $R \equiv I \cup J$ and $I \cap J$ void, I and J are said to be complementary in R.

F) Let $A \subseteq S \in \mathbb{S}$ be nonvoid.

(i) S[A] is the system of strata containing the members of A.

(ii) With $B \subseteq S$, $B \cap S[A]$ is void if B contains no members in the strata S[A] and is otherwise the system of members of B contained in the strata S[A].

(iii) The notation $S[A] \equiv S[B]$ indicates that S[A] and S[B] are the same systems of strata

(iv) These notations are preserved when A is the single S-number a; thus S[a] is the stratum to which a belongs.

G) The notation $C \in \mathbb{M}$ indicates that C is a set, and in this part of the definition, $C \in \mathbb{M}$.

ia) A relationship θ establishing correspondences of the form $a.\theta.b$ between certain members a, b, ... of C is a partial ordering over C. $\mathbb{O}$ is the class of partial orderings and $\mathbb{O}(C)$ is the complete set of partial orderings over C. b) A given partial ordering θ divides the complete system (a,b), (c,d), ...

such that for each $a \in C(\theta)$, $C(\theta|a)$ is contained by one of the $C', C'', \ldots$.
iia) A set $\mathfrak{y} = \{(a,b),(c,d),\ldots\}$ of ordered pairs of members $a, b, \ldots$ of C is a graph over C; a void set of such ordered pairs defines a void graph. $\mathfrak{G}(C)$ is the complete system of graphs over C. b) $\mathfrak{y}$ being the graph of the preceding subclause, $\mathfrak{y}^T$ is void when $\mathfrak{y}$ is void; otherwise $\mathfrak{y}^T = \{(b,a),(d,c),\ldots\}$. c) With $\mathfrak{F}$ and $\mathfrak{y}$ being two graphs over C, $\mathfrak{F} \dagger \mathfrak{y}$ is the complete set of ordered pairs (a,c) for which $b \in C$ exists such that $(a,b) \in \mathfrak{F}$, $(b,c) \in \mathfrak{y}$. d) The complete system $\mathfrak{y}(\theta)$ of ordered pairs belonging to class (1) of clause (iib) above is the graph of $\theta \in \mathfrak{O}(C)$. e) The graph $\mathfrak{y}$ of (a) above being prescribed, the determinative partial ordering θ defined by setting $a \theta b$ if and only if the ordered pair (a,b) belongs to $\mathfrak{y}$ is the ordering of $\mathfrak{y}$. $\mathfrak{O}_{\mathfrak{G}}(C)$ is the complete system of orderings of all graphs in $\mathfrak{G}(C)$. f) O_C is the ordering of the void graph over C; with $B \subseteq C$, $\overline{O}_C(B) < 1_C(B) >$ is the ordering of the graph which consists of the pairs (a,a) for all $a \in B <$ of the pairs (a,b) for all $a, b \in B >$, $\overline{O}_C(C) < 1_C(C) >$ being written simply as $\overline{O}_C < 1_C >$.
(iii) Let $\theta, \psi \in \mathfrak{O}(C)$. (a) The notation $\theta \leq \psi$ indicates that $\mathfrak{y}(\theta) \leq \mathfrak{y}(\psi)$; ψ is then said to include θ. (b) The notation $\theta = \psi$ indicates that $\theta \leq \psi$ and $\psi \leq \theta$; the partial orderings θ and ψ are then said to be equivalent. (c) The notation $\theta < \psi$ indicates that $\theta \leq \psi, \theta \neq \psi$; θ is then said to be subordinate to ψ. (d) A partial ordering

equivalent to $\bar{O}_C\langle 1_C\rangle$ is said to be null <primitive>.
(iva) Let $\theta\in\mathbb{O}(C)$. $t(\theta)$ is the ordering of $\mathfrak{fy}(\theta)^T$. (b) Let $\theta,\psi\in\mathbb{O}(C)$. $\theta+\psi\langle\theta\times\psi\rangle\{\theta-\psi\}$ is the ordering of $\mathfrak{fy}(\theta)\cup\mathfrak{fy}(\psi)\langle\mathfrak{fy}(\theta)\cap\mathfrak{fy}(\psi)\rangle$ $\{\mathfrak{fy}(\theta)\setminus\mathfrak{fy}(\psi)\}$. (c) With $\theta_B\in\mathbb{O}(C)$ defined for all sets $B\in M$ of a prescribed system Λ, $\sum_{B\in\Lambda}\theta_B$ is the ordering of the graph $\cup\mathfrak{fy}(\theta_B)$, the union being formed over all $B\in\Lambda$; $\prod_{B\in\Lambda}\theta_B$ is defined similarly in terms of $\cap\mathfrak{fy}(\theta_B)$. (d) With $\theta\in\mathbb{O}(C)$, $\bar{t}(\theta)=\theta+t(\theta)$ and $t'(\theta)=\theta\times t(\theta)$. (e) Let $\theta,\psi\in\mathbb{O}(C)$. $\theta\mp\psi$ is the ordering of the graph $\mathfrak{fy}(\theta)\mp\mathfrak{fy}(\psi)$; $\theta\mp\theta$ is written as $\bar{2}\theta$.
v) A partial ordering $\theta\in\mathbb{O}(C)$ such that (a) $\bar{O}_C\leq\theta\langle$(b) $t(\theta)\leq\theta\rangle$ $\{$(c) $\bar{2}\theta\leq\theta\}$ is said to be reflexive <symmetric>{transitive} and $\mathbb{O}_R$, $\mathbb{O}_S$ and $\mathbb{O}_T$ are respectively the classes of such orderings. $\mathbb{O}_R(C)$, $\mathbb{O}_S(C)$ and $\mathbb{O}_T(C)$ are the complete subsets of $\mathbb{O}(C)$ belonging to $\mathbb{O}_R$, $\mathbb{O}_S$ and $\mathbb{O}_T$. (d) A determinative partial ordering θ satisfying conditions (a,c) <conditions (a,b,c)> is a skew equivalence relationship <an equivalence relationship> ~~$\mathbb{O}_P(C)$ on~~ over C and $\mathbb{O}_P=\mathbb{O}_R\cap\mathbb{O}_T$ $\langle\mathbb{O}_E=\mathbb{O}_P\cap\mathbb{O}_S\rangle$ is the class of such skew equivalence relationships <such equivalence relationships>; $\mathbb{O}_P(C)$ and $\mathbb{O}_E(C)$ are the complete subsets of $\mathbb{O}(C)$ belonging to $\mathbb{O}_P$ and $\mathbb{O}_E$. (e) A skew equivalence relationship θ such that for each pair $a,b\in C$ either $(a,b)\in\mathfrak{fy}(\theta)$ or $(b,a)\in\mathfrak{fy}(\theta)$ is a complete ordering over C. f) A complete ordering θ over C for which the ordered pairs (a,b) and (b,a), with a,b distinct in C, never occur together in $\mathfrak{fy}(\theta)$ is an irreducible

partial ordering over C.

(vi) Let $\theta \in \mathbb{O}(C)$. (a) A skew equivalence relationship ⟨an equivalence relationship⟩ ψ such that

1) $\theta \leq \psi$ and

2) no $\beta \in \mathbb{O}_p(C)$ ⟨no $\beta \in \mathbb{O}_E(C)$⟩ such that $\theta \leq \beta$ and, for some pair $a, b \in C$, $a . \psi . b$ while the relationship $a . \beta . b$ is false exists is a skew equivalence closure ⟨an equivalence closure⟩ of θ in C.

(b) $o_c(\theta)$ ⟨$e_c(\theta)$⟩ is the ordering of the graph of a skew equivalence closure ⟨an equivalence closure⟩ of θ in C

H) M_E is the class of sets over which an equivalence relationship is defined. When only one set is ~~defined~~ concerned in the use of this notation, the equivalence relationship is denoted simply by $=$. When two or more sets are concerned, the corresponding equivalence relationships are distinguished by the use of bracketed corresponding suffixes; thus with $B, C \in M_E$, the equivalence relationships over B and C are denoted by $=_{(B)}$ and $=_{(C)}$ respectively.

ia) With $B, C \in M_E$, the notation $t : B \to C$ indicates that t is a mapping such that for each $b \in B$ the single value of $t(b)$ is in C, and that when $a =_{(B)} b$, $t(a) =_{(C)} t(b)$. b) With $A, B, C \in M_E$, the notation $\Gamma : A \times B \to C$ indicates that Γ is an operator whose operation for each $a \in A$ and $b \in B$, is expressed in the form $a \Gamma b$, the single value of this expression lying in C, and that when $a =_{(A)} c$, $b =_{(B)} d$ then $a \Gamma b = c \Gamma d$.

(ii) Let $C \in M_E$ and $t: C \to C$. The notation $t \in \mathbb{R}\{C\}\langle t \in ID\{C\}\rangle$ indicates that $t\{t(a)\} = a$ $\langle t\{t(a)\} = t(a)\rangle$ for all $a \in C$; t is then said to be a reflexive $\langle$an idempotent$\rangle$ mapping over C.

(iii) Let $C \in M$. a) Let $\Gamma: C \times C \to C$. The notation $\Gamma \in A_1\{C\}\langle \Gamma \in \mathbb{C}_2\{C\}\rangle$ indicates that $a\Gamma(b\Gamma c) = (a\Gamma b)\Gamma c$ for all $a, b, c \in C$ $\langle a\Gamma b = b\Gamma a$ for all $a, b \in C\rangle$; Γ is then said to be an associative operator $\langle$a commutative operator$\rangle$ over C. b) Let $\Gamma, \Delta: C \times C \to C$ and Θ be a partial ordering over C. The notation $\{\Gamma, \Delta\} \in A_2\{C, \Theta\}$ indicates that $a\Gamma(b\Delta c) . \Theta . (a\Gamma b)\Delta c$ for all $a, b, c \in C$; the operators Γ, Δ are then said to be mutually associative with respect to the partial ordering Θ over C.

(iv) Let $C \in M_E$ and $u, t: C \to C$. The notation $u \in \mathbb{C}_1\{C|t\}$ indicates that $u\{t(a)\} = t\{u(a)\}$ for all $a \in C$; the mappings u and t are then said to commute over C.

(v) Let $C \in M_E$, $t: C \to C$ and $\Gamma: C \times C \to C$. The notation $t \in \mathbb{P}_1\{C|\Gamma\}$ $\langle t \in A\mathbb{C}\{C|\Gamma\}\rangle$ indicates that $t(a\Gamma b) = t(a)\Gamma t(b)\langle t(a\Gamma b) = t(b)\Gamma t(a)\rangle$ for all $a, b \in C$; the mapping t is then said to preserve the operator Γ over $C\langle$to be anti-commutative with respect to the operator Γ over $C\rangle$.

(vi a) Let $C_1, \ldots, C_5 \in M_E$, $\Gamma: C_1 \times C_3 \to C_5$, $\Delta: C_2 \times C_2 \to C_3$, $\Xi: C_4 \times C_4 \to C_5$ and $\Omega: C_1 \times C_2 \to C_4$. The notation ~~$\Gamma_5$~~ $\Gamma, \Omega \in \mathbb{D}\{C_1, C_2, C_3 | \Delta, \Xi\}$ indicates that $a\Gamma(b\Delta c) =_{(C_5)} (a\Omega b)\Xi(a\Omega c)$ for all $a \in C_1$ and $b, c \in C_2$; the operators Γ, Ω

are then said to be distributive with respect to the operators Δ, Ξ over C_1, C_2, C_3. b) Let $C \in \mathbb{M}$, $\Gamma, \Delta : C \times C \to C$ and Θ be a partial ordering over C. The notation $\Gamma \in \mathbb{LD}_2\{C, \Theta | \Delta\}$ indicates that $a\Gamma(b\Delta c) . \Theta . (a\Gamma b)\Delta(a\Gamma c)$ for all $a, b, c \in C$; the operator Γ is then said to be left Θ-distributive with respect to the operator Δ over C. The notation $\Gamma \in \mathbb{RD}_2\{C, \Theta | \Delta\}$ has a similar significance with respect to the relationship $(a\Delta b)\Gamma c . \Theta . (a\Gamma c)\Delta(b\Gamma c)$, the operator Γ then being said to be right Θ-distributive with respect to the operator Δ over C. The notation $\Gamma \in \mathbb{D}_2\{C, \Theta | \Delta\}$ indicates that both $\Gamma \in \mathbb{LD}_2\{C, \Theta | \Delta\}$ and $\Gamma \in \mathbb{RD}_2\{C, \Theta | \Delta\}$, the operator Γ then simply being said to be Θ-distributive with respect to the operator Δ over C.

(vii) Let $C \in \mathbb{M}$, $\Theta \in \mathbb{O}(C)$ and $t : C \to C$. The notation $t \in \mathbb{OP}_1\{C | \Theta\}$ indicates that for all $a, b \in C$ such that $(a, b) \in fy(\Theta)$, $(t(a), t(b)) \in fy(\Theta)$ also; the mapping t is then said to preserve the partial ordering Θ over C.

(viii) Let $C \in \mathbb{M}$, $\Theta \in \mathbb{O}(C)$ and $\Gamma : C \times C \to C$. a) The notation $\Gamma \in \mathbb{OP}_2\{C | \Theta\}$ indicates that for all ordered pairs $(a, c), (b, d)$ of members of C belonging to $fy(\Theta)$, $((a\Gamma b), (c\Gamma d)) \in fy(\Theta)$ also; the operator Γ is then said to preserve the partial ordering Θ over C. b) The notatim $\Gamma \in \overline{\mathbb{OP}_2}\{C | \Theta\}$ indicates that for all ordered pairs (a, b) of members of C belonging to $fy(\Theta)$,

$((a\Gamma c),(b\Gamma c))\in \mathcal{G}(\Theta)$ for all $c\in C$; the operator Γ is then said to preserve the partial ordering Θ completely over C. c) The notation $\Gamma\in \mathbb{P}_{\underline{I}}\{C|\Theta\}$ indicates that for all $a,b,c\in C$ such that both ordered pairs (a,b) and $(a\Gamma b,c)$ belonging to $\mathcal{G}(\Theta)$, $(a,(b\Gamma c))\in\mathcal{G}(\Theta)$ also; the operator Γ is then said to preserve the partial ordering Θ inversively over C.

(ix) Let $C\in M_E$ and C_K be a subset of members of C belonging to a class K. a) Let $t: C\to C\langle\Gamma: C\times C\to C\rangle$. The notation $t\in \mathbb{CP}_1\{C|K\}\langle\Gamma\in\mathbb{CP}_2\{C|K\}\rangle$ indicates that $t: C_K\to C_K$ $\langle$that $\Gamma: C_K\times C_K\to C_K\rangle$; the mapping t $\langle$the operator $\Gamma\rangle$ is then said to preserve the class K in C. (b) Let $t: C\to C$. The notation $t\in \mathbb{U}_1\{C|K\}$ indicates that $t(a)=a$ for all $a\in C_K$; the mapping t is then said to be unitary with respect to the class K over C. (c) Let $\Gamma: C\times C\to C$. The notation $\Gamma\in \mathbb{ID}_2\{C|K\}$ indicates that $a\Gamma a=a$ for all $a\in C_K$; the class K is then said to be idempotent with respect to the operator Γ over C.

(x) In conjoint use of the above notations, operators and partial orderings are separated by commas, as are sets and classes. Thus, for example, with the set $\mathbb{O}(C)$, the operators $+$ and $\times$, and the partial orderings $\leq$ and $=$ defined, the notation $+,\times\in A_1\{\mathbb{O}(C)\}$, $\mathbb{OP}_2\{\mathbb{O}(C)|\leq,=\}$ indicates that $+\in A_1\{\mathbb{O}(C)\}$, $+\in\mathbb{OP}_2\{\mathbb{O}(C)|\leq\}$, ..., $\times\in\mathbb{OP}_2\{\mathbb{O}(C)|=\}$

I i) With $C,D,E\in\mathbb{M}$, for each $x\in C$ let $f(x),g(x)$ be subsets of D and E respectively depending upon x. $\cup\underset{C}{\int} f(x)\underline{d}g(x)=\underset{\mathfrak{G}}{\cup} f(x)$ $\langle\cap\underset{C}{\int} f(x)\underline{d}g(x)=\cap f(x)\rangle\{\cup\underset{C}{\int} f(x)\bar{d}g(x)=\cup f(x)\langle\cap\underset{C}{\int} f(x)\bar{d}g(x)=\cap f(x)\rangle\}$ where the union $\langle$intersection$\rangle$ over D is formed over all $x\in C$ for which $g(x)$ is void in E $\{$is nonvoid in $E\}$.

(ii) Let $\lambda,\rho\in\mathbb{D}(C)$. a) With $\theta\in\mathbb{D}(C)$, $f^{(\lambda,\rho)}(\theta)=\lambda\mp\theta\mp\rho$. b) $\mathbb{D}^{(\lambda,\rho)}(C)=\cup\underset{\mathbb{D}(C)}{\int}\theta\underline{d}\{lg\{f^{(\lambda,\rho)}(\theta)-\theta\}\}$. c) $\leq_{[\lambda,\rho]}\langle\leq_{(\lambda,\rho)}\rangle$ is the partial ordering over $\mathbb{D}(C)$ defined by setting $\theta\leq_{[\lambda,\rho]}\psi\langle\theta\leq_{(\lambda,\rho)}\psi\rangle$ if and only if $\theta\leq f^{(\lambda,\rho)}(\psi)\langle f^{(\lambda,\rho)}(\theta)\leq\psi\rangle$.

(iii) Let $\theta,\psi\in\mathbb{D}(C)$. $\theta[\mp]\psi\langle\theta[\bar{\mp}]\psi\rangle\{\theta(\mp]\psi\}[\theta(\bar{\mp})\psi]$ is the determinative partial ordering over C defined by setting, for each pair $a,b\in C$, $a.\theta[\mp]\psi.b\langle a.\theta[\bar{\mp}]\psi.b\rangle\{a.\theta(\mp]\psi.b\}[a.\theta(\bar{\mp})\psi.b]$ if and only if $R(\theta|a)\cap L(\psi|b)$ is nonvoid $\langle$if and only if $R(\theta|a)\leq L(\psi|b)\rangle\{$if and only if $L(\psi|b)\leq R(\theta|a)\}[$if and only if $R(\theta|a)=L(\psi|b)\}$.

Theorem . Let $C\in\mathbb{M}$.

Aia) Let $\theta,\psi\in\mathbb{D}(C)$. $\theta\leq\psi\langle\theta=\psi\rangle$ if and only if it is known that $a.\theta.b$ for all $a,b\in C$ for which it is known that $a.\psi.b\langle$and conversely$\rangle$. b) $\leq\in\mathbb{D}_p\{\mathbb{D}(C)\}$ and $=\in\mathbb{E}_p\{\mathbb{D}(C)\}$

iia) With $\theta\in\mathbb{D}(C)$, $t(\theta)$ may be defined by setting $b.t(\theta).a$ for

$1 \mp \Theta \mp 1 = 1$. (d) For all $\Theta, \psi \in \mathbb{O}(C)$, $\Theta - \psi \leq \Theta$, $(\Theta - \psi) + \psi = \Theta + \psi$ and $(\Theta + \psi) - \psi \leq \Theta$. (e) If $\Theta \leq \bar{O}$, $\bar{2}\Theta = \Theta$ and if $\bar{O} \leq \Theta$, $\Theta \leq \bar{2}\Theta$.

va) $+, \times \in A_1\{\mathbb{O}(C)\}, \mathbb{C}_2\{\mathbb{O}(C)\}, \mathbb{OP}_2\{\mathbb{O}(C) | \leq, =\}, \mathbb{CP}_2\{\mathbb{O}(C) | \mathbb{O}_R, \mathbb{O}_S\}, \mathbb{ID}_2\{\mathbb{O}(C) | \mathbb{O}\}$. (b) $+ \in \mathbb{D}_2\{\mathbb{O}(C), = | \times\}$. (c) $\times \in \mathbb{D}_2\{\mathbb{O}(C), = | +\}, \mathbb{CP}_2\{\mathbb{O}(C) | \mathbb{O}_T, \mathbb{O}_P, \mathbb{O}_E, \mathbb{O}'\}$.

via) $t \in \mathbb{R}\{\mathbb{O}(C)\}, \mathbb{P}_1\{\mathbb{O}(C) | +, \times\}, \mathbb{AC}\{\mathbb{O}(C) | \mp\}, \mathbb{OP}_1\{\mathbb{O}(C) | \leq, =\}, \mathbb{CP}_1\{\mathbb{O}(C) | \mathbb{O}_R, \mathbb{O}_S, \mathbb{O}_T, \mathbb{O}_P, \mathbb{O}_E, \mathbb{O}'\}, \mathbb{U}_1\{\mathbb{O}(C) | \mathbb{O}_S\}$. (b) $\bar{t}, t' : \mathbb{O}(C) \to \mathbb{O}_S(C)$ and $\bar{t}, t' \in \mathbb{ID}_1\{\mathbb{O}(C)\}, \mathbb{P}_1\{\mathbb{O}(C) | +, \times\}, \mathbb{OP}_1\{\mathbb{O}(C) | \leq, =\}, \mathbb{CP}_1\{\mathbb{O}(C) | \mathbb{O}_R, \mathbb{O}_S, \mathbb{O}'\}, \mathbb{C}_1\{\mathbb{O}(C) | t\}, \mathbb{U}_1\{\mathbb{O}(C) | \mathbb{O}_S\}$. $t' : \mathbb{O}_P(C) \to \mathbb{O}_E(C)$ and $t' \in \mathbb{CP}_1\{\mathbb{O}(C) | \mathbb{O}_T\}$.

viia) $\mp \in A_1\{\mathbb{O}(C)\}, \mathbb{D}_2\{\mathbb{O}(C), = | +\}, \mathbb{D}_2\{\mathbb{O}(C), \leq | \times\}, \mathbb{OP}_2\{\mathbb{O}(C) | \leq, =\}, \mathbb{CP}_2\{\mathbb{O}(C) | \mathbb{O}_R\}, \mathbb{ID}_2\{\mathbb{O}(C) | \mathbb{O}_P, \mathbb{O}_E\}$. (b) If either $\bar{O}_c \leq \Theta + \psi$ or $\bar{O}_c \leq \Theta + \beta$ then $\Theta + (\psi \mp \beta) \leq (\Theta + \psi) \mp (\Theta + \beta)$ and $(\psi \mp \beta) + \Theta \leq (\psi + \Theta) \mp (\beta + \Theta)$.

viii) Let $\Theta \in \mathbb{O}_E(C)$. Θ is equivalent to a uniquely defined partial ordering expressible in the form $\sum_{B \in \Lambda} 1_c(B)$, where Λ is a set of pairwise disjoint ranges such that $C = \cup B(B \in \Lambda)$.

Bia) A skew equivalence closure of $\Theta \in \mathbb{O}(C)$ over C exists. With Θ prescribed, let $K_\Theta(C)$ be the complete system of $\psi' \in \mathbb{O}_P(C)$ for which $\Theta \leq \psi$, and define $\psi \in \mathbb{O}(C)$ by stipulating that

ch pair $a,b\in C$, $a.\psi.b$ if and only if for all $K_\theta(C)$, $a.\psi'.b$. Then ψ is a skew equivalence closure of θ over C. (b) The skew equivalence closures of $\theta\in\mathbb{O}(C)$ over C form an equivalence class: if ψ,ψ' are two such closures, $\psi=\psi'$. c) If ψ is a skew equivalence closure of $\theta\in\mathbb{O}(C)$ and $\xi\in\mathbb{O}_p(C)$ includes θ, then ξ also includes ψ. (d) All skew equivalence closures of $\theta\in\mathbb{O}(C)$ have the same graph: $o_c(\theta)$ is uniquely defined (e) $o_c(\theta)=\bar{O}_c$ if and only if $\theta\leq\bar{O}_c$ (in particular $o_c(\theta)=\bar{O}_c$ if θ is void). iia) $o_c:\mathbb{O}(C)\to\mathbb{O}_p(C)$ and $\theta\in\mathbb{O}_p(C)$ if and only if $o_c(\theta)=\theta$. (b) $o_c\in \mathbb{ID}_1\{\mathbb{O}(C)\}, \mathbb{OP}_1\{\mathbb{O}(C)|\leq,=\}, \mathbb{CP}_1\{\mathbb{O}(C)|\mathbb{O}_R,\mathbb{O}_S,\mathbb{O}_T,\mathbb{O}_P,\mathbb{O}_E,\mathbb{O}'\}, \mathbb{C}_1\{\mathbb{O}(C)|t\}, \mathbb{U}_1\{\mathbb{O}(C)|\mathbb{O}_p\}$. (c) Let $\theta,\psi\in\mathbb{O}(C)$. If $\theta\leq\psi\leq o_c(\theta)$, then $o_c(\psi)=o_c(\theta)$. (d) If $\theta\in\mathbb{O}'(C)$, the existence of a skew equivalence closure of θ may be proved by construction. Select one representative from each set of numbers $a,b,c,\ldots\in C(\theta)$ for which $C(\theta,a)\equiv C(\theta,b)\equiv C(\theta,c)\equiv\ldots$ and let $\hat{C}(\theta)=\{a',b',c',\ldots\}$ be a complete system of representatives extracted from all ranges $C(\theta,d)$ for all members d of $C(\theta)$ in this way. With $g\in\hat{C}(\theta)$, set $\theta_o=1_C\{C(\theta|g)\}\times\theta$. If possible, remove ordered pairs from θ_o by $\{1_C\{C(\theta|g)\}\}$ to produce the graph of a skew

equivalence relationship over $C(\theta|g)$ which includes θ_g. Continue this reduction process until an irreducible partial ordering $\psi_g \in \mathbb{O}_p\{C(\theta|g)\}$ is produced. For each $g \in C \setminus C(\theta)$, let ψ_g be the ordering of the single pair graph (g,g). Set $\Lambda = \hat{C}(\theta) \cup \{C \setminus C(\theta)\}$. Then $o_c(\theta) = \sum_{g \in \Lambda}^{o} \psi_g$.

iiia) With references to skew equivalence closures replaced by references to equivalence closures, and $\mathbb{O}_p(C), o_c(\theta)$ replaced by $\mathbb{O}_E(C), e_c(\theta)$, the results of clause (Bi) hold, as does that of clause (Büa). The results of clause (Büb) hold, mutatis mutandis, concerning e_c with respect to the classes $\mathbb{ID}_1$, $\mathbb{OP}_1$, $\mathbb{CP}_1$ and $\mathbb{C}_1$, but $e_c \in \mathbb{U}_1\{\mathbb{O}(C)|\mathbb{O}_E\}$. The counterpart to clause (Büc) is that if $\theta \leq \psi \leq e_c(\theta)$, then $e_c(\psi) = e_c(\theta)$. Clause (Büd) has a direct counterpart concerning the equivalence closure of $\theta \in \mathbb{O}'(C)$. (b) $e_c\{o_c(\theta)\} = e_c(\theta)$ for all $\theta \in \mathbb{O}(C)$. (c) An equivalence closure of $\theta \in \mathbb{O}(C)$ may be exhibited in the form described in clause (Aviii). Define the set $\hat{C}(\theta)$ as in clause (Büd) and let Λ be the union of the set of ranges $C(\theta, g)$ for all $g \in \hat{C}(\theta)$ and the set of single point ranges g for all $g \in C \setminus C(\theta)$. Then $e_c(\theta) = \sum_{B \in \Lambda}^{} 1_c(B)$, the ranges of Λ being pairwise disjoint and such that $C = \cup B$ $(B \in \Lambda)$.

c) Let $\lambda, \rho \in \mathbb{O}(C)$.

ia) With $\theta, \psi \in \mathbb{O}(C)$, $\theta \leq_{[\lambda,\rho]} \psi$ if and only if for all $(a,d) \in \mathfrak{h}(\theta)$, C contains $b(a,d)$ and $c(a,d)$ such that $(a, b(a,d)) \in \mathfrak{h}(\lambda)$, $(b(a,d), c(a,d)) \in \mathfrak{h}(\psi)$ and $(c(a,d), d) \in \mathfrak{h}(\rho)$. (b) If $\lambda, \rho \in \mathbb{O}_R(C)$ $\langle \mathbb{O}_T(C) \rangle \{\mathbb{O}_P(C)\}$ then $\leq_{[\lambda,\rho]} \in \mathbb{O}_R\{\mathbb{O}(C)\}\langle \mathbb{O}_T\{\mathbb{O}(C)\}\rangle\{\mathbb{O}_P\{\mathbb{O}(C)\}\}$

iia) With $\theta, \psi \in \mathbb{O}(C)$, $\theta \leq_{(\lambda,\rho)} \psi$ if and only if for all $(b,c) \in \mathfrak{h}(\theta)$, C contains $a(b,c), d(b,c)$ such that $(a(b,c), b) \in \mathfrak{h}(\lambda)$, $(c, d(b,c)) \in \mathfrak{h}(\rho)$ and $a(b,c), d(b,c) \in \mathfrak{h}(\psi)$. (b) $\leq_{(\lambda,\rho)} \in \mathbb{O}_R\{\mathbb{O}^{(\lambda,\rho)}(C)\}$.

(c) If $\lambda, \rho \in \mathbb{O}_R(C)$ then $\leq_{(\lambda,\rho)} \in \mathbb{O}_T\{\mathbb{O}(C)\}$

(iii) Let $\lambda' \in \lambda$, $\rho' \leq \rho$. (a) If $\theta \leq_{[\lambda',\rho']} \psi$ then $\theta \leq_{[\lambda,\rho]} \psi$. (b) If $\theta \leq_{(\lambda,\rho)} \psi$ then $\theta \leq_{(\lambda',\rho')} \psi$.

iva) $+ \in \mathbb{OP}_2\{\mathbb{O}(C) | \leq_{[\lambda,\rho]}, \leq_{(\lambda,\rho)}\}$ and $\times \in \mathbb{OP}_2\{\mathbb{O}(C) | \leq_{(\lambda,\rho)}\}$

(b) $t, \bar{t} \in \mathbb{OP}_1\{\mathbb{O}(C) | \leq_{[\lambda,t(\lambda)]}, \leq_{(\lambda,t(\lambda))}\}$ and $t' \in \mathbb{OP}_1\{\mathbb{O}(C) | \leq_{(\lambda,t(\lambda))}\}$

va) $\mathbb{O}^{(\lambda,\rho)}(C)$ is the complete system of partial orderings $\theta \in \mathbb{O}(C)$ for which $\lambda \mp \theta \mp \rho \leq \theta$. (b) $+, \times \in \mathbb{CP}_2\{\mathbb{O}(C) | \mathbb{O}^{(\lambda,\rho)}\}$. (c) Let $\psi \in \mathbb{O}^{(\lambda,\rho)}(C)$. If $\theta \leq_{[\lambda,\rho]} \psi$ then $\theta \leq \psi$. If $\psi \leq \mathfrak{h}$ then $\psi \leq_{(\lambda,\rho)} \mathfrak{h}$. (d) Let $\lambda' \leq \lambda$, $\rho' \leq \rho$. Then $\mathbb{O}^{(\lambda,\rho)}(C) \subseteq \mathbb{O}^{(\lambda',\rho')}(C)$.

vi) Let $\lambda, \rho \in \mathbb{O}_R(C)$. (a) $\mp \in \mathbb{CP}_2\{\mathbb{O}(C) | \mathbb{O}^{(\lambda,\rho)}\}$. (b) Let $\bar{0}_C \leq \lambda' \leq \lambda$, $\bar{0}_C \leq \rho' \leq \rho$. $f^{(\lambda',\rho')} \in \mathbb{U}_1\{\mathbb{O}(C) | \mathbb{O}^{(\lambda,\rho)}\}$ and $\lambda' \mp \theta = \theta \mp \rho' = \theta$ for all $\theta \in \mathbb{O}^{(\lambda,\rho)}(C)$. (c) If, with $\theta, \psi \in \mathbb{O}^{(\lambda,\rho)}(C)$, one of the conditions

(1) $\theta \leq_{[\lambda,\rho]} \psi$, (2) $\theta \leq \psi$, (3) $\theta \leq_{(\lambda,\rho)} \psi$ holds, they all hold.

(vi) Let $\lambda' \leq \lambda, \rho' \leq \rho$ and $\lambda, \rho \in \mathbb{D}_t(C)$. Then $\lambda \mp \rho \in \mathbb{D}^{(\lambda',\rho')}(C)$.

Di) Let $\theta, \psi \in \mathbb{D}(C)$.

$$R\{\theta[\bar{\mp}]\psi|a\} = \cup\int_C x\bar{d}\{R(\theta|a) \cap L(\psi|x)\}$$

$$L\{\theta[\bar{\mp}]\psi|a\} = \cup\int_C x\bar{d}\{R(\theta|x) \cap L(\psi|a)\}$$

Using the notation $\Omega(U, \mathfrak{b}, y; V, \mathfrak{z}, z) = U(\mathfrak{b}|y) \setminus V(\mathfrak{z}|z)$

$$R\{\theta[\underline{\mp})\psi|a\} = \cup\int_C x\underline{d}\,\Omega(R, \theta, a; L, \psi, x)$$

$$L\{\theta[\underline{\mp})\psi|a\} = \cup\int_C x\underline{d}\,\Omega(R, \theta, x; L, \psi, a)$$

$$R\{\theta(\bar{\mp}]\psi|a\} = \cup\int_C x\underline{d}\,\Omega(L, \psi, x|R, \theta, a)$$

$$L\{\theta(\bar{\mp}]\psi|a\} = \cup\int_C x\underline{d}\,\Omega(L, \psi, a|R, \theta, x)$$

Finally

$$R\{\theta(\underline{\mp})\psi|a\} = \cup\int_C x\underline{d}\left[\{R(\theta|a) \cup L(\psi|x)\} \setminus \{R(\theta|a) \cap L(\psi|x)\}\right]$$

$$L\{\theta(\underline{\mp})\psi|a\} = \cup\int_C x\underline{d}\left[\{L(\theta|a) \cup R(\psi|x)\} \setminus \{L(\theta|a) \cap R(\psi|x)\}\right]$$

iia) Let $R(x)$ and $L(x)$ be two subsets of C dependent upon $x \in C$, each being the inverse of the other in the sense that for all $x, y \in C$, $y \in R(x)$ if and only if $x \in L(y)$. Let S and T be two subsets of C. Then $T \subseteq \cup\int_C x\underline{d}\{S \setminus R(x)\}$ if and only if $S \subseteq \cup\int_C x\underline{d}\{T \setminus L(x)\}$. (b) $\{[\underline{\mp}), (\bar{\mp}]\} \in A_2\{\mathbb{D}(C)|=\}$.

Proof. Although for a certain $\theta \in \mathbb{D}(C)$ certain ordered pairs (a, b) of members of C for which it is not known whether

the relationship $a.\theta.b$ is true or false may exist, the system of pairs (a,b) for which it is known with certainty that $a.\theta.b$ is well defined: the graph $\mathfrak{h}(\theta)$ is well defined. The graph of a partial ordering is a set of ordered pairs of members of C. By definition, $\theta \leq \psi$ if and only if $\mathfrak{h}(\theta) \subseteq \mathfrak{h}(\psi)$ or, equivalently, if and only if it is known that $a.\theta.b$ for all $a,b \in C$ for which it is known that $a.\psi.b$. Again $\theta = \psi$ if and only if $\mathfrak{h}(\theta) \equiv \mathfrak{h}(\psi)$: the bracketed remark of clause (Aia) is also valid.

Over sets, the relationship $\subseteq$ is determinative, reflexive and transitive: the partial ordering $\leq$ over $\mathfrak{U}(C)$ possesses these properties and is therefore a skew equivalence relationship. The relationship $\equiv$ over graph sets is ~~reflexive~~ determinative, reflexive, symmetric and transitive: the same is true of the partial ordering $=$ over $\mathfrak{U}(C)$ which is thus an equivalence relationship over $\mathfrak{U}(C)$.

The function $\mathfrak{h}$ of clause (Giid) of Definition establishes a surjection of $\mathfrak{U}(C)$ onto $\mathfrak{G}(C)$. The ordering of a graph, defined in clause (Gie) of Definition, establishes a bijection between $\mathfrak{G}(C)$ and a subsystem $\mathfrak{U}_{\mathfrak{G}}(C)$ of $\mathfrak{U}(C)$. The mapping $\mathfrak{h}: \mathfrak{U}(C) \to \mathfrak{G}(C)$ is a homomorphism by

definition: the operator + over $\mathbb{D}(C)$ is defined in terms of the operator $\cup$ over $\mathbb{G}(C)$, $\theta+\psi$ with $\theta,\psi\in\mathbb{D}(C)$ being a member of $\mathbb{D}_G(C)$. The operators $\times$, $\bar{+}$ and $-$ over $\mathbb{D}(C)$ are similarly defined in terms of operations over $\mathbb{G}(C)$. The mappings $t, \bar{t}, t': \mathbb{D}(C) \to \mathbb{D}_G(C)$ are also defined in terms of operations over $\mathbb{G}(C)$.

The results of clauses (ii), (iii a 1), (iii b 2) and (iii c 1) are reformulations, in terms of properties of partial orderings, of definitions previously presented in terms of graph sets. The converse assertion holds for clauses (iii a 2) and (iii c 2). Membership of the class $\mathbb{D}_S$ is defined by the condition $lg(\theta)^T \subseteq lg(\theta)$. However, this condition holds if and only if $lg(\theta)^T = lg(\theta)$. Firstly, the condition $lg(\theta)^T \subseteq lg(\theta)$ is implied by the condition $lg(\theta)^T = lg(\theta)$. Conversely, assume that $lg(\theta)^T \subseteq lg(\theta)$. If $(a,b)\in lg(\theta)$, $(b,a)^T \in lg(\theta)^T$ and $(b,a)\in lg(\theta)$. Accordingly $lg(\theta)$ consists exclusively of ordered pairs of the form (a,b) and (b,a): $lg(\theta)^T = lg(\theta)$. The result of clause (iii b 1) is a reformulation of the condition defining θ to be a member of the class $\mathbb{D}_S$ directly in terms of properties of θ.

In the remainder of the proof, 0_c, $\bar{0}_c$ and 1_c are written simply as 0, $\bar{0}$ and 1.

Since $\bar{0}\leq\bar{0}$, $\bar{0}\in\mathbb{D}_R(C)$. $lg(\bar{0})$ consists exclusively of the

ordered pairs (a,a) for all $a \in C$. Thus $t(\bar{0}) = \bar{0}$ and $\bar{0} \in \mathbb{O}_S(C)$. $\bar{0} \in \mathbb{O}_T(C)$ by default: $\mathfrak{h}(\bar{0})$ contains no ordered pairs (a,b) with a,b distinct in C; the only ordered pairs of $\mathfrak{h}(\bar{0})$ featuring in a chain of the form (a,b) and (b,c), beginning with the element a, are (a,a) and (a,a); the completion of the general chain by the ordered pair (a,c) is performed in $\mathfrak{h}(\bar{0})$ by the ordered pair (a,a) itself. Hence $\bar{0} \in \mathbb{O}_E(C)$. $\mathfrak{h}(1)$ contains the ordered pairs (a,b) for all $a,b \in C$, in particular the ordered pairs (a,a) for all $a \in C$: $1 \in \mathbb{O}_R(C)$. Corresponding to any $(a,b) \in \mathfrak{h}(1)$, $\mathfrak{h}(1)$ also contains the ordered pair (b,a): $1 \in \mathbb{O}_S(C)$. Finally, corresponding to any two ordered pairs of the form $(a,b), (b,c)$ in $\mathfrak{h}(1)$, $\mathfrak{h}(1)$ also contains the ordered pair (a,c): $1 \in \mathbb{O}_T(C)$ and $1 \in \mathbb{O}_E(C)$.

$\mathfrak{h}(0)$ is the void graph set and is contained by $\mathfrak{h}(\Theta)$ for all $\Theta \in \mathbb{O}(C)$. $\mathfrak{h}(1)$ contains the ordered pairs (a,b) for all $a,b \in C$ and, in particular, contains all ordered pairs belonging to $\mathfrak{h}(\Theta)$ for all $\Theta \in \mathbb{O}(C)$. Hence $0 \leq \Theta \leq 1$. $0+\Theta$ is the ordering of the graph $\mathfrak{h}(0) \cup \mathfrak{h}(\Theta) \equiv \mathfrak{h}(\Theta)$; hence $0+\Theta = \Theta$ and similarly $\Theta+0 = \Theta$. $1 \times \Theta$ is the ordering of the graph $\mathfrak{h}(1) \cap \mathfrak{h}(\Theta)$. Since $\mathfrak{h}(1)$ consists of the ordered pairs (a,b) for all $a,b \in C$, $\mathfrak{h}(1) \cap \mathfrak{h}(\Theta) \equiv \mathfrak{h}(\Theta)$ and hence $1 \times \Theta = \Theta$ and again $\Theta \times 1 = \Theta$.

If Θ is zero, $\bar{0} \mp \Theta$ is also zero. Otherwise, select any ordered pair (b,c) in $fy(\Theta)$. $fy(\bar{0})$ contains only one ordered pair with second member b, namely (b,b); $(b,c) \in fy(\Theta)$ if and only if $(b,c) \in fy(\bar{0}) \mp fy(\Theta)$. Hence $\bar{0} \mp \Theta = \Theta$ and similarly $\Theta + \bar{0} = \Theta$. $0 \times \Theta$ is the ordering of the graph $fy(0) \cap fy(\Theta)$ which is a void graph set: $0 \times \Theta = 0$ and again $\Theta \times 0 = 0$. $fy(0)$ contains no ordered pairs (a,b); for this reason $fy(\bar{0}) \mp fy(\Theta)$ is void and hence $0 \mp \Theta = 0$. Similarly $\Theta \mp 0 = 0$. $1 \mp \Theta$ is the ordering of the graph $fy(1) \cup fy(\Theta) \equiv fy(1)$. Thus $1 + \Theta = 1$; also $\Theta + 1 = 1$. When Θ is nonzero, $fy(\Theta)$ contains at least one ordered pair (b,c). $fy(1)$ contains, in particular, the ordered pairs (a,b) for all $a \in C$; hence $fy(1) \mp fy(\Theta)$ contains the ordered pairs (a,c) for all $a \in C$. $fy(1)$ also contains the ordered pairs (c,d) for all $d \in C$. Thus $\{fy(1) \mp fy(\Theta)\} \mp fy(1)$ contains the ordered pairs (a,d) for all $a,d \in C$ and is accordingly equivalent to $fy(1)$. Thus $(1 \mp \Theta) \mp 1 = 1$. Similarly $1 + (\Theta + 1) = 1$. Over graph sets $fy(\Theta) \setminus fy(\Psi) \leq fy(\Theta)$, $\{fy(\Theta) \setminus fy(\Psi)\} \cup fy(\Psi) \equiv fy(\Theta) \cup fy(\Psi)$ and $\{fy(\Theta) \cup fy(\Psi)\} \setminus fy(\Psi) \leq fy(\Theta)$. The results of clause (A iv d) follow directly. When $\Theta \leq \bar{0}$, $fy(\Theta)$ consists of the ordered pairs (a,a) for all a belonging to a subset of C. $fy(\Theta) \mp fy(\Theta)$ has a similar form and $(a,a) \in fy(\Theta) \mp fy(\Theta)$ if and only if

$(a,a) \in \mathfrak{fy}(\Theta)$. Hence when $\Theta \leq \bar{0}$, $\bar{2}\Theta = \Theta$. When $\bar{0} \leq \Theta$, $\mathfrak{fy}(\Theta)$ contains the ordered pairs (b,b) for all $b \in C$. Hence, selecting any ordered pair $(b,c) \in \mathfrak{fy}(\Theta)$, $\mathfrak{fy}(\Theta) \mp \mathfrak{fy}(\Theta)$ also contains the ordered pair (b,c): $\mathfrak{fy}(\Theta) \subseteq \mathfrak{fy}(\Theta) \mp \mathfrak{fy}(\Theta)$ and hence $\Theta \leq \bar{2}\Theta$.

Over graph sets, the operators $\cup$ and $\cap$ are associative, commutative and idempotent, are distributive with respect to each other, and preserve the partial orderings $\subseteq$ and $\equiv$. Hence $+,\times \in A_1\{\mathbb{D}(C)\}$, $\mathbb{C}_2\{\mathbb{D}(C)\}$, $\mathbb{DP}_2\{\mathbb{D}(C) | \leq, =\}$, $\mathbb{ID}_2\{\mathbb{D}(C) | \mathbb{D}\}$ and $+ \in \mathbb{D}_2\{\mathbb{D}(C), = | +\}$, $\times \in \mathbb{D}_2\{\mathbb{D}(C), = | \times\}$. When $\Theta, \psi \in \mathbb{D}_R(C)$, $\mathfrak{fy}(\Theta)$ and $\mathfrak{fy}(\psi)$ both contain $\mathfrak{fy}(\bar{0})$, $\mathfrak{fy}(\Theta) \cup \mathfrak{fy}(\psi)$ and $\mathfrak{fy}(\Theta) \cap \mathfrak{fy}(\psi)$ both do the same: $+,\times \in \mathbb{CP}_2\{\mathbb{D}(C) | \mathbb{D}_R\}$. When $\Theta \in \mathbb{D}_S(C)$, $(b,a) \in \mathfrak{fy}(\Theta)$ whenever $(a,b) \in \mathfrak{fy}(\Theta)$. If, in addition, $\psi \in \mathbb{D}_S(C)$, $\mathfrak{fy}(\Theta) \cup \mathfrak{fy}(\psi)$ and $\mathfrak{fy}(\Theta) \cap \mathfrak{fy}(\psi)$ also have this property: $+,\times \in \mathbb{CP}_2\{\mathbb{D}(C) | \mathbb{D}_S\}$. If $\Theta, \psi \in \mathbb{D}_T(C)$, $\mathfrak{fy}(\Theta) \cap \mathfrak{fy}(\psi)$ contains the ordered pair (a,c) whenever the pairs (a,b), (b,c) belong to both $\mathfrak{fy}(\Theta)$ and $\mathfrak{fy}(\psi)$: $\times \in \mathbb{CP}_2\{\mathbb{D}(C) | \mathbb{D}_T\}$. The operator $\times$ preserves the classes $\mathbb{D}_P = \mathbb{D}_R \cap \mathbb{D}_T$ and $\mathbb{D}_E = \mathbb{D}_P \cap \mathbb{D}_S$. If $\psi' \subseteq \psi$ then $C(\psi'|a) \subseteq C(\psi|a)$ for all $a \in C$; in consequence, if $\psi \in \mathbb{D}'(C)$, $\psi' \in \mathbb{D}'(C)$ also. For any $\Theta \in \mathbb{D}(C)$, $\Theta \times \psi \subseteq \psi$. If $\Theta, \psi \in \mathbb{D}'(C)$, $\Theta \times \psi \in \mathbb{D}'(C)$ also: $\times \in \mathbb{CP}_2\{\mathbb{D}(C) | \mathbb{D}'\}$. ($\mathbb{D}'$ is a semi-ideal class: if $\psi \in \mathbb{D}'(C)$, $\Theta \times \psi = \psi \times \Theta \in \mathbb{D}'(C)$ for all

$\theta \in \mathfrak{D}(C)$. The operator + does not preserve the class $\mathfrak{W}$. Let $fy(\theta) = \{(a,b),(c,d),\ldots\}$ and $fy(\psi) = \{(b,c),(d,e),\ldots\}$. For each $f \in C(\theta)$, $C(\theta|f) \langle f \in C(\psi), C(\psi|f)\rangle$ may be placed in one of the disjoint ranges $\{a,b\},\{c,d\},\ldots \langle \{b,c\},\{d,e\},\ldots\rangle$: $\theta,\psi \in \mathfrak{D}'(C)$ $fy(\theta+\psi) = \{(a,b),(b,c),(c,d),\ldots\}$. If $\theta+\psi \in \mathfrak{D}'(C)$, $C(\theta+\psi|a)$ is contained by a finite set C' which may be supposed to contain the members $a,b,\ldots,y$ but not z. $C(\theta+\psi|y) \not\subseteq C'$ since $(y,z) \in fy(\theta+\psi)$, $z \in C(\theta+\psi|y)$ and $z \notin C'$. Thus in the specification of $\theta+\psi$ as a member of $\mathfrak{D}'(C)$, $C(\theta+\psi|y)$ belongs to another set C'' containing finitely many members. But C' and C'' then both contain y and are not disjoint: $\theta+\psi \notin \mathfrak{D}'(C)$.) Clause (Iv) has been disposed of.

The properties of the partial ordering $t(\theta)$ are determined by those of the graph set $fy(\theta)^T$. Over graph sets, transposition is a reflexive operation: $t \in R\{\mathfrak{D}(C)\}$. Also $\{fy(\theta) \cup fy(\psi)\}^T \equiv fy(\theta)^T \cup fy(\psi)^T$. Hence $t \in P_1\{\mathfrak{D}(C)|+\}$. That transposition also preserves multiplication over partial orderings is proved in the same way. Since by definition $\theta \mp \psi$ is the ordering of the graph $fy(\theta) \mp fy(\psi)$, $fy(\theta \mp \psi) \equiv fy(\theta) \mp fy(\psi)$. If the ordered pair (c,a) belongs to $fy(\theta \mp \psi)^T$, $(a,c) \in fy(\theta \mp \psi)$ and, for some $b \in C$, $(a,b) \in fy(\theta)$ and $(b,c) \in fy(\psi)$; (c,a) then belongs to $fy(\psi)^T \mp fy(\theta)^T$: $fy(\theta \mp \psi)^T \subseteq fy(\psi)^T \mp fy(\theta)^T$. Conversely, corresponding

to any $(c,a)\in fy(\psi)^T \mp fy(\theta)^T$, $b\in C$ exists such that $(c,b)\in fy(\psi)^T$, $(b,a)\in fy(\theta)^T$. $fy(\theta\mp\psi)$ then contains (a,c) and $(c,a)\in fy(\theta\mp\psi)^T$. Hence $fy(\psi)^T \mp fy(\theta)^T \subseteq fy(\theta\mp\psi)$: $fy(\theta\mp\psi)^T = fy(\psi)^T \mp fy(\theta)^T$. Transposition preserves the operation $\mp$ anti-commutatively over graph sets: $t\in AC\{\mathfrak{D}(C)|\mp\}$. Transposition over graph sets preserves the partial orderings $\leq$ and $\equiv$: $t\in \mathfrak{D}\mathbb{P}_1\{\mathfrak{D}(C)|\leq,=\}$. If the pairs (a,a) for all $a\in C$ belong to $fy(\theta)$, they also belong to $fy(\theta)^T$: t preserves the class $\mathfrak{D}_R$ in $\mathfrak{D}(C)$. $\theta\in\mathfrak{D}_S$ if and only if $t(\theta)=\theta$. In particular, if $\theta\in\mathfrak{D}_S$ then $t(\theta)=\theta\in\mathfrak{D}_S$: $t\in\mathfrak{U}_1\{\mathfrak{D}(C)|\mathfrak{D}_S\}$ and in consequence $t\in\mathbb{CP}_1\{\mathfrak{D}(C)|\mathfrak{D}_S\}$. Supposing that $fy(\theta)^T$ contains the two ordered pairs (c,b) and (b,a), $fy(\theta)$ contains the pairs (a,b) and (b,c) and, when $\theta\in\mathfrak{D}_T$, also the pair (a,c); $fy(\theta)^T$ then contains the pair (c,a): $t\in\mathbb{CP}_1\{\mathfrak{D}(C)|\mathfrak{D}_T\}$. Since $t\in\mathbb{CP}_1\{\mathfrak{D}(C)|\mathfrak{D}_R,\mathfrak{D}_T\}$, t also preserves the class $\mathfrak{D}_P\equiv\mathfrak{D}_R\cap\mathfrak{D}_T$. In addition, $t\in\mathbb{CP}\{\mathfrak{D}(C)|\mathfrak{D}_S\}$: t preserves the class $\mathfrak{D}_E\equiv\mathfrak{D}_S\cap\mathfrak{D}_P$. The range $C(t(\theta))$ of $t(\theta)$ is that of θ; also $C\{t(\theta)|a\}\equiv C\{\theta|a\}$ for each $a\in C\{t(\theta)\}\equiv C(\theta)$. Accordingly, the pairwise disjoint subsets $C',C'',\ldots$ such that for each $a\in C(\theta)$, $C(\theta|a)$ is contained by one of the $C',C'',\ldots$ serving to identify θ as a member of the class $\mathfrak{D}'$ serve also to identify $t(\theta)$ as a member of this class: $t\in\mathbb{CP}_1\{\mathfrak{D}(C)|\mathfrak{D}'\}$. $\bar{t}\{t(\theta)\}=$

$t(\Theta)+t\{t(\Theta)\}$ and, since $t\in P_1\{\mathbb{U}(C)|+\}$, $t\{\bar{t}(\Theta)\}=t(\Theta)+t\{t(\Theta)\}$ also. Hence $\bar{t}\in C_1\{\mathbb{U}(C)|t\}$. That $t'\in C_1\{\mathbb{U}(C)|t\}$ is proved in the same way. The results of clause (Axia) have been disposed of.

Since $t\in P_1\{\mathbb{U}(C)|+\}$, $R\{\mathbb{U}(C)\}$ and $+\in C_2\{\mathbb{U}(C)\}$, it follows that $t\{\Theta+t(\Theta)\}=t(\Theta)+t\{t(\Theta)\}=t(\Theta)+\Theta=\Theta+t(\Theta)$. Thus $\bar{t}$ maps $\mathbb{U}(C)$ into $\mathbb{U}_s(C)$. t is a single valued mapping of $\mathbb{U}(C)$ into $\mathbb{U}(C)$, and the sum of two partial orderings is single valued. Hence $\bar{t}:\mathbb{U}(C)\to\mathbb{U}_s(C)$. That $t':\mathbb{U}(C)\to\mathbb{U}_s(C)$ is proved in the same way. Again, since $+\in \mathbb{ID}_2\{\mathbb{U}(C)|\mathbb{U}\}$, $\bar{t}\{\bar{t}(\Theta)\}=\Theta+t\{\bar{t}(\Theta)\}=\Theta+\Theta+t(\Theta)=\Theta+t(\Theta)=\bar{t}(\Theta)$ and $\bar{t}\in\mathbb{ID}_1\{\mathbb{U}(C)\}$. Similarly $t'\in\mathbb{ID}_1\{\dot{\mathbb{U}}(C)\}$. Since $t\in P_1\{\mathbb{U}(C)|+\}$, $\bar{t}(\Theta+\psi)=\Theta+\psi+t(\Theta+\psi)=\Theta+\psi+t(\Theta)+t(\psi)=\bar{t}(\Theta)+\bar{t}(\psi)$: $\bar{t}\in P_1\{\mathbb{U}(C)|+\}$ also. That $t'\in P_1\{\mathbb{U}(C)|+\}$ and $\bar{t},t'\in P_1\{\mathbb{U}(C)|\times\}$ are proved in the same way. $+\in\mathbb{UP}_2\{\mathbb{U}(C)|\leq\}$ and $t\in\mathbb{UP}_1\{\mathbb{U}(C)|\leq\}$. Hence $\bar{t}\in\mathbb{UP}_1\{\mathbb{U}(C)|\leq\}$. Similarly $t'\in\mathbb{UP}_1\{\mathbb{U}(C)|\leq\}$ and $\bar{t},t'\in\mathbb{UP}_1\{\mathbb{U}(C)|=\}$. Since $+\in\mathbb{CP}_2\{\mathbb{U}(C)|\mathbb{U}_R\}$ and $t\in\mathbb{CP}_1\{\mathbb{U}(C)|\mathbb{U}_R\}$, $t\in\mathbb{CP}_1\{\mathbb{U}(C)|\mathbb{U}_R\}$ also. Similarly $+\in\mathbb{ID}_2\{\mathbb{U}(C)|\mathbb{U}\}$ and $t\in U_1\{\mathbb{U}(C)|\mathbb{U}_s\}$, so that $\bar{t}\in U_1\{\mathbb{U}(C)|\mathbb{U}_s\}$ and in consequence $\bar{t}\in\mathbb{CP}_1\{\mathbb{U}(C)|\mathbb{U}_s\}$. As a simple counter-example given above indicates, the operator + does not preserve the class $\mathbb{U}'$ in $\mathbb{U}(C)$. However, $C\{t(\Theta)|a\}=C\{\Theta|a\}$ for each $a\in C\{t(\Theta)\}=C\{\Theta\}$, so that

$C\{\Theta + t(\Theta)|a\} = C\{\Theta|a\}$ for each $a \in C\{\Theta + t(\Theta)\} \equiv C\{\Theta\}$; t preserves the class $\mathbb{O}'$ in $\mathbb{O}(C)$ in the same way that t does so. Similarly $t' \in \mathbb{CP}_1\{\mathbb{O}(C)|\mathbb{O}_R, \mathbb{O}_S, \mathbb{O}'\}$. Since $t \in \mathbb{CP}_1\{\mathbb{O}(C)|\mathbb{O}_T\}$ and $x \in \mathbb{CP}_2\{\mathbb{O}(C)|\mathbb{O}_T\}$, $t' \in \mathbb{CP}_1\{\mathbb{O}(C)|\mathbb{O}_T\}$ also. Furthermore $t' \in \mathbb{CP}_1\{\mathbb{O}(C)|\mathbb{O}_S\}$ and in consequence t' preserves the class $\mathbb{O}_E = \mathbb{O}_S \cap \mathbb{O}_T$ ~~over~~ in $\mathbb{O}(C)$. Finally, since $t' : \mathbb{O}(C) \to \mathbb{O}_S(C)$ and $t' \in \mathbb{CP}_1\{\mathbb{O}(C)|\mathbb{O}_R, \mathbb{O}_T\}$, $t' \in \mathbb{O}_R(C) \cap \mathbb{O}_T(C) \equiv \mathbb{O}_P(C) \to \mathbb{O}_R(C) \cap \mathbb{O}_S(C) \cap \mathbb{O}_T(C) \equiv \mathbb{O}_E(C)$. Both parts of clause (Avi) have been disposed of.

The properties of the operator $\mp$ over partial orderings are those of this operator over graph sets. It is easily shown that $\mp$ operates associatively over graph sets. Let $\Theta, \Psi, \beta \in \mathbb{O}(C)$. If the graph set $\mathfrak{h}' = \{\mathfrak{h}(\Theta) \mp \mathfrak{h}(\Psi)\} \mp \mathfrak{h}(\beta)$ is void, it is contained by the set $\mathfrak{h}'' = \mathfrak{h}(\Theta) \mp \{\mathfrak{h}(\Psi) \mp \mathfrak{h}(\beta)\}$. Otherwise, select any ordered pair (a,d) in $\mathfrak{h}'$. $c \in C$ exists such that $(a,c) \in \mathfrak{h}(\Theta) \mp \mathfrak{h}(\Psi)$ and $(c,d) \in \mathfrak{h}(\Psi)$. In turn, $b \in C$ exists such that $(a,b) \in \mathfrak{h}(\Theta)$ and $(b,c) \in \mathfrak{h}(\Psi)$. Accordingly $(b,d) \in \mathfrak{h}(\Psi) \mp \mathfrak{h}(\beta)$ and $(a,d) \in \mathfrak{h}''$: $\mathfrak{h}' \subseteq \mathfrak{h}''$. Similarly it is shown that $\mathfrak{h}'' \subseteq \mathfrak{h}'$. The orderings of $\mathfrak{h}'$ and $\mathfrak{h}''$ are equivalent: the operator $\mp$ is associative over $\mathbb{O}(C)$.

The operator $\mp$ preserves the partial ordering $\subseteq$ over graph sets. Let $\Theta, \Psi, \beta \in \mathbb{O}(C)$ and $\mathfrak{h}(\Psi) \subseteq \mathfrak{h}(\beta)$. If $\mathfrak{h}' = \mathfrak{h}(\Theta) \mp \mathfrak{h}(\Psi)$

is void, $\mathfrak{h}' \leq \mathfrak{h}'' = \mathfrak{h}(\theta) \mp \mathfrak{h}(\psi)$. Otherwise, select $(a,c) \in \mathfrak{h}'$, so that $b \in C$ exists for which $(a,b) \in \mathfrak{h}(\theta)$, $(b,c) \in \mathfrak{h}(\psi)$. Since $\mathfrak{h}(\psi) \leq \mathfrak{h}(\beta)$, $(b,c) \in \mathfrak{h}(\beta)$ also, and $(a,c) \in \mathfrak{h}''$. Hence $\mathfrak{h}' \leq \mathfrak{h}''$. If $\psi \leq \beta$, $\mathfrak{h}(\psi) \leq \mathfrak{h}(\beta)$, and if $\mathfrak{h}(\theta) \mp \mathfrak{h}(\psi) \leq \mathfrak{h}(\theta) \mp \mathfrak{h}(\beta)$, $\theta \mp \psi \leq \theta \mp \beta : \mp \in \mathbb{OP}_2\{\mathbb{O}(C) | \leq\}$. By an extension of the preceding argument it is shown that $\cdot \mp \in \mathbb{OP}_2\{\mathbb{O}(C) | =\}$.

To show that the operator $\mp$ is $=-$distributive with respect the operator $+$ over $\mathbb{O}(C)$, let $\theta, \psi, \beta \in \mathbb{O}(C)$ again, and set $\mathfrak{h}' = \mathfrak{h}(\theta) \mp \{\mathfrak{h}(\psi) \cup \mathfrak{h}(\beta)\}$, $\mathfrak{h}'' = \{\mathfrak{h}(\theta) \mp \mathfrak{h}(\psi)\} \cup \{\mathfrak{h}(\theta) \mp \mathfrak{h}(\beta)\}$. If $\mathfrak{h}'$ is void, $\mathfrak{h}' \leq \mathfrak{h}''$. Otherwise, select any ordered pair $(a,c) \in \mathfrak{h}'$, so that $b \in C$ exists such that $(a,b) \in \mathfrak{h}(\theta)$ and (b,c) is in either $\mathfrak{h}(\psi)$ or $\mathfrak{h}(\beta)$. If $(b,c) \in \mathfrak{h}(\psi)$, $(a,c) \in \mathfrak{h}(\theta) \mp \mathfrak{h}(\psi)$ and if $(b,c) \in \mathfrak{h}(\beta)$, $(a,c) \in \mathfrak{h}(\theta) \mp \mathfrak{h}(\beta)$. Hence $(a,c) \in \mathfrak{h}(\theta) \mp \mathfrak{h}(\beta)$, and $\mathfrak{h}' \leq \mathfrak{h}''$. Conversely, if $\mathfrak{h}''$ is void, $\mathfrak{h}'' \leq \mathfrak{h}'$. Otherwise, select $(a,c) \in \mathfrak{h}''$. Either $(a,c) \in \mathfrak{h}(\theta) \mp \mathfrak{h}(\psi)$ or $(a,c) \in \mathfrak{h}(\theta) \mp \mathfrak{h}(\beta)$ and in both cases $(a,c) \in \mathfrak{h}'$, since $\mathfrak{h}(\psi), \mathfrak{h}(\beta) \leq \mathfrak{h}(\psi) \cup \mathfrak{h}(\beta)$ and the operator $\mp$ preserves the ordering $\mp$ over graph sets. Hence $\mathfrak{h}' \leq \mathfrak{h}''$, $\mathfrak{h}' = \mathfrak{h}''$ and $\mp \in \mathbb{D}_2\{\mathbb{O}(C) | \neq\}$. The operator $\mp$ is not $=-$distributive with respect to the operator $\times$ over $\mathbb{O}(C)$. Nevertheless a partial ordering result is available. Let $\theta, \psi, \beta \in \mathbb{O}(C)$. Since $\psi \times \beta \leq \psi, \beta$ and $\mp \in \mathbb{OP}_2\{\mathbb{O}(C) | \leq\}$, it

follows that $\theta\bar{+}(\psi\times\beta)\leq\theta\bar{+}\psi,\theta\bar{+}\beta$. Again, since $\times\in\mathbb{D}_2\{\mathbb{O}(C)|\mathbb{O}\}$, $\mathbb{O}\mathbb{P}_2\{\mathbb{O}(C)|\leq\}$ it follows that $\theta\bar{+}(\psi\times\beta)\leq(\theta\bar{+}\psi)\times(\theta\bar{+}\beta)$, or $\bar{+}\in\mathbb{D}_2\{\mathbb{O}(C),\leq|\times\}$.

