Factorisations of a triangular matrix

$\mathbb{K}$ being a prescribed field, the $n \times n$ matrix with elements $a_{i,j} \in \mathbb{K}$ for which $a_{i,j} = 0$ when $j > i$ is said to be lower triangular. In this section decompositions of such matrices are considered.

To abbreviate the exposition, the following conventions are adopted.

Notation. (1) Angular brackets are used in a way made obvious by the context to combine two statements into one. (2a) $(i:j;k)$ indicates that the integer i should take the values $i = j, \ldots, k$. (b) If the lower limit j is unity, it and the subsequent semi-colon are omitted; thus $(i:k)$ is an abbreviation for $i = 1, \ldots, k$. (c) Conjoint descriptions are separated by a vertical bar; thus $(i:n|j:i)$ is an abbreviation for $i = 1, \ldots, n; j = 1, \ldots, i$. (d) Statements accompanied by null index allocations are to be ignored; thus the allocation $e_k = d_{k+1}$ $(k:r+1;n)$ is not performed if $r = n$.

Special notations for determinants formed from elements of a lower triangular are employed.

Notation. With $n > 0$ a finite integer, let $a_{i,j} \in \mathbb{K}$ $(i:n|j:i)$ be the elements of a lower triangular matrix A. (1) For $(i:n|j:n-i+1)$, $a(i,j)$ is the j^{th} order determinant formed from the array whose τ^{th} row contains the elements $a_{i+\tau-1,\nu}$ $(\nu:j)$ for $(\tau:j)$; when $j<1$, $a(i,j)=1$ $(i:n)$. ~~(2)~~ (2⟨3⟩). For $(i:n|j:n-i+1|k:n)$ $a(i,j)_k$ $\langle a(i,j)^{(k)}\rangle$ is the determinant obtained from the array defining ~~a~~ $a(i,j)$ by replacing the last row ⟨column⟩ by $a_{k,\nu}$ $(\nu:j)$ ⟨by $a_{i+\tau-1,k}$ $(\tau:j)$⟩. (4) For $(i:n-1|j:n-i|k:j)$ $a(i,j|k)$ is the determinant similarly obtained by replacing the elements in row k by $a_{i+j,\nu}$ $(\nu:j)$. (5) For $(i:n|j:n-i+1)$ $a(i,j|(f))$ is the determinant obtained by replacing the elements in the last column by $f_{i+\tau-1}$ $(\tau:j)$, ~~f_τ~~ $f_\tau \in K$ $(\tau:n)$ being given; in particular $a(i,j|(1))$ is that version of $a(i,j|(f))$ obtained by setting $f_\tau=1$ $(\tau:i;i+j-1)$. (6) Assuming $a_{i,i}\neq 0$ $(i:n)$, $a^{(-1)}_{i,i}=1/a_{i,i}$ $(i:n)$ and $a^{(-1)}_{i,j}=-a(1,i-1|j)/a(1,j)$ $(i:2;n|j:i-1)$.

Evidently $a(i,1)=a_{i,1}$ $(i:n)$ and $a(1,j)=\prod_{\nu=1}^{j} a_{\nu,\nu}$ $(j:n)$. Also $a(i,j)_k=0$ $(i:n-1|j:2;n-i+1|k:i;i+j-2)$, $a(i,j)_{i+j-1}=a(i,j)$ $(i:n|j:n-i+1)$, $a(i+1,j)_i=(-1)^{j-1}a(i,j)$ $(i:n-1|j:2;n-i+1)$ and, as may be shown with the help of elementary determinantal identities

$$a(i,j+1)a(i+1,j)_k+a(i,j)a(i+1,j+1)_k=a(i+1,j)a(i,j+1)_k \quad (i:n-1|j:n-i|k:n)$$

Again $a(i,j)^{(k)}=0$ $(i:n-1|j:2;n-i+1|k:j-1)$, $a(i,j)^{(j)}=a(i,j)$ $(i:n|j:n-i+1)$, $a(i,j+1)^{(i+j)}=a_{i+j,i+j}\,a(i,j)$ $(i:n-1|j:n-i)$ and

$$a(i+1,j)a(i,j)^{(k)}+a(i+1,j-1)a(i,j+1)^{(k)}=a(i,j)a(i+1,j)^{(k)} \quad (i:n-1|j:n-i|k:n)$$

Also $a(i,j|j)=a(i,j)_{i+j}$ $(i:n-1|j:n-i)$, $a(i,j|1)=(-1)^{j-1}a(i+1,j)$ $(i:n-1|j:n-i)$ and

$$a(i+1,j)a(i,j+1|k+1)+a(i+1,j+1)a(i,j|k+1)=a(i,j+1)a(i+1,j|k) \quad (i:n-2|j:2;n-i|k:j-1)$$

Furthermore

$$a(i+1,j)a(i,j|(f))+a(i+1,j-1)a(i,j+1|(f))=a(i,j)a(i+1,j|(f)) \quad (i:n-1|j:n-i)$$

Finally it is remarked that if A is nonsingular, $a^{(-1)}_{i,j}$ is the representative element of A^{-1}.

Special notations facilitating descriptions of the structure of a lower triangular matrix are employed.

Notation (1) E_n is the n^{th} order unit matrix. (2) With $d_k\in K$ $(k:m)$, $e_k\in K$ $(k:m-1)$ given, $\beta m\{d_k|e_k\}$ is the m^{th} order lower band matrix A with elements $a_{k,k}=d_k$ $(k:m)$ $a_{k+1,k}=e_k$ $(k:m-1)$ and $a_{i,j}=0$ for other element index numbers. (3) With $x_{i,j}\in K$ $(i:m|j:i)$ given, $L_m\{x_{i,j}\}$ is the m^{th} order lower triangular matrix with elements $x_{i,j}$. $Lm\{x_{i,j}|_r\, x''_{i,j}\}$ is the m^{th} order lower triangular matrix with elements $x'_{i,j}$ $(i:r|j:i)$

