Definition. (i) $\mathbb{P}(j)$ is an additive abelian group $(j=0,1,\dots)$.

(ii) Commutative, ~~and~~ associative ~~and distributive~~ multiplication, which ~~is distributive~~ is defined between elements of the various $\mathbb{P}(\;)$: with $a\in\mathbb{P}(j)$, $b\in\mathbb{P}(k)$, $ab=ba\in\mathbb{P}(j+k)$ ~~with $c\in\mathbb{P}(m)$,~~ $(j,k=0,1,\dots)$ and $(ab)c=a(bc)$ for such products; the distributive law with respect to addition within appropriate $\mathbb{P}(\;)$ is also obeyed: with $a,b\in\mathbb{P}(j)$, $c\in\mathbb{P}(k)$ $(a+b)c=ac+bc$, addition on the left being within $\mathbb{P}(j)$, that on the right within $\mathbb{P}(j+k)$ $(j,k=0,1,\dots)$.

A ~~Numbers~~ number belonging to one of the above $\mathbb{P}$-~~classes~~ is ~~a system defined by these above laws are~~ called a $\mathbb{P}$-number; the $\mathbb{P}(j)$ to which it belongs is its $\mathbb{P}$-class; the aggregate of numbers belonging to all $\mathbb{P}(j)$ $(j=0,1,\dots)$ is a $\mathbb{P}$-system. As is easily demonstrated, if ~~ac~~ $a=0\in\mathbb{P}(j)$, ~~and~~ $b\in\mathbb{P}(k)$, then $ab=0\in\mathbb{P}(j+k)$ $(j,k=0,1,\dots)$; furthermore the distributive law $c(a+b)=ca+cb$ is also ~~held~~ implied by the above hypotheses ~~assumptions~~. The elements of $\mathbb{P}(0)$ form a commutative ring. The assumption that addition between members of disparate $\mathbb{P}$-classes is possible is not made. Also it is not assumed that a unit member of ~~any $\mathbb{P}$-class~~ a $\mathbb{P}$-system exists; neither is it

similar formulae. (The members of the $\mathbb{P}'(j)$ ~~are members~~
may be segregated into equivalence classes, whose members ~~of which~~ have the form $(ac)/(bc)$). The quotients of the $\mathbb{P}'$ form a $\mathbb{P}$ system.

— · —

Much of the work of the following sections is concerned with arrays of $\mathbb{P}$-numbers. The following notations and conventions are introduced. $a(h;m:k;n)$ is the $(m-h+1)\times(n-k+1)$ array whose i^{th} row and j^{th} column jointly contain the element $a_{i+h-1,j+k-1}$ $i=1,\dots,m-h+1$; $j=1,\dots,n-k+1$. $\{a(h_1;m_1:k_1;n_1), d(h_2;m_2:k_2;n_2)\}$, where $m_1-h_1=m_2-h_2$, is the $(m_1-h_1+1)\times(n_1+n_2-k_1-k_2+1)$ compound array obtained by setting the array $a(h_1;m_1:k_1;n_1)$ upon the left of its partner. The bordered arrays $\{a(u;w:$ ~~$\{a(h;m:k;n),$~~ $\{a(h;m:u;u), a(h;m:k;n)\}$, $\{a(h;m:k:n), a(h;m:u;u)\}$ are denoted by the abbreviated forms $a(h;m:u,k;n)$ and $a(h;m:k;n,u)$ respectively; $a(v,h;m:k;n)$ is the compound array obtained by bordering $a(h;m:k;n)$ from above by the row array $a(v;v:k;n)$, $a(h;m:v:k;n)$ is the latter array bordered from below. $a^{[i:]}(h;m:k:n)$

is the $(m-h)\times(n-k+1)$ array obtained by ~~removing~~ deleting the $(i-h+1)$th row of elements $a_{i,j}$ $(j=k,\dots,n)$ from $a(h;m:k;n)$, and $a^{[:j]}(h;m:k;n)$ is the $(m-h+1)\times(n-k)$ array ~~defin.~~ derived by column removal. If, in the above notations, h does not follow a comma and has the value unity, it and the subsequent semi-colon are omitted, the same abbreviation being effected with respect to k (thus the ith column of the array $a(m-1,m+1:n-1,n+1)$ contains the elements $a_{i,j}$ $(j=1,\dots,n-1,n+1)$ for $i=1,\dots,m-1,m+1$); the restriction concerning the precedent comma in the above convention ensures, for example, that the symbol $a(m,u:n)$ is unambiguous: it denotes $a(1;m,u:n)$ and not $a(m,1;u:n)$). The above notations are freely combined; thus: $a^{[i;j]}(u,h;m:w,v)$. If, in statements concerning compound arrays it occurs that ~~for certain~~ for certain values of the indices over the ranges considered, a constituent array becomes empty (e.g. $m<h$ in $a(h;m;n)$) it is to be understood that the constituent array is to be deleted

from the compound for such index values; in this way it becomes unnecessary to write out special statements holding for the index values concerned. In summary, a colon seperates references to row and column subscripts; a semi-colon seperates lower and upper limits of sequence of rows or columns and is omitted, together with the lower limit, if the latter is unity (except in the special case described); a comma follows <precedes> reference to ~~an array or~~ the index of a row which borders from above <below> or of a column which borders from the left <right>; it also follows <precedes> the constituent of a compound array placed on the left <right>, ~~the~~ ~~lower limit and sub of a sequence of rows and columns~~ ~~is and the subsequent semi-colon are omitted +~~

Definition . With $r(i)$ $(i=1,\dots,m)$, and $c(j)$ $(j=1,\dots,n)$ two sequences of (not necessarily distinct) nonnegative integers, $a \in PA(m:n \mid r:c)$ means that the elements of the array $a(m:n)$ are distributed among the various P-classes according to the law $a_{i,j} \in P\{c(i)+r(j)\}$. An array possessing this property is called a P-array.

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~~Write $a(m,n)\in$ $a\in PA(m:n,|c,r)$, $a'\in PA$~~

Two arrays, $a \in PA(m:n_*|c_*,r_*)$ and $a' \in PA(m,n'_*|c'_*,r'_*)$ can be compounded ~~to for~~ in the form $\{a(m:n_*), a'(m,n'_*)\}$ ~~of~~ as a P-array if $r'(i) = r(i) + k$ $(i=1,\dots,m)$ and $k+c'(j)\geq 0$ $(j=1,\dots,n')$; a similar result holds ~~for~~ with regard to vertical adjunction. The array obtained by deleting of one or more rows or columns from a P-array is a P-array. A compound array $\{a(m;n), a^{[:j]}(m;n)\}$ and others of its kind, where a is a P-array, is automatically a P-array, and again a similar remark may be made with regard to vertical adjunction. If $a\in PA(m;n|c,r)$, it is always possible to form P-arrays of the form

$$\begin{Bmatrix} a(m:n), & a^{[:j]}(m;n) \\ O^{[k:]}(m;n), & a^{[k:j]}(m;n) \end{Bmatrix}$$

by taking $O(m;n)$ to be the array ~~of the cla~~ with elements $O\in P\{c(i)+r(j)\}$ $(i=1,\dots,m;\ j=1,\dots,n)$.

~~A single class of P-arrays is represented by many symbols.~~ The classes $PA\{m:n|r:c\}$ and $PA\{m:n|r':c'\}$ where $r'(k)=r(k)+v$, $(k=1,\dots,m)$ $c'(j)=c(j)-v$ $(j=1,\dots,n)$ v being a constant integer, are equivalent. The representation of P-arrays can be made unique by ~~adopting~~ imposing a normalisation condition (for example, that ~~convention of the forms~~ $r(1)=0$ in all representations); ~~that~~ the subsequent theory is then vitiated by tedious auxiliary computations ~~that~~ required to ensure that the ~~convention~~ condition is ~~preserved~~ satisfied. As an alternative to such computations, the convention that a single class of P-arrays is represented by many symbols has been adopted. An equivalence sign is used in statements concerning the ~~order~~ class-defining integers $\{r,c\}$ relating to two classes of P-arrays. Thus, in the ~~statement~~ proposition that the P-arrays $b\in PA\{m:n':r':c'\}$ ~~$b\in PA\{m:n'|r':c'\}$~~ may be adjoined on the ~~right or~~ left or right of the P-array $a\in PA\{m:n|r:c\}$ to form a compound P-array if and only if $r\equiv r'$, the latter statement means that in one of the representations, $PA\{m : n' : r'' : c''\}$ say, of the class to which b belongs, $r(i) = r''(i)$ $(i = 1, \ldots, m)$. ~~In this connection, $-c$ represents the sequence~~ The further convention that a symbol such as $-c$ represents the integer sequence $-c(j)$ $(j = 1, \ldots, n)$ is also adopted.

A distinction between arrays and matrices (to be considered later) is made: an array is merely a display device; a matrix is a mathematical object upon which ~~a~~ certain well-defined operations ~~can~~ may be performed.

Theorem. Let $a(n:n)$ be a P-array $(n>2)$.

(i) Let $\hat{a}$ be the P-array with elements $|a^{[i;j]}(n:n)|$ $(i,j=1,\dots,n)$. With $1<k<n$ let $i(1),\dots,i(k)$ be an increasing sequence of distinct integers taken from the set $1,\dots,n$ and take $I(1),\dots,I(n-k)$ to be the complementary increasing subsequence; define the sequences $j(1),\dots,j(k)$; $J(1),\dots,J(n-k)$ similarly. Let $\hat{a}^{(i;j)}$ be the array formed from the elements lying at the intersections of rows $i(1),\dots,i(k)$ and columns $j(1),\dots,j(k)$ of $\hat{a}$ taken in order, and define $a^{(I;J)}$ similarly. Then

ideals of [illegible]

$$\cancel{|a^{(I;J)}||a|^{k-1} =}\ |\hat{a}^{(i;j)}| = |a^{(I;J)}||\hat{a}|^{k-1}$$

(ii) a) $|a(n-1:n-1)||a(n-2,n:n-2;n)| - |a(n-1:n-2,n)||a(n-2,n:n-1)|$

$$= |a(n-2:n-2)||a(n:n)|$$

b) $|a(2;n:2:n)||a(n-1:n-1)| - |a(n-1:2,n)||a(2;n:n-1)| =$

$$|a(2;n-1:2:n-1)||a(n:n)|$$

c) With a'_i $(i=1,\dots,n)$ a sequence of P-numbers, let $\{a'(n:1), a(n:2,n)\}$ be a P-array and denote it by $a'(n:n)$;

denote $\{a'(n:1), a(n:1;n-1)\}$ by $a''(n:n)$. Then

$$|a'(n:n)||a(n-1:n-1)| - |a(n:n)||a'(n-1:n-1)| = |a''(n:n)||a(n-2:2;n)|$$

Proof. ~~Clause (i) of the theorem~~ For ease in exposition, clause (i) of the theorem is first proved for the case in which $i(m) = j(m) = m$ $(m = 1, 2, \ldots, k)$, $I(m) = J(m) = k+m$ $(m = 1, \ldots, n-k)$

Proof It should first be remarked that $\hat{a}$ is a P-array: if $a \in PA\{n:n|r,c\}$, then $\hat{a} \in PA\{n:n|r',c'\}$ where

$$r'(i) = \sum_{m=1}^{n}{}^{[i]} r(m), \quad c'(j) = \sum_{m=1}^{n}{}^{[j]} c(m) \quad (i,j = 1,\ldots,n),$$

the superscript attached to the summation sign indicating that the term corresponding to the enclosed index is to be omitted.

With $1 \leq k \leq n-1$ let $a(n:n)$ be a P-array, ~~and~~ let $a^{(m)}(n:n-k)$ be the $n \times (n-k-1)$ array obtained ~~from $a(n:n-k)$~~ by deleting the m^{th} column from $a(n:n-k)$, $(m=1,\dots,n-k)$, and let $S^{(m)}$ be the compound $n \times (2n-2)$ array

$\{a^{[m]}(n:n-k), a(n:n-k+1;n), a^{[m+1]}(n:n-k), a(n:n-k+1;n)\}$

$(m=1,\dots,n-k-1)$ and $S^{(n-k)}$ be the $(n-k) \times (2n-1)$ array

$\{a(n-k+1;n:n-k-1), a(n-k+1;n:n-k+1;n)\}$. Construct the $((n-1)(n-k)) \times ((n-1)(n-k))$ P-array A by placing $S^{(m)}$ in the positions jointly occupied by rows $(m-1)n+1$ to mn and columns $(m-1)(n-1)+1$ to $(m+1)(n-1)$ for $(m=1,\dots,n-k-1)$ while $S^{(n-k)}$ ~~occup~~ is placed in the positions jointly occupied by the last $n-k$ rows and $n-1$ columns, all other elements of A being the zero members of their appropriate P-classes. Those elements of A that are not by definition zero occur in a stair-case descending from left to right. The last $n-1$ columns of the slab $S^{(m)}$ overlap the first $n-1$ columns of slab $S^{(m+1)}$, $(m=1,\dots,n^k-k-1)$, the last slab $S^{(n^k-k)}$ being a semi-slab of only $n-1$ columns. The slabs $S^{(n^k-k-1)}$ and $S^{(n^k-k)}$ abut the right edge of A. The overlapped part of slab $S^{(m+1)}$ is simply a copy of that part of $S^{(m)}$ which overlaps it $(m=1,\dots,n^k-k-1)$.

All minors formed from the k last ~~rows, that do not contain~~ ~~are~~ identically zero

The only minors that can be formed from the last k rows of A, ~~that do~~ not containing a column of elements that are by definition zero, ~~an identically zero column~~, are those taken from the last $n-1$ columns of A. These elements that are not defined to be zero of the ~~minors possess cofactors det~~ complementary arrays corresponding to these ~~cofactors~~ minors are also distributed in staircase form, ~~with its~~ composed of the $n-k-2$ slabs $S^{(m)}$ $(m=1,\dots,n-k-2)$ as described above, ~~as described~~ and a last slab obtained from $S^{(n-k-1)}$ by deleting the $n-k$ columns ~~with suffices~~ corresponding to those contained by the minor in question. The last $n-k-1$ columns of the complementary array contain elements not defined to be zero only in the last n rows; these elements belong to the reduced version of $S^{(n-k-1)}$.

Assume that a minor ~~of order~~ formed from the last $n-k$ rows of A does not contain one of the columns, c say, of $a(n-k+1;n:n-k+1;n)$; ~~this column~~ c is according

~~belono~~ is to be found in the reduced version of $S^{(n-k-1)}$ column $j(1)$ say,

and is unaccompanied in the same column, of the complementary array C taken from A, by nonzero elements. The columns $1,\ldots,n-1$ of $S^{(n-k-1)}_{k-1}$ contain a further copy of c ~~of the column in question~~, belonging to column $j(2)$ say, of C. Perform the operation ~~row $j(2)$ =~~ column $j(2)$ = column $j(2)$ − column $j(1)$ upon C. The $j(2)^{th}$ column of C now ~~only~~ contains elements that are not identically zero only in those rows ~~occupied~~ intersected by the slab $S^{(n-k-2)}$, & where c stands ~~the elements of~~ $S^{(n-k-2)}$ alone. ~~situated in the~~ $j(2)^{th}$ column of C ~~being~~ are those of c. ~~By subtracting~~ The $j(2)^{th}$ column of C is subtracted from that column, which contains the further copy of c belonging to $S^{(n-k-2)}_{k-2}$. This process is continued until all elements of one of the first $n-1$ columns of ~~belonging to~~ the first ~~block~~ slab $S^{(1)}$ have been reduced to zero. The remaining elements in this column of C are by definition zero, and ~~are~~ remain unchanged during the above transformations: a fissure has been introduced into the first slab in the staircase of nonzero elements of C ~~has been fractured through~~: ~~the value of $|C|$ is zero~~.

The value of $|C|$ is zero, and this is true of ~~any~~ the determinant formed from the sub-array in A complementary to any minor formed from the last k rows that does not contain all columns of $a(\overset{k+1}{n}-k+1;n:\overset{k+1}{n}-k+1;n)$. Only one such minor contains all required columns: it ~~is~~ belongs to ~~taken from~~ the last $n-k$ columns of A.[91] Let ~~C be~~ the ~~[illegible]~~ array complementary to this minor in A be C. Those elements of C that are not defined to be zero are distributed in staircase form, the last slab, contained in the last n rows and $2n-k-2$ columns being the reduced version of $S^{\overset{k-1}{(n-k-1)}}$,

$\{a^{\overset{k-1}{(n-k-1)}}(n:\overset{k}{n}-k), a(n:\overset{k+1}{n}-k+1;n), a(n:\overset{k-1}{n}-k-1)\}$. ~~Only one~~ From ~~minor~~ the last n rows of C only one minor can be taken that (a) does not possess a column of elements defined to be zero (b) does not possess a complementary array in C possessing a column of elements defined to be zero and (c) does not possess two columns defined to be ~~contain~~ identical ~~elements~~. This minor, ~~is~~ formed from the last n columns of the reduced version of $S^{\overset{k-1}{(n-k-1)}}$, is ~~$\{a(\overset{k}{n}:n-k;n), a(n:\overset{k}{n}-k-1)\}$~~, or $a(n:n)$ in rearranged form.

The complement of this array in C may be considered, and it may be shown that the last n rows of this array contain only one minor with properties (a–c), and that this minor is $\{a(n;k-1;k), a(n:k+1;n), a(n:k-2)\}$ or again $a(n:n)$ in rearranged form. ~~In conclusion,~~ (k.p.9) 20

the general Laplace expansion of $|A|$, involving minors taken successively from the first, second, …, $(n \overset{k-1}{-} k-1)^{th}$ groups of n rows, and also from the last k^{n-k} rows, contains only one term ~~not defined~~ that is not, by ~~implication~~ deduction from the definitions, identically zero; this term is the product of the determinants formed from columns $m, \dots, n+m-1$ of ~~block~~ the slab $S^{(m)}$ $(m = 1, \dots, n \overset{k-1}{-} k - 1)$ (each group of columns being $a(n:n)$ in rearranged form) and also $|a(n \overset{k-1}{-} k+1; n : n \overset{k-1}{-} k+1; n)|$

Before deriving a further expansion of $|A|$, A is first reduced by means of the operations

$$\text{row}(mn+i) = \text{row}(mn+i) - \text{row}((m+1)n+i) \quad \cancel{(m+1)n-k+i} \quad \cancel{(m+2)n-k+i}$$

for $i = 1, \dots, \cancel{k}$ $n \overset{k}{-} k+1, \dots, n$ where in succession $m = n \overset{k-1}{-} k - 1, n \overset{k-2}{-} k - 2, \dots, 1$. ~~The slab of $S^{(m)}$ now has zero elements~~ The block of elements contained in positions $n \overset{k}{-} k+1, \dots, n$ of the last $n-1$ columns of $S^{(m)}$ has been reduced to zero $(m = 1, \dots, n \overset{k-1}{-} k - 1)$. Denote the transformed version of A by A'. From the first $n-1$ columns of A' only $n-k$ minors can be taken that do not possess a column of elements defined to be zero and do not possess a complementary array ~~minor~~ in A' possessing a row of elements either defined to be or having been reduced to zero; these minors are formed from all but one row, $i(i)$ say, of $\{a^{[1]}(n-k:n-k), a(n-k:n-k+1;n)\}$ followed by $\{a^{[1]}(n-k+1;n:n-k), a(n-k+1;n:n-k+1;n)\}$. The first $n-1$ columns of the array complementary to one of the above $n-k$ arrays, $A^{(i(i))}$ say, in A' contain ~~the arrays~~ in succession ~~all~~ the row $a^{(i(i))} = \{a^{(2)}(i(i);i(i):n-k), a(i(i);i(i):n-k+1;n)\}$, and the array ~~$\{a^{(2)}(n;n-k$~~ $\{a^{[2]}(n-k:n-k), a(n-k:n-k+1;n\}$ and further zero elements. From this ~~rows containing~~ above row and array pair, only the ~~$n-k-1$~~ minors can ~~be formed that~~ do not have two rows defined to be ~~contain equal elem~~ identical and do not possess a complementary array in $A^{(i(i))}$ containing a row of zero elements; ~~these minors~~ these are formed from $a^{(i(i))}$ ~~$\{a^{(2)}(i;i:n-k), a(i;i;n-k+1;n)\}$~~ all but one row (row $i(2)$ say) of the array obtained by removing the ith row, $a^{(i(i))}$, from $\{a^{[2]}(n-k:n-k), a(n-k:n-k+1;n)\}$, and $\overset{[2]}{a}\{n-k+1; n : n-k), a(n-k+1; n : n-k+1; n+1)\}$; these minors are $n-k-1$ in number. By repetition of the above argument, it is shown that each chain of minors extracted successively from the first, second, ... groups of $n-1$ columns of A, that do not contain a ~~a minor an identically zero~~ member ~~minor~~ whose determinant is identically zero, is composed of members extracted from sets of successively $n-k, n-k-1, \ldots, 1$ members; the minors in question are those ~~occurring in the array $\{A_{i,j}\}$ adjugate array formed from $a(n-k : n-k)$, which is the complementary array complementary to $a(n-k+1; n : n-k+1; n)$ in $a(n:n)$~~ found in rows i and columns j ~~occurring in positions~~ $(i, j = 1, \ldots, n-k)$ of ~~the array~~ $\overset{k}{\mathrm{adj}}(a)$ ~~adjugate to $a(n:n)$~~. The value of $|A|$ obtained by use of a general Laplace expansion ~~taken~~ based upon minors taken from successive groups of $n-1$ columns is the value of the determinant of the array formed from the first n^k rows and columns of ~~the array~~ $\mathrm{adj}^{(a)}$ ~~adjugate to $a(n:n)$~~.