From clause (Aiv b) $\theta\bar{+}\bar{0}=\theta$ for all $\theta\in\mathbb{O}(C)$. Also $\bar{+}\in\mathbb{O}\mathbb{P}_2\{\mathbb{O}(C)|\leq\}$ so that $\bar{0}=\bar{0}\bar{+}\bar{0}\leq\theta\bar{+}\psi$ in particular when $\bar{0}\leq\theta,\psi:\bar{+}\in\mathbb{C}\mathbb{P}_2\{\mathbb{O}(C)|\mathbb{O}_R\}$. By definition $\theta\bar{+}\theta\leq\theta$ when $\theta\in\mathbb{O}_T(C)$; also, from clause (Aiv b), $\theta\leq\theta\bar{+}\theta$ when $\theta\in\mathbb{O}_R(C)$. Hence $\theta\bar{+}\theta=\theta$ when $\theta\in\mathbb{O}_P(C)\equiv\mathbb{O}_R(C)\cap\mathbb{O}_T(C):\bar{+}\in\mathbb{I}D_2\{\mathbb{O}(C)|\mathbb{O}_P\}$. Since $\mathbb{O}_E\leq\mathbb{O}_P$, $\bar{+}\in\mathbb{I}D_2\{\mathbb{O}(C)|\mathbb{O}_E\}$ also.

To complete the proof of clause (Avii) it is first remarked that if $\mathfrak{h}'=\mathfrak{h}(\theta)\cup\{\mathfrak{h}(\psi)\bar{+}\mathfrak{h}(\beta)\}$ is void, $\mathfrak{h}\leq\mathfrak{h}''=\{\mathfrak{h}(\theta)\cup\mathfrak{h}(\psi)\}\bar{+}\{\mathfrak{h}(\theta)\cup\mathfrak{h}(\beta)\}$. Otherwise, select $(a,c)\in\mathfrak{h}'$. Either $(a,c)\in\mathfrak{h}(\theta)$ or $(a,c)\in\mathfrak{h}(\psi)\bar{+}\mathfrak{h}(\beta)$. In the first case, assuming that $\mathfrak{h}(\bar{0})\leq\mathfrak{h}(\theta)\cup\mathfrak{h}(\psi)$, $(a,c)\in\mathfrak{h}(\theta)\leq\mathfrak{h}(\theta)\cup\mathfrak{h}(\beta)\leq\mathfrak{h}(\bar{0})\cup\{\mathfrak{h}(\theta)\cup\mathfrak{h}(\beta)\}\leq\mathfrak{h}''$; the alternative assumption $\mathfrak{h}(\bar{0})\leq\mathfrak{h}(\theta)\cup\mathfrak{h}(\beta)$ leads to the same result. In the second case, since $\mathfrak{h}(\psi)\leq\mathfrak{h}(\theta)\cup\mathfrak{h}(\psi)$, $\mathfrak{h}(\beta)\leq\mathfrak{h}(\theta)\cup\mathfrak{h}(\beta)$, $(a,c)\in\mathfrak{h}''$ also. Hence $\mathfrak{h}'\leq\mathfrak{h}''$; clause (Aviib) has been disposed of. ("That the operator $\bar{+}$ does not preserve the class $\mathbb{O}'$ in $\mathbb{O}(C)$ is shown by the simple counter-example in which $\mathfrak{h}(\theta)=\{(a,b),(c,d),\ldots\}$, $\mathfrak{h}(\psi)=\{(b,c),(d,e),\ldots\}$; now $\mathfrak{h}(\theta\bar{+}\psi)=\{(a,c),(c,e),\ldots\}$. $\theta,\psi\in\mathbb{O}'(C)$ but $\theta\bar{+}\psi\notin\mathbb{O}'(C)$.)

To prove clause (Aviii) it is first remarked that for each pair $a, c \in C$ either (i) it is known that $a.\Theta.c$ and then $C(\Theta|a) \equiv C(\Theta|c)$ or (ii) it is not known that $a.\Theta.c$ and then $C(\Theta|a)$ and $C(\Theta|c)$ are disjoint in C. For selecting $d \in C(\Theta|c)$ under the conditions (i), it is either known directly that $c.\Theta.d$ or that $d.\Theta.c$ and, since $\Theta \in \Omega_S(C)$ that $c.\Theta.d$ again. The two relationships $a.\Theta.c$ and $c.\Theta.d$ then imply, since $\Theta \in \Omega_T(C)$, that $a.\Theta.d$, so that $d \in C(\Theta|a) : C(\Theta|c) \subseteq C(\Theta|a)$. That $C(\Theta|a) \subseteq C(\Theta|c)$ is proved in the same way. Conversely, supposing that $C(\Theta|a)$ and $C(\Theta|b)$ have a common member d, the two relationships $a.\Theta.d$, $d.\Theta.c$ hold, and, since $\Theta \in \Omega_T(C)$, $a.\Theta.c$. Hence if it is not known that $a.\Theta.c$, $C(\Theta|a)$ and $C(\Theta|b)$ are disjoint in C. Since $\Theta \in \Omega_R(C)$, $a \in C(\Theta|a)$ for all $a \in C$. Accordingly $C \equiv \bigcup C(\Theta|a)$ $(a \in C)$. Unless $\Theta \equiv \overline{O}$, duplication occurs in this representation of C, since $C(\Theta|c) \equiv C(\Theta|a)$ for all $c \in C(\Theta|a)$. Duplicate copies of $C(\Theta|a)$ may be removed from the given representation: $C \equiv \Lambda$ where $\Lambda = \{C', C'', \ldots\}$ and the sets $C', C'', \ldots$ are pairwise disjoint ranges of the form $C(\Theta|a)$. It is not only true (a) that it is not known that $c.\Theta.d$ when c and d belong to distinct members of Λ, but also true (b) that $c.\Theta.d$ for all c, d belonging to the same member

of Λ. For, letting this member be $C'' = C(\theta|b)$ say, c.θ.b and b.θ.d and, since $\theta \in \mathbb{D}_T(C)$, c.θ.d also. Since the operator + is associative and commutative over $\mathbb{D}(C)$, the expression $\sum_{B \in \Lambda} 1_c(B)$ uniquely defines a partial ordering θ′ which also has the properties (a,b) above. Thus for each pair a,b∈C, it is known that a.θ.b if and only if it is known that a.θ′.b : θ=θ′.

The results of clause (Biii a), concerning the equivalence closure of a partial ordering θ, are counterparts to those of clauses (Bi,ii,iii) which concern the skew equivalence closure of θ; the proofs of the former are slight extensions to those of the latter. It suffices to prove the former alone. An equivalence closure of $\theta \in \mathbb{D}(C)$ may be exhibited. There exists at least one equivalence relationship ψ′ such that θ≤ψ′ namely ψ′=1. $K_\theta(C)$, the complete system of $\psi' \in \mathbb{D}_E(C)$ for which θ≤ψ′ is nonvoid. Define $\psi \in \mathbb{D}(C)$ by stipulating that, for each pair a,b∈C, a.ψ.b if and only if for all $\psi' \in K_\theta(C)$, a.ψ′.b. θ≤ψ, since θ≤ψ′ for all $\psi' \in K_\theta(C)$. $\bar{\theta} \leq \psi' < \psi' \in \mathbb{D}_S(C) > \{\psi' \in \mathbb{D}_T(C)\}$ for all $\psi' \in K_\theta(C)$: $\psi \in \mathbb{D}_E(C)$. No $\beta \in \mathbb{D}_E(C)$ such that θ≤β and for some pair a,b∈C, a.ψ.b while the relationship a.β.b is false exists,

since such a ζ belongs to $K_\Theta(C)$ and, in particular, $a.\psi.b$ only if $a.\zeta.b$. ψ is accordingly an equivalence closure of Θ in C.

To demonstrate that the equivalence closures of $\Theta \in \mathbb{D}(C)$ over C themselves form an equivalence class, let ψ, ψ' be two such equivalence closures. ψ, ψ' belong to $K_\Theta(C)$. For all $a, b \in C$, $a.\psi.b$ only if $a.\psi'.b$ and $a.\psi'.b$ only if $a.\psi.b$. Accordingly $\psi = \psi'$. In consequence $\mathfrak{f}(\psi) \equiv \mathfrak{f}(\psi')$: all equivalence closures of $\Theta \in \mathbb{D}(C)$ over C have the same graph: the equivalence closure $e_c(\Theta)$ is uniquely defined as the ordering of this common graph. If $\xi \in \mathbb{D}_E(C)$ includes Θ, $\xi \in K_\Theta(C)$ and, since $a\xi.b$ for all $a, b \in C$ such that $a.\psi.b$, ξ includes ψ.

When $\Theta \leq \bar{O}$, the twin conditions $\Theta \leq \bar{O}$, $\bar{O} \in \mathbb{D}_E(C)$ hold. For any partial ordering ζ such that $a.\zeta.b$ is false when $a.\bar{O}.b$ is true for ~~$a, b \in C$~~ some $a, b \in C$, a relationship of the form $a.\zeta.a$ is false for some $a \in C$, and then $\zeta \notin \mathbb{D}_E(C)$. Accordingly, $\bar{O}$ is an equivalence closure of Θ: $e_c(\Theta) = \bar{O}$ if $\Theta \leq \bar{O}$. Since $\Theta \leq e_c(\Theta)$, $\Theta \leq \bar{O}$ if $e_c(\Theta) = \bar{O}$. The results of clause (Bi) have been obtained.

By definition, e_c maps $\mathbb{D}(C)$ into $\mathbb{D}_E(C)$. $e_c(\Theta)$ is defined uniquely by the system $K_\Theta(C)$. If $\zeta = \Theta$, $K_\zeta(C) \equiv K_\Theta(C)$ and in consequence $e_c(\zeta) = e_c(\Theta)$: $e_c : \mathbb{D}(C) \to \mathbb{D}_E(C)$. When $\Theta \in \mathbb{D}_E(C)$ the twin conditions $\Theta \leq \Theta$, $\Theta \in \mathbb{D}_E(C)$ hold. For any partial

ordering $\beta \in \mathbb{D}_E(C)$ such that $a.\beta.b$ is false when $a.\theta.b$ is true for some $a,b \in C$, the condition $\theta \leq \beta$ is violated. Accordingly θ itself is an equivalence closure of θ: $e_c(\theta)=\theta$ if $\theta \in \mathbb{D}_E(C)$. Since $e_c(\theta) \in \mathbb{D}_E(C)$ by definition, $\theta \in \mathbb{D}_E(C)$ when $e_c(\theta)=\theta$. Clause (Biia) has been disposed of.

Since $e_c(\theta) \in \mathbb{D}_E(C)$ and $e_c(\theta)=\theta$ if $\theta \in e_c(\theta)$, $e_c\{e_c(\theta)\}=e_c(\theta)$ for all $\theta \in \mathbb{D}(C)$: $e_c \in \mathbb{ID}_1\{\mathbb{D}(C)\}$. Let $\theta' \leq \theta$. $e_c(\theta) \in \mathbb{D}_E(C)$ and, since $\theta \leq e_c(\theta)$ and the partial ordering $\leq$ is transitive over $\mathbb{D}(C)$, $\theta' \leq e_c(\theta)$. $e_c(\theta')=\psi'$ where, from the construction given in the proof of clause (Biia), $a.\psi'.b$ for all $a,b \in C$ for which, in particular, $a.e_c(\theta).b$. Hence $e_c(\theta') \leq e_c(\theta)$ if $\theta' \leq \theta$: $e_c \in \mathbb{OP}_1\{\mathbb{D}(C)|\leq\}$. That e_c also preserves the equivalence relationship $=$ over $\mathbb{D}(C)$ is proved in the same way. Again using the transitivity of the partial ordering $\leq$ over $\mathbb{D}(C)$, the condition $\bar{0} \leq e_c(\theta)$ may be deduced from the two conditions $\bar{0} \leq \theta$, $\theta \leq e_c(\theta)$: $e_c \in \mathbb{CR}_1\{\mathbb{D}(C)|\mathbb{O}_2\}$.

To deal with the preservation of the class $\mathbb{D}_S$, it is convenient to consider first the stated result concerning the mapping t. This preserves both the class $\mathbb{D}_E$ and the partial ordering $\leq$ over $\mathbb{D}(C)$: for all $\theta \in \mathbb{D}(C)$, $t\{e_c(\theta)\} \in \mathbb{D}_E(C)$ and $t(\theta) \leq t\{e_c(\theta)\}$. If $\beta' \in \mathbb{D}_E(C)$ exists such that $t(\theta) \leq \beta'$ and for some $b,a \in C$, $b.t\{e_c(\theta)\}.a$ while the relationship $b.\beta'.a$ is false then, setting $\beta=t(\beta')$, $\beta \in \mathbb{D}_E(C)$ exists such that $\theta \leq \beta$ and for some $a,b \in C$, $a.e_c(\theta).b$ while the relationship $a.\beta.b$ is false. From the definition of

$e_c(\Theta)$, no such partial ordering $\flat$ exists. Hence $t\{e_c(\Theta)\} = e_c\{t(\Theta)\}$ and $e_c \in \mathbb{C}_1\{\mathbb{D}(C)|t\}$. That e_c preserves the class $\mathbb{D}_S$ in $\mathbb{D}(C)$ is a simple consequence of the fact that $e_c(\Theta) \in \mathbb{D}_E(C)$ and $\mathbb{D}_S \subset \mathbb{D}_E$. To prove that o_c does the same it is first remarked that the proof just given suffices to show that $t\{o_c(\Theta)\} = o_c\{t(\Theta)\}$ when $\Theta \in \mathbb{D}_S(C)$, $t(\Theta) = \Theta$ and then $t\{o_c(\Theta)\} = o_c(\Theta)$ so that $o_c(\Theta) \in \mathbb{D}_S(C)$ also. Since $e_c(\Theta) \in \mathbb{D}_E(C)$ and $\mathbb{D}_T \subset \mathbb{D}_E$, $e_c \in \mathbb{CP}_1\{\mathbb{D}(C)|\mathbb{D}_T\}$. Since e_c preserves the classes $\mathbb{D}_R, \mathbb{D}_T$ over $\mathbb{D}(C)$, it also preserves the class $\mathbb{D}_P \equiv \mathbb{D}_R \cap \mathbb{D}_T$ in $\mathbb{D}_\Phi(C)$; it also preserves $\mathbb{D}_S$, and hence $\mathbb{D}_E$, in $\mathbb{D}(C)$.

To deal with the preservation of the class $\mathbb{D}'$, it is convenient to deal first with the result of clause (Bid) and its counterpart. The definition of an equivalence closure ψ of $\Theta \in \mathbb{D}(C)$, and by implication that of the corresponding skew equivalence closure, given in the existence proof of clause (Bid), is nonconstructive: ψ is determined by a class to which ψ itself belongs. When $\Theta \in \mathbb{D}'(C)$ the explicit construction of a skew equivalence closure ψ of $\Theta \in \mathbb{D}(C)$ described in clause (Bid), and again by implication the similar construction of an equivalence closure of Θ, are possible; the system $K_\Theta(C)$ may be dispensed with. For each $g \in \hat{C}(\Theta)$, $C(\Theta|g)$ contains only finitely many members of C. The process of reduction of $\flat\{1_c\{C(\Theta|g)\}$ to produce a perfect partial

ordering ψ_g over $C(\theta|g)$ which includes θ_g terminates after finitely many stages for each $g \in \hat{C}(\theta)$. With Λ as described, $\psi = \sum_{g \in \Lambda} \psi_g$ is a skew equivalence relationship over C and includes θ. If β is another skew equivalence relationship over C which includes θ and is such that for some pair $a, b \in C$, $a.\psi.b$ while the relationship $a.\beta.b$ is false then, since the relationship $a.\psi.b$ is false when a and b belong to distinct ranges of the form $C(\theta|g)$, the a, b in question belong to the same range $C(\theta|g)$. For the g in question, set $\beta_g = 1_C\{C(\theta|g)\} \times \beta$. Since $\beta \in \mathbb{Q}_k(C) \cap \mathbb{Q}_t(C)$, β_g is both reflexive and transitive, and therefore (i) a skew equivalence relationship over $C(\theta|g)$. Also, since $\theta \leq \beta$ over C, (ii) β_g contains θ_g in $C(\theta|g)$. ψ_g possesses properties (i, ii) and hence the product $\beta_g \times \psi_g$ formed over $C(\theta|g)$ possesses them. Since $a.\psi_g.b$ for some $a, b \in C(\theta|g)$ for which the relationship $a.\beta_g.b$ is false, the relationship $a.\beta_g \times \psi_g.b$ is also false: ψ_g can be reduced to $\beta_g \times \psi_g$, a skew equivalence relationship over $C(\theta|g)$ which contains θ_g. ψ_g is defined by the property that such a reduction is impossible. Hence, with ψ as defined above, no skew equivalence relationship β over C which includes θ and is such that, for some pair $a, b \in C$, $a.\psi.b$ while the relationship $a.\beta.b$ is false exists: ψ is a skew equivalence closure of θ over C. That all skew equivalence

closures of $\theta \in \mathbb{D}(C)$ have the same graph is proved by considering the product $\psi \times \beta$ of two such closures which both contains θ and is a skew equivalence relationship. Unless $\psi = \beta$ either ψ or β is not minimal in the sense of clause (Gvi2) of Definition . The above theory, mutatis mutandis, may be applied to the equivalence closure of $\theta \in \mathbb{D}(C)$.

That the mappings o_c and e_c preserve the class $\mathbb{D}'$ in $\mathbb{D}(C)$ is a direct consequence of the above theory. The pairwise disjoint subsets $C', C'', \ldots$ serving to define θ as a member of $\mathbb{D}'(C)$ fulfil the same function with regard to $o_c(\theta)$ and $e_c(\theta)$. Since, from the counterpart to clause (Biia), $\theta \in \mathbb{D}_E(C)$ if and only if $e_c(\theta) = \theta$, it follows, in particular, that $e_c \in \mathbb{U}_1\{\mathbb{D}(C) | \mathbb{D}_E\}$. Since $e_c \in \mathbb{DP}_1\{\mathbb{D}(C) | \leq\}$ it follows that if $\theta \leq \psi$ then $e_c(\theta) \leq e_c(\psi)$. Also, since $e_c \in \mathbb{ID}_1\{\mathbb{D}(C)\}$, $e_c(\psi) \leq e_c\{e_c(\theta)\}$ $= e_c(\theta)$ when $\psi \leq e_c(\theta)$. The result of clause (Biic) has been obtained and clauses' (Bii) and (Biiia) have been disposed of.

By definition $\theta \leq o_c(\theta)$. Also $e_c(\theta) \in \mathbb{D}_E(C)$ and, in consequence, $e_c(\theta) \in \mathbb{D}_p(C)$. From clause (Biic), since $\theta \leq e_c(\theta)$, $o_c(\theta) \leq e_c(\theta)$. The counterpart to clause (Biic) then yields the result of clause (Biiib).

The partial ordering $\psi = \sum_{B \in \Delta}{}' 1_c(B)$ defined in clause (Biiic) is in $\mathbb{D}_E(C)$ and contains θ. Suppose that $\beta \in \mathbb{D}_E(C)$ with $\theta \leq \beta$ exists such that for some $a, b \in C$, $a . \psi . b$ while the relationship

$a.\flat.b$ is false. $a.\psi.b$ only when a and b both belong to a common associated subrange, $C(\theta|g)$ with $g \in \hat{C}(\theta)$ say. Either g is a or b, or one of the four pairs of relationships $a.\theta.g$, $b.\theta.g$ or ... or $g.\theta.a$, $g.\theta.b$ holds true. It is easily deduced that if the relationship $a.\flat.b$ is false then either $\theta \not\leq \flat$ or $\flat \notin \mathbb{D}_R(\mathbf{E})$ or $\flat \notin \mathbb{D}_S(C)$ or $\flat \notin \mathbb{D}_T(C)$. The above supposition is untenable and ψ is an equivalence closure of θ over $\mathbb{D}(C)$.

Clause (Cia) ⟨(Ciia)⟩ is simply a reformulation of the unbracketed part ⟨the bracketed part⟩ of clause (Iiic) of Definition in terms of the relevant graph sets. With $\theta, \psi \in \mathbb{D}(C)$, let $\theta \leq_{[\lambda,\rho]} \psi$, so that $\theta \leq \lambda \overline{+} \psi \overline{+} \rho$. If $\bar{0} \leq \lambda, \rho$ then $\theta = \bar{0} \overline{+} \theta \overline{+} \bar{0} \leq \lambda \overline{+} \theta \overline{+} \rho$: if $\lambda, \rho \in \mathbb{D}_R(C)$, $\leq_{[\lambda,\rho]} \in \mathbb{D}_R\{\mathbb{D}(C)\}$. Let $\theta \leq_{[\lambda,\rho]} \psi$ and $\psi \leq_{[\lambda,\rho]} \flat$ so that, in addition, $\psi \leq \lambda \overline{+} \flat \overline{+} \rho$. If $\bar{2}\lambda \leq \lambda$, $\bar{2}\rho \leq \rho$ then $\theta \leq \lambda \overline{+} \psi \overline{+} \rho \leq \lambda \overline{+} \lambda \overline{+} \flat \overline{+} \rho \overline{+} \rho = \bar{2}\lambda \overline{+} \flat \overline{+} \bar{2}\rho \leq \lambda \overline{+} \flat \overline{+} \rho$: if $\lambda, \rho \in \mathbb{D}_T(C)$, $\leq_{[\lambda,\rho]} \in \mathbb{D}_T\{\mathbb{D}(C)\}$. The condition $\lambda, \rho \in \mathbb{D}_P(C) \equiv \mathbb{D}_R(C) \cap \mathbb{D}_T(C)$ implies that $\leq_{[\lambda,\rho]} \in \mathbb{D}_R\{\mathbb{D}(C)\} \cap \mathbb{D}_T\{\mathbb{D}(C)\} \equiv \mathbb{D}_P\{\mathbb{D}(C)\}$. $\leq_{(\lambda,\rho)} \in \mathbb{D}_R\{\mathbb{D}^{(\lambda,\rho)}(C)\}$ from the definition of $\mathbb{D}^{(\lambda,\rho)}(C)$. Let $\theta \leq_{(\lambda,\rho)} \psi$, $\psi \leq_{(\lambda,\rho)} \flat$, so that $\lambda \overline{+} \theta \overline{+} \rho \leq \psi$, $\lambda \overline{+} \psi \overline{+} \rho \leq \flat$. If $\bar{0} \leq \lambda, \rho$ then $\lambda \overline{+} \theta \overline{+} \rho \leq \psi = \bar{0} \overline{+} \psi \overline{+} \bar{0} \leq \lambda \overline{+} \psi \overline{+} \rho \leq \flat$: if $\lambda, \rho \in \mathbb{D}_R(C)$ then $\leq_{(\lambda,\rho)} \in \mathbb{D}_T\{\mathbb{D}(C)\}$. The results of clauses (Cii, ii) have been disposed of.

If $\lambda' \leq \lambda$, $\rho' \leq \rho$ and $\theta \leq \lambda' \mp \psi \mp \rho'$ then $\theta \leq \lambda \mp \psi \mp \rho$. Also if $\lambda \mp \theta \mp \rho \leq \psi$ then $\lambda' \mp \theta \mp \rho' \leq \psi$. The two results of clause (Ciii) have been obtained.

The proofs of the results of clause (iv a) make use of the result that if $\Gamma \in A_2(C)$, $\mathbb{D}_2\{C, \theta|\Delta\}$, $\mathbb{DP}_2\{C|\theta\}$ where $\theta \in \mathbb{D}_1(C)$, then

$$a\Gamma(b\Delta c)\Gamma d.\theta.(a\Gamma b\Gamma d)\Delta(a\Gamma c\Gamma d)$$

for all $a,b,c,d \in C$. For, if $\Gamma \in \mathbb{RD}_2\{C,\theta|\Delta\}$ then $(b\Delta c)\Gamma d.\theta.(b\Gamma d)\Delta(c\Gamma d)$ for all $b,c,d \in C$. Accordingly, since $\Gamma \in \mathbb{DP}_2\{C|\theta\}$,

$$a\Gamma\{(b\Delta c)\Gamma d\}.\theta.a\Gamma\{(b\Gamma d)\Delta(c\Gamma d)\}$$

for all $a,b,c,d \in C$. Also, using the property $\Gamma \in \mathbb{LD}_2\{C,\theta|\Delta\}$,

$$a\Gamma\{(b\Gamma d)\Delta(c\Gamma d)\}.\theta.\{a\Gamma(b\Gamma d)\}\Delta\{a\Gamma(c\Gamma d)\}$$

The stated ~~result~~ relationship follows from use of the properties $\theta \in \mathbb{D}_1(C)$ and $\Gamma \in A_2(C)$. Since $\mp \in A_2\{\mathbb{D}(C)\}$, $\mathbb{D}_2\{\mathbb{D}(C), =|+\}$, $\mathbb{DP}_2\{\mathbb{D}(C)|=\}$ and $= \in \mathbb{D}_1\{\mathbb{D}(C)\}$, it follows that

$$\lambda \mp (\psi + \xi) \mp \rho = (\lambda \mp \psi \mp \rho) + (\lambda \mp \xi \mp \rho)$$

for all $\lambda, \psi, \xi, \rho \in \mathbb{D}(C)$. Thus if $\theta \leq \theta \mp \psi \mp \rho$, $\mu \leq \lambda \mp \xi \mp \rho$ then $\theta + \mu \leq (\lambda \mp \psi \mp \rho) + (\lambda \mp \xi \mp \rho) = \lambda \mp (\psi + \xi) \mp \rho$ or, if $\theta \leq_{[\lambda,\rho]} \psi$, $\mu \leq_{[\lambda,\rho]} \xi$ then $\theta + \mu \leq_{[\lambda,\rho]} (\psi + \xi) : \pm \mathbb{DP}_2\{\mathbb{D}(C)\,\xi|\leq_{[\lambda,\rho]}\}$.

Again, if $\lambda \mp \theta \mp \rho \leq \psi$, $\lambda \mp \mu \mp \rho \leq \xi$ then $\lambda \mp (\theta + \mu) \mp \rho = (\lambda \mp \theta \mp \rho) + (\lambda \mp \mu \mp \rho) \leq \psi + \xi$ or, if $\theta \leq_{(\lambda,\rho)} \psi$, $\mu \leq_{(\lambda,\rho)} \xi$ then $\theta + \mu \leq_{(\lambda,\rho)} (\psi + \xi)$: $+ \in \mathbb{OP}_2\{\mathbb{O}(C) | \leq_{(\lambda,\rho)}\}$. Since $\mp \in \mathbb{D}_2\{\mathbb{O}(C), \leq | x\}$, $\mathbb{OP}_2\{\mathbb{O}(C) | \leq\}$ and $\leq \in \mathbb{O}_1\{\mathbb{O}(C)\}$

$$\lambda \mp (\theta \times \mu) \mp \rho \leq (\lambda \mp \theta \mp \rho) \times (\lambda \mp \mu \mp \rho)$$

"Thus, if $\lambda \mp \theta \mp \rho \leq \psi$, $\lambda \mp \mu \mp \rho \leq \xi$ then $\lambda \mp (\theta \times \mu) \mp \rho \leq \psi \times \xi$: $\times \in \mathbb{OP}_2\{\mathbb{O}(C) | \leq_{(\lambda,\rho)}\}$.

Let $\theta \leq_{[\lambda, t(\lambda)]} \psi$ so that $\theta \leq \lambda \mp \psi \mp t(\lambda)$. Since $t \in \mathbb{OP}_1\{\mathbb{O}(C) | \leq\}$, $A\mathbb{C}\{\mathbb{O}(C) | \mp\}$, $R\{\mathbb{O}(C)\}$, it follows that $t(\theta) \leq \lambda \mp t(\psi) \mp t(\lambda)$: $t \in \mathbb{OP}_1\{\mathbb{O}(C) | \leq_{[\lambda, t(\lambda)]}\}$. Again the relationship $\lambda \mp \theta \mp t(\lambda) \leq \psi$ yields $\lambda \mp t(\theta) \mp t(\lambda) \leq t(\psi)$: $t \in \mathbb{OP}_1\{\mathbb{O}(C) | \leq_{(\lambda, t(\lambda))}\}$. If $\theta \leq_{[\lambda, t(\lambda)]} \psi$ then $t(\theta) \leq_{[\lambda, t(\lambda)]} t(\psi)$ and, since $+ \in \mathbb{OP}_2\{\mathbb{O}(C) | \leq_{[\lambda, t(\lambda)]}\}$, $\bar{t}(\theta) = \theta + t(\theta) \leq_{[\lambda, t(\lambda)]} (\psi + t(\psi)) = \bar{t}(\psi)$ also: $\bar{t} \in \mathbb{OP}_1\{\mathbb{O}(C) | \leq_{[\lambda, t(\lambda)]}\}$. That $\bar{t} \in \mathbb{OP}_1\{\mathbb{O}(C) | \leq_{(\lambda, t(\lambda))}\}$ and $t' \in \mathbb{OP}_1\{\mathbb{O}(C) | \leq_{(\lambda, t(\lambda))}\}$ are proved similarly. The results of clause (Civb) have been obtained.

Clause (Cva) is simply a reformulation of clause (Iiib) of Definition . Assuming that $\lambda \mp \theta \mp \rho \leq \theta$, $\lambda \mp \psi \mp \rho \leq \psi$ it follows from formulae (,) that $\lambda \mp (\theta + \psi) \mp \rho \leq \theta + \psi$,

$\lambda \mp (\theta \times \psi) \mp \rho \leq \theta \times \psi$. Accordingly $+, \times \in \mathbb{CP}_2\{\mathbb{D}(C) \mid \mathbb{D}^{(\lambda,\rho)}\}$. The two relationships $\theta \leq \lambda \mp \psi \mp \rho$, $\lambda \mp \psi \mp \rho \leq \psi$ yield, since $\leq \in \mathbb{D}_\tau\{\mathbb{D}(C)\}$, the relationship $\theta \leq \psi$: if $\psi \in \mathbb{D}^{(\lambda,\rho)}(C)$ and $\theta \leq [\lambda,\rho]\psi$, then $\theta \leq \psi$. Similarly, the relationships $\lambda \mp \psi \mp \rho \leq \psi$, $\psi \leq \beta$ yield the relationship $\lambda \mp \psi \mp \rho \leq \beta$: if $\psi \in \mathbb{D}^{(\lambda,\rho)}(C)$ and $\psi \leq \beta$ then $\psi \leq (\lambda,\rho)\beta$. The last part of clause (Cv) follows from the observation that if $\lambda \mp \theta \mp \rho \leq \theta$ and $\lambda' \leq \lambda$, $\rho' \leq \rho$ then $\lambda' \mp \theta \mp \rho' \leq \theta$ also.

Assuming that $\lambda \mp \theta \mp \rho \leq \theta$, $\lambda \mp \psi \mp \rho \leq \psi$ and $\bar{0} \leq \lambda, \rho$, $\theta \mp \psi = \theta \mp \bar{0} \mp \bar{0} + \psi \leq \theta \mp \rho \mp \lambda \mp \psi$ so that

$$\lambda \mp (\theta \mp \psi) \mp \rho \leq (\lambda \mp \theta \mp \rho) \bar{+} (\lambda \mp \psi \mp \rho) \leq \theta \bar{+} \psi$$

Accordingly, if $\lambda, \rho \in \mathbb{D}_R(C)$ then $\mp \in \mathbb{CP}_2\{\mathbb{D}(C) \mid \mathbb{D}^{(\lambda,\rho)}\}$. Assuming that $\bar{0} \leq \lambda' \leq \lambda$, $\bar{0} \leq \rho' \leq \rho$ and $\lambda \mp \theta \mp \rho \leq \theta$, $\theta = \bar{0} \mp \theta \mp \bar{0} \leq \lambda' \mp \theta \mp \rho' \leq \lambda \mp \theta \mp \rho \leq \theta$. Hence $\lambda' \mp \theta \mp \rho' = \theta$ and $f^{(\lambda',\rho')} \in \mathbb{U}_1\{\mathbb{D}(C) \mid \mathbb{D}^{(\lambda,\rho)}\}$. That $\lambda' \mp \theta = \theta \mp \rho' = \theta$ for all $\theta \in \mathbb{D}^{(\lambda,\rho)}(C)$ is proved in the same way. When $\bar{0} \leq \lambda, \rho$, ~~then~~ $\lambda \mp \theta \mp \rho = \theta$ and $\lambda \mp \psi \mp \rho = \psi$ for all $\theta, \psi \in \mathbb{D}^{(\lambda,\rho)}(C)$. The three conditions of clause (Cvic) are equivalent; clause (Cvi) has been dealt with.

For the proof of clause (Cvii) it suffices to remark that if $\lambda' \leq \lambda$, $\rho' \leq \rho$, $\lambda \mp \lambda \leq \lambda$ and $\rho \mp \rho \leq \rho$ then $\lambda' \bar{+} (\lambda + \rho) \mp \lambda' \leq (\lambda \mp \lambda) + (\rho \mp \rho) \leq \lambda + \rho$.

For each $x \in C$, $x \in R\{\theta[\mp]\psi|a\}$, i.e. $a.\theta[\mp]\psi.x$, if and only if $R(\theta|a) \cap L(\psi|x)$ is nonvoid. The formula for $R\{\theta[\mp]\psi|a\}$ given in clause (Di) is correct; that given for $L\{\theta[\mp]\psi|a\}$ is dealt with similarly. For each $x \in C$, $x \in R\{\theta[\mp)\psi|a\}$, i.e. $a.\theta[\mp)\psi.x$, if and only if $R(\theta|a) \subseteq L(\psi|x)$, i.e. if and only if $R(\theta|a) \setminus L(\psi|x)$ is void. The formula for $R\{\theta[\mp)\psi|a\}$ given in clause (Di) is correct as are, for similar reasons, those given for $L\{\theta[\mp)\psi|a\}$, $R\{\theta(\mp]\psi|a\}$ and $L\{\theta(\mp]\psi|a\}$. For each $x \in C$, $x \in R\{\theta(\mp)\psi|a\}$, i.e. $a.\theta(\mp)\psi.x$, if and only if $R(\theta|a) \doteq L(\psi|x)$, i.e. if and only if $\{R(\theta|a) \cup L(\psi|x)\} \setminus \{R(\theta|a) \cap L(\psi|x)\}$ is void. The formula for $R\{\theta(\mp)\psi|a\}$ given in clause (Di) is correct; that given for $L\{\theta(\mp)\psi|a\}$ is dealt with similarly.

The proof of clause (Diia) is based upon the remark that $S \subseteq R(x)$ for all $x \in T$ if and only if $T \subseteq L(y)$ for all $y \in S$. First, assume that $T \subseteq L(y)$ for all $y \in S$. Select $y \in S$. All $x \in T$ are in $L(y)$: the chosen $y \in S$ lies in $R(x)$ for all $x \in T$. This conclusion holds for all $y \in S$. The condition $T \subseteq L(y)$ for all $y \in S$ implies that $S \subseteq R(x)$ for all $x \in T$. The converse assertion is proved in the same way. $T \subseteq \bigcup_C x \underline{d} \{S \setminus R(x)\}$ if and only if $S \subseteq R(x)$ for all $x \in T$. $S \subseteq \bigcup_C y \underline{d} \{T \setminus L(y)\}$ if and only if $T \subseteq L(y)$ for all $y \in S$. The result of clause (Diia) follows.

The operators $[\mp)$ and $(\mp]$ are mutually associative with respect to the equivalence relationship $=$ over $\mathfrak{D}(C)$ if and only if $\theta[\mp)\{\psi(\mp]\beta\}=\{\theta[\mp)\psi\}(\mp]\beta$ for all $\theta,\psi,\beta\in\mathfrak{D}(C)$, i.e. if and only if, for each triplet $\theta,\psi,\beta\in\mathfrak{D}(C)$, $a.\theta[\mp)\{\psi(\mp]\beta\}.d$ if and only if $a.\{\theta[\mp)\psi\}(\mp]\beta.d$ for each pair $a,d\in C$. The first of the latter conditions holds if and only if $R(\theta|a)\subseteq \bigcup_c \int x\underline{d}\{L(\beta|d)\setminus R(\psi|x)\}$; the second holds if and only if $L(\beta|d)\subseteq \bigcup_c \int x\underline{d}\{R(\theta|a)\setminus L(\psi|x)\}$. That these two conditions are equivalent follows from the result of clause (Diia) by setting $T=R(\theta|a)$, $S=L(\beta|d)$, $R(x)=R(\psi|x)$ and $L(x)=L(\psi|x)$.

Definition (i) Let $S\in\mathfrak{S}$. Let S' be a further system of numbers over which

(a) a determinative partial ordering, denoted by $:\sim:$ is defined

(b) addition is defined between some pairs of numbers, and

(c) multiplication is defined for all pairs.

A relationship $\simeq$ establishing correspondences between elements of S and S' which is such that

(1) each member a of S $\langle a'$ of $S'\rangle$ features in at least one correspondence $a\simeq a'$ and, with $a\simeq a'$, $b\simeq b'$,

(2) $a=b$ in S if and only if $a'\simeq b'$ in S',

(3) if a and b belong to the same S-stratum, the formation of $a'+b'$ is possible in S' and $a+b\simeq a'+b'$, and

(4) $ab \simeq a'b'$

is an isomorphism between S and S'. The notation $S \sim S'$ indicates that such an isomorphism exists.

iia) An isomorphism $S \sim S'$ between two S-rings which is such that, for (j:), all members of S in a fixed stratum $S\{j\}$ correspond to members of S' in a fixed stratum, $S'\{m_j\}$ say, is said to be stratum preserving. (b) If, in the preceding, $m_j = j$ $\langle m_j = -j \rangle$ the isomorphism is said to be a direct $\langle$a reflecting$\rangle$ stratum preserving isomorphism.

(iii) The notation $S \lesssim S''$ indicates that $S \sim S'$ where $S' \leqq S''$.

Theorem . (i) With $S \in \mathfrak{S}$ and $S \sim S'$ (a) the partial ordering over S' featuring in the definition of the isomorphism is an equivalence relationship and (b) S' may be exhibited as an S-ring.

(ii) A stratum preserving isomorphism between two S-rings is either direct or reflecting

(iii) Among S-rings stratum preserving isomorphism is an equivalence relationship

(iv) Let two isomorphisms, $\underline{\sim}$ and $\equiv$, either both direct stratum preserving or both reflecting stratum preserving, subsist between the two S-rings S and S'. Let $\hat{a}, \hat{a}'; \hat{b}, \hat{b}'; \ldots$ be the pairs common to the two systems of pairs $a, a'; b; b'; \ldots$ and

$\tilde{a}, \tilde{a}'; \tilde{b}, \tilde{b}'; \ldots$ featuring in the correspondences $a \simeq a', b \simeq b', \ldots$ and $\tilde{a} \equiv \tilde{a}', \tilde{b} \equiv \tilde{b}', \ldots$ between the members $a, b, \ldots; \tilde{a}, \tilde{b}, \ldots$ of S and $a', b', \ldots; \tilde{a}', \tilde{b}', \ldots$ of S'. Let $\hat{S} \langle \hat{S}' \rangle$ be the system of numbers $\hat{a}, \hat{b}, \ldots$ of $S \langle \hat{a}', \hat{b}', \ldots$ of $S' \rangle$. Then $\hat{S}, \hat{S}' \in \mathfrak{S}$ and $\hat{S} \sim \hat{S}'$, the isomorphism having the same stratum preserving character as that of $\simeq$ and $\equiv$.

Proof. In clause (i) of Definition , the assumption that the partial ordering over S' is determinative gives meaning to the subsequent condition that $a \prec b$ in S if and only if $a' :\sim: b'$ in S'. However, it is possible to show that the partial ordering over S' is an equivalence relationship. With $a \simeq a'$ and $b \simeq b'$, $a' :\sim: b'$ if $a \prec b$. To each $a' \in S'$ at least one $a \in S$ for which $a \simeq a'$ corresponds. $a \prec a$ for all $a \in S$ and hence $a' \prec a'$ for all $a' \in S'$; the partial ordering over S' is reflexive. Again, to any $a', b' \in S'$ members $a, b \in S$ for which $a \simeq a', b \simeq b'$ correspond. If $a' :\sim: b'$ in S', $a \prec b$ in S and, since equality in S is symmetric, $b \prec a$ in S and accordingly $b' :\sim: a'$ in S': the partial ordering in S' is symmetric. Lastly, to any $a', b', c' \in S'$, members a, b, c of S for which $a \simeq a', b \simeq b', c \simeq c'$ correspond. If $a' :\sim: b', b' :\sim: c'$ in S', then $a \prec b, b \prec c$ in S; hence $a \prec c$ in S and $a' :\sim: c'$ in S': the partial ordering over S' is transitive. If, for some $a' \in S'$, $a \simeq a', b \simeq a', \ldots$ then, since $a' :\sim: a'$ in S', $a \prec b, \ldots$ in S; thus all $a, b, \ldots$ in S

corresponding to a' in S' belong to the same stratum. The members of S' may be arranged in strata corresponding to those in which their associates in S belong. In one of these strata, $S'\{j\}$ say, select a', b', c'. $a, b, c \in S\{j\}$ exist such that $a \simeq a'$, $b \simeq b'$, $c \simeq c'$. Since a and b belong to the same S-stratum, the sum a'+b' can be formed in S', and $a+b \simeq a'+b'$. Again, a+b and c belong to the same S-stratum, (a'+b')+c' can be formed in S', and $(a+b)+c \simeq (a'+b')+c'$. Similarly, $a+(b+c) \simeq a'+(b'+c')$. Since the two expressions involving a, b and c are equal in S, addition in $S'\{j\}$ is associative. At least one number in $S'\{j\}$ corresponds to $0_{(j)}$ in $S\{j\}$; call this number in $S'\{j\}$ $0'_{(j)}$. Select $a' \in S'\{j\}$, so that $a \in S\{j\}$ for which $a \simeq a'$ exists. $a+0_{(j)} \simeq a' + 0'_{(j)}$ and, since $a+0_{(j)}=a$ in S, $a'+0'_{(j)}=a' : S'\{j\}$ contains a zero member $0'_{(j)}$. Again, selecting $a' \in S'\{j\}$ and $a \in S\{j\}$ for which $a \simeq a'$, $S\{j\}$ also contains $-a$ for which $a+(-a)=0_{(j)}$ in S. $S'\{j\}$ contains a correspondent, $(-a')$ say, to $(-a)$ and $a+(-a) \simeq a'+(-a')$. Hence to each a' in $S\{j\}$ a number $(-a') \in S'\{j\}$ for which $a'+(-a') \simeq 0'_{(j)}$ corresponds. The numbers in $S'\{j\}$ form an additive Abelian group which is easily shown to be commutative. That multiplication in S' allocates products among strata

according to the rule $a' \in S'\{j\}$, $b' \in S'\{k\}$, $a'b' \in S'\{j+k\}$ and is also distributive with respect to addition may be proved by the use of methods illustrated above. The results of clause (i) have been disposed of.

In clause (ii) of Definition , isomorphism between an S-ring S and a further system S' permitting addition between certain of its members is defined. S' may be a further S-ring; addition within strata of S' is then defined. However, in special cases it may occur that the members of S' are such that addition between distinct strata is also possible; in such a case an isomorphism between S and S' which is not stratum preserving may exist. (It is then a consequence of clause (i) that the members of the S-ring S' may be rearranged to form another S-ring.) Assume that the isomorphism $S \sim S'$ between $S, S' \in \mathbb{S}$ is stratum preserving. Since all members of all strata in S' have pre-images in S, m_j is an integer valued function defined for all integers j and assumes all integer values. Let $a \in S\{j\}$, $b \in S\{k\}$ and $a \simeq a'$, $b \simeq b'$. Then $ab \in S\{j+k\}$, since $S \in \mathbb{S}$ and, since $ab \simeq a'b'$, $a'b' \in S'\{m_{j+k}\}$. However, $S' \in \mathbb{S}$, so that $a'b' \in S'\{m_j + m_k\}$; hence $m_{j+k} = m_j + m_k$ for all finite integers j, k. Setting $k=0$, the relationship $m_j = m_0 + m_j$ reveals that $m_0 = 0$. Setting $k=1$, $m_{j+1} - m_j = m_1$ so that $m_j = j m_1$.

Since m_j assumes all finite integer values m_i can only take the values ± 1. Accordingly either $m_j = j$ or $m_j = -j$. The result of clause (ii) has been disposed of.

Let the S-rings $S, S', \ldots$ be given. For each pair, S and S' say, either a correspondence of the form $a \simeq a'$ between the elements of S and S' with properties described in clauses (i, ii) of Definition exists or no such correspondence exists: the stratum preserving relationship $S \sim S'$ is determinative. The direct correspondence $a \simeq a$ establishes a stratum preserving isomorphism between S and a copy of itself: stratum preserving isomorphism is a reflexive relationship. The correspondences $a \simeq a'$, $b \simeq b'$, ... which establish the isomorphism $S \sim S'$ may be reversed in the form $a' \equiv a$, $b' \equiv b$, .. and in this form establish a correspondence between the elements $a', b', \ldots$ of S' and those, $a, b, \ldots$, of S. Each $a' \in S' \langle a \in S \rangle$ is involved in at least one correspondence of the form $a' \equiv a$. Also, from property (2) of clause (i) of Definition , $a' : \nu : b'$ in S' if and only if $a = b$ in S. Similarly, the correspondence $ab \simeq a'b'$ may be reversed to show that $a'b' \equiv ab$; the reversed correspondence is also preserved during addition. The correspondence $a' \equiv a$ has the properties ascribed to the correspondence $a \simeq a'$ in ~~Defin~~ clause (i) of Definition . If the relationship $\simeq$ is stratum preserving, so is the

relationship $\equiv$: stratum preserving isomorphism is a symmetric relationship. Finally, suppose that the correspondences $a \simeq a'$, $b \simeq b'$, ... establish the isomorphism $S \simeq S'$ as described in Definition, and that the correspondences $a' \equiv a''$, $b' \equiv b''$, ... do the same to establish the isomorphism $S' \simeq S''$. Select $a \in S$. Since $S \simeq S'$, there exists an $a' \in S'$ such that $a \simeq a'$, and in turn, since $S' \simeq S''$, there exists an $a'' \in S''$ such that $a' \equiv a''$; this process is defined for each $a \in S$ and establishes a relationship between the members a of S and certain members a'' of S'', written in the form $a \leftrightarrow a''$, in which all $a \in S$ feature. The same relationship, which is established by the existence of a common member $a' \in S'$ in two relationships of the form $a \simeq a'$, $a' \equiv a''$, may be defined by preliminary selection of $a'' \in S''$: all $a'' \in S''$ feature in the relationships $a \leftrightarrow a''$. If, in conjunction, $a \simeq a'$, $a' \simeq a''$ and $b \simeq b'$, $b' \equiv b''$ then $a = b$ in S if and only if $a' = b'$ in S', and $a' = b'$ in S' if and only if $a'' = b''$ in S''. Hence, if $a \leftrightarrow a''$ and $b \leftrightarrow b''$, $a = b$ in S if and only if $a'' = b''$ in S''. That the relationship $\leftrightarrow$ possesses the further properties with respect to S and S'' attributed to $\simeq$ with respect to S and S' in Definition is verified in the same way. The relationship $\leftrightarrow$ also preserves strata: stratum preserving isomorphism is a transitive relationship. Clause (iii) has been proved.

To prove clause (iv), let the two systems $a, b, \ldots$ and $\hat{a}, \hat{b}, \ldots$ both constitute the complete system of numbers to be found in the stratum $S\{j\}$ of S. Their images under the isomorphisms $\simeq$ and $\equiv$, $a', b', \ldots$ and $\tilde{a}', \tilde{b}', \ldots$ both constitute the complete system of numbers to be found in one of the strata $S'\{j\}$ or $S'\{-j\}$ of S'; assume, for simplicity, the former case to hold (the treatment of the second case is similar). The members of $S'\{j\}$ form an additive Abelian group with zero member $O'_{(j)}$. As was shown in the proof of clause (i), the image of $O_{(j)}$ under the isomorphism $\simeq$ is the zero member of the group $a', b', \ldots$, in this case $O'_{(j)}$. Similarly, the image of $O_{(j)}$ under the isomorphism $\equiv$ is also $O'_{(j)}$. Hence $O_{(j)}$ and $O'_{(j)}$ belong to the systems $\hat{S}$ and $\hat{S}'$ respectively. Let $\hat{a}, \hat{b}, \ldots$ and $\hat{a}', \hat{b}', \ldots$ be the members of $\hat{S}$ and $\hat{S}'$ to be found in $S\{j\}$ and $S'\{j\}$ respectively. Then $\hat{a}+\hat{b} \simeq \hat{a}'+\hat{b}'$ and $\hat{a}+\hat{b} \equiv \hat{a}'+\hat{b}'$: both systems $\hat{a}, \hat{b}, \ldots$ and $\hat{a}', \hat{b}', \ldots$ are closed with respect to addition which, as in the proof of clause (i), may be shown to be associative and commutative. $\hat{O}_{(j)}$ and $\hat{O}'_{(j)}$ belong to these systems. Let $(-\hat{a}) \simeq b'$ and $(-\hat{a}) \equiv c'$. Then $\hat{a}+(-\hat{a}) \simeq \hat{a}+b'$ [illegible]. But $\hat{a}+(-\hat{a}) = O_{(j)}$

so that $\hat{a}+b'=O'_{(j)}$. Similarly, using the isomorphism $\equiv$, $\hat{a}+c'=O'_{(j)}$. Hence $b'=c'$ in S'. The systems $\hat{a},\hat{b},\dots$ and $\hat{a}',\hat{b}',\dots$ contain the numbers $-\hat{a},-\hat{b},\dots$ and $-\hat{a}',-\hat{b}',\dots$ respectively: these systems form additive Abelian groups. Taking $\hat{a}$ and $\hat{b}$ to be any two members of $\hat{S}$, corresponding to $\hat{a}'$ and $\hat{b}'$ in S', $\hat{a}\hat{b}\simeq\hat{a}'\hat{b}'$ and $\hat{a}\hat{b}\equiv\hat{a}'\hat{b}'$: the systems $\hat{S}$ and $\hat{S}'$ are closed with respect to multiplication, products being distributed among strata according to the laws holding for S and S'. In conclusion, $\hat{S}$ and $\hat{S}'$ are S-rings. The stratum preserving isomorphism holding between them is determined by the relationships $\simeq$ and $\equiv$ which jointly hold between the members of $\hat{S}$ and $\hat{S}'$.

Definition. A) With $M,N\subseteq S\in\mathbb{S}$, MN is void when either M or N is void and is otherwise the complete system of products ab each formed from one member $a\in M$ and one member $b\in N$; when $N=M$, MN is written as M^2; when M is the single number a, MN is written as aN.

Bi) A multiplicative S-module is a non-void system $M\subseteq S\in\mathbb{S}$ such that $M^2\subseteq M$. $\mathbb{M}(S)$ is the class of multiplicative S-modules in S.

ii) A multiplicative S-module $M\subseteq S$ for which $M(S\setminus M)\subseteq S\setminus M$ is saturated ⟨for which $(S\setminus M)^2\subseteq S\setminus M$ is independent⟩.

$M_S(S)\langle M_U(S)\rangle$ is the class of saturated ⟨independent⟩ multiplicative $\mathfrak{S}$-modules in S.

iii) With $M \in M(S)$ prescribed, a stratum of S consisting exclusively of members of M is an M-stratum: the notation $S\{j\} \subset M$ indicates that this true of the stratum $S\{j\}$; the notation $S\{j\} \not\subset M$ indicates that at least one member in $S\{j\}$ is not a member of M.

(ii) Let $C, M \leq S$. (a) The inner quotient $C/'M$ of C by M is the largest system $X \leq S$ satisfying the relationship $MX \leq C$. (b) $C/''M = S \setminus \{(S \setminus C)/'M\}$ is the outer quotient of C by M.

ii) With $C \leq S$, $C/'S$ is the partial closure of C in S.

Di) Let $R, C \leq S$ with $R \not\leq C$. (a) $\mathfrak{D}(R,C) = R \cap \{(S \setminus C)/'(R \setminus C)\}$ is the nontorsive part of R with respect to C. (b) $\overline{T}(R,C) = R \cap \{C/''(R \setminus C)\}$. $T(R,C) = \overline{T}(R,C) \setminus C$ is the torsive part of R with respect to C (c) $T_S(C) = \cup\{b \cap C/'b\}$ $(b \in R \setminus C)$ is the self-torsive part of R with respect to C. $\overline{T}_S(R,C) = T_S(R,C) \cup C$.

Theorem (Ai) Let $K \leq M \leq S$ and $L \leq N \leq S$. Then $KL \leq MN$

(ii) Let $L, M, N \leq S$. (a) $L\{M \cup N\} = LM \cup LN$. (b) $L\{M \cap N\} \leq LM \cap LN$.

(iii) If $N \leq S[0]$ then $MN \leq S[0]$

Bi) $0_{(0)}, S[0] \in M(S)$ and $S \in M_S(S) \cap M_U(S)$

(ii) Let $M, N \in M(S)$. Then $MN, M \cap N \in M(S)$

(iii) If $M, N \in M(S)$ are complementary in S, they are both independent multiplicative modules.

(iv) Let $M \in M_c(S)$. (a) If S contains one M-stratum, it also

contains an unbounded sequence of strata which may contain strata not belonging to M, but contains no two neighbouring strata not in M. (b) Let ~~$A \in$~~ S contain an M-stratum and also the strata $S\{h(\tau)\} \notin M\ (\tau : n)$, where the greatest common divisor of the $h(\tau)$ is h. S contains a sequence of M-strata whose orders form an unbounded arithmetic progression with difference h. (c) If $S\{k\} \subset M$ and $k = hj$ with $h \geq 1$, then $S\{j\} \subset M$ also. (d) Let $S\{h(\tau)\} \notin M\ (\tau : n)$, the $h(\tau)$ being as in (b) above. S contains a sequence of strata not in M whose orders form an unbounded arithmetic progression with difference h.

C) In this part, take $M \leq S$ and, where relevant, $N \leq S$, to be nonvoid

i) $C/'M$ $\langle C/''M \rangle$ is the complete system of numbers $a \in S$ such that $ab \in C$ for all $b \in M$ $\langle$for at least one $b \in M\rangle$

iia) $(S \setminus C)/'M \equiv S \setminus (C/''M)$. (b) $C/'b \equiv C/''b$ for all $b \in S$. (c) $C/'M \leq C/''M$.

iiia) k being a fixed finite integer, let $S_k[M]$ be the system of strata $S\{j+k\}$ obtained by displacement from all strata $S\{j\}$ of $S[M]$. Let C contain the zero element in each stratum of $S_k[M]$. Then $C/'M$ contains all elements of $\overline{S[0]}$ in $S\{k\}$, in particular $O(k)$; a fortiori $C/'M$ is nonvoid. (b) If $S[0] \leq C$, $\overline{S[0]} \leq C/'M$. (c) If C contains one member of $S[0]$, $C/''M$ is nonvoid.

iv) $M(C/'M) \leq C \leq (CM)/''M$

v) $(C/'M)(D/'N) \leq S^2 \cap (CD/'MN)$; a similar result holds with regard to the outer quotient.

vi) Let $M \subseteq N, C \subseteq D$. $C/'N \subseteq D/'M$ and $C/''M \subseteq D/''N$

viia) $(C \cap D)/'M = (C/'M) \cap (D/'N)$ (b) $(C \cap D)/''M \subseteq (C/''M) \cap (D/''N)$

viiia) If $C \subset M$, $C/'M \in M(S)$. b) If $C, M \in M(S)$, $C/''M \in M(S)$

c) $\overline{C} \in M(S)$ for all $C \subseteq M$

Dia) Let $R \nsubseteq C$. (a) $D(R,C)$ is composed of two complementary systems. The first, $D'(R,C)$ is the complete system of numbers $a \in R$ such that no $b \in R$ for which $ab \in C$ exists; the second is composed of all $a \in R$ for which $b \in R$ such that $ab \in C$ exist, but all such b are confined to C. (b) $\overline{T}(R,C) \langle \overline{T}_S(R,C) \rangle$ is the complete system of numbers $a \in R \langle a \in R \setminus C \rangle$ such that at least one $b \in R \setminus C$ exists for which $ab \in C \langle$ such that $a^2 \in C \rangle$.

ii) Let $R \nsubseteq C$. (a) The systems $D(R,C)$ and $\overline{T}(R,C)$ are complementary in R. (b) R is composed of four disjoint systems: $D(R,C) \setminus C$, $D(R,C) \cap C$, $C \cap R$ and $T(R,C)$

iii) Set $D' = D(R,A) \cap D(R,B)$. (a) $D' \subseteq D(R, A \cap B)$.

(b) $D' \subseteq D(R, A \cup B)$

iva) $\overline{T}(R, A \cap B) \subseteq \overline{T}(R,A) \cup \overline{T}(R,B)$. (b) $\overline{T}(R, A \cup B) \subseteq \overline{T}(R,A) \cup \overline{T}(R,B)$

v) Let $R \in M(S)$. (a) $D(R,C) \in M(S)$. (b) $D(R,C) \setminus C \in M(S)$ (c) If $C \in M(S)$, $(C \cap R)\overline{T}(R,C) \subseteq \overline{T}(R,C)$ and $C\overline{T}(R,C) \subseteq \overline{T}(R,C)$

vi) Let $R, C \subseteq S$ with $R \nsubseteq C$. a) Let ϕ be a function of two S-numbers such that for all $a, c \in C$ for which $\phi(a,c)$ is defined $\phi(a,c) \in C$. Let R contain a number

$1^{(c)}$ such that either (1) for all $b \in R$ for which $1^{(c)}b \in C$, $b = \phi(1^{(c)}b, u(b))$ for some $u(b) \in C$ or (2) for all $b \in R$, $b = \phi(1^{(c)}b, u(b))$ for some $u(b) \in C$ or (3) $u \in C$ exists such that $b = \phi(1^{(c)}b, u)$ for all $b \in R$ for which $1^{(c)}b \in C$, or (4) $u \in C$ exists such that $b = \phi(1^{(c)}b, u)$ for all $b \in R$. Then $1^{(c)} \in D(R, C)$. b) If R contains a number 1 such that $1a = a$ for all $a \in R$, then $1 \in D(R, C)$. c) Let $C \in M(S)$. Let R contain a number $1^{(c)}$ such that either (1) for all $b \in R$ for which $1^{(c)}b \in C$, $1^{(c)}u(b)b = b$ for some $u(b) \in C$ or (2) for all $b \in R$, $b = 1^{(c)}u(b)b$ for some $u(b) \in C$ or (3) for some fixed $u \in C$, $1^{(c)}ub = b$ for all $b \in R$ such that $1^{(c)}b \in C$ or (4) $u \in C$ and $1^{(c)}$ are complementary factors of the unit member 1 of clause (b), so that $u1^{(c)} = 1$. Then $1^{(c)} \in D(R, C)$. d) Let C be an additive module in the sense that for every pair a, c belonging to the same S-stratum, $a + c \in C$. Let R contain a number $1^{(c)}$ such that (1) for all $b \in R$ for which $1^{(c)}b \in C$, $b = 1^{(c)}b + u(b)$ for some $u(b) \in C$ or (2) for all $b \in R$, $b = 1^{(c)}b + u(b)$ for some $u(b) \in C$ or (3) $u \in C$ exists such that $b = 1^{(c)}b + u$ for all $b \in R$ for which $1^{(b)} \in C$ or (4) $u \in C$ exists such that $b = 1^{(b)}b + u$ for all $b \in R$. Then $1^{(c)} \in D(R, C)$.

Proof. For the proof of clause (Ai) it is remarked that KL is the system of numbers ab where $a \in K$, $b \in L$. When $K \subseteq M$, $L \subseteq N$ this system belongs to the complete system of numbers MN of numbers ab for which $a \in M$, $b \in N$. Using this result, $LM \subseteq L\{M \cup N\}$, $LN \subseteq L\{M \cup N\}$ and $LM \cup LN \subseteq L\{M \cup N\}$. Each $a \in L\{M \cup N\}$ has the form bc where $b \in L$ and c is a member of either M or N. If $c \in M$, $a \in LM$ and if $c \in N$, $a \in LN$; in both cases $a \in LM \cup LN$. Hence $L\{M \cup N\} \subseteq LM \cup LN$ and $L\{M \cup N\} \equiv LM \cup LN$. For the proof of the second result of clause (Aii), $L\{M \cap N\} \subseteq LM$ since $M \cap N \subseteq M$; also $L\{M \cap N\} \subseteq LN$ and in consequence $L\{M \cap N\} \subseteq LM \cap LN$. (That the relationship $L\{M \cap N\} \equiv LM \cap LN$ may be false is shown by the example in which $L \equiv a$, $M \equiv b$, $N \equiv c$ with $b \neq c$ and $ab = ac$; $L\{M \cap N\}$ is void and $LM \cap LN = ab$ is not.) For the proof of clause (Aiii) it has to be shown that with $a \in S\{j\}$, $aO_{(k)} = O_{(j+k)}$. But $aO_{(k)} + aO_{(k)} = a(O_{(k)} + O_{(k)}) = aO_{(k)} . S\{j+k\}$, to which $aO_{(k)}$ belongs, is an additive group. The equation $aO_{(k)} + x = aO_{(k)}$ has a unique solution, namely $x = O_{(j+k)}$. Hence $aO_{(k)} = O_{(j+k)}$.

That $O_{(0)} ; S[0] \in M(S)$ can be verified directly. When $M \equiv S$, $M(S \setminus M) \subseteq S \setminus M$ and $(S \setminus M)^2 \subseteq S \setminus M$, the expressions upon both sides of each relationship representing void systems.