in the first r rows and $x''_{i,j}$ $(i:r+1;m|j:i)$ in the last $m-r$ rows. In the use of such matrix descriptions, it is understood that those of the elements $x'_{i,j}, x''_{i,j}$ not previously defined are to be given the value zero. Thus if $x'_{i,j}$ $(i:r|j:i)$ and $x''_{i,j}$ $(i:r+1;n|j:i-r;i)$ have been defined, the elements in the j^{th} position of the i^{th} row $(i:r+2;m|j:i-r-1)$ of the matrix $L_m\{x'_{i,j}|_r\, x''_{i,j}\}$ is zero. (4) A being a prescribed lower band or lower triangular matrix of order $\geq n-m$, $V_m[A]$ with $0\leq m\leq n$ is that lower triangular matrix having elements $a'_{k,k}=1$ $(k:m)$, $a'_{i,j}=0$ $(i:2;n|j:\min(i-1,m))$ and $a'_{i,j}=a_{i-m,j-m}$ $(i:m+1;n|j:m+1;i)$. (5) For $(k:n-1)$ $A^{[k]}$ is the $(n-k)\times(n-k+1)$ trapezoidal matrix obtained by removing the first k rows and first $k-1$ columns from A. (6) With $1\leq j\leq k$ and A_τ $(\tau:j;k)$ given matrices, $[A_\tau]^k_j$ is the product $A_k \ldots A_{j+1} A_j$; empty products are given the value E_n.

It is remarked that $V_m[A]$ is obtained from A by pushing the latter down its principal diagonal a distance of m elements, adding unit principal diagonal elements in the m such positions vacated and zero elements in the further positions of the first m columns, and possibly truncating the displaced version of A to form, in total, an n^{th} order lower triangular matrix. The V_0-form of A is A itself. In the V_n-form of A, A has been pushed out of sight: $V_n[A] = E_n$. In the special case in which $a_{1,1} = 1$, $V_{n-1}[A] = E_n$ also.

In the foregoing, lower triangular matrices of a certain class are considered.

Definition . An n^{th} order lower triangular matrix for which $a(i,j) \neq 0$ $(i:n|j:n-i+1)$ is said to be decomposable.

(a) To derive the first decomposition of the decomposable lower triangular matrix A, set, for $(r:2;n)$

$$d_k(A;r) = \frac{a(r-1,k)}{a(r,k-1)} \quad (k:n-r+2), \quad e_k(A;r) = \frac{a(r-1,k-1)}{a(r,k-1)} \quad (k:n-r+1)$$

$d_1(A;n+1) = a_{n,1}$, and

$$x_{i,j}(A;r) = \{a(r,j)\}_{i+r-1} \qquad (r:n|i:n-r+1|j:i)$$

Then $d_1(A;r+1) = x_{1,1}(A;r)$ $(r:n)$ and

$$x_{i-1,i-1}(A;r+1)d_i(A;r+1) = x_{i,i}(A;r) \quad (r:n-1|i:2;n-r+1)$$

Also, as may be verified by the use of relationship ( ),

$$x_{i-1,j-1}(A;r+1)d_j(A;r+1) + x_{i-1,j}(A;r+1)e_j(A;r+1) = x_{i,j}(A;r) \quad (r:n-1|i:n-r+1|$$

Accordingly, letting

$$D_r(A) = V_{r-2}\left[\beta_{n-r+2}\{d_k(A;r) \mid e_k(A;r)\}\right] \qquad (r:2;n+1)$$

$$X_r(A) = V_{r-1}\left[L_{n-r+1}\{x_{i,j}(A;r)\}\right] \qquad (r:n)$$

It follows that $X_{r+1}(A)D_{r+1}(A) = X_r(A)$ $(r:n-1)$. Since $X_n(A) = D_{n+1}(A)$

$$X_r(A) = \left[D_\tau(A)\right]_{r+1}^{n+1} \qquad (r:n)$$

and, letting $D_1(A) = \beta_n\{a(1,k-1)^{-1} \mid 0\}$, so that $A = X_1(A)D_1(A)$, the decomposition

$$A = \left[D_\tau(A)\right]_1^{n+1}$$

is obtained.

Again, set

$$d^{(1)}(A;r) = a(1,r)^{-1},\ d^{(k)}(A;r) = \frac{a(k-1,r)}{a(k,r-1)} \qquad (k:2;n-r+1)$$

$$e^{(k)}(A;r) = -\frac{a(k+1,r)}{a(k+1,r-1)} \qquad (k:n-r)$$

for $(r:n)$ and

$$x^{(i,j)}(A;r) = a(i,r)^{(j+r-1)} \qquad (r:n \mid i:n-r+1 \mid j:i)$$

together with $x^{(0,0)}(A;r+1) = 1$, $x^{(i,0)}(A;r+1) = 0$ $(i:n-r)$ all for $(r:n-1)$. Now $d^{(i)}(A;r)x^{(i,i)}(A;r) = x^{(i-1,i-1)}(A;r+1)$ $(r:n-1 \mid i:n-r+1)$, and it follows from formula ( ) that

$$e^{(i)}(A;r)\, x^{(i,j+1)}(A;r) + d^{(i+1)}(A;r)x^{(i+1,j+1)}(A;r) = x^{(i,j)}(A;r+1) \qquad \begin{pmatrix} r:n-1 \mid i:n-r \mid \\ j:0;r \end{pmatrix}$$

Accordingly, letting

$$D^{(r)}(A) = V_{r-1}\left[L_{n-r+1}\{d^{(k)}(A;r)|e^{(k)}(A;r)\}\right] \qquad (r:n)$$

$$X^{(r)}(A) = V_{r-1}\left[L_{n-r+1}\{x^{(i,j)}(A;r)\}\right] \qquad (r:n)$$

it follows that $D^{(r)}(A)X^{(r)}(A) = X^{(r+1)}(A)$ $(r:n-1)$. Since $X^{(1)}(A) = A$, the formula

$$X^{(r)}(A) = \left[D^{(\tau)}(A)\right]_1^{r-1} A \qquad (r:n)$$

is obtained. Since $D^{(n)}(A)X^{(n)}(A) = E_n$,

$$\left[D^{(\tau)}(A)\right]_1^{n} = A^{-1}$$

The elements of the matrices formed from the partial products of the above matrices $\{D_r(A)\}$, and also of the $\{D^{(r)}(A)\}$ with A, may be expressed in simple closed form. For $(r:n+1)$ let

