S: $a = b$ mod if $a - b \in S[0]$

D: system of numbers $d \in S$ such that $da \in S[0]$ ~~if~~ only if $a \in S[0]$

Quotient extension Q of S: ordered pairs $\langle c,d\rangle$ $(d \in D)$. $Q[0]$ all $\langle c,d\rangle \in Q$ with $c \in S[0]$.

Equality $\langle c,d\rangle = \langle e,f\rangle$ iff $cf = de$

Equality reflexive, symmetric transitive ($\langle e,f\rangle = \langle g,h\rangle$:

$cf = de \quad eh = fg \quad cfh = deh = dfg \to ch = dg \ (f \in D)$. $\therefore \langle c,d\rangle = \langle g,h\rangle$

Mult: $\langle c,d\rangle * \langle e,f\rangle = \langle ce, df\rangle$ Addn. $\langle c,d\rangle \pm \langle e,f\rangle = \langle fc \pm ed, df\rangle$

Mult assoc. comm. Addn: comm. assoc, Mult dist. wrt. addn.

If $S\{j\}$ is an ~~A~~ additive Abelian group, $\exists$ corresponding group in $Q\{j\}$ appropriate numbers of form $c \in S\{j+k\}$ $d \in S\{k\}$ all represent zero members of this group. $\langle c,d\rangle \in D(Q)$ iff $c \in D(S)$

—o—

$\langle c,d\rangle = \langle e,f\rangle$ if $\langle c,d\rangle - \langle e,f\rangle \in Q[0]$

(a) ~~itself~~: a representative number class of numbers in Q equal to $\langle ad, d\rangle$ any $d \in D$. class includes $\langle af, f\rangle$ all $f \in D$ $(adf = afd)$

$(a)(b)$ defined by $\langle abdf, df\rangle$

$(a) \pm (b)$ defined by $\langle fad \pm fbd, df\rangle$

$S \subsetneq Q$

—o—

$d \in D$: $dx = c$ is soluble in form $(d)\,x = (c)$ $x \in Q$ $x = \langle c,d\rangle$

if $dy = c$ $(d)(x-y) \in Q[0]$ $x = y$ in Q ‖ $x = \langle y,z\rangle$ satisfies $(d)x = (c)$ iff $dy = cz$

$dx = c \quad fy = e \quad xy$ satisfies $df\,xy = ec$

$x \pm y$ satisfies $df(x \pm y) = fc \pm ed$

Given ~~x satis~~ $x \in Q$ satisfies $dx = c$ $\exists\, y \in Q$ such that $h\,xy = g$. Determine equation satisfied by y

$h(dx)y = dg \quad hcy = dg \quad c \notin D$?

1 Given: $x \in Q$ satisfies $dx = c$. $\exists\ y \in Q$ such that $h(x+y) = \underset{0}{g}$

Determine equation satisfied by y

$dhx + dhy = d\underset{0}{g}$ $\quad dhy = d\underset{0}{g} - ch \quad dh \in D$

—o—

$(b)x = (a)$ $\quad b \notin D$. Suppose solution $x = \langle y, z \rangle \in Q$ exists

$by = za$ $\quad$ if $b\underset{0}{g} \in S[0]$ $\quad by\underset{0}{g} \in S[0] \to za\underset{0}{g} \in S[0] \to a\underset{0}{g} \in S[0]$

$(b)x = (a)$ has a solution in Q only if $\mathcal{O}(b) \subseteq \mathcal{O}(a)$

and then $\mathcal{O}(y) \subseteq \mathcal{O}(a)$ $\quad (y\underset{0}{g} \in S[0] \to a\underset{0}{g} \in S[0])$

suppose u, v exist such that $ub + v \in D$, then $\langle ua, ub+v \rangle$ satisfies $(b)x = (a)$ if $av = 0$. Also $\langle a, b+v \rangle$ does so if $b+v \in D$ and $av = 0$

$(b+v)y = za + vy$ ‖ soluble iff $aD/''b$ nonvoid

$b = de \quad dex = a \quad dey = za$

$ba = (b+v)a \qquad bua = (ub+v)a$

b fixed: $C'(b)$ all v such that either $b+v \in D$ or $\exists u$ for which ~~$av = 0$~~ $ub+v = 0$

$Z(b)$: all a such that $\mathcal{O}(a) \cap C'(b)$ nonvoid

$bua = (ub+v)a \qquad by = za \qquad zbua = (ub+v)by$

$(by)v = 0$ ‖ general solution of $(b)x' = (a)$ $\quad x' = x + t$

$t = \langle e, f \rangle \quad e \in \mathcal{O}(b)$ and then $e \in \mathcal{O}(a)$

N' all b such that $C'(b)$ is nonvoid

$N' \in M(S)$

$(b)x = (a) \quad (d)y = (c) \quad (bd)\,xy = (ac)$?

$ub + v \in D \quad av = 0 \quad sd + t \in D \quad ct = 0$

$usbd + ubt + sdv + tv \in D \quad ac(ubt + sdv + tv) = 0$

① general solution of $(bd)z = (ac)$: $xy + \langle e, f\rangle \quad e \in \mho(bd)$

② {general solution of $(b)x = (a)$}{general solution of $(d)y = (c)$}

$\{x + \langle g, h\rangle\}\{y + \langle i, j\rangle\} \quad g \in \mho(b) \quad i \in \mho(d) \quad h, j \in D$

$xy + \langle uai, (ub+v)j\rangle + \langle scg, (sd+t)h\rangle + \langle ig, jh\rangle$

~~since $ac\,uai = ac\,scg = ac\,ig = 0$ ② ⊆ ①~~

$bdg = 0 \to adg = 0$

③ $(b)(d)(x+y) = (d)(a) + (b)(c)$

if this equation is to be soluble, must have

$(da+bc)\{ubt + sdv + tv\} = 0 \quad \Big\| \quad uasd + \underbrace{uat} + ubsc + \underbrace{scv}$

• when $av = ct = 0$ lhs is $daubt + bcsdv \neq 0$

possibly insoluble ∴ introduce restriction $bv \in 0$

so that ~~bv~~ $av = 0$ $Z(b)$ is simply $a : \mho(b) \subseteq \mho(a)$

N : all b such that $C'(b) \cap \mho(b)$ is nonvoid

$N \in M(S)$ ③ is soluble if for all g such that $bdg = 0$

$(ad+bc)g = 0$ also $bdg = 0 \to adg = 0$ and $bcg = 0$

look at gen. soln for sq.-fra

(i) $x = \langle ua, ub+v \rangle$ general solution of $(b)x' = (a)$ is $x + \langle g, h \rangle$ $g \in \bar{O}(b)$

(ii) $y = \langle sc, sd+t \rangle$ " " " $(d)y' = (c)$ " $y + \langle i, j \rangle$ $i \in \bar{O}(d)$

$z = \langle usac, usbd + ubt + sdv + tv \rangle$ " " " $(bd)z' = (ac)$ " $z + \langle e, f \rangle$ $e \in \bar{O}(bd)$

$f, h, j \in D$

(g.s (i))(g.s. (ii)) : $xy + \langle uai, (ub+v)j \rangle + \langle scg, (sd+t)h \rangle + \langle ig, jh \rangle$ (iii)

$g \in \bar{O}(b)$, $\bar{O}(b) \subseteq \bar{O}(a) \to ag = 0$ also $ci = 0$

any member of set (iii) satisfies $(bd)z' = (ac)$

$z + \langle e, f \rangle$ expressible as member of set (iii)?

given e such that $bde = 0$ $f \in D$ determine g, i such that $bg = di = 0$

and $h, j \in D$ such that

$$\langle e, f \rangle = \langle uai, (ub+v)j \rangle + \langle scg, (sd+t)h \rangle + \langle ig, jh \rangle$$

$$= \langle uaih(sd+t) + scgj(ub+v) + \underbrace{ig(ub+v)(sd+t)}_{igvt}, jh(ub+v)(sd+t) \rangle$$

$$= \langle uaiht + scgjv + igvt, \ldots \rangle$$

$$= uah\,it + scj\,gv + igvt$$

$$f\{uah\,it + scj\,gv + igvt\} = jhe(ub+v)(sd+t)$$

$$= jhe\,vsd + jhe\,tub + jhevt$$

~~fcjg~~ $fcg = hed$ $fai = jeb$ $fig = jhe$

$g = xed$ $i = yeb$ $fcx = h$ $fay = j$ $fxyebd = jh$

$f, h \in D$ but $c \notin D$; must be able to choose g and $x \in D$ such that

$cg = xed$, but c may not be a denominator

$\{(b)(d)\}z' = \cancel{ad}(d)(a) + (b)(c)$ does $z' = x + \langle g, h \rangle + y + \langle i, j \rangle$

satisfy this equation $\langle bd\,ua, ub+v \rangle + \langle bd\,sc, sd+t \rangle$

$(b)(d)z'$ then $\Leftarrow$ $= \langle da(ub+v), ub+v \rangle + \langle bc(sd+t), sd+t \rangle =$

$ad(da+b$

is $\langle ua, ub+v\rangle = \langle u'a, u'b+v'\rangle$ $\quad u'=0 \quad v\in D$

$uu'ab + uav' = uu'ab + u'av$ $\quad uav'=0 \quad$ if $b'\neq 0$ cannot have $u'=0$

~~If $b\in N'$ alone~~ $\quad$ ~~for if $b'\neq 0$~~ for if $u'=0 \quad v'\in D$

$bv'=0 \to b=0$

$b\in N' \quad \exists\, a\in Z'(b) \quad av=0$

then $\langle ua, ub+v\rangle = \langle u'a, u'b+v'\rangle$ $\quad \|\quad$ $b.\ ua,\ ub+v = (a)$ if $av=0$

— o —

with $d\in N$ does $x'=x+\langle g,h\rangle$ $g\in O(b)$ satisfy $(d)(b)x' = (d)(a)$

$O(b)\subseteq O(a) \to O(bd)\subseteq O(ad)$ all $d\in S$

$bdg=0 \to adg=0$ ✓ ④ has a solution

$(d)(b)g\equiv 0 \quad \langle db\,ua,\ ub+v\rangle = (da)$

— o —

above deals with case in which $(a),(b)$ are taken from subring of Q isomorphic to S.