Two ~~equivalent~~ expressions for $|a|$ have now been found: ~~$|a(n-k+1; n : n-k+1; n)||a|^{n-k-1}$ and~~

the first involves $|a(n-k+1; n : n-k+1; n)|$ and a power of $|a|$, and the second, the value of a minor of $\widehat{adj}(a)$; these expressions can ~~to~~ accordingly be equated. ~~A~~ To obtain a more general result ~~is obtained by replacing~~ let letting $I(1), \ldots, I(k)$ ~~; $J(1), \ldots, J(k)$~~ be ~~two~~ an set increasing sequence ~~sequences~~ of distinct integers taken from the set of $1, \ldots, n$, and letting $i(1), \ldots, i(n-k)$ to be the complementary increasing subsequence ~~subset~~, and defining the sets $J(1), \ldots, J(k)$, $j(1), \ldots, j(n-k)$ similarly; let ~~$|a|_{I,J}$~~ ~~$a_{I,J}$~~ $a^{(I,J)}$ be the array formed from the elements lying at the intersections of ~~the $I(m)^{th}$ row and $J(m)^{th}$ column of a (m = 1, ...~~ rows $I(1), \ldots, I(k)$ and columns $J(1), \ldots, J(k)$ of a; and define $adj(a)^{(i,j)}$ $\hat{a}^{(i,j)}$ similarly. Then

$$\cancel{|a^{(I,J)}| = +}$$

$$|adj(a)^{(i,j)}| = |a|^{n-k-1} |a^{(I,J)}| \qquad |\hat{a}^{(i,j)}|$$

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Continuing the process still further, it is shown that only <20|
one minor significant minor can be extracted from the first n rows; it is $\{a(n:2;k), a(n:k+1;n), a(n:1)\}$, again $a(n;n)$ in rearranged form. The significant minors just described ~~or~~ ~~jointly~~ occupy columns and rows $(m-1)n+1$ to mn of A $(m=1,..,k-1)$. Accordingly, the general Laplace expansion of $|A|$ involving minors taken successively from the first, second,... $(k-1)^{th}$ groups of n rows, and also from the last $n-k$ rows, contains only one term that is not, by deduction from the definitions, identically zero; it is

$$\left\{\prod_{m=1}^{k-1} |a(n:m+1;n), a(n:m)|\right\} |a(k+1;n:k+1;n)| =$$

$$(-1)^{\frac{k(k-1)(2k+3n+5)}{6}} |a|^{k-1} |a(k+1;n:k+1;n)|$$

$k, k-1, \dots, 1$ members; the totality ~~the determinants of the minors in question~~ the minors to be found in all chains together form ~~make up~~ the set $a^{[i:j]}(n:n)$ $(i,j=1,\dots,k)$. The general Laplace expansion of $|A|$ based upon minors taken from successive groups of $n-1$ columns has the form

$$\sum \pm |a^{[i(1):1]}(n:n)||a^{[i(2):2]}(n:n)|\dots|a^{[i(k):k]}(n:n)|$$

The sign attached to each term in this sum ~~has~~ must ~~to~~ be determined, and the sum itself must be compared with the simple Laplace expansion of $|\hat{a}|$. The sign attached to each term of a simple Laplace expansion of $|a|$ of the form ( ) may be determined from the sign change experienced by $|a|$ when ~~row $i(1)$ is~~ a is subjected to the row transposition row $i(1)$ into position 1, row $i(2)$ into position 2, ... This principle can be extended to general Laplace expansions. To begin with, the sign attached to the term in expression ( ) for which $i(m)=m$ $(m=1,\dots,k)$ will be determined. By moving rows $2,\dots,n$

into positions $1,\dots,n-1$ and row 1 into position n, the minor $a^{[1:1]}(n:n)$ is brought into the leading position in A'; a sign change of $(-1)^{1(n-1)}$ is thereby induced in $|A'|$; row n now contains the elements $a_{1,1}, a_{1,3},\dots, a_{1,n}$ in the positions intersected by columns $n,\dots,2n-2$. By moving rows $n+3,\dots,2n$ are ~~now moved~~ into positions $n+1,\dots,2n-2$ and rows ~~$2n+$~~ $n+1, n+2$ into positions $n+1,\dots,2n-2$, the minor $a^{[2:2]}(n:n)$ is brought into ~~the~~ rows and columns $n,\dots,2n-2$ of A'; a further sign change of $(-1)^{2(n-2)}$ is thereby induced in $|A'|$; rows $2n-2, 2n$ now contains the elements $a_{i,1}, a_{i,2}, a_{i,4},\dots, a_{i,n}$ in the positions intersected by columns $2n-1,\dots,3n-2$ for $i=1,2$ respectively. $k-1$ stages of this process are carried out, after which the last $n-1$ rows and columns contain $\{a^{[:k]}(k-1:k), a(k-1:k+1;n)\}$ bounded from below by $\{a(k+1;n:k-1), a(k+1;n:k+1;n)\}$, or simply $a^{[k:k]}(n:n)$. The total sign change experienced by $|A'|$ in bringing the minors $a^{[m:m]}(n:n)$ into direct diagonal order is $\prod_{m=1}^{k-1}(-1)^{m(n-m)}$. The sign attached to the term

$$\prod_{m=1}^{k} a^{[k:k]}(n:n)$$

in the Laplace expansion of $|\hat{a}|$ is unity; that attached to this term in the ~~expansion~~ sum $\phi'(\ )$ for $|A'|$ is given by expression ( ). It may be shown by the use similar methods that a corresponding conclusion may be reached with regard to all terms in ~~this cone expansion ( ) is simply~~ the Laplace expansion of $|\hat{a}|$, ~~multiplied th~~ and the sum ( )

In short

$$|A| = \left\{\prod_{m=1}^{k-1} (-1)^{m(n-m)}\right\} |\hat{a}|$$

Comparison of formulae (,) reveals that relationship ( ) holds for the special case in which $i(m) = j(m) = m$ $(m=1,..,k)$

$\tilde{i}(m) = \tilde{j}(m) = k+m$ $(m=1,..,n-k)$

The above special result is applied to the array $\tilde{a}^*(n:n)$, obtained simply by interchanging rows k and $k+1$ of $a(n:n)$. It is possible to present the derived result in terms of a and $\hat{a}$. Firstly $|\tilde{a}^*(n:n)| = -|a(n:n)|$. Secondly the arrays $a^{[k:k]}$ ~~to corresponding to the~~

~~of the form $a^{[i;j]}(n:n)$ derived from $a(n:n)$,~~ $\tilde{a}^{[i:j]}(n:n)$ and $a^{[i;j]}(n:n)$ are related: $\tilde{a}^{[i;j]}(n:n) = -a^{[i;j]}(n:n)$ for $\{i = 1,\dots,k-1,k+2,\dots,n\}$ and $\tilde{a}^{[k+m;j]}(n:n) = a^{[k+1-m;j]}(n,n)$ for $m=0,1$, all for $j=1,\dots,n$. The derived result presented in terms of a and $\hat{a}$, is simply relationship ( ) multiplied on both sides by $(-1)^{k-1}$: relationship ( ) has now been proved for the special case in which $i(m) = m$ $(m=1,\dots,k-1)$, $i(k) = k+1$, $I(1) = k$, $I(m) = k+m$ $(m=2,\dots,n-k)$. It is proved in the same way for the case in which $i(k)$ takes any position in the sequence $k,\dots,n$, and also for the cases in which the $\{i(m)\}$ is any increasing sequence taken from the numbers $1,\dots,n$; the column indices $j(m)$, $J(m)$ can be treated in the same way; clause (i) of the theorem has been demonstrated.

The result of clause (ii a) is simply that of clause (i) with $k=2$, $i(1)=j(1)=n-1$, $i(2)=j(2)=n$; that of clause (iib) is ~~obtained~~ the version obtained by setting $k=2$, $i(1)=j(1)=1$, $i(2)=j(2)=n$. To obtain the result of clause (iic), set in turn ~~$i(1)=$~~ $k=2$ and $i(1)=m$ $(m=1,\dots,n-1)$, $i(2)=n$, $j(1)=1$, $j(2)=r$ in clause (i), thus obtaining the relationships

$$|a^{[m:]}(n:2;n)||a(n-1;n-1)|-|a(n:n)||a^{[m:]}(n-1:2;n-1)|$$
$$=|a^{[m:]}(n:n-1)||a(n-1:2;n)|$$

for $m=1,\dots,n-1$, to which may be appended the identity

$$|a(n-1:2;n)||a(n-1:n-1)|=|a(n-1:n-1)||a(n-1;2;n)|$$

Multiply ~~relationships~~ formulae (,) throughout by $(-1)^{m-1}a'_m$ ~~add~~ $(-1)^{n-1}a'_n$, respectively and add all results obtained. The result is ~~formul~~ relationship ( ).

Naturally the special results of clause (ii) of the above theorem can be derived independently by the use of simpler methods. The $(2n-2)^{th}$ order determinant

$$\begin{vmatrix} a(n:n), a(n;n-2) \\ 0(n-2:n), a(n-2;n-2) \end{vmatrix}$$

where $0(n-2:n)$ is a suitable P-array ~~ent~~ consisting of zero elements has, in particular, two general Laplace expansions. The first, by minors derived from the first $n-1$ columns and cofactors derived from the last $n-1$, contains only two terms not involving determinants which either have ~~identical rows~~ ~~of~~ zero rows or two identical rows. The second, by minors derived from the first $n-1$ rows and cofactors derived from the last $n-2$ has only one such term. Formula ( ) is obtained by equating the two expansions. Formula ( ) is obtained similarly ~~by~~ from two expansions of

$$\begin{vmatrix} a(n:n), & a(n:2;n-1) \\ O(n-2:n) & a(n-2:2;n-1) \end{vmatrix}$$

~~All minors of~~ Using the conventions adopted in clause (iic), and taking $O(n-2:n)$ to be a ~~z~~ suitable zero P-array, it is easily seen that all minors constructed from the first n rows of the determinant obtained by subtracting the last $n-1$ rows of the $(2n-1)^{th}$ order determinant

$$\begin{vmatrix} a'(n:n-1), & a(n:n-1) \\ O(n-1:n-1), & a(n-1:n-1) \end{vmatrix}$$

from the first $n-1$ have zero values; the value of this determinant is accordingly zero. Its Laplace expansion by minors taken from the first n rows has only three terms that are not identically zero. They occur in relationship ( ).

The result of clause (i) of the above theorem may be presented in terms of the adjugate of $|a|$, i.e. the determinant ~~whose element~~ with element $|a^{[j:i]}(n:n)|$ in the i^{th} row and j^{th} column $(i,j=1,\dots,n)$. Clause (i) is accordingly Jacobi's theorem on the adjugate for P-determinants. ~~The result of~~ Clause (iia) is an

(bp 28)

c) With a_i' $(i=1,\dots,n)$ a sequence of P-numbers, let $\{a'(n:1), a(n:2;n)\}$ be a P-array, and denote it by $a'(n:n)$; denote ~~of~~ $\{a'(n:1), a(n:1;n-1)\}$ by $a''(n:n)$. Then

$$|a'(n:n)||a(n-1:n-1)| - |a(n:n)||a'(n-1:n-1)| =$$
$$|a''(n:n)||a(n-1:2;n)|$$

---

extension of a result due to Sylvester, and clauses (iib,c) are versions of results due to Schweins for P-determinants (for references, see [ ]). The method of proof adopted above shows clearly that Sylvester's result and Schweins' first result are special cases of Jacobi's theorem on the adjugate, and that Schweins' second result is a polarised version of that theorem.

~~(ii)~~ (i) Let $a(n,n)$ be a P-array, ~~and~~ $\hat{a}$ be ~~its~~ the adjugate of a. 28

~~Let~~ With $1 \leq k \leq n$ let $i(1),\dots,j(k)$, $i(1),\dots,i(n-k)$ be ~~a sequence of~~ an increasing sequence of distinct integers taken from the set ~~for the range~~ $1,\dots,n$, and let $i(1),\dots,$ ~~together making up the set $1,\dots,n$;~~ $i(n-k)$ be ~~the~~ a complementary ~~set~~ sequence, define ~~and take $I(1),\dots,I(n-k)$ to be the complementary increasing subsequences;~~ the sets sequences $J(1),\dots,J(k)$; $J(1),\dots,J(n-k)$ similarly. Let $a^{(i,j)}$ be the array formed from the elements lying at the intersections of rows $i(1),\dots,i(k)$ and columns $j(1)\dots j(k)$ taken in order of a, and define $\hat{a}^{(I,J)}$ similarly.

Then

$$|\hat{a}^{(i,j)}| = \cancel{|a|^{k-1}}\,|a^{(I,J)}|\,|a|^{k-1}$$

(ii) a) ~~Let a(n~~

$$|a(n-1;n-1)|\,|a(n-2,n:n-2,n)| - |a(n-1:n-2,n)|\,|a(n-2,n:n-1)|$$

$$= |a(n-2:n-2)|\,|a(n;n)|$$

b) ~~$|a(2;n:2;n)|\,|a(n-1:n-1)| - |a(n-1:n-1)|\,|a(2;n-1:2;n-1)| = |a(n-1:2;n)|\,|a(2;n:n-1)|$~~

$$|a(2;n:2;n)|\,|a(n-1:n-1)| - |a(n-1:2;n)|\,|a(2;n:n-1)|$$

$$= a(2;n-1:2;n-1)|\,|a(n:n)|$$

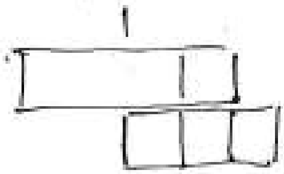
reduced version of

C: slab $n-k-1$ contains

$\{a^{(n-k-1)}(n:n-k), a(n:n-k+1;n), a(n:n-k-1)\}$

minor taken from this slab must contain

$a(n:n-k-1)$ + $k+1$ further columns differing from those of $a(n;n-k-1)$



$1, 2 \ldots, n-k-2, \overbrace{n-k, n-k+1, \ldots, n}^{k+1}$

C′: cofactor of $\{a(n:n-k;n), a(n:n-k-1)\}$ in C

last n rows of C′ contain reduced version of slab $n-k-2$

$a^{(n-k-2)}(n;n-k), a(n:n-k+1;n), a($

first $n-k-2$ columns of $a^{(n-k-1)}(n:n-k)$

cols $n-k-1, \ldots, 2n-k-2$ of $S^{(n-k-1)}$

~~n-k~~ cols $1, \ldots, n-1$

$n-k-1+n-\cancel{n+k}\ldots$

$n-1+n-k-1$

$$\begin{array}{ccc} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{array} \qquad \begin{array}{ccc} a_{2,2}a_{3,3}-a_{2,3}a_{3,2} & a_{2,1}a_{3,3}-a_{2,3}a_{3,1} & a_{2,1}a_{3,2}-a_{2,2}a_{3,1} \\ a_{1,2}a_{3,3}-a_{1,3}a_{3,2} & a_{1,1}a_{3,3}-a_{1,3}a_{3,1} & a_{1,1}a_{3,2}-a_{1,2}a_{3,1} \\ a_{1,2}a_{2,3}-a_{1,3}a_{2,2} & a_{1,1}a_{2,3}-a_{1,3}a_{2,1} & a_{1,1}a_{2,2}-a_{1,2}a_{2,1} \end{array}$$

$$\begin{array}{ccc} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{array} \qquad \begin{array}{ccc} a_{2,2}a_{3,3}-a_{2,3}a_{3,2} & a_{1,3}a_{3,2}-a_{1,2}a_{3,3} & a_{1,2}a_{2,3}-a_{1,3}a_{2,2} \\ a_{2,3}a_{3,1}-a_{2,1}a_{3,3} & a_{1,1}a_{3,3}-a_{1,3}a_{3,1} & a_{1,3}a_{2,1}-a_{1,1}a_{2,3} \\ a_{2,1}a_{3,2}-a_{2,2}a_{3,1} & a_{1,2}a_{3,1}-a_{1,1}a_{3,2} & a_{1,1}a_{2,2}-a_{1,2}a_{2,1} \end{array}$$

$$(a_{2,1}a_{3,3}-a_{2,3}a_{3,1})(a_{1,1}a_{2,2}-a_{1,2}a_{2,1})-(a_{2,1}a_{3,2}-a_{2,2}a_{3,1})(a_{1,1}a_{2,3}-a_{1,3}\ldots$$

term ind. of $a_{2,1}$:

$$-a_{2,3}a_{3,1}a_{1,1}a_{2,2}+a_{2,2}a_{3,1}a_{1,1}a_{2,3}=0$$

$$(a_{1,3}a_{3,2}-a_{1,2}a_{3,3})(a_{1,1}a_{2,2}-a_{1,2}a_{2,1})-(a_{1,2}a_{2,3}-a_{1,3}a_{2,2})(a_{1,2}a_{3,1}-a_{1,1}a_{3,2})$$

1,3 $\cancel{3,2}$ 1,1 2,2 − 1,2 3,3 1,1 2,2 − 1,2 2,3 1,2 3,1 + 1,3 2,2 1,2 3,

+ 1,2 2,3 1,1 3,2 − 1,3 $\cancel{2,2}$ 1,1 3,2

— o —

$$\begin{array}{ccc} -A_{1,1} & -A_{1,2} & -A_{1,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \end{array}$$

$$a_{2,1}\{a_{1,1}a_{2,2}a_{3,3}-a_{1,2}a_{2,1}a_{3,3}+a_{1,2}a_{2,3}a_{3,1}$$
$$-a_{1,1}a_{2,3}a_{3,2}+a_{1,3}a_{2,1}a_{3,2}-a_{1,3}a_{2,2}a_{3,1}\}$$

$$(a_{3,1}a_{2,3}-a_{3,3}a_{2,1})(a_{1,1}a_{3,2}-a_{1,2}a_{3,1})-(a_{3,1}a_{2,2}-a_{3,2}a_{2,1})(a_{1,1}a_{3,3}-a_{1,3}a_{3,1})$$

$$(a_{2,1}a_{3,3}-a_{2,3}a_{3,1})(a_{1,1}a_{3,2}-a_{1,2}a_{3,1})-(a_{2,1}a_{3,2}-a_{2,2}a_{3,1})(a_{1,1}a_{3,3}-a_{1,3}a_{3,1})$$

$1 \leftrightarrow 2$

$$\begin{array}{cccc} A_{2,1} & a_{3,2}a_{2,3}-a_{3,3}a_{2,2} & a_{3,1}a_{2,3}-a_{3,3}a_{2,1} & a_{3,1}a_{2,2}-a_{3,2}a_{2,1} \\ A_{1,1} & a_{1,2}a_{2,3}-a_{1,3}a_{2,2} & a_{1,1}a_{2,3}-a_{1,3}a_{2,1} & a_{1,1}a_{2,2}-a_{1,2}a_{2,1} \\ -A_{3,1} & a_{1,2}a_{3,3}-a_{1,3}a_{3,2} & a_{1,1}a_{3,3}-a_{1,3}a_{3,1} & a_{1,1}a_{3,2}-a_{2,2}\ldots \end{array}$$

$\{a^{[1]}(n:k), a(n:k+1;n), a^{(2)}(n:k), a(n:k+1;n)\}$ $\quad k+1, \dots, n$

$\{a^{[2]}(n:k); a(n:k+1;n)\}, a^{[3]}(n:k), a(n:k+1;n)\}$

$\{a^{(k-2)}(n:k), a(n:k+1;n); a^{(k-1)}(n:k), a(n:k+1;n)\}$

$a^{[i(1):1]}(n:n)$

$\{a^{[k-1]}(n:k), a(n:k+1;n), a^{[k]}(n:k), a(n:k+1;n)\}$

$\{\cancel{a^{[k]}(n:k), a(n:k+1;n)}\}$

$\{a(n:k;k), a(n,k+1;n), a(n:k-1)\}$

$\{a(k+1;n:k-1), a(k+1;n:k+1;$

$a^{[-2]}(i(1),1;n:n)$

$\cancel{a}(\boxed{2}:$

$a^{(k-2)}(n:k), a(n:k+1;n); a^{(k-1)}(n:k)$

$a(n:k;n), a(n:k-1)$

$a^{(k-2)}(n:k), a(n:k+1;n), a^{(k-1)}(n:k), a(n:k+1;n)$

$\{a(n:2;k), a(n:k+1;n), a(n:1)\}$

$\vdots$

$a^{(k-2)}(n:k), a(n:k+1;n), a(n:k-2)$

$a(n:k-1;k), a(n:k+1;n), a(n:k-2)$

$(1,2)\dots(i(1)-1, i(1))(i(1)+1, i(1)+1)\dots(n,n) \quad (i(1),1)\ (1, i$

1 Am Ex TC 100 $\qquad$ RD042 − 148 − 430 } 45
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$(n-m)km$

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$\cancel{i(1), n+1}\ (i(1),1)(1,2),\dots,(i(1)-1,i(1))(i(1)+1,i(1)+1),$

$n\,(i(1),n)(n+1,n+1)\dots(n+i(1)-1,n+i(1)-1)\ (n+i(1)+1,n+i(1))$

$\dots(n+i(2)-1,n+i(2)-2)(n+i(2)+1,n+i(2)-1),\dots(2n,2n-2)$

$(1,1)\dots(i(1)-1,i(1)-1),(i(1)+1,i(1))\dots(n,n-1) \qquad (-1)^{n-i(1)}$

$=$

$(1,2)\dots(i(1)-1,i(1))(i(1)+1,i(1)+1)\dots(n,n)$

$(i(1),1)(1,3),(2,4),\dots(i(1)-1,i(1)+1)(i(1)+1,i(1)+2),\dots$

$(i(2)-1,i(2))(i(2)+1,i(2)+1),\dots,(n,n)$

rows $2,\dots,n$ into positions $1,\dots,n-1$ $\quad (-1)^{(n-1)k}$ $\quad n \left\{ \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} \right.$

$a^{(1)}$ ~~now~~ in rows n, cols $n,\dots,2n-2$ now contain $a_{1,1}, a_{1,3},\dots,a_{1,n}$

rows $n+3,\dots,2n$ into positions $n+1,\dots,2n-2$ $\quad (-1)^{2(n-2)}$
rows $n+1, n+2$ into positions $2n-1, 2n$
rows $2n-1, 2n$ cols ~~$n,\dots,2n-2$~~ $2n-1,\dots,3n-2$ now contain $a_{i,1}, a_{i,2}, a_{i,4}\dots a_{i,n}$ for $i=1,2$ respectively

$a_{2,1}, a_{2,2}$ bded from below by $a(k-1;n:k-1), a(k+1;n:$

$k-1$ stages of this process are carried out
last $n-1$ rows contain $a^{[k]}(k-1:k), a(k-1:k+1,n))$ (C ways)

1,1 1,4 1,5 1,1 1,2 1,3

5,1 5,4 5,5 5,1 5.2 5.3

1 5 1.1 1.3 1.4 1.5 1.1 1.2

5.5 5.1 5.3 5.4 5.5, 5.1 5.2

1.4 1.5 1.1 1.2 1.3 1.~~4~~ 1.5 1.1

$a_{1,1}$ $a_{1,k}$

$a_{k,1}$ $a_{k,k}$ $A_{n,n}$ $A_{n,k+1}$ $A_{n-1,n}$ $A_{k+1,n}$ $A_{k+1,k+1}$

5.4 5.5 5.1 5.2 5.3 5.~~4~~ 5.5 5.1

1.3 1.4 1.5 1.1

(1 2 ... n-1 | 1 ... n-1) ... (1.. , n-1 | 1.. k, k+1,.. n)

(1 2 .. n-2, n | 1 ... n-1)

(1 2 .., k, k+2, ... n | 1 .. n-1) .. (1 .. k, k+2, .. n | 1 .. k, k+2, .. n)

$y + n-(n-k+1)+1 = n-k$

1,1 1,n-k 1,n-k+1 .. 1,n 1,n-2k

n,1 n,n-k n,n-k+1 .. n,n n,n-2k

1,n-k+1 1,n 1,n-2k 1,2n-3k

n,n-k+1 n,n n,n-2k n,2n-3k

1,n-2k+1 .. 1,2n-3k 1,3n-4k

n,n-2+1 ... n,2n-3k

$a(-(m-1)k+1; n$

$\{a(n:[-(m-1)k+1];n), a(n:[-(m+1)k])\}$

in rows $(m-1)n+1, \dots, mn$ columns $(m-1)(n-k)+1, \dots, m(n-k)$

$m = 1, \dots, n-k-1$

1 $\begin{vmatrix} a(n-1;n-1) & 0(n-1;n+1) \\ a(n+1:n-1) & a(n+1:n+1) \end{vmatrix}$

$n=2$

39 $(1,2,3|1,2,3)\ (1,2,3|1,2,4)\ (1,2,3|1,3,4)$ 40
$(1,2,4|1,2,3)\ (1,2,4|1,2,4)\ (1,2,4|1,3,4)$
$(1,3,4|1,2,3)\ (1,3,4|1,2,4)\ (1,3,4|1,3,4)$

$$\begin{matrix} a_{1,1} & 0 & 0 & 0 \\ a_{1,1} & a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a & & \end{matrix}$$