$$y_{i,j}(A;r) = a_{i,j} \quad (i:r-1|j:i),\quad y_{i,j}(A;r) = \frac{a(r,i-r+1)^{(j)}}{a(r,i-r)a(r,i-r+1)} \quad (i:r;n|j:i)$$

so that $y_{i,j}(A;r) = 0$ $(i:r+1;n\,|\,j:i-r)$. Evidently $y_{i,j}(A;r) = y_{i,j}(A;r+1)$ for $(r:2;n\,|\,i:r-1\,|\,j:i)$. Also

$$d_1(A;r+1)y_{r,j}(A;r) = y_{r,j}(A;r+1) \qquad (r:n|j:r)$$

and, from formula ( ),

$$e_{i-r}(A;r+1)y_{i-1,j}(A;r) + d_{i-r+1}(A;r+1)y_{i,j}(A;r) = y_{i,j}(A;r+1) \quad (r:n-1|\,i:r+1;n|j:i-1)$$

while, from formula ( ),

$$d_{i-r+1}(A;r+1)y_{i,i}(A;r) = y_{i,i}(A;r+1) \qquad (r:n|i:r)$$

Thus letting

$$Y_r(A) = L_n\{y_{i,j}(A;r)\} \qquad (r:n+1)$$

$D_{r+1}(A)Y_r(A) = Y_{r+1}(A)$ $(r:n)$. But $Y_1(A) = D_1(A)$. Hence

$$Y_r(A) = [D_i(A)]_1^r \qquad (r:n+1)$$

Formula ( ) expresses the elements of the partial products of the $D_r(A)$ in closed form. In particular, from formulae ( ), $Y_{n+1}(A) = A$, and the result ( ) has been rederived.

Again, for $(r:n)$ let

$$y^{(i,j)}(A;r) = a^{(-1)}_{i,j} \qquad (i:r|j:i)$$

and, for $(i:r+1;n)$,

$$y^{(i,j)}(A;r) = -a(i-r,r|j-i+r+1) \ (j:i-r;i-1), \quad y^{(i,i)}(A;r) = a(i-r,r)$$

Now $y^{(i,j)}(A;r) = y^{(i,j)}(A;r+1)$ $(r:n-1|i:r|j:i)$ and, for $(r:n-1)$

$$d^{(1)}(A;r+1)\,y^{(r+1,j)}(A;r) = y^{(r+1,j)}(A;r+1) \qquad (j:r+1)$$

$$d^{(i-r)}(A;r+1)\,y^{(i,i)}(A;r) = y^{(i,i)}(A;r+1) \qquad (i:r+2;n)$$

and, from formula ( ),

$$e^{(i-r-1)}(A;r+1)\,y^{(i-1,j)}(A;r) + d^{(i-r)}(A;r+1)\,y^{(i,j)}(A;r+1) = y^{(i,j)}(A;r+1) \quad (i:r+2;n|j:i-1)$$

Thus letting

$$Y^{(r)}(A) = L_n\{y^{(i,j)}(A;r)\} \qquad (r:n)$$

$(r:n-1)$, so that $f^{(r)} = Y_{r+1}(A)^{[r]} g\ (r:n)$. Denoting by $F^{(r)}$ the vector with components $f_i\ (i:r)$ and $f_i^{(r)}\ (i:n-r)$ in succession $(r:0;n)$, $F^{(r)} = D_{r+1}(A) F^{(r-1)} = Y_{r+1}(A) g\ (r:n)$.

The matrices $D^{(r)}(A)$ occur in the recursive solution of the system of linear algebraic equations $Ag = f$, f and g being as in the preceding paragraph. Set $g_i^{(0)} = f_i\ (i:n)$ and for $(r:n-i)$ determine

$$g_i^{(r)} = e^{(i)}(A;r) g_i^{(r-1)} + d^{(i+1)}(A;r) g_{i+1}^{(r-1)} \qquad (i:n-r)$$

then $g_i = d^{(1)}(A;i) g_1^{(i-1)}\ (i:n)$. The counterpart to formula ( ) is

$$g_i^{(r)} = \sum_{j=i}^{i+r} y^{(i+r,j)}(A;r) f_j \qquad (r:n-1|i:n-r)$$

With the vectors $g^{(r)}$, $G^{(r)}$ defined in analogy with $f^{(r)}$, $F^{(r)}$ above, $g^{(r)} = D^{(r)}(A)^{[r]} g^{(r-1)} = Y^{(r)}(A)^{[r]} f\ (r:n)$ and $G^{(r)} = D^{(r)}(A) G^{(r-1)} = Y^{(r)}(A) f\ (r:n)$. The numbers $g_i^{(r)}$ may be expressed in determinantal form as

$$g_i^{(r)} = a(i, r+1|(f)) \qquad (r:0;n-1|i:n-r)$$

That determinantal expressions of this form satisfy relationship ( ) may be verified with the help of formula ( ).

A decomposition of a lower triangular matrix into a

$D^{(r+1)}(A)Y^{(r)}(A)=Y^{(r+1)}(A)$ $(r:n-1)$. But $y^{(i,i)}(A;1)=d^{(i)}(A;1)$ $(i:n)$ and $y^{(i+1,i)}(A;1)=e^{(\cancel{i+1},i)}(A;1)$ $(i:n-1)$, the remaining elements of $Y^{(1)}(A)$ being declared zero by default. Thus $Y^{(1)}(A)=D^{(1)}(A)$ and

$$Y^{(r)}(A)=\left[D^{(\tau)}(A)\right]_1^r \qquad (r:n)$$

now formulae (,) express the elements of the partial products of the $D^{(r)}(A)$ in closed form. $Y^{(n)}(A)$ consists exclusively of elements of the form ( ): $Y^{(n)}(A)=A^{-1}$, and in turn the result ( ) has been rederived.