Now consider equation $\langle b,d\rangle x = \langle a,f\rangle$ $\quad d,f\in D \quad b\in N$ $\quad a\in O(b)$

general solution is $x' = \langle d\,ua,\ (ub+v)f\rangle + \langle g,h\rangle$ $\quad g\in O(b)$

$\langle b,d\rangle x = \langle ad(ub+v), (ub+v)fd\rangle = \langle a,f\rangle$

$\langle b,b'\rangle x = \langle a,a'\rangle \quad \langle bd, b'd'\rangle z' = \langle ad, a'd'\rangle + \langle bc, b'c'\rangle$

$\langle d,d'\rangle y = \langle c,c'\rangle \quad z' = x'+y'$ is a solution of this equation

$x' = \langle b'ua, (ub+v)a'\rangle + \langle g,h\rangle \quad g\in O(b)$

$y' = \langle d's c, (sd+t)c'\rangle + \langle i,j\rangle \quad i\in O(d)$

$\langle bd, b'd'\rangle x' = \langle ad(ub+v)b', (ub+v)a'b'd'\rangle = \langle ad, a'd'\rangle$

similarly $z'' = x'y'$ is a solution of $\langle bd, b'd'\rangle z'' = \langle ac, a'c'\rangle$

$\langle bd, b'd'\rangle x'y' = \langle (ub+v)(sd+t)b'd'ac, (ub+v)(sd+t)a'c'b'd'\rangle$

With $d \in N$, general solution of $\langle b,b'\rangle x = \langle a,a'\rangle$

also satisfies $\langle d,d'\rangle\langle b,b'\rangle x = \langle d,d'\rangle\langle a,a'\rangle$.

$N(Q)$: all $\langle b,b'\rangle \in Q$ with $b \in N(S)$.

$\mathcal{O}(Q, \langle b,b'\rangle)$ all $\langle \mathring{o}, \mathring{o}'\rangle$ in Q such that $\langle b,b'\rangle\langle \mathring{o},\mathring{o}'\rangle \in Q[0]$

$Q[0]$ all $\langle a,a'\rangle$ with $a = 0 \in S$

$\langle b,b'\rangle x = \langle a,a'\rangle$ soluble in Q when $\langle b,b'\rangle \in N(Q)$

and $\mathcal{O}(Q, \langle b,b'\rangle) \subseteq \mathcal{O}(Q, \langle a,a'\rangle)$ | $\langle u,b'\rangle\langle b,b'\rangle + \langle v,b'^2\rangle$

$\langle b,b'\rangle \in N(Q)$ iff $b \in N(S)$, $\quad = \langle ub+v, b'^2\rangle \in D(Q)$ if

$\langle u,u'\rangle\langle b,b'\rangle + \langle v,v'\rangle \in D(Q)$ $\quad \langle b,b'\rangle\langle v,v'\rangle \in Q[0]$

$v'ub + u'b'v \in D(S)$ only if. $\quad bv = 0$ in S

$\mathcal{O}(Q, \langle b,b'\rangle) \subseteq \mathcal{O}(Q, \langle a,a'\rangle)$ if $\mathcal{O}(b) \subseteq \mathcal{O}(a)$ in S

—o—

if $b \in N$ is ~~an~~ general solution of $\langle d,d'\rangle x = \langle b,b'\rangle$ in $N(Q)$? no

general solution $= x' = \langle b.sd', (sd+t)b'\rangle + \langle \mathring{o}, \mathring{o}'\rangle$ $\mathring{o} \in \mathcal{O}(d)$

num of x' $\quad bsd'\mathring{o}' + (sd+t)b'\mathring{o} = bsd'\mathring{o}' + tb'\mathring{o}$

$ud\{bsd'\mathring{o}' + tb'\mathring{o}\} = (ub+v)(sd+t)d'\mathring{o}' - (vsd + \underbrace{ubt}_{dt=0 \to bt=0} + vt)d'\mathring{o}'$

$ud\{bsd'\mathring{o}' + tb'\mathring{o}\} + v(sd+t)d'\mathring{o}' = (ub+v)(sd+t)d'\mathring{o}' \in D(S)$

$\{bsd'\mathring{o}' + tb'\mathring{o}'\}v\xi = vtb'\mathring{o}' \neq 0$

Set $(b/d) = \langle sb, sd+t \rangle$ $(a/c) = \langle ya, yc+z \rangle$

$sd+t,\ yc+z \in D$ $dt = cz = 0$ $\mathcal{O}(d) \subseteq \mathcal{O}(b)$ $\mathcal{O}(c) \subseteq \mathcal{O}(a)$

Suppose $b \in N(Q)$. $ub+v \in D$ $bv = 0$

Wish (ad/bc) to be solution of $(b/d)x = (a/c)$

~~$b \in N(S) \to (b/d) \in N(Q)$~~ U lifting factors in S

$UN \subseteq N$ $V \subseteq N$

is $sb \in N(S)$ $u(sb) + sv$ Require $\mathcal{O}(bc) \subseteq \mathcal{O}(ad)$

$udsb = (ub+v)(sd+t) - \cancel{ubt} - vsd - tv$! $\mathcal{O}(d) \subseteq \mathcal{O}(b)$

$udsb + v(sd+t) = (ub+v)(sd+t) \in D(S)$ $\mathcal{O}(c) \subseteq \mathcal{O}(a)$

$sbv(sd+t) = 0$ ∴ $(b/d) \in N(Q)$ $\mathcal{O}(p) \subseteq \mathcal{O}(q)$ $\mathcal{O}(r) \subseteq \mathcal{O}(w)$ $\mathcal{O}(pr) \subseteq \mathcal{O}(qw)$

also require $\mathcal{O}(sb) \subseteq \mathcal{O}(ya)$

i.e. $yag = 0$ whenever $sbg = 0$

to ensure this in definition of $N(S)$, stipulate that

$\mathcal{O}(u) \subseteq \mathcal{O}(v)$ $\mathcal{O}(y) \subseteq \mathcal{O}(c)$ for c to be denominator

$\mathcal{O}(s) \subseteq \mathcal{O}(d)$ " d "

$a \leq b \leq d \leq s$ ~~$c \leq a$~~ $a \leq c$ $c \leq y$ show $ay \leq bs$

$a \leq b$ ~~$a \leq s$~~ $ab \leq s \to a \leq bs$ $\mathcal{O}(y) \subseteq \mathcal{O}(c) \subseteq \mathcal{O}(a)$

$\mathcal{O}(s) \subseteq \mathcal{O}(d) \subseteq \mathcal{O}(b) \subseteq \mathcal{O}(a)$ $\mathcal{O}(bs) \subseteq \mathcal{O}(a^2) \equiv \mathcal{O}(a) \subseteq \mathcal{O}(ay)$

S square free with respect to $\mathcal{O}$: $e^2 = 0$ only if $e = 0$

if $a^2 g = 0$ $(ag)^2 = 0$ ∴ $ag = 0$ $a^2 g = 0$ if and only if $ag = 0$

$\mathcal{O}(a^2) \equiv \mathcal{O}(a)$. ~~$\mathcal{O}(p) \subseteq \mathcal{O}(q)$ $\mathcal{O}(r) \subseteq \mathcal{O}(s$~~ $\mathcal{O}(p) \subseteq \mathcal{O}(q)$ $\mathcal{O}(w) \subseteq \mathcal{O}(z)$

$\mathcal{O}(uw) \subseteq \mathcal{O}(vz)$ $\mathcal{O}(u) \subseteq \mathcal{O}(v)$

$\mathcal{O}(a) \subseteq \mathcal{O}(ab)$ $ag = 0$ $abg = 0$ all $a, b \in S$ $uwg = 0 \to vwg = 0 \to vzg = 0$ $\mathcal{O}(w) \subseteq \mathcal{O}(z)$

~~$\mathcal{O}(b) \subseteq \mathcal{O}(b)$ $\mathcal{O}(c) \subseteq \mathcal{O}(d)$ $\mathcal{O}(ac) \subseteq \mathcal{O}(bd)$ acg~~

general solution of $(b/d)x=(a/c)$ is $\langle yaud, (yc+z)(ub+v)\rangle$ $+\langle g,h\rangle$ 8

$yaud \,,\quad (yc+z)(ub+v)$

$\cancel{g \subseteq sb}$ $g \subseteq \mathcal{O}(sb)=\mathcal{O}(b)$

$(sd+t)ud\,,\ (ub+v)(sd+t)$

$\mathcal{O}(b)\subseteq\mathcal{O}(a)$ $\quad \mathcal{O}(ab)\equiv\mathcal{O}(a)$? $\quad \mathcal{O}(a)\subseteq\mathcal{O}(ab)$

show $\mathcal{O}(ab)\subseteq\mathcal{O}(a)$ $\quad abg=0 \to a^2g=0 \to (ag)^2=0 \to ag=0$

$a\leq b \quad ab=b$ $\qquad \mathcal{O}(b)\subseteq\mathcal{O}(a)$

$\mathcal{O}(s)\subseteq\mathcal{O}(b) \qquad \mathcal{O}(sb)=\mathcal{O}(b)$

$\langle sb\,yaud, (yc+z)(ub+v)(sd+t)\rangle = \langle ya, yc+z\rangle$

{general solution of $dz=b$}x{general solution of $(b/d)x=(a,c)$}

should satisfy $cz'=a$

$\{\langle sb, sd+t\rangle + \langle k,m\rangle\}\{\langle yaud, (yc+z)(ub+v)\rangle + \langle g,h\rangle\}$

$bg=0 \quad dk=0 \to bk=0$

$\langle ua, ub+v\rangle = \langle sa, sb+t\rangle$?

$(a/b)(b/d)=(a/d)$

$ua(sb+t)=sa(ub+v)$

$bS+I \ :\ Z(b) \quad a\in Z(b) \quad x\in S$ s.t. $bx=a \mod I$

$dy=c$

$[a/b]$ all such x

$bcx=ady \qquad bc=ad$

$dy=c \qquad [a/b]=[c/d]$?

$ad(x-y)=0$

$dby=bc=ad$ if $d\in D$ $by=a$ i.e. $[c/d]\subseteq[a/b]$

$d\in D + bc=ad \to [c/d]\subseteq[a/b]$ | $bde=bcf=adf$

$f\in D + de=cf \to [e/f]\subseteq[c/d]$ | $\to be=af$ if $d\in D$

$f\in D \quad de=cf\,, bc=ad \to be=af$?

for which ~~the~~ $d'g = 0$ only when $g = 0$ be nonvoid. The members of D form a multiplicative system: $b', d' \in D \Rightarrow b'd' \in D$

Q complete system of ordered pairs $\langle a, b' \rangle$ ~~of a~~ formed from all $a \in S$, $b' \in D(S)$

Define ~~d~~ equality in Q by stipulating that $\langle a, b' \rangle = \langle c, d' \rangle$ iff $ad' = b'c$ in S

Such equality is evidently reflexive, symmetric and also transitive (. . .)