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,1} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{1,2} \\ a_{1,3} & a_{3,2} & a_{3,3} & a_{3,3} \\ 0 & 0 & a_{1,3} & a_{1,1} \end{matrix} \qquad \begin{matrix} A_{5,5} & A_{5,4} & A_{5,3} & A_{5,2} \\ A_{4,5} & A_{4,4} & A_{4,3} & A_{4,2} \\ A_{3,5} & A_{3,4} & A_{3,3} & A_{3,2} \\ A_{2,5} & A_{2,4} & A_{2,3} & A_{2,2} \end{matrix}$$

$$\begin{matrix} a_{1,1} & 0 & 0 & a_{1,3} \\ a_{1,1} & a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,1} & a_{3,2} & a_{3,3} \end{matrix}$$

$$\begin{matrix} a(n-1;n-1) & 0(n-1:n) & a(n-1:n+1;n+1) \\ a(n+1;n-1) & a(n+1:n) & a(n+1:n+1;n+1) \end{matrix}$$

$$a_{1,1}|A|^2 = \begin{vmatrix} A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,2} & A_{4,3} & A_{4,4} \end{vmatrix} = \begin{vmatrix} A_{4,4} & A_{3,4} & A_{2,4} \\ A_{4,3} & A_{3,3} & A_{2,3} \\ A_{4,2} & A_{3,2} & A_{2,2} \end{vmatrix}$$

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,1} & a_{4,2} \end{matrix} \qquad \begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,1} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,1} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,1} \\ 0 & 0 & a_{1,3} & a_{1,1} \end{matrix}$$

$$\begin{matrix} a_{1,4} & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,1} \\ a_{2,4} & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,1} \\ a_{3,4} & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,1} \\ a_{4,4} & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,1} \\ & & & a_{4,3} & a_{4,4} & a_{4,1} \end{matrix}$$

8

$(1\,2\,3\,4|1\,2\,3\,4)\ (1\,2\,3\,4|1\,2\,3\,5)\ (1\,2\,3\,4|1\,2\,4\,5)\ (1\,2\,3\,4|1\,3\,4\,5)$
$(1\,2\,3\,5|1\,2\,3\,4)\ (1\,2\,3\,5|1\,2\,3\,5)\ (1\,2\,3\,5|1\,2\,4\,5)\ (1\,2\,3\,5|1\,3\,4\,5)$
$(1\,2\,4\,5|1\,2\,3\,4)\ (1\,2\,4\,5|1\,2\,3\,5)\ (1\,2\,4\,5|1\,2\,4\,5)\ (1\,2\,4\,5|1\,3\,4\,5)$
$(1\,3\,4\,5|1\,2\,3\,4)\ (1\,3\,4\,5|1\,2\,3\,5)\ (1\,3\,4\,5|1\,2\,4\,5)\ (1\,3\,4\,5|1\,3\,4\,5)$

the ~~integer sequence~~ set of integers $((-(m-1)k) \bmod(n))+1, \ldots, n, 1, 2, \ldots,$ $2n-2k$

$((-(m+1)k-1) \bmod(n))+1$ contains ~~two copies of~~ the set $1,\ldots,k$ and

$n-k-1$ ~~distinct~~ and all but one member of the

set $k+1,\ldots,n$ The set of $n-k$ integers

$((-(m-1))k \bmod(n))+1, \ldots, (-mk^{-1} \bmod(n))+1$

contains the set $1,\ldots,k$ and all but one member of

the set $k+1,\ldots,n$

wrong

$k=2$ $n=10$

$m=1,\ldots,n-k-1$

$1 \ldots n-k$

$1 \ldots 8$

9 10 $9,10,1,\ldots,6$

17 4

2

2 10

8

$\{a^{(m+1)}(n;n-k), a(n,n-k+1;n)\}$

in $(m)^{th}$ slab ~~lies underneath~~ overlaps same block in $(m+1)^{th}$ block

m^{th} slab: $\{a^{(m)}(n:n-k), a(n:n-k+1;n)\}, a^{(m+1)}(n:n-k), a(n;n-k+1;n)\}$

$2\{n-k-1+\cancel{n}-\cancel{n}+k-\cancel{1}+\cancel{1}\} = 2(n-1)$

$m=1,\ldots,n-k-1$ $(n-k)^{th}$ $\frac{1}{2}$ semi-slab, $\{a^{(n-k)}(\underset{n-k+1;n}{k}:n-k), a(\underset{n-k+1;n}{k};n-k+1;n)\}$

alone

rows $(n-k)n$; cols $(n-k)\{n-k-1+\cancel{n}-\cancel{n}+k-\cancel{1}+\cancel{1}\}$

$n(n-k-1)+k$ $(n-k)n-n+k = (n-k)(n-1)$

$n-k-1+k = n-1$

$\sum_{k=i}^{j} (-1)^k |a(i:i-1,k)|\,|a^{[:k]}(j-1:j)|$ is expansion according to elements from ~~first~~ last row and cofactors from ~~remaining~~ first $i \pm j-1$ of determinant $|\tilde{A}^j|$ say having elements ($|a(i:i-1,k)|$ $k=1,\dots,j$ in ~~first~~ last row) ~~act~~ $a_{i,k}$ in k^{th} position of ~~$(i+1)^{th}$~~ i^{th} row ($i=1,\dots,j-1$ $k=1,\dots,j$)

Let c_k be cofactor of $a_{k,j}$ in ~~last column~~ expansion by elements from last column of $|A'|$.

Perform the operation $\text{row}(j) = \text{row}(j) - \sum_{k=1}^{j-1} c_k \,\text{row}(k)$ upon $|A'|$. ~~This~~ reduces all elements in the last row to zero without changing the value of $|A'|$.

(the first ~~k~~ $i-1$ elements in the last row have zero values)

$\hat{l\pi}\,\tilde{A} = \hat{V}$ $\quad \hat{V}_{i,j} = 0 \quad i<j$

$$\hat{V}_{1,1} = -a_{1,1}\left\{ |a(n-1:n-1)| \right\} \prod_{m=1}^{n-1} |a(m:m)|$$

$$\hat{V}_{i,i} = \prod_{m=1}^{i-1} |a(m:m)|\,(-1)^i a_{1,1}\,|a(n-1:n-1)| \prod_{m=i}^{n-1} |a(m:m)|$$

$$= (-1)^i a_{1,1}\,|a(n-1:n-1)| \prod_{m=1}^{n} |a(m:m)|$$

P-matrices

Definition. $PM\{m:n|r:c\}$ is the class of matrices whose elements ~~are~~ may be displayed in a array of the form $PM\{m:n|r:c\}$; a matrix belonging to such a class is called a P-matrix. the left <right> multiplicate class of $(a) \in PM\{m:n|r:c\}$, $PL(a)$ $\langle PR(a) \rangle$, is the aggregate of classes $PM\{n:u|-c:c'\}$ $\langle PM\{u:m|r':-r\} \rangle$ for $u = 1, 2, \ldots$ and all integer sequences $c'(k)$ $\langle r'(k) \rangle$ $(k = 1, \ldots, u)$

Interior brackets are omitted from ~~the representations of compound and product matrix~~ composite representations: the compound and product matrices $((a), (b))$ and $(a)(b)$ respectively, are denoted by (a, b) and (ab).

The sub-matrices obtained from a given P-matrix by row and column deletion are P-matrices

Two P-matrices $(a) \in PM\{m:n|r:c\}$, $(b) \in PM\{m':n'|r':c'\}$

can ~~form a~~ be compounded in the form (a,b) if and only if $m^* = m'$ and $r \equiv r'$; a similar remark concerns juxtaposition within common columns. The transpose, reciprocal and adjugate of a $\mathbb{P}$-matrix (the elements of each of which are as defined for the corresponding $\mathbb{P}$-array) are $\mathbb{P}$-matrices. ~~$\mathbb{P}$-matrices may be belonging to the same class may be added and subtracted with impunity.~~ ~~When~~ With $d_k \in \mathbb{P}\{d(k)\}$ $(k=1,\dots,n>1)$ a prescribed sequence of $\mathbb{P}$-numbers, there exist unboundedly many classes of the form $\mathcal{PM}\{n:n|r:c\}$ that contain a diagonal matrix whose successive diagonal elements are $\{d_k\}$ and whose remaining elements are the zero members of appropriate $\mathbb{P}$-classes (the classes are constructed by assigning the sequence $r(k)$ $(k=1,\dots,n)$ ~~can be assigned~~ arbitrarily, and taking $c(k)=d(k)-r(k)$ $(k=1,\dots,n)$). $\mathbb{P}$-matrices belonging to the same class may be added and subtracted with impunity. The $\mathbb{P}$-matrix (b) may be

~~pre-multiplie~~ pre-⟨post-⟩ multiplied by the P. matrix (a) if and only if $b \in PL(a)$ ⟨$b \in PR(a)$⟩. With $(a) \in PM\{m:n|r:c\}$, $(b) \in PM\{n:u|-c:c'\}$, $(ab) \in PM\{m:u|r:c'\}$; $(b) \in PL(a)$ if and only if $(a) \in PR(b)$. ~~If~~ Furthermore, if (a) is partitioned by ~~rows~~ columns in the form $(a) = (a(m:j), a(m:j+1;n))$ $(1 \le j < n)$ and (b) is conformably partitioned by ~~columns~~ rows, $(b) = \begin{pmatrix} b(j:u) \\ b(j+1;n:u) \end{pmatrix}$ $(b(j:u))$ preceding $(b(j+1;n:u))$, then $(b(j:u)) \in PL\{a(m:j)\}$, $(b(j+1;n:m)) \in PL\{a(m:j+1;n)\}$. If $(a), (a')$ can be compounded in the form (a, a'),

and

If $(b) \in PL(a)$, $(b)^T$ ~~$\in PR\{(a)^T\}$~~ belongs to the right multiplicative class of $(a)^T$. If $(b) \in PL(a)$, ~~and~~ (a) ~~is~~ is partitioned in the form (a', a'') and (b) in the form $\begin{pmatrix} b' \\ b'' \end{pmatrix}$, b' having as many rows as (a') has columns, then $(b') \in PL(a')$, $(b'') \in PL(a'')$;

conversely, if $(b')\in PL(a')$, $(b'')\in PL(a'')$, and $(a')(a'')$ and $(b'),(b'')$ can be compounded ~~in the forms~~ a indicated to form matrices (a) and (b), then $(b)\in PL(a)$. ~~PL(a)~~ With $(a)\in PM\{m:n|r:c\}$, ~~PL(a)~~ contains ~~a unmo~~ and d_k $(k=1,\dots,n)$ a prescribed sequence of P-numbers, $PL(a)$ contains a unique matrix having the $\{d_k\}$ as diagonal elements and, as nondiagonal elements, ~~of~~ zero members of appropriate P-classes (~~the~~ with $d_k\in P\{d(k)\}(k=1,\dots,n)$, the matrix in question is a member of $PM\{m:n|-c:c'\}$ where $c'(k)=c(k)+d(k)(k=1,\dots,n)$); a similar remark may be made concerning the diagonal matrices in $PR(a)$. If $(a)\in PM\{m:n|r:c\}$, $(a)^T\in PL(a)\langle(a)^T\in PR(a)\rangle$ if and only if all members of $c\langle r\rangle$ are ~~cons~~ equal. If $(b)\in PL(a)$, and $(d)\in PL(b)$, then $(bd)\in PL(a)$ and $(ab)\in PR(d)$. With $(a)\in PM\{m:n|r:c\}$, $(d)\in PM\{m':n'|r':c'\}$, ~~there exists a bridging class of matrices (d) such that (abd) exists;~~

$PM\{n:m'|-c,-r'\}$ is the bridging class of matrices (d) such that ~~ab~~ the product (abd) exists. The tensor product ~~of (a)~~ $(d) = (a \times b)$ of any pair of matrices $(a) \in PM\{m:n|r:c\}$ and $(b) \in PM\{u:v|r':c'\}$ is well defined; ~~without~~ ~~elements is a member of~~ its elements are in order

$d_{(I-1)u+i,(J-1)v+j} = a_{I,J}\, b_{i,j}$ $(I=1,..,m;\ i=1,...,u;\ J=1,...,n;\ j=1,...,v)$,

and $(d) \in PM\{mu:nv|r'':c''\}$ where $r''((I-1)u+i) = r(I)+r'(i)$, $c''((J-1)v+j) = c(J)+c'(j)$ $(I,..,j$ as just related). The component product (d) of ~~any~~ the two homologous matrices $(a) \in PM\{m:n|r:c\}$, $b \in PM\{m:n|r':c'\}$ is well defined; its elements are $d_{i,j} = a_{i,j}\, b_{i,j}$ $(i=1,..,m;\ j=1,..,n)$ and $(d) \in PM\{m:n|r'':c''\}$ where $r''(i) = r(i)+r'(i)$ $(i=1,..,m)$, ~~$c''=c$~~ $c''(j) = c(j)+c'(j)$ $(j=1,..,n)$

PM(m:n|c:r) is the class of matrices whose elements may be ~~are~~ displayed ~~in~~ ~~may be taken from~~ the an array PA(m:n|c:r).

PL(m:n|c:r) <PR(m:n|c:r)> is the aggregate ~~class~~ of classes of the form PM(u:m|r':c') <PM(u:m|r',c')>

where c(k)+r'(k) <r(k)+c'(k)> ~~= v k=1,...,n and v+r(i)+c'(j) ≥ 0~~ is independent of k (k=1,...,n) <m> for u=1,2,... .

PR(m:n|c:r) is the aggregate of classes PM(u:m|r':c')

where r(k)+c'(k) is in

With (a) ∈ PA(m:n|c;r), PL{a} <PR{a}> is

a matrix belonging to such a class is called a P-matrix. If (a) ∈ PM(m:n|c:r) and (b) ∈ PL(m:n|c:r) <PR>, b is said to belong to the left <right> multiplicative class of a.

The ~~rema~~ general remarks made above ~~with~~ concerning P-arrays obtained by row and column deletion, ~~and~~ P-arrays obtained from others by and transposition, composition, and also concerning the adjugate P-array, apply ~~may be made~~ with equal force to and the transposed of a P-matrix and adjugates P-matrices. P-matrices, sub P-matrices and compound P-matrices belonging to the same class may be added and subtracted with impunity. A matrix belonging to the left <right> multiplicative class of ~~a~~ the matrix (a) to may be pre-<post>-multiplied by a to form a product P-matrix. If (b) belongs to the left <right> multiplicative class of (a), (a) belongs to the right <left> " " " (b). The adjugate of a square P-matrix belongs to both its left and right multiplicative classes.

Theorem. Let (a), (b) be two square P- matrices, ~~of the same ord~~ b belonging to the left multiplicative class of a with $(c)=(a)(b)$. Then $|\overset{c}{\cancel{ab}}|=|a||b|$

Proof ~~If b belongs to the left~~ (a) and (b) are necessarily of the same order, n say.

$\delta_k \in \mathbb{P}\{d(k)\}$ $k=1,\dots,m$

$d(k)=r(k)+c(k)$

where

$r'(k)\leq d(k)$ arbitrary; set

$c'(k)=d(k)\cancel{c}-r'(k)$ $k=1,\dots,n$

$c'(k)=d(k)+c(k)+K$

pre diag post diag
scalar mult of arrays

dir. prod. of arrays?

Toeplitz?

diag matrices

$c(k)+r'(k)=K$ $r'(k)=K-c(k)$

$r'(k)\leq d(k)$

$K = \max_k c(k)$. $\forall\ 0 \le r'(k) \le \max$

$\max c(k) - c(k) \le d(k)$ $\quad \max c(k) \le d(k) + c(k)$

$(a) \in PM(m;n \mid r, c)$, and

Let ~~If~~ $\forall D_k \in P\{d(k)\}$ $(k = 1, \dots, n)$ ~~is~~ be a prescribed sequence of P numbers, ~~with~~ and let If ~~$\{\max c(k)\}$~~ $\le C \le d(k) + c(k)$ where $C = \max c(k)$ $(1 \le k \le n)$, then ~~it is possible to construct~~ ~~a diagonal~~ PL(a) contains ~~a~~ one a unique diagonal matrix whose ~~with elements D_k $(k = 1, \dots, n)$~~ with diagonal elements are D_k $(k = 1, \dots, n)$ and whose remaining elements are the zero members of appropriate P-classes (the matrix in question belongs to $PM(n:n \mid r'_z : c'_z)$, where for $z = 0, 1, \dots,$ min

$r'_z(k) = C - c(k) + z$, $c'(k) = \cancel{d(k)}\ c(k) + d(k) - C - z$ for $z = 0, \dots,$ $(k = 1, \dots, n)$)

~~so that where $C' \ge C$ is so chosen that $c'(k) \ge 0$~~

($C\,d(k) = d$ for those index values for which $c(k) = a$). Similarly

if ~~$D \ge$~~ $D'_k \in P\{d'(k)\}$ $(k = 1, \dots, m)$ with $R \le d'(k) + r(k)$ where $R = \max r(k)$ $(1 \le k \le m)$ then PR(a) contains a unique diagonal matrix with diagonal elements D'_k $(k = 1, \dots, m)$ (its class is $PM(m:m \mid r' : c')$ where ~~$c'(k) = R - r(k)$,~~ $r'(k) = r(k) + d(k) - R$, $c'(k) = R - r(k)$ $(k = 1, \dots, m)$) ~~again contain~~ ~~of the D_k must belong to $P(0)$~~)

$C \ge \max\{c(k)\}$ $\quad z = \min \{d(k) + c(k) - C\} \ge 0$

$C' = \min\{d(k)+c(k)\}$ $\quad C \le d(k)+c(k)$

$r_\tau(k) = \tau - c(k) \quad \tau = C, \dots, C'$

$c_\tau(k) = c(k) + d(k) - \tau$

Let $(a) \in PM(m:n|r:c)$ and $D_k \in P\{d(k)\}$ be a prescribed sequence of P-numbers, with $u \le d(k)+c(k)$, where $u = \max c(k)$ $(1 \le k \le n)$. Then, with $v = \min\{d(k)+c(k)\}$, $PL(a)$ contains $v-u+1$ matrices whose diagonal elements are D_k $(k=1,\dots,n)$ and whose remaining elements are the zero members of appropriate P-classes. (The matrices in question belong to $PM(n:n|r_\tau:c_\tau)$ for $\tau = u,\dots,v$, where $r_\tau(k) = \tau - c(k)$, $c_\tau(k) = c(k)+d(k)-\tau$ $(k=1,\dots,n)$)

same class for all τ ?

Square $\mathbb{P}$-matrix with diagonal elements $\mathbb{D}_k \in \mathbb{P}\{d(k)\}$ (46)

$(k=1,\dots,n)$ is member of any class $\mathbb{P}M\{n:n|r;c\}$

for which, with $r(k)$ $(k=1,\dots,n)$ assigned arbitrarily

$c(k) = d(k) - r(k)$ $(k=1,\dots,n)$ ‖ P classes with +ve.
$0 \le r(k) \le d(k)$

~~Diagonal:~~ Matrix to be premultiplied by $(a) \in PM\{m:n|r:c\}$

is of class $PM\{n:n|r',c'\}$ where $c(k)+r'(k) = v$

is independent of k $(k=1,\dots,n)$ ‖ $PM\{n:n|c_v, c'\}$

$\sum_{k=1}^{n}\{r(i)+c(k)+r'(k)+c'(j)\}$ ‖ $c_v(k) = v - c(k)$ $k=1,\dots,n$

$(i,j)^{th}$ element of product matrix is in

$\mathbb{P}\{r(i)+c'(j)+v\}$

prod. mat: $PM\{m:n|r:c_v^*\}$ $c_v^*(j) = c'(j)+v$ $(j=1,\dots,n)$

unbounded number of product classes $-\min\{r(i)+c'(j)\}$

$v = \dots -1, 0, 1, \dots$ if $\mathbb{P}$ class unrestricted // $\hat{c}$ $1\le i\le m; 1\le j\le n$

$v =$ ~~$\min\{c$~~ $\hat{c}, \hat{c}+1, \dots$ $\hat{c} =$ ~~$-\min\{c'(j)\}$ $(1\le j\le n)$~~ if $\mathbb{P}$ class +ve

alt: $PM\{m:n|r_v:c'\}$ $r_v(i) = r(i)+v$ ~~$j=1$~~ $\dots i=1,\dots,m$

~~(a) PM{~~ $(a) \in PM\{m:n \mid r:c\}$, ~~$(b) \in PM\{n:u \mid c_v:c'\}$~~

~~$c_v(k) = v - c(k)\ (k=1,\dots,n)$~~. $PL(a)$ is aggregate of

classes $PM\{n:u \mid c_v:c'\}$ where $c_v(k) = v - c(k)\ (k=1,\dots,n)$

$=$ for $v = \dots -1, 0, 1, \dots$ and $u = 1, 2, \dots$ with ~~for~~ all all integer sequence c'

if $(b) \in PM\{n:u \mid c_v:c'\}$ then $(ab) \in PM\{m:u \mid r''_v:c'\}$

where $r''_v(i) = r(i) + v\ (i = 1,\dots,m)$

$=$

$PR(a)$ is aggregate of classes $PM\{u:m \mid r':r'''_v\}$

where $r'''_v(k) = v - r(k)\ (k=1,\dots,m)$ for $v = \dots -1, 0, 1, \dots$

and $u = 1, 2, \dots$ with all integer sequences $r'(i)\ (i=1,\dots,u)$

$=$

if $(b) \in PM\{u:m \mid r':r''_v\}$ then $(ba) \in PM\{u:n \mid r':c''_v\}$

where $c''_v(j) = c(j) + v\ (i = 1,\dots,m)$

$r'(i) + r'''_v(k) + r(k) + c(j)$

$=$

$PM\{n:n \mid c_v:c'\}$ ~~$c_v(k) = d(k) - r(k) = v - c(k)$~~

~~$d(k) = v - c(k) + r(k)$~~ $c'(k) = d(k) - c_v(k)$

$= d(k) + c(k) - v$

product in $PM\{m:u|r_v'':c'\}$ $r_v''(i) = r(i)+v$

$r_v''(i) = r(i)+v \quad c'(k) = d(k)+c(k)-v$

i.e. $r_v''(i) = r(i)$, $c'(j) = c(j)+d(j)$

PL(a) contains a unique diagonal matrix with diagonal elements $\tilde{d}_k$ $(k=1,\dots,n)$ such that