Thus $S \in M_S(S) \cap M_U(S)$. When $M, N \in M(S)$, $(MN)^2 \equiv M^2N^2 \subseteq MN$. Also, using clause (Aiib), $(M \cap N)^2 \subseteq \{M(M \cap N)\} \cap \{N(M \cap N)\}$ $\subseteq M^2 \cap MN \cap N^2 \subseteq M^2 \cap N^2 \subseteq M \cap N$. Thus $MN, M \cap N \in M(S)$. When $N \equiv S \setminus M$ and $N \in M(S)$, $(S \setminus M)^2 \equiv N^2 \subseteq N \equiv (S \setminus M)$. Thus $M \in M_U(S)$ and $N \in M_U(S)$ for similar reasons. The results of clauses (Bi, ii, iii) have been disposed of.

Suppose that $S\{k\} \subset M$. If the two strata $S\{-h\}$ and $S\{k+h\}$ each possess a number not in M then, contrary to assumption, $S\{k\}$ possesses one number, namely the product of the two numbers in question, not in M. If $S\{-h\} \subset M$ $(h: -k+1;)$ the result of clause (Biva) certainly holds true. To complete the proof of this result, assume that for some $h > -k$, $S\{-h\} \not\subset M$ so that, repeating the above argument with k replaced by $k+h, k+2h, \ldots, S\{k+ih\} \subset M$ $(i: 1;)$. If $S\{-h'\} \not\subset M$ also then, replacing k by $k+ih$ and h by h' in the above argument, $S\{k+ih+jh'\} \subset M$ $(i, j: 1;)$.

In particular, if the sequence of strata $S\{-h\}$ $(h: -k+1;)$ contains two neighbouring strata $S\{-h\}, S\{-h-1\} \not\subset M$ then $S\{k+ih+j(h+1)\} \subset M$ $(i, j; 1)$. All integers $k', k'+1, \ldots$ where $k' = (h-1)(h+1)^2 + h^2$ are uniquely expressible in the form $k+ih+j(h+1)$ where $i = m(h+1) - \omega$ and $j = \omega$ $(m: h+1; | \omega: h(h+1))$ so that for these values of m and ω, $i, j > 0$. In short,

$S\{j\} \subset M$ for all $j \geq k'$, and again the result of clause (Biva) holds true. In the remaining alternative case, namely that in which for no h in the range $(h: -k+1;)$ is it true that $S\{-h\}, S\{-h-1\} \not\subset M$, the result in question also holds.

Since the conditions of clause (Bivb) are unimpaired if zero values are discarded from the integer sequence $\{h(\tau)\}$, it may be assumed that the $h(\tau)$ are nonzero. Replacing i, j by $i(1), i(2)$ and h, h' by $h(1), h(2)$ and extending the above argument, it can be shown that if $S\{k\} \subset M$ and $S\{h(\tau)\} \not\subset M$ $(\tau: n)$, then $S\{k + \sum_{\tau=1}^{n} i(\tau)h(\tau)\} \subset M$ $(\tau: n \mid i(\tau): 1;)$. The greatest common divisor of the $h(\tau)$ being h, the $h(\tau)$ satisfy a relationship of the form $\sum_{\tau=1}^{n} r(\tau)h(\tau) = h$, the $r(\tau)$ being integers for which, by suitably rearranging the $h(\tau)$ if necessary, it may be assumed that $r(\tau) < 0$ $(\tau: n' < n)$ and $r(\tau) > 0$ $(\tau: n'+1; n)$. Set $s = \prod_{\tau=1}^{n} h(\tau)$ and $m' = [(hs)^{-1}\{r(1)h(1)(1-s) + hs\}]$ and let $i(1) = mhs\, h(1)^{-1} + (\omega - 1)r(1)$, $i(\tau) = (\omega - s - 1)r(\tau)$ $(\tau: 2; n')$, $i(\tau) = \omega r(\tau)$ $(\tau: n'+1; n)$ so that for $(m: m'; \mid \omega: s)$ the $i(\tau)$ have positive values. Then

$$k + \sum_{\tau=1}^{n} i(\tau)h(\tau) = k - s\sum_{\tau=2}^{n'} r(\tau)h(\tau) + h\{ms + \omega\}$$

For $(m: m'; \mid \omega: s)$ the integer expressions upon the right hand side traverse an unbounded arithmetic progression with difference h: S contains a sequence of M-strata whose orders form such

a progressim. (It may be remarked that if from the set $\{i(\tau)\}$ an integer other than $i(1)$ is selected for representation differing from $(\omega-s-1)r(\tau)$, a term other than the first is discarded from the sum $\sum_{\tau=1}^{n'} r(\tau)h(\tau)$ in the formula equivalent to ( ). Since s is divisible by h however, an arithmetic progressim distinct from that represented by formula ( ) is not thereby produced.)

Under the conditions of clause (Bivc), either $S\{j\}\subset M$ or $S\{(h-1)j\}\subset M$. Assuming the first alternative to be false, it may be shown in turn that $S\{(h-m)j\}\subset M$ $(m:2;h-2)$ and in consequence that $S\{j\}\subset M$.

The proof of clause (Bivd) is based upon the observation that if $S\{h\}\not\subset M$, $S\{ih\}\not\subset M$ $(i:1;)$ also. The further required arguments are similar to those employed in the proof of clause (Bivb).

That $C/'M$ is the complete system of S-numbers a such that $abe C$ for all $b\in M$ is simply a reformulation of the definition of clause ( ) of Definition. Denote by Y the complete system of numbers $a\in S$ such that $abe C$ for at least one $b\in M$. $S\setminus Y$ is the complete system of numbers $d\in S$ such that $dM\leq S\setminus C$; hence $S\setminus Y=(S\setminus C)/'M$ and $Y=S\setminus\{(S\setminus C)/'M\}=C/''M$. Both results of clause (Ci) have been obtained.

The relationship of clause (Ciia) follows from the definition of $C/''M$. When M consists of the single element b, all $b \in M$ and at least one $b \in M$ are both the single number b. That $C/'M \leq C/''M$ when $C/'M$ is nonvoid follows from the alternative definitions of the systems involved offered by clause (Ci).

Let S_M be that part of S contained in $S[M]$, $\bar{O}_{(k)}$ be the elements of $\overline{S[o]}$ in $S\{k\}$ and $S_{k,M}[o]$ be the members of $S[o]$ in the strata $S_{k_z}[M]$. Then $M\bar{O}_{(k)} \leq S_M \bar{O}_{(k)} = S_{k,M}[o] \leq C$. $C/'M$ is the largest system $X \leq S$ for which $MX \leq C$; evidently $\bar{O}_{(k)} \leq C/'M$. When $S[o] \leq C$, $M\overline{S[o]} \leq S\overline{S[o]} \leq S[o] \leq C$, and $\overline{S[o]} \leq C/'M$. The results of subclauses (Ciiia, b) have been obtained. For the proof of subclause (Ciiic), let M contain the number $b \in S\{j\}$ and C contain $O_{(k)}$. $C/''M$ is the system of S-numbers a such that $ab \in C$ for at least one $b \in M$, and under the stated assumptions $a = O_{(j-k)}$ is one of these numbers: $C/''M$ is nonvoid.

$C/'M$ satisfies the relationship $M(C/'M) \leq C$ by definition. $(CM)/'M$ is the largest system X satisfying the relationship $MX \leq CM$, and evidently $C \leq X$ for this X. The results of clause (Civ) have been derived.

$C/'M$, $D/'N$ are the largest systems X, Y satisfying the

relationships $MX \subseteq C$, $NY \subseteq D$ and then $MNXY \subseteq CD$. Accordingly the system $Z = XY$ satisfies the relationship $MNZ \subseteq CD$; $CD/'MN$, the largest system Z to do so is nonvoid, and $(C/'M)(D/'N) \subseteq (CD)/'(MN)$. Also $(C/'M)(D/'N) \subseteq S^2$.

$C/''M\langle D/''N\rangle$ is the largest system of numbers a_1 $\langle a_2\rangle$ such that $b_1(a_1) \in M \langle b_2(a_2) \in N\rangle$ for which $a_1 b_1(a_1) = c_1(a_1, b_1) \in C \langle a_2 b_2(a_2) = c_2(a_2, b_2) \in D\rangle$ exists. All $b \in MN$ $\langle c \in CD\rangle$ have the form $b_1 b_2 \langle c_1 c_2\rangle$ where $b_1 \in M$, $b_2 \in N$ $\langle c_1 \in C, c_2 \in D\rangle$. The members $a_1 a_2 = a$ of the system $(C/''M)(D/''N)$ satisfy the relationship $b_1(a_1) b_2(a_2) a = c_1(a_1, b_1) c_2(a_2, b_2)$ where $b_1(a_1) b_2(a_2) \in MN$, $c_1(a_1, b_1) c_2(a_2, b_2) \in CD$. The largest system $CD/''MN$ of numbers a satisfying a relationship of the form $ab = c$ where $b \in MN$, $c \in CD$ is accordingly nonvoid and all members of $(C/''M)(D/''N)$ belong to it. Again all members of the latter system belong to S^2. Both results of clause (Cv) have been obtained.

$C/'N$ is the largest system X to satisfy the relationship $NX \subseteq C$. If $M \subseteq N$ and $C \subseteq D$, it follows that $MX \subseteq NX \subseteq C \subseteq D$. Accordingly $Y = C/'N$ satisfies the relationship $MY \subseteq D$ and belongs to the largest system to do so, namely $D/'M$. Each member a of $C/''M$ satisfies a relationship of the form $ab(a) = c(a, b)$ where $b(a) \in M$, $c(a, b) \in C$. When $M \subseteq N$,

$C \subseteq D$, $b(a) \in N$ and $c(a,b) \in D$. Accordingly all members of $C/''M$ belong to $D/''N$. Both results of clause (Cvi) have been disposed of.

$C/'M\,\langle D/'M\rangle\{(C \cap D)/'M\}$ is the largest system $X\langle Y\rangle\{Z\} \subseteq S$ satisfying the relationship $MX \subseteq C\langle MY \subseteq D\rangle \{MZ \subseteq C \cap D\}$. For these X and Y, using clause (Aiib), $M(X \cap Y) \subseteq MX \cap MY \subseteq C \cap D$. Hence $(C/'M) \cap (D/'M) \subseteq (C \cap D)/'M$. Since $C \cap D \subseteq C$, it follows from clause (Cva) that $(C \cap D)/'M \subseteq C/'M$; similarly $(C \cap D)/'M \subseteq D/'M$ and the result of clause (Cviia) follows. That of clause (Cviib) follows from clause (Cvib) again using the relationships $C \cap D \subseteq C, D$.

$X = C/'M$ is the largest system in S satisfying the relationship $MX \subseteq C$. If $C \subseteq M$, $MX^2 \equiv (MX)X \subseteq CX \subseteq MX \subseteq C$. Hence $X^2 \subseteq X$ and $C/'M \in M(S)$, the result of clause (Cviiia). If $C, M \in M(S)$, $C^2 \subseteq C$ and $M^2 \subseteq M$. From clauses (,) it follows that $(C/''M)^2 \subseteq C^2/''M^2 \subseteq C/''M$, so that $C/''M \in M(S)$, the result of clause (Cviiib). Clause (Cviic) is a simple corollary to clause (Cviia) obtained by setting $M = S$.

Let Y be the system of S-numbers a such that, for all $b \in R$, $ab \in C$ only when $b \in C$. For each $y \in Y$, $(R \setminus C)y \subseteq S \setminus C$. $(S \setminus C)/'(R \setminus C)$ is the largest system X for which

$(R\setminus C)X \subseteq (S\setminus C)$. Hence $Y \subseteq (S\setminus C)/'(R\setminus C)$. Select $a \in (S\setminus C)/'(R\setminus C)$. If, with $b \in S$, $ab \in C$, then $b \notin R\setminus C$, since $(R\setminus C)a \subseteq (S\setminus C)$; hence $b \in S\setminus(R\setminus C) = (S\setminus R)\cup(R\cap C)$. Thus for all $b \in R$ for which $ab \in C$, $b \in C$; accordingly $(S\setminus C)/'(R\setminus C) \subseteq Y$ and $(S\setminus C)/'(R\setminus C) = Y$. $R\cap Y$ is the complete system of numbers $a \in R$ with the properties described in clause (Dia), and this system is $D(R,C)$.

The complete system of S-numbers a such that at least one $b \in R\setminus C$ for which $ab \in C$ exists is, by definition, $T' = C/''(R\setminus C)$, and the subsystem belonging to R of these numbers is $R\cap T' = \overline{T}(R,C)$.

$b\cap(C/'b)$ is nonvoid if and only if $b^2 \in C$ and, if nonvoid, is simply b. Hence $U\{b\cap(C/'b)\}$ $(b \in R\setminus C)$ is the complete system of numbers b in $R\setminus C$ for which $b^2 \in C$. All results of clause (Di) have been obtained.

Since the systems $(S\setminus C)/'(R\setminus C)$ and $C/''(R\setminus C) = S\setminus\{(S\setminus C)/'(R\setminus C)\}$ are complementary in S, $D(R,C)$ and $\overline{T}(R,C)$ are complementary in R, the result of clause (Dïa). $D(R,C) = \{D(R,C)\setminus C\}\cup\{D(R,C)\cap C\}$, $\{D(R,C)\setminus C\}$ and $\{D(R,C)\cap C\}$ being disjoint and $\overline{T}(R,C) = T(R,C)\cup\{R\cap C\}$, the systems $T(R,C)$ and $R\cap C$ being disjoint. The result of clause (Dïb) is a corollary to its predecessor.

Select $a \in D' = D(R,A) \cap D(R,B)$ and $b \in R$ such that $ab \in A \cup B$, so that either $ab \in A$ or $ab \in B$. If $ab \in A$, $b \in A$ since $a \in D(R,A)$; similarly $b \in B$ if $ab \in B$. In both cases $b \in A \cup B$. Thus for each $a \in D'$ and for all $b \in R$, $ab \in A \cup B$ only when $b \in A \cup B$. Accordingly all such a belong to $D(R, A \cup B)$. Similarly, when $A \cap B$ is nonvoid, the conditions $b \in R$ and $ab \in A \cap B$ imply that $b \in A$ and $b \in B$, so that $b \in A \cap B$. Hence all such a belong to $D(R, A \cap B)$. Both results of clause (Diii) have been obtained.

The results concerning the transitive systems of clause (Dva) are derived by forming the complements in R of the systems occurring in relationships given in clause (Diii) and using the complementarity relationship of clause (Dii).

$X_R = D(R,C)$ is the largest system satisfying the relationships $X_R \subseteq R$ and $(R \setminus C)X_R \subseteq (S \setminus C)$. Hence $(R \setminus C)X_R \subseteq (R \setminus C)R \subseteq R^2 \subseteq R$ when $R \in M(S)$ and $(R \setminus C)X_R \subseteq R \cap (S \setminus C) = R \setminus C$: X_R is the largest system satisfying the relationship $(R \setminus C)X_R \subseteq R \setminus C$. Then $(R \setminus C)X_R^2 = \{(R \setminus C)X_R\}X_R \subseteq (R \setminus C)X_R \subseteq R \setminus C$, so that $X_R^2 \subseteq X_R$ and $X_R \in M(S)$, the result of clause (Dva). If $D(R,C) \setminus C$ is void, $D(R,C) \setminus C \in M(S)$. Assuming that $D(R,C) \setminus C$ is nonvoid, select $a, b \in D(R,C) \setminus R$ so that $ab \in D(R,C)$. If $ab \in C$, $b \notin C$ exists such that $ab \in C$; accordingly $a \notin D(R,C)$, contrary to assumption.

Hence $ab \in D(R,C) \setminus C$ and $D(R,C) \setminus C \in M(S)$. To deal with the last result of clause (Dv), select $a \in \overline{T}(R,C)$ and $c' \in R \cap C$. $b \in R \setminus C$ for which $ab = c$ exists and, for this b, $(ac')b = cc' \in C$ since $c \in M(S)$. Also $ac' \in R$ since $\overline{T}(R,C) \subseteq R$, $R \cap C \subseteq R$ and $R \in M(S)$. Thus $ac' \in \overline{T}(R,C)$.

If under the conditions of clause (Dva1), $1^{(c)}b \in C$ for no $b \in R$, then $1^{(c)}$ belongs to the system $D'(R,C)$ of clause (Dia) and accordingly $1^{(c)} \in D(R,C)$. Otherwise, supposing that $1^{(c)}b \in C$, $b = \phi(1^{(c)}b, u(b))$ for some $u(b) \in C$. But $\phi(a,c) \in C$ for all $a, c \in C$ for which $\phi(a,c)$ is defined, so that $b \in C$: $1^{(b)}b \in C$ only when $b \in C$ and again $1^{(c)} \in D(R,C)$. For variation (2) it suffices to remark that if for all $b \in R$, $b = \phi(1^{(c)}b, u(b))$ for some $u(b) \in C$, then this condition holds for all $b \in R$ such that $1^{(c)}b \in C$. The result of variation (3) is obtained from that of variation (1) by imposing the further restriction that $u(b) = u$ should be independent of b. Variation (4) is obtained by combining the restrictions imposed in variations (2,3). The result of clause (Dvib) is self evident: if $1b = b$ for all $b \in R$, then $1b \in C$ only when $b \in C$. However, it is also a corollary to clause (Dvia1) obtained by setting $\phi(a,c) = a$ and $1^{(c)} = 1$; now $b = \phi(1^{(c)}b, u(b))$ for all $u(b) \in C$. In the variation of clause (Dvic) $\langle$~~(Dvid)~~$\rangle$ $\phi(a,c)$ is taken to be ac $\langle a+c \rangle$.

Definition. (A) With $A, B \subseteq S$ either both void or such that $S[A] = S[B]$, $A \pm B$ is void when A and B are void and is otherwise the system of combinations $a \pm b$ for all $a \in A, b \in B$ belonging to the same S-stratum in $S[A]$.

(B) A system $C \subseteq S$ which is either void or such that for all $a, b \in C$ belonging to the same S-stratum, $a \pm b \in C$ also, is said to be additively perfect; $AP(S)$ is the class of additively perfect systems in S. $AP'(S)$ is the subclass of systems in $AP(S)$ containing members in all S-strata.

(C) The additive closure $\mathcal{A}(C)$ of a set C of S-numbers is void when C is void and is otherwise that system in $AP(S)$ which

1) contains C and

2) is minimal in the sense that all diminished systems obtained from $\mathcal{A}(C)$ by the removal of one or more numbers either do not contain C or are not additively perfect.

D) With $I, J \subseteq S$

i) $I \wedge J = \mathcal{A}(IJ)$ and

ii) $I \vee J = \mathcal{A}(I \cup J)$

E) With $I \subseteq S$, $a = b \bmod I$ means that $a \in S[b]$ and $a - b \in I$,

Fi) With $B, I \subseteq S$, $b \in B \bmod I$ means that $b' \in B$ exists such that $b = b' \bmod I$

ii) With $B, C, I \subseteq S$ (a) $B \subseteq C \bmod I$ means that $b \in C \bmod I$ for all $b \in B$ (b) $B \subset C \bmod I$ $\langle B = C \bmod I \rangle$ means that $B \subseteq C \bmod I$ and $C \nsubseteq B \bmod I$ $\langle$and $C \subseteq B \bmod I \rangle$.

Theorem Let $S \in \mathbb{S}$.

Aia) $S, S[0] \in AP'(S)$ and for any $C \subseteq S[0]$, $C \in AP(S)$. (b) C contains the zero members of $S[C]$ for all nonvoid $C \in AP(S)$; in particular $S[0] \subseteq C$ for all $C \in AP'(S)$.

(ii) If $A, B \subseteq C \in AP(S)$ and $S[A] \equiv S[B]$ then $A \pm B \subseteq C$

(iii) If $A, B \in AP(S)$ then $A \cap B \in AP(S)$ also

(iva) Let $C \in AP(S)$ and, with $S_k[M]$ as defined in clause (Ciiia) of Theorem, let $S_k[M] \subseteq S[C]$ for some k. Then $C /' M \in AP(S)$. (b) If $C \in AP'(S)$ then $C /' M \in AP'(S)$ also.

(c) If $C \in AP(S)$, $C /'' M$ is nonvoid. (d) (C being nonvoid).

(d) Let $C \subset S$ with $C \in AP(S)$. If S contains a number $1^{(C)}$ which is almost a unit element in the sense that $1^{(C)} b - b \in C$ for all $b \in S$, then $1^{(C)} \in D(S, C)$ and, a fortiori, S is nontorsive with respect to C.

Be) The additive closure $\mathcal{A}(C)$ of $C \subseteq S$ is well defined. Letting $K_C(S)$ be the complete set of additively perfect systems $B \subseteq S$ for which $C \subseteq B$, $\mathcal{A}(C) \equiv \cap B$ $(B \in K_C(S))$.

ii) $C \in AP(S)$ if and only if $C = \mathcal{A}(C)$

iii) With $C \subseteq S$, let $\mathcal{A}(C)$ contain members of a class K

1) to which all members of C belong and which

2) is such that for all $a, b \in K$ belonging to the same S-stratum, $a \pm b \in K$ also.

$\mathcal{A}(C)$ consists exclusively of members of K.

iv) If $B \subseteq C \subseteq S$, $\mathcal{A}(B) \subseteq \mathcal{A}(C)$

v) With $C \subseteq S$, $\mathcal{A}\{\mathcal{A}(C)\} \equiv \mathcal{A}(C)$

vi) If $B \subseteq C \subseteq \mathcal{A}(B)$, $\mathcal{A}(B) \equiv \mathcal{A}(C)$

vii) With $b \in S$, $C \subseteq S$, $\mathcal{A}(bC) \equiv b\mathcal{A}(C) \equiv \mathcal{A}(b)\mathcal{A}(C)$

viii) Let $B, C \subseteq S$. (a) $\mathcal{A}(B)\mathcal{A}(C) \subseteq \mathcal{A}(BC) \equiv \mathcal{A}\{B\mathcal{A}(C)\} \equiv \mathcal{A}\{\mathcal{A}(B)\mathcal{A}(C)\}$ (b) $\mathcal{A}(B) \cup \mathcal{A}(C) \subseteq \mathcal{A}(B \cup C) \equiv \mathcal{A}\{B \cup \mathcal{A}(C)\} \equiv \mathcal{A}\{\mathcal{A}(B) \cup \mathcal{A}(C)\}$

Ci) Let $B, C \subseteq S$. (a) $B \wedge C \equiv C \wedge B \equiv \mathcal{A}(B) \wedge C \equiv \mathcal{A}(B) \wedge \mathcal{A}(C)$

(b) $B \vee C \equiv C \vee B \equiv \mathcal{A}(B) \vee C \equiv \mathcal{A}(B) \vee \mathcal{A}(C)$.

ii) Let $A, B, C \subseteq S$. (a) $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C \equiv \mathcal{A}(ABC)$. (b) $A \vee (B \vee C) \equiv (A \vee B) \vee C \equiv \mathcal{A}(A \cup B \cup C)$

iii) Let $B, C \in AP(S)$. (a) $S[B \vee C] \equiv S[B] \cup S[C]$. (b) In a stratum of $S[B \vee C]$ containing members of $B\langle C \rangle$ alone, $B \vee C$ contains these members and no others; in $S[B] \cap S[C]$ the members of $B \vee C$ have the form $b+c$ where $b \in B$, $c \in C$.

(c) If $S[B] \equiv S[C]$, $B \vee C \equiv B + C$

Di) If $C \in AP(S)$, the relationship $a \equiv b \bmod C$ is an equivalence relationship over $S[C]$.

ii) Let $B \in AP(S)$. Let $\equiv$ be a partial ordering over each stratum of $S[B]$, let B contain at least one pair a, b for which $a \equiv b$ in each stratum of $S[B]$, and let the relationship $\equiv$ be additively uniform over B in the sense that if $a \equiv b$ then, with $u = a - b$, $c \equiv c + u$ for all $c \in B$ belonging to the

stratum containing a and b. Let C be the system of numbers $u = a - b$ obtained from all corresponding ordered pairs a, b for which $a \equiv b$. If $\equiv$ is an equivalence relationship over B then $C \in AP(S)$.

iii) If, with $I, J \subseteq S$, $a \equiv b \bmod I$, then $a \equiv b \bmod J$ also

iv) If, with $I, J \subseteq S$, $a \equiv b \bmod I$ and $a \equiv b \bmod J$, then $a \equiv b \bmod I \cap J$.

Ei) Let $B \subseteq S$, $I \in AP(S)$ and $b \in B \bmod I$. $c \in B \bmod I$ if and only if $b \equiv c \bmod I$.

ii) Let $A, B, C \subseteq S$, $I \in AP(S)$, $A \subseteq B \bmod I$ and $B \subseteq C \bmod I$. Then $A \subseteq C \bmod I$. If either $A \subset B \bmod I$ or $B \subset C \bmod I$, then $A \subset C \bmod I$.

iii) Let $I \subseteq J \subseteq S$, $B, C \subseteq S$ and $B \subseteq C \bmod I$. Then $B \subseteq C \bmod J$ also.

iv) Let $B, C, B', C', I \in S$ be such that $B \subseteq B' \bmod I$, $C \subseteq C' \bmod I$.

(a) $BC \subseteq B'C' \bmod I$ (b) $B \cap C \subseteq B' \cap C' \bmod I$ (c) $B \cup C \subseteq B' \cup C' \bmod I$

v) Let $B, C \subseteq S$, $I \in AP(S)$ and $B \subseteq C \bmod I$. Then $A(B) \subseteq A(C) \bmod I$

vi) Let $B, C, I \in AP(S)$, B and C being disjoint with respect to I, and set $E \equiv B \vee C$. $B \vee C$ is the minimal decomposition of E with respect to I in the sense that if, with $B', C' \in AP(S)$, either $B' \subseteq B$, $C' \subset C \bmod I$ or $B' \subset B$, $C' \subseteq C \bmod I$ then $B' \vee C' \subset E \bmod I$.

Proof. That $S \in AP'(S)$ is a simple consequence of the additive properties of S-numbers, the latter being found in all S-strata. In each S-stratum the zero element is by itself closed with respect to addition and subtraction: $S[0] \in AP'(S)$. That any

system $C \subseteq S[0]$ is additively perfect follows for the same reason. Each system $C \in AP(S)$ contains at least one number a in each stratum of $S[C]$, and also the zero element $a-a$ of that stratum. If, in particular, $C \in AP'(S)$, then $S[0] \subseteq C \in AP'(S)$. The results of clause (Ai) have been obtained.

Let the conditions of clause (Aii) hold. In each S-stratum containing members a of A and b of B, C contains all such pairs a, b. Since $C \in AP(S)$, C also contains all sums and differences $a \pm b$. Hence $A \pm B \subseteq C$.

If $A \cap B$ is void, $A \cap B \in AP(S)$. When $A, B \in AP(S)$, $A \cap B$ is non-void, and $a, b \in A \cap B$, $a \pm b$ belongs to $A\langle B \rangle$ since $A \in AP(S)$ $\langle B \in AP(S) \rangle$; hence $A \cap B \in AP(S)$. The result of clause (Aii) has been disposed of.

Let the conditions of clause (Aiva) hold. Since $C \in AP(S)$, C contains the zero element in each stratum of $S[C]$. From clause ( ) of Theorem , C/M is non-void; it is the largest system $X \subseteq S$ for which $MX \subseteq C$. Let $x_1, x_2 \in C/M$ belong to the same S-stratum; then $Mx_1, Mx_2 \subseteq C \in AP(S)$ and hence, from clause ( ) of Theorem , $Mx_1 \pm Mx_2 = M(x_1 \pm x_2) \subseteq C$; accordingly $C/M \in AP(S)$. When $C \in AP'(S)$, $S[0] \subseteq C$. From clause ( ) of Theorem , $S[0] \subseteq C/M$ and C/M contains members in every S-stratum: $C/M \in AP(S)$. With regard to subclause (Aivc) it suffices to remark that if $C \in AP(S)$,

is nonvoid, C contains at least one member of S[0]. From clause ( ) of Theorem , C/"M is nonvoid. Clause (Aivd) is a corollary to variation (2) of clause (Dvic) of Theorem : an additively perfect system is an additive module in the sense of that clause.

To prove the result of clause (Be) it is first remarked that if C is void, $\mathcal{A}(C)$ is void and therefore well defined. In this case the set $K_c(S)$ contains a void system of numbers. The intersection of a void set with any other is void. Thus $A = \cap B\ (B \in K_c(S))$ is void and represents $\mathcal{A}(C)$. Taking C to be nonvoid, the set $K_c(S)$ contains at least one member, namely S itself.: A is well defined. Since $C \subseteq B$ for all $B \in K_c(S)$, $C \subseteq A$ and A is nonvoid. In a stratum of A containing members select a, b, so that $a, b \in B$ for all $B \in K_c(S)$. Since $B \in \mathcal{AP}(S)$ for all $B \in K_c(S)$, $a \pm b \in B$ for all $B \in K_c(S)$: A is an additively perfect system containing C. Suppose that $A' \in \mathcal{AP}(S)$ such that $C \subseteq A'$ exists for which, for some $a \in S$, $a \in A$ but $a \notin A'$. A' is a member of $K_c(S)$ and, since $a \notin A'$, a is absent from $\cap B\ (B \in K_c(S))$ and hence from A: such a system does not exist: A is an additive closure of C in S. It remains to be shown that there is only one additively perfect system containing C which is minimal in the sense of clause (C 2) of

Definition . Suppose that there are two distinct such minimal systems $\mathcal{A}'(C)$ and $\mathcal{A}''(C)$ and that the stratum $S\{j\}$ contains two distinct sets of numbers $\mathcal{A}'\{j\} \subseteq \mathcal{A}'(C)$, $\mathcal{A}''\{j\} \subseteq \mathcal{A}''(C)$. Since $\mathcal{A}'(C)$, $\mathcal{A}''(C)$ both contain C, $\mathcal{A}'\{j\}$ and $\mathcal{A}''\{j\}$ both contain all members of $C \cap S\{j\}$: $\mathcal{A}''' = \mathcal{A}'\{j\} \cap \mathcal{A}''\{j\}$ does the same. All members of $\mathcal{A}'''$ belong to $\mathcal{A}'\{j\}$; the sums and differences of all members of $\mathcal{A}'''$ belong to $\mathcal{A}'\{j\}$; none belong to $\mathcal{A}''\{j\} \setminus \mathcal{A}'''$; similarly none belong to $\mathcal{A}'\{j\} \setminus \mathcal{A}'''$; they all belong to $\mathcal{A}'''$. Either numbers may be removed from $\mathcal{A}'(C)$ without violation of its minimality property, or the same may be done to $\mathcal{A}''(C)$: $\mathcal{A}'(C)$ and $\mathcal{A}''(C)$ cannot be both distinct and minimal.

Since $\mathcal{A}(C) \in AP(S)$, $C \in AP(S)$ when $C = \mathcal{A}(C)$. Conversely, if $C \in AP(S)$, C contains itself, is additively perfect, and is such that the diminished system obtained by removal of one or more members no longer contains all members of C; hence $C = \mathcal{A}(C)$. Clause (Bii) has been disposed of.

Let $\mathcal{A}(C) = C_1 \cup C_2$ where all members of C_1 belong to K, and none of C_2 do. $C \subseteq C_1$. Furthermore the members of C_1 are closed with respect to addition and subtraction. Unless C_2 is void, $\mathcal{A}(C)$ is not minimal in the sense of clause (C) of Definition .

Clause (Bvii) has been dealt with.

When $B \subseteq C$, $B \subseteq A(C) \in AP(S)$. $A(B)$ may be obtained from $A(C)$ by possible removal of members; hence $A(B) \subseteq A(C)$ the result of clause (Biv).

$A\{A(C)\}$ is that system of numbers which (1) contains $A(C)$, (2) is additively perfect and (3) is minimal in the sense of clause (C) of Definition . $A(C)$ itself possesses properties (1,2): ~~$A(C)$~~ $A\{A(C)\} \subseteq A(C)$. Since $C \subseteq A(C)$, $A(C) \subseteq A\{A(C)\}$ from the preceding result. Hence $A\{A(C)\} = A(C)$. For the result of clause (Bvi) it suffices to remark that when $B \subseteq C$, $A(B) \subseteq A(C)$; and when $C \subseteq A(B)$, $A(C) \subseteq A\{A(B)\} = A(B)$; hence $A(B) = A(C)$.

The results of clauses (Bvii, viii) are clearly correct when C is void. Assume that C is nonvoid. For a prescribed stratum index j, let $C \cap S\{j\} \equiv C_j$ contain the numbers $c, c_0', c_1', \ldots$; $c'', c_0'', c_1'', \ldots$ where $bc' = bc_0' = \ldots$ for all members of the class associated with c', $bc'' = bc_0'' = \ldots$ for the class associated with c'', and so on, $bc', bc'', \ldots$ being distinct. Let $A(C) \cap S\{j\}$ consist of $C_j \cup E_j$ where $E_j = \{e', e_0', e_1', \ldots, e'', e_0'', e_1'', \ldots\}$, the elements of E_j being partitioned into classes as for those of C_j. $bA(C_j)$ may be expressed as $\{bc', bc_0', \ldots, be', be_0' \ldots\}$, a listing which may contain considerable duplication. $bA(C_j)$ evidently (1) contains bC_j and (2) is additively

perfect, since this is true of $\mathfrak{A}(C) \cap S\{j\}$. Hence $\mathfrak{A}(bC_j) \subseteq b\mathfrak{A}(C_j)$. $b\mathfrak{A}(C_j)$ may be expressed without duplication as $\{bc', bc'', \dots, bf', bf'', \dots\}$ where $\{f', f'', \dots\}$ is that subset of $\{e', e'', \dots\}$ obtained from the latter by discarding all $e \in \{e', e'', \dots\}$ for which $be \in \{bc', bc'', \dots\}$. Since $bC_j \subseteq \mathfrak{A}(bC_j) \subseteq b\mathfrak{A}(C_j)$, $\mathfrak{A}(bC_j)$ may be expressed as $\{bc', bc'', \dots, bd', bd'', \dots\}$ where either (a) $\{d', d'', \dots\} = \{f', f'', \dots\}$ or (b) $\{d', d'', \dots\} \subset \{f', f'', \dots\}$. In case (a), $b\mathfrak{A}(C_j)$ contains no numbers that do not belong to $\mathfrak{A}(bC_j)$; in this case $\mathfrak{A}(bC_j) = b\mathfrak{A}(C_j)$.

Supposing case (b) to hold, let $\{f', f'', \dots\} = \{d', d'', \dots\} \cup \{g', g'', \dots\}$ where $\{d', d'', \dots\} \cap \{g', g'', \dots\}$ is void, and let $D_j = \{d', d_0', d_1'', \dots, d'', d_0'', d_1'', \dots\}$ be that subset of E_j from which $\{d', d'', \dots\}$ is derived. Consider the sum x of two members of $C_j \cup D_j$, $x = c_0' + d_1''$ for example. In this case $bx = bc_0' + bd_1'' = bc' + bd'' \in \mathfrak{A}(bC_j)$ since $\mathfrak{A}(bC_j)$ is closed with respect to addition. Thus $x \in \{C_j \cup D_j\}$ since for all $y \in E_j \setminus D_j$, $by = bg$ for some $g \in \{g', g'', \dots\}$ and, for such a g, $bg \notin \mathfrak{A}(bC_j)$. Thus $C_j \cup D_j$ (1) contains C_j and (2) is additively perfect: the numbers $g', g'', \dots$ and their associates may be removed from $\mathfrak{A}(C_j)$ without impairment of the properties (1) and (2): $\mathfrak{A}(C_j)$ is not

minimal in the sense of the definition of additive closure. The supposition that $\{d', d'', \ldots\} \subset \{f', f'', \ldots\}$ is untenable. Thus $\mathfrak{A}(bC_j) = b\mathfrak{A}(C_j)$ for all j such that $S\{j\}$ contains members of C, and $\mathfrak{A}(bC) = b\mathfrak{A}(C)$, the first result of clause (Bvii).

All members of $B\mathfrak{A}(C)$ have a representation of the form bc where $b \in B$, $c \in \mathfrak{A}(C)$. Select any such b and c. Then $bc \in b\mathfrak{A}(C) = \mathfrak{A}(bC) \subseteq \mathfrak{A}(BC)$; hence $B\mathfrak{A}(C) \subseteq \mathfrak{A}(BC)$. Furthermore $\mathfrak{A}\{B\mathfrak{A}(C)\} \subseteq \mathfrak{A}\{\mathfrak{A}(BC)\} = \mathfrak{A}(BC)$. But $BC \subseteq B\mathfrak{A}(C)$, so that $\mathfrak{A}(BC) \subseteq \mathfrak{A}\{B\mathfrak{A}(C)\}$. Accordingly $\mathfrak{A}\{B\mathfrak{A}(C)\} = \mathfrak{A}(BC)$. Furthermore $\mathfrak{A}(BC) = \mathfrak{A}\{\mathfrak{A}(B)C\} = \mathfrak{A}\{\mathfrak{A}(B)\mathfrak{A}(C)\}$ and $\mathfrak{A}(B)\mathfrak{A}(C) \subseteq \mathfrak{A}\{\mathfrak{A}(B)\mathfrak{A}(C)\}$. The results of clause (Bviiia) have been obtained. In the special case in which B consists of only one member b, it is still true that ~~$\mathfrak{A}(bC)$~~ $\mathfrak{A}(b)\mathfrak{A}(C) \subseteq \mathfrak{A}(b)\mathfrak{A}(C)$. However, since $\mathfrak{A}(bC) = b\mathfrak{A}(C)$ and $bc \in \mathfrak{A}(b)$, $\mathfrak{A}(bC) \subseteq \mathfrak{A}(b)\mathfrak{A}(C)$. Hence $\mathfrak{A}(bC) = \mathfrak{A}(b)\mathfrak{A}(C)$; this is the second result of clause (Bvii).

The results of clause (Bviiib) are evidently correct when either B or C is void. Assuming that neither B nor C is void, it is first remarked that, since $C \subseteq \mathfrak{A}(C)$, $\mathfrak{A}(B \cup C) \subseteq$

$\subseteq \mathcal{A}\{B \cup \mathcal{A}(C)\}$. But $B, \mathcal{A}(C) \subseteq \mathcal{A}(B \cup C)$; hence $B \cup \mathcal{A}(C) \subseteq \mathcal{A}(B \cup C)$. Replacing B by $B \cup \mathcal{A}(C)$ and C by $\mathcal{A}(B \cup C)$ in clause (Bvi), it follows that $\mathcal{A}\{B \cup \mathcal{A}(C)\} = \mathcal{A}\{\mathcal{A}(B \cup C)\} = \mathcal{A}(B \cup C)$. Again $\mathcal{A}(B \cup C) = \mathcal{A}\{\mathcal{A}(B) \cup C\} = \mathcal{A}\{\mathcal{A}(B) \cup \mathcal{A}(C)\}$ and $\mathcal{A}(B) \cup \mathcal{A}(C) \subseteq \mathcal{A}\{\mathcal{A}(B) \cup \mathcal{A}(C)\}$.

The first results of clause (Cia) may be written as $\mathcal{A}(BC) = \mathcal{A}(CB) = \mathcal{A}\{\mathcal{A}(B)C\} = \mathcal{A}\{\mathcal{A}(B)\mathcal{A}(C)\}$, and those of clause (Cib) as $\mathcal{A}(B \cup C) = \mathcal{A}(C \cup B) = \mathcal{A}\{\mathcal{A}(B) \cup C\} = \mathcal{A}\{\mathcal{A}(B) \cup \mathcal{A}(C)\}$. In these forms the results are evident consequences of the relationships $BC = CB$, $B \cup C = C \cup B$ and clause (Bviii). Again using the result of clause (Bviiia), $A \wedge (B \wedge C) = \mathcal{A}\{A\mathcal{A}(BC)\} = \mathcal{A}\{\mathcal{A}(ABC)\} = \mathcal{A}(ABC)$; similarly $(A \wedge B) \wedge C = \mathcal{A}(ABC)$. The result of clause (Ciib) is treated in the same way.

The result of clause (Ciiia) is evidently correct as are the initial assertions made in clause (Ciiib). It remains to deal with that concerning those S-strata containing members of both B and C. Let $S\{j\}$ be such a stratum; denote $B \cap S\{j\}$, $C \cap S\{j\}$ by B_j, C_j and the aggregate of numbers $b+c$ where $b \in B_j$, $c \in C_j$ by A_j. The sum and difference of two numbers in A_j have the forms $(b \pm b') + (c \pm c')$

where in addition, $b' \in B_j$, $c' \in C_j$. Since $B_j, C_j \in AP(S)$, $b \pm b' \in B_j$, $c \pm c' \in C_j$. Thus $A_j \in AP(S)$ and, in consequence, $\mathcal{A}(\Lambda_j) = A_j$. Also A_j contains all members of $b \in B_j$, since the zero element of $S\{j\}$ belongs to $C : \{B_j \cup C_j\} \subseteq A_j$. For all $b \in B_j$, $c \in C_j$, $b+c \in \mathcal{A}\{B_j \cup C_j\}$. Hence $B_j \cup C_j \subseteq A_j \subseteq \mathcal{A}\{B_j \cup C_j\}$ and, from clause (Bvi), $\mathcal{A}(A_j) = \mathcal{A}(B_j \cup C_j)$. Hence $A_j = \mathcal{A}(B_j \cup C_j)$ for all j in question. Clause (Ciiib) has been disposed of. Clause (Ciiic) is a corollary to its predecessor.

Under the conditions of clause (Di), all a, b belonging to disparate strata in $S[C]$ obey the relationship $a \neq b \bmod C$. For all a, b belonging to the same stratum of $S[C]$ either $v \in C \in AP(S)$ exists such that $a - b = v$, in which case $a = b \bmod C$, or no such v exists, in which case $a \neq b \bmod C$: for all $a, b \in S[c]$ the relationship $a = b \bmod C$ is determinative. Since $S[0] \cap S[c] \subseteq C$, $a - a \in C$ for all $a \in S[C]$: $a = a \bmod C$ for all $a \in S[C]$; furthermore, if $a - b = v \in C$, $-v \in C$ also; $b = a \bmod C$ if $a = b \bmod C$. Lastly, if a, b belong to the same stratum in $S[C]$ and with $u, v \in C$, $a - b = u$, $b - c = v$, then $a - c = u + v \in C$: the two conditions $a = b \bmod C$, $b = c \bmod C$ imply that $a = c \bmod C$. The proof of clause (Di) has been disposed of.

Let the conditions of clause (Dii) hold. If $u \in C$, B contains, in $S[u]$, a number a for which $a+u \equiv a$. With v a further member of C in $S[u]$, $a+u+v \equiv a+u$. Since the relationship $\equiv$ is transitive, $a+u+v \equiv a$, so that $u+v \in C$ when $u, v \in C$ belong to the same stratum. Also, since $a \equiv a+u$, $-u \in C$ for all $u \in C$, $a-u+v \equiv a$ and $v-u \in C$ when u, v are as described: C is additively perfect.

To dismiss the last two results of clause (D) it suffices to remark that when $I \subseteq J$ and $a-b \in I$, $a-b \in J$ also; again, if $a-b \in I$ and $a-b \in J$, $a-b \in I \cap J$.

If $b \equiv c \bmod I$, a $u \in I$ exists such that $c = b+u$; if $b \in B \bmod I$, $b' \in B$ and $v \in I$ exist such that $b = b'+v$. Hence $c = b'+u+v$ and, when $I \in AP(S)$, $c \equiv b \bmod I$. The converse result that the conditions $b, c \in B \bmod I$ imply that $b \equiv c \bmod I$ is proved in a similar way. The result of clause (Ei) has been obtained.

When $A \subseteq B \bmod I$, $B \subseteq C \bmod I$, for each $a \in A$ $b(a) \in B$ and $u(a,b) \in I$ exist such that $a = b(a)+u(a,b)$; also $c(a,b) \in C$ and $v(a,b,c) \in I$ exist such that $b(a) = c(a,b)+v(a,b,c)$. If in addition $I \in AP(S)$, $w(a,b,c) = u(a,b)+v(a,b,c)$ exists in I such that $a = c(a,b)+w(a,b,c)$.

In short, $a \in C \bmod I$ for all $a \in A$; hence $A \subseteq C \bmod I$. If, in addition, $A \subseteq B$, $B \subset C \bmod I$, then $B \subseteq C \bmod I$ as before, and $A \subseteq C \bmod I$; but now a $c \in C$ exists such that $b = c \bmod I$ for no $b \in B$. Suppose that $A \equiv C \bmod I$. For the c in question, there exists an $a(c) \in A$ such that $a(c) = c \bmod I$. Since $A \subseteq B \bmod I$, there exists a $b(a,c) \in B$ such that $a(c) = b(a,c) \bmod I$, and hence $c = b(a,c) \bmod I$, in conflict with the above prescription of c. The supposition that $A \equiv C \bmod I$ is untenable, and $A \subset C \bmod I$. The case in which $A \subset B$ is dealt with similarly, to complete the proof of clause (Ei).

If, to each $b \in B$, a $c(b) \in C$ corresponds such that $b = c(b) \bmod I$, then $b = c(b) \bmod J$ for all $I \subseteq J$: the result of clause (Eii) has been obtained. The three results of clause (Eiv) are similar, as are their proofs. It suffices for the purposes of illustration to show that if $B \subseteq B'$, $C \subseteq C' \bmod I$, then $B \cup C \subseteq B' \cup C' \bmod I$. If $d \in B \cup C$, either $d \in B$ or $d \in C$. In the first case there exists a $d'(d) \in B'$ such that $d = d'(d) \bmod I$; also $d'(d) \in B' \cup C'$. The second case is dealt with similarly: for all $d \in B \cup C$, $d \in B' \cup C' \bmod I$.

Under the conditions of clause (Ev), $\mathcal{A}(B)$ is composed of two systems B' and B'': B' is the system of numbers b' such that there exists a $c \in \mathcal{A}(C)$ for which $b' = c \bmod I$,

and B'' is the system of numbers b'' such that for no $c \in \mathcal{A}(C)$ is $b'' = c \bmod I$. Since $B \leq C \bmod I$, all members of B belong to the first class. Select $b_1', b_2' \in B'$ in the same S-stratum with $b_1' = c_1 + u_1$, $b_2' = c_2 + u_2$ where $c_1, c_2 \in C$ and $u_1, u_2 \in I$. Since $c_1 \pm c_2 \in \mathcal{A}(C)$, $u_1 \pm u_2 \in I$ and $b_1' \pm b_2' = c_1 \pm c_2 + u_1 \pm u_2$ the members of the first class are closed with respect to addition and subtraction. If the second class is nonvoid, $\mathcal{A}(B)$ is not minimal in the sense of clause (C) of Definition. Hence to each $b \in \mathcal{A}(B)$ there corresponds a $c(b) \in \mathcal{A}(C)$ such that $b = c(b) \bmod I$; hence $\mathcal{A}(B) \leq \mathcal{A}(C) \bmod I$.

Turning to the last result of the theorem, it is first remarked that since $B' \leq B$, $C' \leq C \bmod I$, $B' \cup C' \leq B \cup C \bmod I$ and $\mathcal{A}(B' \cup C') \leq \mathcal{A}(B \cup C) \bmod I$; hence $B' \vee C' \leq E \bmod I$. Consider the case in which $B' \subset B \bmod I$, so that there exists a $b \in B$ such that $b - b' \in I$ for no $b' \in B'$. Since $b \in B$, this b is also in E. Since $B', C' \in AP(S)$, all members of $E' = B' \vee C'$ have a representation $b' + c'$ with $b' \in B'$, $c' \in C'$. Suppose that for some $t \in I$, $b' \in B'$ and $c' \in C'$ exist such that $b + t = b' + c'$. Since $b' \in B' \subset B \bmod I$, $c' \in C' \leq C \bmod I$, u, v exist such that $b' = b'' + u$, $c' = c'' + v$ where $b'' \in B$, $c'' \in C$ and then $b - b'' = c'' + u + v - t$. Since $B \in AP(S)$, $b - b'' \in B$. Similarly $u + v - t \in I = B \cap C \leq C$ so that $c'' + u + v - t \in C$. Thus $w = b - b'' \in B \cap C = I$ and $b - b' = b - b'' + b'' - b' = w - u \in I$. A $b' \in B'$ exists

for which $b-b' \in I$, conflicting with the above prescription of b. The supposition that $b \in E'$ mod I is untenable. Hence $E' \subseteq E$ mod I and there exists a $b \in E$ such that for no $b' \in E'$ is $b-b' \in I$; thus $E' \subset E$ mod I.

For the sake of completeness it is remarked, concerning the first set of results in clause (Bviii a), that the relationship $\mathfrak{A}(B)\mathfrak{A}(C) \subset \mathfrak{A}(BC)$ is possible: taking $B\langle C \rangle$ to be the system of numbers b_i $(i:m)$ $\langle c_j\ (j:n) \rangle \in S\{0\}$ and addition in $S\{0\}$ to be of finite characteristic, the members of $\mathfrak{A}(B)\langle \mathfrak{A}(C) \rangle$ have representations of the form $\sum_{i=1}^{m} \mu_i b_i \langle \sum_{j=1}^{n} \nu_j c_j \rangle$, the $\mu_i \langle \nu_j \rangle$ being integers. The members of $\mathfrak{A}(B)\mathfrak{A}(C)\langle \mathfrak{A}(BC) \rangle$ have representations of the form $\sum_{i=1}^{m} \sum_{j=1}^{n} \mu_i \nu_j (b_i c_j)$ $\langle \sum_{i=1}^{m} \sum_{j=1}^{n} r_{i,j} (b_i c_j) \rangle$, the $r_{i,j}$ again being integers. When $m,n > 1$ and $m+n > 4$, double integer sequences $\{r_{i,j}\}$ having no representation of the form $\{\mu_i \nu_j\}$ exist: $\mathfrak{A}(BC)$ contains numbers not in $\mathfrak{A}(B)\mathfrak{A}(C)$. Again, clause (Ci) does not have a direct counterpart concerning the intersection of two systems. That the relationship $\mathfrak{A}(B \cap C) = \mathfrak{A}(B \cap \mathfrak{A}(C))$ may be false is shown by taking $B \subseteq \mathfrak{A}(C) \setminus C$ where $C \subseteq \mathfrak{A}(C)$: $B \cap C$ is void, as is $\mathfrak{A}(B \cap C)$, but $\mathfrak{A}(B \cap \mathfrak{A}(C))$ is non-void. Clause (Cii) does not have a similar counterpart for the

same reason: the relationship $\mathcal{A}\{A \cap \mathcal{A}(B \cap C)\} = \mathcal{A}\{A \cap B \cap C\}$ is false when $B \cap C \subset \mathcal{A}(B \cap C)$ and $A \subseteq \mathcal{A}(B \cap C) \setminus (B \cap C)$.

Again, in connection with the last clause of the theorem, it is remarked that a system $E \in \mathrm{AIP}(S)$ may have more than one decomposition of the form $E = X \vee Y$ with $X, Y \subsetneq E$, and that even with X fixed it can occur that $X \vee Y = X \vee Z$ with $Y \neq Z$. For example, considering an S-ring whose members have a representation of the form (a,b), where (a,b) are least nonnegative residues mod (2) and arithmetic operations are performed upon components independently (so that $(a_1,b_1)+(a_2,b_2) = (a_1+a_2, b_1+b_2)$ for example) with all strata of S except S_0 of null, let X, Y, Z be the systems containing the zero element $(0,0)$ together with $(1,0)$ for X, $(0,1)$ for Y and $(1,1)$ for Z, and let E be the system containing all four given elements. Then $X, Y, Z, E \in \mathrm{AIP}(S)$ and $E = X \vee Y = X \vee Z = Y \vee Z$.

A constructive definition of $\mathcal{A}(I)$, viable when S contains only a finite number of elements in each S-stratum, may be given. Let $A_0 = \mathbb{C}$ and for $n = 0, 1, \ldots$ let B_n be the system of sums and differences of all pairs of numbers belonging to the same S-stratum in A_n. If $B_n \setminus A_n$ is void, $\mathcal{A}(C) = A_n$; otherwise let $A_{n+1} = A_n \cup B_n$. Subject to termination, this definition of $\mathcal{A}(C)$ is consistent with that of clause (C) of

Definition . Assuming termination at the nth stage as described, $A_n \equiv C \cup B_0 \cup \dots \cup B_{n-1}$, so that $C \subseteq A_n$. Also, since $B_n \setminus A_n$ is void, A_n contains all sums and differences of its that can be formed from its members. Hence $A_n \in AP(S)$. Use of the minimal property of $\mathcal{A}(C)$ yields the relationship $\mathcal{A}(C) \subseteq A_n$. However $C \subseteq \mathcal{A}(C)$. Since $\mathcal{A}(C) \in AP(S)$, $B_0 \in \mathcal{A}(C)$ also, so that $I \cup B_0 \subseteq \mathcal{A}(C)$ and, as is easily shown by induction, $A_n \subseteq \mathcal{A}(C)$. Hence $A_n \equiv \mathcal{A}(C)$: the two definitions are consistent.

Definition . (Ai) With $R, C \subseteq S \in \mathbb{S}$ and $C \setminus S[0]$ nonvoid, $K(R,C) = R \cap [\cap (bS)\ (b \in C \setminus S[0])]$ is the composite part of R with respect to C; $\overline{K}(R,C) = K(R,C) \cup S[0]$.

(ii) With $C \in S$ (a) $\min C = SC$; (b) $\liminf C = S^2 \cap C$.

Bi) A semi S-ideal is a nonvoid system $C \subseteq S \in \mathbb{S}$ such that $\min C \subseteq C$; $SI(S)$ is the class of semi S-ideals in S.

iia) A semi S-ideal C for which $C \equiv \min C \langle C \equiv \bar{C} \rangle$ is said to be minimal $\langle$closed$\rangle$ in S; $SI_{min}(S) \langle SI_c(S) \rangle$ is the class of minimal $\langle$closed$\rangle$ semi S-ideals in S. (b) An S-ring for which $T(S,C) \langle \overline{T}_S(S,C) \rangle \{D(S,C)\}$ is void is said to be torsion free $\langle$square-free$\rangle$ $\{$torsive$\}$ with respect to C; an S-ring S for which $D(S,C)$ is nonvoid is said to be nontorsive with respect to C.

iii) Let $S[o] \subseteq C \subseteq S$. (a) $\max_{SI} C$ ⟨$\max_{\min SI} C$⟩ is that semi S-ideal ⟨that minimal semi S-ideal⟩ $B \subseteq C$ for which no other $B' \in SI(S)$ ⟨no other $B' \in SI_{\min}(S)$⟩ with $B' \subseteq C$ and $B' \setminus B$ nonvoid exists. (b) With $\overline{S[o]} \subseteq C \subseteq S$, $\max_{cSI} C$ is defined, with $B' \in SI_c(S)$, in a similar way.

iv) With $A \subseteq C \in SI(S)$ ⟨$A \subseteq C \in SI_{\min}(S)$⟩{$A \subseteq C \in SI_c(S)$} $\min_{SI} A$ ⟨$\min_{\min SI} A$⟩{$\min_{cSI} A$} is that semi S-ideal ⟨minimal semi S-ideal⟩{closed semi S-ideal} B for which (a) $A \subseteq B \subseteq C$ and (b) no $B' \in SI(S)$ ⟨$B' \in SI_{\min}(S)$⟩ {$B' \in SI_c(S)$} such that $A \subseteq B'$ and $B \setminus B'$ is nonvoid exists.

Ci) An S-ideal is an additively perfect semi S-ideal; $I(S) \equiv SI(S) \cap AP(S)$ is the class of S-ideals in S.

ii) A closed S-ideal and the class $I_c(S)$ of closed S-ideals in S are defined as in clause (Biii); an S-ring which is torsion free, square free, torsive or nontorsive with respect to $I \in \mathbb{I}(S)$ is defined as in clause (Büa) above.

iiia) Let $S[o] \subseteq \bar{J} \subseteq S$ ⟨$\overline{S[o]} \subseteq \bar{J} \subseteq S$⟩. $\max_I \bar{J}$ ⟨$\max_{cI} \bar{J}$⟩ is that S-ideal ⟨that closed S-ideal⟩ $I \subseteq \bar{J}$ for which no other $I' \in \mathbb{I}(S)$ ⟨$I' \in \mathbb{I}_c(S)$⟩ with $I' \subseteq \bar{J}$ and $I' \setminus I$ nonvoid exists. (b) $\min_I \bar{J}$ ⟨$\min_{cI} \bar{J}$⟩ is the S-ideal ⟨closed S-ideal⟩ defined in analogy with $\min_{SI} C$ ⟨$\min_{cSI} C$⟩ in clause (Biv) above.

Certain S-rings possess ideals which are immediately visible. For example, let S be a directly decomposable S-ring, each number $b \in S\{j\}$ having a representation $b=\{b^{(k)}\}_{(k:n)}$ where $b^{(k)} \in S^{(k)}$ $(k:n)$, the component S-rings $\{S^{(k)}\}$ being devoid of proper divisors of zero. I can be taken to be a subset of numbers b for which $b^{(k(\tau))}=0 \in S^{(k(\tau))}\{j\}$ $(\tau:n' \leq n)$, the $k(\tau)$ being distinct integers belonging to the set $1,\ldots,n$. In this case $T(S,I)$ is the set of numbers b for which $b^{(h(\tau))}=0 \in S^{(h(\tau))}\{j\}$, $b^{(m(\tau))} \neq 0 \in S^{(m(\tau))}\{j\}$, $\{h(\tau)\}$ and $\{m(\tau)\}$ being complementary nonvoid subsets of $\{k(\tau)\}$, and $D(S,I)$ is the set of numbers b for which $b^{(k(\tau))} \neq 0 \in S^{(k(\tau))}\{j\}$ $(\tau:n')$, with $(j:)$ in both cases.

Again with $\mathbb{P}$ the S-ring of quintics over a commutative ring as described above, q_τ being a fixed irreducible quintic selected from the stratum $\mathbb{P}\{i(\tau)\}$ $(\tau:n)$ and $q=\prod_{\tau=1}^{n} q_\tau$, I can be taken to be the set of quintics aq for all $a \in \mathbb{P}$. In this case $T(\mathbb{P},I)$ is the set of quintics $b=a\prod_{\tau=1}^{n'} q_{k(\tau)}$, a possessing no q_τ as a factor, where $n'<n$, the $k(\tau)$ again being distinct integers belonging to the set $1,\ldots,n$, and $D(P,I)$ is the set of quintics possessing no q_τ as a factor.

Theorem (Ai) $K(R,C)$ is the complete system of numbers $a \in R$ such that each $b \in C \setminus S[0]$ is partnered by a number $c(a,b) \in S$ for which $a = bc(a,b)$.

ii) For any nonvoid system $C \subseteq S$, $\bar{K}(S,C) \in \mathbb{I}(S)$.

iii) If $R \subseteq R'$ and $C' \subseteq C$ with $C' \nsubseteq S[0]$, $K(R,C) \subseteq K(R',C')$.

Bi) $\mathbb{I}(S) \subseteq SI(S) \subseteq M(S)$

ii) $S \in Sl_c(S)$, $S[0] \in Sl_{min}(S)$ and $S[0] < \overline{S[0]} >$ belongs to every semi S-ideal $<$ to the partial closure of every semi S-ideal $>$.

iii) If $C \in Sl(S)$, $\mathcal{A}(C) \in \bar{\mathbb{I}}(S)$

C) Let $C \in Sl(S)$

i) $\bar{T} \langle \bar{T}_S \rangle (S,C) \in Sl(S)$

ii) $\min C$, $\lim\inf C$, $\bar{C} \in Sl(S)$

$S[0] \subseteq \min C \subseteq \lim\inf C \subseteq C \subseteq \bar{C} \subseteq \bar{T}_S(S,C) \subseteq \bar{T}(S,C)$

and, when $C \nsubseteq S[0]$, $S[0] \subseteq \bar{K}(S,S) \subseteq \bar{K}(S,C) \subseteq \min C$ also

iiia) $\min \mathcal{A}(C) \subseteq \mathcal{A}(\min C)$. (b) $S\mathcal{A}(\bar{C}) \subseteq \mathcal{A}(C)$. (c) $S^2\mathcal{A}(\bar{C}) \subseteq \mathcal{A}(\min C)$. (d) $\min(\bar{C}) \subseteq C \subseteq \overline{(\min C)}$ (e) $\bar{C} = \overline{(\lim\inf C)}$

(f) $\lim\inf(\lim\inf C) = \lim\inf C$ (g) If C is minimal $\langle$closed$\rangle$ in S then $\min(\min C) = C \langle \bar{\bar{C}} = C \rangle$ and $\min C \langle \bar{C} \rangle$ is minimal $\langle$closed$\rangle$ in S

iva) If S is nontrivial with respect to C, the multiplicative modules $\bar{T}(S,C)$ and $D(S,C)$ are both independent in S, $D(S,C)$ being saturated in S. (b) S is composed of three

disjoint systems: $D(S,C)$, $T(S,C)$ and C.
va) If S is nontorsive with respect to C, C is closed in S. (b) If C is open in S, S is torsive with respect to C and $T(S,C)$, $T_S(S,C)$ are both nonvoid. (c) If S is torsion free with respect to C, $T_S(S,C)$ is void, and $\min C = \lim \inf C$. (d) If S contains a number 1_C such that $1_C C = C$, then C is minimal in S and for any system $B \subseteq S$ containing 1_C, $BC = C$. (e) If $C \subset S$ and S contains a unit element 1 such that $1a = a$ for all $a \in S$, S is nontorsive with respect to C, $1 \in D(S,C)$ and C is both minimal and closed in S.
vi) With $B \subseteq S$, $BC \subseteq SI(S)$. If $C \in SI_{min}(S)$, $BC \in SI_{min}(S)$ also. In both cases $BC \subseteq \min C$
viia) Let $M \subseteq S$ be nonvoid. Then $C/'M, C/''M \in SI(S)$ and $\bar{C} \subseteq C/'M \subseteq C/''M$. (b) Let $M \subseteq C$ be nonvoid. Then $C/'M = C/''M = S$. (c) With $C, D \in SI(S)$, the various inner and outer quotients $C/'M, \ldots, D/''N$ considered in clauses (Cv-Cvii) of Theorem are semi S-ideals and the results stated in those clauses hold for them.
(viiia) With $C \in SI(S)$ $\langle B, C \in SI_{min}(S) \rangle$ $BC, B \cap C, B \cup C \in SI(S)$ $\langle SI_{min}(S) \rangle$ and $BC \subseteq \overline{BC} \subseteq B \cap C \subseteq B \cup C$. (b) Let $B, C \in SI_c(S)$. Then $B \cap D \in SI_c(S)$.
(ixe) The semi S-ideals $\max_{SI} C$, $\max_{\min SI} C$, $\max_{cSI} C$, $\min_{SI} A$, $\min_{\min SI} A$, $\min_{cSI} A$ are well defined.

x) Let $B \subseteq B' \bmod C$. For all $A \subseteq S$, $AB \subseteq AB' \bmod C$ also

D) Let $I \in \mathbb{I}(S)$. With results concerning $\min C$, $\liminf C$ and minimal semi S-ideals deleted, C replaced by I and $\mathrm{SI}(S)$ replaced by $\mathbb{I}(S)$, clause (Bii) and clauses (ii–v) of part C hold with regard to S-ideals. To the counterpart to clause (Cve) the result may be added that if $I \subset S$ and S contains a number $1^{(I)}$ which is almost a unit element in the sense that $1^{(I)}b - b \in I$ for all $b \in S$, then $1^{(I)} \in \mathbb{D}(S, I)$ and a fortiori S is nontorsive with respect to I and I is closed in S. The counterpart to clause (Cvi) is: if $b \in S$ then $bI \in \mathbb{I}(S)$. With regard to clause (Cvii) with M nonvoid, $I/'M \in \mathbb{I}(S)$, and $I/''M \in \mathbb{I}(S)$ if $M \in \mathbb{M}(S)$. Mutatis mutandis, the further results of clause (Cvii) hold. The counterpart to clause (Cviii) is: with $J \in \mathbb{I}(S)$ also, $I \wedge J$, $I \cap J$, $I \vee J \in \mathbb{I}(S)$ and $I \wedge J \subseteq I \cap J \subseteq I + J \equiv I \vee J$. That to clause (Cix) is that the ideals $\max_I \bar{J}$, $\max_{cI}$, $\min_I \bar{J}$, $\min_{cI} \bar{J}$ are well defined. Clause (Cx) holds when $I \in \mathbb{I}$ simply because $I \in \mathrm{SI}(S)$, but in this case an additional result holds: let $a = b \bmod I$; $a \in \mathbb{D}(S, I)$ if and only if $b \in \mathbb{D}(S, I)$.