Define addition ~~and subtraction~~ in on Q by setting $\langle a, b' \rangle + \langle c, d' \rangle = \langle ad' + b'c, b'd' \rangle$ and multiplication by $\langle a, b' \rangle \langle c, d' \rangle = \langle ac, b'd' \rangle$. Addition and multiplication on Q are associative and commutative and multiplication is distributive with respect to addition. ~~The ses most~~ Let $Q[0]$ be the ~~set~~ complete syst of Q, numbers of the form $\langle a, d' \rangle$ with $a = 0$ in S. All members of $Q[0]$ are equal in Q

A number functions as an additive zero in Q if and only if it is in $Q[0]$.

Q is a commutative ring. It contains a nonvoid system D(Q) of numbers $\langle d, d' \rangle$ for which $\langle d, d' \rangle \langle g, g' \rangle \in S[0]$ if and only if $\langle g, g' \rangle \in S[0]$; D(Q) being the complete system of such numbers $\langle a, b' \rangle \in D(Q)$ iff $a \in D(S)$. The members of D(Q) form a multiplicative system.

~~In Q~~, Let (a) be the class of Q-numbers equal to $\langle ad', d' \rangle$ for some $d' \in D(S)$. Since $\langle ab', b' \rangle = \langle ad', d' \rangle$ for all $b', d' \in D(S)$, the class (a) is independent of the number $d' \in D(S)$ used in the above definition. Taking any $\langle c, d' \rangle \in (a)$, $\langle e, f' \rangle \in (b)$, stipulate that $(a) = (b)$ if and only if $\langle c, d' \rangle = \langle e, f' \rangle$ in Q. Such equality is independent of the selected representatives $\langle c, d' \rangle$ and $\langle e, f' \rangle$. Define the sum $(a) + (b)$ to be the class of Q-numbers equal to $\langle ce, d'f' \rangle$. ~~Taking~~ ~~$\langle g, h' \rangle$ to~~ All members $\langle g, h' \rangle$ of this sum class are equal to each member of the class (ab)

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With C_1 and C_2 two such classes of ~~Q-numbers~~, take any $\langle c, d' \rangle \in C_1$, $\langle e, f' \rangle \in C_2$. Stipulate that $C_1 = C_2$ if and only if $\langle c, d' \rangle = \langle e, f' \rangle$ in Q. Define the ~~product~~ sum $C_1 + C_2$ to be the class of Q-numbers equal to $\langle c, d' \rangle + \langle e, f' \rangle$, and define the product class $C_1 C_2$ ~~in t~~ similarly in terms of $\langle c, d' \rangle \langle e, f' \rangle$.

Let C be a class of Q-numbers all equal to each other in Q, C being such that no Q-number not in C is equal in Q to a member of C.

.. definition. $(a) = (b)$ if and only if $a = b$ in S. Furthermore $(a) + (b) = (a+b)$ and ~~(a+b)~~ $(a)(b) = (ab)$. The classes (a) form a commutative ring ~~in~~ isomorphic to S.

If S contains three numbers a, b, c such that $a = bc$, the equation

$$bx = a$$

has at least one solution in S, namely c. If no such number c exists, equation has no solution in S. However, if b is Q-number in the class (b') where $b' \in D(S)$, and a'' is a Q-number belonging to the class (a), the equation $bx = a''$ has a solution in Q, namely $\langle a, b' \rangle$. The complete system $\{x\}$ $\{a'', b\}$ of Q-number solutions of the equation $bx = a''$ is the complete system of Q-numbers equal to $\langle a, b' \rangle$.

With d a Q-number in the class (d') where $d' \in D(S)$ and c'' a Q-number belonging to the class (c), the equation $dy = c''$ has the Q-number solution $\langle c, d' \rangle$. The equation

$bdz = (a''d + b c'')$ also has a solution, and $\{a'', b\} + \{c'', d\} = \{a''d + bc'', bd\}$. Similarly $\{a'', b\}\{c'', d\} = \{a''c'', bd\}$.

The above results can be extended to the case in which the Q-number a'' featuring in the equation $b''x'' = a''$ is a general Q-number, ~~not necessarily taken~~ $\langle e, f' \rangle$ say. With b'' taken from the class (b') with $b' \in D(S)$, ~~the Q-number $\langle e, f' b' f' \rangle$ is a solution of this equation and again~~ the complete system of solutions to this equation is the complete system of Q-numbers equal to $\langle e, b'f' \rangle$. Also b'' may be taken to be any number in $\in D(Q)$, $\langle g', h' \rangle$ say, with $g', h' \in D(S)$ and, with a'' taken from the class (a), the complete system of solutions to the equation $b''x'' = a''$ is ~~$\langle ah', g' \rangle$~~ the complete system of Q-numbers equal to $\langle ah', g' \rangle$. These two extensions may be combined. Mutatis mutandis, the results concerning solutions to equations obtained by adding and multiplying two equations may also be extended.

$\{a'', b''\}$ for sys.? $[a, b']$? for Q-nu?

Under certain conditions, the system Q may be used to derive solutions to the equation $b''x'' = a''$ when $a'' \in (a)$, $b'' \in (b)$ and $b \notin D(S)$. It is first remarked that if this ~~condition~~ equation has a solution, then, since $a''g'' = x''b''g''$, $a''g'' \in Q[D]$ whenever $b''g'' \in Q[D]$. Thus, denoting the complete system of numbers $g'' \in Q$ for which $a''g'' \in Q[D]$ by $\Theta(Q|a'')$, the condition $\Theta(Q|b'') \subseteq \Theta(Q|a'')$ is necessary for and this condition holds if and only if [illegible].

solution to be possible. It is also ~~necessary~~ required that b should be associated with a member of $D(S)$ (in the previous discussion, b simply ~~is~~ a member of $D(S)$). Denote by $N''(Q)$ the complete system of numbers $b \in S$ such that either (1) a direct complement $v \in S$ exists such that $b+v \in D(S)$ or (2) a lifting factor $u \in \sigma S$ and a displaced complement $v \in S$ exist for which $ub+v \in D(S)$; ~~Sup~~ Let $b'' = \langle bd', d' \rangle$, and ~~&~~ $a'' = \langle ad', d' \rangle$ $(d' \in D(S))$; ~~If~~ ~~the~~ $x'' = \langle ua, ub+v \rangle$ is ~~to be~~ a solution ~~&~~ of the equation $b''x'' = a''$ ~~then~~ if and only if $uab = a(ub+v)$, i.e. $av = 0$. Thus ~~if $C''(b) \cap \mathcal{O}(a)$ is nonvoid~~ denoting the complete system ~~numerator set $Z(S|b)$~~ of numbers a for which $\mathcal{O}(b) \subseteq \mathcal{O}(a)$ and $C''(b) \cap \mathcal{O}(a)$ is nonvoid, the equation $b''x'' = a''$ has, ~~the special $\langle ua, ub+v \rangle$~~ ~~a solution in Q when $b \in N(S)$ and $a \in Z(S|b)$ (the case in which b satisfies~~

{denote the complete system of numbers v featuring in the preceding relationships by $C''(v)$

when $b \in N(S)$ and $a \in Z(S|b)$, ~~the~~ and v is so chosen that $av = 0$, the special solution $\langle ua, ub+v \rangle$ (the case in which b satisfies condition (1) above and the special solution is $\langle a, b+v \rangle$ is dealt with similarly}. $\{a, b\}$, the complete

~~Denote the class of Q-numbers equal to $\langle ua, ub+v \rangle$ by $[a/b]$~~

system of Q-number solutions of the equation $BX = A$, is the ~~complete system of Q-num~~ union of all systems of Q-numbers equal to (when $g=0$ for all $g\in\bar{O}(b)$) $\langle ua, ub+v\rangle + \langle g, h'\rangle$ for each $g\in\bar{O}^{g}(b)$ and $h'\in D(S)$. Unless $b\in D$, all members of $\{a,b\}$ are not mutually equal ~~to~~ in Q.

Denote the class of Q-numbers equal to the special solution $\langle ua, ub+v\rangle$ by $[a/b]$. This class is independent of the lifting factor-displaced complement pair (u,v) serving to define it. For suppose s,t to be another such pair, so that $sb+t\in D(S)$, and $at=0$, in S. Then, since $ua(sb+t) = sa(ub+v)$, $\langle ua, ub+v\rangle = \langle sa, sb+t\rangle$.

~~The members of $N''(Q)$ form a multiplicative system~~

Supposing that $ub+v, sd+t\in D(S)$, $(ub+v)(sd+t) = (us)bd + (ubt+sdv+vt)\in D(S)$ also. ~~Thus the members of $N''(Q)$ form a multiplicative system, the numbers us and ubt+sdv+vt functioning~~ Thus if the numbers u,v and s,t function as lifting factor-displaced complement pairs for the members b and d of $N(Q)$, the numbers us, $ubt+sdv+vt$ function as such a pair for the product bd: the members of $N''(Q)$ form a multiplicative system.

Let a,b,u and v be four S-numbers ~~with the properties~~ possessing the properties described above, and c,d,s and t be a similar set of numbers, so that $sd+t\in D(S)$, $\bar{O}(d)\le\bar{O}(c)$ and $ct=0$. $bd\in N''(Q)$, from the preceding paragraph. If $\bar{O}(b)\le\bar{O}(a)$ and $\bar{O}(d)\le\bar{O}(c)$ then ~~$\bar{O}(bd)\le\bar{O}(ac)$ also~~: for any $g\in S$ such that $bdg=0$, $adg=0$ since $\bar{O}(b)\le\bar{O}(a)$ and $acg=0$ since $\bar{O}(d)\le\bar{O}(c)$: $\bar{O}(bd)\le\bar{O}(ac)$. Also $ac(ubt+sdv+vt)=0$, since $av=at=0$. In conclusion $bd\in N''(Q)$ and $ac\in Z''(S|bd)$. The ~~class of~~ Q-number class

$[ac/bd]$ is well defined, and $[a/b][c/d] = [ac/bd]$. Also, ~~denoting the union of all the sets of Q numbers equal to each product formed from one member of $\{a,d\}$ and one member~~ letting $\{a,b\}\times\{c,d\}$ be the complete set ~~theoretic product~~ of products formed from one member of $\{a,b\}$ and $\{c,d\}$ for each such pair and ~~$\{\{a,b\}\times\{c,d\}\}$ to be~~ denoting the union of the sets of Q-numbers equal to each member of $\{a,b\}\times\{c,d\}$ by $\{\{a,b\}\times\{c,d\}\}$,

x.7 $\{ac, bd\} \equiv \{\{a,b\}\times\{c,d\}\}$. With the denominator set $N''(S)$ in S, and the numerator set $Z''(S \mid b)$ associated with $b \in S$ defined as above, the classes $[a/b]$ obey the multiplicative law $[a/b][c/d] = [ab/cd]$ and the systems $\{a,b\}$ behave in a similar way.