~~(aD)~~ (ad) exists $(d) \in PM\{n:n|\begin{smallmatrix} r'':c'' \\ \cancel{c_v:c'} \end{smallmatrix}\}$

where $c_v(k) = \cancel{v}-c(k) \quad c'(k) = d(k)+c(k)\cancel{-v}$

$r''(k) = -c(k) \quad c''(k) = d(k)+c(k)$

$(a) \in PM\{m:n|r,c\} \quad (b) \in PM\{n:u|c_v:c'\}$

$c_v(k) = v-c(k) \quad (k=1,\dots,n)$

$(a) = (\{a(m:j), a(\overset{m:j+1;n}{\cancel{j+1:n}})\}) \quad (b) = \left(\begin{Bmatrix} b(j:u) \\ b(j+1;n:u) \end{Bmatrix}\right)$

since $(b(j:u)) \in PL\{(a(m:j)\}$ ✓

$(b(j+1;n:m)) \in PL\{a(m:j+1;n)\}$

$(b(j:u)) \in PM\{j:u|c_v:c'\}$ $c_v(k) = v-c(k)$ $(k=1,\dots,j)$

$(b(j+1;n:u)) \in PM\{n-j:u|\tilde{c}_v:c'\}$ $c_v(k) = v-c(k+j)$

$k=1,\dots,n-j$

$(a) \in PM\{m:n \mid r:c\}$ $(b) \in PM\{u,v \mid r':c'\}$

$(a \dot{\times} b) = c \in PM\{mu:nv \mid r'', c''\}$

$c_{\cancel{Iu+i}\,(I-1)u+i,\,(J-1)v+j}$

$\in r(I) + c(J) + r'(i) + c'(j)$

$r''((I-1)u+i) = r(I) + r'(i)$ $(I = 1, \ldots, m;\ i = 1, \ldots, u)$

$c''((J-1)v+j) = c(J) + c'(j)$ $(J = 1, \ldots, n;\ j = 1, \ldots, v)$

—o—

~~at~~ $(a) \in PM\{m:n \mid r:c\}$ $(b) \in PM\{m:n \mid r':c'\}$

$c_{i,j} \in r(i) + c(j) + r'(i) + c'(j)$

$(a (\times) b) \in PM\{m:n \mid r'', c''\}$

$r''(i) = r(i) + r'(i)$ $(i = 1, \ldots, m)$ $c''(j) = c(j) + c'(j)$ $(j = 1, \ldots, n)$

—·—

if $(a), (a')$ can be compounded in the form (a, a')

~~(b)~~, b' have same number of columns $b \in PL(a)$, $b' \in PL(a')$

then $\begin{pmatrix} b \\ b' \end{pmatrix} \in PL\{(a, a')\}$

$\exists (a, a') \equiv r(i) = r'(i)$

$(a) \in PM\{m:n|r:c\}$ $\quad (\tilde{a}^*)\{m:\tilde{n}^*|r:\tilde{c}^*\}$

$(b) \in PM\{n:u|c_v : c'\}$ $\quad c_v(k) = v - c(k) \; (k=1,\dots,n)$

$(\tilde{b}^*) \in PM\{\tilde{n}^*:u|\tilde{c}_v^* : c^*\}$ $\quad \tilde{c}_v^*(k) = \tilde{v}^* - \tilde{c}(k) \; (k=1,\dots,\tilde{n})$

$\binom{b}{\tilde{b}} \in PL\{(a,\tilde{a})\}$

$(a,\tilde{a}) \in PM\{m: n+\tilde{n} | r, \hat{c}\}$ $\quad \hat{c}(k) = c(k) \; k=1,\dots,n$

$\hat{c}(k) = \tilde{c}(k-n) \; k=n+1,\dots,n+\tilde{n}$

$\binom{b}{\tilde{b}} \in PM\{n+\tilde{n} : u | \hat{c}_v : c'\}$ $\quad \hat{c}_v(k) = c_v(k) \; k=1,\dots,n$

$\hat{c}_v(k) = \tilde{c}_v(k-n) \; k=n+1,\dots,n+\tilde{n}$

does v' exist such that $\hat{c}_v(k) = v' - c(k)$

If $(a) \in PM\{m:n|r:c\}$, $(\tilde{a}) \in PM\{m:\tilde{n}|r:\tilde{c}\}$

and $(b) \in PL(a)$ is of the class $(b) \in PM\{n:u|c_v:c'\}$

while $(\tilde{b}) \in PL(\tilde{a})$ is of the class $(\tilde{b}) \in PM\{\tilde{n}:u|\tilde{c}_v:c'\}$

Where ~~such that~~ in the range $c_v(k) + c(k) = \tilde{c}_v(k) + \tilde{c}(k)$ for one $k \vee (1 \le k \le n)$ then $\binom{b}{\tilde{b}} \in PL\{(a,\tilde{a})\}$

If $(b) \in PL(a)$ then $a^T \in PL(b^T)$?

Theorem. Let (a), (b) be two square $\mathbb{P}$-matrices, with $(b) \in PL(a)$. Then $|ab| = |a||b|$

Proof. (a) and (b) are necessarily of the same order. The result of the theorem is correct for first order matrices, and is assumed to be true for matrices of order n. Let (a), (b) be of order $n+1$. Denoting the cofactor of the j^{th} element in the last row of $|ab|$ by C_j

$$|ab| = \sum_{j=1}^{n+1} \left\{ \sum_{m=1}^{n+1} a_{n+1,m} b_{m,j} \right\} C_j$$

or, reversing the order of summation

$$|ab| = \sum_{m=1}^{n+1} (-1)^{m+n+1} a_{n+1,m} \, |c^{(m)}|$$

where

$$c_{i,j}^{(m)} = \sum_{k=1}^{n+1} a_{i,k} b_{k,j}$$

for $i = 1, \dots, n$; $j = 1, \dots, n+1$, and $\cancel{c_{i,j}^{(m)}}$ $c_{n+1,j}^{(m)} = b_{m,j}$ $(j = 1, \dots, n+1)$.

Set $d_{i,j}^{(m)} = c_{i,j}^{(m)} - a_{i,m} c_{n+1,j}^{(m)}$ $(i = 1, \dots, n;\ j = 1, \dots, n+1)$ and $d_{n+1,j}^{(m)} = c_{n+1,j}^{(m)}$ $(j = 1, \dots, n+1)$, so that $|c^{(m)}|$ in ( ) may be replaced by $|d^{(m)}|$. The cofactor of $b_{m,h}$ in the last row of $d^{(m)}$ is the value (with suitable sign attached) of a determinant whose elements are

$$d^{(m)}_{i,j} = \sum_{k=1}^{n+1}{}^{[m]} a_{i,k} b_{k,j} \quad (j=1,\dots,h-1), \quad d^{(m)}_{i,j} = \sum_{k=1}^{n+1}{}^{[m]} a_{i,k} b_{k,j+1} \quad (j=h,\dots,n)$$

for $i=1,\dots,n$. ~~Sth~~ (the terms corresponding to the index value $k=m$ in the above sums are to be omitted). The elements of this determinant are those of the ~~nth order square~~ ~~matrix product~~ product of the two n^{th} order square P-matrices $(a^{[n+1:m]}_{(n+1:n+1)})$ and $(b^{[m:h]}_{(n+1:n+1)})$ (~~which~~, since $(b) \in PL(a)$, this product is well defined). Applying the result assumed with regard to n^{th} order matrices to the above product, and denoting the cofactor of $a_{n+1,j}$ in the ~~last row~~ expansion of $|a|$ by $\mathfrak{a}^{(n+1,j)}$ ~~$(j=1,\dots,n+1)$~~ and that of $b_{m,j}$ in the expansion of $|b|$ by $\mathfrak{b}^{(m,j)}$ $(j=1,\dots,n+1)$ it is found that ~~the complete expansion of according~~ ~~to~~ formula ( ) reduces to

$$|ab| = \sum_{m=1}^{n+1}{}' a_{n+1,m} \sum_{h=1}^{n+1}{}' b_{m,h} \mathfrak{a}^{(n+1,m)} \mathfrak{b}^{(m,h)}$$

$$= \sum_{m=1}^{n+1}{}' a_{n+1,m} \mathfrak{a}^{(n+1,m)} \sum_{h=1}^{n+1}{}' b_{m,h} \mathfrak{b}^{(m,h)} = |a||b|.$$

The required result has been demonstrated for square matrices of order $n+1$, and is hence generally true.

The array adjugate to $a \in PA\{n:n|\mathfrak{r},\mathfrak{c}\}$, i.e the array whose elements are $(-1)^{i+j}|a^{[j:i]}(n:n)|$ is a P-array of the class $PA\{n:n|\mathfrak{r}',\mathfrak{c}'\}$ where $\mathfrak{r}'(i) = \sum_{k=1}^{n}{}^{[i]} c(k)$, $c'(j) = \sum_{k=1}^{n}{}^{[j]} r(j)$ $(i,j=1,\ldots,n)$ the superscript attached to the ~~with~~ $\cancel{R = \sum_{k=1}^{n} r(k),\ C = \sum_{k=1}^{n} c(k),\ r'(i) = C - c(i),\ c'(i) = R - r(i)}$ $\cancel{(i,j=1,\ldots,n)}$ summation sign indicating that the term corresponding to the ~~with~~ enclosed ~~suffix is to~~ index is to be omitted.

Theorem. ~~Let (a(n:n)) be a P matrix~~ $a \in PM\{n:n|r:c\}$
(i) ~~Deta~~ $v(1) = r(1)+c(1)$, $\cancel{v(k) = \sum_{u=1}^{k-1}}$ set
(ii) ~~With~~ Set $v(k) = \sum_{u=1}^{k-1} u\{r(k-u)+c(k-u)\}$

$$\pi_k^{[h]}(a) = \prod_{m=1}^{k-1}{}^{[h]} |a(m:m)| \quad \text{for } k=2,\ldots,n. \text{ Set}$$

for $k=2,\ldots,n$, where the term corresponding to the ~~suffix~~ index value $m=h$ ~~being~~ is to be omitted from the product when $1 \le h \le k$, and define $\pi_k(a)$ by use of a similar formula without possible omission of a term. Let (π'), (π'') be the Where they occur, delete empty products from the following formulae ~~diagonal~~ matrices belonging to the classes $PM\{n:n|r+v:-r\}$ $PM\{n:n|-c:c+v\}$ respectively ~~which has diagonal elements~~ for which $\pi'_{1,1} = \pi''_{1,1} = a_{1,1}$, $\pi'_{k,k} = \pi''_{k,k} = \pi_{k+1}(a)$ $(k=2,\ldots,n)$ ~~, all other elements~~

1 ~~being the zero members of their appropriate P-classes.~~

Let (ℓ) be the lower triangular matrix in ⟨54|

$PM\{n:n|r+v:-r\}$

~~[illegible]~~ for which $\ell_{1,1} = a_{1,1}$ ~~and~~ and

$$\ell_{i,j} = |a(j-1, i:j)| \pi^{[j]}_{i \,[illegible]}(a)$$

when $i = 2, \ldots, n$; $j = 1, \ldots, i-1$, and $\ell_{i,i} = \pi_{i\,[illegible]}(a)$ $(i = 2, \ldots, n)$, ~~all elements above the principal diagonal being~~ ~~the~~ ~~zero members of their appropriate P-classes.~~ Let (w) be the upper triangular matrix in $PM\{n:n|r:c+v\}$

for which $w_{1,1} = a^2_{1,1}$, and

$$w_{i,j} = |a(i:i-1, j)| \pi^{[i-1]}_{j\,[illegible]}(a)$$

when $j = 2, \ldots, n$; $i = 2, \ldots, j$ and $w_{1,j} = a_{1,j}\pi_{j\,[illegible]}(a)$ $(j = 2, \ldots, n)$, (All elements not belonging to, and ~~lying above~~ lying below the principal diagonals of (π), (ℓ) and (w) ~~below the~~ respectively are the zero members of their appropriate P-classes). Then

⟨60| $(\pi)(\pi' a \pi'') = (\ell w)$

border

[illegible] with $\tilde{v}(1) = \sum_{u=2}^{n} \{r(u)+c(u)\}$, set $\tilde{v}(k) = \sum_{u=2}^{n} (u-1)\{r(u)+c(u)\} + (k-1)\sum \{r(u)+c(u)\}$ // for $k = 2, \ldots, n$. $u = k$

(iii) ~~Set~~ Let $\tilde{a}_{i,j} = (-1)^{i+j} a^{[j:i]}(n:n)$ $(i,j = 1,\dots,n)$. Set

$$\tilde{\pi}^{[k]}_k(a) = \prod_{m=2}^{k} |a^{[k]}(m;n:m;n)|$$

(ii) Set $\tilde{r}(k) = \sum_{u=1}^{n}{}^{'[k]} c(u)$, $\tilde{c}(k) = \sum_{u=1}^{n}{}^{[k]} v(u)$ [for $k=1,\dots,n$] and, with

for $k = 2,\dots,n$, and define $\tilde{\pi}_k(a)$ similarly, and preserve the conventions described above with regard to formula ( ).

Let $(\tilde{\pi}')$, $(\tilde{\pi}'')$ be the diagonal matrices belonging to the classes $PM\{n:n|\tilde{r}+\tilde{v}:-\tilde{r}\}$, $PM\{n:n|-\tilde{c}:\tilde{c}+\tilde{v}\}$

for which $\tilde{\pi}'_{1,1} = \tilde{\pi}''_{1,1} = |a(2;n:2;n)|$, $\tilde{\pi}'_{k,k} = \tilde{\pi}''_{k,k} = (-1)^{k(k),n}\tilde{\pi}_k(a)$ $(k=2,\dots,n)$

Let $(\tilde{\ell})$ be the lower triangular matrix in

$PM\{n:n|\tilde{r}+\tilde{v}:-\tilde{r}\}$

for which $\tilde{\ell}_{1,1} = |a(n:n)||a(2;n:2;n)|$, and

$$\tilde{\ell}_{i,1} = \tilde{\pi}_i(a)\, |a^{[:i]}(2;n:n)| \leftarrow \tilde{\pi}_i(a)\,?$$

(for $i=2,\dots,n$) and

$j=n$? spec. formula remark

$$\tilde{\ell}_{i,j} = |a^{[:i]}(j+1;n:j;n)||a(n:n)|\,\tilde{\pi}^{[j]}_i(a)$$

for $i=2,\dots,n$; $j=2,\dots,i$. Let $(\tilde{w})$ be the upper triangular matrix in $PM\{n:n|-\tilde{r}:\tilde{c}+\tilde{v}\}$

for which $\tilde{w}_{1,1} = |a(2;n:2;n)|$, $\tilde{w}_{j,j} = \tilde{\pi}_j(a)$ $(j=2,\dots,n)$ and

$$\tilde{w}_{i,j} = |a^{[j:]}(i;n:i+1;n)|\,\tilde{\pi}_j^{[i+1]}(a)$$

for $j=2,\dots,n$; $i=1,\dots,j-1$. Then

$$(\tilde{\pi}\,\tilde{a}\,\tilde{\pi}) = (\tilde{l}\,\tilde{u})$$

p 71 ← 59

Proof. ~~With regard to the result of~~ When $1<i\leq j\leq n$, ~~exist and~~ the formula

$$\sum_{m=k}^{i} l_{i,m} w_{m,j} = |a(k-1,i:k-1,j)|\,\pi_{i}(a)\,\pi_{j}^{[k-1]}(a)$$

is evidently correct when $k=i$. Assume it to be correct for some k in the range $2\leq k\leq i$. Then

$$l_{i,k-1} w_{k-1,j} + \sum_{m=k}^{i} l_{i,m} w_{m,j} =$$

$$\pi_i^{[k-1]}(a)\,\pi_j^{[k-2]}(a)\{|a(k-2:k-1)|\,|a(k-1:k-2,j)|$$

$$+|a(k-1,i:k-1,j)|\,|a(k-2:k-2)|\}$$

Formula ( ) ~~with $k=2$ in H~~ of Theorem , ~~with $k=2$ in the notation of that theorem,~~ and with $a(k-1,i:k-1,j)$

in place of $a(n:n)$, with $k=2$ in the notation of that theorem
and ~~$i(1),\ldots,i(h-2)$~~ with $i(1)=1,\ldots, i(h-2)=h-2$, $I(1)=h-1$, $I(2)=h$?
and similar definitions of the integers $\{j(k), J(k)\}$
may be applied to the expression enclosed in braces
in formula ( ), which then reduces to

$$|\cancel{a(h-2,i:h-2,j)}|\,|a(h-1:h-1)|\,|a(h-2,i:h-2,j)|$$

In short formula ( ) is correct when h is replaced by $h-1$. Using the special formula given for $u_{1,j}$, it is shown in the same way that for $1<i\leq j\leq n$

$$\sum_{m=1}^{i} l_{i,m}\, u_{m,j} = \pi_i(a)\, a_{i,j}\, \pi_j(a)$$

That this formula is also correct when ~~$i>j$~~ $j<i\leq n$ is demonstrated in the same fashion; that it is correct when $i=1$ is evident from inspection of the ~~appropriate~~ formulae for $l_{i,1}$ and $u_{1,j}$. Formula ( ) has been verified.

$\Rightarrow$ b.p. 42 <60>

With regard to the proof of the result of clause (ii), the formula corresponding to ( ), holding for $1<j\leq i$, is

$$\sum_{m=h}^{j}{}' \, l_{i,m} \tilde{u}_{m,j} = |a^{[j:i]}(h;n:h;n)|\,|a(n:n)| \, \overset{a}{\pi}_j(a) \, \tilde{\pi}_i^{[h]}(a)$$

~~This relationship is proved, and subsequent use is made of it,~~ exactly as in the preceding proof.

and that corresponding to ( ) is

$$\tilde{l}_{i,h-1} \tilde{u}_{h-1,j} + \sum_{m=h}^{j}{}' \, \tilde{l}_{i,m} \tilde{u}_{m,j} =$$

$$\tilde{\pi}_i^{[h-1]}(a) \, \tilde{\pi}_j^{[h]}(a) \, |a(n:n)| \{ |a(h-1;n:h-1;n)|\,|a^{[j:i]}(h;n:h;n)| + |a^{[:i]}(h;n:h-1;n)|\,|a^{[j:]}(h-1;n:h;n)| \}$$

Formula ( ), now with $a(n:n)$ replaced by $a(h-1;n:h-1;n)$, and $k=2$ and $i(1)=h,\dots,i(j-h)=j-1$, $i(j-h+1)=j+1$, $i(n-h)=n$; $I(1)=h-1$, $I(2)=j$; $j(1)=h,\dots,j(i-h)=i-1$, $j(i-h+1)=i+1,\dots,j(n-h)=n$; $J(1)=h-1$, $J(2)=i$ is used to reduce the expression enclosed in braces in formula ( ) to

$$|a^{[j:i]}(h-1;n:h-1;n)|\,|a(h;n:h;n)|.$$

The remainder of the proof, including that of the uniqueness of the triangular decomposition ( ), is as for that of clause (i).

$$\tilde{l}_{i,i} = |a(i+1;n:i+1;n)| \, \prod_{m=2}^{i-1} |a(m:n:m:n)| \, |a(n:n)|$$

$$\tilde{u}_{i,i} = \tilde{\pi}_i(a) = \prod_{m=2}^{i} |a(m;n:m;n)|$$

$$l_{1,1} = |a(n:n)| \, |a(2;n:2:n)|$$

— o —

The matrix $(\tilde{\pi}'\tilde{a}\tilde{\pi}'')$ possesses only one decomposition of the form $(\tilde{l}\tilde{u})$ as described in which the lower $\langle$upper$\rangle$ triangular matrix $(\tilde{l})\langle(\tilde{u})\rangle$ has diagonal elements $\{\tilde{l}_{i,i}\ \langle\tilde{u}_{i,i}\rangle\}$ as given if and only if $|a(m;n:m;n)|$ is not a divisor of zero for $m = 1,\dots,n$

— o —

$\pi'_{1,1} = \pi''_{1,1} = a_{1,1}$ $\quad \pi'_{k,k} = \pi''_{k,k} = \pi_k(a)$ $(k=2,\dots,n)$

$$\pi_k^{[k]}(a) = \prod_{m=1}^{k-1}{}^{[k]} |a(m:m)|$$

$l_{1,1} = a_{1,1}$ $\quad l_{i,i} = \pi_i(a)$ $\quad l_{i,j} = |a(j-1,i;j)|/\pi_i^{[j]}(a)$

$u_{1,1} = a_{1,1}^2$ $\quad u_{1,j} = a_{1,j}\pi_j(a)$ $\quad u_{i,j} = |a(i;i-1,j)|/\pi_j^{[i-1]}(a)$

$\pi'_{k,k} = \pi''_{k,k} = \pi_{n-k+1}(a')$ $k=1,\dots,n-1$ $\quad \pi'_{n,n} = \pi''_{n,n} = a'_{1,1}$

$U_{i,j} = |a'(n-j,n-i+1:n-j+1)|/\pi_{n-i+1}^{[n-j+1]}(a')$

$u_{i,i} = |a(i:i)| \cdots$

unique l and u (with $\& u_{i,i}$ as described) iff

$|a(m:m)|$ not divisor of zero $m = 1, \ldots, n$

If and only if $|a(m:m)|$ is not a divisor of zero

for $m = 1, \ldots, n-1 \langle n \rangle$ the matrix $(\pi' a \pi'')$ possesses

only one triangular decomposition of the form (lu) as described

~~in which the lower~~ ~~with the~~ diagonal elements $\{l_{i,i} \langle u_{i,i} \rangle\}$ as

given

in which the lower ⟨upper⟩ triangular matrix (l)⟨(u)⟩ has

if and only if, when $n=1$ $a_{1,1}$ is not a divisor of zero, and when $n>1$

and $|a(m:m)|$ is not a divisor of zero for $m = 1, \ldots, n-1 \langle n \rangle$

— · —

Concerning the statements made with regard to ~~concerning~~ the uniqueness of the triangular decomposition ( ), it is first remarked that they are evidently correct when $n=1$. It is now assumed that the condition that $|a(m:m)|$ is not a divisor of zero for $m = 1, \ldots, n-1$ suffices to ensure that the decomposition ( ) with diagonal

$$l(h+1;h+1:h)(u(h:h)) = \pi'_{h+1,h+1}(a(h+1;h+1:h))(\pi''(h:h))$$

$$l(h+1;h+1:h)(u(h:h+1;h+1) + l_{h+1,h+1}u_{h+1,h+1} =$$

$$\pi'_{h+1,h+1}a_{h+1,h+1}\pi''_{h+1,h+1}$$

---

elements $\{l_{i,i}\}$ as given is uniquely determined with n replaced by h, for some h in the range $1 \le h < n$. All matrices occurring in the version of formula ( ) ~~relating~~ obtained by replacing n by $h+1$ may be partitioned by separation of the last row and column (thus (l) becomes $(l(h:h))$ bounded on the right by ~~the column vector $l($~~ a zero vector, and from below by the row vector $l(h+1;h+1:h)$, together with a further diagonal element $l_{h+1,h+1}$). The new version of formula ( ) is can thus ~~be express~~ equivalent to the four sets of equations

$$(l(h:h))(u(h:h)) = (\pi'(h:h))(a(h:h))(\pi''(h:h))$$
$$(l(h:h))(u(h:h+1;h+1)) = (\pi'(h:h))a(h:h+1;h+1)\pi''_{h+1,h+1}$$

$$(l(h+1;h+1:h))(u(h:h)) = \pi'_{h+1,h+1}(a(h+1;h+1:h))(\pi''(h:h))$$

$$(l(h+1:h+1:h))(u(h:h+1;h+1)) + l_{h+1,h+1}\,u_{h+1,h+1} =$$

$$\pi'_{h+1,h+1}\,a_{h+1,h+1}\,\pi''_{h+1,h+1}$$

With the diagonal elements $\{l_{i,i}\}$ as given, there is, by assumption, only one pair of matrices $(l(h:h)),(u(h:h))$ satisfying equation ( ). Equation ( ) is a set of h linear algebraic equations used to determine the components of the column vector $(u(h:h+1;h+1))$; ~~the since~~ $\cancel{|l(h:h)| = \prod_{i=1}^{h} l_{i,i}}$ ~~is not a factor of zero, the~~ a solution, as described above, exists and, since $|l(h:h)| = \prod_{i=1}^{h} l_{i,i}$ is not a ~~factor~~ divisor of zero, this solution is the only possible one. Similarly, relationship ( ) determines $(l(h+1;h+1:h))$ uniquely. Lastly, since $l_{h+1,h+1}$ as given is not a ~~factor~~ divisor of zero, $u_{h+1,h+1}$ is uniquely determined by relationship ( ). In conclusion the condition that $|a(m:m)|$ is not a divisor of zero for $m=1,\ldots,n-1$ suffices to ensure that the decomposition ( ) with elements $\{l_{i,i}\}$ as given is unique. If $|a(m:m)|$ is not a divisor of zero for $m=1,\dots,h-1$, but $|a(h:h)|$ is, ~~a divisor of~~ then $l_{h+1,h+1}$ is a divisor of zero, and many choices of $u_{h+1,h+1}$ other than that stated in the theorem, are possible: the decomposition ( ) is no longer unique. The statement made concerning the diagonal elements $\{u_{i,i}\}$ is verified in the same way.