The matrices $D_r(A)$ occur in the recursive formation of the vector $f=Ag$, f and g being two column vectors with components f_i and g_i respectively. Set $f_i^{(0)}=g_i/a(1,i-1)$ $(i:n)$ and for $(r:n-1)$ determine

$$f_i^{(r)}=e_i(A;r+1)f_i^{(r-1)}+d_{i+1}(A;r+1)f_{i+1}^{(r-1)} \qquad (i:n-r)$$

then $f_i=d_1(A;i+1)f_1^{(i-1)}$ $(i:n)$. In terms of the components of the matrices $Y_r(A)$

$$f_i^{(r)}=\sum_{j=i}^{i+r} y_{i+r,j}(A;r+1)g_j \qquad (r:0;n-1|i:n-r)$$

For $(r:0;n)$ denote by $f^{(r)}$ the column vector with components $f_i^{(r)}$ $(i:n-r)$; recursion may then be written as $f^{(r)}=D_{r+1}(A)^{[r]}f^{(r-1)}$

product of band matrix factors is not, of course, unique. Between two factors $D_{r+1}(A)$, $D_r(A)$ occurring in the decomposition of A, for example, a term of the form $E^{-1}E$, where E is a suitable diagonal matrix, may be inserted; the factors then occur in modified form $D_{r+1}(A)E^{-1}$ and $ED_r(A)$. In the above theory, that normalisation which results in the expression of the numbers $d_k(A;r), \dots, e^{(k)}(A;r)$ as simple quotients of determinants has been chosen. For other normalisations, the expressions defining these numbers have more complicated form. For example, by combining $D_{n+1}(A)D_n(A)$ and $D_2(A)D_1(A)$ as simple band matrices, A may be expressed as the product of $n-1$ band matrices, but the elements of the two matrices just mentioned have forms differing from those of the remaining factors. Some formulae arising from normalisations alternative to that considered in detail above will be given.

b) In the first of the alternative normalisations to be considered, the principal diagonal elements $d_k(A;r)$ and $d^{(k)}(A;r)$ of the band matrices $D_r(A)$ and $D^{(r)}(A)$ with $r \geq 2$

are taken to be unity, the matrices $D_1(A)$ and $D^{(1)}(A)$ being diagonal matrices. Set

$$e_k(A;r) = \frac{a(r,k)\,a(r-1,k-1)}{a(r,k-1)\,a(r-1,k)} \qquad (r:2;n|k:n-r+1)$$

$$D_1(A) = \beta_n\{a_{k,k}|0\},\quad D_r(A) = V_{r-2}\left[\beta_{n-r+2}\{1|e_k(A;r)\}\right] \qquad (r:2;n)$$

$$X_r(A) = V_{r-1}\left[L_{n-r+1}\left\{\frac{a(r,j)_{i+r-1}}{a(r,j)}\right\}\right] \qquad (r:n)$$

$$y_{i,j}(A;r) = \frac{a(r,i-r+1)^{(j)}}{a(r,i-r)} \qquad (r:n|i:r+1;n|j:i)$$

and $Y_r(A) = L_n\{a_{i,j}|_r\, y_{i,j}(A;r)\}$ $(r:n)$. Then

$$X_{r+1}(A)D_{r+1}(A) = X_r(A) \qquad (r;n-1)$$

$X_r(A) = [D_{=}(A)]^{n}_{r+1}$ $(r:n-1)$ and $A = [D_{=}(A)]^{n}_{1}$. Also $D_{r+1}(A)Y_r(A) = Y_{r+1}(A)$ $(r:n-1)$, $Y_r(A) = [D_{=}(A)]^{r}_{1}$ $(r:n)$ and $Y_n(A) = A$. Lastly, $X_r(A)Y_r(A) = A$ $(r:n)$.

Again, set

$$e^{(k)}(A;r) = -\frac{a(k+1,r-1)\,a(k,r-2)\,a_{k+r-2,k+r-2}}{a(k+1,r-2)\,a(k,r-1)\,a_{k+r-1,k+r-1}} \qquad (r:2;n|k:n-r+1)$$

$$D^{(1)}(A) = \beta_n\{a_{k,k}^{-1}|0\},\quad D^{(r)}(A) = V_{r-2}\left[\beta_{n-r+2}\{1|e^{(k)}(A;r)\}\right] \qquad (r:2;n)$$

$$X^{(r)}(A) = V_{r-1}\left[L_{n-r+1}\left\{\frac{a(i,r)^{(j+r-1)}}{a(i,r-1)\,a_{i+r-1,i+r-1}}\right\}\right] \qquad (r:2;n)$$

$$y^{(i,j)}(A;r) = -\frac{a(i-r+1,r-1|j-i+r)}{a(i-r+1|r-1)} \quad (j:i-r+1;i-1),\quad y^{(i,i)}(A;r) = a_{i,i}^{-1}$$

for $(i:r+1;n)$ and $Y^{(r)}(A) = L_n\{a^{(-1)}_{i,j} |_r y^{(i,j)}(A;r)\}$ $(r:n)$

Then $X^{(r+1)}(A) = D^{(r+1)}(A)X^{(r)}(A)$ $(r:n-1)$, $X^{(r)}(A) = [D^{(2)}(A)]_1^r$ $(r:n)$ and $[D^{(2)}(A)]_1^n = A^{-1}$. Also $D^{(r+1)}(A)Y^{(r)}(A) = Y^{(r+1)}(A)$ $(r:n-1)$, $Y^{(r)}(A) = [D^{(2)}(A)]_1^r$ $(r:n)$ and $Y^{(n)}(A) = A^{-1}$. Lastly $X^{(r)}(A) = Y^{(r)}(A)A$ $(r:n)$.

The matrices $D_r(A)$ occur in the formation of the vector $f = Ag$, where f and g are as described above. Set $f_i^{(1)} = a_{i,i}g_i$ $(i:n)$ and for $(r:n-1)$ determine

$$f_i^{(r+1)} = f_i^{(r)} + d_i(A;r+1)f_i^{(r)} \qquad (i:n-r)$$

then $f_i = f_1^{(i)}$ $(i:n)$. Now

$$f_i^{(r)} = \sum_{j=i}^{i+r-1} y_{i+r-1,j}(A;r)g_j \qquad (i:n-r+1)$$

With $f^{(r)}$ and $F^{(r)}$ defined as above, $f^{(r+1)} = D_{r+1}(A)^{[r]}f^{(r)}$ $(r:n-1)$ and $f^{(r)} = Y_r(A)^{[r]}g$ $(r:n)$; also $F^{(r+1)} = D_{r+1}(A)F^{(r)}$ $(r:n-1)$, $F^{(r)}(A) = Y_r(A)g$ $(r:n)$ and $F^{(n)} = f$.