Unfortunately it is not true ~~that, if~~ the fractions ~~classes~~ $[a/b]$ and $[c/d]$ ~~are~~ being so well defined, that their sum is $[\{ad+bc\}/bd]$. ~~is likewise,~~ The latter fraction may not be well defined. It is still true, as is required, that if $b, d \in N''(S)$, $bd \in N''(S)$, also. It is also true that if $O(b) \subseteq O(a)$ and $O(d) \subseteq O(c)$, then $O(bd) \subseteq O(ad+bc)$; for if $bdg = 0$ then $adg = 0$ since $O(b) \subseteq O(a)$ and $bcg = 0$ since $O(d) \subseteq O(c)$, so that $(ad+bc)g = 0$. But with a, b, u, v and c, d, s, t two sets of numbers ~~$(ad+bc)=$~~ as described above, $(ad+bc)(ubt+sdv+vt)$ ~~$=0$~~ $= adubt + bcsdv$, since $av = ct = 0$, and it may not be true that $adubt + bcsdv = 0$ as is required ~~for~~ to establish the existence of the fraction ~~class~~ $[ad+bc/bd]$.

By suitably restricting the denominator set $N(S)$, it is possible to ensure that ~~the sums of two~~ fractions ~~is a fraction~~ are added in accordance with the law $[a/b]+[c/d]=[(ad+bc)/bd]$. Let $N'(S)$ be that subset of ~~$N''(S)$~~ members $b \in N''(S)$ for which, in both cases (1,2) above, ~~the complement~~ v is an orthogonal complement in the sense that $bv=0$. When

Let $C'(b)$ be the complete system of such orthogonal complem. to v.

~~b is confined to $N'(S)$,~~ the numerator set $Z'(b)S|b)$ ~~assoc of b~~ is simply ~~that~~ the complete set of numbers a for which $O(b) \subseteq O(a)$. The further condition that $C''(b) \cap O(a)$ should be nonvoid is automatically ~~full~~ satisfied, since for $C'(b) \subseteq C''(b)$ and $b \in N'(S)$, $C'(v) \subseteq O(b)$ and accordingly $C'(b) \subseteq O(a)$, and $O(a)$, in containing 0, is always nonvoid. That the fraction $[a/b]$ is again independent

Now any $v \in C'(b)$ may be used in the definition of the fraction

of the lifting factor-orthogonal displaced complement pair $\{u,v\}$ serving to define it is shown exactly as above

$[a/b]$, since the two conditions $bv=0$, $O(b) \subseteq O(a)$ automatically imply the required condition $av=0$

Letting $b,d \in N'(S)$ and $(u,v) \in UV'(b)$, $(s,t) \in UV'(d)$,

~~Taking u,v and s,t to be lifting factor, orthogonal displacement pairs associated with b and d in N'(S),~~

us and $ubt+sdv+vt$ function as ~~such a pair for bd,~~ a lifting factor-displaced complement pair for bd, as was shown above; but now ~~since~~ the auxiliary conditions $bv=dt=0$ imply that $bd(ubt+sdv+vt)=0$: bd is in $N'(S)$: the members of $N'(S)$ form a multiplicative system.

~~That the fractions obey~~ The multiplicative law $[a/b][c/d]=[ac/bd]$ for fractions and the relationship ~~and that solution systems satisfy the rel~~ ~~{ac,d~~ $\{\{a,b\} \times \{c,d\}\} = \{ac,bd\}$ for solution systems holding for all $b, d \in N''(S)$ and $a \in Z(b)$, $c \in Z(d)$ hold, in particular, for all $b, d \in N'(S)$

$\subseteq N''(S)$ and $a \in Z(b)$, $c \in Z(d)$.

When $b, d \in N'(S)$ and $a \in Z(b)$, ~~b~~ $c \in Z(d)$, the fraction $[ad+bc\,/\,bd]$ is well defined. Firstly $bd \in N'(S)$. Secondly for any $g \in S$ for which $bdg = 0$ in S, $adg = 0$ since $O(b) \subseteq O(a)$ and $bc = 0$ since $O(d) \subseteq O(c)$, so that $(ad+bc) = 0$: $O(bd) \subseteq O(ad+bc)$ and accordingly $(ad+bc) \in Z(bd)$. Supposing that $(u,v) \in UV'(b)$ and $(s,t) \in UV'(d)$, $(us,\ ubt + sdv + vt) \in UV'(b)$. $\langle ua, ub+v \rangle$ and $\langle sc, sd+t \rangle$ are the special solutions defining the fractions $[a/b]$ and $[c/d]$ and $\langle \cancel{usac}\, us(ad+bc), \cancel{usbd+} (ub+v)(sd+t) \rangle$ is that defining the fraction $[ad+bc/bd]$

replace $+, \times$ by ~~eq~~ binary operators Γ, Δ; "$C/'M$, $C/''M$ become $C(\Gamma^{-1})M$ $C[\Gamma^{-1}]M$. // partial Γ-closure of C: $C(\Gamma^{-1})S$

Reorder theory

multiplicative module becomes Γ-module (?: confusion with module

~~Γ~~ domain of Γ-integrity: mult. Γ module M such that for all $a, b \in M$ ~~∃x~~ there exists an $x \in M$ for which $a = b\Gamma x$ ~~≠a~~

Γ-covering of A: smallest multiplicative module $\langle$smallest domain of Γ integrity$\rangle$ containing A.

~~$D(\Gamma; R, C)$~~ $a = b \bmod(\Gamma, C)$: ~~if ∃x∈C~~ if and only if C contains an x such that $a = b\Gamma x$ (Γ-reduction of $S_{\Gamma,C}$ of W

$D(\Gamma; R, C)$ all d such that $da = db \bmod(\Gamma, C)$ ~~if~~ only if $a = b \bmod(\Gamma, C)$ all $a, b \in W$. ($D := R$ rigid font)

$T(\Gamma; R, C)$ all $a \notin C$ for which $b(a) \notin C$ exists such that $a\Gamma b(a) \in C$

~~partial Γ-closure of C: $C(\Gamma^{-1})S$~~

semi Γ-ideal in W $S\Gamma C \subseteq C$

orthogonal system, ... $O\langle P\rangle\{Z\}C(\Gamma; R, C|b)$

square free semi Γ-ideal

(Γ, Δ)-ideal in S semi Γ-ideal, also Δ domain of integrity

square-free (Γ, Δ)-ideal

$(\Gamma, \Delta; I)$ ideal reduction of W

denominator with respect to $(\Gamma, \Delta; I)$ in W ($N(\Gamma, \Delta; I)$ }, $N := D$)

(Γ, Δ, I) quotient (rational) extension of W.

$Q: \{\Gamma, \Delta; W, I\} \to \{\Gamma, \Delta; Q_{(\Gamma,\Delta;I)}(W), Q_{(\Gamma,\Delta;I)}(0)\}$

W system of [illegible]

$\{\Gamma, \Delta$ operations; C system of sets; I Γ, Δ ideal in C

$Q : W^{(3)} \to W^{(3)}$

Formulate theory in terms of notations $\mathbb{D}, \mathbb{CP}, \ldots$

$C(\Gamma^{-1})M = q_M(C) = q^{(C)}(M)$ $q_M \in \mathbb{DP}$, $\{C, = | C\}$ etc.

Present theory in terms of set with partial ordering λ

(λ not necessarily equivalence relationship)

mapping $t, C_1 \to C_2$: if $c = t(b)$ then $d = t(a)$ for all

a, d such that $a.\lambda.b$, $c.\rho.d$

t may be written as $b.t.c$

theory of partial orderings presented in terms of

$\mathbb{D}^{(\lambda,\rho)}(C)$ (possibly not only one set C; $a.\theta.b$ with $\forall a \in C_1, b \in C_2$) ~~$a.\theta.b$~~

operations defined in terms of partial orderings

e.g. $a\theta.b$ means $b = c\Gamma a$ (c, Γ constants)

$A \subseteq_{(\Gamma, B)} C$: for each $a \in A$ corresponds at least one pair $b(a) \in B$ $c(a,b) \in C$

~~K completely ordered set~~ such that $a \Gamma \xi b(a) = c(a,b)$.