Q-numbers

Definition . $Q\{j|k\}$ is the class of ordered pairs (a/b), where $a \in P\{j\}$, $b \in P\{k\}$; a member of such a class is called a Q-number. The sum and difference of $(a/b) \in Q\{j|k\}$, $(c/d) \in Q\{h|i\}$ are defined when $i+j=h+k$, and then $(a/b) \pm (c/d)$ are ~~defined to be~~ $(ad \pm bc)/bd$. The product of ~~the above two~~ the above two Q-numbers is defined, without restriction to be $(ac)/(bd)$. The Q-number a/b is said to be well defined if b is not a divisor of zero. The relationship $(a/b)=(c/d)$ ~~means that (~~ concerning the above Q-numbers means that (a) both Q-numbers are well defined, (b) $i+j=h+k$ and (c) $ad-bc=0 \in \mathbb{P}\{i+j\}$. ~~A system of P-numbers~~ is embedded within it's associated Q-numbers system by (a) adjoining to $\mathbb{P}\{0\}$ an identity element 1 (if one such does not exist) for which $1a=a1=a$ for all numbers a of the $\mathbb{P}$-system, and (b) giving each $\mathbb{P}$-number the Q-number representation $(a/1)$.

If a P-system is such that $\mathbb{P}\{0\}$ contains an identity element 1 for which $1a = a1 = a$ for all numbers a of the system, the P-system is embedded within its ~~Q-sys~~ associated Q-system by giving each P-number a the Q-number representation $(a/1)$. If the P-system does not contain ~~such~~ an identity element with the above multiplicative properties, one such ~~with the above multiplicative~~ is adjoined to $\mathbb{P}\{0\}$, the associated Q-system is extended by the inclusion of all ~~representation~~ numbers $(a/1)$ for all a of the original

A P-system and its associated system of Q-numbers are embedded in a joint PQ-system in the following way: ~~the arithmetic~~ operations ~~defined~~ involving pairs of numbers both from $\mathbb{P}$ or both from Q are retained; the sum and difference of $a \in \mathbb{P}\{\xi i\}$ and ~~b/c~~ $(b\backslash c) \in Q(j|k)$ are defined when $k\xi = i + kj$, and then $a \pm (b\backslash c) =$ ~~$(b\backslash c) \pm a = (ac \pm b$~~ ~~(addition being commutative)~~ $b\backslash(ab \pm c)$, $(b\backslash c) \pm a = b\backslash(c \pm ab)$ ~~with a similar definitions of $(b\backslash c) \pm a$~~;
the product of ~~ba~~ the above two numbers is defined

without restriction to be $b\backslash(ca)$. The relationships $a=(b\backslash c)$ and $(b\backslash c)=a$ concerning the 4 above numbers means that (i) c is not a ~~pos~~ divisor of zero (ii) $k:=i+j$, $j=i+k$ and (iii) $ba-c=0\in\mathbb{P}\{i{:}j{:}k\}$.

With $r(i)$ $(i=1,\dots,m)$ and $c(j)$ $(j=1,\dots,n)$ two sequences of root integers, $a\in QA\{m:n|r:c\}$ means that the elements of the array $a(m:n)$ are distributed among the various Q-classes according to the laws $a_{i,j}\in$ ~~$\mathbb{P}\{c(i)$~~ $Q\{k(i,j)\,|\,k(i,j)+c(i)+r(j)\}$, $k(i,j)$ $(i=1,\dots,m; j=1,\dots,n)$ being some double sequence of integers. $QM\{m:n|r:c\}$ is the class of matrices whose elements may be displayed in an array of the form $QA\{m:n|r:c\}$; a matrix of such a class is called a Q-matrix.

Subject to restrictions similar to those outlined above in connection with $\mathbb{P}$-matrices ~~above~~, sums, differences and products of $\mathbb{P}$-, Q- and PQ-matrices are also defined. A matrix equation of the form $(a\ddot{x})=b$, where (a) and (b) are $\mathbb{P}$-matrices, and $\ddot{x}$ is a Q-matrix is to be interpreted as a system of equations each involving a component of (ax) and a corresponding component of (b), and each interpreted as an equation involving elements of a PQ-system. A similar convention governs further equations involving P-, Q- and PQ-matrices.

Although the arithmetic operations defined for a Q-~~numbers~~ system permit the generation of numbers that are not well-defined, statements involving equations between such numbers ~~have~~ may not, within the above definition of equality, be made. Thus if ~~a is ∞~~ the P-number a is not a divisor of zero, ~~then~~ and x is a well-defined Q-number, then $(a\backslash a)x = x$; but if x is not ~~well~~ defined this statement cannot be made. ~~Naturally the theory of Q-numbers is direct~~ bp. 83

If, $(\tilde{a})$ being the adjugate of the square P-matrix (a), $b \in LM(\tilde{a})$, and $|a|$ is not a divisor of zero, then the matrix equation $(ax)=(b)$ has a unique Q-matrix solution (x): $\{\tilde{a}b_{i;j}^{c_{i;j}}\}$ being the components of $(\tilde{a}b)$, the components of (x) are $\{|a|\backslash \tilde{a}b_{i;j}^{c_{i;j}}\}$.

Subject to appropriate conditions, the P-matrix $(a) \in PM\{n:n|\mathfrak{c}:\mathfrak{c}\}$ ~~has a direct dec~~ may be directly

With $r(i)$ $(i=1,\dots,m)$ and $c(j)$ $(j=1,\dots,n)$ two sequences

of integers, $QM\{m:n|r:c\}$ is the class of matrices (a) whose elements $a_{i,j}$ $(i=1,\dots,n; j=1,\dots,n)$ are Q-numbers distributed ~~among the various Q-classes~~ according to the law $a_{i,j} \in Q\{k(i,j) | k(i,j)+r(i)+c(j)\}$, $k(i,j)$ $(i=1,\dots,m; j=1,\dots,n)$ being a double sequence of integers which depends upon (a). With $(a) \in PM\{n:n|r:c\}$ and $|a|$ not a divisor zero $(a)^{-1} \in QM\{n:n|-c:-r\}$ is the Q-matrix with elements

Again with ~~$r(i)$~~ r, c as above $PQM\{m:n|r:c\}$ is the class of matrices (a) whose elements ~~are either P or Q-numbers~~ $a_{i,j}$ $(i=1,\dots,n; j=1,\dots,n)$ ~~with~~ are either P or Q-numbers, with $a_{i,j} \in P\{r(i)+c(j)\}$ if $a_{i,j}$ is a P-number, and $a_{i,j} \in Q\{k(i,j) | k(i,j)+c(i)+r(j)\}$ if $a_{i,j}$ is a Q-number, k being as described above in connection with Q-matrices. $I \in PQM\{n:n|r:-r\}$ is ~~the~~ a matrix of the indicated class whose diagonal elements are all 1 and whose remaining elements are zero members of their appropriate P-class.

1 denotes any Q-number of the form $(a \backslash a)$ where a is not a divisor of zero. With a not a divisor of zero, a^{-1} denotes any Q-number of the form ~~b~~ $((ab) \backslash b)$ where b is not a divisor of zero.

$\{(|a| \backslash \tilde{a}_{j,i}\}$, where $(\tilde{a})$ is the adjugate of (a).

decomposed (without the accompaniment of multiplying factors of the form π', π' as given above) in terms (9) of triangular PQ-matrices. Assuming that $|a(i,i)|$ is not a divisor of zero ($i=1,\dots,n$), let the components of $(l) \in PQM\{n:n|r:-r\}$, $(d) \in PQM\{n:n|r,c\}$ $(u) \in PQM\{n:n|-c:c\}$ be as follows

$\pm r(i) - r(j)$

$$l_{i,j} = (|a(j:j)| \backslash |a(j-1,i:j)| \quad (i=1,\dots,n:\ j=1,\dots,i)$$

$d_{1,1} = a_{1,1}$ and

$$d_{i,i} = (|a(i-1:i-1)| \backslash a(i:i)|) \quad (i=2,\dots,n)$$

$$u_{i,j} = (|a(i:i)| \backslash |a(i:i-1,j)| \quad (j=1,\dots,n;\ i=1,\dots,j)$$

the ~~undefined~~ elements not defined above being in each case zero members of their appropriate P-classes; then $(a) = (ldu)$. Assuming that $|a(i;n:i;n)|$ is not a divisor of zero ($i=1,\dots,n$) define the components of (U, D, L) ~~(U), (D), (L)~~, of the same classes as (l,d,u) above by means of the formulae

$$U_{i,j} = (|a(j;n:j;n)| \backslash |a(i,j+1;n:j;n)|) \quad (j=1,\dots,n;\ i=1,\dots,j)$$

$$D_{i,i} = (|a(i+1;n:i+1;n)| \backslash |a(i;n:i;n)|) \quad (i=1,\dots,n-1)$$

with $D_{n,n} = a_{n,n}$ and

$> r(j) + r(i)$

$$L_{i,j} = (|a(i;n:i;n)| \backslash |a(i;n:j,i+1;n)|) \quad (i=1,\dots,n;\ j=1,\dots,i)$$

undefined elements being zero, ~~as above~~; then

$$(a) = (UDL).$$

U: $c(j)-c(i)$

[Subject to ~~Under~~ the same assumptions, with $n>1$, define the components of $(\tilde{U},\tilde{D},\tilde{L})$ ~~of the same classes as (l,d,u) above~~ u,d,l by use of the $(\tilde{U}) \in PQM\{n:n|-c:c\}$, $\tilde{D} \in PQM\{n:n|-c:-r\}$, $\tilde{L} \in PQM\{n:n|r:-r\}$ formulae $U_{1,1} = (a_{1,1} \backslash a_{1,1})$ and

$$\tilde{U}_{i,j} = (-1)^{i+j}(|a(j-1:j-1)| \backslash |a^{[:i]}(j-1;j)|) \quad (j=2,\dots,n;\ i=1,\dots,j)$$

~~$L_{1,1} = (a_{1,1} \backslash a_{1,1})$ and~~

$$\tilde{D}_{i,i} = (|a(i:i)| \backslash |a(i-1:i-1)||a(n:n)|) \quad (i=1,\dots,n)$$

$\tilde{L}_{1,1} = (a_{1,1} \backslash a_{1,1})$ and

$$\tilde{L}_{i,j} = (-1)^{i+j}(|a(i-1:i-1)| \backslash |a^{[j:]}(i:i-1)|) \quad (i=2,\dots,n;\ j=1,\dots,i)$$

undefined elements being zero as above; then

$$(\tilde{a})^{-1} = (\tilde{U}\tilde{D}\tilde{L})$$

~~$(\tilde{a})$ being the matrix adjugate to (a).~~

Again subject to the same assumptions, ~~with~~ with $n>1$, define the components of $(\tilde{l},\tilde{d},\tilde{u})$ of the same classes as (l,d,u) above by use of the formulae

$$\tilde{l}_{i,j} = (-1)^{i+j}\left(|a(j+1;n:j+1;n)|\backslash|a^{[:i]}(j+1;n:j;n)|\right) \quad \cancel{i=1,\dots,n-1;\, j=1,\dots,i}$$

$(j=1,\dots,n-1;\, i=j,\dots,n)$ 71

$$\tilde{l}_{n,n} = \cancel{(|a(n:n)|}(a_{n,n}\backslash a_{n,n}) \text{ and}$$

$$\tilde{d}_{i,i} = \left(|a(j;n:j;n)|\backslash|a(j+1;n:j+1;n)||a(n:n)|\right) \quad (i=1,\dots,n)$$

$$\tilde{u}_{i,j} = \left(|a(i+1;n:i+1;n|\backslash(-1)^{i+j}|a^{[j:]}(i;n:i+1;n)|\right)$$

$(i=1,\dots,n-1;\, j=i,\dots,n)$

with $\tilde{u}_{n,n} = (a_{n,n}\backslash a_{n,n})$, ~~and~~ undefined elements being zero; then

$$(a)^{-1} = (\tilde{l}\tilde{d}\tilde{u}).$$

Subject to the stated conditions, the above decompositions are unique: ~~with $l_{i,i} = u_{i,i} = 1$ $(i=1,\dots,n)$~~, ~~the~~ (l,d,u) are the only lower triangular, diagonal and upper triangular matrices with $l_{i,i} = u_{i,i} = 1$ $(i=1,\dots,n)$ that satisfy equation ( ); and similar remarks may ~~can~~ be made concerning formulae (, ,). ~~The matrices occurring in the above decompositions~~ Subject to the existence conditions stated in each case, the matrices in the above decompositions satisfy the relationships

$$(d\tilde{D}) = \cancel{(\tilde{D}d)} = (D\tilde{d}) = \cancel{(\tilde{d}D)}$$

$$(l\tilde{L}) = (\tilde{L}l) = (L\tilde{u}) = (\tilde{u}U) = I \in QM\{r:-r\}$$

$$(u\tilde{U}) = (\tilde{U}u) = (\tilde{l}L) = (L\tilde{l}) = I \in QM\{c:-c\}$$

$$(\tilde{D}d) = (\tilde{d}D)$$

If (a) is symmetric then, again subject to the stated existence conditions, $(u) = (\ell)^T, (\tilde{U}) = (\tilde{L})^T$, ~~$(U) = (L)$~~, $(\tilde{u}) = (\tilde{\ell})^T, U = (L)^T$.

The Q-classes to which the elements in the above formulae belong can easily be determined by inspection. Thus $\ell_{i,j} \in Q\left\{ \sum_{k=1}^{j} \{r(k)+c(k)\} \setminus \sum_{k=1}^{j-1} \{r(k)+c(k)\} + r(i) + c(j) \right\}$

and so on.

Subject to ~~appropr~~ suitable conditions, bilinear forms whose coefficients are $\mathbb{P}$-numbers, may be directly decomposed (without the intervention of multiplying factors) to diagonal forms involving π', π'' as given above) to diagonal forms involving array such that Q-numbers. With $\{a\}$ an $n \times n$ $\mathbb{P}$-array such that $|a(i:i)|$ is not a divisor of zero $(i = 1, \ldots, n)$ and $\{x_i\}$ and $\{y_j\}$ the ~~elements of~~ row and column $\mathbb{P}$-number elements of ~~(x) and (y) columns~~ and vectors in $R(a)$ and $L(a)$ respectively, we obtain Q-numbers $\{X_k\}$, $\{Y_k\}$ by setting

$$X_k = \sum_{i=k}^{n} \ell_{i,k} x_i \qquad Y_k = \sum_{j=k}^{n} u_{k,j} y_j$$

for $k = 1, \ldots, n$; then

$$A(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i y_j = \sum_{k=1}^{n} d_{k,k} X_k Y_k$$

the coefficients ℓ, u and d being as defined by formulae (-).

$(x)(a)(y) = (xl)\,d\,(uy) = X\,d\,Y$ $\qquad |a(i:i)| \notin \bar{D} \quad (i=1,\dots,n)$ 73

$\cancel{X_j = \sum_{k=1}^{k} l_{k,j}\, x_k} \qquad \cancel{Y_k = \sum_{j=1}^{k} u_{k,j}\, y_j} \qquad X_k = \sum_{i=k}^{n} l_{i,k}\, x_i$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\, x_i y_j = \sum_{k=1}^{n} d_{k,k} X_k Y_k \qquad Y_k = \sum_{j=k}^{n} u_{k,j}\, y_j$$

$$\sum_{k=1}^{n} a_{k,k}\, x_k y_k + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{i,j}\, x_i y_j = \sum_{k=1}^{n} d_{k,k} X_k^2$$

$(x) \in R(a) \quad (y) \in L(a)$ $\qquad \sum x_k \tilde{y}_k = \sum X_k \tilde{Y}_k$

$\sum \tilde{x}_k y_k = \sum \tilde{X}_k Y_k$

$(x)(a)(y) = xUDLy = XDY$ $\qquad = \sum_{k=1}^{n} D_{k,k} X'_k Y'_k$

$X'_k = \sum_{i=1}^{k} U_{i,k}\, x_i \qquad Y'_k = \sum_{j=1}^{k} L_{k,j}\, y_j$

$x'(a)^{-1}y' = x'\tilde{l}\tilde{d}\tilde{u}y' \qquad \cancel{X'(a)^{-1}Y'} \qquad x'(a)^{-1}y' = x'\tilde{U}\tilde{D}\tilde{L}y'$

$l\tilde{L} = U\tilde{u} = 1$

$u\tilde{U} = \tilde{l}L = 1$

$xU = \hat{X} \quad Ly = \hat{Y}$ $\qquad \sum x_k \tilde{y}_k = \sum X'_k \tilde{Y}'_k$

$xl = X \quad uy = Y$ $\qquad \sum \tilde{x}_k y_k = \sum \tilde{X}'_k Y'_k$

$x'\tilde{l} = X' \quad \tilde{u}y = Y'$

$x'\tilde{U} = \hat{X}' \quad \tilde{L}y' = \hat{Y}'$ $\qquad x' \in \{?:c\}$

$xl\tilde{L}y' = X\hat{Y}' \qquad xy' = X\hat{Y}' \qquad (a)^{-1} \in M\{-c:-r\} \qquad y'\{?r:?c\}$

$xy' \qquad x \in M\{?:-r\} \qquad y \in M\{-c:?\}$

$X = xl \quad Y = uy \quad \tilde{X} = \tilde{x}\tilde{U} \quad \tilde{Y} = \tilde{L}\tilde{y}$

$X' = xU \quad Y' = Ly \quad \tilde{X}' = \tilde{x}\tilde{l} \quad \tilde{Y}' = \tilde{u}\tilde{y}$

Similarly, with ~~$(\tilde x)$ and $(\tilde y)$ being~~ row and column vectors

P-number

in $R(\tilde a)$ and $L(\tilde a)$ respectively, let

$$\tilde X_k = \sum_{i=1}^{k} \tilde U_{i,k}\,\tilde x_i \qquad \tilde Y_k = \sum_{j=1}^{k} \tilde L_{k,j}\,\tilde y_j$$

for $k=1,\dots,n$; then

$$A^{(-1)}(\tilde x,\tilde y) = \sum_{i=1}^{n}\sum_{j=1}^{n} a^{(-1)}_{i,j}\, x_i\, y_j = \sum_{k=1}^{n} \tilde D_{k,k}\,\tilde X_k^{*}\,\tilde Y_k$$

where ~~$a^{(-1)}_{i,j}$~~ the $a^{(-1)}_{i,j}$ are the elements of $(a)^{-1}$, and the coefficients $\tilde U$, $\tilde L$, $\tilde D$ are as defined by formulae (-). Assuming that $|a(i;n:i;n)|$ is not a divisor of zero, ~~and setting~~ $(i=1,\dots,n)$, and setting

$$X'_k = \sum_{i=1}^{k} U_{i,k}\,x_i \qquad Y'_k = \sum_{j=1}^{k} L_{k,j}\,y_j$$

for $k=1,\dots,n$

$$\cancel{\sum_{i=1}^{n}}\cancel{\sum_{j=1}^{n}} A(x,y) = \sum_{k=1}^{n} D_{k,k}\,X'_k\,Y'_k$$

the coefficients U, L and D being given by formulae ( ).

Similarly, letting

$$\tilde X'_k = \sum_{i=k}^{n} \tilde l_{i,k}\,\tilde x_i\,, \qquad \tilde Y'_k = \sum_{j=k}^{n} \tilde u_{k,j}\,y_j$$

for $k=1,\dots,n$, ~~then~~

$$A^{(-1)}(\tilde{x},\tilde{y}) = \sum_{k=1}^{n} \tilde{d}_{k,k} \tilde{X}_k' \tilde{Y}_k'$$

the coefficients $\tilde{l}, \tilde{u}, \tilde{d}$ being as given by formulae ( ).

The P-numbers $x_k, \ldots, \tilde{y}_k$ and Q-numbers $X_k, \ldots, \tilde{Y}_k'$ occurring in the above ~~formulae~~ relationships are connected by means of the following invariance formulae

$$\sum_{k=1}^{n} x_k \tilde{y}_k = \sum_{k=1}^{n} X_k \tilde{Y}_k = \sum_{k=1}^{n} X_k' \tilde{Y}_k'$$

$$\sum_{k=1}^{n} \tilde{x}_k y_k = \sum_{k=1}^{n} \tilde{X}_k Y_k = \sum_{k=1}^{n} \tilde{X}_k' Y_k'$$

~~summation~~

When $\{a\}$ is symmetric, the x_k and y_k may be taken to be identical ~~reference to the elements $y_k, Y_k, \ldots$ may be discarded~~; and formula ( ) evolves to

$$A(x,x) = \sum_{k=1}^{n} a_{k,k} x_k^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{i,j} x_i x_j = \sum_{k=1}^{n} d_{k,k} X_k^2$$

and formulae (, ,) may also be presented in terms of sums of squares. The above invariance formulae reduce to

$$\sum_{k=1}^{n} x_k \tilde{x}_k = \sum_{k=1}^{n} X_k \tilde{X}_k = \sum_{k=1}^{n} X_k' \tilde{X}_k'$$

If P possesses proper divisors of zero, 1 is not an identity: there exist Q-numbers x ~~exist~~ for which the statement $1x = x$ cannot be made. In allocation statements such as $a := 1$, 1 can be given the simplest ~~in~~ representation that the context permits; subsequent equations involving a remain valid for all ~~bor~~ permissible allocations.