Proof. To prove the result of clause (Ai), denote the complete system of numbers $a \in S$ such that each $b \in C \setminus S[0]$ is partnered by a number $c(a,b) \in S$ for which $a = bc(a,b)$ by Y, and set

$Z = \cap(bS)$ $(b \in C \setminus S[0])$. For each $a \in Z$, $c(a,b) \in S$ such that $a = bc(a,b)$ for all $b \in C \setminus S[0]$ exists; hence $Z \subseteq Y$. For each $a \in Y$, $a \in bS$ for all $b \in C \setminus S[0]$, so that $a \in Z$; hence $Y \subseteq Z$ and $Y = Z$. But the system of numbers described in clause (Ai) is $R \cap Y$, and $K(R,C) = R \cap Z$.

Concerning clause (Aii), it is first remarked that $\bar{K}(S,C)$ is well defined, reducing to $S[0]$ when $K(S,C)$ is void. For any $d \in S$ and $a \in S[0]$, $da \in S[0]$. For any $d \in S$, $a \in K(S,C)$ and $b \in C \setminus S[0]$, $c(a,b) \in S$ exists such that $da = b\{dc(a,b)\}$; hence $da \in K(S,C)$ also. In short $\bar{K}(S,C) \in SI(C)$. For any $a, a' \in \bar{K}(S,C)$ and $b \in C$, $c(a,b)$ and $c'(a',b)$ exist such that $a \pm a' = b\{c(a,b) \mp c'(a',b)\}$. Hence $K(S,C) \in AP(S)$ and $K(S,C) \in \mathbb{I}(S)$.

$K(R,C)$ is the system of numbers $a \in R$ such that corresponding to each number $b \in C \setminus S[0]$ a number $c(a,b)$ for which $a = bc(a,b)$ may be found in S. When $C' \subseteq C$ and $R \subseteq R'$, the systems of numbers $b \in C \setminus S[0]$, $c(a,b) \in S$ serves also to define each $a \in K(R,C)$ as a member of $K(R',C')$.

The first result of clause (Bi) follows from the definition $\underline{I}(S) = SI(S) \cap AP(S)$; the second from the remark that for any system C satisfying the relationship $SC \subseteq C$, $C^2 \subseteq C$ also. $S \in S$ satisfies the relationship $S^2 \subseteq S$, so that $S \in \underline{SI}(S)$; hence $S \in SI_c(S)$. $S[0]$ satisfies the relationship $SS[0] = S[0]$ so that $S[0] \in SI_{min}(S)$. A semi S-ideal C, by definition, contains at least one number,

$b \in S\{i\}$ say. Since $SC \subseteq C$, C also contains $O_{(k-i)}\, b = O_{(k)}$ $(k \geq 1)$ and hence $S[0] \subseteq C$ for all $C \subseteq Sl(S)$. $\overline{S[0]}$ satisfies the relationship $S\overline{S[0]} \subseteq S[0]$. Since $S[0] \subseteq C \in Sl(S)$ and the closure $\bar{C}$ of C is the largest system $X \subseteq S$ for which $SX \subseteq C$, $\overline{S[0]} \subseteq \bar{C}$ for all $C \in Sl(S)$. The results of clause (Bii) have been dealt with.

To prove the result of clause (Biii) define, among the members of $\mathcal{A}(C)$, the class K of numbers a such that $aS \subseteq \mathcal{A}(C)$. Since $C \subseteq \mathcal{A}(C)$, $C \in K$. Let b, c in K belong to the same S-stratum. Since $ab, ac \in \mathcal{A}(C)$ for all $a \in S$, and $\mathcal{A}(C) \in AP(S)$, $a(b \pm c) = ab \pm ac \in \mathcal{A}(C)$ for all $a \in S$, and $(b \mp c) \in K$. From clause ( ) of Theorem , $S\mathcal{A}(C) \subseteq \mathcal{A}(C)$ and hence $\mathcal{A}(C) \in Sl(S)$ or, since $\mathcal{A}(C) \in AP(S)$, $\mathcal{A}(C) \in \mathbb{I}(S)$.

The abbreviations $D, \mathbf{I}, \bar{T}, \bar{T}_S, \bar{\bar{T}}_S$ are used for $D(S,C), \ldots,$ $\bar{T}_S(S,C)$ in the proof of part C. To derive the results of clause (Ci), let $a \in \bar{\bar{T}}$ and $b \in S$. If $a \in C$, $ab \in C \subseteq \bar{\bar{T}}$. If $a \notin C$, $a \in \bar{T}$ and $c \notin C$ exists such that $ac \in C$. For this c, $(ab)c \in C$. Either $ab \in C$ or $ab \in \bar{T}$. In conclusion $S\bar{\bar{T}} \subseteq \bar{\bar{T}}$. Similarly when $a \in \bar{\bar{T}}_S$ either $ab \in C$ or $ab \notin C$ but $(ab)^2 \in C$, for all $b \in S$. Hence $\bar{\bar{T}}_S \in Sl(S)$.

To prove the results of clause (Cii) it is first remarked that, since $SC \subseteq C$, $S \min C = S^2 C \subseteq SC \equiv \min C$ so that $\min C \in Sl(S)$. Also $S(\liminf C) \equiv S(S^2 \cap C) \equiv S^3 \cap SC \subseteq S^2 \cap C = \liminf C$,

so that $\liminf C \in SI(S)$. $\bar{C}$ is the largest system X of S-numbers satisfying the relationship $SX \subseteq C$ and then $S(SX) \subseteq SC \subseteq C$, so that $SX \subseteq X$ and $\bar{C} \in SI(S)$. Since $\min C \in SI(S)$, $S[0] \subseteq \min C$ from clause (Bii). Again $SC \subseteq S^2$ and $SC \subseteq C$ so that $\min C = SC \subseteq (S^2 \cap C) = \liminf C$. Also $\liminf C = S^2 \cap C \subseteq C$. Since $SC \subseteq C$, C belongs to the largest system X satisfying the relationship $SX \subseteq C$; hence $C \subseteq \bar{C}$. If $\bar{C} \equiv C$ then $\bar{C} \subseteq C \cup T_S \equiv \bar{T}_S$. When $C \subset \bar{C}$, $(\bar{C} \setminus C)^2 \subseteq S\bar{C} \subseteq C$. Each $a \in \bar{C} \setminus C$ does not belong to C but is such that $a^2 \in C$; thus $\bar{C} \setminus C \in \bar{T}_S$ and $\bar{C} \subseteq \bar{T}_S \cup C \equiv \bar{\bar{T}}_S$. $\bar{T}_S$ is by definition a subsystem of $\bar{T}$. For the case in which $C \neq S[0]$, it is first remarked that $S[0] \subseteq \bar{K}(S,S)$ by definition, and that $\bar{K}(S,S) \subseteq \bar{K}(S,C)$ from clause (Aiii). If $K(S,C)$ is nonvoid, each of its members a may be expressed in the form $bc(a,b)$ where $b \in C \setminus S[0]$ and $c(a,b) \in S$. Hence $\bar{K}(S,C) \subseteq SC = \min C$.

Using clauses (,) of Theorem , firstly $\min \mathcal{A}(C) \equiv S\mathcal{A}(C) \subseteq \mathcal{A}\{S\mathcal{A}(C)\} \equiv \mathcal{A}(SC) = \mathcal{A}(\min C)$ and secondly $S\mathcal{A}(\bar{C}) \subseteq \mathcal{A}\{S\mathcal{A}(\bar{C})\} \equiv \mathcal{A}\{S\bar{C}\} \subseteq \mathcal{A}(C)$, the first two results of clause (Ciii); the third follows from the relationship $S^2\mathcal{A}(\bar{C}) \subseteq S\mathcal{A}(C)$. For the first result of clause (Ciiid), $\min(\bar{C}) \equiv S\bar{C} \subseteq C$; also $\overline{\min C} \equiv \overline{SC}$ is the complete system $X \subseteq S$ for which $SX \subseteq SC$, and all members of C belong to X.

For the result of clause (Ciiie) $\overline{\lim\inf C}$ is the complete system X of S-numbers for which $SX \subseteq C \cap S^2$; $\bar{C}$ is the complete system $Y \subseteq S$ for which $SY \subseteq C$. But $SY \subseteq S^2$, so that $SY \subseteq C \cap S^2$: the characterisations of X and Y are equivalent. The result of clause (Ciiif) follows directly from the relationship $S^2 \cap \{S^2 \cap C\} \equiv S^2 \cap C$. Since it may occur that $S(SC) \equiv S^2C \subset SC = \min C$ it is not always true that $\min(\min C) \equiv \min C$. However, when $SC \equiv C$, $S(SC) \equiv SC \equiv C$ and $\min C$ is minimal. Similarly, X being the largest system in S for which $SX \subseteq C$, it may occur that the largest system $Y \subseteq S$ for which $SY \subseteq X$ contains numbers not belonging to X and then $\bar{\bar{C}} \neq \bar{C}$. However, when $X \equiv C$, $Y \equiv C$ also and then $\bar{C} = C$ and, since $\bar{\bar{C}} \equiv \bar{C}$, $\bar{C}$ is closed. The results of clause (Ciiig) have been derived.

From clause (Dva) of Theorem , $D \in M(S)$, since $S \in M(S)$. From clauses (Bi) and (Ci), $\bar{T} \in M(S)$. From clause (Diia) of Theorem , D and $\bar{T}$ are complementary in S and, from clause (Biii) of the same theorem, both multiplicative modules are independent. Since $\bar{T} \in SI(S)$, $ab \in \bar{T}$, i.e, $ab \notin D$, when either a or b is in $S \setminus D$; D is also saturated. Clause (Civd) has been disposed of. From clause (Cii), $C \subseteq \bar{T}$. Hence $D \setminus C \equiv D$, $D \cap C$ is void and $C \cap S \equiv \mathbf{0}$. The result of clause (Ciib) follows from clause (Diib) of Theorem .

To obtain the result of clause (Cva) it is first remarked that $\bar{C}$ satisfies the relationship $S\bar{C}\subseteq C$. If S contains a number d such that $da\in C$ only when $a\in C$ then, since $d\bar{C}\subseteq C$, all members a of $\bar{C}$ belong to C: $\bar{C}\subseteq C$. But $C\subseteq\bar{C}$, so that $C=\bar{C}$. That $\mathfrak{D}$ is void when $C\subset\bar{C}$ is a simple consequence of the preceding result. In the proof of clause (Cii) it was shown that $\bar{C}\setminus C\subseteq\overline{T_S}$, so that T_S is nonvoid when $C\subset\bar{C}$ and, a fortiori, T is nonvoid. Clause (Cvb) has been disposed of. The first result of clause (Cvc) follows from the observation that, since $T_S\subseteq T$, T_S is void when T is void. $S^2\equiv SC\cup S(S\setminus C)\equiv SC\cup(S\setminus C)^2\cup(S\setminus C)C$. Since $S\setminus C\subset S$ $(S\setminus C)C\subset SC$ and $S^2\equiv SC\cup(S\setminus C)^2$, so that $C\cap S^2\equiv(C\cap SC)\cup\{C\cap(S\setminus C)^2\}$ or, since $SC\subseteq C$ when $C\in SI(S)$, $C\cap S^2\equiv SC\cup\{C\cap(S\setminus C)^2\}$. When T is void, $C\cap(S\setminus C)^2$ is also void. Hence $\liminf C=C\cap S^2\equiv SC=\min C$, the second result of clause (Cvc). If the number 1_C of clause (Cvd) exists, $C\equiv 1_C C\subseteq SC=\min C$. From the two relationships $C\subseteq\min C$ and $\min C\subseteq\liminf C\subseteq C$, it follows that $\min C\equiv\liminf C$. If $1_C\in B$, then $C\equiv 1_C C\subseteq BC$. But $BC\subseteq SC\subseteq C$. Hence $BC\equiv C$. That $1\in\mathfrak{D}$ for the unit element of clause (Cve) is a consequence of clause ( ) of Theorem; this unit element plays the rôle of 1_C in the preceding clause. Thus, from clauses (Cva,d), C is both minimal and

closed in S.

When $B \subseteq S$ and $SC \subseteq C$, $S(BC) = B(SC) \subseteq BC$ so that $BC \in Sl(S)$. In particular, when $SC = C$, $S(BC) = BC$ and $BC \in Sl_{min}(S)$. In both cases $BC \subseteq SC = \min C$. Clause (Cvi) has been dealt with. $C \in Sl(S)$ contains $S[0]$. $S[0] \subseteq S[0]/'M$ for any nonvoid $M \subseteq S$, so that $C/'M$ contains $S[0]$ and is thus nonvoid when $C \in Sl(S)$. $C/'M$ is the largest system $X \subseteq S$ for which $MX \subseteq C$ and then $MSX \subseteq SC \subseteq C$ when $C \in Sl(S)$. Hence $SX \subseteq X$ and $C/'M \in Sl(S)$. Since $M \subseteq S$, $\bar{C} = C/'S \subseteq C/'M$. $C/'M$ has been shown to be nonvoid and, since $C/'M \subseteq C/''M$, $C/''M$ is nonvoid also. It is the system of numbers $b \in S$ such that $a(b) \in M$, $c(a,b) \in C$ for which $a(b)b = c(a,b)$ exist. For any $e \in S$, $a(b)(eb) = e\,c(a,b) \in C$. Thus $S(C/''M) \subseteq C/''M$ and accordingly $C/''M \in Sl(S)$. Also $\bar{C} \subseteq C/'M \subseteq C/''M$. The results of clause (Cviia) have been obtained. When $M \subseteq C$, $MS \subseteq SC \subseteq C$. Thus $C/'M = S$ and, since $C/'M \subseteq C/''M$, $C/''M = S$ also, the result of clause (Cvii b). The results of clauses (Cv-vii) of Theorem hold, in particular, when $C, D \in Sl(S)$ when, from clause (Cviia) of the present theorem, $C/'M, \ldots, D/''N$ are semi S-ideals. Clause (Cviic) has been disposed of.

That $BC \in Sl(S)\langle Sl_{min}(C)\rangle$ when $B, C \in Sl(S)\langle Sl_{min}(S)\rangle$ follows from clause (Cvi). When $B, C \in Sl(S)$, $S\{B \cap C\} = SB \cap SC$

$\subseteq B\cap C$, so that $B\cap C\in Sl(S)$. In particular, $S\{B\cap C\}\equiv B\cap C$ when $B,C\in Sl_{min}(S)$. The union $B\cup C$ is treated in the same way. $X=(\overline{B\cap C})$ is the larges system in S for which $SX\subseteq B\cap C$. When $B,C\in Sl(S)$, $B\cap C\in Sl(S)$ and $B\cap C\subseteq X$. Also $SX\subseteq B,C$ so that $X\subseteq \overline{B},\overline{C}$. When $B,C\in Sl_c(S)$, $\overline{B}\equiv B$ and $\overline{C}\equiv C$ and then $X\subseteq B\cap C$. Hence $(\overline{B\cap C})\equiv B\cap C$, the result of clause (Cviiib).

Concerning the existence of $max_{Sl}C$ and $max_{minSl}C$ it is first remarked that C contains a semi S-ideal ⟨a minimal semi S-ideal⟩, namely $S[0]$. Suppose that A, A' satisfying the conditions ascribed to B in clause (Biiia) of Definition exist, with $A\neq A'$. In particular $A, A'\subseteq C$ and $A,A'\in Sl(S)\langle A,A'\in Sl_{min}(S)\rangle$. Set $B'=A\cup A'$. Then $B'\subseteq C$ and, from clause (Bviiia) $B'\in Sl(S)\langle B'\in Sl_{min}(S)\rangle$. But either $B'\setminus A$ or $B'\setminus A'$ is nonvoid. A and A' do not both satisfy the required conditions: $max_{Sl}\{0\}\langle max_{minSl}C\rangle$ is uniquely defined. If $\overline{S[0]}\subseteq C$, C contains the closed semi S-ideal $\overline{S[0]}$ and the existence of $max_{CSl}C$ is proved in the same way. Concerning the existence of $min_{Sl}A$, it is first remarked that the system C of clause (Biv) of Definition certainly contains a semi S-ideal to which all members of A belong, namely C itself. Suppose that B, B'' with $B\neq B''$ both satisfy the conditions attributed

to B in clause (Biv) of Definition , so that $B, B'' \in SI(S)$, $A \subseteq B, B'' \subseteq C$. Set $B' = B \cap B''$ so that, from clause (Cviiia), $B' \in SI(S)$ and also $A \subseteq B'$, $B' \subseteq C$. Either $B \setminus B'$ or $B'' \setminus B'$ is nonvoid. B and B'' do not both possess the minimality condition attributed to B: $\min_{SI} A$ is well defined. The two further results of clause (Cix) are, using the results concerning $B \cap B''$ when $B, B'' \in SI_{min}(S)$ or $B, B'' \in SI_c(S)$ of clauses (Cviiia,b), proved in the same way.

The result of clause (Cx) follows from the multiplicative properties of $C \in SI(S)$: if to $b \in B$ there corresponds a $b'(b) \in B'$ such that $b - b'(b) = c \in C \in SI(S)$, then $ab - ab'(b) = ac \in C$ also for all $a \in A$, where A is any system in S.

To justify the statement made in the second sentence of part D, four ancillary results must be established. The first is that when $I \in \mathbb{I}(S)$, $\overline{I} \in \mathbb{I}(S)$ also. But $\overline{I} = I'/S \in SI(S) \cap AP(S)$, from clause (Cviia) of the present theorem and clause ( ) of Theorem ; hence $\overline{I} \in \mathbb{I}(S)$. The remaining three are that $S \in I_c(S)$, $S[0] \in \mathbb{I}(S)$ and $\overline{S[0]} \in I_c(S)$. Since $S \in AP(S)$ and $S \in SI_c(S)$, $S \in \mathbb{I}_c(S)$. Again, $S[0] \in AP(S) \cap SI(S)$ so that $S[0] \in I(S)$. Lastly, since $S[0] \in AP(S)$, $\overline{S[0]} = S[0]/S \in AP(S)$ also. Furthermore $\overline{S[0]} \in SI_c(S)$ so that $\overline{S[0]} \in \mathbb{I}_c(S)$. The remark concerning the number $1^{(I)}$ is a corollary to clause ( ) of Theorem , since $I \in AP(S)$. From clause ( )

of Theorem , $bI \in AP(S)$ if $I \in AP(S)$ and, from clause (Cvii) of the present Theorem , $bI \in SI(S)$ if $I \in SI(S)$. Hence $bI \in \mathbb{I}(S)$ if $I \in \mathbb{I}(S)$. Let $M \leq S$ be nonvoid. From clause (Cviiia) , $I/'M, I/''M \in SI(S)$ if $I \in SI(S)$. From clause ( ) of Theorem , if $I \in AP(S)$, $I/'M \in AP(S)$ and, if $M \in M(S)$, $I/''M \in AP(S)$. The remark made in part (D) concerning the ideals $I/'M$ and $I/''M$ has been justified.

Concerning the systems $I \wedge J, \ldots, I \vee J$ it is first remarked that $I, J \in SI(S)$ and, from clauses (Cvi, Biii) in turn IJ, $I \wedge J = \mathcal{A}(IJ) \in SI(S)$. Since $\mathcal{A}(IJ) \in AP(S)$, $I \wedge J \in \mathbb{I}(S)$. Again, $I, J \in AP(S)$ and, from clause (Cviiia) of the present Theorem and clause ( ) of Theorem , $I \cap J \in SI(S) \cap AP(S)$. From clauses (Cviiia, Bii), $I \cup J \in SI(S)$ and $I \vee J = \mathcal{A}(I \cup J) \in SI(S)$. Since $\mathcal{A}(I \cup J) \in AP(S)$, $I \vee J \in \mathbb{I}(S)$. All sums occurring in $\mathcal{A}(IJ) = I \wedge J$ are representations of numbers belonging to both I and J: $I \wedge J \leq I \cap J$. Also $I \cap J \leq I \cup J \leq \mathcal{A}(I \cup J) = I \vee J$. Lastly, from clause ( ) of Theorem , $\mathcal{A}(I \cup J) = I + J$, since $I, J \in AP(S)$.

That the ideals $\max_I J$ and $\max_{cI} J$ exist is demonstrated by methods used for the corresponding results concerning $\max_{SI} C$ and $\max_{cSI} C$ in clause (Cix). Now J contains $S[0] \in \mathbb{I}(S)$. Suppose that A, A' satisfying the conditions ascribed to I in clause ( ) of Definition exist, with

$A \neq A'$. Set $I' = A \vee A'$. Since $A, A' \subseteq J$, $A \cup A' \subseteq J$ and, from clause ( ) of Theorem , $A \vee A' = \mathcal{A}(A \cup A') \subseteq \mathcal{A}(J) = J$. Also, from the above, $A \vee A' \in \mathbb{I}(S)$. Again, either $I' \setminus A$ or $I' \setminus A'$ is nonvoid, and A, A' do not both satisfy the conditions attributed to I: $\max_I J$ is well defined. That $\max_{cI} J$ is well defined is proved in the same way.

The proof that the S-ideals $\min_I J$ and $\min_{cI} J$ exist is also similar to that of the corresponding semi S-ideals $\min_{SI} A$ and $\min_{cSI} A$ of clause (Cix). Now, from clause (Cviii b) and the above, $B \cap B'' \in \mathbb{I}(S) \langle \mathbb{I}_c(S) \rangle$ when $B, B' \in \mathbb{I}(S) \langle \mathbb{I}_c(S) \rangle$.

Concerning the final result of part (D), it is first remarked that if $a = b \bmod I$, $a - b = u \in I$. $ac = (b+u)c = v \in I$ only if $bc = v - uc \in I$ or, when $b \in D$, only when $c \in I$. Hence $a \in D$ if $b \in D$. The converse assertion is proved in the same way.

When $I \in \mathbb{I}(S)$, $\min I$ and $\liminf I$ are semi S-ideals but may not be S-ideals. However, it is possible to set $\min' I = \mathcal{A}(SI)$ and $\liminf' I = \mathcal{A}\{S^2 \cap I\}$; $\min' I$ and $\liminf' I$ are then S-ideals for which $\min' I \subseteq \liminf' I \subseteq I$.

The results of "Theorem , holding for general additively perfect systems, naturally have application to S-ideals. For example when $I, J, K \in \mathbb{I}(S)$, the ideal $I \vee J \vee K$ is well defined without bracketing, and so on.

Definition Let $I \in \mathbb{I}(S)$

(i) $(a)_I$ is the class of S-numbers b for which $b = a \bmod I$; such a class is an S_I-number.

(ii) With a' any S-number belonging to $(a)_I$, and b' any member of $(b)_I$, (a) $(a)_I = (b)_I$ if and only if $a' = b' \bmod I$; (b) $(a)_I (b)_I$ is the class of numbers c' such that $c' = a'b' \bmod I$; (c) when a and b belong to the same S-stratum, $(a)_I \pm (b)_I$ are the classes of S-numbers c' for which $c' = a' \pm b' \bmod I$.

(iii) $S_I\{j\}$ is the stratum of S_I-numbers $(a)_I$ with $a \in S\{j\}$, and that member of this stratum for which $a \in I$ is denoted by $(0)_{I(j)}$ $(j:)$. S_I is the complete system of strata $S_I\{j\}$ $(j:)$; it is the I-reduction of S. An I-reduction of S for some S-ideal I is an I-reduction of S.

Theorem. (i) $S \in \mathbb{S}$ and $I \in \mathbb{I}(S)$ uniquely determine the ideal reduction S_I of S with respect to I, and $S_I \in \mathbb{S}$.

(ii) $S_{S[0]} \sim S$ and S_S is a null S-ring

(iii) In S_I, $\mathfrak{D}\langle T\rangle\{T_S\}(S_I, S_I[0])$ is the complete system of numbers $(a)_I$ with $a \in \mathfrak{D}\langle T\rangle\{T_S\}(S, I)$.

Proof. Each class $(a)_I$ is uniquely determined by a and I: the elements of S_I are well determined. To complete the proof of clause (i) it has to be shown that the equivalence relationship and arithmetic operations of clauses (iia-c) of Definition are

unequivocally defined in the sense that they do not depend upon the choices of a', b' in $(a)_I, (b)_I$ respectively, and that the set of classes and relationships between them determine an S-ring.

Suppose that $(a)_I, (b)_I$ contain a'', b'' as well as a', b' so that $u', \ldots, v'' \in I$ exist such that $a-a'=u'$, $a-a''=u''$, $b-b'=v'$, $b-b''=v''$. If $a'=b' \bmod I$, $w \in I$ exists such that $a'-b'=w$ and then $a''-b''=w+u'-v'+v''-u'' \in I$ since $I \in AP(S)$, so that $a''=b'' \bmod I$. Similarly it follows that $a''=b'' \bmod I$ if and only if $a'=b' \bmod I$. Again if $c'=a'b' \bmod I$, $x \in I$ exists such that $c'-a'b'=x$ and then $c'-a''b''=x+b(u''-u')-a(v''-v')+u'v'-u''v'' \in I$ since $1 \in I(S)$. Thus $c'=a''b'' \bmod I$ if and only if $c'=a'b' \bmod I$. With regard to clause (iic) of Definition, it is similarly shown that $c'=a''\pm b'' \bmod I$ if and only if $c'=a' \pm b' \bmod I$.

Every stratum $S\{j\}$ contains at least one element of I, namely $0_{(j)}$: all strata $S_I\{j\}$ are nonvoid. With respect to the members of a stratum $S_I\{j\}$, equality is determinative, reflexive, symmetric and transitive; the members form an additive Abelian group with zero member $0_{I(j)}$. Multiplication of S_I-numbers is associative and commutative, and distributive with respect to addition.

When $I=S[0]$, $(a)_I$ contains but one member, namely a itself. The equivalence relationship and arithmetic operations for the $(a)_I$ described in clauses (iia-c) of Definition are, in

this case, precisely those holding over S-itself. Hence $S_{S[0]} \sim S$. When $I \equiv S$, every member of $S\{j\}$ is in $O_{I(j)}$ $(j:)$: S_S is a null S-ring.

Given (a) and S-ring S' isomorphic to S_I, (b) a single fixed representative S-number taken from each distinct S_I-number class and (c) the S-ideal I, it is possible to establish an isomorphism between the system of distinct members of S and a system of ordered pairs of elements taken from S' and I. Let $a', b', \ldots \in S'$ correspond to the distinct S_I-number classes $(a'')_I, (b'')_I, \ldots$ and select fixed numbers a from $(a'')_I$, b from $(b'')_I, \ldots$ arbitrarily. Any $c \in S\{j\}$ belongs to one and only one of the classes $(a'')_I, (b'')_I, \ldots$ in $S_I\{j\}$, to $(a'')_I$ say. Set $c \sim (a', u)$ where $u = c - a \in I$, so that $u \in I\{j\}$; c uniquely determines the pair (a', u). With $a' \in S'\{j\}$, $u \in I\{j\}$ the ordered pair (a', u) uniquely determines the S-number $c = a + u$ in $S\{j\}$. With $a_1', a_2' \in S'\{j\}$, $u_1, u_2 \in I\{j\}$ and $a_1' + a_2' = a_3'$ in $S\{j\}$, addition between ordered pairs of the form (a', u) is defined by

$$(a_1', u_1) + (a_2', u_2) = (a_3', u_1 + u_2 + a_1 + a_2 - a_3)$$

(Since $a_1 + a_2 - a_3$ in S is a member of I, the sum is also an ordered pair of the form (a', u) with, in particular, $u \in I$.) With $c_1 \sim (a_1', u_1)$, $c_2 \sim (a_2', u_2)$, so that $c_1 = a_1 + u_1$, $c_2 = a_2 + u_2$, the term on the right hand side of the above relationship corresponds to the S-number $a_3 + u_1 + u_2 + a_1 + a_2 - a_3 = c_1 + c_2$: the correspondence of the form $c \sim (a', u)$ is preserved during addition. With $a_1' \in S'\{j\}$,

$a_2' \in S'\{k\}$, $u_1 \in I\{j\}$, $u_2 \in I\{k\}$ and $a_1'a_2' = a_3' \in S'\{j+k\}$ multiplication between ordered pairs of the form (a', u) is defined by

$$(a_1', u_1)(a_2', u_3) = (a_3', (u_1 + a_1)(u_2 + a_2) - a_3)$$

Again, since $a_1a_2 - a_3 \in I$, the second member of the pair on the right hand side of this relationship is in I. The pair itself corresponds to $a_3 + c_1c_2 - a_3 = c_1c_2$: the correspondence of the form $cv(a', u)$ is preserved during multiplication.

For the sake of completeness, it is mentioned that other S-rings are generated by ordered pairs (a', u) of numbers taken from S' and I. For example, addition and multiplication may be defined in terms of operations upon components independently. Thus $(a_1', u_1)(a_2', u_2) = (a_1'a_2', u_1u_2)$ for multiplication

More generally a suitable mapping $\phi : S' \times S' \to I$ may be introduced and addition defined by the relationship

$$(a_1', u_1) + (a_2', u_2) = (a_3', u_1 + u_2 + \phi(a_1', a_2'))$$

$\phi(a_1', a_2') = a_1 + a_2 - a_3$ in the first of the above cases and $\phi(a_1', a_2') = 0$ in the second. Multiplication may be treated in the same way.

Definition. Let $I, J \in \mathbb{I}(S)$.

(i) J_I is the system of S_I-number classes $(a)_I$ that contain a member $a' \in J$

(ii) $(a)_{I,J}$ is the class of S_I-numbers $(b)_I$ for which $(b)_I - (a)_I \in J_I$; such a class is an $S_{I,J}$ number.

(iii) With $(a')_I$ any S_I-number belonging to $(a)_{I,J}$ and $(b')_I$ any member of $b_{I,J}$, $(a)_{I,J} = (b)_{I,J}$ if and only if $(a')_I = (b')_I \bmod J_I$; arithmetic operations involving $S_{I,J}$ numbers are, mutatis mutandis, as described in clauses (iii b, c) of Definition.

(iv) $S_{I,J}$ is the complete system of strata $S_{I,J}\{j\}$ constructed as in clause (iii) of Definition.

Theorem. (i) J_I is an S-ideal in S_I.

(ii) Let $K = I \vee J$. (a) $(a)_{I,J}$, defined as a class of classes of S-numbers is a class of S-numbers; the members of the latter class are those of the S_K-number class $(a)_K$. (b) $(a)_{I,J}$ and $(a)_{J,I}$ contain the same S-numbers. (c) If $J \leq I$, $(a)_{I,J}$ and $(a)_I$ contain the same S-numbers. (d) The S-rings $S_{I,J}$, $S_{J,I}$ and S_K are isomorphic.

Proof. It is easily verified that J_I is an S_I-ideal in S_I. Taking b' to be a representative member of $(b)_I$ in S_I, and a' to be a member of $(a)_I$ in J, the product $(a)_I (b)_I$ contains $a'b' \in J$: if $(a)_I \in J_I$ then $(b)_I (a)_I \in J_I$ for all $(b)_I$ in S_I. That the members of J_I are closed with respect to addition and subtraction within their strata is verified in the same way.

According to clause (ii) of Definition, $(b)_I \in (a)_{I,J}$ if and only if there is one member a' of $(a)_I$ and one member b' of $(b)_I$ such that $a' - b' = v \in J$. If this condition holds, all members b'' of $(b)_I$ belong to the class of S-numbers $(a)_{I,J}$. Also $a - a' =$

$u' \in I$, $b - b'' = u'' \in I$. Accordingly, the S-number b'' belongs to $(a)_{I,\bar{J}}$ if and only if $u', u'' \in I$ and $v \in \bar{J}$ exist such that $a - b'' = u' + v + u''$ or, since $I \in AP(S)$, if and only if $u \in I, v \in \bar{J}$ exist such that $a - b'' = u + v$. From clause ( ) of Theorem , $K = I \vee \bar{J}$ is the aggregate of numbers $u+v$ where $u \in I, v \in \bar{J}$ belong to the same S-stratum. Thus b'' belongs to the class of S-numbers associated with $(a)_{I,\bar{J}}$ if and only if $w \in K$ exists such that $a - b'' \in K$. The result of clause (iia) has been dealt with. That of clause (iib) follows from the relationship $I \vee \bar{J} = \bar{J} \vee I$, and that of clause (iic) from the remark that $I \vee \bar{J} = I$ if ~~and~~ $\bar{J} \subseteq I$. The results of clauses (iia,b) establish a one to one correspondence between the numbers of the systems $S_{I,\bar{J}}$, $S_{\bar{J},I}$ and S_K which is preserved during addition and subtraction within strata and during multiplication; the last result of the Theorem has been disposed of.

Definition . A) Let $U, V \subseteq S$, U being numerid, and $F(b)$ be a system of numbers which depends upon $b \in S$.

(i) $F(U) \equiv V$ means (1) that $F(b) \equiv V$ for all $b \in U$ and (2) that for all $b \in S \setminus U$, $F(b) \not\equiv U$

(ii) $V \subseteq F(U) \langle F(U) \subseteq V \rangle$ means (1) that $V \subseteq F(b) \langle F(b) \subseteq V \rangle$ for all $b \in U$ and (2) that to each $a \in S \setminus V$ when $S \not\equiv V \langle$to each $a \in V \rangle$ corresponds at least one $b \in U$ such that $a \in S \setminus F(b) \langle a \in F(b) \rangle$

B) Let $C \leq S$

(i) $O(S,C|b) = C/'b$ is the system orthogonal to b with respect to C

(ii) $P(S,C|b) = b(C/'b)$ is the projection of b into C

(iii) $Z(S,C|b)$, defined to be b when $C/'b$ is void and $\cap(C/'c)$ $(c \in C/''b)$ otherwise, is the numerator system of b with respect to C

(i) $a \leq b \bmod C$ means that $a \in Z(S,C|b)$

(ii) $a \equiv b \bmod C$ means that $a \leq b, b \leq a \bmod C$

(iii) $E(S,C|b)$ is the system of numbers a for which $a \equiv b \bmod C$

(iv) $\overline{T}'(S,C) \langle \overline{D}(S,C) \rangle$ is the complete system of numbers $b \in S$ for which $D\{Z(S,C|b), C\}$ is void $\langle$nonvoid$\rangle$; $\overline{T}'(S,C) = \overline{T}'(S,C) \setminus C$

(ii) An S-ring S for which $\overline{T}'(S,C) \langle \overline{D}(S,C) \rangle$ is void is said to be locally nontorsive $\langle$locally torsive$\rangle$ with respect to C

Again the sets $O\langle Z \rangle(S,I|b)$ in a directly decomposable S-ring have immediately visible forms. Supposing that each $b \in S\{j\}$ has the representation $b = \{b^{(k)}\}_{(k:n)}$ with $b^{(k)} \in S^{(k)}\{j\}$ $(k:n)$, the $S^{(k)}$ being torsion free, take I to be the subset of numbers c for which $c^{(k(\tau))} = 0 \in S^{(k(\tau))}\{j\}$ $(\tau : n' \leq n)$ and b to have the above representation with $b^{(m(\omega))} = 0 \in S^{(m(\omega))}\{j\}$ $(\omega : n'' < n')$, $\{m(\omega)\}$ being a subset of $\{k(\tau)\}$, and $b^{(h(\omega))} \neq 0$

$\in S^{(h(\omega))}\{j\}$ $(\omega : n'-n'')$, $\{h(\omega)\}$ being the set complementary to $\{m(\omega)\}$ in $\{k(\tau)\}$. $\mathfrak{O}(S,I|b)$ is the system of S-numbers g for which $g^{(h(\omega))} = 0 \in S^{(h(\omega))}\{j\}$ $(\omega : n'-n'')$; $\mathfrak{Z}(S,I|b)$ is the system of S-numbers a for which $a^{(n(\omega))} = 0 \in S^{(m(\omega))}\{j\}$ $(\omega : n'')$, the numbers in both cases possibly having other zero components.

Taking S to be the S-ring of quintics over a commutative ring, I to be the set of quintics $a\prod_{\tau=1}^{n} q_\tau$ for all $a \in S$, where q_τ is a fixed quintic selected from the stratum $S\{i(\tau)\}$ $(\tau : n)$, and b to be of the form $b = a\prod_{\tau=1}^{n'} q_{k(\tau)}$, $\{k(\tau)\}$ being a subset of the integers $1, \ldots, n$ and a having no q_τ as a factor, $\mathfrak{O}(S,I|b)$ is the system of quintics g of the form $d\prod_{\tau=1}^{n-n'} q_{m(\tau)}$ for all $d \in S$, $\{m(\tau)\}$ being the subset complementary to $\{k(\tau)\}$ in $1, \ldots, n$, and $\mathfrak{Z}(S,I|b)$ is the set of quintics of the form $d\prod_{\tau=1}^{n'} q_{k(\tau)}$ for all $d \in S$.

Theorem. Ai) If $F(U) \equiv V$, U is the largest system in S over which the condition $F(b) \equiv V$ for all $b \in U$ holds: $F(b) \equiv V$ if and only if $b \in U$.

iia) If $V \subseteq F(U) \langle F(U) \subseteq V \rangle$ V is the largest $\langle$smallest$\rangle$ system in S for which the relationship $V \subseteq F(b) \langle F(b) \subseteq V \rangle$ for all $b \in U$ holds. (b) $V \subseteq F(U) \langle F(U) \subseteq V \rangle$ if and only if $V \equiv \cap F(b)$ $(b \in U)$

$\langle V \equiv \cup F(b)\ (b \in U) \rangle$.

(iii) If $F(U) \equiv V$, $V \subseteq F(U) \subseteq V$

(iv) If $F(b) \equiv V$ for all $b \in U_1$ and $V \subseteq F(U_2) \langle F(U_2) \subseteq V \rangle$ then $V \subseteq F(U_1 \cup U_2) \langle F(U_1 \cup U_2) \subseteq V \rangle$

B) Let $C \subseteq S$.

ia) $\theta(S, C|b)$ is the complete system of numbers $a \in S$ for which $ab \in C$. (b) $a \in \theta(S, C|b)$ if and only if $b \in \theta(S, C|a)$. (c) If $A, B \subseteq S$ are nonvoid and $AB \subseteq C$, $A\langle B \rangle \subseteq \theta(S, C|b)$ for all $b \in B \langle b \in A \rangle$ and $\theta(S, C|b)$ is nonvoid for all $b \subseteq A \cup B$. (d) $\overline{C} \subseteq \theta(S, C|b)$ for all $b \in S$.

iia) If $\overline{C}$ is nonvoid, $\theta(S, C|\overline{C}) \equiv S$ (b) If $\overline{C} \subset S$, $\overline{C} \subseteq \theta(S, C|S \setminus \overline{C})$

c) If S is not torsion free with respect to C, $\theta(S, C|b) \subseteq \overline{T}(S, C)$ for all $b \in \overline{T}(S, C)$. (d) If S is nontorsion with respect to C, $\theta(S, C|b) \subseteq C$ for all $b \in D(S, C)$

iii) $\theta(S, C|b) \equiv \theta(S, C|\text{min inf } C)$ for all $b \in S$

iv) Let $B \subseteq S$. (a) $\theta(S, B \cap C|b) \subseteq \theta(S, B|b) \cap \theta(S, C|b)$ for all $b \in S$

(b) $\theta(S, B|b) \cup \theta(S, C|b) \subseteq \theta(S, B \cup C|b)$ for all $b \in S$. (c) If $B \subseteq C$ $\theta(S, B|b) \subseteq \theta(S, C|b)$ for all $b \in S$.

v) Let $C \in S1(S)$. (a) $\theta(S, C|b) \in S1(S)$ and $C \subseteq \theta(S, C|b)$ for all $b \in S$. (b) If S is not torsion free. with respect to C, $\theta(S, C|T(S, C)) \subseteq \overline{T}(S, C)$. (c) If S is nontorsion with respect to C, $\theta(S, C|D(S, C)) \equiv C$. (d) $\theta(S, C|a) \cup \theta(S, C|b) \subseteq \theta(S, C|ab)$ for all $a, b \in S$.

vi) Let $I \in \mathbb{I}(S)$. (a) $\theta(S, I|b) \in \mathbb{I}(S)$ and $I \subseteq \theta(S, I|b)$ for all $b \in S$.

(b) $\mathcal{O}(S,I|a) \vee \mathcal{O}(S,I|b) \leq \mathcal{O}(S,I|ab)$ for all $a,b \in S$. (c) For all $a,b,x,y \in S$ such that $axc \in S[by]$, $\mathcal{O}(S,I|a) \cap \mathcal{O}(S,I|b) \leq \mathcal{O}(S,I|axc+by)$. (d) If $J \in \mathbb{I}(S)$ also, $\mathcal{O}(S, I \wedge J|b) \leq \mathcal{O}(S, I \cap J|b)$ and $\mathcal{O}(S,I|b) \cup \mathcal{O}(S,J|b) \leq \mathcal{O}(S, I \vee J|b)$ for all $b \in S$. (e) If $a \equiv b \bmod I$ then $\mathcal{O}(S,I|a) \equiv \mathcal{O}(S,I|b)$ and, for all $c \in S$, $a \in \mathcal{O}(S,I|c)$ if and only if $b \in \mathcal{O}(S,I|c)$.

(i) $P(S,C|b)$ is the complete system of numbers c of C having a representation of the form $c = ab$ where $a \in S$.

(iia) If $\bar{C} \setminus S[0]$ is nonvoid, $K(S,\bar{C}) \leq P(S,C|\bar{C} \setminus S[0])$. (b) If C is nonvoid $P(S,C|C) \leq C \cap \min C$ (c) If S is not null and C is nonvoid, $C \cap K(S,S) \leq P(S,C|S \setminus S[0])$. (d) If S is not null, $P(S,C|S \setminus S[0]) \leq \liminf C$. (e) $P(S,C|S) \leq \liminf C$

(iii) $P(S,C|b) \equiv P(S, \liminf C|b)$ for all $b \in S$.

(iv) With $\mathcal{O}(S, B \cap C|b)$, ... replaced by $P(S, B \cap C|b)$, ... the results of clause (Biv) hold, as do similarly modified versions of clauses (Bvb, vib, c, d); also, when $C \in \mathbb{SI}(S)$, $P(S,C|b) \in \mathbb{SI}(S)$ for all $b \in S$, the system $C \cap \min C$ in clause (Ciib) may be replaced by $\min C$ and if, in addition, $C \setminus S[0]$ is nonvoid, $C \cap K(S,S)$ in clause (Ciic) may be replaced by $K(S,S)$; furthermore, $P(S,I|b) \in \mathbb{II}(S)$ for all $b \in S$ when $I \in \mathbb{II}(S)$.

Dia) For all $b \in S$, $Z(S,C|b)$ is, when C/b is void, simply b, and otherwise the complete system of numbers $a \in S$ such that $ac \in C$ for all $c \in S$ for which $bc \in C$ <such that

$\mathcal{O}(S,C|b) \subseteq \mathcal{O}(S,C|a)\rangle$. (b) $b \in Z(S,C|b)$ for all $b \in S$. (c) $a \in Z(S,C|b)$ if and only if $Z(S,C|a) \subseteq Z(S,C|b)$. (d) $\bar{C} \subseteq Z(S,C|b)$ for all $b \in S$.

(iia) If $\bar{C}$ is nonvoid, $Z(S,C|\bar{C}) = \bar{C}$. (b) If $\bar{C}$ is nonvoid, $\bar{C} \subseteq Z(S,C|S)$. (c) If S is not torsion free <not square free> with respect to C, $Z(S,C|b) \setminus C \subseteq T\langle T_S\rangle(S,C)$ for all $b \in T\langle T_S\rangle(S,C)$

iii) $Z(S,C|b) = Z(S, \liminf C|b)$ for all $b \in S$

iva) $Z(S,C|a)Z(S,C|b) \subseteq Z(S,C|ab)$ for all $a,b \in S$ (b) If $ab \in C$, $Z(S,C|a\langle b\rangle) \subseteq \mathcal{O}(S,C|b\langle a\rangle)$

(v) With $B \subseteq S$, $Z(S,B|b) \cap Z(S,C|b) \subseteq Z(S,B \cap C|b)$ and $Z(S,C|b) \cap Z(S,C|b) \subseteq Z(S,B \cup C|b)$ for all $b \in S$

(vi) Let $C \in SI(S)$. (a) $Z(S,C|b) \in SI(S)$ and $C \subseteq Z(S,C|b)$ for all $b \in S$. (b) $\bar{C} \subseteq Z(S,C|\bar{T}\langle \bar{T}_S\rangle(S,C)) \subseteq \bar{T}\langle \bar{T}_S\rangle(S,C)$. (c) If S is not torsion free <not square free> with respect to C, $Z(S,C|T\langle T_S\rangle(S,C)) \subseteq \bar{T}\langle \bar{T}_S\rangle(S,C)$. (d) If S is nontorsive with respect to C, $Z(S,C|D(S,C)) = S$. (e) $Z(S,C|ab) \subseteq Z(S,C|a) \cap Z(S,C|b)$ for all $a,b \in S$

(vii) Let $I \in \mathbb{I}(S)$. (a) $Z(S,I|b) \in \mathbb{I}(S)$ for all $b \in S$. (b) $Z(S,I|a) \cup Z(S,I|b) \subseteq Z(S,I|ab) \subseteq Z(S,I|a) \cap Z(S,I|b)$ for all $a,b \in S$. (c) If either (1) $v \in S$ exists such that $b+v \in D(S,I)$ or (2) $u_1, v \in S$ exist such that $u_1 b + v \in D(S,I)$ or (3) $u_1, u_2, v \in S$ exist such that

$u_1 b + u_2 v \in D(S, I)$ then $\theta(S, I | b\langle v\rangle) \subseteq Z(S, I | v\langle b\rangle)$. (d) If one of the preceding conditions (1, 2, 3) holds and also $bv \in I$ then $\theta(S, C | b\langle v\rangle) \equiv Z(S, I | v\langle b\rangle)$. (e) If $a \equiv b \bmod I$ then $Z(S, I | a) \equiv Z(S, I | b)$ and, for all $c \in S$, $a \in Z(S, C | b)$ if and only if $b \in Z(S, I | c)$.

Ei(a) $a \leq a \bmod C$ for all $a \in S$. (b) If $a \leq b$, $b \leq c \bmod C$ then $a \leq c \bmod C$ (c) If $a \leq b \bmod C$ then $ac \leq bc \bmod C$ for all $c \in S$. (d) If $a \leq b$, $c \leq d \bmod C$ then $ac \leq bd \bmod C$ (e) Let $b \in D(S, C)$. If $ab \leq c \bmod C$ and (α) $C/'C$ is nonvoid, then $a \leq c \bmod C$; if $a \leq c \bmod C$ and (β) $C/'bc$ is nonvoid, then $a \leq bc \bmod C$. (f) The relationship $a \equiv b \bmod C$ is an equivalence relationship over S. (g) $a \equiv b \bmod C$ if and only if $Z(S, C | a) \equiv Z(S, C | b)$. (h) $b \in E(S, C | b) \subseteq Z(S, C | b)$ for all $b \in S$. (i) If $Z(S, C | b)$ is nontorsive with respect to C, $E(S, C | b) \subseteq D\{Z(S, C | b), C\}$.

ii) Let $C \in \mathfrak{I}(S)$. (a) If $a \leq c \bmod C$ then $ab \leq c$ for all $b \in S$. (b) If $b \in D(S, C)$, $a \leq b \bmod C$ for all $a \in S$. (c) Conditions (α, β) may be discarded from clause (Eid). (d) If $Z(S, C | b)$ is nontorsive with respect to C, $E(S, C | b) \equiv D\{Z(S, C | b), C\}$. (e) Let $Z(S, C | a)$ and $Z(S, C | b)$ be nontorsive with respect to C. Either (1) $D\{Z(S, C | a), C\}$ and $D\{Z(S, C | b), C\}$ are disjoint in S or (2) $Z(S, C | a) \equiv Z(S, C | b)$ and, in particular, $D\{Z(S, C | a), C\} \equiv D\{Z(S, C | b), C\}$. (f) If $b \in D(S, C)$ then $D\{Z(S, C | b), C\} \equiv$

$D(S,C)$.

iii) Let $I \in \mathbb{I}(S)$. If $a=b \bmod I$ then $a \equiv b \bmod I$

Fia) $\bar{C} \setminus C \subseteq T_S(S,C) \subseteq T'(S,C) \subseteq T(S,C)$. (b) $D(S,C) \subseteq \bar{D}'(S,C)$. (c) $\bar{T}'(S,C)$ and $\bar{D}(S,C)$ are complementary in S; $T'(S,C)$ and $C \langle T'(S,C)$ and $\bar{D}(S,C) \rangle \{T'(S,C)$ and $D(S,C)\}$ are disjoint in S. (d) If $\bar{C} \setminus C$ is nonvoid, $\bar{C} \subseteq \bar{T}'(S,C)$ and $\bar{C}$ and $\bar{D}(S,C)$ are disjoint in S. (e) If $\bar{C} \subseteq C$, $\bar{C} \subseteq \bar{D}(S,C)$ and $\bar{C}$ and $\bar{T}'(S,C)$ are disjoint in S.

(ii) Let $C \in S1(S)$. (a) If C is not maximal in S, C and $\bar{D}(S,C)$ are disjoint in S and $\{\bar{D}(S,C) \cup C\} \in M(S)$. (b) If C is maximal in S, C and $\bar{T}'(S,C)$ are disjoint in S, $\bar{T}'(S,C) = \bar{T}(S,C)$, $\bar{D}(S,C)$ and $T(S,C)$ are complementary in S, and $\bar{D}(S,C) \in M(S)$.

Proof. The abbreviation D for $D(S,C)$ is used; the symbols $\bar{T}$, T_S, T, $\bar{T}_S$, T', $\bar{T}'$ and $\bar{D}$ have similar meanings. $\theta(b)$ is used as an abbreviation for $\theta(S,C|b)$ and $\theta(S,I|b)$, the context clearly indicating which is meant; $P(b)$ and $Z(b)$ have similar meanings. $D(b)$ and $E(b)$ are abbreviated forms of $D\{Z(S,C|b),C\}$ and $E(S,C|b)$. $a \leq b \bmod C$ and $a \equiv b \bmod C$ are written simply as $a \leq b$ and $a \equiv b$. The abbreviated forms $D(b),\ldots,a \equiv b$ are also used as described with C replaced by I, it again being clear from the context which abbreviation is meant.

The results of clause (iia) are directly implied by conditions

(1,2) of clause (Ai) of Definition . If $V \subseteq F(U) \langle F(U) \subseteq V \rangle$ and V' is taken to be $V \cup a \langle V \setminus a \rangle$ for some $a \in S \setminus V \langle a \in V \rangle$ there is a $b \in U$ such that $a \notin F(b) \langle a \in F(b) \rangle$ and for this b, $V' \not\subseteq F(b)$ $\langle F(b) \not\ni V' \rangle$; the results of clause (Aiia) follows. For the proof of the unbracketed result of clause (Aiib), let $V' = \cap F(b)$ $(b \in U)$. The relationship $V \subseteq F(b)$ holding for all $b \in U$ implies that $V \subseteq V'$. Conversely, the condition that for each $a \in S \setminus V$ at least one $b \in U$ such that $a \in S \setminus F(b)$ exists implies that $S \setminus V \subseteq \cup \{S \setminus F(b)\}$ $(b \in U)$ or, forming the complement in S on both sides, that $V' \subseteq V$. Thus $V \equiv V'$ as stated. Again, in the case of the notation $F(U) \subseteq V$, let $V'' = \cup F(b)$ $(b \in U)$. Now the relationship $F(b) \subseteq V$ holding for all $b \in U$ implies that $V'' \subseteq V$. Conversely, the condition that for each $a \in V$ at least one $b \in U$ such that $a \in F(b)$ exists implies that $V \subseteq V''$, so that in turn $V \equiv V''$. If $F(U) \equiv V$, $F(b) \equiv V$ for all $b \in U$ and $V \equiv \cap F(b)$ $(b \in U)$ so that, from the preceding clause, $V \subseteq F(U)$. That the relationship $F(U) \equiv V$ implies that $F(U) \subseteq V$ is proved in the same way. The two conditions $F(b) \equiv V$ for all $b \in U_1$ and $V \equiv \cap F(b)$ $(b \in U_2)$ $\langle V \equiv \cup F(b)$ $(b \in U_2) \rangle$ imply that $V = \cap F(b)$ $(b \in U_1 \cup U_2)$ $\langle V \equiv \cup F(b)$ $(b \in U_1 \cup U_2) \rangle$, the unbracketed result of clause (Aiv).

Clause (Bia) is simply a corollary to clause ( ) of Theorem , with $M = b$. Clause (Bib) follows directly from its predecessor. If

$AB \subseteq C$ and $b \in B$, then $bA \subseteq C$ and $A \in \mathcal{O}(b)$ for all $b \in B$. A and B can be interchanged in this result and, since A and B are nonvoid, $\mathcal{O}(b)$ is nonvoid for all $b \in A \cup B$. If $\bar{C}$ is void, clause (Bid) is correct; otherwise this clause is a corollary to its predecessor, obtained by setting $A = S$, $B = \bar{C}$. When $\bar{C}$ is nonvoid, $bS \subseteq C$, i.e. $\mathcal{O}(b) \equiv S$ for all members b of $\bar{C}$. $\bar{C}$ is the complete system of numbers b for which $bS \subseteq C$: if $b \notin \bar{C}$, $\mathcal{O}(b) \neq S$. Hence $\mathcal{O}(\bar{C}) \equiv S$, the result of clause (Biia). From clause (Bid) $\bar{C} \subseteq \mathcal{O}(S, \bar{C} | b)$ for, in particular, all $b \in S \setminus \bar{C}$. If $a \notin \bar{C}$, S contains a number b for which $ab \notin C$. This b is not in $\bar{C}$, since $Sb \in C$ for all $b \in \bar{C}$. Hence if $a \notin \bar{C}$, $b \in S \setminus \bar{C}$ for which $a \notin \mathcal{O}(b)$ exists: $\bar{C} \subseteq \mathcal{O}(S \setminus \bar{C})$, the result of clause (Biib). If $b \in T$ and $a \in \mathcal{O}(b)$, either $a \in C$ or a is not in C but is partnered by a number also not in C, namely b, for which $ab \in C$ and in this case $a \in T$. Hence $\mathcal{O}(b) \subseteq T$ for all $b \in T$, the result of clause (Biic). If $b \in D$, $ab \in C$ only when $a \in C$: $\mathcal{O}(b) \subseteq C$ for all $b \in D$, the result of clause (Biid).

$\mathcal{O}(b)$ is the largest system $Y \subseteq S$ for which $bY \subseteq C$, and $\mathcal{O}(S, \liminf C | b)$ is the largest system $Y' \subseteq S$ for which $bY' \subseteq C \cap S^2$. Since $C \cap S^2 \subseteq C$, $Y' \subseteq Y$. For any $a \in Y$, $ba \in C$ and $ba \in S^2$; hence $ba \in C \cap S^2$ and $Y \subseteq Y'$. Thus $Y \equiv Y'$, the result of clause (Biii).

The results of clause (Biv) are special cases of those of clauses (,) of Theorem obtained by setting $M = b$. The first

result of clause (Bva) is obtained ~~by~~ from clause ( ) of Theorem in the same way. Since $C \subseteq \bar{C}$ when $C \in SI(S)$ the second result follows from clause (Bid) of the present theorem. From clause (Bic) $O(b) \subseteq \bar{T}$ for all $b \in T$. If $a \in C$, $a \in O(b)$ for all $b \in T$, from the preceding clause. If $a \in \bar{T}$, T also contains a number b for which $ab \in C$; for this $b \in \bar{T}$, $a \in O(b)$. Hence $O(T) \subseteq \bar{T}$, the result of clause (Bvb). From clause (Bva), $C \subseteq O(b)$ for, in particular, all $b \in D$, and from clause (Biid) $O(b) \subseteq C$ for all $b \in D$. Hence $O(b) = C$ for all $b \in D$. If $b \notin D$, i.e. $b \in C \cup T$, T contains a number a such that $ab \in C$ and accordingly $O(b) \neq C$. Hence $O(D) = C$, the result of clause (Bvc). Clause (Bvd) is a corollary to clause ( ) obtained by setting $M = a$, $N = b$.

The first result of clause (Bvia) is a corollary to clause ( ) of Theorem and the second, since $I \in \mathbb{I}(S)$, follows from its counterpart in clause (Bva) of the present theorem. Since, when $O(a), O(b) \in \mathbb{I}(S)$, $O(a), O(b) \in AP(S)$ in particular, $O(a) \vee O(b)$ is, from clause ( ) of Theorem , the system of numbers $u+v$ where $u \in S[v]$ and $u \in O(a)$, $v \in O(b)$. For such a number $u+v$, $ab(u+v) \in I$, since $I \in I(S)$. Hence $O(a) \vee O(b) \subseteq O(ab)$. For the following result of clause (Bvic) it suffices to remark that if $c \in O(a) \cap O(b)$ then $ac, bc \in I$, and for all x, y such that $axc \in S[by]$, $(ax+by)c \in I$ so that $c \in O(axc+by)$. From clause ( ) of Theorem , $I \wedge J \subseteq I \cap J$ when

$I, J \in \mathbb{I}(S)$; the first result of clause (Brid) is a consequence of clause (Bivc) of the present theorem. Again, since $I \cup J \leq \mathfrak{A}(I \cup J) = I \vee J$ when $I, J \in \mathbb{I}(S)$, the second result of clause (Brid) follows from clauses (Bivb,c) of the present theorem. If $a = b \bmod I$, $u = b - a \in I$. If $c \in \bar{O}(a)$, $ac = v \in I$ and then $bc = uc + v \in I$ so that $c \in \bar{O}(b)$ also. Hence $\bar{O}(a) \leq \bar{O}(b)$. a and b can be interchanged in this argument, so that $\bar{O}(a) = \bar{O}(b)$, the first result of clause (Brie). If $bc = w \in I$ then $ac = v - uc \in I$. Thus if $b \in \bar{O}(c)$, $a \in \bar{O}(c)$ for all $c \in S$ and conversely, yielding the second result.

$\bar{O}(b)$ is the largest system X of S-numbers satisfying the relationship $bX \leq C$. Accordingly all members of $P(b) = b\bar{O}(b)$ lie in C; no $a \in S \setminus \bar{O}(b)$ such that $ab \in C$ exists: all members of C having a representation ab where $a \in S$ belong to P(b), the result of clause (Ci). If $K(S, \bar{C})$ is void, $K(S, \bar{C}) \leq P(b)$ for all $b \in \bar{C} \setminus S[0]$. Assuming $K(S, \bar{C})$ to be nonvoid, let $a \in K(S, \bar{C})$. Corresponding to each $b \in \bar{C} \setminus S[0]$, $c(a,b) \in S$ such that $a = bc(a,b)$ exists. From clause (Biia) of the present theorem, $\bar{O}(b) = S$ for all $b \in \bar{C}$. Thus $a = b\bar{O}(b)$ for all $b \in \bar{C}$, or $K(S, \bar{C}) \leq P(b)$ for all $b \in \bar{C} \setminus S[0]$. For any $a \notin K(S, \bar{C})$, $b \in \bar{C}$ exists such that no $c(a,b) \in S$ can be found for which $a = bc(a,b)$. For this b, all $a \in P(b)$ have the form bc where $c \in S$. Hence $a \notin P(b)$ for this b. Thus $K(S, \bar{C}) \leq P(\bar{C} \setminus S[0])$, the result of clause (Ciia). All members of P(b) have the form ab where $a \in S$. Thus for all

$b \in C$, $P(b) \subseteq SC = \min C$. From clause (Ci), $P(b) \subseteq C$ for, in particular, all $b \in C$. Thus $P(b) \subseteq C \cap \min C$ for all $b \in C$. Any $c \in C \cap \min C$ has the form ab where $a \in S$, $b \in C$ and is in C. For the number b in question, $c \in P(b)$. Hence $P(C) \subseteq C \cap \min C$, the result of clause (Cii b). If $C \cap K(S,S)$ is void, $C \cap K(S,S) \subseteq P(b)$ for all $b \in S \setminus S[0]$. If $C \cap K(S,S)$ is nonvoid, fix $b \in S \setminus S[0]$ and select any $a \in C \cap K(S,S)$. Since $a \in K(S,S)$, $c(a,b) \in S$ exists such that $a = bc(a,b)$; furthermore $a \in C$, so that $a \in O(b)$: all $a \in C \cap K(S,S)$ belong to $P(b)$: $C \cap K(S,S) \subseteq P(b)$ for all $b \in S \setminus S[0]$. When $a \notin C \cap K(S,S)$, $a \notin K(S,S)$. For any $a \notin K(S,S)$, $b \in S \setminus S[0]$ exists such that no $c(a,b) \in S$ can be found for which $a = bc(a,b)$. For this b, all $a \in P(b)$ have the form bc where $c \in S$. Hence $a \notin P(b)$ for this b and $C \cap K(S,S) \subseteq P(S \setminus S[0])$, the result of clause (Cii c). For each $b \in S$, all members of $P(b)$ lie in C and also have the form $abc \in S^2$. Thus $P(b) \subseteq C \cap S^2 = \liminf C$ for all $b \in S$. Select $c \in C \cap S^2$. c has the form $c = ab$ and lies in C; from clause (Ci), $c \in P(b)$ for the b in question. Thus $P(S) \subseteq \liminf C$, the result of clause (Cii d).

The result of clause (Ciii) follows directly from that of clause (Biii) and the definition $P(b) = bO(b)$.