~~$C(x_1, \ldots, x_n)$ set with $x_1, \ldots, x_n \in K$~~ $\leq K$ ~~a set of objects~~

n systems of binary operations $\Gamma(x,y), \Delta(x,y), \ldots, \Omega(x,y)$

depending on $x, y \in K$

$\Xi(x,y)$ r^{th} operation

$\tau_{i,j}(x_1, \ldots, x_{n+1}) \in K : K^{n+1} \to K$ $\binom{i;j}{r : n}$

5 pages on

$a \in C(x_1 \ldots x_{r-1}, x_r, x_{r+1} \ldots x_n)$

$b \in C(x_1 \ldots x_{r-1}, y_r, x_{r+1} \ldots x_n)$

if defined

~~$\forall$ $a \,\Xi_1(x_r, y_n)\, b \in C(\tau_{r,1}(x_1 \ldots x_{r-1}, x_r, y_r, \ldots x_n) \ldots \tau_{r,n}(x_1 \ldots x_{r-1}, x_r, y_r, n_r)$~~

Replace operations on objects belonging to sets by operations on sets : $a \,\overline{\boxplus}\, b$ by $A \rightrightarrows B$

Decomposition also based on $q = z_1 + j z_2$, q quaternions, $z_1 z_2$ complex numbers

$\bar{q} = z_1 - j z_2 \quad q + \bar{q} \to W' \subseteq W.$

Examples of distributive law

Γ = multiplication $\triangle$ = exponentiation $a \Gamma b = ab$, $a \triangle b = b^a$

$\triangle$ left distributive with respect to Γ

$(bc)^a = (b^a)(c^a)$: $a \triangle \{b \Gamma c\} = (a \triangle b) \Gamma (a \triangle c)$

since $a^{bc} \neq a^b a^c$ $\{b \Gamma c\} \triangle a \neq (b \triangle a) \Gamma (c \triangle a)$
not right distributive

$(a^b)^c = a^{bc} \quad c \triangle \{b \triangle a\} = (c \Gamma b) \triangle a$

$a \triangle b = aba^{-1} \quad a \Gamma b = ab$

$\triangle$ left distributive with respect to Γ

$a \triangle \{b \Gamma c\} = abca^{-1} \quad (a \triangle b) \Gamma (a \triangle c) = aba^{-1}aca^{-1} = abca^{-1}$

since $(b \Gamma c) \triangle a = bca(bc)^{-1} = bcac^{-1}b^{-1}$ and

$(b \triangle a) \Gamma (c \triangle a) = bab^{-1}cac^{-1}$: not ~~left~~ right distributive

$a \triangle (b \triangle c) = abcb^{-1}a^{-1}$

$(a \triangle b) \triangle a = aba^{-1}cab^{-1}a^{-1}$

$a \triangle (b \triangle a) = (a \triangle b) \triangle a$; $\triangle$ flexible

$\cancel{a^2 ca^{-2} = a(aca^{-1})a^{-1}} \quad a \triangle (a \triangle c) = a \triangle \{(a \triangle a) \triangle c\}$ ~~$a \triangle \{a \triangle (a \triangle a)\}$~~

$a \triangle a = a$

In cases in which $D(S, I)$ is void, consider quotient extension

Stratification

K a reference set ; nonvoid $K_r \subseteq K$ $(r:n)$

$\underset{\sim}{x}$ the sequence $(x_1, \ldots, x_n)$ $(x_r \in K_r \ (r:n))$. K^n the space of $\underset{\sim}{x}$

$x^{[r]}$ the sequence $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$ $\{x_i \in K_i \ (i:n(i \neq r))\}$

$K^{[r]}$ the space of $x^{[r]}$

$x^{[r]}(u) \in K^n$ the sequence $x_1, \ldots, x_{r-1}, u, x_{r+1}, \ldots, x_n$ $\{$
  $(x_i \in K_i \ (i:n\,(i \neq r)), u \in K_r)$

$C(\underset{\sim}{x})$ a set of objects placed at $\underset{\sim}{x} \in K^n$

$\Gamma, T_1 ; \Delta, T_2 ; \ldots ; \Omega, T_n$ n systems of operators - system of coordinate function pairs

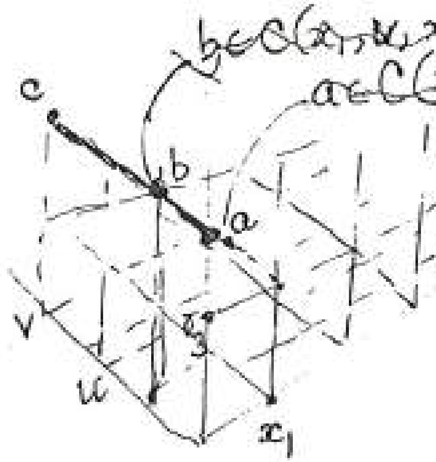
(i) (a) The r^{th} operator system Ξ is composed of subsystems $\Xi_1(x^{[r]})$ $(x^{[r]} \in K^{[r]})$ whose general members are $\Xi_1(x^{[r]}|u,v)$ where $\{u, v \in K_r\}$.

The r^{th} coordinate function system T_r is composed of ~~subsystems $T_r(x^{[r]})$ $(x^{[r]} \in K_r)$ of sequences of~~ coordinate mappings $\tau_r(x^{[r]}|u,v)$: $K_r \times K_r \to K_r$

With $a \in C(x^{[r]}(u))$, $b \in C(x^{[r]}(v))$

$a\,\Xi_1(x^{[r]}|u,v)\,b \in C(x^{[r]}(\tau_r(x^{[r]}|u,v)))$ $(u, v \in K_r)$

For fixed $x^{[r]} \in K^{[r]}$, the operators of the subsystem $\Xi_1(x^{[r]})$ operate upon objects belonging to sets lying in a curve $C(x^{[r]}(u))$ $(u \in K_r)$ in the space of sets $C(\underset{\sim}{x})$, and produce an object belonging to a set lying on this curve



$a \in C(x_1, u, x_3)$ $b \in C(x_1, v, x_3)$

$c = a \,\underline{4}\, b \,\cancel{a} \in C(x_1, \tau_2(x_1, x_3 | u, v), x_3)$

Example: K nonnegative integers, $K_1 = \{1\}$, $K_2 = K$

$C(x_1, x_2)$ set of x_2^{th} degree quintics

$a \Gamma b \langle a \Delta b \rangle$ addition $\langle$multiplication$\rangle$ of quintics

$\Gamma : \Gamma(x_2)$ system of operations of adding two x_2^{th} degree quintics; $\Gamma(x_2 | u, v)$ formation of sum of two x_2^{th} degree quintics ($\Gamma(x_2)$ contains only one member in this case)

$\tau_1 = (x^{[1]} | u, v) = u$ $\quad x^{[1]}(\tau_1(x^{[1]} | u, v)) = (u, x_2)$ ~~(u,v=1)~~

$a \in C(u, x_2)$ $b \in C(v, x_2)$ $a \Gamma(x_2 | u, v) b \in C(u, x_2)$ ~~(u,v=1)~~ $(u, v = 1, x_2 \in$

$\Delta : \Delta(x_1)$ system of operations of multiplying quintics. $\Delta(x_1 | u, v)$ formation of product of u^{th} and v^{th} degree quintics.

$\tau_2(x^{[2]} | u, v) = u + v$ $\quad x^{[2]}(\tau_2, (x^{[2]} | u, v)) = (x_1, u + v)$

$a \in C(x_1, u)$ $b \in C(x_1, v)$ $a \Delta(x_1 | u, v) b \in C(x_1, u + v)$

$(x_1 = 1, u, v \in K)$

ii) (a) as for (i).

The r^{th} coordinate function system T_r is composed of subsystems $T_r(x^{[r]})$ $(x^{[r]} \in K^{[r]})$ of sequences of coordinate mappings

$\tau_{r,j}(x^{[r]})$ $K_r \times K_r \to K_j$ $(j:n)$

With $a \in C(x^{[r]}(u))$, $b \in C(x^{[r]}(v))$

$a\,\Xi(x^{[r]}|u,v)\,b \in C(\tau_{r,1}(x^{[r]}|u,v), \dots, \tau_{r,n}(x^{[r]}|u,v))$

For fixed $x^{[r]} \in K^{[r]}$ the operators of the subsystem $\Xi(x^{[r]})$ operate upon objects belonging to sets lying in a curve $C(x^{[r]}(u))$ $(u \in K_r)$ in the space of sets $C(\underset{\sim}{x}^{\varepsilon})$ and produce an object which ~~may possibly not does not lie~~ belonging to a set which possibly does not lie on this curve.

Example: K: positive integers $K_r = K(r:2)$

$C(x_1, x_2)$ set of $x_1 \times x_2$ ~~integer~~ matrices

$a \Gamma b$ juxtaposition: rows of a followed by those of b

$a \Delta b$: $a^T b$

Γ: $\Gamma(x^{[1]})$ system of operations of juxtaposing x_2 columned matrices; $\Gamma(x_2|u,v)$ is the operation of juxtaposing $u \times x_2$ and $v \times x_2$ matrices

$T_1(x^{[1]})$ contains the two mappings $\tau_{1,1}(x_2|u,v) = u+v$ and $\tau_{1,2}(x_2|u,v) = x_2$ // With $a \in C(u, x_2)$ and $b \in C(v, x_2)$ $a\Gamma(x_2|u,v)\,b \in C(u+v, x_2)$

Δ: $\Delta(x^{[2]})$: system of operations of forming the product of ~~an~~ a matrix with x_1 columned ~~matrix~~ and ~~an~~ a matrix with x_1 rowed ~~matrices~~; $\Delta(x_1|u,v)$ is the operation of forming the product of ~~an~~ two matrices of order $u \times x_1$ and $x_1 \times v$

$T_2(x^{[2]})$ contains the two mappings $\tau_{2,1}(x_1|u,v)=u$ and $\tau_{2,2}(x_1|u,v)=v$

With $a\in C(x_1,u)$ and $b\in C(x_1,v)$, $a\Delta(x_1|u,v)b\in C(u,v)$

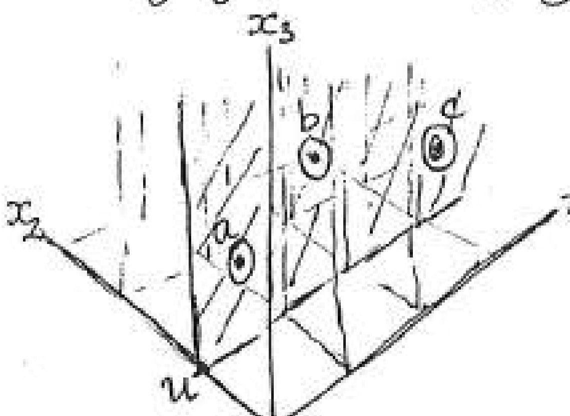
iii) The r^{th} operator system Ξ_1 is composed of subsystems $\Xi_1(u)$ ($u\in K_r$) whose general members are $\Xi_1(x^{[r]},y^{[r]}|u)$ where $x^{[r]},y^{[r]}\in K^{[r]}$

The r^{th} coordinate function system T_r is composed of subsystems $T_r(u)$ ($u\in K_r$) of sequences of coordinate mappings $\tau_{r,j}(u)$: $K^{[r]}\times K^{[r]}\to K_j$ $(j:n)$

With $a\in C(x^{[r]}(u))$, $b\in C(y^{[r]}(u))$

$a\Xi_1(x^{[r]},y^{[r]}|u)b\in C(\tau_{r,1}(x^{[r]},y^{[r]}|u),\dots,\tau_{r,n}(x^{[r]},y^{[r]}|u))$

For a fixed $u\in K_r$, the operators of the subsystem $\Xi_1(u)$ operate upon objects belonging to sets lying on a hypersurface $C(z^{[r]}(u))$ ($z^{[r]}\in K^{[r]}$) in the space $C(x)$ of sets, and produce an object belonging to a set lying in this surface.