To solve the equation $Ag = f$ by use of the matrices $D^{(r)}(A)$, set $g_i^{(1)} = f_i/a_{i,i}$ $(i:n)$ and, for $(r:n-1)$, evaluate

$$g_i^{(r+1)} = g_{i+1}^{(r)} + e^{(i)}(A;r+1)g_i^{(r)} \qquad (i:n-r)$$

then $g_i = g_1^{(i)}$ $(i:n)$. Now, for $(r:n)$

$$g_i^{(r)} = \sum_{j=i}^{i+r-1} y^{(i+r-1,j)}(A;r) f_j \qquad (i:n-r+1)$$

and, defining the vectors $g^{(r)}, G^{(r)}$ as in the treatment of normalisation (a), $g^{(r+1)} = D^{(r+1)}(A)^{[r]} g^{(r)}$ $(r:n-1)$ and $g^{(r)} = Y^{(r)}(A)^{[r]} f$ $(r:n)$; also $G^{(r)} = Y^{(r)}(A) f$ and $G^{(n)} = g$ again. The $g_i^{(r)}$ have the determinantal representations

$$g_i^{(r)} = \frac{a(i,r\,|\,(f))}{a(i,r-1)\, a_{i+r-1,i+r-1}} \qquad (r:n\,|\,i:n-r+1).$$

(c) In the next normalisation to be considered, the sub-diagonal elements $e_k(A;r)$ and $e^{(k)}(A;r)$ of the band matrices $D_r(A)$ and $D^{(r)}(A)$ are taken to be 1 and -1 respectively. Set

$$d_k(A;r) = \frac{a(r,k)\,a(r+1,k-2)}{a(r,k-1)\,a(r+1,k-1)} \quad (r:n-1\,|\,k:n-r+1), \quad d_1(A;n) = a_{n,1}$$

$$D_r(A) = V_{r-1}[\beta_{n-r+1}\{d_k(A;r)\,|\,1\}] \quad (r:n)$$

$$X_r(A) = V_{r-1}\left[L_{n-r+1}\left\{\frac{a(r+1,j)_{i+r-1}}{a(r+1,j-1)}\right\}\right] (r:n-1), \quad X_n(A) = V_{n-1}[L_1\{a_{n,1}\}]$$

$$y_{i,j}(A) = \frac{a(r+1,i-r)^{(j)}}{a(r+1,i-r)} \quad (r:n-1\,|\,i:r+1;n\,|\,j:i)$$

and $Y_r(A) = L_n\{a_{i,j}\,|_r\, y_{i,j}(A;r)\}$ $(r:n)$. Then $X_{r+1}(A) D_{r+1}(A) = X_r(A)$

$(r:n-1)$, $X_r(A) = [D_\tau(A)]_{r+1}^{n}$ $(r:n-1)$ and $A = [D_\tau(A)]_1^{n}$.

Also $D_{r+1}(A)\,Y_r(A) = Y_{r+1}(A)$ $(r:n-1)$, $Y_r(A) = [D_\tau(A)]_1^{r}$ $(r:n)$ and $Y_n(A) = A$. Lastly $X_r(A)\,Y_r(A) = A$ $(r:n)$

Again, for $(r:n)$ set

$$d^{(1)}(A;r) = \frac{a(2;r-1)}{a(1,r)}, \quad d^{(k)}(A;r) = \frac{a(k-1,r)\,a(k+1,r-1)}{a(k,r)\,a(k,r-1)} \quad (k:2;n-r+1)$$

$$D^{(r)}(A) = V_{r-1}[\beta_{n-r+1}\{d^{(k)}(A;r)|-1\}]$$

$$X^{(r)}(A) = V_{r-1}\left[L_{n-r+1}\left\{\frac{a(i,r)^{(j+r-1)}}{a(i+1,r-1)}\right\}\right]$$

with $X^{(n+1)}(A) = E_n$. For $(r:n|i:r+1;n)$ set

$$y^{(i,j)}(A;r) = -\frac{a(i-r,r|j-i+r+1)}{a(i-r+1,r}\,(j:i-r;i-1), \quad y^{(i,i)}(A;r) = \frac{a(i-r,r)}{a(i-r+1,r)}$$

and $Y^{(r)}(A) = L_n\{a^{(-1)}_{i,j}|_r\, y^{(i,j)}(A;r)\}$ $(r:n)$. Then $X^{(r+1)}(A) = D^{(r)}(A)\,X^{(r)}(A)$ $(r:n-1)$, $X^{(r)}(A) = [D^{(\tau)}(A)]_1^{r-1} A$ $(r:n)$, $[D^{(\tau)}(A)]_1^{n} = A^{-1}$. Also $Y^{(r+1)}(A) = D^{(r+1)}(A)\,Y^{(r)}(A)$ $(r:n-1)$, $Y^{(r)}(A) = [D^{(\tau)}(A)]_1^{r}$ $(r:n)$ and $Y^{(n)}(A) = A^{-1}$. Now $X^{(r+1)}(A) = Y^{(r)}(A)\,A$ $(r:n)$.

To form the product $f = Ag$, set $f_i^{(1)} = g_i$ $(i:n)$ and ~~determine~~ for $(r:n-1)$ determine

$$f_i^{(r+1)} = d_{j+1}(A;r)\,f_{i+1}^{(r)} + f_i^{(r)} \qquad (i:n-r)$$

then $f_i = d_1(A;i) f_1^{(i)}$ $(i:n)$. Now

$$f_i^{(r+1)} = \sum_{j=i}^{i+r} y_{i+r,j}(A;r) g_j \qquad (r:n-1 \mid i:n-r)$$

and, with $f^{(r)}, F^{(r)}$ as above, $f^{(r+1)} = D_r(A)^{[r]} f^{(r)}$ $(r:n-1)$ and $f^{(r+1)} = Y_r(A)^{[r]} g$ $(r:n-1)$; also $F^{(r+1)} = D_r(A) F^{(r)}$ $(r:n-1)$, and $F^{(r+1)} = Y_r(A) g$ $(r:n-1)$.