The results obtained from clauses (Biv, v b, vi b, c, d) by use of the modifications suggested in clause (Civ) are derived by multiplying the relationships concerned throughout

by b. Since $\breve{O}(b) \in \mathfrak{I}(S)$ for all $b \in \mathfrak{B}$, $K(b) = b\breve{O}(b) \in \mathfrak{I}(S)$ for all such b, from clause ( ) of Theorem . In this case, from clause ( ) of Theorem , $\min C \subseteq C$ so that $C \cap \min C$ may be replaced in clause (Ciib) simply by $\min C$; again, if $C \setminus S[b]$ is nonvoid, $K(S,S) \subseteq C$ from clause ( ) of Theorem , and $C \cap K(S,S)$ in clause (Ciic) may be replaced by $K(S,S)$. Lastly, since $\breve{O}(b) \in \mathfrak{I}(S)$ for all $b \in S$ when $I \in \mathfrak{I}(S)$, $P(b) = b\breve{O}(b) \in \mathfrak{I}(S)$ for all $b \in S$ also, from clause ( ) of Theorem .

That $Z(b)$ is simply b when $C'b$ is void is stated in clause (Biii) of Definition ; otherwise let $Z'(b)$ be the complete system of numbers $a \in S$ for which $ac \in C$ for all $c \in S$ such that $bc \in C$. $c \in \breve{O}(b)$ if and only if $bc \in C$. By definition, $Z(b) = \bigcap \breve{O}(c)$ $(c \in \breve{O}(b))$. For any $a \in Z(b)$, $ac \in C$ for all $c \in \breve{O}(b)$ i.e. for all c such that $bc \in C$. Hence $Z(b) \subseteq Z'(b)$. If $a \in Z'(b)$, $a \in \breve{O}(c)$ for all $c \in \breve{O}(b)$. Hence $a \in \bigcap \breve{O}(c)$ $(c \in \breve{O}(b))$ and $Z'(b) \subseteq Z(b)$. Accordingly $Z(b) = Z'(b)$. If $ac \in C$ for all $c \in S$ such that $bc \in C$ then all $c \in \breve{O}(b)$ belong to $\breve{O}(a)$, or $\breve{O}(b) \subseteq \breve{O}(a)$; the converse assertion is also true. Clause (Dia) has been dealt with. If $\breve{O}(b)$ is void, $b \in Z(b)$; if $\breve{O}(b)$ is nonvoid, $a = b$ is one of the numbers a described in the preceding clause. In either case $b \in Z(b)$. If $Z(a) \subseteq Z(b)$ then $a \in Z(a) \subseteq Z(b)$. If $a \in Z(b)$, $ac \in C$ for all $c \in S$ for which $bc \in C$; if $a' \in Z(a)$, $a'c \in C$ for all $c \in S$ for which $ac \in C$. Thus for all $c \in S$ for which $bc \in C$, $a'c \in C$; all $a' \in Z(a)$ belong

to $Z(b)$ when $a \in Z(b)$, the result of clause (Dic). If $\bar{C}$ is void, $\bar{C} \subseteq Z(b)$ for all $b \in S$. Otherwise, from clause (Büa), $O(a) = S$ for all $a \in \bar{C}$ and accordingly $O(b) \subseteq O(a)$ for all such a: $\bar{C} \subseteq Z(b)$ for all $b \in S$, the result of clause (Did).

From clause (Büa), $O(a) = S$ if and only if $a \in \bar{C}$, when $\bar{C}$ is nonvoid. Thus when $b \in \bar{C}$, $C/'b = S$ is nonvoid, and $O(b) \subseteq O(a)$ if and only if $a \in \bar{C}$, so that $Z(b) = \bar{C}$ for all $b \in \bar{C}$. Since $b \in Z(b)$ for all $b \in S$, $Z(b) \neq \bar{C}$ whenever $b \notin \bar{C}$. Hence $Z(\bar{C}) = \bar{C}$, the result of clause (Düa). From clause (Did) $\bar{C} \subseteq Z(b)$ for all $b \in S$. Select an $a \in \bar{C}$, so that a $c \in S$ such that $ac \notin C$ exists, and hence $c \notin O(a)$. However, from clause (Büa), $O(b) = S$, i.e. $c \in O(b)$ for all $b \in \bar{C}$. For the selected a, $O(b) \subseteq O(a)$ for all $b \in \bar{C}$ and, since $\bar{C}$ has been assumed nonvoid, S contains such a b. Hence $\bar{C} \subseteq Z(S)$, the result of clause (Düb). When $b \in T$, there exists a $c \notin C$ such that $bc \in C$. For all $a \in Z(b)$, $ac \in C$ also; for these $a \in Z(b) \setminus C$, $a \in S \setminus C$ and hence $a \in T$. Accordingly, $Z(b) \setminus C \subseteq T$ for all $b \in T$. If $b \in T_S$, $b \notin C$ but $b^2 \in C$. For all $a \in Z(b)$, the condition $b^2 \in C$ implies in turn that $abc \in C$ and $a^2 \in C$. For these $a \in Z(b) \setminus C$, $a \in S \setminus C$ and $a^2 \in C$, so that $a \in \bar{T}_S$. Accordingly $Z(b) \setminus C \subseteq \bar{T}_S$ for all $b \in T_S$. The two results of clause (Düc) have been disposed of.

The largest system X satisfying the condition $bX \subseteq C$ is

void if and only if the largest system X' satisfying the condition $bX' \subseteq S^2 \cap C$ is void. When $C/'b$ is void, $Z(b)$ and $Z(S, \lim\inf C|b)$ both reduce to b. Otherwise, from clause (Biii), the condition of the form $\bar{O}(b) \subseteq \bar{O}(a)$ a to be a member of $Z(b)$ is equivalent to the condition defining a to be a member of $Z(S, \lim\inf C|b)$. Clause (Diii) has been derived.

If $Z(a)$ and $Z(b)$ reduce to a and b respectively, $Z(a)Z(b) = ab \in Z(ab)$ from clause (Dib). Assuming that neither of these reductions occurs, let $c \in Z(a)$ and $d \in Z(b)$. The condition $abg \in C$ implies, since $c \in Z(a)$ that $cbg \in C$ and, since $d \in Z(b)$, that $cdg \in C$. Hence $cd \in Z(ab)$, or $Z(a)Z(b) \subseteq Z(ab)$. If $Z(a)$ reduces to a, but $Z(b)$ does not reduce to b, the members of $Z(a)Z(b)$ have the form ad, where $d \in Z(b)$ and again, by use of an argument similar to that just employed, $Z(a)Z(b) \subseteq Z(ab)$; the reduced form of $Z(b)$ is dealt with similarly. The result of clause (Dwa) has been obtained. If $ab \in C$, $C/'a$ and $C/'b$ are both nonvoid. For all $c \in Z(a)$, $bc \in C$ and $c \in \bar{O}(b)$. Hence $Z(a) \subseteq \bar{O}(b)$. Similarly $Z(b) \subseteq Z(a)$. The results of clause (Bivb) have been obtained.

If either $Z(S, B|b)$ or $Z(S, C|b)$ reduces to b, $Z(S, B|b) \cap Z(S, C|b)$ reduces to b; both results of clause (Dv) are then true. Otherwise let $a \in Z(S, B|b) \cap Z(S, C|b)$; with $g \in S$ the condition $bg \in B \cap C$ then implies, since $a \in Z(S, C|b)$ and $bg \in B$, that $ag \in B$ and,

since $a \in Z(S, C|b)$ and $bg \in C$, that $ag \in C$: $ag \in B \cap C$ for all $g \in S$ such that $bg \in B \cap C$. The first result of clause (Dv) has been obtained. Again with a as described, suppose that $bg \in B \cup C$ for some $g \in S$. If $bg \in B$, $ag \in B$; if $bg \in C$, $ag \in C$; in either case $ag \in B \cup C$: for all $g \in S$ such that $bg \in B \cup C$, $ag \in B \cup C$. The second result of clause (Dv) has been obtained.

When $C \in SI(S)$, $S[0] \subseteq C \subseteq \bar{C}$, $\bar{C}$ is nonvoid and $C/'b$ is nonvoid for all $b \in S$. If $a \in Z(b)$, $ac \in C$ for all $c \in S$ for which $bc \in S$ and, since $C \in SI(S)$, $acd \in C$ for all $d \in S$ whenever $bc \in S$. Thus $SZ(b) \subseteq Z(b)$ and accordingly $Z(b) \in SI(S)$ Also, from clause (Dva), $Z(b)$ contains $C \subseteq \bar{C}$ for all $b \in S$. The results of clause (Dvia) have been obtained. From clause (Did) $\bar{C} \subseteq Z(b)$ in particular for all $b \in \bar{\bar{T}}$. If T and D are void, $\bar{\bar{T}} = C = S$ and $\bar{C} \subseteq Z(\bar{\bar{T}})$. Otherwise, if $a \in \bar{C}$, S contains a number c such that $ac \notin C$, so that $c \notin O(a)$. However, from clause (Biia), $O(b) = S$ for all $b \in \bar{C}$ and; when $C \in SI(S)$, $\bar{C} \subseteq \bar{\bar{T}}$ from clause ( ) of Therem . For all such values of $b \in \bar{\bar{T}}$, $O(b) \subseteq O(a)$ and $a \notin Z(b)$. In consequence $\bar{C} \subseteq Z(\bar{\bar{T}})$. Since $Z(b) = \bar{C} \subseteq \bar{\bar{T}}$ for all $b \in C \subseteq \bar{C}$ when $C \in SI(S)$, $Z(b) \subseteq \bar{\bar{T}}$ for all $b \in C$. Any $b \in T$ is partnered by a number $c \in \bar{T}$ for which $bc \in C$ and, for all $a \in Z(b)$, $ac \in C$ also; either $a \in C \subseteq \bar{\bar{T}}$ or $a \notin C$, partnered by $c \notin C$ for which $ac \in C$, lies in $T \subseteq \bar{\bar{T}}$. Hence $Z(b) \subseteq \bar{\bar{T}}$ for all $b \in T$ and, since $Z(b) \subseteq \bar{\bar{T}}$ for all $b \in C$,

$Z(b) \subseteq \overline{T}$ for all $b \in T \cup C = \overline{T}$. Each $a \in T$ belongs to a system $Z(b)$ with $b \in \overline{T}$, namely, since $a \in Z(a)$, $Z(b)$ with $b = a$. Hence $Z(\overline{T}) \subseteq \overline{T}$. The proof of the result $\overline{C} \subseteq Z(\overline{T}_S)$ is as for its counterpart $\overline{C} \subseteq Z(\overline{T})$ above. Let $b \in T_S$. If $a \in Z(b)$ is in C then $a \in \overline{T}_S$ also. If $b \in T_S$ and $a \in Z(b)$ is not in C, the conditions $b^2 \in C$, $a \in Z(b)$ applied twice yield $ab \in C$ and $a^2 \in C$, so that $a \in T_S$. Hence $Z(b) \subseteq \overline{T}_S$ for all $b \in T_S$ and, again since $Z(b) \subseteq \overline{T}_S$ for all $b \in C$, $Z(b) \subseteq \overline{T}_S$ for all $b \in T_S \cup C = \overline{T}_S$. As before, each $a \in \overline{T}_S$ belongs to $Z(b)$ with $b = a \in \overline{T}_S$, and $Z(\overline{T}_S) \subseteq \overline{T}_S$. The results of clause (Dvib) have been obtained. For the results of clause (Dvic), it has been shown ~~that~~ above that $Z(b) \subseteq \overline{T} \langle \overline{T}_S \rangle$ for all $b \in T \langle b \in T_S \rangle$. Now if $a \in \overline{T}$ is in C, $a \in Z(b)$ for all $b \in T$. Otherwise if $a \in \overline{T}$ is in T, a is in $Z(b)$ with $b = a \in T$ and $Z(\overline{T}) \subseteq \overline{T}$. The bracketed result is treated in the same way. From clause (Bvc), $O(b) = C$ for all $b \in D$. Also, from clause (Did), $C = \overline{C} \subseteq O(a)$ for all $a \in S$ when $C \in Sl(S)$ and D is non-void. Hence when $b \in D$, $O(b) \subseteq O(a)$ for all $a \in S$; accordingly $Z(b) = S$ for all $b \in D$. If $b \notin D$, $b \in \overline{T}$ and, from clause (Dvib), $Z(b) \subseteq \overline{T}$, or $Z(b) \neq S$, for all $b \notin D$. Thus $Z(D) = S$, the result of clause (Dvid). The conditions $a \in Z(a)$ and $Z(a) \in Sl(S)$ imply that $ab \in Z(a)$ and, from clause (Dic), that $Z(ab) \subseteq Z(a)$ for all $a, b \in S$. Similarly, $Z(ab) \subseteq Z(b)$ or $Z(ab) \subseteq Z(a) \cap Z(b)$ for all $a, b \in S$, the result of clause (Drie)

Since $I \in \mathfrak{I}(S)$ when $I \in \mathbb{I}(S)$, $Z(b) \in \mathfrak{I}(S)$ for all $b \in S$ from clause (Dvia). If a, d belong to the same S-stratum and are such that $ac, dc \in I$ for all $c \in S$ such that $bc \in I$, then, since $I \in AP(S)$ when $I \in \mathbb{I}(S)$, $(a \pm d)c \in I$ also for all such c: $Z(b) \in AP(S)$ and hence $Z(b) \in \mathbb{I}(S)$ for all $b \in S$ when $I \in \mathbb{I}(S)$, the result of clause (Dviia). From clause (Dvia), $Z(a)Z(b) \leq Z(ab)$. Hence when $Z(ab) \in AP(S)$, $Z(a) \wedge Z(b) = \mathcal{A}\{Z(a)Z(b)\} \leq \mathcal{A}\{Z(ab)\}$ $\equiv Z(ab)$, the first result of clause (Dviib). The second result follows directly from clause (Dvie), since $I \in \mathfrak{I}(S)$ in this case. For the proof of clause (Dviic) suppose that conditions (3) obtain (the treatment of the other cases is simpler). Select $a \in \mathcal{O}(b)$, so that ~~$\mathcal{O}(b) \in \mathfrak{I}(S)$~~ since $\mathcal{O}(b) \in \mathfrak{I}(S)$, $au_1 bc \in I$ for all $c \in S$. For all $c \in S$ such that $vc \in I$, $au_2 vc \in I$ also, since $I \in \mathfrak{I}(S)$, and hence $(u_1 b + u_2 v)ac \in I$, since $I \in AP(S)$. However $u_1 b + u_2 v \in D$, so that $ac \in I$: for all $c \in S$ such that $vc \in I$, $ac \in I$ for all $a \in \mathcal{O}(b)$: $\mathcal{O}(b) \leq Z(v)$. Similarly $\mathcal{O}(v) \leq Z(b)$. If $bv \in I$, $Z(v) \leq \mathcal{O}(b)$ and $Z(b) \leq \mathcal{O}(v)$ from clause (Dvib). Thus if one of the conditions (1,2,3) of clause (Dviic) holds, and $bv \in I$, $\mathcal{O}(b) \equiv Z(v)$ and $\mathcal{O}(v) \equiv Z(b)$, the result of clause (Dviid). Since I/a is nonvoid when $I \in \mathbb{I}(S)$, the condition establishing c as a member of $Z(a)$ is $\mathcal{O}(a) \leq \mathcal{O}(c)$. From the first result of clause (Bvie), $\mathcal{O}(b) \equiv \mathcal{O}(a)$ when $a = b \bmod I$. Hence c is a member of $Z(a)$ if and only if it is a member of $Z(b)$. The first result of clause (Dviid) has been obtained. For the

second it suffices to remark that $a \in Z(c)$ if and only if $\delta(c) \leq \delta(a)$, so that when $a \equiv b \bmod I$ and in consequence $\delta(a) = \delta(b)$, $a \in Z(c)$ if and only if $b \in Z(c)$ for all $c \in S$.

Clause (Eia) is simply clause (Dib) expressed in another notation. If $Z(b)$ reduces to b, there is only one $a \in S$ for which $a \leq b$, namely $a = b$, and then the conditions $a \leq b$, $b \leq c$ imply that $a \leq c$. The case in which $Z(c)$ reduces to c is dealt with similarly. Otherwise the conditions $a \leq b$, $b \leq c$ imply, from clause (Dic), that $Z(a) \leq Z(b)$, $Z(b) \leq Z(c)$ so that $Z(a) \leq Z(c)$ and in turn that $a \leq c$, the result of clause (Eib). If $Z(b)$ reduces to b, the only $a \leq b$ is $a = b$ and then, since $bc \leq bc$, $ac \leq bc$ for all $c \in S$. Otherwise the conditions $a \leq b$, $bcg \in C$ imply that $acg \in C$. Hence if $a \leq b$, $ac \leq bc$ for all $c \in S$, the result of clause (Eic). If $Z(b)$ reduces to b, so that $a = b$ when $a \leq b$, the condition $c \leq d$ implies, from the preceding clause, that $ac \leq bd$. That the conditions $a \leq b$, $c \leq d$ imply that $ac \leq bd$ when $Z(d) \equiv d$ is shown in the same way. Otherwise the condition $bdg \in C$ implies, when $a \leq b$, that $adg \in C$ and, when $c \leq d$, that $acg \in C$. The result of clause (Eid) follows. If $C'c \langle C'bc \rangle$ is nonvoid $Z(c) \langle Z(bc) \rangle$ is the system of numbers a such that for all $g \in S$ for which $cg \in C \langle bcg \in C \rangle$ $ag \in C$ also. If $ab \leq c$, $abg \in C$ whenever $cg \in C$. If $b \in D$, the condition $abg \in C$ ensures that $ag \in C$; thus $a \in Z(c)$ if $ab \leq c$. Again if $b \in D$, the condition $bcg \in C$ ensures that $cg \in C$. Thus, if $a \leq c$, for all $g \in S$

such that $bcg \in C$, $ag \in C$ also. Thus $a \in Z(bc)$ if $a \leq c$. The two results of clause (Eie) have been obtained. For all pairs a, b of S-numbers it is either true or untrue that both $a \leq b$ and $b \leq a$. The partial ordering $a \equiv b$ is determinative. From clause (Eia), $a \leq a$ so that $a \equiv a$; the partial ordering $\equiv$ is reflexive. The conditions $a \leq b$, $b \leq a$ imply that $b \leq a$, $a \leq b$; thus if $a \equiv b$, $b \equiv a$ also; the partial ordering $\equiv$ is symmetric. Lastly the conditions $a \leq b$, $b \leq a$, $b \leq c$, $c \leq b$ imply, from clause (Eib), that $a \leq c$, $c \leq a$; hence if $a \equiv b$ and $b \equiv c$, $a \equiv c$; the partial ordering $\equiv$ is transitive. The result of clause (Eif) has been obtained. From clause (Dic), $a \leq b$ and $b \leq a$ if and only if $Z(a) \leq Z(b)$ and $Z(b) \leq Z(a)$, ie if and only if $Z(a) = Z(b)$, the result of clause (Eig). For all $a = b$, $a \in Z(b)$ so that $E(b) \leq Z(b)$. Also from clause (Eia), $b = b$ so that $b \in E(b)$. The results of clause (Eih) have been obtained. If $D(b)$ is nonvoid, $Z(b)$ contains a number d such that either (1) there is no $c \in Z(b)$ for which $dc \in C$ or (2) for all $a \in Z(b)$ such that $dc \in C$, $c \in C$ also. Select $a \in E(b)$, so that $d \in Z(a) = Z(b)$, from clause (Eig). If $c \in Z(a) = Z(b)$ exists such that $ac \in C$, then $dc \in C$ also. If d satisfies (1) of the preceding conditions, no such c exists and $a \in D(b)$. If d satisfies condition (2), $c \in C$ and again $a \in D(b)$. Hence all $a \in E(b)$ belong to $D(b)$, the result of clause (Ei,i).

Clause (Eiia) is a corollary to clause (Diva). From clause ( ), $Z(b) = S$ for all $b \in D$, and from clause (Did), $C \leq \bar{C} \leq Z(a)$ for all $a \in S$.

Hence $Z(a) \leq Z(b)$ or, from clause (Dic), $a \leq b$, for all $a \in S$ when $b \in D$, the result of clause (Eiib). When $C \in SI(S)$, C/a contains C, and is accordingly nonvoid, for all $a \in S$. Thus when $C \in SI(S)$, the conditions (α, β) in clause (Eid) may be discarded. Assuming $Z(b)$ to be nontorsive with respect to C, select $a \in D(b)$ so that in particular $a \leq b$. For all $g \in S$ such that $ag \in C$, $abg \in C$ also, since $C \in SI(S)$, and since $b \in Z(b)$ and $Z(b) \in SI(S)$, $bg \in Z(b)$ so that, since $a \in D(b)$, $bg \in C$. Hence $b \leq a$ and accordingly $a = b : D(b) \leq E(b)$. From clause (Ei, i), $E(b) \leq D(b)$ when $Z(b)$ is nontorsive with respect to C: subject to the latter condition $E(b) = D(b)$, the result of clause (Eiid). The result of clause (Eiie) is a corollary to its predecessor. If $D(a)$ and $D(b)$ possess a common element d, then, from the preceding clause, $d \in E(a)$ and $d \in E(b)$, so that $a = d = b$ and $Z(a) = Z(b)$. The conditions defining numbers in $E(a)$ as members of the subsystem $D(a)$ depend upon C but not upon a: when $E(a) = E(b)$, $D(a) = D(b)$ also. From clause (Dvid), $Z(b) = S$ for all $b \in D$ when $C \in SI(S)$. The conditions defining membership of D are those defining membership of $D(b)$; hence $D(b) = D$, the last result of clause (Eii).

The result of clause (Eiii) follows from the first result of clause (Driid) and clause (Dib): if $a = b \bmod I$, $Z(a) = Z(b)$ and since $a \in Z(a)$, $b \in Z(b)$, $a \leq b$ and $b \leq a$. (That, when a and b

belong to the same S-stratum, the condition $a \equiv b \bmod I$ does not imply that $a = b \bmod I$ can be seen by considering the case in which all strata of S are simply copies of the field of rational numbers and $I \in S[0]$. All nonzero members of any stratum of S belong to D and $a \equiv b$ for all $a, b \in D$; but it is not true that $a - b = 0$ for all such a, b.)

That $\bar{C} \setminus C \subseteq \bar{T}_S$ follows from clause ( ) of Theorem . When $\bar{T}_S$ is void, $\bar{T}_S \subseteq \bar{T}'$. Otherwise, if $Z(b)$ contains a nonzero system $D(b)$ such that for all $d \in D(b)$ either no $a \in Z(b)$ such that $da \in C$ exists, or for all $a \in Z(b)$ for which $da \in C$, $a \in C$, then, from clause (Ei, i) b is a member of the system $D(b)$. If $b \in \bar{T}_S$, $b \notin C$ but $b^2 \in C$; hence b does not qualify for membership of $D(b)$, and $D(b)$ is accordingly void. In this case $b \in \bar{T}'$ and hence $\bar{T}_S \subseteq \bar{T}'$. If $\bar{T}'$ is void, $T' \subseteq \bar{T}$. Otherwise, for each $b \in \bar{T}'$, $D(b)$ is void and hence there exists at least one number $a \in Z(b) \setminus C$ for which $ab \in C$; accordingly $b \in \bar{T} : T' \subseteq \bar{T}$. The results of clause (Fia) have been obtained. If D is void, $D \subseteq \bar{D}$. Otherwise, let $b \in D$; if no $a \in S$ exists for which $ab \in C$, no such $a \in \bar{Z}(b) \subseteq S$ exists and $b \in D(b)$; if, for all $a \in S$ such that $ba \in C$, $a \in C$, then for all $a \in Z(b) \subseteq S$, $ba \in C$ only when $a \in C$ and again $b \in D(b)$. D is accordingly a subsystem of $\bar{D}$, the result of clause (Fib). Since $T' = \bar{T}' \setminus C$, T' and C are disjoint; $\bar{T}'$ is a subsystem of $\bar{T}'$ so that T' and $\bar{D}$ are disjoint; from the preceding clause,

$\mathfrak{D}$ is a subsystem of $\mathfrak{D}$, so that $\mathfrak{T}'$ and $\mathfrak{D}$ are disjoint. All results of clause (Fic) have been obtained. If $\bar{C}\setminus C$ is nonvoid, $\bar{C}$ is nonvoid and, from clause (Diia), $Z(b)=\bar{C}$ for all $b\in\bar{C}$. When $b\in\bar{C}$, $ba\in C$ for all $a\in S$, in particular for $a\in\bar{C}\setminus C$. It is not true that for all $a\in Z(b)$ for which $ab\in C$, $a\in C:\mathfrak{D}(b)$ is void. Thus when $\bar{C}\setminus C$ is nonvoid, $\bar{C}\leq\bar{\mathfrak{T}}'(S,C)$, the first result of clause (Fid); since $\bar{\mathfrak{T}}'$ and $\bar{\mathfrak{D}}$ are complementary in S, the second follows. If $\bar{C}$ is void, $\bar{C}\leq\bar{\mathfrak{D}}$. Otherwise, if $\bar{C}\leq C$, $Z(b)=\bar{C}$ for all $b\in\bar{C}$ again, but now all members of $Z(b)$ are in C. $ab\in C$ for all $a\in Z(b)$ and all such a are in C. In this case b is a nontransitive element of $Z(b)$ with respect to C. Hence if $\bar{C}\leq C$, $\bar{C}\leq\bar{\mathfrak{D}}$, the first result of clause (Fie); again, since $\bar{\mathfrak{T}}'$ and $\bar{\mathfrak{D}}$ are complementary in S, the second follows.

If $C\in SI(S)$ and C is not maximal in S, $\bar{C}\setminus C$ is nonvoid, so that from clause (Fid), $C\subset\bar{C}$ and $\bar{\mathfrak{D}}$ are disjoint in S. The second result of clause (Fiia) holds whether C is maximal or not. If either of $a, b\in\bar{\mathfrak{D}}\cup C$ is in C, $ab\in C\leq\bar{\mathfrak{D}}\cup C$. Assume that neither is in C. If $ab\in C$ then $ab\in\bar{\mathfrak{D}}\cup C$ again. Otherwise for all $c\leq a$, $ac\in C$ only when $c\in C$ and similarly with regard to b. For any $c\leq ab$, $c\leq a$ and $c\leq b$, since $C\in SI(S)$; also $bc\leq a$. Thus for any $c\leq ab$, the condition $abc\in C$ implies, since $a\in\bar{\mathfrak{D}}$ and $bc\leq a$, that $bc\in C$ and in turn, since $b\in\bar{\mathfrak{D}}$ and $c\leq b$, that $c\in C\leq\bar{\mathfrak{D}}\cup C$. Hence for all $a, b\in\bar{\mathfrak{D}}\cup C$, $ab\in\bar{\mathfrak{D}}\cup C$.

The results of clause (Fiia) have been obtained. If C is maximal in S, $\bar{C}=C$, so that, from clause (Fie), $C=\bar{C}$ and $\bar{T}'$ are disjoint in S, $T'=\bar{T}'\setminus C=\bar{T}'$, and from clause (Fic), $T'=\bar{T}'$ and $\bar{D}$ are complementary in S. Now $C=\bar{C}\subseteq\bar{D}$ so that $\bar{D}\cup C=\bar{D}$ and, from the above proof, $\bar{D}\in M(S)$. The last results of the theorem have been derived.

The relationships $V\subseteq F(U)$ and $F(U)\subseteq V$ are sharp with respect to V in the senses described in clause (Aiia) of the above theorem. Nevertheless they imply nothing concerning extremal properties of U. If, however, to the definition of the relationship $V\subseteq F(U)\langle F(U)\subseteq V\rangle$ the condition that for each $d\in U\{d\notin U\}$ there exists a $c(d)\notin F(d)\langle c(d)\in F(d)\rangle$ such that $c(d)\in F(b)\langle c(d)\notin F(b)\rangle$ for all $b\neq d$ in U {all b in U} then, with V prescribed, U is the smallest {largest} system over which the relationship $V\subseteq F(U)\langle F(U)\subseteq V\rangle$ holds; also with $d\in U\{d\notin U\}$, $V\subset[V\cup c(d)]\subseteq[\cap F(b)\ (b\in U\setminus d)]$ $\{[\cap F(b)\ (b\in U\cup d)]\subseteq[V\setminus c(d)]\subset V\}\langle[\cup F(b)\ (b\in U\setminus d)]$ $\subseteq V\setminus c(d)\subset V\{V\subset[V\cup c(d)]\subseteq[\cup F(b)\ (b\in U\cup d)]\}\rangle$.

A converse to the result of clause (Aiii) may be stated as follows: if $V\subseteq F(U)\subseteq V$ and for each $d\notin U$, $F(d)$ either contains an element $c(d)$ not contained by one of the $F(b)$ for $b\in U$ or lacks an element $c(d)$ contained by one of these $F(b)$, then $F(U)=V$. The condition additional to $V\subseteq F(U)\subseteq V$

in this result is, in particular, implied by either of the conditions indicated by use of the braces in the preceding paragraph.

The notations described in part A of Definition relate a system derived from a single system $F(b)$ as b ranges over a further system U to a constant system V; naturally two systems $F(b)$ and $G(b)$ can be related in an analogous way.

Theorem. A i) $S \in \mathrm{II}_{QF}(S)$

ii) If S is nontorsive with respect to C, $\overline{T}(S,C) \in SI_{QF}(S)$

iii) If S is torsion-free with respect to $C \in SI(S)$, $C \in SI_{QF}(S)$

iv) Let $C \subset S$ and $C \in SI(S)$. $C \in SI_{QF}(S)$ if and only if $S = \overline{D}(S,C) \cup C$

B) Let $C \in SI_{QF}(S)$

i) C is closed in S

ii) Let $M \in M(S)$. M and C/M are disjoint with respect to $C \cap M$ in S.

iii) If $ab \ldots cx^2y^2 \ldots z^2 \in C$ then $ab \ldots cxy \ldots z \in C$

iva) $O(S,C|b) \in SI_{QF}(S)$ for all $b \in S$. (b) For all $b \in S \setminus C$, $O(S,C|b)$ and $Z(S,C|b)$ are disjoint with respect to C. (c) $D\{Z(S,C|b),C\} \equiv E\{Z(S,C|b),C\}$ for all $b \in S \setminus C$. (d) $Z(S,C|ab) = Z(S,C|a) \cap Z(S,C|b)$. (e) For all $a \in S$, $Z(S,C|a^m) \equiv Z(S,C|a^n)$ $(m,n:;)$

va) Let $a \leq b \bmod C$. $ab \in C$ if and only if $a \in C$. (b) If $a \leq b$, $ab \leq c \bmod C$ then $a \leq bc \leq c \bmod C$. (c) If $a \leq b \bmod C$ then $a \equiv ab \bmod C$. (d) Let $a \leq b \bmod C$. $ab \in \overline{T}(S,C)$ if and only if $a \in \overline{T}(S,C)$

vi) If $D \in \mathfrak{Sl}_{QF}(S)$ also, then [illegible] $\in \mathfrak{Sl}_{QF}(S)$

C) Let $\bar{I} \in \bar{\mathbb{I}}_{QF}(S)$

i) $\bar{O}\langle Z\rangle(S, \bar{I} \mid b) \in \bar{\mathbb{I}}_{QF}(S)$ for all $b \in S$. (b) If $a, c \leq b \bmod \bar{I}$ and $ab = bc \bmod \bar{I}$, then $a = c \langle a \equiv c\rangle \bmod \bar{I}$.

ii) If $\bar{J} \in \bar{\mathbb{I}}_{QF}(S)$ then $\bar{I} \cap \bar{J} \in \mathfrak{Sl}_{QF}(S)$.

Proof. The abbreviations $D, \bar{D}, T, T_S, O(b), Z(b)$ and $a \leq b$ are for $D(S,C), \bar{D}(S,C), T(S,C), T_S(S,C), O(S,C|b), Z(S,C|b)$ and $a \leq b \bmod C$ are adopted for the proofs of parts A and B.

For the proof of clause (Ai), it suffices to remark that $S \in \mathbb{I}(S)$ and $T_S(S,C)$ is void when $S = C$. When $C \in \mathfrak{Sl}(S)$, $\bar{T} \in \mathfrak{Sl}(S)$ also, from clause ( ) of Theorem . Furthermore, from clause( ) of Theorem , if S is nontorsive with respect to C, $\bar{T}$ and D are complementary in S and $\bar{T} \cup D = S$. Thus any $b \notin \bar{T}$ belongs to D. But $D \in M(S)$ so that b^2 is then in D and therefore not in $\bar{T}$. S possesses no $b \notin \bar{T}$ such that $b^2 \in \bar{T}$ and accordingly $\bar{T} \in \mathfrak{Sl}_{QF}(S)$. The assertion of clause (Aiii) is a simple consequence of the fact that $T_S \subseteq T$; T_S is void when T is void. For the proof of clause (Aiv), first assume that $S = \bar{D} \cup C$. For each $b \notin C$ and all $a \leq b$, $ab \in C$ only when $a \in C$. In particular, since $a \leq a$, for all $a \in S$, $a^2 \in C$ only when $a \in C$. S has no elements that are self torsive with respect to C: $C \in \mathfrak{Sl}_{QF}(S)$. Conversely, assume that $C \in \mathfrak{Sl}_{QF}(S)$ and choose $b \in S \setminus C$. For any $a \leq b$, the condition $ba \in C$ implies that $a^2 \in C$ or, since $C \in \mathfrak{Sl}_{QF}(S)$, that $a \in C$. Hence all $b \in S \setminus C$

are members of $\mathfrak{D}$.

From clause ( ) of Theorem , $C \subseteq \bar{C} \subseteq C \cup T_S$. When T_S is void, $C = \bar{C}$, the result of clause (Bi). $M \cap (C/M)$ is the complete system X of S-numbers in M such that $MX \subseteq C$. Since $C \in Sl(S)$, $M(C \cap M) \subseteq MC \subseteq C$; accordingly $C \cap M \subseteq X$. Since $X \subseteq M$ and $M^2 \subseteq M$, $b^2 \in C$ for all $b \in X$ and, since $C \in Sl_{QF}(S)$ $b \in C$ for all such b. Hence $X \subseteq C \cap M$ and $X = C \cap M$. The result of clause (Bii) has been obtained. Since $C \in Sl(S)$, $(ab_* \dots c)(ab \dots cx^2y^2 \dots z^2) = (ab \dots cxy \dots z)^2 \in C$ when $ab \dots cx^2y^2 \dots z^2 \in C$. If in addition $C \in Sl_{QF}(S)$, $ab \dots cxy \dots z \in C$. When $b \in C$, $O(b) = S$ and accordingly $O(b) \in Sl_{QF}(S)$. When $b \in S \setminus C$ and $x^2 \in O(b)$, $bx^2 \in C$ so that $bx \in C$ and $x \in O(b)$: $O(b) \in Sl_{QF}(S)$ for all $b \in S$. $b \in O(b)$ if and only if $b^2 \in C$ or, when $C \in Sl_{QF}(S)$ if and only if $b \in C$. The results of clauses (Biii, iv) have been dealt with.

When $b \in C$, $Z(b) = C \in Sl_{QF}(S)$. Let $b \in S \setminus C$. If $x^2 \in Z(b)$ then $x^2 y \in C$, i.e. $xy \in C$, whenever $bx \in C$, so that $x \in Z(b)$: $Z(b) \in Sl_{QF}(S)$ for all $b \in S$. If $a \in O(b)$, $ab \in C$; this condition, when $a \in Z(b)$, implies that $a^2 \in C$ or, when $C \in Sl_{QF}(S)$, that $a \in C$. Hence $O(b) \cap Z(b) \subseteq C$. But $\bar{C} \subseteq O(b) \cap Z(b)$ for all $b \in S$ when $C \in Sl(S)$, and $C = \bar{C}$ when $C \in Sl_{QF}(S)$. Thus $C \subseteq O(b) \cap Z(b)$ for all $b \in S \setminus C$. In conclusion $C = O(b) \cap Z(b)$ for all $b \in S \setminus C$,

the result of clause (Bvb). From the result of clause (Bvc), $Z(b)$ is nontorsive with respect to C for all $b \in S \setminus C$. That $D(b) \equiv E(S, C|b)$ for all such b follows from clause ( ) of Theorem . Clause (Bvc) has been disposed of. If $x \in Z(a) \cap Z(b)$ is in C, $x \in Z(ab)$ also, since $C \subseteq Z(ab)$ for all $a, b \in S$. If a or b is in C, ab is in C and $Z(a) \cap Z(b) \equiv Z(ab) \equiv C$, since $Z(b) \equiv C$ for all $b \in C$. Assume that $a, b \in S \setminus C$. If $x \in Z(a) \cap Z(b)$ the condition $abg \in C$ implies, since $x \in Z(a)$ that $xbg \in C$ and, since $x \in Z(b)$ that $x^2 g \in C$ and, since $C \in Sl_{QF}(S)$ that $xg \in C$; x is characterised as a member of $Z(ab)$: $Z(a) \cap Z(b) \subseteq Z(ab)$. However $Z(ab) \subseteq Z(a) \cap Z(b)$ from clause ( ) of Theorem . The result of clause (Bvd) follows. When $m = n$, $a^m = a^n \in Z(a^n)$ from clause ( ) of Theorem and, when $m > n$, $a^m = a^{m-n} a^n \in Z(a^n)$ from clause ( ) of that Theorem. Thus, from clause ( ) of Theorem , $Z(a^m) \subseteq Z(a^n)$ when $m \geq n$. The relationship $a^{n-r} \leq a^n$ is true when $r = 0$. Assume it to be true for some r in the range $0 \leq r \leq n-2$. From clause (Bvib) to be proved below, the relationships $a^{n-r-1} \leq a$, $a^{n-r} = (a^{n-r-1})a \leq a^n$ imply that $a^{n-r-1} \leq a^n$. Hence $a^{n-r} \leq a^n$ for $(r:0;n-1)$ or $a^m \leq a^n$ when $m < n$ and, from clause ( ) of Theorem , $Z(a^m) \subseteq Z(a^n)$ when $m < n$. Thus in view of the result proved above, $Z(a^m) \subseteq Z(a^n)$ for $(m, n:;)$. Interchanging m and n

the result of clause (Bve) is obtained.

For the proof of clause (Bvia) it is first remarked that, since $C \in Sl(S)$, $ab \in C$ if $a \in C$ for all $b \in S$. If $b \in C$, the condition $a \leq b$ implies that $a \in C$ also. When $b \in S \setminus C$, $Z(b)$ is nontorsive with respect to C, from clause (Aiv), $b \in D(b)$ from clause ( ) of Theorem , and for all $a \leq b$, $ab \in C$ only when $a \in C$. If $a \leq b$ and $ab \leq c$ then, from clause ( ) of Theorem $a^2 b \leq bc$. Accordingly, for all $g \in S$ such that $bcg \in C$, $a^2 bg \in C$ or, since $C \in Sl_{QF}(S)$, $abg \in C$ and, since $a \leq b$, $a^2 g \in C$ and again $ag \in C$. Hence $a \leq bc$. Also $bc \leq c$ for all $b \in S$. The result of clause (Bvib) has been derived. For that of clause (Bvic) it is remarked that $a \leq a$ and, if $a \leq b$, $a^2 \leq a \leq b$. Using the preceding result with b, c replaced by a, b it follows that $a \leq ab$. But $ab \leq a$ for all $b \in S$ and accordingly $a \equiv ab$ when $a \leq b$. For the proof of clause (Bvid) first assume that $a \in \overline{T}$, so that $a \notin C$. Since $Z(b) \equiv C$ when $b \in C$, $Z(b)$ cannot contain $a \in \overline{T}$. Accordingly the conditions $a \leq b$, $a \in \overline{T}$ imply that $b \notin C$. Also, since $a \leq b$, $ab \in C$ only when $a \in C$, so that if $a \in \overline{T}$, $ab \notin C$. $a \in \overline{T}$ is accompanied by a number $c \in \overline{T}$ for which $ac \in C$, and accordingly $ab \notin C$ is accompanied by $c \notin C$ for which $abc \in C$. Hence $ab \in \overline{T}$. If $ab \in \overline{T}$, $a \notin C$, S also contains a number $d \notin C$ such that $abd \in C$. But $ad \leq a \leq b$, so that $ad \in C$. Accordingly $a \in \overline{T}$.

To derive the result of clause (Bvii) it is first remarked

that if $B \equiv C \equiv S$, $B \cap C \equiv S \in SI_{QF}(S)$ from clause (Ai). If $B \equiv S$, $C \subset S$ then $B \cap C \equiv C \in SI_{QF}(S)$, the case in which $B \subset S$, $C \equiv S$ being dealt with similarly. Otherwise, any $b \notin B \cap C$ either belongs to $S \setminus B$ or $S \setminus C$. If $b \in S \setminus B$, $b^2 \in S \setminus B \subseteq S \setminus (B \cap C)$, the case in which $b \in S \setminus C$ being dealt with in the same way. Thus if $b \in S \setminus (B \cap C)$, $b^2 \in S \setminus (B \cap C)$. Finally, $B \cap C \in SI(S)$ when $B, C \in SI(S)$, from clause ( ) of Theorem . Thus $B \cap C \in SI_{QF}(S)$.

From clause ( ) of Theorem , $O\langle Z \rangle (S, I|b) \in \mathbb{I}(S) \in AP(S)$ when $I \in \mathbb{I}(S)$ for all $b \in S$, and from clauses (Bvia, va) of the present theorem, $O\langle Z \rangle (S, I|b) \in SI_{QF}(S)$ when $I \in SI_{QF}(S)$, also for all $b \in S$. The result of clause (Cia) follows. Concerning the result of clause (Cib), it is remarked that if $ab = ac \bmod I \in \mathbb{I}_{QF}(S)$, then $a \in S[c]$ and $b(a-c) \in I$. Furthermore, if $a, c \leq b$, $a - c \leq b \bmod I$ also, since $Z(S, I|b) \in \mathbb{I}(S) \in AP(S)$. Hence $a - c \in I$, or $a = 0 \bmod I$. The further result $a \equiv c \bmod I$ is a consequence of clause ( ) of Theorem .

For the last result of the theorem, it suffices to remark that if $I, J \in \mathbb{I}_{QF}(S)$ then, from clause ( ) of Theorem , $I \cap J \in AP(S)$ since $I, J \in AP(S)$, and from clause (Bvii) of the present theorem, $I \cap J \in SI_{QF}(S)$ since $I, J \in SI_{QF}(S)$.

Definition . Let $I \in \mathbb{I}(S)$ and S be nontorsive with respect to I.

Ai) A denominator in S with respect to I is an S-number b for which either ~~(a) $v \in S[b]$ such that (a1) $bv \in D(S, I)$~~

(a) $v \in S[b]$ such that (a1) $b+v \in D(S,I)$ and (a2) $bv \in \underline{I}$ exists

or

(b) u for which (b1) $b \leq u \bmod I$ and $v \in S[ub]$ for which (b2) $ub+v \in D(S,I)$ and (b3) $bv \in I$ exists.

(ii) $N(S,I)$ is the complete system of denominators in S with respect to I.

(iiia) A number v possessing the properties described in (ia) above is a direct orthogonal complement of b. (b) A number u possessing the properties described in (ib) above is a lifting factor of b, and the corresponding number v is a displaced orthogonal complement of b.

(iv) With $b \in N(S,I)$, $\mathcal{O}_c(S,I|b)$ is the complete system of direct and displaced orthogonal complements of b.

B) $N'(S,I)$ is that subsystem of $N(S,I)$ defined by replacing the condition (b1) by $u \in D(S,\bar{I})$.

C) Associate with each $b \in N(S,I)$ its two complete systems $V_{b,I}$ of direct orthogonal complements v and $U_{b,I}$ of lifting factor-displaced orthogonal complement pairs (u,v). With $J \in \mathbb{I}(S)$, $N(S;I,J)$ is the complete system of $b \in N(S,I) \cap N(S,J)$ for which $\{V_{b,I} \cap V_{b,J}\} \cup \{U_{b,I} \cap U_{b,J}\}$ is nonvoid.

Theorem. Let $I \in \mathbb{I}(S)$ and S be nontorsive with respect to I Aia) $\{I \cup D(S,I)\} \subseteq N(S,I) \subseteq \{I \cup \bar{D}(S,I)\}$. (b) For all $b \in N(S,I)$ $\mathcal{O}_c(S,I|b) \subseteq N(S,\bar{I})$.

ii) $N(S,I) \in \mathbb{M}(S)$

iii(a) Let $d \in D(S,I)$. $bd \in N(S,I)$ if and only if $b \in N(S,I)$. (b) Let $a = b \bmod I$. $a \in N(S,I)$ if and only if $b \in N(S,I)$.

iv) Let $I \in \mathbb{I}_{QF}(S)$. If $b \leq d \bmod I$ and $bd \in N(S,I)$ then $b \in N(S,I)$.

Bia) For a fixed finite integer k, let $S[D]_k$ be the system of displaced strata $S\{j+k\}$ obtained from the strata $S\{j\}$ of $S[D(S,I)]$. Let I_D be the subsystem of members of I taken from each stratum $S\{k\}$ for those values of k for which $S[D]_k$ and $S[D(S,I)]$ have at least one stratum in common. Then $\{I_D \cap D(S,I)\} \subseteq N'(S,I) \subseteq \{I \cup \bar{D}(S,I)\}$. (b) The preceding result with I_D replaced by $S\{0\} \cap I$ holds. (c) If $\bar{D}(S,I)$ possesses members in every stratum of S, $I_D = \bar{I}$.

ii) The ~~clau~~ results of clauses (Aib-iv) hold if $N(S,I)$ is replaced consistently by $N'(S,I)$.

c) Let $\bar{J} \in \bar{\mathbb{I}}(S)$ and S be nontorsive with respect to $\bar{J}$.

ia) If $D(S,I) \cap D(S,\bar{J})$ is nonvoid (in particular, if S contains a unit element 1 such that $1a = a$ for all $a \in S$)

$$(I \cap J) \cup \{D(S,I) \cap D(S,J)\} \subseteq N(S;I,\bar{J}) \subseteq N(S;I \cap \bar{J})$$

(b) For all $b \in N(S;I,\bar{J})$, $V_{b,I} \cap V_{b,\bar{J}} \subseteq N(S;I,\bar{J})$. Denoting the displaced orthogonal complements belonging to the system of lifting factor-orthogonal complement pairs $U_{b,I} \cap U_{b,\bar{J}}$ by $V'_{b;I,\bar{J}}$, $V'_{b,I,\bar{J}} \subseteq N(S;I,\bar{J})$ also for all $b \in N(S;I,\bar{J})$

(ii) $N(S;I,\bar{J}) \in \mathbb{M}(S)$.

iiia) Let $d \in D(S,I) \cap D(S,\bar{J})$. $bd \in N(S;I,\bar{J})$ if and only if $b \in N(S;I,\bar{J})$

(b) Let $a = b \bmod I \cap \bar{J}$. $a \in N(S;I,\bar{J})$ if and only if $b \in N(S;I,\bar{J})$.

(iv) Let $I, J \in \mathcal{I}_{QF}(S)$. If $b \leq d \bmod I$, $b \leq d \bmod \bar{J}$ and $bd \in N(S;I,\bar{J})$ then $b \in N(S;I,\bar{J})$.

Proof. The abbreviations $D, \bar{D}, N, N', Z(b)$ and $a \leq b$ for $D(S,I)$, $\bar{D}(S,I)$, $N(S,I)$, $N'(S,I)$, $Z(S,I|b)$ and $a \leq b \bmod I$ are adopted during the proofs of parts (A) and (B). In all cases in which b is taken to be a denominator, it is assumed that conditions (Aib) of Definition hold; the simpler conditions (Aia) are correspondingly easier to deal with. It is remarked that, since S is assumed to be nontorsive with respect to I, $\bar{\bar{I}} = \bar{I}$.

Since S is nontorsive with respect to I, D contains at least one number, $v \in S\{j\}$ say. Let $b \in S\{k\}$ belong to I. With $u = 0_{(j-k)}$ $Z(u) = I$ so that $b \leq u$. Also $ub+v \in D$ and $bv \in I$, so that $b \in N$ for all $b \in I$. If v is taken to be the zero element in $S[b]$ when $b \in D$, b satisfies the conditions of (Aia) of Definition. All $b \in D$ belong to N. Assuming that $b \in N \setminus I$ and $c \leq b$, the condition $bv \in I$ implies that $cv \in I$. If $bc \in I$ then $(ub+v)c \in I$ and, since $ub+v \in D$, $c \in I$. Hence when $b \in N \setminus I$, $bc \in I$ only when $c \in I$ for all $c \in Z(b)$. $D\{Z(b), I\}$ contains b and is nonvoid. All $b \in N \setminus I$ belong to $\bar{D}$. Clause (Aia) has been dealt with. The number v satisfying the relationships $v+(ub) \in D$, $v(ub) \in I$ is a denominator according to clause (Aia) of Definition. The result of clause (Aib)

follows.

Let $b,d \in N$ so that in conjunction with the numbers u,v associated with b, numbers y,z for which $d \le y$, $yd+z \in D, dz \in I$ are associated with d. Since $b \le u$ and $d \le y$, $bd \le uy$ from clause ( ) of Theorem . Since $D \in M(S)$, $(uy)bd + ubz + ydv + vz = (ub+v)(yd+z) \in D$. Lastly, since $bv, dz \in I$, $bd(ubz+ydv+vz) \in I$. Accordingly $bd \in N$ if $b,d \in N$, the result of clause (Aii).

Since $D \subseteq N$, $d \in N$ when $d \in D$. Also $N \in M(S)$, so that $bd \in N$ when $d \in D, b \in N$. Conversely, assuming that $u,v \in S$ exist such that $bd \le u$, $u(bd)+v \in D$, $bdv \in I$. From clause ( ) of Theorem , the two conditions $d \in D$, $bd \le u$ imply that $b \le ud$. Also, since $d \in D$, $bv \in I$ if $bdv \in I$. Accordingly b satisfies the conditions $b \le ud$, $(ud)b+v \in D$, $bv \in I$: if $d \in D$ and $bd \in N$, $b \in N$ also. Clause (Aiia) has been dealt with. Assuming b to be a denominator, $ua+v$ $= ub+v+(b-a)v$ and when $a = b \bmod I$, $ua+v = ub+v \bmod I$ so that, from clause ( ) of Theorem , the condition $ub+v \in D$ implies that $ua+v \in D$. Since $a = b \bmod I$, $a \le u$ when $b \le u$; also $av \in I$ when $bv \in I$. Hence $a \in N$ when $b \in N$. The converse assertion is proved in the same way. Both parts of clause (Aii) have been proved.

If k' is one of the integer values of k for which $S[D]_k$ and $S[D]$ have at least one stratum, $S\{j+k'\}$ say, in common, D contains two members $u \in S\{j\}$ and $v \in S\{j+k'\}$. Let $b \in S\{k'\}$ belong to I. Then $u \in D$, $ub+v \in D$ and $bv \in I$, so that $b \in N'$ for all such b: $I_D \in N'$. The relationships $D \subseteq N'$ and $N' \setminus I \subseteq \overline{D}$

are obtained by methods sufficing for the proofs of their counterparts in clause (Aia). There is certainly one value of k for which $S[D]_k$ and $S[D]$ have a common stratum, namely $k=0$; the result of clause (Bib) follows. If D possesses members in every stratum of S, $S[D]_k$ and $S[D]$ have a common stratum for all finite integer values of k; clause (Bic) has been dealt with.

The results of clause (Bi) are obtained by slight modification, where necessary, of the arguments used in the proofs of their counterparts in part (A). The counterpart to clause (Aib) requires no modification. For that to clause (Aii), the conditions $b \leq u, d \leq y$ are replaced by $u, y \in D$ when, since $D \in M(S)$, $uy \in D$ also. For that to clause (Aiii a), the condition $bd \leq u$ is replaced by $u \in D$ and then the condition $u(bd)+v \in D$ is rewritten in the form $(ud)b+v \in D$ where, since $u, d \in D$, $ud \in D$, to show that $b \in N'$. For that to clause (Aiii b), the condition $u \in D$ is imposed and the proof that $a \leq u$ is simply discarded. In the counterpart to clause (Aiv), the condition $b \leq d$ must be replaced by $d \in D$; the result is then included in the counterpart to clause (Aiii a).

The abbreviations D_I, D_J, N_I, N_J and $N_{I,J}$ for $D(S,I)$, $D(S,\bar{J})$, $N(S,I)$, $N(S,\bar{J})$ and $N(S;I,\bar{J})$ are adopted for the proof of part C.

For the proof of clause (Cia) it is remarked that D_I and $D_{\bar{J}}$ possess a common member, $v \in S[j]$ say. It follows from the proof of clause (Aia) that the numbers $u = O(j-k)$,

v functions as lifting factor and displaced orthogonal complement identifying $b \in S\{j\} \cap I \cap \bar{J}$ as a member of both N_I and N_J. Hence $(I \cap \bar{J}) \subseteq N_{I,\bar{J}}$. Again, if $b \in D_I \cap D_J$ belongs to $S\{j\}$, $v = Q_{(j)}$ functions as a direct orthogonal complement identifying b as a member of both N_I and N_J. Hence $D_I \cap D_J \subseteq N_{I,\bar{J}}$. Suppose that the lifting factor u and displaced orthogonal complement v identify b as a member of both N_I and N_J. Then $b \in Z(S, I|u) \cap Z(S, \bar{J}|u) \subseteq Z(S, I \cap \bar{J}|u)$ from clause ( ) of Theorem, $ub + v \in D_I \cap D_J \subseteq D(S, I \cap \bar{J})$ from clause ( ) of Theorem, and $bv \in I \cap \bar{J}$. Thus u,v identify b as a member of $N(S, I \cap \bar{J})$.

If $v \in V_{b,I} \cap V_{b,\bar{J}}$, $b + v \in D_I \cap D_J$. b functions as a direct orthogonal complement identifying v as a member of both N_I and N_J. Hence $v \in N_{I,\bar{J}}$. Again if the lifting factor u and displaced orthogonal complement v identify b as a member of both N_I and N_J, $ub + v \in D_I \cap D_J$ and ub functions as a direct orthogonal complement identifying v as a member of $N_{I,\bar{J}}$. Clause (Cib) has been disposed of.

Clause (Cii) is demonstrated by adapting the proof of clause (Aii). If the lifting factor, displaced orthogonal complement pairs u,v and y,z serve to identify b and d

as members of both N_I and N_J, then the pair $uy, ubz+ydv+vz$ serves to identify bd as a member of both N_I and N_J. Accordingly, if $b, d \in N_{I,\bar{J}}$, $bd \in N_{I,\bar{J}}$ also: $N_{I,\bar{J}} \in M(S)$.

For the proof of clause (Ciii), that of clause (Aiii) is adapted. Assuming that $b \in N_{I,\bar{J}}$, $d \in D_I \cap D_J$ is, from clause (Cia) in $N_{I,\bar{J}}$ also and, from the result of the preceding clause, $bd \in N_{I,\bar{J}}$ also. Conversely, assuming that the lifting factor, displaced orthogonal complement pair u, v identify bd as a member of both N_I and N_J, the pair ud, v identifies b as a member of both systems.

Again adapting the proof of clause (Aiv) for that of clause (Civ), let the lifting factor, displaced orthogonal complement pair which identifies b as a member of N_I and N_J be u, v. If $a-b \in I \cap \bar{J}$, the conditions $ub+v \in D_I$ and $ub+v \in D_J$ imply that $uba+v \in D_I$ and $ua+v \in D_J$ also. Again, since $a \equiv b \bmod \bar{I}$, $a \equiv b \bmod \bar{J}$, $a \equiv u \bmod I$ and $a \equiv u \bmod J$ when $b \equiv u \bmod I$ and $b \equiv u \bmod J$. Also $av \in I$ and $av \in J$ when $bv \in I$ and $bv \in J$. In short, the pair u, v identifies a as a member of both N_I and N_J and accordingly $a \in N_{I,J}$.

Lastly, in the similar adaptation of the proof of clause (Aiv), the pair u, v which identifies bd as a member of both N_I and N_J also identifies b as a member of these systems, and the result of clause (Civ) has been obtained.

In $S \notin \mathbb{S}$, an equation of the form $bx=a$ is soluble only in the case in which $a=bc$ for some $c \in S$; in this case one solution is $x=c$ and the general solution is $x=c+g$ where g is any member of $\ddot{O}(S, S[o] \mid b)$ lying in the stratum to which c belongs. It is possible, under certain conditions, to construct an extension, the quotient extension, of S in which equations of the form $bx=a$ are soluble under more relaxed conditions.

Definition. Let $I \in \mathbb{I}_{QF}(S)$, S being nontorsive with respect to I.

(i) With $b', d' \in D = D(S, I)$, the two ordered pairs $\langle a', b' \rangle$ and $\langle c', d' \rangle$ of S-numbers are equivalent with respect to I if and only if $a'd' = b'c' \bmod I$; $\langle a', b' \rangle \equiv \langle c', d' \rangle \bmod I$ denotes such equivalence.

(ii) The notation $b/a \bmod I$ is used to indicate firstly that $b \in N(S, I)$ and secondly that $a \in Z(S, I \mid b)$.

(iii) Let $b/a \bmod I$, so that, in particular, either (1) $v \in S(b)$ $u \in S$ such that $b \equiv u \bmod I$ exists for which $b+v \in D$, $bv \in I$ or (2) ~~$v \in b$~~ and $v \in S(ub)$ exist for which $ub+v \in D$, $bv \in I$. $[a/b]_I$ is the class of

ordered pairs of S-numbers $\langle a, b' \rangle$ with $b \in I$ such that $\langle a, b+v \rangle \equiv \langle a', b' \rangle \bmod \underline{I}$ in case (1) and $\langle ua, ub+v \rangle \equiv \langle a', b' \rangle \bmod \underline{I}$ in case (2), where v in case (1) and u, v in case (2) are any S-numbers possessing the required properties with respect to b in each case. Such a class is a Q-number. The notation $\langle a', b' \rangle \in [a/b]_{\underline{I}}$ indicates that a, b, a', b' are related as above; the ordered pair $\langle a', b' \rangle$ is said to represent the Q-number $[a/b]_{\underline{I}}$.

(iv) With $b/a \bmod \underline{I}$ and $d/c \bmod \underline{I}$, and $\langle a', b' \rangle$ any S-number pair representing $[a/b]_{\underline{I}}$ and $\langle c', d' \rangle$ any pair representing $[c/d]_{\underline{I}}$

a) $[a/b]_{\underline{I}} = [c/d]_{\underline{I}}$ if and only if $\langle a', b' \rangle \equiv \langle c', d' \rangle \bmod \underline{I}$.

b) $[a/b]_{\underline{I}}\,[c/d]_{\underline{I}}$ is the class of ordered pairs equivalent to $\langle a'c', b'd' \rangle$

c) When ad and bc belong to the same stratum, $[a/b]_{\underline{I}} \pm [c/d]_{\underline{I}}$ are the classes of ordered pairs equivalent to $\langle a'd' \pm b'c', b'd' \rangle$.

(v) For a fixed integer j, $Q\{j\}$ is the class of Q-numbers $[a/b]_{\underline{I}}$ with $a \in S\{j+k\}$, $b \in S\{k\}$ $(k \gg)$. $Q(S, \underline{I})$ is the complete system of classes $Q_{\underline{I}}\{j\}$ $(j \gg)$; it is the quotient extension of S with respect to $\underline{I}$.

(vi) Let $S\{k\}$ be the stratum to which $D \in D$ belongs; any Q-number $[a/b]_I$ with $a \in S\{j+k\}$ a member of I is (e.g. $a = 0_{(j+k)}$) is denoted by $0_{I(j)}$ $(j>)$.

(vii) The Q-number $[ad/d]_I$ with $d/a \bmod I$ is written simply as $[a]_I$.

Theorem. With $I \in \mathbb{I}_{QF}(S)$, let S be nontorsive with respect to I. (ia) $Q_I = Q(S,I) \in S$. b) The complete system of numbers $0_{I(j)}$ in Q_I constitutes $Q_I[0]$, and $Q_I[0] \in \mathbb{I}_{QF}(Q_I)$. c) All numbers of the form $[a/b]_I$ with $a \in S$, $b \in D = D(S,I)$ feature in Q_I; and every Q-number is equal to one of these numbers; with $a \in S$, $b \in D$, $[a/b]_I$ is represented directly by the S-number ordered pair $\langle a,b \rangle$. d) The complete system of numbers of the form $[a/b]_I$ in Q_I with $(1)\langle(2)\rangle\{(3)\}$ $a \in D\langle T = T(S,I)\rangle \{N = N(S,I)\}$ constitutes $D_I = D(Q_I, Q_I[0])\langle T_I = T(Q_I, Q_I[0])\rangle \{N_I = N(Q_I, Q_I[0])\}$; this system includes all numbers of the form $[a]_I$ with $a \in D\langle T\rangle\{N\}$; of the numbers $[a]_I$ only those for which $a \in D\langle T\rangle\{N\}$ belong to this system.

(ii) Let (α) $A = [a]_I$ or (β) $A = [a/c]_I$ and (α') $B = [b]_I$ or (β') $B = [b/d]_I$ all in Q_I. (a1$\langle 2\rangle$) $0_I(B) \equiv 0(Q_I, Q_I[0]|B)$ $\langle Z_I(B) \equiv Z(Q_I, Q_I[0]|B)\rangle$ is the complete system of numbers A of the form (β) with $a \in 0(b) \equiv 0(S,I|b)\langle Z(b) \equiv Z(S,I|b)\rangle$; this

system includes all numbers of the form (α) with $a \in U(b)$ $\langle Z(b) \rangle$ and of the numbers of the form (α) only those with $a \in O(b) \langle a \in Z(b) \rangle$ belong to this system. (b) If (1) $b \mid a \bmod I$ $\langle$(2) $B \mid A \bmod Q_I[o]\rangle$ the equation $BX = A$ has a unique solution in $Z_1(B)$ namely $X = [a/b]_I$ in case (α, α'), $X = [ad/bc]_I$ in case (β, β') and similarly in cases (α, β') and (α', β). This solution X also belongs to $Z_I(A) \subseteq Z_I(B)$. Furthermore the general solution of the equation $BY = A$ in Q_I is $Y = X + G$ where G is any member of $O_I(B)$ lying in the Q-stratum to which X belongs.

Proof. The abbreviations Q_I, D, I, N, D_I, I_I, N_I, $O(b)$, $Z(b)$, $O_I(B)$, $Z_I(B)$ used in the statement of the theorem are retained. Relationships ~~of the~~ such as $\langle a', b' \rangle \equiv \langle c', d' \rangle \bmod I$ and $a \leq b \bmod I$ are written simply as $\langle a', b' \rangle \equiv \langle c', d' \rangle$ and $a \leq b$ respectively. At certain points in the following, the S-number b features as a member of N; numbers $u, v \in S$ such that $b \leq u$, $ub + v \in D$, and $bv \in I$ are then tacitly associated with b. The simpler case in which $v \in S$ for which $b + v \in D$, $bv \in I$ exists is correspondingly easier to deal with. Similarly, when $c \in N$, numbers $w, x \in S$ such that $c \leq w$, $wc + x \in D$ and $cx \in I$ are associated with c; also, numbers $y, z \in S$ such that $d \leq y$, $yd + z \in D$ and $dz \in I$ are associated

with d when $d \in \mathrm{IV}$.