$a\in C(x_1,u,x_3)$, $b\in C(y_1,u,y_3)$

$c=a\Xi_1(x_1,x_3;y_1,y_3;u)\,b\in$

$C(\tau_{2,1}(x^{[2]},y^{[2]}|u),\dots,\tau_{2,3}(x^{[2]},y^{[2]}|u))$

(In exposition, consider reversing order (i, ii) since example of (i) (with $K_1=\{1\}$) is so simple. Further example of (i): ~~I~~ space with inner product in K field stratification (J,K)), Rationalisation with respect to a single operation Γ ($a'b$ expressed as ab in the following) (14)

D: the rigid part of W: all $d \in W$ such that, for all $a, b \in W$, $da = db$ only if $a = b$

L: a left Γ-centre in W is a nonvoid set $L \subseteq W$ for which

(i) $a(bc) = (ab)c$ for all $a \in L$ and $b, c \in W$ and

(ii) $ab = ba$ for all $a, b \in L$

$S(\Gamma, W; L, D)$, the Γ-subtrahend system in W with respect to L and D, is the system of numbers $b \in W$ with each of which may be associated a nonvoid set ~~S = S(C(b) =~~ $\mathfrak{C}(\Gamma, W; L, D \mid b) \subseteq W$ of left-centralising factors ~~$x \in W$~~ such that

(i) $xb \in D \cap L$ for all $x \in \mathfrak{C}(b)$ and

(ii) $(xb)x' = (x'b)x$ for all $x, x' \in \mathfrak{C}(b)$

An ordered pair $[a, b]$ of W-numbers, with b a member ~~of the~~ subtrahend in $S(\Gamma, W; L, D)$ is a Γ-difference

Equality: $[a, b] = [c, d]$ if and only if $x \in \mathfrak{C}(b)$ and $y \in \mathfrak{C}(d)$ exist such that $(xb)(yc) = (yd)(xa)$.

輔系 If W contains one number a satisfying condition (Li), it possesses a left centre.

Since $aa = aa$, a by itself is such a centre (The largest system in W of numbers a satisfying condition (i) alone may not satisfy (ii): more than one L may exist.) ~~The centre C for which $a(bc) = (ab)c$ for all $a, c \in C$, $b \in W$ and $ab = ba$ for all $a \in C$ $b \in W$ is uniquely defined.~~

If $a, b \in L_\xi$, then $a(bc) = b(a_\xi c)$ (α)

$$a(bc) = (ab)c \ [a \in L(i)] = (ba)c \ [a, b \in L(ii)] = b(ac) \ [b \in L(i)]$$

Equality between differences is independent of the $x \in C(b)$ and $y \in C(d)$ taken to define such equality.

$$[a, b] = [c, d] \to (xb)(yc) = (yd)(xa)$$

Let $x' \in C(b)$ also : $(x'b)\{(xb)(yc)\} = (x'b)\{(yd)(xa)\}$

$$\text{l.h.s.} = (xb)\{(x'b)(yc)\} \quad (xb, x'b \in L \ (\alpha))$$

$$\text{r.h.s.} = (yd)\{(x'b)(xa)\} \quad (x'b, yd \in L \ (\alpha))$$

$$= (yd)[\{(x'b)x\}a] \quad (x'b \in L(i))$$

$$= (yd)[\{(xb)x'\}a] \quad (x, x' \in C(b) \ (ii))$$

$$= (yd)\{(xb)(x'a)\} \quad (xb \in L(i))$$

$$= (xb)\{(yd)(x'a)\} \quad (xb, yd \in C \ (\alpha))$$

~~the~~ $(xb)\{(x'b)(yc)\} = (xb)\{(yd)(x'a)\}$ &

$$(x'b)(yc) = (yd)(x'a) \qquad (xb \in D)$$

$$(x'b)\{(xb)(yc)\} = (xb)\{(x'b)(yc)\}$$

$$(x'b)\{(yd)(xa)\} = (xb)\{(yd)(x'a)\}$$

$x \Leftrightarrow y$ etc

$\cancel{(y'd)\{(xb)(yc)\} =}$

$$(y'd)\{(yd)(xa)\} = (yd)\{(y'd)(xa)\}$$

$$(y'd)\{(xb)(yc)\} = (yd)\{(xb)(y'c)\}$$

$$(xb)(yc) = (yd)(xa) \to (xb)(y'c) = (y'd)(xa).$$

equality between differences is reflexive

$$(xb)(xa) = (xb)(xa) \to [a,b] = [a,b]$$

is symmetric

$$(xb)(yc) = (yd)(xa) \to (yd)(xa) = (xb)(yc)$$

$$[a,b] = [c,d] \to [c,d] = [a,b]$$

is transitive

$$[a,b] = [c,d],\ [c,d] = [e,f]$$

$$(1)\quad (xb)(yc) = (yd)(xa) \qquad (2)\quad (yd)(ze) = (zf)(yc)$$

$$(zf)\times(1)\quad (zf)\{(xb)(yc)\} = (zf)\{(yd)(xa)\}$$

$$(3)\qquad (xb)\{(zf)(yc)\} = (yd)\{(zf)(xa)\}\ (xb, yd, zf \in L(ii))$$

$$(xb)\times(2)\quad (xb)\{(yd)(ze)\} = (xb)\{(zf)(yc)\}$$

$$(yd)\{(xb)(ze)\} = (yd)\{(zf)(xa)\}\ ((3) \text{ and } yd, xb \in L(ii))$$

$$(xb)(ze) = (zf)(xa)\ (yd \in D) \to [a,b] = [e,f].$$

— o —

L is a Γ-module

$x, y \in L;\ b, c \in W''$ $(xy)(bc) = x\{y(bc)\}\ (x \in L(i)) =$

$= x\{(yb)c\}\ (y \in L(i)) = \{x(yb)\}c\ (x \in L(i)) = \{(xy)b\}c\ (x \in L(i))$

$z \in L.$ $(xy)z = x(yz)\ (x \in L(i)) = x(zy)\ (z, y \in L(ii))$

$= (xz)y\ (x \in L(i)) = (zx)y\ (x, z \in L(ii)) = z(xy)\ (z \in L(i))$

If $x \in L \cap D$ and $y \in D$ then $xy \in D$

Suppose $(xy)a = (xy)b$. Then $x(ya) = x(yb)\ (x \in L(i))$ $ya = yb\ (x \in D)$

and $a = b$ $(y \in D)$

$$(x'b)\{(yd)(xa)\} = (xb)\{(yd)(x'a)\}$$

$$(x'b)\{(xb)(yc)\} = (xb)\{(x'b)(yc)\}$$

$x \Leftrightarrow y$ etc

$$(y'd)\{(xb)(yc)\} = (yd)\{(xb)(y'c)\}$$

$$(y'd)\{(yd)(xa)\} = (yd)\{(y'd)(xa)\}$$

$$(yd)(xa) = (xb)(yc) \rightarrow (y'd)(xa) = (xb)(y'c)$$

$$(xb)(xa) = (xb)(xa) \rightarrow [a,b] = [a,b]$$

$$(yd)(xa) = (xb)(yc) \rightarrow (xb)(yc) = (yd)(xa)$$

$$[a,b] = [c,d] \rightarrow [c,d] = [a,b]$$

$[a,b] = [c,d] \quad [c,d] = [e,f]$

① $(yd)(xa) = (xb)(yc) \quad (zf)(yc) = (yd)(ze)$

$(zf) \times$ ① $\quad (zf)\{(yd)(xa)\} = (zf)\{(xb)(yc)\}$

$a, b \in L$

$$a(bc) = (ab)c = (ba)c = b(ac) \quad ②$$

$(zf)\{(yd)(xa)\} = (yd)\{(zf)(xa)\}$ & $zf, yd \in L$

$(zf)\{(xb)(yc)\} = (xb)\{(zf)(yc)\}$ $\quad zf, xb \in L$

$$(yd)\{(zf)(xa)\} = (xb)\{(zf)(yc)\}$$

$(xb) \times$ ② $\quad (xb)\{(zf)(yc)\} = (xb)\{(yd)(ze)\}$

$(xb)\{(yd)(ze)\} = (yd)\{(xb)(ze)\}$ $\quad xb, yd \in L$

$$(yd)\{(zf)(xa)\} = (yd)\{(xb)(ze)\}$$

$(zf)(xa) = (xb)(ze)$ $\quad (yd \in D$

$$[a,b] = [e,f]$$

$D \cap L$ ~~is a~~ (the rigid part of the left centre L) is a Γ-module.

$$x, y \in D\cap L \to x, y \in L \to xy \in L$$
$$\text{"} \quad \to x \in L \cap D,\ y \in D \to xy \in D \Big\} \ xy \in D \cap L$$

— o —

Let $b \in S$ and x be a fixed member of $C(b)$. Then $xb \in S$ and xb ~~is associated with~~ possesses a set $C(xb)$ of left-centralising factors for which $xC(b) \subseteq C(xb)$

For all $x' \in C(b)$ $x'b \in D \cap L$. Also $xb \in D \cap L$. Hence $(x'b)(xb) \in D \cap L$

$(xb, x'b, x''b \in L(ii))$ ✓

Let $x', x'' \in C(b)$. Then $\{(x''b)(xb)\}(x'b) = (x'b)\{(xb)(x''b)\}$
$$= \{(x'b)(xb)\}(x''b)$$
$(x'b \in L(i))$

~~A The~~ The left-centralised form of the difference $[a, b]$ is $[xa, xb]$, where $x \in C(b)$ is a Γ

All left-centralised forms of $[a, b]$ are equal to $[a, b]$

$$(xb)\{(xb)(xa)\} = \{(xb)(xb)\}(xa) \quad (xb \in L(i))$$

Thus, taking $x \in C(b)$ and $(xb) \in C(xb)$ to establish equality $[a, b] = [xa, xb]$.

Consider defining $D(b)$ to be that part of W such that $bc = dc$ only when $a = c$ for all $a, b \in D(b)$ (in analogy with $Z(S, 1|b), 1)$)

If $L(ii)$ is replaced by $ab = ba$ for all $a \in L$ $b \in W$ then $(xb)(ac) = a\{(xb)c\}$, $[\{(xb)a\}\{(xb)c, (xb)^2\}] = [(xb)(ac), xb]$

the correspondence $a \to [(xb)a, xb]$ is preserved during multiplication defined by $[a,b][c,d] = [(xa)(uc), (xb)(ub)]$

complete writing out proof of ordering theorem

change ξ, ζ to λ, ρ also perfect ordering to skew equivalence

In proof of existence of $A\{C\}$, define $A\{C\}$ to be intersection of all systems in $AP\{S\}$ ~~included~~ containing C

Ideal ~~cl~~ closure similarly treated.