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$a \in P\{j\}$, $b \in P\{k\}$, $(a|b)$ set of Q numbers x satisfying

$ax = b$ $\quad (a|b) \pm (c|d) = (ac \mid bc \pm ad)$ $\quad (a|b)(c|d) = (ac|bd)$

$(a|b) \in QS\{j|k\}$ $\quad (c|d) \in QS\{h|i\}$ $\quad (i + j = h + k)$

$(c|d) + (e|f) = (ce \mid de + cf)$

$\{(a|b) + \{(c|d) + (e|f)\} = (a(ce) \mid b(ce) + a(de) + a(cf))$

$\{(a|b) + (c|d)\} + (e|f)\} = (ace \mid bce + ade + acf)$

a = 2 0 0 $\quad$ b = 1 0 0 $\quad$ c = 1 0 3 $\quad$ d = 1 0 2

ac 2 0 0 $\quad$ bc = 1 0 0 $\quad$ ad = 2 0 0 $\quad$ $(ac \mid bc + ad) = \frac{3}{2}$ x y

$(a|b) = \frac{1}{2}$ x y $\quad$ $(c|d) = 1$ x $\frac{2}{3}$

$$a_{1,1}|a(n-1:n-1)|$$

$$\sum_{k=j}^{i} l_{i,k}\hat{\pi}_{k,k}\tilde{L}_{k,j} - \Big\{(-1)^j |a(j-1,i:j)|\pi_i^{[j]}(a)\tilde{\prod}_j(a)$$

$$+\sum_{k=j+1}^{i-1}(-1)^k |a(k-1,i:k)|\pi_i^{[k]}(a)|a^{[j:]}(k:k-1)|\tilde{\prod}_j^{[k-1]}(a)$$

$$+(-1)^i\pi_i(a)|a^{[j:]}(i:i-1)|\tilde{\prod}_j^{[i-1]}(a)\Big\}$$

i^{th} term $+(i-1)^{th}$:

$$(-1)^i\Big\{\pi_i(a)|a^{[j:]}(i:i-1)|\tilde{\prod}_j^{[i-1]}(a) - |a(i-2,i:i-1)|\pi_i(a)$$
$$|a^{[j:]}(i-1:i-2)|\tilde{\prod}_j^{[i-2]}(a)\Big\}$$

$$=(-1)^i\tilde{\prod}_j^{[i-1]}(a)\,\pi_i^{[i-2]}(a)$$

$$\Big\{|a(i-2:i-2)||a^{[j:]}(i:i-1)| - |a(i-2,i:i-1)||a^{[j:]}(i-1:i-2)$$

$$|a^{[i-1:]}(i-1:i-2)|a^{[j:]}(i:i-1)| - |a^{[i-1:]}(i:i-1)||a^{[j:]}(i-1:i-$$

$$\tilde{L}\hat{\pi}l:\ \sum_{k=j}^{i}\tilde{L}_{i,k}\hat{\pi}_{k,k}l_{k,j} = a_{1,1}|a(n-1:n-1)|$$

$$\Big\{|a^{[j:]}(i:i-1)|\tilde{\prod}_j^{[i-1]}(a)\,\pi_j(a)$$

$$+\sum_{k=j+1}^{i-1}|a^{[k:]}(i:i-1)|\tilde{\prod}_k^{[i-1]}(a)|a(j-1,k:j)\pi_k^{[j]}(a)$$

$$+\tilde{\prod}_i(a)|a(j-1,i:j)|\pi_i^{[j]}(a)$$

$$\Big\{|a^{[j:]}(i:i-1)| \prod_{m=j}^{n-1}{}^{[i-1]} |a(m:m)| \prod_{m=1}^{j-1} |a(m:m)|$$

$$+\sum_{k=j+1}^{i-1} |a^{[k:]}(i:i-1)| \prod_{m=k}^{n-1}{}^{[i-1]} |a(m:m)|\, |a(j-1,k:j)| \prod_{m=1}^{k-1}{}^{[j]} |a(m:m)|$$

$$+\prod_{m=i}^{n-1} |a(m:m)|\, |a(j-1,i:j)| \prod_{m=1}^{i-1}{}^{[j]} |a(m:m)| \Big\}$$

$$= \prod_{m=1}^{n-1}{}^{[i-1,j]} |a(m:m)| \Big\{ \sum_{k=j}^{i} |a^{[k:]}(i:i-1)|\, |a(j-1,k:j)| \Big\}$$

$$u\hat{\pi}\tilde{U} = \hat{\pi}\tilde{U}u = \tilde{U}u\hat{\pi} = v$$

$$\tilde{L}\hat{\pi}l = l\tilde{L}\hat{\pi} = \hat{\pi}l\tilde{L} = v$$

$$\hat{\pi}l\tilde{L}\tilde{U}u\hat{\pi} = v^2 \qquad \tilde{L}\hat{\pi}lu\hat{\pi}\tilde{U} = v^2$$

$$\tilde{L}\hat{\pi}\pi' a \pi'' \hat{\pi}\tilde{U} = v^2$$

$$\tilde{\Pi}_{1,1}\tilde{\pi}_{1,1} = -\prod_{m=2}^{n} |a(m;n:m;n)|\,|a(2;n:2;n)|$$

$$\Pi_{k,k}\tilde{\pi}_{k,k} = \prod_{m=k+1}^{n} |a(m;n:m;n)|\,(-1)^{k}\prod_{m=2}^{k} |a(m;n:m;n)|$$

$$\Pi_{n,n}\tilde{\pi}_{n,n} = a_{n,n}\prod_{m=2}^{n} |a(m;n:m;n)|\,(-1)^{n}$$

$$\hat{\Pi}_{1,1} = -a_{n,n} \qquad \hat{\Pi}_{k,k} = (-1)^{k} a_{n,n}\,|a(2;n:2;n)|\,\cancel{\prod_{m=2}^{n} |a(m;n:m;n)|}$$

$$\hat{\Pi}_{n,n} = (-1)^{n}\,|a(2;n:2;n)|$$

$$L\hat{\Pi}\tilde{\ell} = V \qquad V_{i,j} = 0 \quad i<j$$

$$V_{1,1} = |a(2;n:n)|\,a_{n,n}\,|\cancel{a(n:n)}|\,|a(2;n:2;n)|\prod_{m=3}^{n} |a(m;n:m;n)|$$

$$= -a_{n,n}\,|a(\overset{2;n:2;n}{n:n})|\prod_{m=1}^{n} |a(m;n:m;n)|$$

$$V_{i,i} = (-1)^{i}\,|a(i;n:i;n)|\prod_{m=i+2}^{n} |a(m;n:m;n)|\,a_{n,n}\,|a(2;n:2;n)|$$

$$|a(i+1;n:i+1;n)|\,|a(n:n)|\prod_{m=2}^{i-1} |a(m;n:m;n)|$$

$$= (-1)^{i} a_{n,n}\,|a(2;n:2;n)|\prod_{m=1}^{n} |a(m;n:m;n)|$$

$$V_{n,n} = (-1)^n a_{n,n}^2 \, |a(2;n:2;n)| \, |a(n:n)| \prod_{m=2}^{n-1} |a(m;n:m;n)|$$

$$= (-1)^n a_{n,n} \, |a(2;n:2;n)| \prod_{m=1}^{n} |a(m;n:m;n)|$$

$$L \hat{\Pi} \tilde{\ell} = \hat{\Pi} \tilde{\ell} L = \tilde{\ell} L \hat{\Pi} = V$$

$$\tilde{u} \hat{\Pi} U = \hat{\Pi} U \tilde{u} = U \tilde{u} \hat{\Pi} = \hat{V}$$

$$\hat{V}_{1,1} = |a(2;n:2;n)| \, a_{n,n} \prod_{m=2}^{n} |a(m;n:m;n)|$$

$$\hat{V}_{i,i} = (-1)^i \prod_{m=2}^{i} |a(m;n:m;n)| \prod_{m=i+1}^{n} |a(m;n:m;n)| \, a_{n,n} |a(2;n:2;n)|$$

$$= (-1)^i a_{n,n} \, |a(2;n:2;n)| \prod_{m=2}^{n} |a(m;n:m;n)|$$

$$\hat{V}_{n,n} = \prod_{m=2}^{n} |a(m;n:m;n)| \, (-1)^n \, |a(2;n:2;n)| \, a_{n,n}$$

$$(-1)^i \tilde{\pi}_i(a) \Pi_i^{[j]}(a) \, |a(i;n:j,i+1;n)|$$

$i < j$

$$\cancel{(-1)^i} a_{n,n} |a(2;n:2;n)| \left\{ \tilde{\pi}_i(a) \Pi_i(a) (-1)^i \right.$$

$$\sum_{k=i}^{j} \tilde{u}_{i,k} \hat{\Pi}_{k,k} U_{k,j} = \cancel{\prod_{m=2}} \tilde{\pi}_i(a) (-1)^i a_{n,n} |a(2;n:2;n)| \Pi_i(a)$$

$$+ \sum_{k=i+1}^{j-1} |a^{[k:j]}(i;n:i+1;n)| \tilde{\pi}_k^{[i+1]}(a) \, |a(k,j+1;n:j;n)| \, \Pi_k^{[j]}(a) (-1)^k$$

$$\left. + |a^{[j:]}(i;n:i+1;n)| \tilde{\pi}_j^{[i+1]}(a) \, \Pi_j(a) (-1)^j \right\}$$

$$a_{n,n} |a(2;n:2;n)| \prod_{m=2}^{n}{}^{[i+1,j]} |a(m;n:m;n)|$$

$$\left\{ \sum_{k=i}^{j} (-1)^k |a^{[k:]}(i;n:i+1;n)||a(k,j+1;n:j;n)| \right\}$$ ✓

$$V_{k,k} = (-1)^k a_{n,n} |a(2;n:2;n)| \prod_{m=1}^{n} |a(m;n:m;n)|$$

$$\hat{V}_{k,k} = (-1)^k a_{n,n} |a(2;n:2;n)| \prod_{m=2}^{n} |a(m;n:m;n)|$$

$$\hat{\Pi}_{1,1} = -a_{n,n}, \hat{\Pi}_{k,k} = (-1)^k a_{n,n} |a(2;n:2;n)| (k=2,\dots,n-1)$$

$$\hat{\Pi}_{n,n} = (-1)^n |a(2;n:2;n)|$$

$$L\hat{\Pi}\tilde{\ell} = \hat{\Pi}\tilde{\ell}L = \tilde{\ell}L\hat{\Pi} = V, \tilde{u}\Pi U = \hat{\Pi} U u = U\tilde{u}\hat{\Pi} = \hat{V}$$

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$$a'(n:n) \equiv \{a'(n:1), a(n:2;n)\},\ a''(n:n) \equiv \{a'(n:1), a(n:1;n-1)\}$$

$$|a'(n;n)||a(n-1:n-1)\}| - |a(n:n)||a'(n-1:n-1)\} =$$

$$|a''(n:n)||a(n-1;2;n)|$$

— —

$$|a(i-2:i-2)||a^{[j:]}(i:i-1)| + |a(i-2,i:i-1)||a^{[j:]}(i-1:i-2)|$$

$$|a^{[:j]}(i-1:i)||a(i-2:i-2)| + |a^{[:j]}(i-2:i-1)||a(i-2,i:i-1)|$$

$$|a(i-1;i-2,i)|$$

$$|a_{1,1}$$

$$a_{1,i}\ a_{1,2} \dots a_{1,j-1}\ a_{1,j+1} \dots a_{1,i-2}\}\ a \dots a_{1,j+2}\ a_{1,i-1}$$

$$a_{1,j}\ a_{1,1}\ a_{1,2} \dots a_{1,j-1}\ a_{1,j+1} \dots a_{1,i-2}$$

$$a'_{1,1}\ a_{1,2} \dots a_{1,n} \quad a_{1,1}\ a_{1,2} \dots a_{1,n-1}$$

$$a_{1,1}\ a_{1,2} \quad a_{1,n}\} \quad a'_{1,1}\ a_{1,2} \quad a_{1,n-1}$$

$$a'_{1,1}\ a''_{1,1}\ a_{1,2} \dots a_{1,n-1} \quad a_{1,2} \dots a_{1,n}$$

$$a_{1,j}\ a_{1,1} \dots a_{1,j-1}\ a_{1,j+1} \dots a_{1,i-2}\ a_{1,i} \qquad (-1)^{j-1}$$

$$a_{1,i-1}\ a_{1,1} \dots a_{1,j-1}\ a_{1,j+1} \dots a_{1,i-2} \qquad (-1)^{i-2}$$

$$a_{1,i-1}\{a_{1,1} \dots a_{1,j-1}\ a_{1,j+1} \dots a_{1,i-2}\}a_{1,i} \qquad (-1)^{i-2}$$

$$a_{1,j}\ a_{1,1} \qquad a_{1,i-2} \qquad (-1)^{j-1}$$

$a'_{1,1} \equiv a_{1,i-1} \quad a_{1,1} \equiv a_{1,j}$

$|a_{1,i-1}\, a_{1,j}\, a_{1,1} \ldots a_{1,j-1}\, a_{1,j+1} \cdots a_{1,i-2}| \; |a_{1,1} \overset{a_{1,j-1}\, a_{1,j+1}}{\cdots} a_{1,i-2}\, a_{1,i}|$

$|a(i-2:i-2)|\,|a^{[j:]}(i:i-1)| - |a(i-2,i:i-1)|\,|a^{[j:]}(i-1:i-2)|$

$= |a(i-1:i-1)|\,|a^{[j,:i-1]}(i-2:i)|$

$(-1)^i l_{i,i} \tilde{L}_{i,j} + (-1)^{i-1} l_{i,i-1} \tilde{L}_{i-1,j}$

$= (-1)^i \tilde{\Pi}_j(a)\, \pi_i^{[i-2]}(a)\, |a^{[j,:i-1]}(i-2:i)| \qquad |a^{[:j]}(i-2:i-2,i)| \quad |a^{[:j]}(i-1:i-1,i|$

$(-1)^i \Pi_j(a)\, \pi_i^{[i-1]}(a)\, a^{[:j,i]}(i-1:i)$

$\sum_{m=k}^{i} (-1)^m l_{i,m} \tilde{L}_{m,j} = (-1)^k \tilde{\Pi}_j(a)\, \pi_i^{[k-1]}(a) |a^{[:j,k]}(k-1:k-1,i)|$

$|a(j-1,i:j)|\, \pi_i^{[j]}(a)\, \tilde{\Pi}_j - \tilde{\Pi}_j(a)\, \pi_i^{[j]} |a^{[:j]}(j:j,i)$

$\sum_{m=j}^{i} (-1)^m l_{i,m} \tilde{L}_{m,j} = 0 \qquad j < i$

— o —

$a(i-1:i-1;i-1)$ in place of $a'(n:1)$ $\quad \| a(k-1:k-1;k-1)$

$a(i-1:j;j)$ in place of $a(n:1)$ $\quad a(k-1:j;j)$

$a^{[:j]}(i-1:i-2,i)$ in place of $a(n:2;n)$ $\quad \| a^{[:j]}(k-1:k-2,i)$

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$$a'(n:n) \to \{a(k-1:k-1;k-1), a^{[:j]}(k-1:k-2,i)\} = (-1)^{k-3} a^{[:j]}(k-1:k-4,i)$$

$$a(n:n) \to \{a(k-1:j;j), a^{[:j]}(k-1:k-2,i)\} = (-1)^{j-1} a(k-1:k-2,i)$$

$$a(n-1:n-1) \to \{a(k-2:j;j), a^{[:j]}(k-2:k-2)\} = (-1)^{j-1} a(k-2:k-2)$$

$$a'(n-1:n-1) \to \{a(k-2;k-1;k-1), a^{[:j]}(k-2:k-2)\} = (-1)^{k-3} a^{[:j]}(k-2:k-1)$$

$$a(n-1:2;n) \to a^{[:j]}(k-2:k-2,i)$$

$$a''(n:n) \to \{a(k-1:k-1;k-1), a(k-1:j;j), a^{[:j]}(k-1:k-2)\}$$

$$= (-1)^{j-1+k-2} a(k-1:k-1)$$

$$|a^{[:j]}(k-1:k-1,i)||a(k-2;k-2)| - |a(k-1:k-2,i)||a^{[:j]}(k-2:k-1)|$$

$$= -|a(k-1:k-1)||a^{[:j]}(k-2:k-2,i)|$$

$$(-1)^{k-1} l_{i,k-1} \tilde{L}_{k-1,j} = |a(k-2,i:k)| \,\pi_i^{[k-1]}(a)\, |a^{[j:]}(k-1:k-2)| \,\tilde{\prod}_j^{[k-2]}(a)$$

$$= (-1)^{k-1} |a(k-1:k-2,i)||a^{[:j]}(k-2:k-1)| \prod_{m=1}^{i-1}{}^{[k-1]} |a(m:m)| \prod_{m=j}^{n-1}{}^{[k-2]} |a(m;m)|$$

$$\sum_{m=k}^{i} (-1)^m l_{i,m} \tilde{L}_{m,j} = (-1)^{k} \tilde{\prod}_j(a)\, \pi_i^{[k-1]}(a)\, |a^{[:j]}(k-1:k-1,i)| \checkmark$$

$$(-1)^{k-1} \cdot \mathrm{sum} = \pi_i^{[k-1]}(a) \,\tilde{\prod}_j^{[k-2]}(a)$$

$$\cancel{|a(k-1:k-1)||a^{[:j]}(k-1:k-1,i)| -}$$

$$|a(k-1:k-2,i)||a^{[:j]}(k-2:k-1)| - |a^{[:j]}(k-1:k-1,i)||a(k-2;k-2)|$$

$$\mathrm{sum} = (-1)^{k-1} \pi_i^{[k-1]}(a)\, \tilde{\prod}_j^{[k-2]}(a)\, |a^{[:j]}(k-2,k-2,i)|$$

$(-1)^i l_{i,i} \tilde{L}_{i,i}$

$i=1: \ -a_{1,1} \prod_{m=1}^{n-1} |a(m:m)|$

$(-1)^i \prod_{m=1}^{i-1} |a(m:m)| \prod_{m=i}^{n-1} |a(m:m)| = (-1)^i \prod_{m=1}^{n-1} |a(m:m)| \quad (i=2,\dots,n-1)$

$i=n \ (-1)^n \prod_{m=1}^{n-1} |a(m:m)| \, |a(n-1:n-1)|$

—o—

$\tilde{U}u$

$\sum_{m=k}^{j} \tilde{U}_{i,m} u_{m,j} \overset{(-1)^m}{=} \cancel{\sum_{m=k}^{j} |a^{[:i]}(k-1:k)|}$

$\sum_{m=k}^{j} |a^{[:i]}(m-1:m)| \, |a(n:n)| \{ \tilde{\prod}_i^{[m]}(a) \, |a(m:m-1,j)| \pi_j^{[m-1]}(a) (-1)^m$

$k=j: \ |a^{[:i]}(j-1:j)| \, |a(n:n)| \tilde{\prod}_i^{[j]}(a) \cancel{|a(j:j)|} \, \pi_j^{[j-1]}(a) (-1)^j$

ⓐ $\tilde{U}_{i,k-1} u_{k-1,j} \overset{(-1)^{k-1}}{=} |a(n:n)| \, |a^{[:i]}(k-2:k-1)| \tilde{\prod}_i^{[k-1]}(a) \, |a(k-1:k-2,j)| \pi_j^{[k-2]}(a) \, (-1)^{k-1}$

ⓑ $\text{sum} \equiv |a(n:n)| \tilde{\prod}_i(a) \, |a^{[:i]}(k-1:k)^{k-1,j}| \pi_j^{[k-1]}(a) \times (-1)^k$ ✓

ⓐ+ⓑ $= |a(n:n)| \tilde{\prod}_i^{[k-1]}(a) \, \pi_j^{[k-2]}(a) (-1)^{k-1}$

$\{ |a^{[:i]}(k-2:k-1)| \, |a(k-1:k-2,j)| - |a^{[:i]}(k-1:k)^{k-1,j}| \, |a(k-2:k-2)|$

$= (-1)^{k-1} |a(n:n)| \tilde{\prod}_i(a) \, \pi_j^{[k-2]} |a^{[:i]}(k-2:k-2;j)|$

$k=i+1: \ (-1)^{i+1} |a(n:n)| \tilde{\prod}_i(a) |a(i:i-1,j)| \pi_j^{[i]}(a)$

$(-1)^i \tilde{U}_{i,i} u_{i,j} = (-1)^i \, \cancel{|a(i-1:i-1)|} \, |a(n:n)| \tilde{\prod}_i^{[i]}(a) \, |a(i:i-1,j)| \, \pi_j^{[i]}(a)$

$(-1)^i \tilde{U}_{i,i} u_{i,i}$

$$i=1: -|a(n:n)| \prod_{m=2}^{n-1} |a(m:m)| \, a_{1,1}^2 = -a_{1,1} \cancel{|a(n:n)|} \prod_{m=1}^{n} |a(m:m)|$$

$$(-1)^i |a(i-1:i-1)| |a(n:n)| \tilde{\overline{\Pi}}_i^{[i]}(a) \, |a(i:i)| \pi_i^{[i-1]}(a)$$

$$= (-1)^i \prod_{m=1}^{n} |a(m:m)| \,,$$

$$i=n: (-1)^n |a(n-1:n-1)| |a(n:n)| |a(n:n)| \overset{?}{\pi}_n^{[n-1]}(a)$$

$$\overset{?}{=} (-1)^n |a(n:n)| \prod_{m=1}^{n} |a(m:m)|$$

—o—

$\tilde{\ell} L$

$$\sum_{m=k}^{i} \tilde{\ell}_{i,m} L_{m,j}^{(-1)^m} = \sum_{m=k}^{i} |a^{[:i]}(m+1;n:m;n)| |a(n:n)| \tilde{\pi}_i^{[m]}(a)$$

$$|a(m;n:j,m+1;n)| \overline{\Pi}_j^{[m+1]}(a) \, (-1)^m$$

$$k=i: \cancel{|a(i+1;n:i+1;n)|} |a(n:n)| \prod_{m=2}^{i-1} |a(m;m:m;n)| (-1)^i$$

$$|a(i;n:j,i+1;n)| \overline{\Pi}_j^{[i+1]}(a)$$? replace by i+1 m. ← i-k+2

$$\text{sum} = \overline{\Pi}_j(a) \tilde{\pi}_i^{[k]}(a) \, |a(n:n)| \, |a^{[:i]}(k;n:j,k;n)| \overset{?}{}(-1)^k$$

$(k-1)^{th}$ term:

$$(-1)^{k-1} |a^{[:i]}(k;n:k-1;n)| |a(n:n)| \, | \, |a(k-1;n:j,k;n)| \tilde{\pi}_i^{[k-1]}(a) \overline{\Pi}_j^{[k]}(a)$$

$$\tilde{\pi}_i^{[k]}(a) \overline{\Pi}_j^{[k-1]}(a) |a(n:n)| \Big\{ |a(k-1;n:k-1;n)| |a^{[:i]}(k;n:j,k;n)|$$

$$- |a(k-1;n:j,k;n)| |a^{[:i]}(k;n:k-1;n)| \Big\} (-1)^k$$

$a'_{1,1} \equiv a_{1,k-1}$ ~~$a_{2,n'}\cdots a$~~ $a_{1,2}\cdots a'_{1,n-1} \equiv a_{1,k}\cdots a_{1,i-1}\, a_{1,i+1}\cdots a_{1,n},\ a_{1,n'} \equiv a_{1,i}$ 89