The proof of clause (ia) is carried out by demonstrating certain auxiliary propositions in succession.

(A) Equivalence between ordered pairs of S-numbers is perfect. Either $(a'd' - b'c') \in I$ or $(a'd' - b'c') \notin I$: two ordered pairs $\langle a', b' \rangle$ and $\langle c', d' \rangle$ of S-numbers either are or are not equivalent. Since $S[0] \in I$, $a'd' - a'b' \in I$: such equivalence is reflexive; since $c'b' - a'd' \in I$ when

$a'd'-c'b' \in \underline{I}$, it is also symmetric. To show that it is also transitive, it is remarked that if $\langle a',b' \rangle \equiv \langle c',d' \rangle, \langle c',d' \rangle \equiv \langle e',f' \rangle$ then $a'd', b'c'$ belong to the same S-stratum, as do $c'f', d'e'$, and $(a'd'-b'c'), (c'f'-d'e') \in \underline{I}$. Accordingly $f'(a'd'-b'c')+b'(c'f'-d'e') = d'(a'f'-b'e') \in \underline{I}$ and, since $d' \in \underline{D}$, $(a'f'-b'e') \in \underline{I}$; hence $\langle a',b' \rangle \equiv \langle e',f' \rangle$.

(B) Should cases (1) and (2) in clause (iii) of Definition both apply to the number b, $[a/b]_{\underline{I}}$ is independent of the case taken as a basis for the definition of this class. Furthermore, should such choice be available, the class $[a/b]_{\underline{I}}$ depends neither upon the choice of $v \in \overline{O}_c'(S, I \mid b)$ in case (1) nor upon the choices of the multiplier u and the orthogonal complement v in case (2). Since equality between ordered pairs of S-numbers is symmetric and transitive, it suffices to show that if $v, v'' \in S(b)$ exist such that $b+v$, $b+v'' \in \underline{D}$ and $bv, bv'' \in \underline{I}$, then $\langle a, b+v \rangle \equiv \langle a, b+v'' \rangle$, that if v with the stated properties exists, together

with u', v' such that $b \le u'$, $u'b+v' \in D$ and $bv' \in I$ then $\langle a, b+v\rangle \equiv \langle u''a, u''b+v''\rangle$ and that if $u, u'' \in D$, $v, v'' \in S$ exist such that $b \le u$, $b \le u''$ and $ub+v, u''b+v'' \in D$ and $bv, bv'' \in I$ then $\langle ua, ub+v\rangle \equiv \langle u''a, u''b+v''\rangle$. With regard to the last result, if $v, v'' \in \mathcal{O}(b)$ and $a \in Z(b)$ it follows that $av, av'' \in I$ and in consequence that $ua(u''b+v'') = u''a(ub+v) \bmod I$. The further possibilities are dismissed in the same way.

(c) The definition of equality between the two Q-numbers $[a/b]_I$ and $[c/d]_I$ is independent of the selection of the representative S-number pairs $\langle a', b'\rangle$ and $\langle c', d'\rangle$ from the Q-number classes involved; the same holds with regard to the definitions of the product and sum and difference of two Q-numbers. The first of these propositions is a direct consequence of the symmetric and transitive nature of equivalence between S-number pairs. To demonstrate the corresponding result with respect to the formation of a product it has to be shown that if $\langle a', b'\rangle \equiv \langle a'', b''\rangle$, $\langle c', d'\rangle \equiv \langle c'', d''\rangle$ then any pair $\langle e', f'\rangle$ equivalent to $\langle a'c', b'd'\rangle$ is equivalent to any pair $\langle e'', f''\rangle$ equivalent to $\langle a''c'', b''d''\rangle$ or, in view of

the transitive nature of equivalence between ordered pairs of S-numbers, simply that $\langle a'c', b'd' \rangle \equiv \langle a''c'', b''d'' \rangle$. The two equations $a'b'' = a''b'$, $c'd'' = c''d' \bmod I$ lead, after multiplication, directly to the result $a'c'b''d'' = a''c''b'd' \bmod I$, establishing the required relationship. After multiplicatim throughout by $d'd''$ and $b'b''$ respectively, the above two equatims become $a'd'b''d'' = a''d''b'd'$, $c'b'b''d'' = c''b''b'd' \bmod I$, so that $(a'd' \pm c'b')b''d'' = (a''d'' \pm c''b'')b'd' \bmod I$ or $\langle a'd' \pm c'b', b'd' \rangle \equiv \langle a''d'' \pm c''b'', b''d'' \rangle$. Hence, with regard to the formatin of a sum or difference, any S-number pair equivalent to $\langle a'd' \pm c'b', b'd' \rangle$ is equivalent to $\langle a''d'' \pm c''b'', b''d'' \rangle$.

(D) The ordered pair $\langle a'c', b'd' \rangle$ taken to represent the product of the two Q-numbers $[a/b]_I$ and $[c/d]_I$ is a member of a Q-number class, namely $[ac/bd]_I$. Similarly the ordered pairs $\langle a'd' \pm b'c', b'd' \rangle$ taken to represent the sum and difference of these two Q-numbers are members of the Q-number classes $[ad \pm bc / bd]_I$. Since $b, d \in \mathbb{N}$, $bd \in \mathbb{N}$ also. Furthermore, the conditions $a \leq b$ ~~$ac \in Z(b)$~~, $c \leq d$ ~~$c \in Z(d)$~~ imply that ac ~~$\in Z(b)Z(d)$~~ $\in Z(bd)$, from clause ( ) of Theorem . Hence $bd/ac \bmod I$ and the Q-number class ~~to~~ $[ac/bd]_I$ is well defined. Using the conventims concerning the denominators b and d stated above,

$\langle a', c' \rangle \equiv \langle ua, ub+v \rangle$ and $\langle c', d' \rangle \equiv \langle yc, yd+z \rangle$ and, from (C) above, $\langle a'c', b'd' \rangle \equiv \langle uyac, uybd+ud+yb+vz \rangle$. The last ordered pair represents $[ac/bd]_I$, so that $\langle a'c', b'd' \rangle$ is a member of the Q-number class $[ac/bd]_I$. With regard to the formation of the sum or difference, the conditions $a \leq b$, $c \leq d$ imply, from clause ( ) of Theorem , that $ad, bc \leq bd$, so that $ad \pm bc \in Z(bd)$ and the Q-number class $[(ad \pm bc)/bd]_I$ is well defined. The rest of the argument is as above.

(E) In each ordered pair $\langle a', b' \rangle$ representing $[a/b]_I$, $a' \equiv a \bmod I$. $\langle a', b' \rangle \in [a/b]_I$ if and only if $a'(ub+v) = ab' \bmod I$ where, in particular, $ub+v, b' \in D$. This relationship implies, from clause ( ) of Theorem , that $a'(ub+v) \equiv ab' \bmod I$ so that, in particular, $a'(ub+v) \leq ab'$ and in consequence $a'(ub+v) \leq a$. Since $ub+v \in D$ it follows from clause ( ) of Theorem that $a' \leq a$. Again, the relationships $ab' \leq a'(ub+v)$, $b' \in D$ lead to the complementary result $a \leq a'$. Hence $a' \equiv a \bmod I$. (Not every ordered pair $\langle a', b' \rangle$ for which $a' \equiv a \bmod I$ belongs to a given Q-number class $[a/b]_I$; a' is subject to a

further restriction involving the denominator by

(F) For a fixed value of j, all Q-numbers denoted by $O_{I(j)}$ are equal; the ordered pair $\langle a', b' \rangle$ represents $O_{I(j)}$ if and only if when $a' \in S\{j+k\}$, $b' \in S\{k\}$ for some $k \in I$ and $a' \in I$. Taking $[a/b]_I$ and $[c/d]_I$ to be two Q-number classes denoted by $O_{I(j)}$, $a, c \in I$. From clause ( ) of Theorem , it follows that, since S is monotonic, $Z(a) \equiv Z(c) \equiv I$. If $\langle a', b' \rangle \in [a/b]_I$ and $\langle c', d' \rangle \in [c/d]_I$, $a', b' \in I$ and in consequence $a'd' - b'c' \in I$: the two Q-numbers in question are equal. Any ordered pair $\langle a', b' \rangle$ with $b' \in S\{k\}$, $a' \in S\{j+k\}$ for some k represents $[a'/b']_I$; if, in addition, $a' \in I$, this Q-number is denoted by $O_{I(j)}$. Conversely, if $\langle a', b' \rangle$ represents a number denoted by $O_{I(j)}$, $b' \in S\{k\}$ and $a' \in S\{j+k\}$ for some k and, since $a' \leq a$ [a/b]_I, $a' \in I$.

(G) Within each Q-stratum, Q-numbers form an additive Abelian group, the zero member of which is the relevant number of the form $O_{I(j)}$. It follows from the definition of equivalence between Q-number classes that such equivalence can only hold if the two classes belong to the same stratum; that is, within a stratum, determinative

reflexive, symmetric and transitive follows from the fact that equivalence between ordered pairs of S-numbers possesses these properties. Addition between Q-number classes is only possible if they belong to the same Q-stratum. To show that such addition is associative, let $[a/b]_I$, $[c/d]_I$ and $[e/f]_I$ all in the same Q-stratum be represented by $\langle a', b' \rangle$, $\langle c', d' \rangle$ and $\langle e', f' \rangle$ respectively. The components of any representative pair $\langle g', h' \rangle$ taken from the Q-number class defined by $\{[a/b]_I + [c/d]_I\} + [e/f]_I$ satisfies the relationship $(a'd'f' + b'c'f' + b'd'e')h' = b'd'f'g'$ mod I; $\langle g', h' \rangle$ is equivalent to the pair $\langle a'd'f' + b'c'f' + b'd'e', b'd'f' \rangle$. Any representative pair taken from the Q-number class obtained by changing the order of additions is equivalent to the same pair. With $\langle a', b' \rangle \in [a/b]_I$ and $\langle c', d' \rangle$ representing any number $[c/d]_I$ of the form $0_{I(j)}$ belonging to the Q-stratum containing $[a/b]_I$, $[a/b]_I + [c/d]_I$ is represented by $\langle a'd' + b'c', b'd' \rangle$. However, $c' \in I$; and with this condition $\langle a'd' + b'c', b'd' \rangle = \langle a', b' \rangle$. Thus $0_{I(j)}$ functions as an additive zero element in its stratum. Each Q-number $[a/b]_I$ has an additive inverse, namely $[-a/b]_I$. That addition is commutative

can be verified directly.

(H) Multiplication of Q-numbers is associative and commutative, the product of $[a/b]_I$ in the jth stratum and $[c/d]_I$ in the j'^{th} stratum belonging to the $(j+j')^{th}$ stratum of Q_I; multiplication is distributive with respect to addition. Taking $\langle a', b' \rangle$, $\langle c', d' \rangle$ and $\langle e', f' \rangle$ to represent $[a/b]_I$, $[c/d]_I$ and $[e/f]_I$ respectively, $\{[a/b]_I [c/d]_I\}[e/f]_I$ is represented by the pair $\langle (a'c')e', (b'd')f' \rangle$ which also represents $[a/b]_I \{[c/d]_I [e/f]_I\}$. That multiplication of Q-numbers is commutative is easier to verify. Concerning the distribution of products within strata, it is supposed that with b, u, v related as described at the outset of the proof of this clause, $b \in S\{k\}$, $u \in S\{k''\}$ so that $v \in S\{k+k''\}$ with $a \in S\{j+k\}$, $\langle a', b' \rangle$ represents $[a/b]_I$ if ~~and only if~~ $uab = a'(ub+v) \mod I$. If this relationship is satisfied with $b' \in S\{k'\}$, a' lies in the stratum $S\{k''+j+k+k'-k-k''\}$ $= S\{j+k'\}$. Conversely, if $\langle a', b' \rangle$ with $a' \in S\{j+k'\}$, $b' \in S\{k'\}$ represents $[a/b]_I$ and $b \in S\{k\}$, then $a \in S\{j+k\}$ and $[a/b]_I$ belongs to the jth stratum of Q_I. Let $\langle a', b' \rangle$, $\langle c', d' \rangle$ represent $[a/b]_I$, $[c/d]_I$ in the jth and j'^{th} strata of Q_I respectively, with $b' \in S\{k'\}$, $d' \in S\{k''\}$, then $a' \in S\{j+k'\}$,

$c' \in S\{j'+k''\}$. The product of these two Q-numbers is represented by $\langle a'c', b'd' \rangle$ where $a'c' \in S\{j+j'+k'+k''\}$, $b'd' \in S\{k+k''\}$. Accordingly this product lies in the $(j+j')^{th}$ stratum of Q_I. Lastly, with $dee \in S(cf)$ and $[a/b]_I$, $[c/d]_I$, $[e/f]_I$ represented by $\langle a', b' \rangle$, $\langle c', d' \rangle$, $\langle e', f' \rangle$ respectively, both $[a/b]_I\{[c/d]_I + [e/f]_I\}$ and $[a/b]_I[c/d]_I + [a/b]_I[e/f]_I$ are represented by $\langle a'(c'f'+d'e'), b'd'f' \rangle$.

(I) The Q-number $[a]_I$ defined by use of the expression $[ad/d]_I$ with $d|a \bmod I$ is independent of the S-number d; Q-numbers of this form for all $a \in S$ are present in Q_I; these numbers form a subsystem of those represented in the form $[c/d]_I$. In addition to the number d used to define $[a]_I$, let d'' with $d''|a \bmod I$ be used for the same purpose; in analogy with the S-numbers y, z associated with the denominator d, let $y'', z'' \in S$ exist such that $d \leq y''$, $y''d''+z'' \in D$, $d''z'' \in I$. Since $a \equiv d \bmod I$, and $dz \in I$, $az \in I$ also; similarly $az'' \in I$. The two Q-numbers $[ad/d]_I$ and $[ad''/d'']_I$ are represented by the ordered pairs $\langle yad, yd+z \rangle$ and $\langle y''ad'', y''d''+z'' \rangle$. Since $yad(y''d''+z'') = y''ad''(yd+z) \bmod I$, these two pairs are equivalent; the Q-numbers

represented are equal. Since D is nonvoid, contains $d \in D$ such that $d \in N$ and $ad \in Z(d)$, or $d | a \bmod I$, for all $a \in S$. Hence numbers $[a]_I$ for all $a \in S$ are present in Q_I. Since $[ad/d]_I$ can be written as $[c/d]_I$ with $c = ad$, the numbers $[a]_I$ form a subsystem of Q_I.

It has now been shown that Definition offers an unequivocal description of a mathematical system and that this system is an S-ring.

That the numbers $0_{I(j)}$ in Q_I function as the zero element in each stratum was demonstrated in (F) and (G) above: $Q_I[0]$ is the complete system of such numbers. To show that $Q_I[0] \in \mathbb{I}_{QF}(Q_I)$, suppose that $([a/b]_I)^2$ represented by $\langle u^2a^2, (ub+v)^2 \rangle$ belongs to $Q_I[0]$, so that $a^2u^2 \in I$. Since $I \in \mathbb{I}_{QF}(S)$, $ua \in I$ and in turn, since $a \leq b \leq u$, $a \in I$. Hence $[a/b]_I \in Q_I[0]$ whenever $([a/b]_I)^2 \in Q_I[0]$.

When $b \in D$, $b \in N$ and $a \leq b$ for all $a \in S$. Hence for all $a \in S$, $b \in D$, $b | a \bmod I$ and $[a/b]_I$ exists in Q_I. When $b \in D$, case (1) of Definition applies, with $v \in S[0] \cap S(b)$, so that $\langle a, b \rangle$ represents $[a/b]_I$ directly. With $b | a \bmod I$, $\langle ua, ub+v \rangle \in [a/b]_I$ and $\langle ua, ub+v \rangle \in [ua/(ub+v)]_I$ also. Thus $[a/b]_I = [a'', b'']_I$ where, in particular, $b'' = (ub+v) \in D$.

Every Q-number is equal to one of the special Q-numbers considered in clause (ic).

For the result of clause (id1) it is remarked that since $Z(a) \equiv S$ when $a \in D$, and $Z(a) \leq Z(b)$, $b \in D$ in this case

and $[a'/b']_I$ is represented [illegible]. The product of $[a/b]_I$ and any number $[c/d]_I$ represented by $\langle c', d' \rangle$ is represented by $\langle ac', bd' \rangle$. This product belongs to $Q_I[0]$ only if $c' \in I$ and then $[c/d]_I \in Q_I[0]$. Hence all Q-numbers $[a/b]_I$ with $a \in D$ belong to D_I. If $a \notin D$, S contains a number $z \notin I$ such that $az \in I$. Q_I contains $[z/d]_I$ where $d \in D$, represented by $\langle z, d \rangle$ and, since $z \notin I$, $[z/d]_I \notin Q_I[0]$. The product $[a/b]_I [z/d]_I$ is represented by $\langle az, bd \rangle$ and belongs to $Q_I[0]$. No Q-number $[a/b]_I$ with $a \notin D$ belongs to D_I.

Concerning the ~~bracketed result above~~ result of clause (id2), let $a \in T$, so that $c \notin I$ such that $ac \in I$ exists. For some d for which $d/a \bmod I$ (e.g. $d \in D$) the Q-number $[c/d]_I$ is well defined and is not a member of $Q_I[0]$. With $[a/b]_I$, $[c/d]_I$ represented by $\langle ua, ub+v \rangle$, $\langle yc, yd+z \rangle$ respectively, where b, u, v and d, y, z are related as described at the outset of the proof, the product $[a/b]_I [c/d]_I$ is represented by $\langle uyac, (ub+v)(yd+z) \rangle$. Since $ac \in I$ this product is, however, in $Q_I[0]$ and hence $[a/b]_I \in T_I$. Conversely, suppose that $[a/b]_I \in T_I$ with $[a/b]_I [c/d]_I \in Q_I[0]$

where $[c/d]_I \notin Q_I[0]$. With the above representations $uyac \in I$. Since $a \leq b \leq u$, $c \leq d \leq y$, so that $ac \leq uy$, and $I \in \mathbb{I}_{QF}(S)$, it follows that $ac \in I$. Since $[c/d]_I \notin Q_I[0]$, $c \notin I$ and accordingly $a \in T$.

The proof of ~~the result of~~ clause (id 3) makes use of the result of clause (iia 2). It is convenient to prove clause (iia) immediately. Of the two forms of A and B given in clause (ii), (β, β') are, since they are the more complicated, selected for examination.

The Q-number $[a/c]_I$ is represented by the ordered pair of S-numbers $\langle wa, wc+x \rangle$, w and x being related to c as described above, and $B = [b/d]_I$ is represented by the pair $\langle yb, yd+z \rangle$, y and d being related to d as stated at the outset of the proof. AB is accordingly represented by the ordered pair $\langle wyab, (wc+x)(yd+z) \rangle$. $AB \in Q_I[0]$ if and only if $wyab \in I$ or, since $a \leq c \leq w$, $b \leq d \leq y$ and hence $ab \leq wy$, if and only if $ab \in I$. Thus all numbers of the form (β) for which $a \in \breve{O}(b)$ belong to $\breve{O}_I(B)$, and for all numbers of the form (β) belonging to $\breve{O}_I(B)$, $a \in \breve{O}(b)$. If $a \in \breve{O}(b)$, $ad \in \breve{O}(b)$ also and $[ad/d]_I = [a]_I \in \breve{O}_I(b)$. Conversely, when ~~ad~~ $a \leq d$,

$ab \leq d$ also, so that if $abd \in I$, $ab \in I$ since $I \in \underline{\mathbb{I}}_{QF}(S)$. Thus $[a]_I \in \mathcal{O}_I(B)$ only when $a \in \mathcal{O}(b)$.

To deal with the result of clause (iia2), take G to be a Q-number represented by the pair $\langle g', h' \rangle$. AG and BG are represented by the pairs $\langle wag', h'(wc+x) \rangle$ and $\langle ybg', h'(yd+z) \rangle$ respectively. If $a \in Z(b)$, $ag' \in I$ for all g' such that $bg' \in I$. If $A \in Z_I(B)$, $AG \in Q_I[o]$ for all G such that $BG \in Q_I[o]$ or, equivalently, $wag' \in I$ for all g' such that $ybg' \in I$. Since $ag' \leq a \leq c \leq w \langle bg' \leq b \leq d \leq y \rangle$ $AG \in Q_I[o] \langle BG \in Q_I[o] \rangle$ if and only if $ag' \langle bg' \rangle \in I$. Accordingly $a \in Z(b)$ if and only if ~~$A \in Z_I(B)$. Conversely, the condition $ad \leq b$ implies~~ $A \in Z_I(B)$. If $a \in Z(b)$, $ad \in Z(b)$ so that $[ad/d]_I \equiv [a]_I \in Z_I(B)$. Conversely, the conditim $ad \leq b$ implies, when $a \leq d$, that $a \leq b$ from clause ( ) of Theorem, so that $[a]_I \in Z_I(B)$ only when $a \in Z(b)$.

With the ~~bracketed~~ result of clause (iia2) in hand, it is possible to prove the result of clause (iia3) by showing that $B = [b/d]_I \in N_I$ if and only if $b \in N$. Since $d \in N$, S-numbers y, z related to d as described at the outset of the proof exist. Assume that $b \in N$, so

that the existence of $u, v \in S$ related to b as described may also be assumed. Since $ub+v \in D$, the Q-number $U = [ud/(ub+v)]_I$ exists. Furthermore, since $b \leq u$ and $b \leq d$, $b \in Z(ud)$ from clause ( ) of Theorem . Using the result of clause (iia2) just obtained, it follows that $B \in Z_I(U)$, or $B \leq U \bmod Q_I[0]$. Again since $ub+v \in D$ the Q-number $V = [v/(ub+v)]_I$ exists. Using the representations $\langle ud, ub+v \rangle$, $\langle yb, yd+z \rangle$ and $\langle v, ub+v \rangle$ of U, B and V respectively, and the fact, implied by the relationships $dz \in I$, $b \leq d$ that $ubz \in I$, the representation $\langle (ub+v)(yd+z), (ub+v)(yd+z) \rangle$ of $UB+V$ is obtained. Since $(ub+v)(yd+z) \in D$, it follows from the result of clause (iia1) that ~~ubz~~ $UB+V \in D_I$. It has been shown that if $b \in N$ then $B \in N_I$.

Conversely, assume that $U, V \in Q_I$ such that $B \leq U \bmod Q_I[0]$, $UB+V \in D_I$ and $BV \in Q_I[0]$ exist. Let $\langle u', y' \rangle \in U$ and $\langle v', d' \rangle \in V$. Since $\langle yb, yd+z \rangle \in B$ and $B \in Z_I(U)$, $yb \in Z(u')$. From clause ( ) of Theorem , the relationships $yb \leq u'$, $b \leq y$ imply that $b \leq uy'$; since $b \leq d$ also, ~~$b \leq$~~ $\leq u'yd'$. $UB+V$ is represented by the

S-number ordered pair $\langle ub+v, y'd'(yd+z)\rangle$ where $v = v'y'(yd+z)$ and, since $UB+V \in D_I$, $ub+v \in D$. Since $BV \in Q_I[0]$, $ybv' \in I$ and, since $b \leq d \leq y$, $bv' \leq y$ and in consequence $bv' \in I$ and $bv \in I$. It has been shown that if $B \in N_I$ then $u, v \in S$ for which $b \leq u$, $ub+v \in D$ and $bv \in I$ exist, or that $b \in N$

Concerning the concluding remarks of clause (id) it has to be shown, ~~that~~ since $[a]_I$ is $[ad/d]_I$ for some $d \in N$ such that $a \leq d$, that with these conditions upon a and d, $ad \in D\langle T\rangle\{N\}$ if and only if $a \in D\langle T\rangle\{N\}$. The condition $a \leq d$ implies that $d \in D$ when $a \in D$, and then $ad \in D$. If $a \notin D$, so that either $a \in T$ or $a \in I$, either $ad \in T$ or ~~ad~~ $ad \in I$ for all $d \in S$, so that $ad \notin D$. The two required results concerning T and N follow directly from clause ( ) of Theorem and clause ( ) of Theorem respectively.

Clause (i) having been dismissed in its entirety, the proof continues with consideration of clause (iib). The condition $b|a \bmod I$ implies in part that $a \leq b$. This condition, together with the condition $a \leq c$ imposed by the existence of A lead, by use of the result of

clause ( ) of Theorem , to the relationship $a \leq bc$, so that $ad \leq bc$. Since $b, c \in N$, $bc \in N$ also. The Q-number $X = [ad/bc]_I$ is well defined, and the ordered pair $\langle uwad, (ub+v)(wc+x)\rangle$ represents it. Also BX is represented by the ordered pair $\langle uwyabd, (ub+v)(wc+x)(yd+z)\rangle$ which, as is easily demonstrated, is equivalent to $\langle wa, wc+x\rangle$ representing A. Furthermore, since $ad \in Z(a) \subseteq Z(b)$, X belongs to $Z_I(A) \subseteq Z_I(B)$.

To show that X is the only solution of the equation $BY = A$ in $Z_I(B)$, let X, X' be two solutions of the equation $BY = A$ in $Z_I(B)$; then $B(X - X') \in Q_I[0]$ and, since $Q_I[0] \in \mathbb{I}_{QF}(Q_I)$ and $X, X' \in Z_I(B)$, $X = X' \bmod Q_I[0]$. Accordingly X is the only solution of the equation $BY = A$ in $Z_I(B)$.

With regard to the general solution Y of this equation in Q_I, set $G = Y - X$, so that $BG \in Q_I[0]$. Each solution $Y \in Q_I$ of the equation $BY = A$ has the form $Y = X + G$ where $Y \in O_I(B)$ belongs to the same Q-stratum as X, and all numbers Y of this form satisfy the equation.

For the result of clause (iib2), it is remarked that $b|a \bmod I$ if and only if $b \in N$ and $a \in Z(b)$. From

clause (id3), if $B \in N_I$ then $D \in N$ and, from clause (id2) if $A \in Z_I(B)$, $a \in Z(b)$. Hence if $B/A \bmod Q_I[o]$, $b/a \bmod I_S$ the required result is a consequence of its predecessor. (Slightly more than is required, namely that for the numbers A, B of clause (ii b), $B/A \bmod Q_I[o]$ if and only if $b/a \bmod I$, is easily demonstrated.)

The quotient extension of an S-ring S is constructed by taking a Q-number to be a class of S-number pairs and defining arithmetic operations upon Q-numbers in terms of operations upon representative members of classes by use of a process, analogous to that which defines rational numbers in terms of the rational integers, which takes torsive elements of S into account. The second members of the ordered S-number pairs are confined to the nontorsive part D of S; this restriction in turn imposes upon S the condition that it should be nontorsive. The denominators of Q-numbers may be a torsion elements of S; nevertheless the restriction upon b such denominators that they that it should be accompanied either by an orthogonal complement v in isolation or by such a v and a factor u has been imposed. The numerator a of a Q-number is confined to the

numerator set $Z(b)$ of ers corresponding denominator b. Finally, the restriction $b \leq u \bmod I$ has been imposed upon the factor u that accompanies the denominator b when such a factor does so, and the condition $I \in \mathbb{I}_{QF}(S)$ has also been imposed. It is perhaps prudent to indicate why these restrictions have been introduced.

First, if members of T are permitted to function as second members of ordered pairs of S-numbers, and $b \in N \setminus I$, the stratum $Q_I\{0\}$ contains two Q_I-numbers represented by $\langle b, b \rangle$ and $\langle v, v \rangle$ where $b, v \notin I$, $bv = v'' \in I$. The product C of these two Q-numbers is represented by $\langle v'', v'' \rangle$. With a any number in $S\{j\}$, the further product $C[a]_I$ is represented by $\langle v'', v''' \rangle$ where $v'', v''' \in I$. All members of $Q_I\{j\}$ are equal to this product. Either some stratum of Q_I contains two distinct numbers and a further number equal to both, in which case equality between Q-numbers is nontransitive, or in every stratum of Q_I all numbers are equal, in which case Q_I is null. Thus all second members of ordered pairs of S-numbers are confined to $\mathfrak{D}$. To avoid confusion, it is remarked that members b of I are, if in N, permitted to be denominators of

Q-numbers. The restriction $a \in Z(b)$ imposed upon the numerator of the Q-number $[a/b]_I$ then implies that $a \in I$: all Q-numbers with denominators in I are members of $Q_I[0]$.

Secondly, if $\langle a', b' \rangle \in [a/b]_I$ with $b \leq u \bmod I$, $ub+v \in D$, $bv \in I$ then $a'(ub+v) = uab'$. Since $a' \leq a \leq b \bmod I$, $a'v \in I$ and hence, for all $\langle a', b' \rangle \in [a/b]_I$, $ab' = a'b \bmod I$. The suggestion naturally forompts itself that all numbers in S should be permitted to be denominators, that reference to further S-numbers u, v should be abandoned, and that the conditions for membership of the ordered pair $\langle a', b' \rangle$ in the class $[a/b]_I$ should be, with $b' \in D$, simply $ab' = a'b \bmod I$. All former members are retained, but the relaxation of conditions for membership permits entry into the class $[a/b]_I$ of further ordered pairs. If $\langle a', b' \rangle \in [a/b]_I$ according to the relaxed membership condition, and $\langle a', b' \rangle \equiv \langle a'', b'' \rangle$ then $ab' = a'b$, $a'b'' = a''b' \bmod I$. The first formula yields the relationships $ab' \leq a'b \leq a' \bmod I$, so that $a'b'b'' \leq a' \bmod I$. The second formula yields the relationships $a''b' \leq a'b'' \leq a' \bmod I$ and now $a'b'b \leq a' \bmod I$. The two formulae ~~yield~~ together yield the relationship

$a'b'ab'' = a'b'a''b \bmod \mathfrak{I}$ or, since $b'ab'' = b'a''b \bmod \mathfrak{I}$ and again, since $b' \in D$, $ab'' = a''b \bmod \mathfrak{I}$. In conclusion, if $\langle a', b' \rangle \in [a/b]_I$ according to the relaxed membership condition, and $\langle a'', b'' \rangle \equiv \langle a', b' \rangle$ then $\langle a'', b'' \rangle \in [a/b]_I$ according to the same condition. If, however, $\langle a', b' \rangle, \langle a'', b'' \rangle \in [a/b]_I$, so that $ab' = a'b$, $ab'' = a''b \bmod \mathfrak{I}$, it cannot be concluded that $a'b'' = a''b' \bmod \mathfrak{I}$ or that $\langle a', b' \rangle \equiv \langle a'', b'' \rangle$; indeed, a simple counter example is easily provided. Let v exist such that $bv \in \mathfrak{I}$, $b+v \in D$ so that $b \in N$, and let $a \equiv b \bmod \mathfrak{I}$. Set $a' = a$, ~~$b' = b+v$~~, $b' = b'' = b+v$, $a'' = a+v$. Then $ab' = a'b = ab'' = a''b = ab \bmod \mathfrak{I}$, so that $\langle a', b' \rangle, \langle a'', b'' \rangle \in [a/b]_I$. But $a'b'' = ab \bmod \mathfrak{I}$ while $a''b' = ab + v^2 \bmod \mathfrak{I}$. Unless $v^2 \in \mathfrak{I}$, $\langle a', b' \rangle \not\equiv \langle a'', b'' \rangle$. If the membership condition is relaxed as indicated above, equality between Q-number classes based upon comparison of representative ordered pairs of S-numbers ceases even to be determinative. When checking the relationship $[a/b]_I \equiv [a'/b']_I$ it may be found that some pairs are equivalent while others are not.

Thirdly, one at least of the consequences of abolishing

the restriction $a \in Z(b)$ upon the numerator of the Q-number $[a/b]_I$ is that this class is no longer independent of $v \in \mathcal{O}(b)$ alone or of the numbers $u \in S$ such that $b \leq u \bmod I$ and $v \in \mathcal{O}(b)$ associated with the denominator b. If u', v' are two numbers similar to u, v and $a \in Z(b)$ then $\langle ua, ub+v\rangle \equiv \langle u'a, u'b+v'\rangle$, since $ua(u'b+v') - u'a(ub+v) = uav' - u'av \in I$ when $bv, bv' \in I$ and in consequence $av, av' \in I$. When $a \notin Z(b)$ the last relationship may cease to be valid and the two ordered pairs $\langle ua, ub+v\rangle$ and $\langle u'a, u'b+v\rangle$ may not be equivalent. A simple example of such nonequivalence is easily given. Suppose that $u, v \in S$ such that $b \leq u \bmod I$, $ub+v \in D (d \in I)$ and $bv \in I$ exist. Since $ub \leq ub \bmod I$, it follows from clause ( ) of Theorem that $b \leq u^2 b \bmod I$. Also $(u^2 b)b + v^2 = d^2 \in D$ and $bv^2 \in I$. Hence $u' = u^2 b$ and $v' = v^2$ may be taken in the above argument. Suppose that although $b \leq u \bmod I$ and $bv \in I$, $uv \notin I$ (for example, that $u \in D$ and $v \notin I$). Now $uav' - u'av = uav^2$. Set $a = uv$. If $uav^2 \in I$ then, since $I \in \mathbb{I}_{QF}(S)$, $uv \in I$ contrary to assumption; hence $uav^2 \notin I$ and, for $a = uv$, $\langle ua, ub+v\rangle \not\equiv \langle u'a, u'b+v'\rangle$.

Although, as pointed out above, it is necessary to associate with each denominator b either an orthogonal complement v in isolation, or such a complement together with a factor u, much can be proved without use of the assumption that $b \equiv u \bmod I$ and of the further condition that the ideal I should be square free. Defining the system $N' \leq S$ by deleting the condition that $b \equiv u \bmod I$ from the definition of N, N' remains a multiplicative module (if, corresponding to b and d, u, v and y, z exist such that $ub+v, yd+z \in D$ and $bv, dz \in I$, then, corresponding to bd, $u'' = uy$ and $v'' = ubz + ydv + vz$ exist such that $u''bd + v'' \in D$ and $bdv'' \in I$).

Using N' in place of N, relinquishing from the definition of the quotient extension that $I \in \mathfrak{I}_{QF}(S)$, denoting by Q_I'' the resulting quotient extension of S, and denoting by $D_I'', \bar{I}_I''$ the systems in Q_I'' analogous to $D_I, \bar{I}_I$ in Q_I, it may still be shown that $Q_I'' \in \mathfrak{S}$, again with the complete system of numbers $o_{I(j)}$ in Q_I'' forming $Q_I''[o]$, and that each Q''-number is equal to a number of the form $[a/b]_I$ with $b \in D$. It may also be shown that $[a/b] \in D_I''$ if and only if $a \in D$; also all numbers $[a/b]_I$ with $a \in \bar{I}$ belong to $\bar{I}_I''$. Furthermore all numbers $[a]_I$ with $a \in D$ belong to D_I'' and only such numbers

do so; all numbers $[a]_I$ with $a \in \bar{T}$ belong to the complete system of numbers $[a/b]_I$ with $a \in \bar{T}$.

In the proof of Theorem the joint assumptions that b and u should satisfy the relationship $b \equiv u \bmod I$ and that $I \in \mathbb{I}_{QF}(S)$ are used to show that $Q_I[o] \in \mathbb{I}_{QF}(S)$ and that $[a/b]_I \in \bar{T}_I$ only if $a \in \bar{T}$, so that the complete system of numbers $[a/b]_I$ with $a \in \bar{T}$ constitutes $\bar{T}_I$. These assumptions are used again to show that of the numbers $[a]_I$ ~~with~~ only those for which $a \in \bar{T}$ belong to $\bar{T}_I$. With $B = [b/d]_I$ or $B = [b]_I$, they are also used to ~~show that~~ establish the equivalence of $\bar{O}_I(B)\langle Z_I(B)\rangle$ and the systems of Q-numbers $[a/c]_I$ with $a \in \bar{O}(b)\langle Z(b)\rangle$ and, subsequently, the similar equivalence of N_I and the system of numbers $[a/b]_I$ with $a \in N$: they are used again to show that $[a]_I \in N_I$ if and only if $a \in N$.

A further quotient extension Q_I' of S may be defined by using the system N' in place of N, but retaining the condition $I \in \mathbb{I}_{QF}(S)$. When $c|a, d|b \bmod I$, the Q'-numbers $A = [a/c]_I$ and $B = [b/d]_I$ are well defined; also, in particular, $a \equiv c \bmod I$. If, in addition $b|a \bmod I$, then $a \equiv b \bmod I$, so that $a^2 \equiv bc \bmod I$. Since, in the definition of Q_I', the assumption $I \in \mathbb{I}_{QF}(S)$ has been retained,

$a \equiv bc \bmod I$ and hence $ad \equiv bc \bmod I$. Furthermore $bc \in N$, since $b, c \in N'$, so that $bc \mid ad \bmod I$. In short, the number $X = [ad/bc]_I$ is well defined in Q'_I; it satisfies the equation $BX = A$. The general solution of the equation $BY = A$ in Q'_I has the form $Y = X + G$, where $G \in \mathfrak{o}(B)$ belongs to the Q'-stratum in which X occurs.

The joint assumptions concerning the numerator b and the ideal I are used to show that $Y = X$ is the only solution of the equation $BY = A$ in $Z_I(B) \subseteq Q_I$ and that this solution lies in $Z_I(A)$.

Theorem . (i) $Q_I \equiv Q(S, I)$ exists if and only if $Q_I \equiv Q(S_I, S_I[0])$ exists.

(ii) If these S-rings exist (a) they are isomorphic, $[a/b]_I \cong [(a)_I/(b)_I]_{S_I[0]}$ being the relationship determining the isomorphism and (b) numbers of the form $[a]_I$ in S_I and $[(a)_I]_{S_I[0]}$ in $\hat{Q}_I$ form sub S-rings isomorphic to S_I, $[a]_I \cong (a)_I$ and $[(a)_I]_{S_I[0]} \cong (a)_I$ being the relationships establishing these isomorphisms.

Proof. Q_I exists if and only if $\mathfrak{D} \equiv \mathfrak{D}(S, I)$ is nonvoid and $1 \in \mathbb{I}_{QF}(S)$; a similar remark concerning the existence of $\hat{Q}_I$ may be made. From clause ( ) of Theorem , $\mathfrak{D}$ is nonvoid if and only if $\hat{\mathfrak{D}}_I \equiv \mathfrak{D}(S_I, S_I[0])$ is nonvoid and, from clause ( ) of Theorem , $I \in \mathbb{I}_{QF}(S)$ if and only if $S_I[0] \in \mathbb{I}_{QF}(S_I)$. The result of clause (i) follows immediately.

The equivalence between relationships mod I in S and relationships mod $S_I[0]$ in S_I, established in the proof of Theorem , is again invoked. If for all $g \in S$ such that $bg \in I$, $ag \in I$ also, then for all $(g)_I \in S_I$ such that $(b)_I (g)_I \in S_I[0]$, $(a)_I (g)_I \in S_I[0]$, and conversely. Hence $a \in Z(b) \equiv Z(S, I | b)$ if and only if $(a)_I \in \hat{Z}_I((b)_I) \equiv Z(S_I, S_I[0] | (b)_I)$. If $u, v \in S$ exist such that $b \in Z(u)$,

$ub+v \in D$, $bv \in I$ then $(u)_I, (v)_I \in S_I$ exist such that $(b)_I \in \hat{Z}_I((u)_I)$, $(u)_I(b)_I + (v)_I \in \hat{D}_I$, $(b)_I(v)_I \in S_I[0]$ and conversely. Hence $b \in N \equiv N(S,I)$ if and only if $(b)_I \in \hat{N}_I \equiv N(S_I, S_I[0])$. (The simpler case in which $b \in N$ is unaccompanied by a factor u is dealt with more easily.) Combining the two results just derived, $b|a \bmod I$ if and only if $(b)_I | (a)_I \bmod S_I[0]$. Expressed in another way, the number $A = [a/b]_I$ exists in Q_I if and only if the number $\hat{A} = [(a)_I / (b)_I]_{S_I[0]}$ exists in $\hat{Q}_I$. Furthermore, since $A \in Q_I[0]$ if and only if $a \in I$, $(a)_I \in S_I[0]$ if and only if $a \in I$, and $\hat{A} \in \hat{Q}_I[0]$ if and only if $(a)_I \in S_I[0]$, it follows that $A \in Q_I[0]$ if and only if $\hat{A} \in \hat{Q}_I[0]$.

The principle used above may be invoked still further to show that there is a complete correspondence between ordered pairs of S-numbers and S_I-numbers in corresponding Q_I and $\hat{Q}_I$ number classes. $\langle a', b' \rangle \in A$ if and only if $\langle (a')_I, (b')_I \rangle \in \hat{Q}_I$. For, with b related to u, v as described above, $\langle a', b' \rangle \in A$ if and only if $ua b' = a'(ub+v) \bmod I$, a relationship which holds if and only if its counter-part with $u, \ldots, v$ replaced by $(u)_I, \ldots, (v)_I$ holds with respect to $\bmod S_I[0]$ in S_I,

that is, if and only if $\langle (a')_I, (b')_I \rangle \in A$. There is a complete correspondence between the construction of Q_I and that of $\hat{Q}_I$. Thus if, with $C = [c/d]_I$, $\langle c', d' \rangle \in C$, then $A = C$ in Q_I if and only if $a'd' \equiv b'c' \bmod I$; with $\hat{C} = [(c)_I / (d)_I]_{S_I[0]}$, $\langle (c')_I, (d')_I \rangle \in \hat{C}$, and $\hat{A} = \hat{C}$ in $\hat{Q}_I$ if and only if $(a')_I (d')_I \equiv (b')_I (c')_I \bmod S_I[0]$. Hence if $A \simeq \hat{A}$, $C \simeq \hat{C}$, $A = C$ in Q_I if and only if $\hat{A} = \hat{C}$ in $\hat{Q}_I$. Again, if $S(ad) = S(bc)$, $B = A \pm C$ is represented by $\langle a'd' \pm c'd', b'd' \rangle$, $\hat{B} = \hat{A} \pm \hat{C}$ is similarly represented, and $B \simeq \hat{B}$. Products are treated in the same way. The correspondence of the form $A \simeq \hat{A}$ is preserved during addition, subtraction and multiplication. Q_I and $\hat{Q}_I$ are isomorphic and, in particular, $\hat{Q}_I$ may be exhibited as an S-ring.

Within Q_I, let $\tilde{Q}_I$ be the system of numbers of the form $[a]_I$. Each number $[a]_I$ of $\tilde{Q}_I$ has a representation of the form $[ab/b]_I$ where b is any member of D, and such a representation has in turn a further representation as an ordered pair $\langle ab, b \rangle$ of S-numbers. ~~With $S(a) = S(c)$, $[a]_I \pm [b]_I \to$~~ $[a]_I [b]_I$ has the representation $\langle acb^2, b^2 \rangle$ and, with $S(a) \equiv S(c)$, $\langle (a \pm c) b^2, b^2 \rangle \in [a]_I \pm [b]_I$. The members of $\tilde{Q}_I$ are closed with respect to addition

and subtraction, where these are possible, and to multiplication. The numbers $[a]_I$ with $a \in I$ play the rôle of zero members of $\tilde{Q}_I$. The properties of equivalence, addition, subtraction and multiplication over $\tilde{Q}_I$ are those of these operations over Q_I. Hence $\tilde{Q}_I \in \mathfrak{S}$.

Set $A = [a]_I$, $C = [c]_I$ in Q_I and $\tilde{A} = (a)_I$, $\tilde{C} = (c)_I$ in S_I; these assignments induce a correspondence of the form $A \simeq \tilde{A}$. A and C have the representations $[ab/b]_I$ and $[cb/b]_I$ where b is any member of D. $A = C$ in Q_I if and only if $b(a-c) \in I$ or, since $b \in D$, if and only if $a = c \bmod I$. $\tilde{A} = \tilde{C}$ in S_I if and only if $a = c \bmod I$. Hence $A = C$ in Q_I if and only if $\tilde{A} = \tilde{C}$ in S_I. With $S(a) \equiv S(b)$, $A \pm C$ are represented by the ordered pairs $\langle (a \pm c) b^2, b^2 \rangle$ of S-numbers, so that $A \pm C = [\xi a \pm b^3]_I$. Within S_I, $\tilde{A} \pm \tilde{C} = (a \pm c)_I$. Hence $A \pm C \simeq \tilde{A} \pm \tilde{C}$. Similarly, $AC \simeq \tilde{A}\tilde{C}$. In conclusion $\tilde{Q}_I$ and S_I are isomorphic S-rings.

Within $\hat{Q}_I$ the numbers of the form $[(a)_I]_{S_I[0]}$ have a representation of the form $[(a)_I (b)_I / (b)_I]_{S_I[0]}$ where $(b)_I \in \hat{D}_I$. The correspondence $[ab/b]_I \simeq [(a)_I (b)_I / (b)_I]_{S_I[0]}$ between the members of Q_I and $\hat{Q}_I$ induces a correspondence between $[a]_I \simeq [(a)_I]_{S_I[0]}$ between the members of $\tilde{Q}_I$ and

the subsystem $\bar{Q}_I$ of $\hat{Q}_I$ of numbers of the form $[(a)_I]_{S_I[0]}$. The former correspondence, that between members of Q_I and $\hat{Q}_I$, is preserved during addition, subtraction and multiplication; the latter correspondence is preserved in the same way. $\tilde{Q}_I$ is isomorphic to $\bar{Q}_I$. Since $\tilde{Q}_I \sim S_I$, $\bar{Q}_I$ is isomorphic to S_I, $[(a)_I]_{S_I[0]} = (a)_I$ being the relationship establishing this isomorphism.

Definition. Let $I \in \mathbb{I}_{QF}(S)$, $J \in \mathbb{I}(S)$.

(i) $\mathcal{J}_I$ is the system of Q_I-number classes that contain an ordered pair $\langle a', b' \rangle$ of S-numbers for which $a' \in J$.

(ii) $[a/b]_{I,J}$ and $[a]_{I,J}$ are the classes of Q_I-numbers B such that $[a/b]_I - B \in \mathcal{J}_I$ and $[a]_{I,J} - B \in \mathcal{J}_I$ respectively; such classes are $Q_{I,J}$-numbers; the notation $A \in Q_{I,J}(S)$ indicates that A is a $Q_{I,J}$-number.

(iii) With A', B' any Q_I-numbers belonging to $A, B \in Q_{I,J}(S)$ respectively, $A = B \bmod \mathcal{J}_I$ if and only if $A' - B' \in \mathcal{J}_I$; arithmetic operations involving $Q_{I,J}$-numbers are, mutatis mutandis, as described in clauses (iii b, c) of Definition.

(iv) $Q_{I,J}$ is the complete system of classes $Q_{I,J}\{i\}$ constructed as in clause (iii) of Definition.

Theorem. Let $I \in \mathbb{I}_{QF}(S)$, $J \in \mathbb{I}(S)$.

) $\mathcal{J}_I$ is an S-ideal in Q_I.

(ii) $Q_{I,J}$ is an S-ring.

(iii) Within $Q_{I,J}$, the numbers of the form $[a]_{I,J}$ constitute an independent S-ring.

(iv) With $(b)_I$, $(a)_{I,J}$ as described in Definition, $[b]_I \in [a]_{I,J}$ if $(b)_I \in (a)_{I,J}$.

(v) Set $K = J/''D(S,I)$. If $(b)_I \in (a)_{I,K}$, $[b]_I \in [a]_{I,K}$.

Proof. The proof that $\mathfrak{J}_I$ is an S-ideal in Q_I is similar to that of clause (i) of Theorem using, in place of an S-number b' to represent an S_I-number $(b)_I$, an ordered pair $\langle a', b' \rangle$ to represent a Q_I-number $[a/b]_I$. That $Q_{I,J}$ is an S-ring and contains numbers of the form $[a]_{I,J}$ as a sub S-ring are results that again are easily verified.

$[a]_{I,J}$ is the class of Q_I-numbers A' ~~for which~~ such that for at least one S-number pair $\langle x, y \rangle$ representing $[a]_I - A'$, $x \in J$. The pair $\langle x, y \rangle$ represents $[a]_I - [b]_I$ if and only if $y(a-b) - x \in I$. Accordingly $[b]_I \in [a]_{I,J}$ if and only if $v \in I$, $x \in J$ and $y \in D = D(S,I)$ can be found for which $y(a-b) - x = v$.

$(a)_{I,K}$ is the class of S_I-numbers $(b)_I$ such that for at least one S-number w representing $(a)_I - (b)_I$, $w \in K$. w represents $(a)_I - (b)_I$ if and only if $a - b - w \in I$. Accordingly, $(b)_I \in (a)_{I,K}$ if and only if $u \in I$ and $w \in K$ can be found such that

$a-b-w=u$.

When $K=J/''D$, $w\in K$ if and only if $yw\in J$ for at least one $y\in D$. Multiplication of the equation $a-b-w=u$ throughout by such a y reveals that $x=yw\in J$ and $v=yu\in I$ satisfying the conditions ensuring that $[b]_I\in[a]_{I,J}$ have been found. The result of clause (v) has been obtained. With K as just defined, $J\subseteq K$: the result of clause (iv) is a corollary to that of clause (v).

Definition .(i) An S-ring S whose quotient extension with respect to S[0] is isomorphic to S itself is a quotient S-ring; the notation $S \in \mathbb{Q}$ indicates that S is a quotient S-ring. (ii) With $S \in \mathbb{Q}$, $Q_0 = Q(S, S[0])$, $b \in N(S, S[0])$, $a \in S$ such that $a \prec b \bmod S[0]$ and a_0, b_0 being the members of Q_0 for which $a \simeq a_0$, $b \simeq b_0$, the member of S corresponding to $[a_0/b_0]_{S[0]}$ in Q_0 is written simply as a/b. b) With m, n positive integers, b^m/b^n is written as b^r where $r = m - n$. c) b^0 is written as 1_b and, when $b \in D(S, S[0])$, 1_b is written as 1.

In the following theorem, it is shown that quotient S-rings exist; and it is shown that for each denominator b the number 1_b is a unique local unit element for the corresponding numerator system, that these elements form a multiplicative S-module, and that 1 is a unit element for S itself; it is also shown that when b is a denominator, each element of S may uniquely be expressed as the sum of a member of the numerator system $Z(S, S[0]|b)$ and the complementary orthogonal system $O(S, S[0]|b)$ and that S itself is the S-ideal compound $Z(b) \vee O(b)$ of these two systems.

Theorem .(i) The quotient extension of an S-ring is a quotient S-ring.

(ii) Let $b \in N(S \in \mathfrak{Q}_s, S[o])$. (a) for a fixed r, the number $b^r \in S$ is independent of m and n occurring in clause (ii) of Definition . (b) $b^r \equiv b \bmod S[o]$ $(r;)$. (c) For all $v \in O_c(S, S[o]$ b) the numbers 1_v in S are equal.

(iiia) With $b \in N(S \in \mathfrak{Q}, S[o])$, $\S 1_b a = a$ for all $a \in Z(b) = Z(S, S[o]|b)$ and 1_b is the only element of $Z(b)$ to possess this property; furthermore, with $v \in O_c(S, S[o]|b)$, $1_b + 1_v = 1$. (b) $1a = a$ for all $a \in S$. (c) The numbers 1_b form a multiplicative $\mathfrak{S}$-module in S and, with $c \in N(S, S[o])$, $1_b 1_c = 1_{bc}$.

(iv) Let $b \in N(S \in \mathfrak{Q}, S[o])$ and $v \in O_c(S, S[o]|b)$. (a) Each number d of S has a unique decomposition of the form $d = d_b + d_v$ where $d_b \leq b, d_v \leq v \bmod S[o]$; furthermore, the constituents d_b, d_v are expressible in the form $d_b = x(d,b)b$, $d_v = y(d,v)v$ where $x(d,b)\langle y(d,v)\rangle$ is a uniquely determined member of $Z\langle O\rangle(S, S[o]|b)$; in particular, d may be divided by b to yield the quotient $x(d,b,v)$ unique in $x(d,b,v)$ unique in $Z(S, S[o]|b)$ and remainder d_v unique in $O(S, S[o]|b)$ in the sense that $d = x(d,b,v)b + d_v$. (b) $S = \{Z(S, S[o]|b) \vee O(S, S[o]|v)\}$.

Proof. Since the existence of the quotient extension Q_I of an S-ring S with respect to an S-ideal I in S is assumed, (a) $I \in \overset{I_{QF}(S)}{\cancel{QF(S)}}$ and (b) S possesses at least one element $b' \in \mathbb{D}(S, I)$. Consequently (a) (from clause ( ) of Theorem) $Q_I[0] \in \overset{\mathbb{I}_{QF}}{\cancel{QF}}(Q_I)$ and (b) (from clause ( ) of that theorem), Q_I possesses at least one element, namely $B' = [b']_I$ in $\mathbb{D}(Q_I, Q_I[0])$: the construction of $Q_{\cancel{Q}} \equiv \mathbb{Q}(Q_I, Q_I[0])$ is possible. $Q_{\cancel{Q}}$ contains the system Q' of numbers of the form YB'/B' where $Y \in Q_I$; it will be shown that $Q_{\cancel{Q}} \cong Q'$. Taking Y to be $(y/z)_I$, YB'/B' is the class of classes of ordered pairs of S-numbers equivalent to $\langle xy, xz+t \rangle$ where $z \leq x \bmod I$, $zt \in I$. Q' is the system of such classes of classes for all $y, z \in S$ such that $z | y \bmod I$. Q_0 also contains all numbers A/B for which $B | A \bmod Q_I[0]$. Taking A and B to have the relatively complicated forms $(2, 2')$ of clause ( ) of Theorem and $a, \ldots, d, e, f, u, v$ to be ~~as~~ related as in the proof clause ( ) of that theorem, A/B is the class of classes

of ordered pairs of S-numbers equivalent to $\langle ue, ue bc + buf + cev + vf \rangle$ where $b \leq u$, $c \leq e \bmod I$ and $bv, cf \in I$. The ordered pair just given has the form $\langle xy, xz+t \rangle$ with $x = ue$, $y = ad$, $z = bc$, $t = buf + cev + vf$, where $z \leq x \bmod I$, $zt \in I$; it is one of the ordered pairs serving to characterise membership of Q'. The same conclusion may be drawn with regard to the quotient A/B for which A or B have the simpler forms described in clause ( ) of Theorem : all members of Q belong to Q'. The relationship $Y_1 \simeq Y'B'/B'$ establishes an isomorphism between Q_I and Q'. The result of clause (i) follows.

In the remainder of the proof, $D(S, S[0])$, $N(S, S[0])$, $Q(S, S[0])$, $O(S, S[0]|b)$ and $Z(S, S[0]|b)$ are written as D, N, Q_0, $O(b)$ and $Z(b)$ respectively, and $a \leq b \bmod S[0]$, $a \equiv b \bmod S[0]$ expressed as $a \leq b$, $a \equiv b$.

For the proof of clause (iia) it is first remarked that, since $N \in M(S)$, $b^n \in N$ when $b \in N\,(n:)$. Also, from clause ( ) of Theorem , $b^m \equiv b^n\,(m, n:)$ so that, in particular, $b^m \leq b^n$ and the quotients b^m/b^n are well defined $(m, n:)$. Supposing that $u, v \in S$ exist such that $b \leq u$, $bv \in S[0]$,

$ub+v \in D$, it follows that $b^n \leq u^n$, $b^n v^n \in S[0]$ and $u^n b^n + v^n = (ub+v)^n \in D$. Accordingly b^m/b^n corresponds to a member of Q_0 represented by the ordered pair $\langle u^n b^m, u^n b^n + v^n \rangle$. Since $u^{n+1} b^{m+1}(u^n b^n + v^n) - u^n b^m (u^{n+1} b^{m+1} + v^{n+1}) = u^n b^m v^n (ubv - v) \in S[0]$, $b^m/b^n = b^{m+1}/b^{n+1}$ in $S(m,n:)$. Use of this result yields that of clause (iia).

$b \in S$ corresponds to a member of Q_0 represented by an ordered pair of the form $\langle db, d \rangle$ where $d \in D$. For any $f(\in S) \simeq f_0 (\in Q_0)$ where f_0 is represented by $\langle g, h \rangle$ in Q_0, the condition $bf \in S[0]$ implies in turn that $dbg \in S[0]$, $bg \in S[0]$, $u^n b^m g \in S[0]$ $(m:)$ and $(b^m/b^n) f \in S[0]$. Conversely, if $(b^m/b^n) f \in S[0]$ then $u^n b^m g \in S[0]$ so that $(ub)^{m+n} g \in S[0]$ and, since $1_{QF}(S)$, $ubg \in S[0]$ and, since $b \leq u$, $bg \in S[0]$, $dbg \in S[0]$ and in consequence $bf \in S[0]$. Accordingly $b^m/b^n = b(m,n:)$, the result of clause (iib).

For the proof of clause (iic) suppose that $v, w \in O_c(S, S[0]|b)$ and that $u, u' \in S$ for which $b \leq u, u' \bmod S[0]$ exist such that $ub+v, u'b+w \in D$, $bv, bw \in S[0]$. From clause ( ) of Theorem , $v, w \in N$ and v/v and w/w correspond to two members of Q_0 represented by the pairs $\langle v, ub+v \rangle$ and $\langle w, u'b+w \rangle$ respectively. Since $(u'b+w)v = (ub+v)w = wv$, these two pairs are equivalent:

$d_b + d_v$ where $d_b = 1_b d$, $d_v = 1_v d$. Since $1_b\langle 1_v\rangle \in Z(b\langle v\rangle)$, $d_b \leq b$, $d_v \leq v \bmod S[b]$. The equation $xb = d_b$ has a unique solution $x(d,b)$ in $Z(b)$. Similarly, from clause ( ) of Theorem , $Z(v) \equiv \mathcal{O}(b)$: the equation $yv = d_v$ has a unique solution $y(d,v)$ in $\mathcal{O}(b)$. That d permits the unique decomposition $d = x(d,b)b + b_v$ is a consequence of the preceding results. Since every member d of S may be expressed as the sum of $d_b \in Z(b)$ and $d_v \in \mathcal{O}(b)$, $S \subseteq Z(b) \vee \mathcal{O}(b)$. But $Z(b) \vee \mathcal{O}(b) \subseteq S$: clause (W b) has been demonstrated.

Definition. (i) $\mathfrak{J}$, $\mathfrak{K}$ and $\mathfrak{R}$ being three systems of $\mathbb{S}$-numbers such that for all $d \in \mathfrak{K}$ and $a \in \mathfrak{R}$, $ad \in \mathfrak{J}$ only when $a \in \mathfrak{J}$, the system $\mathfrak{K}$ is said to be integral with respect to $\mathfrak{J}$ in $\mathfrak{R}$; the notation $\mathfrak{K} \in \mathbb{G}(\mathfrak{R}, \mathfrak{J})$ is used to indicate that this is so.

(ii) $\mathfrak{J}$ and $\mathfrak{R}$ being two systems of $\mathbb{S}$-numbers such that $\mathfrak{R} \setminus \mathfrak{J} \in \mathbb{G}(\mathfrak{R}, \mathfrak{J})$, the system $\mathfrak{R}$ is said to be a domain of integrity with respect to $\mathfrak{J}$; the notation $\mathfrak{R} \in \mathbb{IB}(\mathfrak{J})$ denotes this fact.

(iii) An $\mathbb{S}$-ring $\mathbb{S}$ for which $\mathbb{S} \in \mathbb{IB}(\mathbb{S}[o])$ is an $\mathbb{S}$-domain of integrity, and the notation $\mathbb{S} \in \mathbb{IB}$ indicates that this is true.

In the following theorem, it is first shown that systems of $\mathbb{S}$-numbers possessing the properties stated in the above definitions exist; conditions that suffice and are necessary for a system to possess such properties are given, and relationships involving two systems which suffice to ensure that if the first possesses certain of the above properties, the other also does so, are given.

Theorem .(i) Let $C \in \mathbb{S}\mathbb{I}(S)$ (a) $\mathbb{D}(S,C)$ being nonvoid, $\mathbb{D}(S,C) \in \mathbb{G}(S,C)$. (b) $S \setminus C$ being nonvoid, $T(S,C)$ is nonvoid if and only if $S \in \mathbb{I}\mathbb{B}(C)$

(ii) S being non-null, T is void if and only if $S \in \mathbb{I}\mathbb{B}$.

(iii) $K \in \mathbb{G}(R,J)$ if and only if $\{R \setminus J\} \leq S \setminus J$. $R \in \mathbb{I}\mathbb{B}(J)$ if and only if $\{R \setminus J\}^2 \leq S \setminus J$. $S \in \mathbb{I}\mathbb{B}$ if and only if $\{S \setminus S[b]\}^2 \leq S \setminus S[b]$.

(iv) If $K \in \mathbb{G}(R,J)$, $K' \leq K$ and $R' \leq R$, then $K' \in \mathbb{G}(R',J)$. If $R \in \mathbb{I}\mathbb{B}(J)$ and $R' \leq R$ then $R' \in \mathbb{I}\mathbb{B}(J)$ also. If $S \in \mathbb{I}\mathbb{B}$ and $S' (\leq S) \in \mathbb{S}$ then $S' \leq \mathbb{I}\mathbb{B}$.

Proof. By definition, $\mathbb{D}(S,C)$ is that system of $\mathfrak{S}$-numbers b for which $ab \in \mathbf{C}$ only when $a \in \mathbf{C}$, and this is the property that characterises $\mathbb{D}(S;C)$ as being integral with respect to C in S. If $S \in \mathbb{IB}(C)$, no $a \notin C$ and $d \in S \setminus C$ exist for which $ad \in C$; the system of such numbers a, by definition $\mathbb{T}(S,C)$, is void. Conversely, if $\mathbb{T}(S,C)$ is void, for all $d \in S \setminus C$ and $\mathfrak{S}$-numbers a, $ad \in C$ only when $a \in C$, and $S \setminus C \in \mathbb{G}(S,C)$. Taking C to be $S[0]$, the result of clause (ii) is obtained

If $K\{R \setminus \mathfrak{J}\} \leq S \setminus J$, then no $d \in \mathfrak{D}K$ and $a \in R \setminus \mathfrak{J}$ exist such that $ad \in \mathfrak{J}$; alternatively, the only $a \in R$ for which $ad \in \mathfrak{J}$ are $a \in \mathfrak{J}$. Conversely if, for all $d \in \mathfrak{D}$ and $a \in R$, $ad \in \mathfrak{J}$ only when $a \in \mathfrak{J}$, then for all $d \in \mathfrak{D}$ and $a \in R \setminus \mathfrak{J}$, $ad \in \mathfrak{J}$ and hence $ad \in S \setminus \mathfrak{J}$. The second result of clause (iii) is obtained by setting $K = R \setminus \mathfrak{J}$, and the third by taking $R = S$ and $\mathfrak{J} = S[0]$.