To solve the system of equations $Ag = f$, set $g_i^{(1)} = f_i$ $(i:n)$ and, for $(r:n-1)$ determine

$$g_i^{(r+1)} = d^{(i+1)}(A;r) g_{i+1}^{(r)} - g_i^{(r)} \qquad (i:n-r)$$

then $g_i = d^{(1)}(A;i) g_1^{(i)}$ $(i:n)$. Now

$$g_i^{(r+1)} = \sum_{j=i}^{i+r} y^{(i+r,j)}(A;r) f_j \qquad (r:n-1 \mid i:n-r)$$

and, with $g^{(r)}, G^{(r)}$ as above, $g^{(r+1)} = D^{(r)}(A)^{[r]} g^{(r)}$ $(r:n-1)$ and $g^{(r+1)} = Y^{(r)}(A)^{[r]} f$ $(r:n-1)$; also $G^{(r+1)} = D^{(r)}(A) G^{(r)}$ $(r:n-1)$ and $G^{(r+1)} = Y^{(r)}(A) f$ $(r:n-1)$. The $g_i^{(r)}$ have the determinantal expressions

$$g_i^{(r)} = \frac{a(i,r \mid (f))}{a(i+1,r-1)} \qquad (r:n \mid i:n-r+1)$$

Before describing two further normalisations, definitions of two classes of lower triangular matrices are given.

Definition. A lower triangular matrix $L_n\{a_{i,j}\}$ for which

$$\sum_{j=1}^{i} a_{i,j} = 0 \qquad (i:2;n)$$

<for which

$$\sum_{j=1}^{i} a_{i,j} = 1 \qquad (i:n)>$$

is said to be annihilative <permanent>.

d) If A is annihilative, and all components of the vector g are equal, all but the first of the components of the vector $f = Ag$ are zero. The elements $d_k(A;r), e_k(A;r)$ of those factor band matrices $D_r(A)$ that are not simple diagonal matrices may be so determined that relationships of the form

$$e_k(A;r) + d_{k+1}(A;r) = 0$$

obtain consistently; all such matrices $D_r(A)$ are then annihilative. Particular interest attaches to the case in which A itself is annihilative; for such a matrix, the special relationship

$$a_{1,1}\, a(2;j-1) = (-1)^{j-1} a(1,j) \qquad (j:n)$$

holds. The inverse of A is not, of course, annihilative; nevertheless it may be expressed in terms of annihilative factors.

Assuming A to be decomposable and annihilative, set

$$d_1(A;r)=1\ (r:2;n),\quad d_k(A;r)=-\frac{a(r,k-1)a(r,k-2)}{a(r+1,k-2)a(r-1,k-1)}\quad (r:2;n|k:2;n-r+2)$$

$$e_k(A;r)=\frac{a(r,k)a(r,k-1)}{a(r+1,k-1)a(r-1,k)}\quad (r:2;n|k:n-r+1)$$

$$D_1(A;r)=\beta_n\{a_{1,1}|0\},\quad D_r(A)=V_{r-2}[\beta_{n-r+2}\{d_k(A;r)|e_k(A;r)\}]\ (r:2;n)$$

$$X_r(A)=V_{r-1}\left[L_{n-r+1}\left\{(-1)^{j-1}\frac{a(r+1,j-1)a(r,j)_{i+r-1}}{a(r,j-1)a(r,j)}\right\}\right]\quad (r:n)$$

$$y_{i,j}(A;r)=(-1)^{i-r}\frac{a(r,i-r+1)^{(j)}}{a(r+1,i-r)}\qquad (r:n-1|i:r+1;n|j:i-r+1;i)$$

and $Y_r(A)=L_n\{a_{i,j}|_r\, y_{i,j}(A;r)\}$ $(r:n)$. Then $X_r(A)=X_{r+1}(A)D_{r+1}(A)$ $(r:n-1)$, $X_r(A)=[D_z(A)]_{r+1}^{n}$ $(r:n-1)$ and $A=[D_z(A)]_1^n$. Also $D_{r+1}(A)Y_r(A)=Y_{r+1}(A)$ $(r:n-1)$, $Y_r(A)=[D_z(A)]_1^r$ $(r:n)$ and $Y_n(A)=A$. Lastly $X_r(A)Y_r(A)=A$ $(r:n)$.

Again, for $(r:2;n)$ set

$$d^{(1)}(A;r)=1,\quad d^{(k)}(A;r)=\frac{a(k,r-1)a(k-1,r-1)}{a(k,r-2)a(k-1,r)}\quad (k:2;n-r+2)$$

$$e^{(k)}(A;r)=-\frac{a(k+1,r-1)a(k,r-1)}{a(k+1,r-2)a(k,r)}\qquad (k:n-r+1)$$

$$D^{(r)}(A)=V_{r-2}[\beta_{n-r+2}\{d^{(k)}(A;r)|e^{(k)}(A;r)\}]\qquad \cancel{(r:2;n)}$$

and for $(r:n)$ set

$$X^{(r)}(A)=V_{r-1}\left[L_{n-r+1}\left\{\frac{a(i,r)^{(j+r-1)}}{a(i,r}\right\}\right]\qquad \cancel{(r:n)}$$

$$y^{(i,j)}(A;r) = -\frac{a(i-r+1,r-1|j-i+r)}{a(i-r+1,r)} \quad (j:i-r+1;i-1), \quad y^{(i,i)}(A;r) = a_{i,i}^{-1}$$

and $Y^{(r)}(A) = L_n\{a^{(-1)}_{i,j}|_r\, y^{(i,j)}(A;r)\}$. Then $X^{(r+1)}(A) = D^{(r+1)}(A)\,X^{(r)}(A)$ $(r:n-1)$, $X^{(r)}(A) = [D^{(2)}(A)]_1^r A$ $(r:n)$ and $[D^{(2)}(A)]_1^n = A^{-1}$. Also $Y^{(r+1)}(A) = D^{(r+1)}(A)\,Y^{(r)}(A)$ $(r:n-1)$, $Y^{(r)}(A) = [D^{(2)}(A)]_1^r$ $(r:n)$ and $Y^{(n)}(A) = A^{-1}$. Now $X^{(r)}(A) = Y^{(r)}(A)A$ $(r:n)$.