$a_{1,1} \equiv a_{1,j}$ 88

$$|a'(n:n)| \to \{a(\overset{k;n,k-1}{\cancel{k-1;n}}:k-1;k-1),\ a^{[i]}_k(\overset{k;n,k-1}{\cancel{k-1;n}}:k;n,i)\}$$

$$(-1)^{n-k+1+n-i}|a(k-1;n:k-1;n)|$$

$$|a(n:n)| \to \{a(\overset{k;n;k-1}{\cancel{k-1;n}}:j;j),\ a^{[:i]}(\overset{k;n;k-1}{\cancel{k-1;n}}:k;n,i)\}$$

$$(-1)^{n-k+1+n-i}|a(k-1;n:j,k;n)|$$

$$|a(n-1:n-1)| \to \{a(\underset{k;n}{\cancel{k-1;n-1}}:j;j)\ a^{[:i]}(\underset{k;n}{\cancel{k-1;n-1}}:k;n) \qquad |a^{[:i]}(k;n:j,k;n)|$$

$$|a'(n-1:n-1)| \to \{a(\overset{k;n}{\cancel{k-1;n-1}}:k-1;k-1),\ a^{[:i]}(\overset{k;n}{\cancel{k-1;n-1}}:k;n)\}$$

$$|a^{[:i]}(k;n:k-1;n)|$$

$$a(n-1:2;n) \to \{\cancel{a(k-1;n}\ a^{[:i]}(\overset{k;n}{\cancel{k-1;n-1}}:k;n,i)\}(-1)^{n-i}|a(k;n:k;n)$$

$$a''(n:n) \to \{a(k;n,k-1:k-1;k-1),\ a(k;n,k-1:j;j),\ a^{[:i]}(k;n,k-1:k;n)$$

$$(-1)^{n-k+1+1}|a^{[i]}(k-1;n:j,k-1;n)|$$

$$\cancel{(-1)^{i+k+1}}|a(k-1;n:k-1;n)||a^{[:i]}(k;n:j,k;n)|$$

$$-(-1)^{i+k+1}|a(k-1;n:j,k;n)||a^{[:i]}(k;n:k-1;n)| =$$

$$-(-1)^{i+k+2}|a^{[i]}(k-1;n:j,k-1;n)||a(k:n:k;n)|$$

p.90 $\to (-1)^{k-1}\prod_j(a)\,\overset{\sim}{\pi}{}_i^{[k-1]}|a(n:n)||a^{\widehat{[:i]}\,i+1}(k-1;n:j,k-1;n)|$

$k=j+1$: sum $= (-1)^{j+1}\prod_j(a)\,\overset{\sim}{\pi}{}_i^{[j+1]}(a)\,|a(n:n)||a^{\widehat{[:i]}\,i+1}(j+1;n:j;n)|$ $\Big\}+$

$(-1)^{j}\tilde{l}_{j,j}L_{j,j} = (-1)^{j}|a^{[:i]}(j+1;n:j;n)||a(n:n)|\overset{\sim}{\pi}{}_i^{[j]}|a(j;n:j;n)|\prod_j^{[j+1]}(a)$ $\Big\}=0$

$(-1)^i \tilde{l}_{i,i} L_{i,i}$

$$i=1: -|a(2;n:2;n)|^2 |a(n:n)| \prod_{m=3}^{n} |a(m;n:m;n)| = \cancel{|a(2;n:2;n)| \prod}$$

$$= -|a(2;n:2;n)| \prod_{m=1}^{n} |a(m;n:m;n)|$$

$$(-1)^i |a(i+1;n:i+1;n)| |a(n:n)| \prod_{m=2}^{i-1} |a(m;n:m;n)| |a(i;n:i;n)| \prod_{m=i+2}^{n} |a(m;n:m;n)|$$

$$= (-1)^i \prod_{m=1}^{n} |a(m;n:m:n)| \quad (i=2,\dots,n-1)$$

$$i=n: (-1)^n |a(n:n)| \prod_{m=2}^{n-1} |a(m;n:m;n)| \, a_{n,n}^2 = (-1)^n a_{n,n} \prod_{m=1}^{n} |a(m;n:m;n)|$$

—o—

$U\tilde{u}$

$$\sum_{m=k}^{j} U_{i,m} \tilde{u}_{m,j} = \sum_{m=k}^{j-1} (-1)^m |a(i,m+1;n:m;n)| \prod_{i}^{[m]}(a) \cdot |a^{[j:]}(m;n:m+1;n)| \tilde{\Pi}_j^{[m+1]}(a)$$

$$+ |a(i,j+1;n:j;n)| \prod_i^{[j]}(a) \tilde{\Pi}_j(a) (-1)^j$$

$$\text{sum} = \prod_i^{[k]}(a) \underbrace{\tilde{\Pi}_j(a)}_{? \leftarrow j-k+2} |a^{[j:]}(i,k;n:k;n)| (-1)^k$$

$(k-1)^{th}$ term:

$$(-1)^{k-1} |a(i,k;n:k-1;n)| |a^{[j:]}(k-1;n:k;n)| \prod_i^{[k]}(a) \tilde{\Pi}_j^{[k-1]}(a)$$

$$(-1)^k \prod_i^{[k]}(a) \tilde{\Pi}_j^{[k-1]}(a) \Big\{ |a(k-1;n:k-1;n)| |a^{[j:]}(i;k;n:k;n)|$$

$$- |a(i,k;n:k-1;n)| |a^{[j:]}(k-1;n:k;n)| \Big\}$$

$$= (-1)^{k-1} \prod_i^{[k-1]}(a) \tilde{\Pi}_j(a) |a^{[j:]}(i,k-1;\cancel{k-1};n:k-1;n)|$$

$\cancel{k=j+1 : sum = (-1)^j |a(i,j+1;n:j;n)||a(j+1;n:j+1;n)|}$

$\cancel{(-1)^{j+1} \Pi_i^{[j+1]}(a) \tilde{\pi}_j(a) |a(i,j+1;n:}$

$k=i+1 \quad sum = \cancel{\frac{1}{a}}(-1)^{i+1} \Pi_i^{[i+1]}(a) \tilde{\pi}_j(a) |a^{[j:]}(i;n:i+1;n)|$

$(-1)^i U_{i,i} \tilde{u}_{i,j} = \Pi_i(a) |a^{[j:]}(i;n:i+1;n)| \tilde{\pi}_j^{[i+1]}(a) (-1)^i$

$(-1)^i U_{i,i} \tilde{u}_{i,i}$:

$i=1: -\prod_{m=2}^{n} |a(m;n:m;n)| \, |a(2;n:2;n)|$

$(-1)^i \prod_{m=i+1}^{n} |a(m;n:m;n)| \prod_{m=2}^{i} |a(m;n:m;n)| = (-1)^i \prod_{m=2}^{n} |a(m;n:m;n)|$

$(i=2,\dots,n-1)$

$i=n: \; (-1)^n a_{n,n} \prod_{m=2}^{n} |a(m;n:m;n)|$

$\frac{a}{a}x = 0 \quad = x = 0$: $\quad a_{2,1} \frac{a_{1,1}}{a_{1,1}}$

$d_{2,2} = \frac{a(2;2)}{a_{1,1}}$

$\cancel{l_{1,1}} = \quad l_{2,1} = \frac{a_{2,1}}{a_{1,1}} \quad l_{2,2} = 1 \quad u_{1,1} = 1 \quad d_{1,1} = a_{1,1}$

$u_{1,2} = \frac{a_{1,2}}{a_{1,1}} \quad u_{2,2} = 1$

$\frac{a_{2,1}}{a_{1,1}} \cdot a_{1,1} \frac{a_{1,2}}{a_{1,1}} + 1 \cdot \frac{(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})}{a_{1,1}}$

$\frac{a_{1,1}}{a_{1,1}^2} \cdot a_{1,1} a_{2,2}$

$\hat{\pi}_{1,1} = -|a(n-1:n-1)|$ $\hat{\pi}_{n,n} = (-1)^n a_{1,1}$

$u\hat{\pi}\tilde{U} = dv$

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$\hat{\pi}_{k,k} = (-1)^k a_{1,1} |a(n-1:n-1)|$

$$\sum_{k=i}^{j} u_{i,k} \hat{\pi}_{k,k} \tilde{U}_{k,j} = 0 \quad (j>k)$$

$d_{i,i} = |a(n:n)|$

$$u_{1,1} \hat{\pi}_{1,1} \tilde{U}_{1,1} = -a_{1,1} |a(n-1:n-1)| \prod_{m=1}^{n} |a(m:m)|$$

$$u_{i,i} \hat{\pi}_{i,i} \tilde{U}_{i,i} = (-1)^i a_{1,1} |a(n-1:n-1)| \prod_{m=1}^{n} |a(m:m)| = d_{i,i} v_{i,i}$$

$$\sum_{k=j}^{i} \tilde{L}_{i,k} \hat{\pi}_{k,k} l_{k,j} = 0 \quad (i>j)$$

$$\tilde{L}_{1,1} \hat{\pi}_{1,1} l_{1,1} = -a_{1,1} |a(n-1:n-1)| \prod_{m=1}^{n-1} |a(m:m)|$$

$\tilde{L}\hat{\pi} l = v$

$$\tilde{L}_{i,i} \hat{\pi}_{i,i} l_{1,1} = (-1)^i a_{1,1} |a(n-1:n-1)| \prod_{m=1}^{n-1} |a(m:m)| = v_{i,i}$$

$$\hat{\Pi}_{1,1} = -a_{n,n} \quad \hat{\Pi}_{n,n} = (-1)^n |a(2;n:2;n)|$$

$$\hat{\Pi}_{k,k} = (-1)^k a_{n,n} |a(2;n:2;n)|$$

$L\hat{\Pi}\tilde{l} = dV$

$$\sum_{k=j}^{i} L_{i,k} \hat{\Pi}_{k,k} \tilde{l}_{k,j} = 0 \quad (i>j)$$

$$L_{1,1} \hat{\Pi}_{1,1} \tilde{l}_{1,1} = -a_{n,n} |a(2;n:2;n)| \prod_{m=1}^{n} |a(m;n:m;n)|$$

$$L_{i,i} \hat{\Pi}_{i,i} \tilde{l}_{i,i} = (-1)^i a_{n,n} |a(2;n:2;n)| \prod_{m=1}^{n} |a(m;n:m;n)| = d_{i,i} V_{i,i}$$

$$\sum_{k=i}^{j} \tilde{u}_{i,k} \hat{\Pi}_{k,k} U_{k,j} = 0 \quad (i<j)$$

$\tilde{u}\hat{\Pi}U = V$

$$\tilde{u}_{i,i} \Pi_{i,i} U_{i,i} = (-1)^i a_{n,n} |a(2;n:2;n)| \prod_{m=2}^{n} |a(m;n:m;n)| = V_{i,i}$$

$$\sum_{m=j}^{i}(-1)^m l_{i,m}\tilde{L}_{m,j}=0\ (j<i)$$

$$-l_{1,1}\tilde{L}_{1,1}=-a_{1,1}\prod_{m=1}^{n-1}|a(m;m)| \qquad (-1)^n l_{n,n}\tilde{L}_{n,n}=(-1)^n\prod_{m=1}^{n-1}|a(m;m)|\,|a(n-1:n-1)|$$

$$(-1)^i l_{i,i}\tilde{L}_{i,i}=(-1)^i\prod_{m=1}^{n-1}|a(m;m)| \equiv \delta_{i,i} \qquad \hat{\Pi}\delta=v$$

$$\sum_{m=i}^{j}(-1)^m\tilde{U}_{i,m}u_{m,j}=0\ (i<j)$$

$$-\tilde{U}_{1,1}u_{1,1}=-a_{1,1}\prod_{m=1}^{n}|a(m:m)| \qquad (-1)^n|a(n:n)|\overset{?!}{}\prod_{m=1}^{n}|a(m:m)|=(-1)^n\tilde{U}_{n,n}u_{n,n}$$

$$(-1)^i\tilde{U}_{i,i}u_{i,i}=(-1)^i\prod_{m=1}^{n}|a(m:m)|=d_{i,i}\delta_{i,i} \qquad \hat{\Pi}d\delta=dv$$

$$\sum_{m=j}^{i}(-1)^m\tilde{l}_{i,m}L_{m,j}=0\ (j<i) \qquad \hat{\Pi}d\Delta=dV$$

$$-\tilde{l}_{1,1}L_{1,1}=-|a(2;n:2;n)|\prod_{m=1}^{n}|a(m;n:m;n)|$$

$$(-1)^n\tilde{l}_{n,n}L_{n,n}=(-1)^n a_{n,n}\prod_{m=1}^{n}|a(m;n:m;n)|$$

$$(-1)^i\tilde{l}_{i,i}L_{i,i}=(-1)^i\prod_{m=1}^{n}|a(m;n:m:n)| \equiv d_{i,i}\Delta_{i,i}$$

$$\sum_{m=i}^{j}(-1)^m U_{i,m}\tilde{u}_{m,j}=0\ (i<j) \qquad \hat{\Pi}\Delta=V$$

$$-U_{1,1}\tilde{u}_{1,1}=-|a(2;n:2;n)|\prod_{m=2}^{n}|a(m;n:m;n)|$$

$$(-1)^n U_{n,n}\tilde{u}_{n,n}=(-1)^n a_{n,n}\prod_{m=2}^{n}|a(m;n:m;n)|$$

$$(-1)^i U_{i,i}\tilde{u}_{i,i}=(-1)^i\prod_{m=2}^{n}{}'|a(m;n:m;n)| \equiv \Delta_{i,i}$$

replace $\tilde{l}_{i,j}, \tilde{u}_{i,j}, \tilde{L}_{i,j}\ \tilde{U}_{i,j}$ by $(-1)^{i+j}\tilde{l}_{i,j}$ etc

remove signs from all other terms

$u\hat{\pi}\tilde{U} = \hat{\pi}\delta d\ \delta\hat{\pi}d \qquad \pi' a \pi'' = lu$

$\tilde{L}\hat{\pi}l = \delta\hat{\pi} \qquad \Pi' a \Pi'' = UL$

$L\hat{\Pi}\tilde{l} = \Delta\hat{\Pi}d \qquad \tilde{\pi}'\tilde{a}\tilde{\pi}'' = \tilde{l}\tilde{u}$

$\tilde{u}\hat{\Pi}U = \Delta\hat{\Pi} \qquad \tilde{\Pi}'\tilde{a}\tilde{\Pi}'' = \tilde{U}\tilde{L}$

$l\tilde{L} = \delta$

$\tilde{U}u = \delta d \qquad l\tilde{L}\hat{\pi} = \delta\hat{\pi} = \text{const diag}$

$\tilde{l}L = \Delta d \qquad \hat{\pi}\tilde{U}u = \hat{\pi}\delta d =$ "

$U\tilde{u} = \Delta \qquad \hat{\Pi}\tilde{l}L = \hat{\Pi}\Delta d$ "

$U\tilde{u}\hat{\Pi} = \hat{\Pi}\Delta$ "

$\delta\hat{\pi} = \text{const diag}$

$\Delta\hat{\Pi} =$ " $\qquad \hat{\pi}\tilde{\Pi}'\tilde{a}\tilde{\Pi}''\hat{\pi} = \hat{\pi}\tilde{U}\tilde{L}\hat{\pi}$

$\tilde{L}\hat{\pi} = \hat{L} \quad \hat{\pi}\tilde{U} = \hat{U} \quad \hat{\Pi}\tilde{l} = \hat{l} \quad \tilde{u}\hat{\Pi} = \hat{u} \quad \delta\hat{\pi} = \hat{\delta} \quad \Delta\hat{\Pi} = \hat{\Delta}$

$\hat{\Pi}\tilde{\pi}' = \hat{\pi}' \quad \hat{\pi}\tilde{\Pi}' = \hat{\Pi}'$

interchange $\hat{\pi}, \hat{\Pi}$?

$u\hat{U} = \hat{\delta}\,\cancel{\ }d \quad \hat{L}l = \hat{\delta}\,\cancel{\ } \quad L\hat{l} = \hat{\Delta}\,\cancel{\ }d \quad \hat{u}U = \hat{\Delta}\,\cancel{\ }$

$l\hat{L} = \hat{\delta}\,\cancel{\ } \quad \hat{U}u = \hat{\delta}\,\cancel{\ }d \quad \hat{l}L = \hat{\Delta}\,\cancel{\ }d \quad U\hat{u} = \hat{\Delta}\,\cancel{\ }$

$\hat{l}\hat{u} = (\hat{\Pi}\tilde{\pi}')\tilde{a}(\tilde{\pi}''\hat{\Pi}) \quad (\hat{\pi}\tilde{\Pi}')\tilde{a}(\tilde{\Pi}''\hat{\pi}) = \hat{U}\hat{L}$

$\hat{l}\hat{u} = \hat{\pi}'\tilde{a}\hat{\pi}'' \qquad \hat{\Pi}'\tilde{a}\hat{\Pi}'' = \hat{U}\hat{L}$

$lu = \pi' a \pi'' \qquad UL = \Pi' a \Pi''$

simplification of ~~formulae~~ appearance of formulae

$$\pi_k(a) = \prod_{m=1}^{k} |a(m:m)$$

$$\pi'_{k,k} = \pi''_{k,k} = \pi_k(a) \quad l_{i,i} = \pi_i(a) \quad i=1,\dots,n$$

$(k=1,\dots,n)$ $\quad$ $l_{i,j}$ as before, also $u_{i,j}$

$$u_{1,j} = a_{1,j}\,\pi_j(a)\ (j=1,\dots,n) \quad (\pi' a \pi'') = (lu)$$

but products involved in all formulae are of higher order

$$\tilde{L}\hat{\pi}\pi' a \pi'' \hat{\pi}\tilde{U} = \tilde{L}\hat{\pi} l\, u \hat{\pi}\tilde{U} = (\delta\hat{\pi} d)\hat{\delta\pi}$$

$$\tilde{u}\hat{\Pi}\Pi' a \Pi''\hat{\Pi}\tilde{l} = \tilde{u}\hat{\Pi} U\, L\hat{\Pi}\tilde{l} = (\Delta\hat{\Pi})^2 d$$

$$L\hat{\Pi}\tilde{\pi}'\tilde{a}\tilde{\pi}''\hat{\Pi}U = L\hat{\Pi}\tilde{l}\,\tilde{u}\hat{\Pi}U = (\Delta\hat{\Pi})^2 d$$

$$u\hat{\pi}\tilde{\Pi}'\tilde{a}\tilde{\Pi}''\hat{\pi} l = u\hat{\pi}\tilde{U}\,\tilde{L}\hat{\pi} l = (\delta\tilde{\pi})^2 d$$

$$\tilde{\Pi}''\hat{\pi} l = \tilde{\lambda} \quad u\hat{\pi}\tilde{\Pi}' = \tilde{\omega} \quad \Pi''\hat{\Pi}\tilde{l} = \hat{\lambda} \quad \tilde{u}\hat{\Pi}\Pi' = \omega$$

$$L\hat{\Pi}\tilde{\pi}' = \tilde{\Lambda} \quad \tilde{\pi}''\hat{\Pi}U = \tilde{\Omega} \quad \tilde{L}\hat{\pi}\pi' = \Lambda \quad \pi''\hat{\pi}\tilde{U} = \Omega$$

$$(\Lambda a \Omega) = (\tilde{\omega}\tilde{a}\tilde{\lambda}) = \hat{\delta}^2 d,\ (\omega a \lambda) = (\tilde{\Omega}\tilde{a}\tilde{\Lambda}) = \hat{\Delta}^2 d$$

$$(\hat{\delta} d \hat{\delta}) \qquad (\hat{\Delta} d \hat{\Delta})$$

$\tilde{\lambda}, \tilde{\Lambda}$, λ, Λ lower triangular $\quad \omega\ \Omega, \tilde{\omega}\ \Omega\ \tilde{\Omega}$ upper triangular

$$x\pi' a \pi'' y = XY \quad X = xl \quad Y = uy \quad Y = DX^T \quad x = y^T$$

$$x\Pi' a \Pi'' y = \hat{X}\hat{Y} \quad \hat{X} = x\hat{U} \quad \hat{Y} = Ly \quad uy = Dl^T x^T$$

$$u = Dl^T$$

$$\pi' a \pi'' = lDl^T$$

$u_{1,1} = a_{1,1}^2 \quad u_{1,j} = a_{1,j}\pi_j(a) \quad u_{i,j} = |a(i:i-1,j)| / \pi_j^{[i-1]}$

$l_{1,1}^T = a_{1,1} \quad l_{i,i}^T = \pi_i(a) \quad l_{i,j}^T = |a(i-1,j:i)| / \pi_j^{[i]}$

$i=1 \quad u: \; a_{1,1}^2 \quad a_{1,j}\, a_{1,1} \ldots |a(j-1:j-1)|$

$l: \; a_{1,1} \quad a_{j,1} \; |a(2:2)| \ldots a(j-1:j-1)|$

$i=2 \quad u: \quad |a(2:1,j)| \; |a(2:2)| \ldots |a(j-1:j-1)|$

$l: \quad a_{1,1}, \; |a(1,j:2)| \; a_{1,1} \; |a(3:3)| \ldots |a(j-1:j-1)|$

$$a(i,j) = a(j,i)^T \quad \begin{pmatrix} 1 & & \\ & a_{1,1} & \\ & & |a(i-1:i-1)| \end{pmatrix} u = \begin{pmatrix} a_{1,1} & \\ & \ddots \\ & & |a(i:i)| \end{pmatrix} l^T$$

$$d\{|a(i:i)|^{-1}\}\big((d\{a(i:i)\})\,l\big).\; d\{|a(i-1:i-1)|\}^{-1}\big(d\{a(i-1:i-1)\}\big)u\big)$$

$$= \pi' a \pi'' \quad d\{a(i:i)\}\,\pi' a \pi' = \acute{l}\big(d\{|a(i-1:i-1)|\}\big)^{-1}\acute{u} \quad \acute{u}^T = \acute{l}$$

$$\pi' a \pi'' = l \begin{pmatrix} a_{1,1} & & \\ & \frac{|a(2:2)|}{a_{1,1}} & \\ & & \ddots \end{pmatrix} l^T$$

$$\pi_n \pi' a \pi'' \pi_n = l\,\pi_n \big(\;''\; \big) \pi_n\, l^T \qquad |a(i:i)|\,\pi_n(a)\,\pi_n^{[i-1]}(a) = d_{i,i} \; (i=1,\ldots,n)$$

$$\rightarrow \xi a \xi^T = l d l^T$$

$$\beta_{k,k} = \pi_n(a)\pi_k(a) \quad \beta_{1,1} = \pi_n(a) a_{1,1} \quad \beta' a \beta'' = l d l^T$$