When $K' \leq K$ and $R' \leq R$, $K'\{R' \setminus \mathfrak{J}\} \leq K\{R \setminus \mathfrak{J}\}$; the condition $K\{R \setminus \mathfrak{J}\} \leq S \setminus \mathfrak{J}$ then implies that $K'\{R' \setminus \mathfrak{J}\} \leq S \setminus \mathfrak{J}$. The second result of clause (iv) is obtained by taking $K = R \setminus \mathfrak{J}$, $K' = R' \setminus \mathfrak{J}$ and the third by setting $R = S$, $R' = S'$

Definition (i) K, L and R being three systems of S-numbers such that for each $a \in K$ and $c \in L$ a $b(a,c) \in R$ <a unique $b(a,c) \in R$> exists such that $a\,b(a,c) = c$, L is said to be divisible <uniquely divisible> by K in R; the notation $K|L\{R\}$ <$K|'L\{R\}$> indicates that this is so.

(ii) If, in the preceding clause, $L = R$, L is said to be divisible <uniquely divisible> by K, and the notation $K|L$ <$K|'L$> denotes this fact.

(iii) If $L \subseteq R$, $K|'L\{R\}$ and, in clause (i) $b(a,c) \in L$ for all $a \in K$, $c \in L$, L is said to be locally divisible by K in R; the notation $K|''L\{R\}$ indicates that this is so

(iv) L and R being two systems of S-numbers such that $L|L\{R\}$ <$L|'L\{R\}$>, L is said to be self-divisible <uniquely self-divisible> in R; the notation $L|\{R\}$ <$L|'\{R\}$> indicates that this is so.

(v) L and R being two systems of S-numbers such that $L|L''\{R\}$, L is said to be locally self-divisible in R; the notation $L|''\{R\}$ indicates that this is so.

(vi) J and R being two systems of S-numbers such that $R\backslash J|\{R\}$ <$R\backslash J|'\{R\}$>, R is said to be a division system

⟨a unique division system⟩ with respect to J; the notation $R \in \mathbb{DS}(J)$ ⟨$R \in \mathbb{DS}'(J)$⟩ indicates that this is so.

(vii) An S-ring for which $S \in \mathbb{DS}'\{S[0]\}$ is a division S-ring; the notation $S \in \mathbb{DR}$ indicates this fact.

With L' being the largest system $L \subseteq S$ such that $C \setminus S[0] \mid L\{S\}$, the composite part $K(R,C)$ of R with respect to C is $R \cap L'$.

In the following theorem, it is shown that systems of S-numbers possessing the properties stated above exist, results concerning related systems are derived, and special results concerning multiplicative S-modules, semi S-ideals and S-ideals possessing divisibility properties are given.

Theorem. (i) Let $S \in \mathbb{Q}$. (a) Nth $b \in \mathbb{N}(S, S[0])$ and $a \leq b \bmod S[0]$, $bD \mid \mathbb{Z}(S, S[0] \mid a)\{S\}$, $bD \mid'' \mathbb{Z}(S, S[0] \mid a)\{\mathbb{Z}(S, S[0] \mid b)\}$ and $bD \mid' \mathbb{Z}(S, S[0] \mid a)$. (b) $D \mid' S$. (c) $D \mid'' \{S\}$ and $bD \equiv D$ for all $b \in D$. (d) $S \in \mathbb{DR}$ if and only if T is void.

(ii) If $K \mid L\{R\}$, then $aR \supseteq L$ for all $a \in K$, and $KR \supseteq L$. If $L \mid \{R\}$ then $aR \supseteq L$ for all $a \in L$ and $LR \supseteq L$. If $R \in \mathbb{DS}(\mathcal{T})$ then $aR \supseteq R \setminus \mathcal{T}$ for all $a \in R \setminus \mathcal{T}$ and $\{R \setminus \mathcal{T}\}R \supseteq R \setminus \mathcal{T}$. If $S \in \mathbb{DR}$, $S \in \mathbb{IB}$ also.

(iii) If $K \mid L\{R\}$, $K \supseteq K'$, $L \supseteq L'$ and $R \subseteq R'$, then $K' \mid L'\{R'\}$. If $K \mid' L\{R\}$, then $K \mid' L$. If $K \mid' L\{R\}$, $K \supseteq K'$ and $L \supseteq L'$, then $K' \mid' L'\{R\}$. If $L \mid \{R\}$, $L \supseteq L'$ and $R \subseteq R'$, then $L' \mid \{R'\}$. If $R \in \mathbb{DS}(\mathcal{T})$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $R \in \mathbb{DS}(\mathcal{T}')$. If $S \in \mathbb{DR}$ and $S'(\in S) \subseteq S$, $S' \in \mathbb{DR}$.

(iv) Let $M(\in M_S(S)) \mid \{S\}$, then $M \mid \{M\}$, $bM \equiv M$ for all $b \in M$ and $M^2 = M$.

(va) If $K \mid C$ where $C \in \mathbb{SI}(S)$ then $aC = C$ for all $a \in K$ and $KC \equiv C$. (b) If $C \in \mathbb{SI}\{S \in \mathbb{Q}\}$ then $D \mid'' C\{S\}$

(vi) Let $M(\in M(S)) \mid \{S\}$ and $C \in \mathbb{SI}(S)$. Then $C \mid'' M = C \mid' M$.

(vii) If $b \in \mathbb{N}(S, S[0] \in \mathbb{QF})$ and $D \mid \mathcal{T}(\in \mathbb{I}(S))\{\mathcal{T}\}$ then $\mathbb{Z}(S, S[0] \mid b) \subseteq \mathbb{Z}(S, \mathcal{T} \mid b)$.

proof. When $\mathfrak{o} \in \Psi$, $\psi_0 \equiv Q(S, S[\mathfrak{o}])$ exists uniquely, with

$a, b (\in S) \simeq a_0, b_0 (\in Q_0)$, $\mathfrak{Z}(S, S[\mathfrak{o}] | a\langle b \rangle) \sim \mathfrak{Z}(Q_0, Q_0[\mathfrak{o}] | a_0 \langle b_0 \rangle)$ and if $b \in \mathfrak{N}(S, S[\mathfrak{o}])$, $b_0 \in \mathfrak{N}(Q_0, Q_0[\mathfrak{o}])$. To each $c_0 \in \mathfrak{Z}(a_0) = \mathfrak{Z}(S, S[\mathfrak{o}] | a_0)$ there corresponds a unique $x_0 \in \mathfrak{Z}(b_0) = \mathfrak{Z}(S, S[\mathfrak{o}] | b_0)$ such that $b_0 x_0 = a_0$. Similarly, there exists a unique $y_0 \in S$ such that $d_0 y_0 = x_0$, where $d_0 \simeq d (\in D)$. ~~and this y be~~ Since $S \sim Q_0$, the equation $bdy = c$ possesses a unique solution $y \in S$ and this y belongs to $\mathfrak{Z}(b_0)$; but in the above, $x_0 \in \mathfrak{Z}(a_0)$ so that $y \in \mathfrak{Z}(a_0)$ also: since $S \sim Q_0$, the equation $bdy = c$ possesses a unique solution $y \in \mathfrak{Z}(b)$, and this y belongs to $\mathfrak{Z}(a)$. Hence $bD |'' \mathfrak{Z}(a) \{\mathfrak{Z}(b)\}$. The less informative result $bD^{\alpha} | \mathfrak{Z}(a)$ is a simple corollary to that just derived. Since $\mathfrak{Z}(b) \leq S$, $bD | \mathfrak{Z}(a) \{S\}$ (but in this case, if $v \in S\{y\}$, $y \notin S[\mathfrak{o}]$, by $\in S[\mathfrak{o}]$, then $bd(y+v) = c$ also, so that divisibility with respect to S is nonunique. Taking $a = b = 1$, the result of clause (ib) is obtained.

In the special case in which $b \in D$, $\mathfrak{Z}(b) = S$. To each $a \in S$ there corresponds a unique $x = x(a, b) \in S$ such that

$bx=a$. This is, in particular, true when $a(\in U)\simeq a_0(\in U_0$ $\equiv D(Q_0, Q_0[0])$. This x corresponds to $x_0=[a_0/b_0]$ in Q_0 which, since $a_0\in D_0$, is also a member of D_0, so that $x\in D$: D is locally self-divisible. The analysis may be extended to all $a\in D$: $bD\supseteq D$. But D is a multiplicative S-module, so that $bD\subseteq D$ and $bD=D$.

When T is void, all members of $S\setminus S[0]$ belong to D, and in this case $S\setminus S[0]$ is uniquely divisible by itself: S is a division ring. Conversely, if the equation $bx=a$ has a unique solution $x=x(a,b)$ for all pairs $a,b\in S\setminus S[0]$, then for no $b\in S\setminus S[0]$ is it possible that $by\in S[0]$ when $y\in S[0]$, for otherwise $x=x(a,b)+y$ would be a second solution, x accordingly being nonunique: T is void.

If $K|L\{R\}$ and $a\in K$ is fixed, every member c of L has a representation ab for some $b\in R$; $aR\supseteq L$ and accordingly $KR\supseteq L$. Taking $K=L$ and then $L=R\setminus T$, the second and third results of clause (ii) are obtained. If $S\in DR$, the equation $bx=a$ has a unique solution when $a,b\in S\setminus S[0]$: if $v\in S\setminus S[0]$ were to exist such that $bv\in S[0]$ when $b\in S\setminus S[0]$, $x+v$ would be another solution:

no such v exists: T is void and $S \in \mathbb{D}$.

If, for each $a \in K$ and $c \in L$ a $b(a,c) \in R$ exists such that $ab(a,c) = c$, the same assertion remains valid with regard to systems $K' \subseteq K$, $L' \subseteq L$ and $R' \subseteq R$. The second result of clause (iii) is a simple consequence of the definitions involved, and the remaining results may be treated in the same way as the first.

If $M \in \mathbb{M}(S)$ is divisible in S, the equation $bx = a$ possesses a solution $x = x(a,b) \in S$ for each pair $a, b \in M$. If M is, in addition, saturated in S, $bx \notin M$ when $b \in M$, $x \notin M$, and hence $x(a,b) \in M$: M is divisible in M. Letting a range over M, it follows that $bM \supseteq M$. But M is a multiplicative S-module, so that $bM \subseteq M$; hence $bM \equiv M$ for all $b \in M$ and the result of clause (iv) follows.

For the result of clause (va), it suffices to remark that, since $K|C$, with $a \in K$ fixed there exists a $b(c) \in C$ for which $ab(c) = c$ for all $c \in C$: $aC \supseteq C$. Since $C \in S\mathbb{I}(S)$, $aC \subseteq C$: hence $aC \equiv C$ for all $a \in K$, and $KC \equiv C$. When $S \in Q$, $D|'S\{S\}$ and, with $C \subseteq S$, $D|'C\{S\}$: for each $d \in D$ and $a \in C$, there exists a unique $x \in S$ such that $dx = a$. Let $Q_0 = Q(S, S[0])$, $d, x, a (\in S) \simeq d_0, x_0, a_0 (\in Q_0)$, $D \sim D_0 \equiv D(Q_0, Q_0[0])$,

and $C \vee C_0' \in \mathfrak{Q}_0$ so that $a_0 \in \mathfrak{V}$, x_0 is unique in $\mathfrak{a}_0$, $\mathfrak{a}_0 \in \mathfrak{o}$ and, if $C \in \mathfrak{SI}(S)$, $\mathfrak{C}_0 \in \mathfrak{SI}(\mathfrak{Q}_0)$. Then $d_0 x_0 = a_0$ and $\mathfrak{Q}_0$ contains a number $1/d_0$, so that $x_0 = (1/d_0) a_0$. ~~and $\mathfrak{Q}_0$ contains a number $1/d_0$, so that x_0~~ Since $a_0 \in C_0 \in \mathfrak{SI}(\mathfrak{Q}_0)$, $x_0 \in C_0$ also, and $x \in C$. Thus $D|''C\{S\}$, the result of clause (vb).

$C/''M$ is the system, known to be nonvoid when C is a semi S-ideal, of S-numbers b for which at least one $a \in M$ exists such that $ab = c \in C$. Taking $M \in \mathfrak{M}(S)$ to be divisible in S, and d to be any member of M, there exists an $x \in S$ such that $ax = d$. Thus b satisfies the condition $db = cx \in C$ for all $d \in M$, and accordingly $C/''M \leq C/'M$. But $C/''M \geq C/'M$, and the result of clause (vi) follows.

With regard to the proof of clause (vii) it is first remarked that if I, J are two S-ideals, with $I \leq J$, it is by no means clear that $\mathfrak{Z}(S, I|b) \leq \mathfrak{Z}(S, J|b)$. The members a of $\mathfrak{Z}(S, I|b)$ are characterised by the property that, $\mathfrak{G}$ being the largest system of S-numbers for which $b\mathfrak{G} \leq I$, $a\mathfrak{G} \leq I$ also. It may occur that for such an a, g' exists such that $bg' \in J$ but $ag' \notin J$ and

accordingly $a \notin \mathbb{Z}(S, J \mid b)$. Subject to the restriction $1 \equiv 0 \pmod{J}$ and the further conditions of clause (vi) the result in question may however be proved.

Under the conditions of clause (vii), S possesses a denominator b as described, $Q_0 = Q(S, S[0])$ may be constructed, and $S \subseteq Q_0$. Let $a, b (\in S) \simeq A = [ad/d]_0$, $B = [bd'/d']_0 (\in Q_0)$ where $d, d' \in S$. $a \in \mathbb{Z}(S, S[0] \mid b)$ if and only if $A \in \mathbb{Z}(Q_0, Q_0[0] \mid B)$. Let J_0 be the reduction in Q_0 of J with respect to $S[0]$ in S. If $a \in J$, then $ad \in J$ when $d \in D$, so that $A \in J_0$. Conversely, if $A \in J_0$, so that $ad \in J$ for some $d \in D$, then, since $D \mid J \{J\}$, $a \in J$: in conclusion $a \in J$ if and only if $A \in J_0$. It follows that with a, b related to A, B as described above, $a \in \mathbb{Z}(S, J \mid b)$ if and only if $A \in \mathbb{Z}(Q_0, J_0 \mid B)$: If $a \in \mathbb{Z}(S, S[0] \mid b)$ then $A \in \mathbb{Z}(Q_0, Q_0[0] \mid B)$ and there is an $X \in \mathbb{Z}(Q_0, Q_0[0] \mid B)$ such that $A = XB$; thus for all G for which $BG \in J_0$, $AG = XBG \in J_0$ also: $A \in \mathbb{Z}(Q_0, J_0 \mid B)$ and $a \in \mathbb{Z}(S, J \mid b)$. The required result has been obtained.

Definition .(i a) A semi $\mathbb{S}$-ideal each of whose elements c has a representation $c = bd(c)$, where $d(c) \in \mathbb{S}_0$ and $b \in \mathbb{S}$ is fixed, is a principal semi $\mathbb{S}$-ideal; such a semi $\mathbb{S}$-ideal is denoted by $C(b)$, and $\mathbb{PS}(S)$ is the class of principal semi $\mathbb{S}$-ideals in $\mathbb{S}$. (b) A principal semi $\mathbb{S}$-ideal $C(b)$ for which, in the preceding, $b \in C(b)$ and for each $c \in C(b)$ a unique $d(c) \in C(b)$ such that $c = bd(c)$ exists is an isolated principal semi $\mathbb{S}$-ideal; $\mathbb{PS}_I(S)$ is the class of isolated principal semi $\mathbb{S}$-ideals in $\mathbb{S}$. (c) The unique principal semi $\mathbb{S}$-ideal $C(b) = b\mathbb{S}$ is said to be maximal; $\mathbb{PS}_M(S)$ $\langle \mathbb{PS}_{M,I}(S) \rangle$ is the class of maximal $\langle$maximal isolated$\rangle$ principal semi $\mathbb{S}$-ideals in $\mathbb{S}$.

(ii) Let $S \leq S'$ $(S, S' \in \mathbb{S})$. A semi $\mathbb{S}$-ideal $C(b) \leq S$ for which $C(b) \leq C'(b') \in \mathbb{PS}_I(S')$, where $b = b' (\in S')$ is an indirect principal semi $\mathbb{S}$-ideal; the class of such semi $\mathbb{S}$-ideals in $\mathbb{S}$ is denoted by $\mathbb{PS}'(S)$.

(iii) Mutatis mutandis, the above definitions serve for those of a principal $\mathbb{S}$-ideal $I(b)$, of an isolated principal $\mathbb{S}$-ideal, of the maximality of $I(b)$, and

of an indirect principal ideal. In analogy with the above, the classes of principal $\mathfrak{S}$-ideals considered are denoted by $\mathbb{P}(S)$, $\mathbb{P}_I(S)$, $\mathbb{P}_M(S)$, $\mathbb{P}_{M,I}(S)$ and $\mathbb{P}'(S)$.

It is possible that $C(b) \in \mathbb{PS}(S)$ is not maximal: for example, $C(ab) \in \mathbb{PS}_M(S)$ also has the form $C(b) \in \mathbb{PS}(S)$ but a may be such, e.g. when $b \notin S[a]$, $ab \in S[a]$, that $C(b) \notin \mathbb{PS}_M(S)$. Concerning the above definition of an indirect principal semi $\mathfrak{S}$-ideal it is remarked that, with $C(b) \subset C'' \subset S'$, S' may contain an element a for which $aC'' \leq C''$, so that C'' itself is not a semi $\mathfrak{S}$-ideal in S'; furthermore, although for each $c' \in C'(b')$ a $d'(c') \in S'$ exists such that $c' = d'(c')b'$, it may occur that when $c' \geq c \in C(b)$, no $d'(c')$ with the required properties corresponds to any element $d(c) \in S$, and in such a case $C(b) \notin \mathbb{PS}(S)$. Lastly it is remarked that by definition $C(b) \in \mathbb{PS}_M(S)$ and $I(b) \in \mathbb{P}_M(S)$ are identical.

In the following theorem, the connection between a numerator system $\mathbb{Z}(S, S[a] \mid b)$ and a principal ideal $I(b)$ is established, and it is shown that under suitable conditions these two systems are the same; explicit

representations of $\perp(a) \wedge \perp(b)$ and $\perp(a) \vee \perp(b)$ in terms of principal ideals are derived and, again under suitable conditions, it is shown that the former is $I(a) \cap I(b)$; an explicit representation of $I(ab)/I(b)$ is also given.

Theorem (ia) Let $b \in \mathbb{N}(S \in \mathbb{Q}, S[0])$; then $\mathcal{O}\langle\mathbb{Z}\rangle(S, S[0]|b) \in \mathbb{P}_{M,I}(S)$. (b) Let $b \in \mathbb{N}(S, I \in \mathbb{QF})$; then $\mathcal{O}\langle\mathbb{Z}\rangle(S_I, S_I[0]|b) \in \mathbb{P}'(S_I)$.

(iia) Let $C(b) \in \mathbb{PS}(S)$ and $C' \in \mathbb{SI}(S)$; then $C(b) \leq \mathbb{Z}(S, C'|b)$

b) Let $b \in \mathbb{N}(S \in \mathbb{Q}, S[0])$ and $I(b) \in \mathbb{P}_M(S)$; then $I(b) = \mathbb{Z}(S, S[0]|b) \in \mathbb{P}_{M,I}(S)$

(iiia) Let $I(a), I(b) \in \mathbb{P}(S)$; then $I(a) \wedge I(b) = I(ab) \in \mathbb{P}(S)$.

(iiib) If $I(a), I(b) \in \mathbb{P}'(S)$, then $I(a) \wedge I(b) \in \mathbb{P}'(S)$. (c) If $a, b \in \mathbb{N}(S \in \mathbb{Q}, S[0])$ and $I(a), I(b) \in \mathbb{P}_M(S)$ then $I(a) \wedge I(b) = I(ab) = \mathbb{Z}(S, S[0]|ab) \in \mathbb{P}_{M,I}(S)$. (d) Let $b \in \mathbb{N}(S \in \mathbb{Q}, S[0])$ and $I(a) \in \mathbb{P}(S)$, $I(b) \in \mathbb{P}_M(S)$; then $I(a) \wedge I(b) = I(a) \cap I(b)$

(iva) Let $I(a), I(b) \in \mathbb{P}(S)$ and set $J = I(a) \vee I(b)$. (a) All elements of J have representations of the form $ax+by$ $(x, y \in S)$. (b) Let $S \in \mathbb{Q}$ and $a, b \in \mathbb{N}(S, S[0])$ with $t\langle v\rangle \in \mathbb{O}_c(S, S[0]|a\langle b\rangle)$; then $J \leq I(av+bt+ab) \in \mathbb{P}_I(S)$ and if $I(a), I(b) \in \mathbb{P}_M(S)$, then $J = \mathbb{Z}(S, S[0] | (a +_b t)) \in \mathbb{P}_{M,I}(S)$. (c) Let $D|I(a), I(b) \in \mathbb{P}_M(S)$ and $v \in \mathbb{O}_c(S, S[0]|b)$; then $\{v I(a)\} \vee I(b) = J$. If, in

addition, $S[0] \in \mathbb{I}_{QF}(S)$, $I \equiv \vee I(a)$ is the [illegible]
of the equation $I' \vee I(b) \equiv J$ in the sense that for any S-ideal $I'' \subset \vee I(a)$, $I'' \vee I(b) \subset J$: furthermore, the decomposition $\vee I(a) \vee I(b)$ is a minimal binary decomposition of J in the sense that if $K' \leq \vee I(a)$, $K'' \leq I(b)$ are two S-ideals, with either $K' \subset \vee I(a)$ or $K'' \subset I(b)$, then $K' \vee K'' \subset J$. (d) Let $I(a) \in \mathbb{P}_m(S)$, $D \mid I(a)$ and $v \in \mathbb{U}_c(S, S[0] \mid b)$. Then the decomposition $\vee I(a) \vee bI(a) \equiv I(a)$ is a minimal binary decomposition of $I(a)$ in the sense of the preceding clause. (e) Let $I(a), I(b) \in \mathbb{P}_m(S)$, $D \mid I(a)$, $D \mid I(b)$ and $t\langle v \rangle \in \mathbb{U}_c(S, S[0] \mid a\langle b \rangle)$. The two decompositions $\vee I(a) \vee I(b)$ and $I(a) \vee tI(b)$ are minimal binary decompositions of J in the sense of clause (c), and $\vee I(a) \vee I(ab) \vee tI(b)$, where $I(ab) \in \mathbb{P}_m(S)$, is a minimal ternary decomposition of J in the same sense.

(va) Let $I(a), I(b) \in \mathbb{P}(S)$ and $I(ab) \in \mathbb{P}_m(S)$: then $\mathbb{O}(S, S[0] \mid b) \vee I(a) \leq I(ab) / I(b)$. (b) Let $a, b \in N(S \neq \emptyset, S[0])$, $I(a), I(b) \in \mathbb{P}_m(S)$ and $I(ab) \in \mathbb{P}(S)$: then $\mathbb{O}(S, S[0] \mid b) \vee I(a) \geq$

$I(ab) \equiv / I(b)$. (c) If, in the preceding clause $I(ab) \in \Pi_M(S)$ also, then $I(ab)/I(b) \equiv O(S, S[o] | b) \vee I(a) \equiv$

? $(\exists a+1)(\exists b) Z(S, S[o] | a+1, v, t)$ where $t_{\langle v \rangle} \in \mathfrak{U}_C(S, S[o] | a \langle b \rangle)$

Proof. Under the conditions of clause (ia), let $a, b, \ldots (\in S) \simeq a_0, b_0, \ldots (\in Q_0 = Q(S, S[o])$. In particular, $b_0 \in N(Q_0, Q_0[o])$ and $Z(S, S[o] \mid b) \simeq Z(Q_0, Q_0[o] \mid b_0)$. Each member a_0 of $Z(b_0)$ has a unique representation $a_0 = x_0 b_0$ with $x_0 \in Z(b_0)$. All members of $Z(b)$ have similar representations: $Z(b) \in P_{M,I}(S)$. Since $b \in N(S, S[o])$, a $v \in S$ exists such that $O(b) \equiv O(S, S[o] \mid b) \equiv Z(S, S[o] \mid v)$ $\equiv Z(v)$ (clause ( ) of Theorem). The result just derived may be applied to $Z(v)$ to show that $O(b) \in P_{M,I}(S)$.

For the proof of clause (ib), let $[a]_I, [b]_I, \ldots (\in S_I)$ $\simeq A, B, \ldots (\in Q_I \equiv Q(S, I))$. If $b \in N(S, I)$ then $B \in N(Q_I, Q_I[o])$. Since $Q_I \in Q$, $Z(B) \equiv Z(Q_I, Q_I[o] \mid B) \in P_I(Q_I)$. Also $Z([b]_I) \equiv Z(S_I, S_I[o] \mid [b]_I) \subseteq Z(B)$. Hence $Z([b]_I) \in P'(S_I)$. $O(S_I, S_I[o] \mid [b]_I)$ is treated in the same way.

Under the assumptions of clause (iia), each $c \in C(b)$ has a representation $c = bd(c)$. For each such c, $cg = d(c) b_g \in C'$ whenever $b_g \in C'$; hence $c \in Z(S, C' \mid b)$.

From clause (ia), $\mathbb{Z}(b) = \mathbb{Z}(S, S[0]|b) \in \mathbb{P}_I(S)$ under the conditions of clause (iib): each ~~$\in$~~ $c \in \mathbb{Z}(b)$ has a representation $c = d(b,c)\, b$. All numbers of the form db with $d \in S$ belong to $I(b) \in \mathbb{P}_M(S)$. Thus $I(b) \geq \mathbb{Z}(b)$. But $I(b) \in \mathbb{PS}(S)$ and $S[0] \in \mathbb{SI}(S)$; from clause (iia), $I(b) \in \mathbb{Z}(b)$. Thus $I(b) = \mathbb{Z}(b)$. It has been assumed that $I(b) \in \mathbb{P}_M(S)$ and shown that $\mathbb{Z}(b) \in \mathbb{P}_I(S)$: both are in $\mathbb{P}_{M,I}(S)$.

The members of $I(a), I(b) \in \mathbb{PS}(S)$ have the representations ca, db respectively, with $c, d \in S$. All members of $I(a)I(b)$ have ~~the~~ representations of the form eab with $e \in S$. The additive closure $A = A\{I(a)I(b)\}$ $= I(a) \wedge I(b)$ is that system of S-numbers that (a) contains all members of $I(a)I(b)$, (b) is closed with respect to addition and subtraction and (c) is such that the system obtained by removal of one or more members of A no longer possesses both properties (a) and (b). Let $A = A_1 \cup A_2$, A_1 consisting of numbers of the form eab, A_2 of numbers not of this form. A_1 alone possesses properties (a, b): $A_1 \cup A_2$ is not A unless A_2 is void: the result of clause (iiia) has

been obtained.

If $\underline{I}(a), I(b) \subseteq I'(a'), I'(b') (\in \mathbb{P}(S'))$ where $S \subseteq S'$ and $a, b (\in S) \simeq a', b' (\in S')$, the result just obtained may be applied to $I'(a'), I'(b')$; $\underline{I}(a) \wedge I(b) \subseteq I'(a') \wedge I'(b')$, all elements of $I'(a') \wedge I'(b')$ having representations of the form $e'a'b'$ with $a', b' \in S'$ fixed. The result of clause (iiib) follows. Under the conditions of clause (iiic), $\underline{I}(a), \underline{I}(b)$ are, from clause (iib), $\mathcal{Z}(S, S[0]|a, b)$ respectively. $a \in I(a)$ and $cb \in I(b)$ for all $c \in S$. From clause (iiia), $I(a) \wedge I(b)$ has the form $I(ab)$ and $cab \in I(ab)$ for all $c \in S$; hence $I(ab) \in \mathbb{P}_M(S)$. Furthermore $ab \in N(S, S[0])$ and hence $I(ab) \equiv \mathcal{Z}(S, S[0]|ab) \in \mathbb{P}_{M,I}(S)$, from clauses (ia, iib).

It is possible that, I and J being two S-ideals, $I \wedge J$ and $I \cap J$ should be distinct. For example, the strata of S consisting simply of copies of the least nonnegative residues mod 4, and $\underline{I}$ being composed of the elements 0, 2, $I \cap I$ is I but $I \wedge I = S[0]$. In the special case considered in clause (iiid) however, the above two S-ideals are equivalent. Since $\mathcal{Z}(b) \equiv \mathcal{Z}(S, S[0]|b)$ contains an element 1_b which belongs to the ~~I(b) better since the~~ $I(b)$ since

the latter is maximal, such that $1_b c = c$ for all $c \in \mathcal{Z}(b)$ and hence for all $c \in I(b)$ and, in particular for all $c \in I(b) \cap I(a)$. Such a c also belongs to $I(a)$. Thus each $c \in I(a) \cap I(b)$ may be expressed as the product of an element of $I(a)$, namely c itself, and one of $I(b)$, namely 1_b. Hence $I(a) \cap I(b) \subseteq I(a) \wedge I(b)$. But, from clause ( ) of Theorem , $I(a) \wedge I(b) \subseteq I(a) \cap I(b)$ and the required result follows.

With regard to clause (iva), all elements of $I(a) \cup I(b)$ are of the form xa or yb. It may easily be shown, as in the treatment of $I(a) \wedge I(b)$ in the proof of the preceding clause, that all elements of $\mathcal{A}\{I(a) \cup I(b)\}$ have representations of the form $xa + yb$. These representations may be nonunique. For example, when ab belongs to both $I(a)$ and $I(b)$, and hence to $\bar{J}$, $x \in S[o]$, $y = a$ and $x = b$, $y \in S[o]$ may be taken. Under the conditions of clause (ivb) however, a unique representation may be given. With a, b being denominators as described, tv is also one and, since $S \in \mathbb{Q}$, every S-number d has a unique representation of the form $d_1 tv + d_2(av + bt + ab)$ with $d_1 \in \mathcal{Z}(S, S[o] \mid tv)$, $d_2 \in \mathcal{Z}(\$av+bt+ab) \equiv \mathcal{Z}(S, S[o] \mid av+bt+ab)$

Since $tvd \in S[o]$ for all $d \in J$, $d_1(tv) \in S[o]$ for such d; since $S[o] \in QF$, $d_1 uv \in S[o]$ and each $d \in J$ has a unique representation of the form $d_2(av+bt+ab)$ with $d_2 \in \mathcal{Z}(av+bt+ab)$ and accordingly all such d belong to a principal ideal $I(av+bt+ab)$ which, from clause (iia) is contained by $\mathcal{Z}(av+bt+ab)$. Conversely, taking any $d = d_2(av+bt+ab) \in \mathcal{Z}(av+bt+ab)$ and setting $x = d_2(v+b)$, $y = d_2 u$, d is expressed as $xa+yb$, the sum of two members of $I(a)$ and $I(b)$ respectively when these principal ideals are maximal. Hence $\mathcal{Z}(av+bt+ab) \subseteq J$ and in consequence $J = \mathcal{Z}(av+bt+ab)$. However, with $a, b, t, v (\in S) \to a_o, b_o, t_o, v_o$ $(\in Q_o = Q(S, S[o]))$, $(1/(b_o+v_o)) \in D(Q_o, Q_o[o])$ so that ~~$\mathcal{Z}(a_o v_o + b_o t_o$~~ from $\mathcal{Z}(Q_o, Q_o[o] | a_o v_o + b_o t_o + a_o b_o) \equiv \mathcal{Z}(Q_o, Q_o[o] | a_o + (t_o b_o/(b_o+v_o)))$. Presenting this result in terms of S, the last result of clause (ivb) is obtained. (When $b=a$, $1_b t \in S[o]$ and $J \equiv I(a) \vee I(b) \equiv I(a)$; when $b=t$, $J \equiv I(a) \vee I(b) \equiv I(a+t) \equiv S$, in agreement with clause ( ) of Theorem .)

If $v \in \mathfrak{D}_c(S, S[o] | b)$, either $u \in D$ exists such that $ub+v \in D$, $bv \in S[o]$, or simpler conditions, whose treatment is correspondingly simpler, obtain. Assuming the stated conditions to hold, and noting that $D | I(a)$, it follows

from clause ( ) of "Theorem that $(ub+v)\perp(a) \equiv \perp(a)$: if the members of $I(a)$ are given by expressions of the form xa with x ranging over a system X of S-numbers, they are also given by $x'(ub+v)a$ with x' ranging over X. Each member of $vI(a) \vee I(b)$ may be expressed as $d = xva + yb$ where $x \in X, y \in S$. But $d = xa(ub+v) + (y - xau)b$: d has been expressed as the sum of an element of $I(a)$ and one of $I(b)$, since the latter is maximal. Conversely, each member of $I(a) \vee I(b)$ may be expressed as $xa + yb = x'(ub+v)a + yb$ $= xv'a + (y + x'au)b$, a member of $vI(a) \vee I(b)$. Hence $vI(a) \vee I(b) \equiv J$, the first result of clause (ivc).

The members of $vI(a)$ have representations of the form $c = xva$ where x ranges over a system X of S-numbers; those of an $\mathfrak{S}$-ideal $I'' \subset vI(a)$ have representations of the form $c'' = x''va$, where the x'' form a subsystem X'' of X. Take $c' = x'va$ to be a member of $vI(a) \setminus I''$; it is in $vI(a)$ and hence in $\{vI(a)\} \vee I(b)$. All members of $I'' \vee I(b)$ have the form $c'' + d$, where $c'' \in I''$, $d \in I(b)$, or $x''va + yb$ where $x'' \in X''$,

$y \in S$. Assuming that c' is one of these numbers, so that $c' = c'' + d$, it follows that $x'va = x''va + yb$ and $(x'-x'')^2 v^2 a^2 = (x'-x'')vayb \in S[0]$ or, when $S[0] \in \mathbb{QF}$, that $c' = c''$. But $c' \notin I''$, $c'' \in I''$. From this contradiction it follows that no member of $vI(a) \setminus I''$ is a member of $I'' v I(b)$. Hence $I'' v I(b) \subset J$. (It is remarked that $I' \equiv I(a) v I(b)$ is the maximal solution of the equation $I' v I(b) \equiv I(a) v I(b)$.) Again, taking $J'' \subset I(b)$ to be an ideal whose members are represented by expressions of the form $d'' = y''b$, with y'' belonging to a subsystem Y'' of S, and $d' = y'b$ to be in $I(b) \setminus J''$, the equation $y'b = xva + y''b$ must be satisfied if d' is to be a member of $vI(a) v J''$. Now $(y'-y'')^2 b^2 \in S[0]$, so that $d' = d''$ where $d' \in I(b) \setminus J''$, $d'' \in J''$ and again $vI(a) v I''_c$ J. With K', K'' as prescribed at the end of clause (ivc), either $K' v K'' \subseteq I'' v I(b)$ or $K' v K'' \subseteq vI(a) v J''$ and the desired result follows from those derived above.

If $I(a) \in \mathfrak{P}_m(S)$ and $I(ab) \equiv bI(a)$, $I(ab) \in \mathfrak{P}_m(S)$ also. Replacing the ideal $I(b)$ of clause (ivc) by $I(ab)$, the minimal decomposition $vI(a) v bI(a)$ of $I(a) v bI(a)$ is

obtained. However, $bI(a) \subseteq I(a)$, so that $\nu I(a) \vee bI(a)$ is a minimal decomposition of $I(a)$, the result of clause (ivd).

The first results of clause (ive), concerning the binary decompositions of J, are simple corollaries to clause (ivc); replacing $I(a)$ in the decomposition $I(a) \vee tI(b)$ by its minimal decomposition $\nu I(a) \vee bI(a)$ of clause (ivd), and expressing $bI(a)$ in its alternative maximal form $I(ab)$, the required result concerning ternary decomposition is obtained.

Under the conditions of clause (va), $I(ab)/I(b)$ is the system of S-numbers d such that $cd \in I(ab)$ for all $c \in I(b)$, all members c of $I(b)$ having a representation $x(c)b$. For all $c \in I(b)$ and $d' \in O(b) \equiv O(S, S[0]|b)$, $cd' = x(c)bd' \in S[0] \subseteq I(ab)$. Hence $O(b) \subseteq I(ab)/I(b)$. Also any $d'' \in I(a)$ has a representation $y(d'')a$, so that $cd'' = x(c)y(d'')ab \in I(ab)$, since $I(ab) \in \mathfrak{P}_m(S)$. Any number belonging to the additive closure of the systems of numbers d' and d'' also belongs to $I(ab)/I(b)$ – the result of clause (va). (If $a \in N(S \in \mathfrak{Q}, S[0])$, $O(S, S[0]|a) \in \mathfrak{P}_{M,I}(S)$ has the form $I(a)$; in this case

$\mathfrak{O}(b) \vee \mathfrak{Z}(S, S[0] | a) \leq \perp(ab)/\perp(b)$.) If a and $D \in \mathbb{N}(S \in \mathfrak{Q}, S[0])$, $t, v \in S$ exist such that $at, bv \in S[0]$, $ab \in \mathbb{N}(S, S[0])$ also and, in particular, any $d \in I(ab)/I(b)$ has a representation $d = d_1 ab + d_2(av + bt + tv)$. For this d, $cd \in I(ab)$ for all $c \in I(b)$. Since the members of $I(ab)$ are multiples of ab, $ctd \in S[0]$ for all $c \in I(b)$, in particular for $c = tb$ which belongs to $I(b)$ when $I(b)$ is maximal. Thus $t^2bd = d_2 t^3 b^2 \in S[0]$ and, since $S[0] \in \mathfrak{OF}$, $d_2 tb \in S[0]$: d has a representation $d = (d_1 b + d_2 v)a + (d_2 t)v$, the sum of a member of $I(a)$ when $I(a)$ is maximal and one of $\mathfrak{O}(b)$. Hence $I(ab)/I(b) \leq \mathfrak{O}(b) \vee I(a)$ as stated in clause (vb). Using the result of clause (va), the first result of clause (vc) is obtained, the second following from clause (ivb).

Definition. (i) Two S-rings S' and S'' are compounded to form a stratified system S in the following way: for (j:) $S\{j\}$ is the stratum containing all ordered pairs $a=(a', a'')$ of numbers $a' \in S'\{j\}$, $a'' \in S''\{j\}$. Addition, subtraction and multiplication over S are defined in terms of these operations over S' and S'' taken in isolation; thus with $b=(b', b'') \in S\{k\}$, ab is $(a'b', a'', b'')$ and when $k=j$ $a \pm b = (a' \pm b', a'' \pm b'')$. Finitely many S-rings are compounded in the same way.

(ii) An S-ring S' is directly decomposed in the following way: each number $a' \in S'$ is given a representation $a' = \bigcup_{k=1}^{n} a^{(k)}$ where (a) a' determines the $a^{(k)}$ uniquely and the sequence $\{a^{(k)}\}$ is associated with one a' alone; (b) the system $S^{(k)}$ of numbers $a^{(k)}$ being disposed in strata $S^{(k)}\{j\}$ corresponding to the strata $S'\{j\}$ of S' in which associated a' lie, addition and subtraction within strata $S^{(k)}\{j\}$ are either

(1) defined in terms of these operations over S, so that for $a' \in S'(b')$, $a' \pm b' = \bigcup_{k=1}^{n} (a^{(k)} \pm b^{(k)})$, the correspondence of (a) being preserved or (2) already defined and such that, by means of the formulae just given, the correspondence of (a) is preserved $(k:n)$; (c) multiplication within $S^{(k)}$ is either (1) defined in terms of this operation over S', so that $a'b' = \bigcup_{k=1}^{n} a^{(k)} b^{(k)}$ the correspondence of (a) being preserved or (2) already defined and such that the correspondence of (a) is preserved $(k:n)$. The notation $S' \leq \bigcup_{k=1}^{n} S^{(k)}$ indicates that the S-ring S' may be decomposed in terms of the systems $S^{(k)}$ as just described; an S-ring permitting such a decomposition is said to be directly decomposable into an n-component S-ring

(iii) I' being a subsystem of numbers of $S' \leq \bigcup_{k=1}^{n} S^{(k)}$, the notation $I' \leq \bigcup_{k=1}^{n} I^{(k)}$ $(S' \simeq I')$ indicates that $I^{(k)}$ is the subsystem of members $a^{(k)}$ of $S^{(k)}$ featuring in the representations $a' = \bigcup_{k=1}^{n} a^{(k)}$ of the members a' of $I'(k:n)$.

(iva) With $S' \leq \ddot{\bigcup_{k=1}} S^{(k)}$, $S'' = [\ddot{\bigcup_{k=1}} S^{(k)}]$ is the complete compound of $S^{(k)}(k:n)$ as defined in (i) above, and with $I' \leq \overset{n}{\underset{k=1}{\cup}} I^{(k)}$ $(S' \simeq I')$, $[\overset{n}{\underset{k=1}{\cup}} I^{(k)}]$ is the subsystem of members of S'' obtained by completely compounding the $I^{(k)}$ in the same way. (b) The stratum $S'\{j\} \in S[I']$ for which $S'\{j\} \cap I' \equiv S'\{j\} \cap I''$ is said to be complete with respect to the decomposition of I' in terms of the systems $S^{(k)}$ $(k:n)$. (c) The system of numbers S'' is called the complete extension of the $\mathfrak{S}$-ring S' with respect to the decomposition $\overset{n}{\underset{k=1}{\cup}} S^{(k)}$. (d) An $\mathfrak{S}$-ring S' for which $S' \equiv S''$ is said to be complete with respect to this decomposition.

(v) An $\mathfrak{S}$-ring S that is isomorphic to an $\mathfrak{S}$-ring S' which is directly decomposable into an $\mathfrak{S}$-ring with n components is said to be decomposable into an $\mathfrak{S}$-ring with n components; with the decomposition of S' being complete, S is said to be completely decomposable into an $\mathfrak{S}$-ring with n-components.

Before discussing the processes of composition and decomposition, the natures of the derived number systems are established. It is clear that every $\mathfrak{S}$-ring S permits the trivial decomposition $S \leq \overset{n}{\underset{k=1}{\cup}} S^{(k)}$ in which $S^{(k)} \equiv S$

(k:n) and in the correspondence $a = \bigcup_{k=1} a^{(k)}$, $a^{(k)} = a$ (k:n), combinations involving $a^{(k)}$ deriving from distinct elements of S not arising. If, in this case, each stratum of S contains a finite number of elements, the corresponding stratum of each $S^{(k)}$ contains no fewer. For nontrivial decompositions in which combinations involving $a^{(k)}$ deriving from distinct elements of such S' do arise, strata of one or more of the $S^{(k)}$ contain fewer elements than do the corresponding S'-strata. This is true, for example, of the direct decomposition of an S-ring S' which is itself the complete compound of a number of S-rings. In this case a decomposition $S' = [\bigcup_{k=1}^{n} S^{(k)}]$ is immediately available: the $S^{(k)}$ are simply the S-rings from which S' is compounded. That decompositions intermediate to the trivial and the complete are possible is established by the following example in which the ~~S'-number have~~ elements of S' have the form $a' = (a_1, \ldots, a_4)$, the a_m being integers mod 2, arithmetic operations being performed only upon corresponding

components (so that $(u_1,\dots,u_4)=(v_1,\dots,v_4)$ [illegible]

and so on). All $S\{j\}$ with $j<0$ are null strata; $S\{0\}$ consists of the two numbers $(0,0,0,0)$ and $(1,1,1,1)$; the only nonzero number of $S\{2j+1\}$ is $(0,0,0,1)$ $(j=0;)$; $S\{2\}$ consists of the numbers of $S\{1\}$ together with the six others $(1,1,0,1)$, $(1,0,1,1)$, $(0,1,1,1)$, $(1,1,0,0)$, $(1,0,1,0)$, $(0,1,1,0)$ (these eight numbers form an additive Abelian group); $S\{2j\}$ ~~is complete $(j=2)$.~~ involves all sixteen possible combinations of $a_k=0,1$ $(k=4)$. Clearly a decomposition $S'\subseteq \overset{4}{\underset{k=1}{\cup}} S^{(k)}$ is available, $S^{(k)}$ consisting of the numbers a_k $(k=4)$. The strata $S'\{j\}$ with $j<0$ are complete, being null strata. $S'\{0\}$ is incomplete ($S^{(k)}\{0\}$ contains the numbers 0 and 1 $(k=4)$ but all possible combinations of these numbers do not occur in $S'\{0\}$). $S'\{2j+1\}$ is complete $(j=0)$: $S^{(k)}\{2j+1\}$ contains the number 0 alone $(k=3)$ and $S^{(4)}\{2j+1\}$ contains the numbers 0 and 1, all two possible combinations of these numbers occurring in $S'\{2j+1\}$ $(j=0;)$. $S'\{2\}$ is incomplete and $S'\{2j\}$ is complete

In this case S' and $[\bigcup_{k=1}^{\top} S^{(k)}]$ differ in the strata of orders 0 and 2.

'Theorem (i) 'The compound of two or more S-rings is an S-ring.

(ii) If $S \subseteq \bigcup_{k=1}^{n} S^{(k)}$, $S^{(k)}$ is an S-ring $(k : n)$

(iii) 'The complete component extension of an S-ring is an S-ring.

Prof. Within each stratum of the system S of numbers defined in terms of the members of S' and S'' of clause ( ) of Definition , addition is defined, the sum of any two numbers being present in the relevant stratum of S. Such addition is associative and commutative. A zero element, defined in terms of the appropriate zero elements of S' and S'' is also present in each stratum: the members of each stratum of S form an additive Abelian group. The product of any pair of members of S is present in S; in particular, if $a \in S\{j\}$, $b \in S\{k\}$ then $ab \in S\{j+k\}$. Multiplication within S is associative and commutative. In conclusion $S \in \mathbb{S}$.

Under the condition of clause (ii), take $a' \in S'$ to be represented by $\bigcup_{k=1}^{n} a^{(k)}$ and b', c' to be similarly

represented, and let m be [illegible]

To any pair $a^{(m)}, b^{(m)} \in S^{(m)}\{j\}$ correspond the numbers $a, b \in S\{j\}$ from which they are obtained. The number $c = a + b$ is present in $S\{j\}$, and consequently the number $c^{(m)}$ is present in $S^{(m)}\{j\}$. Addition within $S^{(m)}\{j\}$ is defined by setting $c^{(m)} = a^{(m)} + b^{(m)}$. Such addition is, as a consequence of the properties of addition within $S\{j\}$, associative and commutative. To each pair $c^{(m)}, a^{(m)} \in S^{(m)}\{j\}$ a $b^{(m)}$ satisfying the foregoing relationship exists in $S^{(m)}\{j\}$: the members of $S^{(m)}\{j\}$ constitute an additive Abelian group. Multiplication within $S^{(m)}$ may similarly be defined in terms of multiplication over S and have similar properties: The members of $S^{(m)}$ form an S-ring.

The result of clause (iii) is a simple consequence of its two predecessors.

The simple example, given earlier in this section, of an $\mathfrak{S}$-ring composed of numbers of the form $(a^{(1)},\ldots,a^{(4)})$ the $a^{(k)}$ being residues mod (2), sufficed to draw attention to the existence of $\mathfrak{S}$-rings with incomplete strata; it possesses two such strata. It is a simple matter to construct directly decomposable $\mathfrak{S}$-rings possessing unboundedly many incomplete strata: take $S^{(k)}$ $(k:n)$ to be n isomorphic $\mathfrak{S}$-rings and let $S' \leq \bigcup_{k=1}^{n} S^{(k)}$ where, in the representations $a' = \bigcup_{k=1}^{n} a^{(k)}$ of the numbers of S', only the $a^{(k)}$ related by the correspondence that establishes the isomorphism between the $S^{(k)}$ occur together. S' is incomplete: all possible combinations of the $a^{(k)}$ in corresponding strata do not occur together in the representations of the numbers $a' \in S'$; indeed, S' is isomorphic to each of the $S^{(k)}$. The example just described may be given even more degenerate form by taking each stratum of the $S^{(k)}$ to be a copy of a single commutative ring G_0; all strata of all $S^{(k)}$ are then similar. Construction of decomposable

S-rings that are not degenerate as just described is, however, possible.

Theorem . Directly decomposable S-rings that are not isomorphic to any of their constituent S-rings and possess arbitrarily many dissimilar incomplete strata exist.

Proof. An S-ring possessing the properties described above will be constructed.

Let $p(\nu) \geq 1$ $(\nu: 0; n-1)$ and $q(0) \geq 1$ be integers and set $q(k) = q(0) \prod_{\nu=0}^{k-1} p(\nu)$ $(k: n)$. For $(k: 0; n)$, $\Psi^{(k)}$ is the set of $q(k)$ integers $\psi(k,\nu) = \nu \prod_{\omega=k}^{n-1} p(\omega)$ $(\nu: 0; q(k)-1)$. $\Psi^{(k)} \leq \Psi^{(k+1)}$ $(k: 0; n-1)$. Arithmetic operations upon the integers $\psi(k,\nu)$ are carried out with respect to the modulus $\varphi(n)$. The members of $\Psi^{(k)}$ then form a commutative ring isomorphic to that of the residue classes mod $q(k)$ $(k: 0; n)$. Systems $S\{j\}$ $(j;)$ of sequences a of nonnegative integers $a^{(k)}$ $(k: n)$ expressed in the form $a = \{a^{(k)}\}_{(k:n)}$ are constructed as follows. For $j < 0$, $S\{j\}$ contains the ~~$q(0)$ seq~~ single sequence a for which $a^{(k)} = 0$ $(k: n)$. $S\{0\}$ contains the $q(0)$ sequences a for which $a^{(k)} = \psi(0,\nu)$ $(k: n)$ are equal $(\nu: 0; q(0)-1)$. For j in the range $1 \leq j \leq n$,

$S\{j\}$ contains $r(j)=\prod_{k=1}^{j} q(k)$ sequences $a(\tau)=\{a^{(k)}(\tau)\}_{(k:n)}$ $(\tau:0;r(j)-1)$. For these values of τ, numbers $\nu(k,\tau)$ in the range $0\leq\nu(k,\tau)\leq q(k)-1$ are first derived from the decomposition $\tau=\nu(1,\tau)+\sum_{k=2}^{j} r(k-1)\,\nu(k,\tau)$ and then $a^{(k)}(\tau)=\psi(k,\nu(k,\tau))$ $(k:j)$. If $j<n$, $a^{(j+1)}(\tau)$ is set equal to $\sum_{k=1}^{j} a^{(k)}(\tau)$. If $j<n-1$, the further values $a^{(k)}(\tau)=0$ $(k:j+2;n)$ are assigned. For $j>n$, $S\{j\}$ is simply a copy of $S\{n\}$.

Arithmetic operations upon the sequences a are carried out with respect to components independently. The single member systems $S\{j\}$ with $j<0$ form additive Abelian groups. All components of the sequences a in $S\{0\}$ being equal members of $\Psi^{(0)}$, the members of $S\{0\}$ form an additive Abelian group, that of the residue classes mod $q(0)$. Let j be fixed in the range $1\leq j\leq n$. With $1\leq k\leq j$, the component values $c^{(k)}$ of the $c=\{c^{(k)}\}_{(k:n)}\in S\{j\}$ range through the set $\Psi^{(k)}$. $S\{j\}$ contains all c corresponding to all possible combinations of such component values. In particular, with $a,b\in S\{j\}$, $S\{j\}$ contains c for which $c^{(k)}=a^{(k)}+b^{(k)}$ $(k:j)$. Also, when $j<n$, $a^{(j+1)}+b^{(j+1)}=\sum_{k=1}^{j}\{a^{(k)}+b^{(k)}\}$ and, for the c in question,

$c^{(j+1)} = \sum_{k=1}^{j} c^{(k)}$. Lastly, $a^{(k)} + b^{(k)} = c^{(k)} = 0$ $(k : j+2; n)$ for all $a, b, c \in S\{j\}$ when $j < n-1$. $S\{j\}$ is closed with respect to addition and, as is easily verified, constitutes an additive Abelian group $(j : n)$. This is true of all $S\{j\}$ $(j;)$ since $S\{j\}$ with $j > n$ is a copy of $S\{n\}$.

The components of the product of $a \in S\{m\}$ and $b \in S\{j\}$ are all zero when either $m < 0$ or $j < 0$: the product is to be found in $S\{m+j\}$.

The components $a^{(k)}$ $(k : n)$ of $a \in S\{0\}$ belong to $\Psi^{(0)}$ and are all equal to $a^{(1)}$. When $j < n$, the component $b^{(k)}$ of $b \in S\{j\}$ lies in $\Psi^{(k)}$ $(k : j)$ and $\Psi^{(0)} \leq \Psi^{(k)}$ $(k : j)$. c with $c^{(k)} = a^{(1)} b^{(k)} \in \Psi^{(k)}$ $(k : j)$ exists in $S\{j\}$. $c^{(j+1)} = a^{(1)} \sum_{k=1}^{j} b^{(k)} = a^{(1)} b^{(j+1)}$ for this c. Also, when $j < n-1$, $c^{(k)} = a^{(1)} b^{(k)} = 0$ $(k : j+2; n)$. The product of $a \in S\{0\}$ and $b \in S\{j\}$ is to be found in $S\{j\}$ $(j : n)$; this result holds more generally for $(j;)$.

Since $\Psi^{(k)} \leq \Psi^{(k+1)}$ $(k : m-1)$ and $a^{(m+1)} = \sum_{k=1}^{m} a^{(k)}$, it follows that $a^{(m+1)} \in \Psi^{(m)}$ for all $a \in S\{m\}$ $(m : n-1)$. With $1 \leq m \leq j < n-1$, let $a \in S\{m\}$, $b \in S\{j\}$. $S\{m+j\}$ contains c corresponding to all combinations of $c^{(k)} \in \Psi^{(k)}$ $(k : m+1)$. In particular, $S\{m+j\}$ contains all c with $c^{(k)} = a^{(k)} b^{(k)}$ $(k : m+1)$ and in particular, that c with $c^{(k)} = 0$ $(k : m+2; n)$. The product

of $a \in S\{m\}$ and $b \in S\{j\}$ is to be found in $S\{m+j\}$ when $1 \leq m \leq j < n-1$; this result holds for $(m,j:)$.

Multiplication between members of the $S\{j\}$ is distributive with respect to addition within the $S\{j\}$. The $S\{j\}$ are strata of an $\mathfrak{S}$-ring S. With k fixed in the range $1 \leq k \leq n$, the components $a^{(k)}$ of $a \in S\{j\}$ belong to a stratum $S^{(k)}\{j\}$ of an $\mathfrak{S}$-ring $S^{(k)}$. Evidently $S \leq \bigcup_{k=1}^{n} S^{(k)}$.

When $j \geq n$, $S\{j\}$ contains $r(n)$ numbers, but $S^{(k)}\{j\}$ only $q(k)$ numbers $(k:n)$: when either $q(0) \geq 1$ or one of the $p(\omega) > 1$, S is not isomorphic to any of the $S^{(k)}$.

Numbers a corresponding to all combinations of the components $a^{(k)} \in S^{(k)}\{j\}$ $(k:n)$ occur in $S\{j\}$ when $j < 0$ ($S^{(k)}\{j\}$ then contains only one number $(k:n)$) and also when $j \geq n$: the strata $S\{j\}$ with $j < 0$ and $j \geq n$ are complete. Only combinations of components $a^{(k)}$ for which all $a^{(k)}$ are equal occur in the members of the stratum $S\{0\}$, which is, when $q(0) > 1$, incomplete. If $q(1) > 1$, the strata $S\{j\}$ for $(j:n-1)$ are also incomplete. $S\{j\}$ contains sequences a corresponding to

all combinations of $a^{(k)} \in \Psi^{(k)}$ for $(k:j)$. The components $a^{(j+1)}$ range over all numbers in $\Psi^{(j)}$, but do not do so independently of their predecessors; they are fixed by a condition of the form $a^{(j+1)} = \sum_{k=1}^{j} a^{(k)}$. ($S\{j\}$ contains $r(j)$ numbers; in $S' = [\bigcup_{k=1}^{n} S^{(k)}]$, $S'\{j\}$ contains $q(j)r(j)$ numbers $(j:n-1)$.) When $p(0)>1$, the strata $S\{j\}$ $(j:0;n)$ contain unequal numbers of sequences: the incomplete strata $S\{j\}$ $(j:0;n-1)$ are dissimilar. In the above analysis, n may be taken arbitrarily large; the theorem has been proved.

A simple illustration of the above theory is provided by the example in which $n=3$, $q(\omega)=2$ $(\omega:0;3)$. In particular $S\{0\}$ contains $\{0,0,0\}, \{1,1,1\}$; $S\{1\}$ contains $\{0,0,0\}, \{1,1,0\}$; $S\{2\}$ contains $\{0,0,0\}, \{1,0,1\}, \{0,1,1\}, \{1,1,0\}$. Again with $n=3$ and $q(0)=q(1)=2, q(2)=q(3)=4$, $S\{0\}$ contains $\{0,0,0\}, \{2,2,2\}$; $S\{1\}$ contains $\{0,0,0\}, \{2,2,0\}$; $S\{2\}$ contains $\{0,0,0\}, \{2,0,2\}, \{0,1,1\}, \{2,1,3\}, \{0,2,2\}, \{2,2,0\}, \{0,3,3\}, \{2,3,1\}$.

the $\mathcal{S}$-ring S being indirectly decomposed [illegible]

$S \sim S' \simeq \bigcup_{k=1}^{n} S^{(k)}$, and the decomposition of $I \subseteq S$ being subordinate to that of S, so that $I \sim I' \subseteq \bigcup_{k=1}^{n} I^{(k)}$ where $I' \subseteq S'$ and $I^{(k)} \subseteq S^{(k)}$ $(k:n)$, properties of I with respect to S are, subject to appropriate conditions, also possessed by $I^{(k)}$ with respect to $S^{(k)}$ $(k:n)$. Conversely, if $I^{(k)}$ possesses certain properties with respect to $S^{(k)}$ $(k:n)$, it may be demonstrated that again subject to suitable conditions, I possesses them with respect to S. Results of this nature are the subject of the following theorem.

Theorem. In the following, $S \in \mathcal{S}$ is decomposed in the form $S \sim S' \subseteq \bigcup_{k=1}^{n} S^{(k)}$. Where relevant, the decompositions of $D, I, J, M \subseteq S$ are subordinate to that of S, so that $D \sim D' \subseteq \bigcup_{k=1}^{n} D^{(k)}$ where $D' \subseteq S'$ and $D^{(k)} \subseteq S^{(k)}$ $(k:n)$; $I', I^{(k)}$ $(k:n)$, $J', J^{(k)}$ $(k:n)$, M' and $M^{(k)}$ $(k:n)$ are related to I, J and M respectively in a similar way. The notation $\delta(S) \in \Delta$ indicates that the decomposition

of S is complete, so that $U \equiv [\bigcup_{k=1}^{n} D]$; the notations $\delta(D) \in \Delta$, $\delta(I) \in \Delta$, $\delta(J) \in \Delta$ and $\delta(M) \in \Delta$ have similar meanings. Again where relevant, $a \in S$ is taken to correspond to $a' = \{a^{(k)}\}_{(k:n)}$; $b \in S$ is related to $b^{(k)}$ $(k:n)$ in the same way.

A) Let $\delta(S), \delta(I) \in \Delta$. $D(S,I)$ is nonvoid if and only if (a) for some m in the range $1 \leq m \leq n$, m distinct integers $k(\tau)$ in the range $1 \leq k(\tau) \leq n$ exist such that $D(S^{(k(\tau)}, I^{(k(\tau))})$ is nonvoid $(\tau:m)$ while $S^{(k)} \equiv I^{(k)}$ $(k:n\ (k \neq k(\tau)\ (\tau:m)))$; if these conditions are satisfied, $D(S,I) \sim E' \equiv [\bigcup_{k=1}^{n} E^{(k)}]$ where $E^{(\rho)} \equiv D(S^{(\rho)}, I^{(\rho)})$ $(\rho = k(\tau)\ (\tau:m))$ and $E^{(k)} \equiv I^{(k)}$ $(k:n\ (k \neq k(\tau)\ (\tau:m)))$ and $\delta(D(S,I)) \in \delta(S)$.

Bi) If $M \in \mathbb{M}(S)$ then $M^{(k)} \in \mathbb{M}(S^{(k)})$ $(k:n)$ and if also $\delta(S), \delta(M) \in \Delta$ and $M \in \mathbb{M}_\delta(S)$ then $M^{(k)} \in \mathbb{M}_\delta(S^{(k)})$ $(k:n)$

(ii) If $\delta(M) \in \Delta$ and $M^{(k)} \in \mathbb{M}(S^{(k)})$ $(k:n)$ then $M \in \mathbb{M}(S)$ and if also $M^{(k)} \in \mathbb{M}_\delta(S^{(k)})$ $(k:n)$ then $M \in \mathbb{M}_\delta(S)$.

(Ci) Let $I \cap J$ be nonvoid and $I \cap J \sim K' \leq \bigcup_{k=1}^{n} K^{(k)}$ where $\delta(I \cap J) \in \delta(S)$. (a) $K^{(k)} \leq I^{(k)} \cap J^{(k)}$ $(k:n)$. (b) If $\delta(I), \delta(J) \in \Delta$, $K^{(k)} = I^{(k)} \cap J^{(k)}$ $(k:n)$ and $I \cap J \sim [\bigcup_{k=1}^{n} K^{(k)}]$.

(ii) Let $I \cup J \langle IJ \rangle \sim K' \leq \bigcup_{k=1}^{n} K^{(k)}$ where $\delta(I \cup J) \langle \delta(IJ) \rangle \in \delta(S)$. Then $K^{(k)} \equiv I^{(k)} \cup J^{(k)} \langle I^{(k)} J^{(k)} \rangle$ $(k:n)$.

I. (iii) Let $I/''M\langle I/'M\rangle \sim K \subseteq \bigcup_{k=1} K$ be nonvoid and $\delta(I/''M)\langle\delta(I/'M)\rangle$ etc. $\in \delta(S)$. (a) $K^{(k)} \subseteq I^{(k)}/''M^{(k)}\langle I^{(k)}/'M^{(k)}\rangle$ $(k:n)$. (b) If $\delta(S), \delta(I) \in \Delta$ and $S[0] \subseteq I$ then $K^{(k)} \equiv I^{(k)}/''M^{(k)}\langle I^{(k)}/'M^{(k)}\rangle$ $(k:n)$.

(c) If $\delta(S), \delta(I), \delta(M) \in \Delta$ then the preceding result (b) holds; also $I/''M\langle I/'M\rangle \sim \left[\bigcup_{k=1}^{n} I^{(k)}/''M^{(k)}\right]\left\langle\left[\bigcup_{k=1}^{n} I^{(k)}/'M^{(k)}\right]\right\rangle$.

(Di) If $I \in \mathbb{AP}(S)$ then $I^{(k)} \in \mathbb{AP}(S^{(k)})$ $(k:n)$.

(ii) If $\delta(I) \in \Delta$ and $I^{(k)} \in \mathbb{AP}(S^{(k)})$ $(k:n)$ then $I \in \mathbb{AP}(S)$.

(Ei) Let $\mathcal{A}(\mathcal{J}) \sim K' \subseteq \bigcup_{k=1}^{n}$ with $\delta(\mathcal{J}) \in \delta(S)$. Then $K^{(k)} \equiv \mathcal{A}(\mathcal{J}^{(k)})$ $(k:n)$.