To form the product $f = Ag$, set $f_i^{(1)} = a_{i,1}g_i$ $(i:n)$ and for $(r:n-1)$ determine

$$f_i^{(r+1)} = e_i(A;r+1)\{f_i^{(r)} - f_{i+1}^{(r)}\} \qquad (i:n-r)$$

then $f_i = f_1^{(i)}$ $(i:n)$. Now

$$f_i^{(r)} = \sum_{j=i}^{i+r-1} y_{i+r-1,j}(A;r)\, g_j \qquad (r:n|i:n-r+1)$$

and, with $f^{(r)}, F^{(r)}$ defined as above, $f^{(r+1)} = D_{r+1}(A)^{[r]} f^{(r)}$ $(r:n-1)$ and $f^{(r)} = Y_r(A)^{[r-1]} g$ $(r:n)$; also $F^{(r+1)} = D_{r+1}(A)F^{(r)}$ $(r:n-1)$ and $F^{(r)} = Y_r(A)\, g$ $(r:n)$.

To solve the system of equations $Ag = f$, set $g_i^{(1)} = f_i/a_{i,1}$ $(i:n)$ and for $(r:n-1)$ determine

$$g_i^{(r+1)} = d^{(i+1)}(A;r+1)\{g_{i+1}^{(r)} - g_i^{(r)}\} \qquad (i:n-r)$$

then $g_i = g^{(i)}$ $(i:n)$. Now

$$\delta_i^{(r)} = \sum_{j=i}^{i+r-1} y^{(i+r-i,j)}(A;r) f_j \qquad (r:n|i:n-r+1)$$

and, with $\delta^{(r)}$, $\mathcal{G}^{(r)}$ as defined above, $\delta^{(r+1)} = D^{(r+1)}(A)^{[r]} \delta^{(r)}$ $(r:n-1)$ and $\delta^{(r+1)} = Y^{(r+1)}(A)^{[r]} f$ $(r:n-1)$. Also $\mathcal{G}^{(r)} = Y^{(r)}(A) f$

Y? $(r:2;n)$. The $\delta_i^{(r)}$ have the determinantal representations

$$\delta_i^{(r)} = \frac{a(i,r|(f))}{a(i,r)} \qquad (r:n|i,n-r+1)$$

e) If A is permanent, and all components of the vector δ are equal, to S say, all components of the vector $f = A\delta$ are also equal to S. By stipulating that the elements $d_k(A;r)$, $e_k(A;r)$ of the factor matrices $D_r(A)$ that are not simple diagonal matrices should consistently satisfy relationships of the form

$$e_k(A;r) + d_{k+1}(A;r) = 1$$

all such matrices are rendered permanent. Again, particular interest attaches to the case in which A itself is permanent; for such a matrix, the special relationships

$$a(1,j|(1)) = a(1,j) \ (j:n), \quad a(2,j) + a_{j+1,j+1} a(2,j-1) = a(2,j|(1)) \ (j:n-1)$$

hold; the inverse of A, although not permanent, may be expressed in terms of permanent factors.

Definition. An n^{th} order decomposable lower triangular matrix A for which $a(r,k|(1)) \neq 0$ $(r:n|k:2;n-r+1)$ is said to be permanently decomposable.

Assuming A to be permanently decomposable, set

$$d_k(A;r) = \frac{a(r,k|(1))a(r+1,k-2)}{a(r+1,k-1|(1))a(r,k-1)} \qquad (k:n-r)$$

$$e_k(A;r) = \frac{a(r,k|(1))a(r+1,k)}{a(r+1,k|(1))a(r,k)} \qquad (k:n-r-1)$$

$$D_r(A) = V_{r-1}[\beta_{n-r+1}\{d_k(A;r)|e_k(A;r)\}]$$

all for $(r:n-1)$, and

$$X_r(A) = V_{r-1}\left[L_{n-r+1}\left\{\frac{a(r,j|(1))a(r,j)_{i+r-1}}{a(r,j)a(r,j-1)}\right\}\right] \qquad (r:n)$$

$$y_{i,j}(A;r) = \frac{a(r+1,i-r)^{(j)}}{a(r+1,i-r|(1))} \qquad (r:n-1|i:r+1,n-r+1|j:i)$$

and $Y_r(A) = L_n\{a_{i,j}|_r\, y_{i,j}(A;r)\}$ $(r:n-1)$. Then

$$X_{r+1}(A)D_r(A) = X_r(A) \qquad (r:n-1)$$

$X_r(A) = [D_\tau(A)]_r^{n-1}$ $(r:n-1)$ and $A = [D_\tau(A)]_1^{n-1}$. Also $D_{r+1}(A)Y_r(A) = Y_{r+1}(A)$ $(r:n-2)$, $Y_r(A) = [D_\tau(A)]_1^{r}$ $(r:n-1)$ and $Y_{n-1}(A) = A$. Lastly $Y_r(A)X_{r+1}(A) = A$ $(r:n-1)$.

Again, set

$$d^{(1)}(A;r) = 1,\ d^{(k)}(A;r) = \frac{a(k,r|(1))a(k-1,r)}{a(k-1,r|(1))a(k,r-1)} \qquad (k:2,n-r+1)$$

$$e^{(k)}(A;r) = -\frac{a(k,r|(1))a(k+1,r)}{a(k,r+1|(1))a(k+1,r-1)} \qquad (k:n-r)$$

$$D^{(r)}(A) = V_{r-1}\left[\beta^{\delta}_{2n-r+1}\{d^{(k)}(A;r)\,|\,e^{(k)}(A;r)\}\right],$$

$$X^{(r)}(A) = V_{r-1}\left[L_{n-r+1}\left\{\frac{a(i,r)^{(r+j-1)}}{a(i,r|(1))}\right\}\right]$$

all for $(r:n)$, so that $X^{(1)}(A) = A$,

$$y^{(i,j)}(A;r) = -\frac{a(i-r,r\,|\,j-i+r+1)}{a(i-r,r+1\,|(1)}\quad (j:i-r;i-1),\ y^{(i,i)}(A;r) = \frac{a(i-r,r)}{a(i-r,r+1|(1))}$$

for $(r:n-1|i:r+1;n)$ and $Y^{(r)}(A) = L_n\{a^{(-1)}_{i,j}\,|_r\,y^{(i,j)}(A;r)\}^{\delta}$ $(r:n)$.