—o—

$$L_{n,j} = a_{n,j}\,\Pi_j(a) \quad L_{n,n} = a_{n,n}^2 \quad L_{i,j} = |a(i;n;j,i+1;n)| / \overline{\Pi}_j^{[i+1]}$$

$$U_{i,i}^T = \Pi_i(a) \quad U_{n,n}^T = a_{n,n} \quad U_{i,j}^T = |a(j,i+1;n:i;n)| / \Pi_j^{[i]}(a)$$

$$\Pi' a \Pi' = UL, \left\langle \frac{a(i+1 \overset{n}{;} i+1)}{a(i;i)} \right\rangle L = U^T$$

$$\Pi' a \Pi'' = U \left\langle \frac{a(i:i)^{;n;n}}{a(i+1:i+1)_{;n\ ;n}} \right\rangle U^T$$

$$\overline{\Pi}_{1} \Pi' a \Pi'' \overline{\Pi}_{1} = U \left\langle \frac{a(i;i)}{a(i;n:i;n)} \overline{\Pi}_{1}(a) \overline{\Pi}_{1}^{[i+1]}(a) \right\rangle U^T \quad \overline{\Pi} \equiv \prod_{k=2}^{n} |a(k;n:k;n)| = \overline{\Pi}_1(a)$$

$$\Xi_{k,k} = \overline{\Pi}_1(a) \Pi_k(a) \quad k = 1, \ldots, n-1 \;\; \& \;\; \Xi_{n,n} = \overline{\Pi}_1(a) a_{n,n}$$

$$D_{i,i} = |a(i:i)| \overline{\Pi}_1(a) \overline{\Pi}_1^{[i+1]}(a) \quad i = 1, \ldots, n$$

$$\Xi' a \Xi'' = U D U^T$$

$$\tilde{u}_{1,1} = |a(2;n:2;n)| \quad \tilde{u}_{i,i} = \Pi_i(a) \quad i = 2, \ldots, n \quad \tilde{u}_{i,j} = |a^{[j:]}(i;n:i+1;n)| \tilde{\Pi}_j^{[i+1]}(a)$$

$$\tilde{l}^T_{1,1} = |a(2;n:2;n)|^2 \quad \tilde{l}^T_{1,j} = |a^{[:j]}(2;n:n)| \tilde{\Pi}_j(a) \Leftarrow (-1)^{i+j}$$

$$\tilde{l}^T_{i,j} = |a^{[:j]}(i+1;n:i;n)| \, |a(n:n)| \tilde{\Pi}_j^{[i]}(a)$$

$$\tilde{\Pi}' \tilde{a} \tilde{\Pi}'' = \tilde{l} \tilde{u} \quad \left\langle \frac{|a(i+1;n:i+1;n)|}{|a(i;n:i;n)|} \right\rangle \tilde{u}_{i,j} = l_{i,j}$$

$$\overline{\Pi}_1 \tilde{\Pi}' \tilde{a} \tilde{\Pi}'' \overline{\Pi}_1 = \tilde{l} \tilde{d} \tilde{l}^T = \tilde{\xi}' \tilde{a} \tilde{\xi}''$$

$$\tilde{\xi}'_{1,1} = \overline{\Pi}_1(a) |a(2;n:2;n)| \quad \tilde{\xi}'_{k,k} = \overline{\Pi}_1(a) \tilde{\Pi}_k(a) \quad k = 2, \ldots, n$$

$$\tilde{d}_{i,i} = |a(i;n:i;n)| \overline{\Pi}_1(a) \overline{\Pi}_1^{[i+1]}(a) = \Xi_i \, D_{i,i}$$

~~also:~~ $\cancel{\tilde{l}_{i,j} = \left\langle \frac{|a(i+1;n:i+1;n)|}{|a(i;n:i;n)|} \right\rangle \tilde{u}^T_{i,j}}$

$$\tilde{l}_{1,1} = |a(2;n:2;n)|^2 \quad \tilde{l}_{i,1} = |a^{[:i]}(2;n:n)| \tilde{\Pi}_i(a) \quad \tilde{l}_{i,j} = |a^{[:i]}(j+1;n:j;n)| \, |a(n:n)| \tilde{\Pi}_i^{[j]}(a)$$

$$\tilde{u}^T_{1,1} = |a(2;n:2;n)| \quad \tilde{u}^T_{i,i} = \tilde{\Pi}_i(a) \quad \tilde{u}^T_{i,j} = |a^{[i:]}(j;n:j+1;n)| \tilde{\Pi}_i^{[j+1]}(a)$$

$$\tilde{l}_{i,j} = \left\langle \frac{|a(j+1;n:j+1;n)| \, |a(n:n)|}{a(j;n:j;n)} \right\rangle \tilde{u}^T_{i,j}$$

$$\tilde{d}_{1,1} = |a(2;n:2;n)|\overline{\Pi}_1(a)^2 \quad \tilde{d}_{k,k} = |a(k+1;n:k+1;n)||a(n:n)|\overline{\Pi}_1^{[k]}(a)$$
$(k=2,\dots,n)$

$$\tilde{\xi}'_{k,k} = \overline{\Pi}_1(a)\tilde{\pi}_k(a) \quad k=2,\dots,n \quad \tilde{\xi}'_{1,1} = |a(2;n:2;n)|\overline{\Pi}_1(a)$$

$$\tilde{\xi}'\tilde{a}\tilde{\xi}'' = \tilde{u}^T\tilde{d}\tilde{u}$$

$$\tilde{U}_{i,n} = |a^{[:i]}(n-1:n)|\overline{\tilde{\Pi}}_i(a) \quad \tilde{U}_{n,n} = |a(n-1:n-1)|^2$$

$$\tilde{U}_{i,j} = |a^{[:i]}(j-1,j)||a(n:n)|\overline{\tilde{\Pi}}_i^{[j]}(a)$$

$$L^T_{i,i} = \overline{\tilde{\Pi}}_i(a) \quad \tilde{L}^T_{n,n} = |a(n-1:n-1)|$$

$$L^T_{i,j} = |a^{[i:]}(j:j-1)|\overline{\tilde{\Pi}}_i^{[j-1]}(a)$$

$$\tilde{U}_{i,j} = \left\langle \frac{|a(n:n)||a(j-1:j-1)|}{|a(j:j)|} \right\rangle L_{i,j}$$

$$\tilde{D}_{k,k} = \pi_n(a)\pi_n^{[k]}(a)|a(n:n)||a(k-1:k-1)| \quad k=1,\dots,n-1$$

$$\tilde{D}_{n,n} = \pi_n(a)^2|a(k-1:k-1)|$$

$$\tilde{\Xi}'_{k,k} = \tilde{\Pi}_k(a)\pi_n(a) \quad (k=1,\dots,n-1) \quad \tilde{\Xi}'_{n,n} = |a(n-1:n-1)|\pi_n(a)$$

$$\tilde{\Xi}\tilde{a}\tilde{\Xi} = \tilde{L}\tilde{D}\tilde{L}^T$$

$$l_{1,1} = \frac{a_{1,1}}{a_{1,1}} \qquad l_{i,j} = \frac{|a(j-1,i:j)|\,\Pi_i^{[j]}(a)}{\Pi_i(a)} \qquad l_{i,i} = \frac{\Pi_i(a)}{\Pi_i(a)}$$

$$u_{1,1} \equiv \frac{a_{1,1}^2}{a_{1,1}} \qquad u_{1,j} \equiv \frac{a_{1,j}\,\Pi_j(a)}{\Pi_j(a)} \qquad u_{i,j} = \frac{|a(i:i-1,j)|\,\Pi_j^{[i-1]}(a)}{\Pi_j(a)}$$

$$u_{i,i} = \frac{|a(i:i)|}{|a(i-1:i-1)|} \qquad u_{i,j} = \frac{|a(i:i-1,j)|}{|a(i:i)|} \qquad d_{i,i} = \frac{|a(i:i)|}{|a(i-1:i-1)|}$$

$$l_{i,j} = \frac{|a(j-1,i:j)|}{|a(j:j)|} \qquad u_{i,j} = \frac{|a(i:i-1,j)|}{|a(i:i)|} \qquad d_{i,i} = \text{"}$$

$a = ldu$ $\quad a \in PM\{n:n|r:c\}$ $\quad l \in QM\{n:n|r:-r\}$

— ∘ — $\quad d \in QM\{n:n|r:c\}$ $\quad u \in QM\{n:n|-c:c\}$

r, n fixed

$$U_{i,i} \equiv \frac{\Pi_i(a)}{\Pi_i(a)} \qquad D_{n,n} \equiv \frac{a_{n,n}}{a_{n,n}} \qquad U_{i,j} \equiv \frac{|a(i,j+1;n:j;n)|\,\Pi_i^{[j]}(a)}{\Pi_i(a)}$$

$$L_{n,j} \equiv \frac{a_{n,j}\,\Pi_j(a)}{\Pi_j(a)} \qquad L_{n,n} \equiv \frac{a_{n,n}^2}{a_{n,n}} \qquad L_{i,j} \equiv \frac{|a(i;n:j,i+1;n)|\,\Pi_j^{[i+1]}(a)}{\Pi_j(a)}$$

m ~~…~~ $= n+p+1$

$$L_{i,i} \equiv \frac{|a(i;n:i;n)|}{|a(i+1;n:i+1;n)|}$$

$(6x)$ mod

$$U_{i,j} \equiv \frac{|a(i,j+1;n:j;n)|}{|a(j;n:j;n)|} \qquad L_{i,j} \equiv \frac{|a(i;n:j,i+1;n)|}{|a(i;n:i;n)|}$$

$$D_{i,i} \equiv \frac{|a(i;n:i;n)|}{\cancel{|a(i;n:i;n)|}\;|a(i+1;n:i+1;n)|}$$

$a = UDL$ $\quad U \in QM\{n:n|$ $\quad D \in QM\{n:n|$ $\quad L \in QM\{n:n|$

$$|a'(n:n)||a(n-1;n-1)| - |a(n;n)||a'(n-1:n-1)| = |a''(n,n)|$$

$$\cancel{|a(n:2}$$

$$|N(n:1;1)a(n:2;n)| \qquad |a(n-1:2;n)|$$

$$a'(n;n-1)\; a(n:n-1)$$

$$0'(n-1:n-1)a(n-1:n-1)$$

$$\frac{a'(n:n)}{a(n:n)} = \frac{a'_{1,1}}{a_{1,1}} + \sum_{m=2}^{n} \frac{|a''(m:m)||a(m-1:2;m)|}{|a(m:m)||a(m-1:m-1)|}$$

$$\frac{|a(k:k)|}{|a(2;k:2;k)|} - \frac{|a(k-1:k-1)|}{|a(2;k-1:2;k-1)|} = -\frac{|a(k-1:2;k)||a(2;k:k-1)|}{|\ldots}.$$

$$\frac{a(2:2)}{a(2;2:2;2)} - \frac{a(1:1)}{a(2;\phi:2;\phi)} = -\frac{|a(1:2;2)||a(2;2:1)}{}$$

$$\frac{a(2:2)}{a_{2,2}} = a_{1,1} - \frac{\cancel{|a(1:2;}\; a_{1,2}\, a_{2,1}}{a_{2,2}}$$

$$\frac{a(n:n)}{a(2;n:2;n)} = a_{1,1} - \frac{a_{1,2}a_{2,1}}{a_{2,2}} - \sum_{k=3}^{n} \frac{|a(k-1:2;k)||a(2;k:k-1)|}{|a(2;k:2;k)||a(2;k-1:2;k-1)|}$$

$$\cancel{a_{i,j}} \qquad a(n:n) := a(m-1, i : m-1, j)$$

$$a_{i,j} = \frac{a_{i,1}a_{1,j}}{a_{1,1}} + \sum_{k=3}^{\min(i,j)+1} \frac{|a(k-2,i:k-1)||a(k-1:k-2,j)|}{|a(k-1:k-1)||a(k-2:k-2)|}$$

a

$$\begin{vmatrix} a(n+1:n+1) & a(n+1,n-1) \\ O(n-1:n+1) & a(n-1,n-1) \end{vmatrix}$$

$$|a(n+1:n+1), a(n+1,n-1) / O(n-1:n+1), a(n+1:n-1)|$$

$$|a(n,n)||a(n-1,n+1:n-1,n+1)| - |a(n:n-1,n+1)||a(n-1,n+1:n)|$$

$$= |a(n-1:n-1)||a(n+1:n+1)|$$

$$|a'(n:n-1), a(n:n-1) / O'(n-1:n-1), a(n-1:n-1)|$$

$$|a'(n;n)||a(n-1:n-1)| - |a(n;n)||a'(n-1;n-1)| = |a''(n;n)|$$

$$|a(n-1:2;n)|$$

$$\cancel{|a(n;n-1)}$$

$$\left|\begin{vmatrix} a(n:n) & a(n:2;n-1) \\ O(n-2,n) & a(n-2:2;n-1) \end{vmatrix}\right| \quad \frac{a'(1:1)}{a(1:1)}$$

$$\cancel{|a(n;n)||a(2;n:2;n)}$$

$$|a(n:n)||a(2;n-1:2;n-1)| - |a(2;n:2;n)||a(n-1:n-1)| =$$

$$-|a(n-1:2;n)||a(2;n:n-1)| \qquad (*)$$

$$\frac{|a'(m;m)|}{|a'(m-1;m-1)|} - \frac{|a(m;m)|}{|a(m-1;m-1)|} = |a''(m:m)||a(m-1$$

$$\frac{|a'(m;m)|}{|a(m;m)|} - \frac{|a'(m-1:m-1)|}{|a(m^{-1}:m^{-1})|} = \frac{|a''(m,m)||a(m-1:2;m)|}{|a(m:m)||a(m-1:m-1)|}$$

$L_{i,j}$: $\cancel{\frac{a(j-2,i:j-1)}{a(j-1:j-1)}}$ $\frac{|a(j-1,i:j)|}{|a(j:j)|}$, $L_{i,1} = \frac{a_{i,1}}{a_{1,1}}$ $L_{k,i} = 1$

$U_{i,j}$: $\frac{|a(i:i-1,j)|}{|a(i-1:i-1)|}$ $U_{1,j} = a_{1,j}$ $U_{j,j} = \frac{a(j:j)}{a(j-1:j-1)}$

$\cancel{L'_{i,1} = a_{i,1}}$ $U'_{1,j} = a_{1,j}$

$$\sum_{k=1}^{\min(i,j)} L_{i,k} U_{k,j} = a_{i,j}$$

$U'_{2,2} = |a(2:2)|$ $L'_{i,1} = a_{i,1} \pi_i^{(1)}(a)$

$$L'_{i,j} = \frac{|a(j-1,i:j)|}{|a(j,j)|} \prod_{k=1}^{i} |a(k,k)| = |a(j-1,i:j)| \pi_i^{(j)}(a) \checkmark$$

$$U'_{i,j} = \frac{|a(i:i-1,j)|}{|a(i-1:i-1)|} \prod_{k=1}^{j-1} |a(k,k)| = |a(i:i-1,j) \pi_{j-1}^{(i)}(a) \checkmark$$

$$\sum_{k=1}^{\min(i,j)} L'_{i,k} U'_{k,j} = \pi_i(a) \pi_{j-1}(a) a_{i,j} \qquad \cancel{L'_{1,1} U'_{1,1} = a^2_{1,1}}$$

$$L'_{2,2} = |a(1:2:2)| \, |a(1,1)| = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,1} \end{vmatrix} a_{1,1}$$

$$U'_{2,2} = |a(2:1,2)| \qquad \sum_{1}^{1} L'_{i,1} U'_{1,1} = \pi_i(a) a_{i,1}$$

$$\begin{vmatrix} a_{1,1} & 0 \\ a_{2,1} & \cdots \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,1} \end{vmatrix} \end{vmatrix} \begin{vmatrix} a_{1,1} & a_{1,2} \\ 0 & \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,1} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} a_{1,1} & a_{1,1} a_{1,2} \\ a_{2,1} a_{1,1} & a_{1,1}^2 \cdots a_{2,2} \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{a_{2,1}}{a_{1,1}} & 1 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} \\ & \frac{|a(2:2)|}{a_{1,1}} \end{pmatrix} \checkmark$$

$$\frac{|a(k;k)|}{|a(2;k:2;k)|} - \frac{|a(k-2:k-2)|}{|a(2;k-2:2;k-2)|} = -\frac{|a(k-1:2;k)||a(2;k:k-1)|}{|a(2;k:2;k)||a(2;k-1;2;k-1)|} - \frac{|a(k-2:2;k-1)||a(2;k-1:k-2)|}{|a(2;k-1:2;k-1)||a(2;k-2;2;k-2)|} \tag{62}$$

$$|a(2;k-2:2;k-2)||a(2;k-1:2;k-1)||a(k;k)|$$

$$-|a(2;k-1:2;k-1)||a(2;k-2:2;k-2)||a(k-2;k-2)|$$

$$= -|a(2;k-2:2;k-2)||a(k-1:2;k)||a(2;k:k-1)|$$

$$-|a(2;k:2;k)||a(k-2;2;k-1)||a(2;k-1:k-2)|$$

eliminate $|a(k-1:k-1)|$ from (*) and (*) with k replaced by $k-1$

$|a(k-2:k-2)|$

The determinants $|a'(k-m:k-m)|$ $(m=1,\dots,k-2)$ may be eliminated from the $k-2$ relationships obtained by replacing a by a' and n by $k, k-1, \dots, 3$ ~~and~~ in (*) and the special version ~~obtained~~ holding when $n=2$. A relationship between $a'_{1,1}$ and $|a'(k:k)|$, ~~is obtain~~ which involves none of the eliminated determinants, is obtained. It is now supposed that the a'

trip you would permit me, after telephoning for
confirmation, to drive out

is a subarray of a further array a' specified by the equation $a'(k;k) = \cancel{a'(k-1,i;k-1,j)}\ a^*(i,1;k-1:j,1;k-1)$; the ~~rela~~ derived relationship may be reformulated so as to involve $a^*_{i,j}$ and $a^*(i,1;k-1:j,1;k-1)$. The reformulated relationship may be described in the following way. Set

$$\Pi_i^{(j)}(a) = \prod_{m=1}^{i\ (j)} |a(m,m)|$$

where the bracketed j indicates that the factor corresponding to $m=j$ is to be omitted; and let $\Pi_i(a)$ be the corresponding product without omission of terms.

(the classes to which the zero elements of these matrices belong is not of immediate concern)

Let and $A=(a(n:n))$, then $\Pi A \Pi = LU$ — special conv.

$$L_{i,1} = a_{i,1}\Pi_i^{(1)}(a) \qquad U_{1,j} = a_{1,j}\Pi_j(a)$$

$$L_{i,j} = |a(j-1,i:j)|\Pi_i^{(j)}(a), \quad U_{i,j} = |a(i:i-1,j)|\Pi_j^{(i-1)}(a)$$

(~~zero~~ empty products are to be omitted from the formulae).

Then

$$\sum_{k=1}^{\min(i,j)} L_{i,k}U_{k,j} = \Pi_i(a)\,\Pi_j(a)\,a_{i,j}$$

If L, & U are the lower and upper triangular matrices with elements $L_{i,j}$ $(i=1,\dots,n;\ j=1,\dots,i)$ and $U_{i,j}$ $(i=1,\dots,n;\ j=i,\dots,n)$ respectively and Π is the diagonal matrix with elements $\Pi_k(a)$ $(k=1,\dots,n)$ (the classes to which the

56 Def P-numbers (= S-numbers) 104

61 Def of $a(h;m:k;n)$

63 $a(h_1;m_1:k_1;n_1), a'(h_2;m_2:k_2;n_2)\}$

$a(h;m:u,k;n)$ $a(h;m:k;n,u)$

~~a(v,v:k;n)~~ $a(v,h;m:k;n)$ $a(h;m,v;k;n)$.

$a^{[i:]}(h;m:k;n)$ $a^{[:j]}(h;m:k;n)$

omission of $h=1$ ($k=1$?)

63 P-array $a \in PA(m;n|r:c)$ PM??

64 compound deletion...

65 $a \in PM\{n;n|r;c\}$

$v(1)=r(1)+c(1)$ $v(k)=\sum_{u=1}^{k-1} u\{r(k-u)+c(k-u)\}$ $k=2,..,n$

$\pi_k^{[h]}(a)=\prod_{m=1}^{k-1}{}^{[h]} |a(m;m)|$ $k=2,..,n$

$\pi_k(a)$ similarly without ommission of term in $m=h$ v?

π' π'' diagonal matrices in $PM\{n:n|r+v:-r\}$ $PM\{n:n|-c:c+v\}$

$\pi'_{1,1}, \pi''_{1,1}=a_{1,1}$ $\pi'_{k,k}=\pi''_{k,k}=\pi_k(a)$ $k=2,...n$

(l) lower triangular matrix in $PM\{n:n/r+v:-r\}$

$l_{1,1}=a_{1,1}$ $l_{i,i}=\pi_i(a)$ $i=2,..,n$ $l_{i,j}=|a(j-1,i:j)|\pi_i^{[j]}(a)$

$i=2,..,n$ $j=1,..i-1$

(u) upper triangular matrix in $PM\{n:n/r:c+v\}$

$u_{1,1}=a_{1,1}^2$ $u_{1,j}=a_{1,j}\pi_j(a)$ $j=2,..,n$ $u_{i,j}=|a(i;i-1,j)|\pi_j^{[i-1]}(a)$

$j=2,..,n$; $i=2,..,j$

$(lu)=(\pi' a \pi'')$

bp 42

The matrix $(\pi' a \pi'')$ possesses only one decomposition of the form (lu) as described in which the lower <upper> triangular matrix $(l)\langle(u)\rangle$ has diagonal elements $l_{i,i}\ \langle u_{i,i}\rangle$ as given iff when $n=1$ $a_{1,1}$ is not a divisor of zero and when $n\neq 1$ $|a(m,m)|$ is not a divisor of zero for $m=1,\ldots,n-1\ \langle n\rangle$

(ii) $\tilde{v}(1)=\sum_{u=2}^{n}\{r(u)+c(u)\}$ $\tilde{v}(k)=\sum_{u=2}^{k-1}(u-1)\{r(u)+c(u)\}+(k-1)\sum_{u=k}^{n}\{r(u)+c(u)\}$ $(k=2,\ldots,n)$

$\tilde{a}_{i,j}=(-1)^{i+j}a^{[j:i]}(n:n)\quad i,j=1,\ldots,n$

$\tilde{r}(k)=\sum_{u=1}^{n}{}^{[k]}c(u)\quad \tilde{c}(k)=\sum_{u=1}^{n}{}^{[k]}v(u)$

$\tilde{\pi}'\quad\tilde{\pi}''$

$\tilde{l}\quad\tilde{u}$

bp71 remark on uniqueness

p70 bp42, 41, 40, 39

P numbers 56

Parray 63 ~~61-63~~

notation $a(\ldots)$ 61-63 64

Determinant ?

Jacobi + Schweins ~~bp30 bp84~~ bp 25, 24 bp24 / pp 30 – 38 bp 38''

P matrix bp62 – bp 58 bp 35 – bp30 bp28

prod bp 57 + bp 56

LU dec of $\pi a \pi'$ 65-68 69-71 bp 42 bp42 – 39

Q-numbers bp 95 – bp 90

LU dec of $a\ Pa^{-1}$ bp80 – bp84

Q-numbers bp 95, bp 94 bp 93 bp 83 bp 92 – bp 87 bp55 bp 84 (statement of th.)

bp90 – bp 84

bp 64 remark on PA bp 63

bp 14 for $a_{i,j}=\ldots$

P_matrices bp62 – bp 58

56, 61 – 64 ~~30~~ 27 – 38

65 – 71

(b) ∈ PL(a) |ab| = |a| |b| bp 57 – bp 56

27 – 38 56 61–71 80 ~~[illegible]~~ – 101''

~~bp 29 + bp25 same result?~~

bp 95 – bp 83, bp 71, bp 64 – bp 56 bp 38''

bp 42 – bp39 bp 35 – bp 30 bp28 ~~bp 25, bp24~~

bp 25, 24 Jacobi on adj, Schweins, Sylvester

bp 27 – bp 14 bp12 bp 10 – bp8

Proof bp24 pp 30, 31, 32; 33, 34, 35, ~~36, 37~~ bp 38'' 35, 36 37 38

bp 35 – bp 30, bp 28

adjunction bp 15 mult x x bp 16 also bp 17 bp 18 bp 19 bp 23 – bp 20

P_matrices & + relationships