(ii) Let $\mathcal{J}$ contain the zero element in each stratum containing in wh. it cont. elements of $\mathcal{J}$ and $\delta(\mathcal{J}) \in \Delta$. Then $\mathcal{A}(\mathcal{J}) \sim K' \equiv \left[\bigcup_{k=1}^{n} \mathcal{A}(\mathcal{J}^{(k)})\right]$.

(Fi) If $I \in \mathbb{SI}(S)\langle\mathbb{I}(S)\rangle$ then $I^{(k)} \in \mathbb{SI}(S^{(k)})\langle\mathbb{I}(S^{(k)})\rangle$ $(k:n)$.

(ii) If $\delta(I) \in \Delta$ and $I^{(k)} \in \mathbb{SI}(S^{(k)})\langle\mathbb{I}(S^{(k)})\rangle$ $(k:n)$, then $I \in \mathbb{SI}(S)\langle\mathbb{I}(S)\rangle$.

G) Let $I \in \mathbb{I}(S)$.

(i) Let $\mathcal{O}(S, I|b) \sim K' \subseteq \bigcup_{k=1}^{n} K^{(k)}$ where $\delta(\mathcal{O}(S, I|b)) \in \delta(S)$.

(a) $K^{(k)} \subseteq \mathcal{O}(S^{(k)}, I^{(k)}|b^{(k)})$ $(k:n)$. (b) If $\delta(S), \delta(I) \in \Delta$ then $K^{(k)} \equiv \mathcal{O}(S^{(k)}, I^{(k)}|b^{(k)})$ $(k:n)$ and $\mathcal{O}(S, I|b) \sim \left[\bigcup_{k=1}^{n} K^{(k)}\right]$

(ii) Let $\delta(S), \delta(I) \in \Delta$ and $\mathbb{Z}(S, I|b) \sim K' \subseteq \bigcup_{k=1}^{n} K^{(k)}$ where $\delta(\mathbb{Z}(S, I|b)) \in \delta(S)$. Then $K^{(k)} \equiv \mathbb{Z}(S^{(k)}, I^{(k)}|b^{(k)})$ $(k:n)$ and $\mathbb{Z}(S, I|b) \sim \left[\bigcup_{k=1}^{n} K^{(k)}\right]$.

H) Let $\delta(I)\in\Delta$. $I\in\mathbb{I}_{QF}(S)$ if and only if $I\in\mathbb{I}_{QF}(S)(k:n)$

I) Let $\delta(S),\delta(I)\in\Delta$. $b\in\mathbb{N}(S,I)\langle b|a \bmod I\rangle$ if and only if conditions (a) of clause (A) hold and $b^{(\nu)}\in\mathbb{N}(S^{(\nu)},I^{(\nu)})\langle b^{(\nu)}|a^{(\nu)} \bmod I^{(\nu)}\rangle$ $(\nu=k(\tau)\,(\tau:m))$.

J) Let $\delta(S),\delta(I)\in\Delta$. $Q_I=\mathbb{Q}(S,I)$ exists if and only if for some m in the range $1\le m\le n$, m distinct integers $k(\tau)$ in the range $1\le k(\tau)\le n$ exist such that $Q^{(\tau)}=\mathbb{Q}(S^{(\nu)},I^{(\nu)})$ $(\nu=k(\tau)\,(\tau:m))$ exist, while $S^{(k)}=I^{(k)}$ $(k:n\,(k\neq k(\tau)\,(\tau:m)))$. If this condition holds, $Q_I\sim\left[\bigcup_{\tau=1}^{m}Q^{(\tau)}\right]$, the relationship between the members $\{(a/b)_I\}$ of Q_I and those $\{(a^{(\nu)}/b^{(\nu)})_{I^{(\nu)}}\,(\nu=k(\tau))\}$ of $Q^{(\tau)}$ establishing this isomorphism being that for each $b\in\mathbb{N}(S,I)$ and each $a\in\mathbb{Z}(S,I|b)$,

$$(a/b)_I\simeq\{(a^{(k(\tau))}/b^{(k(\tau))})_{I^{(k(\tau))}}\}_{(\tau:m)}.$$

K) Let $\delta(\mathfrak{J})\in\Delta$.

(i) If $\delta(S)\in\Delta$, $S[0]\le\mathfrak{J}$ and $D\in\mathbb{G}(S,\mathfrak{J})$ then $D^{(k)}\in\mathbb{G}(S^{(k)},\mathfrak{J}^{(k)})\,(k:n)$.

(ii) If $D^{(k)}\in\mathbb{G}(S^{(k)},\mathfrak{J}^{(k)})\,(k:n)$, then $D\in\mathbb{G}(S,\mathfrak{J})$.

(i) If $I \in \mathbb{P}(S) \langle \mathbb{P}_I(S) \rangle$ has the form $I(b)$, $I' \in \mathbb{P}(S_k)$ $\langle \mathbb{P}_I(S^{(k)}) \rangle$ ~~and has (k:~~ and has the form $I(b^{(k)})$ $(k:n)$.

(ii) If $\delta(S), \delta(I) \in \Delta \langle \delta(I) \in \Delta \rangle$ and $I^{(k)} \in \mathbb{P}(S^{(k)})$ $\langle \mathbb{P}_I(S^{(k)}) \rangle$ has the form $I(b^{(k)})$ $(k:n)$, then $I \in \mathbb{P}(S)$ $\langle \mathbb{P}_I(S) \rangle$ has the form $I(b)$.

Proof. The conventions stated at the commencement of the theorem are retained in the proof and, in particular, the components of the numbers $c', d', u', v' \in \mathbb{S}'$ to which $c, d, u, v \in S$ correspond are indicated by use of the convention relating to a, b. Also h is tacitly taken to be a fixed integer in the range $1 \leq h \leq n$. Furthermore, where relevant in the following, the notation $a \simeq a^{(h)}$ indicates that $a(\in S) \simeq a'(\in S')$ where $a' = \{a^{(k)}\}_{(k:n)}$ with $a^{(k)} \in S^{(k)}[0]$ $(k:n(k \neq h))$; the further notations $c \simeq c^{(h)}, \ldots$ have similar meanings.

To prove part (i), first suppose that the stated conditions concerning $S^{(k)}$ and $I^{(k)}$ are satisfied.

Since $\delta(S) \in \Delta$, S contains a number $a \simeq a' = \{a^{(\nu)}\}_{(k:n)}$ ($\in S'$) with $a^{(\nu)} \in D^{(\tau)} = D(S^{(\nu)}, I^{(\nu)})$ ($\nu = k(\tau)\,(\tau:m)$) and $a^{(k)} \in I^{(k)}$ ($k:n\ (k \neq k(\tau)\,(\tau:m))$). If b is such that $ab \in I$, then $a^{(k)} b^{(k)} \in I^{(k)}$ ($k:n$). In particular, $b^{(\nu)} \in I^{(\nu)}$ ($\nu = k(\tau)\,(\tau:m)$); also $b^{(k)} \in I^{(k)}$ for the remaining values of k, since $S^{(k)} \equiv I^{(k)}$ for these values. Since $\delta(I) \in \Delta$, $b \in I$: the only $b \in S$ for which $ab \in I$ are members of I: the existence of one member a of $D(S, I)$ has been established: $D(S, I)$ is nonvoid.

To show that the existence of $D(S, I)$ implies the stated conditions, it is first remarked that if $S^{(k)} \equiv I^{(k)}$ ($k:n$), all members of S belong to I and $D(S, I)$ is void. Hence ~~$m = 0$. Now suppose that for some μ in the range~~ for $m > 0$ of the $\{S^{(k)}\}$, $S^{(k(\tau))} \not\equiv I^{(k(\tau))}$ ($\tau:m$) and $S^{(k)} \equiv I^{(k)}$ ($k:n\ (k \neq k(\tau)\,(\tau:m))$). Now suppose that for a fixed τ in the range $1 \leq \tau \leq m$, $D^{(\tau)}$ is void, so that, since $S^{(k(\tau))} \not\equiv I^{(k(\tau))}$, $T^{(\tau)} \equiv T(S^{(k(\tau))}, I^{(k(\tau))})$ is nonvoid. Suppose that S contains a such that $ab \in I$ only when $b \in I$. Since $\delta(S) \in \Delta$, S contains a

number $b \simeq b^{\cdots} \in T^{\cdots}$, and this $b \notin I$. Clearly $a^{(k(z))} \notin I^{(k(z))}$, since otherwise $ab \in I$ with $b \notin I$, b being as just described. If $a^{(k(z))} \in T^{(z)}$, there is a $b^{(k(z))} \in T^{(z)}$ for which $a^{(k(z))} b^{(k(z))} \in I^{(k(z))}$ and hence a ~~$b \in I$~~ $b \simeq b^{(k(z))}$ for which $a^{(k)} b^{(k)} \in I^{(k)}$ $(k:n)$ or, since $\delta(I) \in \Delta$, a $b \in S \setminus I$ for which $ab \in I$. Accordingly ~~no such~~ S contains no ~~such~~ a such that $ab \in I$ only when $b \in I$: $D(S, I)$ is void if for ~~some~~ any z in the range $1 \leq z \leq m$, $D^{(z)}$ is void.

Since $\delta(S) \in \Delta$, S contains $E \sim E'$ where E' is the complete compound of the subsets $E^{(k)}$ of $S^{(k)}$ $(k:n)$ defined in clause (A). For any $a \in E$, $ab \in I$ only when $a^{(\lambda)} b^{(\lambda)} \in I^{(\lambda)}$ $(\lambda = k(z)(z:m))$ a condition which, since $a^{(k(z))} \in D^{(z)}$ $(z:m)$ confines $b^{(\lambda)}$ to $I^{(\lambda)}$ $(\lambda = k(z)(z:m))$ and hence b to I, and accordingly specifies a as a member of $D(S, I)$. Hence $E \subseteq D(S, I)$. Conversely, for any $a \in D(S, I)$, $ab \in I$ only when $b \in I$, so that since $\delta(I) \in \Delta$, $a^{(\lambda)} b^{(\lambda)} \in I^{(\lambda)}$ only when $b^{(\lambda)} \in I^{(\lambda)}$ $(\lambda = k(z)(z:m)))$. $a^{(k(z))}$ is specified as a member of $D^{(z)}$ $(z:m)$

and since $\delta(E) \in \Delta$, $a \in E$. Hence $\mathfrak{D}(S,I) \subseteq E$ and $\mathfrak{D}(S,I) = E$ as required.

To prove clause (Bia) let $a^{(h)}, b^{(h)}$ be two members of $M^{(h)}$. M' contains two numbers a, b with h^{th} components $a^{(h)}, b^{(h)}$ respectively. Since $M \in \mathfrak{M}(S)$, $ab \in M$ also and in consequence $a^{(h)} b^{(h)} \in M^{(h)}$. Hence $M^{(k)} \in \mathfrak{M}(S^{(k)})$ $(k:n)$. If $S^{(k)} = M^{(k)}$ $(k:n)$, $M^{(k)}$ is by definition saturated in $S^{(k)}$ $(k:n)$ and the second result of clause (Bia) follows without further restriction upon M and S. Discounting this case, select an h for which $S^{(h)} \setminus M^{(h)}$ is nonvoid, and $a^{(h)} \in S^{(h)} \setminus M^{(h)}$. Since $\delta(S) \in \Delta$, S contains a number $a \simeq a' \in S'$ for which $a^{(k)} \in M^{(k)}$ $(k:n\,(k \neq h))$ and with h^{th} component $a^{(h)}$: this number is in $S \setminus M$ and there exists a $b \in M$ such that $ab \notin M$. But $a^{(k)} b^{(k)} \in M^{(k)}$ $(k:n\ (k \neq h))$. Since $\delta(M) \in \Delta$ and $ab \notin M$, $a^{(h)} b^{(h)} \notin M^{(h)}$: $a\, b^{(h)} \in M^{(h)}$ exists such that $a^{(h)} b^{(h)} \notin M^{(h)}$: for all k such that $S^{(k)} \setminus M^{(k)}$ is nonvoid, $M^{(k)}$ is saturated.

Under the conditions of clause (Bii), $a^{(k)}, b^{(k)} \in M$ for any pair $a, b \in M$. Then $ab \simeq c' \in S$ with components $c^{(k)} = a^{(k)} b^{(k)} \in M^{(k)}$ $(k:n)$. Since $\delta(M) \in \Delta$, $c \in M$ and $M \in \mathfrak{M}(S)$. Furthermore, if $a \notin M$ there is an h for which $a^{(h)} \in S^{(h)} \setminus M^{(h)}$. If $M^{(h)}$ is saturated in $S^{(h)}$, $M^{(h)}$ contains a $b^{(h)}$ for which $a^{(h)} b^{(h)} \notin M^{(h)}$.

For any $b \in M$ with h^{th} component $b^{(h)}$, $ab \notin M$ and M is accordingly saturated in S.

For the proof of clause (Cia) it suffices to remark that if for some h, $a^{(h)} \in K^{(h)}$, there exists an $a \in I \cap J$ with h^{th} component $a^{(h)}$. Since $a \in I$, $a^{(h)} \in I^{(h)}$; similarly $a^{(h)} \in J^{(h)}$. Hence $a^{(h)} \in I^{(h)} \cap J^{(h)}$ for all such $a^{(h)}$ and $K^{(h)} \leq I^{(h)} \cap J^{(h)}$; incidentally, $I^{(h)} \cap J^{(h)}$ is nonvoid. With regard to clause (Cib), suppose that $a^{(k)} \in I^{(k)} \cap J^{(k)}$ $(k:n)$ so that $a^{(k)} \in I^{(k)}$ $(k:n)$. Since $\delta(I) \in \Delta$, I' contains $\{a^{(k)}\}_{(k:n)}$; J' does the same. Hence $a' \in I' \cap J'$, $a^{(k)} \in K^{(k)}$ $(k:n)$, $I^{(k)} \cap J^{(k)} \leq K^{(k)}$ $(k:n)$ and, using the preceding result, $I^{(k)} \cap J^{(k)} = K^{(k)}$ $(k:n)$. The preceding argument relates to all combinations of $a^{(k)} \in I^{(k)} \cap J^{(k)} = K^{(k)}$ and establishes that each member of $[\bigcup_{k=1}^{n} K^{(k)}]$ corresponds to a member a of $I \cap J$.

Turning to the unbracketed result of clause (Cii), let $a^{(h)} \in K^{(h)}$ for some h. $I' \cup J'$ contains an a with h^{th} component $a^{(h)}$. Either $a \in I$, so that $a^{(h)} \in I^{(h)} \leq I^{(h)} \cup J^{(h)}$ or $a \in J$ and the same conclusion follows. Hence $K^{(h)} \leq I^{(h)} \cup J^{(h)}$. Conversely, if $a^{(h)} \in I^{(h)} \cup J^{(h)}$, either $a^{(h)} \in I^{(h)}$, $I' \cup J' \supseteq I'$ contains

a number with h component a and $a \in K$, or $a \in J$ and the same conclusion holds: $I^{(k)} \cup J^{(k)} \subseteq K^{(k)}$ and $I^{(k)} \cup J^{(k)} \equiv K^{(k)}$ $(k:n)$.

The bracketed result of clause (Cii) is treated in the same way. If $c^{(h)} \in K^{(h)}$, $a' \in I'$ and $b' \in J'$ exist such that $c^{(h)} = a^{(h)} b^{(h)}$. But $a^{(h)} \in I^{(h)}$, $b^{(h)} \in J^{(h)}$ so that $c^{(h)} \in I^{(h)} J^{(h)}$ and $K^{(k)} \subseteq I^{(k)} J^{(k)}$ $(k:n)$. Similarly, if $c^{(h)} \in I^{(h)} J^{(h)}$, $a^{(h)} \in I^{(h)}$ and $b^{(h)} \in J^{(h)}$ exist such that $c^{(h)} = a^{(h)} b^{(h)}$. I', J' contain a', b' with h^{th} components $a^{(h)}$, $b^{(h)}$ respectively. $I'J'$ contains a number $a'b'$ with h^{th} comp... $c^{(h)} \in K^{(h)}$: $I^{(h)} J^{(h)} \subseteq K^{(h)}$ and in consequence $K^{(h)} \equiv I^{(h)} J^{(h)}$.

For the proof of the unbracketed result of clause (Ciia), the existence of $b^{(h)} \in K^{(h)}$ for some h implies that $I'/''M'$ contains b' with h^{th} component $b^{(h)}$. $a' \in M'$ and $c' \in I'$ exist such that $a'b' = c'$. Thus $a^{(h)} \in M^{(h)}$ and $c^{(h)} \in I^{(h)}$ exist such that $a^{(h)} b^{(h)} = c^{(h)}$ and accordingly $b^{(h)} \in I^{(h)}/''M^{(h)}$. Hence $K^{(k)} \subseteq I^{(k)}/''M^{(k)}$ $(k:n)$. Equally, with regard to the bracketed result, the existence of $b^{(h)} \in K^{(h)}$ implies that of b' such that $a'b' = c'(a') \in I'$ for all $a' \in M'$; the above argument may be repeated.

Under the assumptions of clause (Ciiib) suppose that $b^{(h)} \in I^{(h)}/''M^{(h)}$ for some h, so that $a^{(h)} \in M^{(h)}$ and $c^{(h)} \in I^{(h)}$ exist such that $a^{(h)} b^{(h)} \in c^{(h)}$. M' contains a number a' with ~~the~~ h^{th} component $a^{(h)}$. Since $\delta(S) \in \Delta$, S contains $b \simeq b^{(h)}$ and since $\delta(I) \in \Delta$ and $S[o] \subseteq I$, I contains $c \simeq c^{(h)}$. Hence $a \in M$, $b \in S$ and $c \in I$ exist such that $ab = c$; b is accordingly identified as a member of I^M, and $b^{(h)} \in K^{(h)}$. Thus $I^{(k)}/''M^{(k)} \subseteq K^{(k)}$ and $I^{(k)}/''M^{(k)} \equiv K^{(k)}$ $(k:m)$. The same b can be taken when $a^{(h)} b^{(h)} = c^{(h)}(a^{(h)})$ for all $a^{(h)} \in M^{(h)}$; b is now shown to be a member of $I/'M$.

Assuming the ~~existence~~ conditions of clause (Ciiic) to hold, select $b^{(k)} \in I^{(k)}/''M^{(k)}$ $(k:n)$, so that $a^{(k)} \in M^{(k)}$ and $c^{(k)} \in I^{(k)}$ exist such that $a^{(k)} b^{(k)} = c^{(k)}$ $(k:n)$. Since $\delta(M), \delta(I), \delta(S) \in \Delta$, $a \in M$ and $c \in I$, and S contains b such that $ab = c$, b is accordingly identified ~~as a~~ member of I^M, so that $b^{(k)} \in K^{(k)}$ $(k:n)$ and $I^{(k)}/''M^{(k)} \subseteq K^{(k)}$ $(k:n)$; hence $I^{(k)}/''M^{(k)} \equiv K^{(k)}$ $(k:n)$. Again a similar result holds when $b^{(k)}$ is such that numbers

$c^{(k)}(a^{(k)}) \in I^{(k)}$ exist for which $a^{(k)} b^{(k)} = c^{(k)}(a^{(k)})$ for all $a^{(k)} \in M^{(k)}$ $(k:n)$. The foregoing arguments apply to all combinations of numbers $b^{(k)}$ in $I^{(k)}/''M^{(k)}$ and $\tilde{I}^{(k)}/'M^{(k)}$ respectively: the last result of clause (Ciii) follows.

To prove clause (Di), let $a^{(h)}, b^{(h)}$ be two members of $I^{(h)}$. I' contains two numbers a', b' with h^{th} components $a^{(h)}, b^{(h)}$ respectively. Since $I \in AP(S)$, $a' \pm b' \in I'$ and accordingly $a^{(h)} \pm b^{(h)} \in I^{(h)}$. Hence $I^{(k)} \in AP(S^{(k)})$ $(k:n)$. Conversely, select $a, b \in I$. Since $I^{(k)} \in AP(S^{(k)})$ $(k:n)$, $a^{(k)} \pm b^{(k)} \in I^{(k)}$ $(k:n)$ also, and if $\delta(I) \in \Delta$, $a \pm b \in I$, or $I \in AP(S)$, the result of clause (Dii).

To prove clause (Ei) it is first remarked that, since $J \subseteq A(J)$, $J^{(k)} \leq K^{(k)}$ $(k:n)$. Also $A(J) \in AP(S)$ so that from clause (Di), $K^{(k)} \in AP(S^{(k)})$ $(k:n)$. But $A(J^{(k)})$ is the minimal additively perfect system in $S^{(k)}$ containing $J^{(k)}$ $(k:n)$. Hence $A(J^{(k)}) \leq K^{(k)}$ $(k:n)$. Let A be the subsystem of numbers in $A(J)$ with h^{th} component belonging to $A(J^{(h)})$, so

that, if $a \in \mathcal{A}(J) \setminus A$, $a^{(n)} \notin \mathcal{A}(I^{(n)})$ and if $a^{(h)} \in \mathcal{A}(I^{(h)})$, $a \in A$. If $A = \mathcal{A}(J)$, no $a \in \mathcal{A}(J)$ has an hth component not in $\mathcal{A}(J^{(h)})$. Hence $K^{(h)} = \mathcal{A}(J^{(h)})$. Suppose that $A \subset \mathcal{A}(J)$. Since $J^{(h)} \subseteq \mathcal{A}(J^{(h)})$, $J \subseteq A$. However, $A \notin AP(S)$, since $\mathcal{A}(J)$ is the minimal system in $AP(S)$ containing J, and $\mathcal{A}(J) \setminus A$ is nonvoid. Thus $a, b \in A$ exist such that $a+b = c \notin A$ (or $a-b = c \notin A$, a case treated similarly). Thus $a^{(h)} + b^{(h)} = c^{(h)} \notin \mathcal{A}(J^{(h)})$. But $a^{(h)}, b^{(h)} \in \mathcal{A}(J^{(h)})$ and $\mathcal{A}(J^{(h)}) \in AP(S^{(h)})$, so that $a^{(h)} + b^{(h)} \in \mathcal{A}(J^{(h)})$. From this contradiction it follows that the supposition that $A \subset \mathcal{A}(J)$ is untenable. Thus $K^{(k)} = \mathcal{A}(J^{(k)})$ $(k:n)$.

Under the conditions of clause (Eii), let $I = [\bigcup_{k=1}^{n} \mathcal{A}(J^{(k)})]$. Since $J^{(k)} \subseteq \mathcal{A}(J^{(k)})$ $(k:n)$ and $\delta(I) \in \Delta$, $J \subseteq I$. Also, since $\delta(I) \in \Delta$ and $\mathcal{A}(J^{(k)}) \in AP(S^{(k)})$ $(k:n)$, $I \in AP(S)$ from clause (Dii). Hence $\mathcal{A}(J) \subseteq I$, since $\mathcal{A}(J)$ is the minimal additively perfect system in S containing J. Since J contains the zero element in every stratum in which elements of J are present and $\delta(J) \in \Delta$,

$\mathfrak{J}$ contains the system $\mathfrak{J}_h$ of all numbers $a \simeq a^{(h)}$ with $a^{(h)} \in \mathfrak{J}^{(h)}$. Since $\mathfrak{J}^{(h)} \subseteq \mathfrak{J}$, $\mathcal{A}(\mathfrak{J})$ contains $\mathfrak{I}_h \equiv \mathcal{A}(\mathfrak{J}_h)$, the system of all numbers $b \simeq b^{(h)} \in \mathcal{A}(\mathfrak{J}^{(h)})$. Any member c of $\mathfrak{I}$ can be represented as the sum of members of $\mathfrak{I}_k$ $(k:n)$. Since $\mathcal{A}(\mathfrak{J}) \in \mathcal{AP}(\mathfrak{S})$, c belongs to $\mathcal{A}(\mathfrak{J})$ and hence $\mathfrak{I} \subseteq \mathcal{A}(\mathfrak{J})$ or, since $\mathcal{A}(\mathfrak{J}) \subseteq \mathfrak{I}$, $\mathcal{A}(\mathfrak{J}) = \mathfrak{I}$.

For the proof of the unbracketed result of clause (Fi), it is remarked that to any $a^{(h)} \in \mathfrak{S}^{(h)}$, $b^{(h)} \in \mathfrak{I}^{(h)}$ correspond two numbers $a' \in \mathfrak{S}'$, $b' \in \mathfrak{I}'$ with h^{th} components $a^{(h)}$, $b^{(h)}$ respectively. Since $\mathfrak{I} \in \mathbb{S}\mathbb{I}(\mathfrak{S})$, $a'b' \in \mathfrak{I}'$, $a^{(h)} b^{(h)} \in \mathfrak{I}^{(h)}$ and accordingly $\mathfrak{I}^{(k)} \in \mathbb{S}\mathbb{I}(\mathfrak{S}^{(k)})$ $(k:n)$. If, furthermore, $\mathfrak{I} \in \mathbb{I}(\mathfrak{S})$ then $\mathfrak{I} \in \mathcal{AP}(\mathfrak{S})$ and, from clause (Ri), $\mathfrak{I}^{(k)} \in \mathcal{AP}(\mathfrak{S}^{(k)})$ $(k:n)$ so that $\mathfrak{I}^{(k)} \in \mathbb{I}(\mathfrak{S}^{(k)})$ $(k:n)$.

Subject to the assumptions relating to the unbracketed result of clause (Fii), select any $a \in \mathfrak{S}$, $b \in \mathfrak{I}$ so that $ab \simeq \{a^{(k)} b^{(k)}\}_{(k:n)}$. Since $\mathfrak{I}^{(k)} \in \mathbb{S}\mathbb{I}(\mathfrak{S}^{(k)})$ $(k:n)$, $a^{(k)} b^{(k)} \in \mathfrak{I}^{(k)}$ $(k:n)$ and, since $\delta(\mathfrak{I}) \in \Delta$, $ab \in \mathfrak{I}$. Hence $\mathfrak{I} \in \mathbb{S}\mathbb{I}(\mathfrak{S})$. For the bracketed result, $\mathfrak{I}^{(k)} \in \mathbb{I}(\mathfrak{S}^{(k)})$ and hence $\mathfrak{I}^{(k)} \in \mathcal{AP}(\mathfrak{S}^{(k)})$ $(k:n)$. Since $\delta(\mathfrak{I}) \in \Delta$, it follows that

$I \in AIP(S)$ and hence, since $I \in \mathfrak{S}II(S)$ also, that $I \in II(S)$.

Subject to the conditions of clause (Gi) let $a^{(h)} \in K^{(h)}$. S contains a number a with h^{th} component $a^{(h)}$ such that $ab \in I$. Accordingly $a^{(h)}b^{(h)} \in I^{(h)}$ and $a^{(h)} \in O(b^{(h)}) \equiv O(S^{(h)}, I^{(h)} | b^{(h)})$. The result of clause (Gia) follows immediately. Subject to the further assumption that $\delta(S) \in \Delta$, select $a^{(h)} \in O(b^{(h)})$ so that S contains $a \simeq a^{(h)}$. For this a, $a^{(k)}b^{(k)} \in S^{(k)}[o]$ $(k:n\ (k \neq h))$ and $a^{(h)}b^{(h)} \in I^{(h)}$. Since $S^{(k)}[o] \subseteq I^{(k)}$ $(k:n)$, $ab \in I$ when $\delta(I) \in \Delta$. Thus $a^{(h)} \in K^{(h)}$ and in consequence $O(b^{(k)}) \subseteq K^{(k)}$ $(k:n)_k$ or, in view of the preceding result, $O(b^{(k)}) \equiv K^{(k)}$ $(k:n)$. Select any sequence $a^{(k)} \in K^{(k)} \equiv O(b^{(k)})$ $(k:n)$. Since $\delta(S) \in \Delta$, S contains a with these components and, since $\delta(I) \in \Delta$, $ab \in I$. Hence all $a \in K \cap K' \equiv [\bigcup_{k=1}^{n} K^{(k)}]$ belong to $O(b) \equiv O(S, I | b)$, and since $O(b) \subseteq \bigcup_{k=1}^{n} K^{(k)}$, $O(b) \cap K'$ as stated in clause (Gib).

Under the conditions of clause (Gii), select $a^{(h)} \in \mathbb{Z}(b^{(h)}) \equiv \mathbb{Z}(S^{(h)}, I^{(h)} | b^{(h)})$ so that $a^{(h)}_{\sigma}b^{(h)}_{\sigma} \in I^{(h)}$ whenever $b^{(h)}_{\sigma} \in I^{(h)}$. Since $\delta(S) \in \Delta$, S contains $a \simeq a^{(h)}$. Since $\delta(I) \in \Delta$, $a_{\sigma} b_{\sigma} \in I$ if and only if $a^{(h)}_{\sigma}b^{(h)}_{\sigma} \in I^{(h)}$ for all $\sigma \in S$. If σ is such that $b_{\sigma} \in I$,

then $b^{(h)}g^{(h)} \in I^{(h)}$, $a^{(h)}g^{(h)} \in I^{(h)}$, $ag \in \underline{I}$, $a \in Z(b) \equiv Z(\underline{S}, \underline{I} | b)$ and $a^{(h)} \in K^{(h)}$. Hence $Z(b^{(h)}) \subseteq K^{(h)}$ $(k:n)$. Conversely, select $a^{(h)} \in K^{(h)}$ and any $g^{(h)} \in S^{(h)}$ for which $b^{(h)}g^{(h)} \in I^{(h)}$ so that, since $\delta(S), \delta(I) \in \Delta$, S contains $g \simeq g^{(h)}$ for which $bg \in I$. Since $a^{(h)} \in K^{(h)}$, $Z(b)$ contains a number a with h^{th} component $a^{(h)}$, and $ag \in I$ also, so that $a^{(h)}g^{(h)} \in I^{(h)}$. Hence $a^{(h)}g^{(h)} \in I^{(h)}$ whenever $b^{(h)}g^{(h)} \in I^{(h)}$. Thus $a^{(h)} \in Z(b^{(h)})$. $K^{(h)} \subseteq Z(b^{(h)})$ or, since $Z(b^{(h)}) \subseteq K^{(h)}$, $K^{(k)} \equiv Z(b^{(k)})$ $(k:n)$. Now select $a^{(k)} \in K^{(k)} \equiv Z(b^{(k)})$ $(k:n)$. Since $\delta(S) \in \Delta$, S contains a. Select any g such that $bg \in I$ so that $b^{(k)}g^{(k)} \in I^{(k)}$ $(k:n)$. For such a g, $a^{(k)}g^{(k)} \in I^{(k)}$ $(k:n)$ and, since $\delta(I) \in \Delta$, $ag \in I$. Thus all $a \simeq a' \in K' \equiv \left[\bigcup_{k=1}^{n} K^{(k)}\right]$ belong to $Z(b)$. But $K' \subseteq \bigcup_{k=1}^{n} K^{(k)}$ and hence $Z(b) \simeq \left[\bigcup_{k=1}^{n} K^{(k)}\right]$.

To prove the result of clause (H), suppose first that $I \in \underline{\mathbb{I}}_{QF}(S)$ and that $(a^{(h)})^2 \in I^{(h)}$. Since $\delta(I) \in \Delta$, I contains $a^2 \simeq (a^{(h)})^2$ and therefore a, from which $a^{(h)} \in I^{(h)}$. Hence $I^{(k)} \in \underline{\mathbb{I}}_{QF}(S^{(k)})$ $(k:n)$. Conversely, assuming that $I^{(k)} \in \underline{\mathbb{I}}_{QF}(S^{(k)})$ $(k:n)$ suppose that $a^2 \in I$, so that $(a^{(k)})^2 \in I^{(k)}$ $(k:n)$, $a^{(k)} \in I^{(k)}$ $(k:n)$ and, since $\delta(I) \in \Delta$, $a \in I$. Hence $I \in \underline{\mathbb{I}}_{QF}(S)$.

To prove the unbracketed result of clause (L), assume first that $\delta(S), \delta(I) \in \Delta$ and that conditions (a) of clause (A) hold: $\mathbb{D}(S,I)$ is nonvoid and has the structure described in clause (A). Furthermore $I^{(k)} = S^{(k)}$ for $k \neq k(\tau)$, so that automatically $I^{(k)} \in \mathbb{I}_{QF}(S^{(k)})$ $(k:n(k \neq k(\tau)(\tau:m)))$. Also assuming that $b^{(\omega)} \in \mathbb{N}(S^{(\omega)}, I^{(\omega)})$ $(\omega = k(\tau)(\tau:m))$ it follows firstly that $I^{(k)} \in \mathbb{I}_{QF}(S^{(k)})$ for $(k:n)$ so that, from the preceding clause $I \in \mathbb{I}_{QF}(S)$, and secondly that $u^{(\omega)} \in \mathbb{D}^{(\tau)} \equiv \mathbb{D}(S^{(\omega)}, I^{(\omega)})$, $v^{(\omega)} \in S^{(\omega)}$ exist such that $u^{(\omega)} b^{(\omega)} + v^{(\omega)} \in \mathbb{D}^{(\tau)}$, $b^{(\omega)} v^{(\omega)} \notin \mathbb{I}^{(\omega)}$ $(\omega = k(\tau)(\tau:m))$, or that simpler conditions whose treatment is correspondingly simpler obtain. Select $u^{(k)}, v^{(k)} \in I^{(k)} \equiv S^{(k)}$ $(k:n(k \neq k(\tau)(\tau:m)))$ arbitrarily. Since $\delta(S) \in \Delta$, S' contains u', v' with components $u^{(k)}, v^{(k)}$ ~~respe~~ $(k:n)$ respectively, and, from clause (A), $u \in \mathbb{D} \equiv \mathbb{D}(S,I)$. Furthermore, making use of the same clause, $ub+v \in \mathbb{D}$. Also, since $\delta(I) \in \Delta$, $bv \in \mathbb{I}$. All conditions ensuring that $b \in \mathbb{N}(S,I)$ are satisfied. Conversely, assume that $b \in \mathbb{N}(S,I)$. Since $I \in \mathbb{I}_{QF}(S)$ and $\delta(I) \in \Delta$, it follows from clause (G) that $I^{(k)} \in \mathbb{I}_{QF}(S^{(k)})$ $(k:n)$. Also, since $\delta(S), \delta(I) \in \Delta$, $\mathbb{D}$ is nonvoid, $\mathbb{D}^{(\tau)} \equiv \mathbb{D}(S^{(\omega)}, I^{(\omega)})$ is nonvoid $(\omega = k(\tau)(\tau:m))$

for some m in the range $1 \leq m \leq n$, and $S \equiv \bot\ (k:n\ (k \neq k(\tau)(\tau:m)))$. Furthermore $u \in \mathfrak{D}$, $v \in S$ exist such that $ub+v \in \mathfrak{D}$, $bv \in I$, or simpler conditions hold. Thus $u^{(\sigma)} \in \mathfrak{D}^{(\tau)}$, $v^{(\sigma)} \in S^{(\sigma)}$ exist $\forall$ such that $u^{(\sigma)} b^{(\sigma)} + v^{(\sigma)} \in \mathfrak{D}^{(\tau)}$, $b^{(\sigma)} v^{(\sigma)} \in I^{(\sigma)}$ $(\sigma = k(\tau)(\tau:m))$. Now all conditions ensuring that $b^{(\sigma)} \in \mathfrak{N}(S^{(\sigma)}, I^{(\sigma)})$ $(\sigma = k(\tau)(\tau:m))$ are satisfied.

The bracketed result of clause (I) is an extension of its unbracketed counterpart. $b|a \bmod I$ if and only if $b \in \mathfrak{N}(S,I)$ and $a \in \mathfrak{Z}(S, I|b)$; the notation $b^{(\sigma)}|a^{(\sigma)} \bmod I^{(\sigma)}$ has a similar meaning. Thus if conditions (a) of clause (A) hold and $b^{(\sigma)}|a^{(\sigma)} \bmod I^{(\sigma)}$ $(\sigma = k(\tau)(\tau:m))$, $b \in \mathfrak{N}(S,I)$ from the above proof and $a^{(\sigma)} \in \mathfrak{Z}(S^{(\sigma)}, I^{(\sigma)}|b^{(\sigma)})$ $(\sigma = k(\tau)(\tau:m))$. But $S^{(k)} \equiv I^{(k)}$ $(k:n\ (k \neq k(\tau)(\tau:m)))$ so that all $b^{(k)} \in S^{(k)}$ with $k \neq k(\tau)(\tau:m)$ lie in $I^{(k)}$, $\mathfrak{Z}(S^{(k)}, I^{(k)}|b^{(k)}) \equiv S^{(k)}$ for such k, and $a^{(k)} \in \mathfrak{Z}(S^{(k)}, I^{(k)}|b^{(k)})$ $(k:n)$. Since $\delta(S), \delta(I) \in \Delta$, it follows from clause (Gii) that $a \in \mathfrak{Z}(S,I|b)$, and hence that $b|a \bmod I$. Conversely, the assumption that $b|a \bmod I$ implies firstly, from the above proof, that conditions (a) of clause (A) hold, with $b^{(\sigma)} \in \mathfrak{N}(S^{(\sigma)}, I^{(\sigma)})$ $(\sigma = k(\tau)(\tau:m))$ and secondly, since $a \in \mathfrak{Z}(S, I|b)$ that, again by use of clause (Gii) that $a^{(\sigma)} \in \mathfrak{Z}(S^{(\sigma)}, I^{(\sigma)}|b^{(\sigma)})$ $(\sigma = k(\tau)(\tau:m))$

so that $b^{(\nu)} | a^{(\nu)} \bmod \mathfrak{I}^{(\nu)}$ $(\nu = k(\tau)(\tau:m))$.

$Q_I = Q(S,I)$ exists if and only if $D = D(S,I)$ is nonvoid; a similar remark concerning the existence of $Q^{(\tau)} = Q(S^{(\omega)}, I^{(\omega)})$ $(\nu = k(\tau)(\tau:m))$ may be made. The first result of clause (J) is a consequence of clause (A). Assume the existence of Q_I and Q_τ $(\tau:m)$. From the bracketed result of clause (I), $(a/b)_I$ exists in Q_I if and only if $(a^{(\omega)}/b^{(\omega)})_{I^{(\omega)}}$ exists in $Q^{(\tau)}$, where $\nu = k(\tau)(\tau:m)$: a one to one correspondence between $(a/b)_I$ and the sequence $\{(a^{(\omega)}/b^{(\omega)})_{I^{(\omega)}}\}$ is thereby established. $(a/b)_I$ exists if and only if $u \in D$, $v \in S$ exist such that $ub+v \in D$, $bv \in I$ (or simpler conditions whose treatment is correspondingly simpler obtain) and, in turn if and only if $u^{(\omega)} \in D^{(\tau)} = D(S^{(\omega)}, I^{(\omega)})$, $v^{(\omega)} \in S^{(\omega)}$ exist such that $u^{(\omega)}b^{(\omega)} + v^{(\omega)} \in D^{(\tau)}$, $b^{(\omega)}v^{(\omega)} \in I^{(\omega)}$ $(\nu = k(\tau)(\tau:m))$. The ordered pair $\langle c/d \rangle$ of S-numbers belongs to $(a/b)_I$ if and only if $d \in D$ and $(ub+v)c - ad = t \in I$; these conditions are satisfied if and only if $d^{(\omega)} \in D^{(\tau)}$ and $(u^{(\omega)}b^{(\omega)} + v^{(\omega)})c^{(\omega)} - a^{(\omega)}u^{(\omega)}d^{(\omega)} = t^{(\omega)} \in I^{(\omega)}$ $(\nu = k(\tau)(\tau:m))$ or, in turn, if and only if $\langle c^{(\omega)}, d^{(\omega)} \rangle$ belongs to $(a^{(\omega)}/b^{(\omega)})_{I^{(\omega)}}$ $(\nu = k(\tau)(\tau:m))$. A one to one correspondence

7 between $\langle c,d\rangle$ and the sequence $\langle c^{(\omega)},d^{(\omega)}\rangle$ of ordered pairs of numbers in $Q^{(\tau)}$ $(\nu=k(\tau)\,(\tau:m))$ is thereby established. This correspondence, as is easily verified, is preserved during addition, subtraction and multiplication; the correspondence between $(a/b)_I$ and the sequence $\{\{a^{(\omega)}/b^{(\omega)}\}_{I^{(\omega)}}\}$ is thereby preserved in the same way. All combinations of $b_\nu \in N(S^{(\omega)},I^{(\omega)})$, $a^{(\nu)}\in Z(S^{(\omega)},I^{(\omega)}|b^{(\nu)})$ $(\nu=k(\tau)\,(\tau:m))$ in which the b_ν occur independently and all possible a_ν determined by b_ν may be chosen feature in this correspondence. Hence $Q_I \sim [\bigcup_{\tau=1}^{m} Q^{(\tau)}]$.

To deal with the result of clause (ki), let $d^{(h)}, v^{(h)}$ be elements of $D^{(h)}, J^{(h)}$ respectively. D' contains d' with h^{th} component $d^{(h)}$ and, since $\delta(J)\in\Delta$ and $S[0]\subseteq J$, J contains $v\simeq v^{(h)}$. Since $\delta(S)\in\Delta$, S contains $a\simeq a^{(h)}$ for any $a^{(h)}\in S^{(h)}$. For any such $a^{(h)}$, the condition $ad=v$ implies that $a\in J$, since D is integral with respect to J in S. Thus for any $d^{(h)}\in D^{(h)}$, $v^{(h)}\in J^{(h)}$ and $a^{(h)}\in S^{(h)}$, the condition $a^{(h)}d^{(h)}=v^{(h)}$ implies that $a^{(h)}\in J^{(h)}$. Accordingly $D^{(x)}$ is integral with respect to $J^{(x)}$ in $S^{(x)}$ $(x:n)$

Under the assumptions of clause (Kii), any $a' \in S'$, $d' \in D'$, $v' \in J'$ have components $a^{(k)} \in S^{(k)}$, $d^{(k)} \in D^{(k)}$, $v^{(k)} \in J^{(k)}$ $(k:n)$ respectively. If $ad = v \in J$, $a^{(k)} d^{(k)} = v^{(k)}$ $(k:n)$ and $a^{(k)} \in J^{(k)}$ $(k:n)$. Since $\delta(J) \in \Delta$, $a \in J$. Hence D is integral with respect to J in S.

To prove the unbracketed result of clause (L) it is first remarked that, from clause (Fi), if $I \in \underline{\underline{I}}(S)$, $I^{(k)} \in \underline{\underline{I}}(S^{(k)})$ $(k:n)$. Select $c^{(h)} \in I^{(h)}$. I contains c with h^{th} component $c^{(h)}$, and if I is a direct principal ideal of the form $I(b)$, $c = ab$ for some $a \in S$. Thus $c^{(h)} = a^{(h)} b^{(h)}$ for some $a^{(h)} \in S^{(h)}$, and $I^{(h)}$ has the form $I(b^{(h)})$. Conversely, with regard to unbracketed results of clause (Lii), the assumptions $\delta(I) \in \Delta$, $I^{(k)} \in \underline{\underline{I}}(S^{(k)})$ $(k:n)$ imply, from clause (Fii) that $I \in \underline{\underline{I}}(S)$. Furthermore, if $c \in I$ and $I^{(k)}$ is a direct principal ideal in $S^{(k)}$ of the form $I(b^{(k)})$, $c^{(k)} \in I^{(k)}$ has a representation $c^{(k)} = a^{(k)}(b^{(k)}) b^{(k)}$ with $b^{(k)}$ fixed in $S^{(k)}$ and $a^{(k)}(c^{(k)}) \in S^{(k)}$ $(k:n)$. When $\delta(S) \in \Delta$, S' contains a' with components $a^{(k)}(c^{(k)})$ $(k:n)$, and $c = ab$ for some $a \in S$ with $b \in S$ fixed. Hence I is a direct principal

ideal in S of the form $I(b)$. If, in addition, I is an isolated principal ideal in $S^{(k)}$ of the form $I(b^{(k)})$ $(k:n)$, $a^{(k)}(c^{(k)})$ and $b^{(k)}$ in the preceding both belong to $I^{(k)}$ and the assumption $\delta(I)\in\Delta$ alone suffices to ensure that all $c\in I$ have a representation of the form $c=a(c)b$ with $b\in I$ fixed and $a(c)\in I$, so that I is an isolated principal ideal in S.

Conditions which ensure that properties of subsystems, $I^{(k)}$ say, of $\mathfrak{S}$-rings $S^{(k)}$ $(k:n)$ are transmitted to the subsystem $I'\equiv[\bigcup_{k=1}^{n} I^{(k)}]$ belonging to the complete compound $S'\equiv[\bigcup_{k=1}^{n} S^{(k)}]$ of the $\mathfrak{S}$-rings $S^{(k)}$ $(k:n)$ may be obtained as corollaries to certain of the results of the above theorem. With S' a complete compound as described, S has a decomposition $\delta(S)\in\Delta$ in which the $S^{(k)}$ feature as component $\mathfrak{S}$-rings. The clauses of the above theorem dealing with properties of I that may be deduced from those of the $I^{(k)}$ yield the desired results. Thus, from clause (Fii) for example, if $I^{(k)}$ is an $\mathfrak{S}$-ideal in $S^{(k)}\in\mathfrak{S}$ $(k:n)$, I' is an $\mathfrak{S}$-ideal in the complete compound S' of the $S^{(k)}$, and so on.

Definition .(a)H S-ring which possesses n elements x_k (k:n), where $1<n<\infty$, for which $x_k^2 = x_k \notin S[0]$ and $2x_j x_k \in S[0]$ when $j \neq k$ (k:n), with $\sum_{k=1}^{n} x_k = x$ $\in \mathbb{D}(S)$ is said to be rationally decomposable into an S-ring with n components.

(ii) An S-ring which possesses this property for some finite n is rationally decmposable

(iii) A decomposable S-ring which is isomorphic to a directly decomposable S-ring whose components $S^{(k)}$ (k:n) are not rationally decomposable is said to have a finite basis of order n.

It is evident from the relatimship $x_k^2 = x_k$ that the numbers x_k of clause (i) of the above definitim belong to the stratum $S\{0\}$ of S.

Theorem (i) An $\mathfrak{S}$-ring that is rationally decomposable into an n-dimensional $\mathfrak{S}$-ring is completely decomposable into one.

(ii) Let the $\mathfrak{S}$-ring S be rationally decomposable, containing the n elements $x^{(k)}$ $(k:n)$ of Definition (i) and let I be a subsystem of elements of S such that $ax^{(1)}+\ldots+bx^{(n)}\in I$ for all $a,\ldots,b$ belonging to the same stratum of I. With $S\sim S'\equiv[\bigcup_{k=1}^{n} S^{(k)}]$ being the decomposition determined by the elements $x^{(k)}$, $I\sim I'\equiv[\bigcup_{k=1}^{n} I^{(k)}]$ where $S'\simeq I'$.

(iii) Let the $\mathfrak{S}$-ring S be rationally decomposable into an n-dimensional $\mathfrak{S}$-ring, with $S\sim S'$, and $S'\equiv[\bigcup_{k=1}^{n} S^{(k)}]$ being the complete decomposition resulting from this condition; let I be an $\mathfrak{S}$-ideal in S. $I\sim I'\equiv[\bigcup_{k=1}^{n} I^{(k)}]$ where $S'\simeq I'$.

proof. The numbers $a=xa$ for [illegible] ideal S' in S; hence $S'\in\mathfrak{S}$. Each number of S' has a representation $a'=\sum_{k=1}^{n} a^{(k)}$ where $a^{(k)}=x^{(k)}a$ $(k:n)$. This representation is preserved during addition and subtraction and also during multiplication since, when $b'=\sum_{k=1}^{n} b^{(k)}$ where $b^{(k)}=x^{(k)}b$ $(k:n)$, $a'b'=\sum_{k=1}^{n} a^{(k)}b^{(k)}$: S' is directly decomposable; $S'\leq\bigcup_{k=1}^{n} S^{(k)}$ where $S^{(k)}=x^{(k)}S$ $(k:n)$. Only one number a' is determined by the mapping $a'=xa$. Since $x\in\mathfrak{D}(S)$, the relationship $xa=xb$ implies that $a=b$: each number a' is the image of only one $a\in S$ under the mapping $a'=xa$. Since $x^2=x$, $a'b'=xab$ for all a,b all $ab\in S$. Furthermore $a'\pm b'=x(a\pm b)$ for all a,b belonging to the same S-stratum: $S\sim S'$. S is therefore decomposable. Furthermore, for all $a,\ldots,b$ belonging to the same stratum, S contains the number $ax_1+\ldots+bx_n$ and, since $x_kx=x_k$ $(k:n)$, S' does the same. All numbers with components $a^{(1)},\ldots,b^{(n)}$ for all $a^{(1)}\in S^{(1)},\ldots,b^{(n)}\in S^{(n)}$ feature in S': S' has the complete direct decomposition $S'\equiv[\bigcup_{k=1}^{n} S^{(k)}]$ and S is accordingly completely decomposable into an n-dimensional S-ring.

With I as described in clause (ii), let $I \sim I'' \subseteq \bigcup_{k=1}^{n} I^{(k)}$, where $I'' \subseteq S'$. Since $S' = [\bigcup_{k=1}^{n} S^{(k)}]$, $I \subseteq S'$ where $I' = [\bigcup_{k=1}^{n} I^{(k)}]$. Evidently $I'' \subseteq I'$. Select any member z of I. $a, \ldots, b \in S$ exist such that $z = ax^{(1)} + \ldots + bx^{(n)}$. Since I contains all numbers of this form, $z \in I$ and hence $I' \subseteq I''$: $I' = I''$ and $I \sim I'$ as required. When $I \in \mathbb{I}(S)$, $ax^{(1)}, \ldots, bx^{(n)} \in I$ for all $a, \ldots, b \in I$; also $ax^{(1)} + \ldots + bx^{(n)} \in I$ for all $a, \ldots, b$ belonging to the same stratum of I. The result of clause (iii) is a corollary to that of clause (ii).

Theorem (i) A rational $\mathfrak{S}$-ring n rationally decomposable into an $\mathfrak{S}$-ring with n components if and only if it possesses $n-1$ denominators $b(\nu)(\nu:n-1)$ for which $b(1)\notin S[0]$, $b(\nu)\equiv b(\nu+1) \bmod S[0]\ (\nu:n-2)$, $b(n-1)\notin D(S,S[0])$.

(ii) Let the rational $\mathfrak{S}$-ring S be rationally decomposable into an $\mathfrak{S}$-ring with n components, with $S\cup S'\equiv\left[\bigcup_{\tau=1}^{m} S^{(1,\tau)}\right]\left(S^{(1,\tau)}\in\mathfrak{S}\ (\tau:m)\right)$; let the preceding statement also hold with m replaced by n and $S^{(1,\tau)}$ by $S^{(2,\tau)}$.

a) If $m<n$, at least one of the $S^{(1,\tau)}$ is rationally decomposable.

b) If $m=n$ either (1) one of the $S^{(1,\tau)}$ and one of the $S^{(2,\tau)}$ are rationally decomposable or (2) $S^{(2,\tau)}\sim S^{(1,\nu(\tau))}\ (\tau:n)$, $\nu(1),\dots,\nu(n)$ being a rearrangement of the integers $1,\dots,n$.

(iii) If the rational S-ring S possesses n elements $x_k\ (k:n)$ as described in Definition (i) and for

every $a \in S$ such that $ax_k \notin S[0]$, $abx_k \in S[0]$ only when $bx_k \in S[0]$ $(k:n)$ then S has a finite basis of order n.

Proof. $D(S, S[0])$ and $N(S, S[0])$ are written as D and N respectively.

Assuming S to possess $n-1$ denominators as described in clause (i), it is further supposed that S possesses $n-1$ pairs of numbers $u(\omega), v(\omega)$ $(\omega: n-1)$ such that $b(\omega) \leq u(\omega) \bmod S[0]$, $b(\omega)v(\omega) \in S[0]$, $u(\omega)b(\omega) + v(\omega) = d(\omega) \in D$ $(\omega: n-1)$. $b(1) \in N$ and $b(1) \notin S[0]$ by assumption. Since $v(\omega) \in N$ $(\omega: n-1)$ and $N \in M(S)$, $b(\omega)v(\omega-1) \in N$ $(\omega: 2; n-1)$. If $b(\omega)v(\omega-1) \in S[0]$ for some ω in the range $2 \leq \omega \leq n-1$, $b(\omega) \in O(S, S[0] \mid v(\omega-1)) \equiv Z(S, S[0] \mid b(\omega-1))$ and $b(\omega) \leq b(\omega-1) \bmod S[0]$, contrary to assumption. Also $v(n-1) \in N$ and, since $b(n-1) \notin D$, $v(n-1) \notin S[0]$. The numbers $x_1 = 1_{b(1)}$, $x_\omega = 1_{b(\omega)v(\omega-1)}$ $(\omega: 2; n-1)$, $x_n = 1_{v(n-1)}$ are defined in S and are not members of $S[0]$. Since $b(\tau) \leq b(\omega) \bmod S[0]$ $(\tau: \omega)$ and $b(\omega)v(\omega) \in S[0]$, $b(\tau)v(\omega) \in S[0]$ $(\omega: n-1 \mid \tau: \omega)$. Thus $x_1 x_\omega \in S[0]$ $(\omega: 2; n)$. When $1 < j, k < n$, $x_j x_k$ is expressible as a fraction whose numerator is $b(k)v(j-1)\,b(j)v(k-1)$. When $j \neq k$ either $k \leq j-1$ or $j \leq k-1$ and in consequence $x_j x_k \in S[0]$ when $1 < j, k < n$, $j \neq k$. Since $1_{b(\tau)v(n-1)} \in S[0]$, $x_\tau x_n \in S[0]$ $(\tau: n-1)$. In conclusion $x_j x_k \in S[0]$ when $1 \leq j, k \leq n$, $j \neq k$.

The numbers x_ν are now written as $x_\nu^{(n)}$ and, with $X^{(n)}=\sum_{\nu=1}^{n} x_\nu^{(n)}$ and $D^{(n)}=\prod_{\nu=1}^{n-1} d(\nu)$ it will be shown that $D^{(n)}X^{(n)}=D^{(n)}$. This result is correct when $n=2$, since $D^{(2)}x_1^{(2)}=u(1)b(1)$ and $D^{(2)}x_2^{(2)}=v(1)$. The result is assumed true as stated, and a further denominator $b(n)\notin D$ for which $b(n-1)<b(n)$ is introduced, and it is supposed that $u(n), v(n)\in S$ exist such that $u(n)\,b(n)+v(n)=d(n)\in D$ and $b(n)v(n)\in S[0]$. Now $x_\nu^{(n+1)}=x_\nu^{(n)}(\nu:n-1)$ but $x_n^{(n+1)}=1_{b(n)v(n-1)}$ and $x_{n+1}^{(n+1)}=1_{v(n)}$. Accordingly

$$D^{(n+1)}X^{(n+1)}=d(n)D^{(n)}\{X^{(n)}-1_{v(n-1)}+1_{b(n)v(n-1)}+1_{v(n)}\}$$

But $1_{b(n)v(n-1)}d(n)=1_{v(n-1)}1_{b(n)}d(n)=1_{v(n-1)}u(n)\,b(n)$ and $1_{v(n)}d(n)=v(n)$. Also, since $b(\tau)v(\nu)\in S[0]$ $(\nu:n|\tau:\nu)$, $1_{v(n-1)}D^{(n)}=\prod_{\nu=1}^{n-1} v(\nu)$ and $v(n)D^{(n)}=\prod_{\nu=1}^{n} v(\nu)$. Hence $D^{(n+1)}X^{(n+1)}=D^{(n+1)}$ and the required result follows by induction. Since $D\in M(S)$, $D^{(n)}\in D$ and $X^{(n)}=1\in S$. S possesses n nonzero orthogonal idempotent elements as described in Definition (i) and is accordingly rationally decomposable into an S-ring with n components.

Assuming S to possess n numbers x_τ $(\tau:n)$

with the properties described in Definition (i), set $b(\nu)=\sum_{\tau=1}^{\nu} x_\tau$, $v(\nu)=\sum_{\tau=\nu+1}^{n} x_\tau$ $(\nu : n-1)$, so that $b(\nu)v(\nu) \equiv S[0]$, $b(\nu)+v(\nu)=x=\sum_{\tau=1}^{n} x_\tau \in D$, and accordingly $b(\nu)\in N$ $(\nu : n-1)$. Since $b(\nu)=b(\nu)\,b(\nu+1)$, $b(\nu)\leq b(\nu+1)$ mod $S[0]$ $(\nu : n-2)$. Also $x_{\nu+1}\,b(\nu+1)=x_{\nu+1}\notin S[0]$, but $x_{\nu+1}\,b(\nu)\in S[0]$ so that $b(\nu+1)\not\equiv b(\nu)$ mod $S[0]$ and accordingly $b(\nu)<b(\nu+1)$ mod $S[0]$ $(\nu : n-2)$. Finally $b(1)=x_1\notin S[0]$ and, since $b(n-1)\,x_n\in S[0]$ with $x_n\notin S[0]$, $b(n)\notin D$. S possesses $n-1$ denominators as described in clause ( ).

Under the conditions of clause ( ), S possesses m elements y_τ $(\tau : m)$ with properties similar to those ascribed to x_k in Definition (i) and in particular $\sum_{\tau=1}^{m} y_\tau = y \in D$. S also possesses n elements x_k as described in that definition. Let k be fixed in the

range $1 \le k \le n$. If $x_k y_\tau \in S[0]$ $(\tau : m)$, $x_k y \in S[0]$ also; $x_k \notin S[0]$ and hence $y \notin D(S)$, contrary to assumption. Thus for $(k : n)$, $x_k y_{\tau(k,j)} \notin S[0]$ $(j : \nu(k) \ge 1)$ with $x_k y_\tau \in S[0]$ when $\tau \notin \{\tau(k,j)\}$. The $\tau(k,j)$ can only take m values. Hence $x_\omega, \dots, x_\mu$ with distinct suffixes $\omega, \dots, \mu$ and y_τ exist such that $x_\omega y_\tau = \dots = x_\mu y_\tau \notin S[0]$ and $x_k y_\tau \in S[0]$ when $x_k \neq x_\omega, \dots, x_\mu$. The elements of $S^{(1,\tau)}$, the τth component S-ring in the system to which S is rationally decomposed by use of the numbers y_τ, have the form $a y_\tau$ for all $a \in S$, and among them feature the nonzero orthogonal idempotent elements $x_\omega y_\tau, \dots, x_\mu y_\tau$. Furthermore $(x_\omega + \dots + x_\mu) a y_\tau = x a y_\tau \in S[0]$ only when $a y_\tau \in S[0]$; hence $x_\omega + \dots + x_\mu \in D(S^{(1,\tau)}, S^{(1,\tau)}[0])$: $S^{(1,\tau)}$ may be rationally decomposed.

In the above argument the system of integers $\tau(k,j)$ contains each of the integers $1, \dots, m$ at least once (if μ in the range $1 \le \mu \le m$ is missing from the $\tau(k,j)$, $x y_\mu \in S[0]$ and $x \notin D(S, S[0])$. When $m = n$, two cases can occur: either $\nu(k) > 1$ for some k in

the range $1 \leq k \leq n$, and then both one of the υ and one of the $S^{(2,\tau)}$ can be rationally decomposed as described above, or $\nu(k) = 1$ $(k:n)$ and the $\nu(k) = \tau(k;1)$ simply form a rearrangement of the numbers $1, \ldots, n$. The members of $S^{(2,\tau)}$ have the form ax_τ, and those of $S^{(1,\nu(\tau))}$ the form $ay_{\nu(\tau)}$, where $a \in S$. But $y(ax_\tau) = x(ay_{\nu(\tau)})$: each element ax_τ of $S^{(2,\tau)}$ determines a unique member $ay_{\nu(\tau)}$ of $S^{(1,\nu(\tau))}$ and conversely. The last result of clause (ii) has been obtained.

If S is as described in clause (ii), it is isomorphic to a compound system $\bigcup_{k=1}^{n} S^{(k)}$ for which every member of $S^{(k)}$ has the form ax_k with $a \in S$ $(k:n)$. Furthermore $(ax_k)(bx_k) \in S^{(k)}[0]$ only when ax_k or bx_k are in $S^{(k)}[0]$: $T(S^{(k)}, S^{(k)}[0])$ is empty and $S^{(k)}$ possesses no denominators of the form described in clause ( ): $S^{(k)}$ is not rationally decomposable $(k:n)$: S has a finite basis of order n.

The existence of a system of denominators as described in clause (i) of the above theorem ensures that the $\mathfrak{S}$-ring concerned is rationally decomposable. Other denominator systems suffice for the same purpose. For example, let the rational $\mathfrak{S}$-ring S contain $\sum_{\omega=1}^{m}\{n(\omega)-1\}$ denominators $b_{\omega,\nu}$ $(\omega:m\,|\,\nu:n(\omega)-1)$ for which $b_{\omega,\nu}<b_{\omega,\nu+1} \bmod S[0]$ $(\omega:m\,|\,\nu:n(\omega)-2)$ with $b_{\omega,1}\notin S[0]$, $b_{\omega,n(\omega)-1}\notin D(S,S[0])$ and, with $b_{\omega,\nu}u_{\omega,\nu}\in S[0]$, $b_{\omega,\nu}+u_{\omega,\nu}\in D(S,S[0])$ $(\omega:m\,|\,\nu:n(\omega)-1)$ let $b_{\omega,1}b_{\tau,1}, u_{\omega,n(\omega)-1}u_{\tau,n(\tau)-1}\notin S[0]$ $(\omega:m-1\,|\,\tau:\omega+1;m)$ and $b_{\omega,1}u_{\tau,n(\tau)-1}\notin S[0]$ $(\omega,\tau:m(\omega\neq\tau))$. Let the denominators $b_{\omega,\nu}$ and companion numbers $u_{\omega,\nu}$ in S correspond to $B(\omega,\nu)$ and $U(\omega,\nu)$ in $Q\equiv Q(S,S[0])$, so that the properties of $B(\omega,\nu), U(\omega,\nu)$ in Q are similar to those $b_{\omega,\nu}, u_{\omega,\nu}$ in S. For $(\omega:m)$ set $B_{\omega}^{(1)}=1_{B(\omega,1)}$, $B_{\omega}^{(\xi)}=1_{B(\omega,\xi)\,U(\omega,\xi-1)}$ $(\xi:2;n(\omega)-1)$ $B_{\omega}^{(n(\omega))}=1_{U(\omega,n(\omega)-1)}$ and, with j having the decomposition

$$j=\xi(j,1)+\sum_{\omega=2}^{m}\left[\left\{\xi(j,\omega)-1\right\}\prod_{\tau=1}^{\omega-1}n(\tau)\right]$$

with $1 \leq \xi(j,\omega) \leq n(\omega)$ set $X_j = \prod_{\omega=1}^{m} B_\omega^{(\xi(j,\omega))}$ for $(j : n = \prod_{\omega=1}^{m} n(\omega))$. The numbers X_j $(j:n)$ possess the properties attributed to the x_k in Definition (i) and accordingly S is rationally decomposable into a system with $n = \prod_{\omega=1}^{m} n(\omega)$ components. As clause ( ) of the above theorem indicates, S also contains a system of $\prod_{\omega=1}^{m} n(\omega) - 1$ denominators with the properties described in that clause.