In theorem stating that S rationally decomposable iff no denominators $b_\nu \le b_{\nu+1}$ .. exist change treatment to case in which $b \le u$ in definition of denominator ✓

Provide simple numerical example involving S_k = integers mod 2 to illustrate case in which $S \subseteq \cup S_k$ and not $S = [\cup S_k]$ ✓

Define $S \setminus D$ as essentially singular part.

Draft introductory section dealing with rationalisation of commutative ring; identification of quotients $(a/b)_{\{0\}_S}$ $(a/b|\{0\}_S)(\{0\}_S|a/b)$ with solution of $bx=a$. $N'(S,\{\{0\}_S)$ all b such that u, v exist for which $ub+v \in D(S,\{0\}_S)$. $a \in Z'(S,\{0\}_S|b)$ iff $O(b) \le O(a)$ and $av=0$ (If ~~D~~ $(\{0\}_S|b)x = (\{0\}_S|a)$ then $O(b)\le O(a)$) If $a(ua/ub+v) = (ya/yb+z)$ then must ~~then~~ have $av = az = 0$; $C'(b)$: all v such that $b+v \in D$ or $ub+v\in D$)

Extend to treatment of $bx = a+u$, $u\in I$ $cbx = ca+cu \to I$ semi-ideal, addition of equations $\to I$ ideal.

In the case in which $D(S,S[0])$ is void, $b\notin S[0]$ consider $S_b = Sb$ $b \in D(S_b, S_b[0])$? $b \le u$ mod $S[0]$, $\to b\in D(S_u,S_u[0])$?

Consider $bx=a$ mod $I(0), I(1),\dots$ and $(a/b)_{I(0)} \cap (a/b)_{I(1)} \cap \dots$

Illustrate with ~~$I(0)$~~ mod $I(0)$ = mod $p(0)$, ... ~~$p(0)$~~ a, b, .. $p(0)$, ... integers

also $b(1)x = a(1)$, $b(2)x=a(2)$, ...

Also $ubx = ua$, $vbx = va$, ... $\to (ua/ub), (va/vb), \dots$ ✓

Extend treatment of complex decomposition ~~R~~ distributive involution $R = a+\bar a\ (a\in S)$ R mult & add. system

Discussion of $bx=a$ first then consider a, b row ~~vectors~~ x column matrix

S commutative ring, zero element 0.

$b \in S$. $\mathfrak{O}(S|b)$ complete system of numbers $c \in S$ for which $bc = 0$

$x \in S$. Set $a = bx$. Equation $by = a$ has a solution in S, namely x.

General solution $y = x + z$ for all $z \in \mathfrak{O}(S|b)$

Construction of system $Q(S|\{0\})$ in which equation $by = a$ has a solution when no $x \in S$ exists such that $a = bx$.

Assumptions: (i) nontorsive part $D(S|\{0\})$, system of numbers b such that $bc = 0$ only when $c = 0$, nonvoid (ii) $x^2 = 0$ only when $x = 0$.

$Q(S|\{0\})$: system of ordered pairs $\langle a, b\rangle$ with $a \in S$, $b \in D(S|\{0\})$.

equality: $\langle a, b\rangle = \langle c, d\rangle$ iff. $ad = bc$

addition: $\langle a, b\rangle + \langle c, d\rangle = \langle ad + bc, bd\rangle$

multiplication: $\langle a, b\rangle \langle c, d\rangle = \langle ac, bd\rangle$

$D(S|\{0\})$ multiplicative system: $b, d \in D \to bd \in D$

$(bdg = 0 \to dg = 0 \ (b \in D) \to g = 0 \ (d \in D))$

equality reflexive and symmetric.

$\langle a, b\rangle = \langle c, d\rangle \to ag = 0$ iff. $cg = 0$

$(cg = 0 \to bcg = 0 \to adg = 0 \ (ad = bc) \to ag = 0 \ (d \in D)$. Conversely$)$

equality transitive

$\langle a, b\rangle = \langle c, d\rangle$, $\langle c, d\rangle = \langle e, f\rangle$ $\quad ad = bc$, $ed = fc$, $af.dc = be.dc$

$dc(af - be) = 0 \to c(af - be) = 0 \ (d \in D) \xrightarrow{af} (af - be) = 0 \ (\uparrow) \to af(af - be) = 0$

similarly $\uparrow$ with a, b, c, d replaced by c, d, e, f. $be(af - be) = 0$

$(af - be)^2 = 0 \to af - be = 0 \ ((ii))$. $\to \langle a, b\rangle = \langle e, f\rangle$

sum and product in $Q(S|\{0\})$ (D multiplicative)

addition associative and commutative.

all numbers $\langle 0,d\rangle$ function as ~~addt~~ zero with respect to addition and are all equal. Each $\langle a,b\rangle$ has additive inverse $\langle -a,b\rangle$. Members of $Q(S|\{0\})$ form additive Abelian group. Multiplication distributive with respect to addition: $Q(S|\{0\})$ is a ring.

With $d,f\in D$, $\langle ad,d\rangle=\langle af,f\rangle$

Denote numbers of form $\langle ad,d\rangle$ in $Q(S|S\{0\})$ by $(a)_{S\{0\}}$

Correspondence $a\in S\to(a)_{S\{0\}}$ preserved under addition and multiplication. System of numbs $(a)_{S\{0\}}$ in $Q(S|S\{0\})$ form a ring isomorphic to S.

Nontrivial part $D(Q(S|S\{0\})|Q\{0\})$ of $Q(S|S\{0\})$ nonvoid, and $\langle a,b\rangle\in D(Q)$ iff $a\in D(S)$.

$\langle x,z\rangle^2=0\in Q$ iff $x=0$ in S

The equation $(b)_{S\{0\}}X=(a)_{S\{0\}}$ with $b\in D(S|S\{0\})$ has a unique solution in $Q(S|S\{0\})$ namely $X=\langle a,b\rangle$

($\langle bf,f\rangle.\langle a,b\rangle=\langle bfa,bf\rangle=(a)_{S\{0\}}$ 3. $(b)_{S\{0\}}(X-Y)=0\in Q$

$X=Y$ in Q)

The equation $BY=A$ for any $A\in Q$, $B\in D(Q)$ also has a unique solution

$B=\langle b,d\rangle$ $A=\langle e,f\rangle$ $Y=\langle ed,bf\rangle$

Weakly Singular part $N'(S|S\{0\})$ of S: all b such that either (i) $v\in S$ exists such that $b+v\in D(S)$ or (ii) $u,v\in S$ exist such that $ub+v\in D(S)$. $C(b)$: complete system of numbers featuring in (i) and (ii).

$b\in N'(S|S\{0\})$. Weak Numerator set $Z'(S|b)$ of b: all a such that $O(a)\cap C(b)$ nonvoid

For $b \in N'(S|S\{0\})$ and a such that $O(S|a) \cap C(b)$ nonvoid

set $(a/b)_{S\{0\}} = \langle a, b+v \rangle$ if v satisfies condition (i)

$(a/b)_{S\{0\}} = \langle ua, ub+v \rangle$ in case (ii).

$(a/b)_{S\{0\}}$ is uniquely defined, being independent of choice of v in case (i) and of u, v in case (ii).

(e.g. in case (ii) suppose also that $yb+z \in D(S)$, $az=0$ then since $az = av = 0$, $(yb+z)au = (ub+v)ay$ and hence $\langle au, ub+v \rangle = \langle ay, yb+z \rangle$)

For such b and a, the equation $(b)_{S\{0\}} Y = (a)_{S\{0\}}$ has a solution in $Q(S|S\{0\})$ namely $X = (a/b)_{S\{0\}}$

$(\langle bf, f \rangle \langle au, ub+v \rangle = \langle afub, f(ub+v) \rangle = \langle af(ub+v), f(ub+v) \rangle = (a)_{S\{0\}})$

With $C = \langle c, d \rangle$, $O(Q|C)$ is the complete system of numbers $\langle e, f \rangle$ with $e \in O(S|c)$ in $Q(S|S\{0\})$.

General solution of $(b)_{S\{0\}} Y = (a)_{S\{0\}}$ is $X + G$ where $G = O(Q|B)$ $B = (b)_{S\{0\}}$. i.e $G = O(Q|(b)_{S\{0\}})$.

$N'(S|S\{0\})$ is a multiplicative system in S: if $b, d \in N'$ then $bd \in N'$

$(ub+v, yd+z \in D \rightarrow uy\, bd + ubz + ydv + vz \in D)$

$a \in Z'(S|b) \rightarrow ac \in Z'(S|b)$ all $c \in S$. $b, d \in N'(S|S\{0\})$ and $a \in Z'(S|b)$, $c \in Z'(S|d) \rightarrow ac \in Z'(S|bd)$ and

$(a/b)_{S\{0\}} (c/d)_{S\{0\}} = (ac/bd)_{S\{0\}}$

$ub+v \in D$, $av = 0$; $yd+z \in D$, $cz = 0$

$\langle ua, ub+v \rangle \langle yc, yd+z \rangle = \langle uy\, ac, uy\, bd + (ubz + ydv + vz) \rangle, ac(\ldots) = 0$

If $b, c, d \in N'$ and $a \in Z'(b)$, $b \in Z'(c)$, $b \in Z'(d)$ then $ad \in Z'(bc)$

and $X = (ad/bc)_{S\{0\}}$ satisfies the equation $(b/d)_{S\{0\}} Y = (a/c)_{S\{0\}}$

(let $ub+v,\ wc+x,\ yd+z \in D$ with $av = ax = bz = 0$.

Since $ad(ubx + wcv + vx) = 0$, $(ub+v)(wc+x) = uwbc + (ubx + wcv + vx)$

$ad \in Z'(bc)$ and $X = \langle uwad,\ uwbc + ubx + wcv + vx \rangle$

also $wa(yd+z)(ub+v) = uwab(yd+z)$ $(av=0)$ $= uwyabd$ $(bz=0)$

thus $\langle uwyabd,\ (yd+z)(wc+x)(ub+v) \rangle = \langle wa, wc+x \rangle$

and $(b/d)_{S\{0\}} X = (a/c)_{S\{0\}}$)

General solution of $(b/d)_{S\{0\}} Y = (a/c)_{S\{0\}}$ is $X + G$ where

$G \in O(Q \mid (b)_{S\{0\}})$

Unfortunately, if $b, d \in N'$ and $a \in Z'(b)$, $c \in Z'(d)$ it may not be true that $ad + bc \in Z'(bd)$

(With $ub+v,\ yd+z \in D$, $av = cz = 0$, bd satisfies the relationship $uy\,bd + ubz + ydv + vz \in D$ but

$(ad+bc)(ubz + ydv + vz) = db(auz + cvy)$ may not be zero)

If $(b)_{S\setminus\{0\}}\, Y = (a)_{S\setminus\{0\}}$ has a solution $\langle x, y\rangle$ in $Q(S, S\setminus\{0\})$

$ay = bx$ $\quad bz = 0 \to az = 0$ i.e. ~~$a \leq$~~ $O(b) \leq O(a)$ necessary condition

also $O(x) \leq O(a)$ $\quad ub + v \in D$ $\quad uay = ubx$ $\quad xz = 0 \to az = 0$

$(ub+v)x = uay + vx$ $\quad \Big| \quad b + v = y \quad x = a \quad ay = ab + av = ab \to av = 0$

given $y \in D$ $x \in S$ satisfying $ay = bx$ determine $f, g \in D$ $u, v \in S$

such that $fx = gua$ $\quad fy = g(ub+v)$

take $y = g$ $f = ub + v$ $\quad x(ub+v) = uya = ubx \to xv = 0$

If u, v exist such that $ub + v \in D$, Y exists for all a & b

$by = bx$

---

suggest define N' by $ub + v \in D$ and use $a \leq b$ alone

as $ub + v$, $yb + z$ $\quad ua(yb+z) = ya(ub+v)$? $\quad uaz = yav$?

need $O(b) \leq O(a)$ and $O(a) \cap C(b)$ nonvoid

$b + x = a$ $\quad [a, b]$ $\quad (a - b)$ addition associative $\quad \Big|$ x for $+$ $\quad ad = cb \quad cf = ed \quad af = be$? $\quad adcf = cbed$

$[a, b] = [c, d]$ iff $a + d = c + b$ $\quad [a, b] = [a, b]$

$[c, d] = [a, b]$ iff $c + b = a + d$ $\quad [a, b] = [c, d] \Leftrightarrow [c, d] = [a, b]$

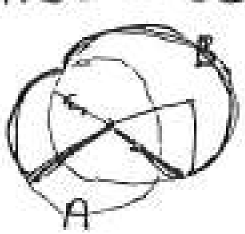
$[a, b] = [c, d]$, $[c, d] = [e, f] \to [a, b] = [e, f]$?

$a + d = c + b \quad c + f = e + d \to a + f = b + e$?

$[a, b] = [a + x, b + x]$? $\quad a + b + x =$ ~~$b + a + x$~~ ? $a + x + b$

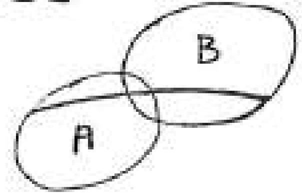
x must belong to set $a + x = x + a$ all $a \in S$?

$A \cup D = C \cup B \quad C \cup F = E \cup D \quad A \cup F = B \cup E$?



require commutative addition

and $a + b = c + b$ iff $a = c$



$A \cup F \cup C \cup D$

$= B \cup E \cup C \cup D$

$a \leq b$ means $O(S|b) \leq O(S|a)$ ‖ $N(S|S\{0\})$ singular part

$N(S)$ all b such that either (i) $v \in O(b)$ exists such that $b+v \in D$ or
(ii) ~~...~~ u such that $b \leq u$ and $v \in O(b)$ exist such that $ub+v \in D$

$Z^{\$}(S|b)$ a such that $a \leq b$ ‖ $O_c(b)$ system of such v

$O_c(b) \subseteq C(b) \cap O(b)$

in $C(b)$ u unrestricted $\therefore \leq$ $\qquad N(S) \subseteq N'(S)$

$a \in Z^{\$}(S|b)$: $O_c(b) \leq O(b) \cap C(b) \leq O(a) \cap C(b)$

$a \in Z^{\$}(S|b) \to O(b) \leq O(a)$ ‖ since $O_c(b)$ nonvoid $O(a) \cap C(b)$ nonvoid

$a \in Z'(S|b)$ $\qquad Z(S|b) \leq Z'(S|b)$.

$b \in N(S|S\{0\})$ $a \in Z(S, S\{0\}|b)$ $(b)_{S\{0\}} Y = (a)_{S\{0\}}$ has solution

$X = (a/b)_{S\{0\}}$ as before; general solution as before

$N(S\{, S\{0\})$ multiplicative system

$ub+v,\ yd+z \in D$ $\quad b \leq u,\ d \leq y$ $\quad uyg = 0 \to byg = 0\ (b \leq u) \to bdg = 0\ (d \leq y)$

$bd \leq uy$ $\quad uy\,bd + ubz + ydv + vz \in D$ $\quad bd\,(ubz + ydv + vz) = 0$

$a \in Z(S, S\{0\}|b)$ $c \in Z(d)$ $\to ac \leq bd$ $\quad ac \in Z(bd)$

$(a/b)\,(c/d) = (ac/bd)$ as before | $bcg = 0 \to acg = 0 \to ag = 0$

~~b, c, d~~ $a \leq b$ $\quad a \leq c$ $\quad b \leq d$ | $a \leq bc \to ad \leq bc$

$ad \leq bc$ ‖| $X = (ad/bc)_{S\{0\}}$ satisfies $(b/d)_{S\{0\}} Y = (a/c)_{S\{0\}}$

general solution as before

$a \leq c$ $\quad b \leq c$ $\quad a+b \leq c$

$a \leq b$ $\quad c \leq d$ $\quad ad + bc \leq bd$ $\qquad cg = 0 \to ag,\ bg = 0 \to a+b$

now $a/b \pm c/d = (ad \pm bc)/bd$

~~Q_b subsy~~ $b \in N$ fixed Q_b subsystem : $(a/b)_{S\{0\}}$ all ~~...~~ $a \leq b$

$a \times b = b^a \quad a \times c = c^a \qquad b+c = bc$

$(a \times b) + (a \times c) = b^a c^a = (bc)^a = a \times (b+c)$

$a \times 1 = 1$ all $a \quad 1 \times b = b$ all b

+ rationalised $[a, b] = a/b \qquad [c, d] = c/d$

$a \times [c, d] = \left(\frac{c}{d}\right)^a = \frac{c^a}{d^a} = [a \times c, a \times d]$

$[a, b] \times [c, d] = \quad \left(\frac{c}{d}\right)^a \Big/ \left(\frac{c}{d}\right)^b$

try $\left(\frac{c}{d}\right)^{\frac{a}{b}}$

$[a, b] \times [c, d] = \langle [a \times c, a \times d], [b \times c, b \times d] \rangle$

$\langle a, c \rangle \times b = \langle a \times b, c \times d \rangle$

$\frac{a-b}{c-d} \qquad \{a, b, c, d\} \quad \left\{\frac{a-b}{c-d}\right\}\left\{\frac{a'-b'}{c'-d'}\right\} \quad \underline{aa' + bb' - a'b - ab'}$

$\{a, b, c, d\} \times \{a', b', c', d'\} = \{a \times a' + b \times b', a' \times b + a \times b', c \times c' + d \times d', c' \times d + c \times$

$\cancel{a+b} \quad (a')^a (b')^b \big/ (a')^b (b')^a \quad (a'/b')^{a-b}$

b fixed $Q_b(S|S\{0\})$ subsystem in $Q(S|S\{0\})$ of numbers of form $(a/b)_{S\{0\}}$ ~~fa~~ with $a \in \mathbb{Z}'(S|b)$.

$ub+v \in D$ $av=0$ $yb+z \in D$ ~~az~~ $=0$ (a/b) (c/b).

$\langle ua, ub+v\rangle \langle yc, yb+z\rangle = \langle uyac, \underbrace{(ub+v)y}\, b + (ub+v)z\rangle$

$\langle ua, ub+v\rangle + \langle yc, yb+z\rangle = \langle ua(yb+z) + yc(ub+v), \underbrace{(ub+v)y}\, b + (ub+v)z\rangle$

$uyb^2 + \cancel{}\ (uz+yv)b + vz$

$\langle bf, f\rangle \langle x, y\rangle = \langle af, f\rangle$

$\langle bfx, fy\rangle = af\cdot fy - f\cdot bfx = 0$

$ay - bx = 0$ $y \in D$

$ub+v \in D$ $av=0$ $ub+v=d$ $ad = uab$

$uyac = fg$ $(ub+v)(yb+z) = f\cancel{b}\cancel{+}h$ $gh=0$ $ay = bx$ $bg=0 \to gc=0$

$uyach = 0$ $fgb = g(ub+v)(yb+z) = uyacb$

$av=0$

$g = \langle uyabc, (ub+v)(yb+z)\rangle = \langle acy(ub+v), (ub+v)(yb+z)\rangle$

$= \langle acy, yb+z\rangle = \langle acu, ub+v\rangle$ $ub+v$

$x = ua$
$y = ub+v$
$av=0$

$uyach = 0$ $uyac(ub+v)(yb+z) = uyabcf$

Q_b system of solutions of $bx=a$ all possible a

$bx=a$ $by=c$ $b(x+y) = a+c$ | $b.(bxy) = ac$

$b^2xy = ac$ | $ua(yb+z) + yc(ub+v) = fg$ |

given $y \in D$ $x \in S$ satisfying $ay = bx$ $f, g \in D$ u, v can be found such that $x = ua$
$y = ub+v$
$av=0$

$fgb = g(ub+v)(yb+z) = \{ua(yb+z) + yc(ub+v)\}b$

$= (ub+v)(yb+z)\{a+c\}$ $g = (a+c)_{\{0\}}$

$fx = gua$
$fy = g(ub+v)$

Q_b not multiplicative : ~~bxy satis~~ with $bx=a$ $by=c$

bxy satisfies $b(bxy) = ac$ but xy may not satisfy equation $\mathcal{D}$

$|(ub+v)x = yua = ubx$

form $bxy = e$ with $C(b) \cap \mathcal{O}(e)$ nonvoid

Q_b not additive $x+y$ satisfies $b(x+y) = a+c$ and possibly $C(b) \cap \mathcal{O}(a+c)$ void