Then $X^{(r+1)}(A) = D^{(r)}(A)X^{(r)}(A)$ $(r:n-1)$, $X^{(r)}(A) = [D^{(z)}(A)]_1^{-r-1}$ $(r:n)$ and $[D^{(z)}(A)]_1^{-n-1} = A^{-1}$. Also $Y^{(r+1)}(A) = D^{(r+1)}(A)Y^{(r)}(A)$ $(r:n-1)$, $Y^{(r)}(A) = [D^{(z)}(A)]_1^{-r}$ $(r:n-1)$ and $Y^{(n-1)}(A) = A^{-1}$. Now $X^{(r+1)}(A) = Y^{(r)}(A)A$ $(r:n-1)$.

To form the product $f = Ag$, set $f_i^{(1)} = g_i$ $(i:n)$ and for $(r:n-1)$ determine

$$f_i^{(r+1)} = \{1-e_i(A;r)\}f_{i+1}^{(r)} + e_i(A;r)\,f_i^{(r)} \qquad (i:n-r)$$

then $f_i = f_1^{(i)}$ $(i:n)$. Now

$$f_i^{(r+1)} = \sum_{j=i}^{i+r} y_{i+r,j}(A;r)\,g_j \qquad (r:n-1|i:n-r)$$

and, with $f^{(r)}$, $F^{(r)}$ as defined above, $f^{(r+1)} = D_r(A)^{[r]}f^{(r)}$ $(r:n-1)$, $f^{(r+1)} = Y_r(A)^{[r]}g$ $(r:n-1)$, $F^{(r+1)} = D_r(A)F^{(r)}$ and $F^{(r+1)} = Y_r(A)g$ $(r:n-1)$.

To solve the system of equations $A\mathring{g} = f$, set $g_i^{(1)} = f_i$ $(i:n)$ and for $(r:n-1)$ determine

$$\mathring{g}_i^{(r+1)} = \{1-e^{(i)}(A;r)\}\, g_{i+1}^{(r)} + e^{(i)}(A;r)\, g_i^{(r)} \qquad (i:n-r)$$

then $\mathring{g}_i = g_i^{(i)}$ $(i:n)$. Now

$$\mathring{g}_i^{(r+1)} = \sum_{j=i}^{i+r} y^{(i+r,j)}(A;r) f_j \qquad (r:n-1|i:n-r)$$

and, with $\mathring{g}^{(r)}, G^{(r)}$ defined as above, $\mathring{g}^{(r+1)} = D^{(r)}(A)^{[r]}\mathring{g}^{(r)}$ $(r:n-1)$ and $\mathring{g}^{(r+1)} = Y_1^{(r)}(A)^{[r]} f$ $(r:n-1)$; also $G^{(r+1)} = D^{(r)}(A)\, G^{(r)}$ $(r:n-1)$, $G^{(r+1)} = Y_1^{(r)}(A) f$ $(r:n-1)$. The $\mathring{g}_i$ have the determinantal expressions

$$\mathring{g}_i^{(r)} = \frac{a(i,r|(f))}{a(i,r|(1))} \qquad (r:n|i:n-r+1)$$

The elements $d_k(A;r), \ldots, e^{(k)}(A;r)$ of the various band matrices occurring in the above exposition have been expressed as determinantal quotients. They may in each case be determined by the use of simple recursive algorithms. For example, for the normalisation (b) above, set $d_1(A;1) = a_{1,1}$ and, for $(r:n-1)$ determine

$$d_{r+1}(A;1) = a_{r+1,r+1} \quad , \quad e_1(A;r+1) = a_{r+1,1} / a_{r,1}$$

and, if $r>1$,

$$x_{r+1,j}(A;1)=a_{r+1,j}/d_j(A;1) \qquad (j:r)$$

and for $(m:2;r)$

$$x_{r-m+2,1}(A;m)=x_{r-m+3,1}(A;m-1)/e_1(A;m)$$

with, if $m<r$,

$$x_{r-m+2,j}(A;m)=\{x_{r-m+3,j}(A;m-1)-x_{r-m+2,j-1}(A;m)\}/e_j(A;m)$$
$$(j:2;r-m+1)$$

and finally

$$e_{r-m+2}(A;m)=x_{r-m+3,r-m+2}(A;m-1)-x_{r-m+2,r-m+2}(A;m)$$

Use of the above formulae results in the determination of $d_{r+1}(A;1)$ and $e_k(A;r-k+2)$ $(k:r)$, use being made of numbers $d_m(A;1)$ $(m:r)$ and $e_k(A;m-k+2)$ $(m:r-1|k:m)$ previously determined, and auxiliary numbers $x_{i,j}(A;m)$. The $x_{i,j}(A;m)$ are elements of the matrices $X_m(A)$; the numbers $x_{i,j}(A;m)$ computed during one stage, stage r say, of the above process are not used in the implementation of the next, stage $r+1$. The above factorisation of a matrix A may be carried out in situ, the elements $a_{r+1,j}$ $(j:r+1)$ being progressively displaced by $e_k(A;r-k+2)$ $(k:r)$ and $d_{r+1}(A;1)$.

To determine the elements $d^{(k)}(A;1)$, $e^{(k)}(A;r)$, again in the normalisation of (b), set $d^{(k)}(A;1) = 1/a_{k,k}$ $(k:2)$, $e^{(1)}(A;2) = -a_{2,1}/a_{1,1}$ and $x^{(1,1)}(A;2) = 1$. Then, for $(r:2;n-1)$ determine $d^{(r+1)}(A;1) = 1/a_{r+1,r+1}$ and

$$x^{(r+1,j)}(A;1) = a_{r+1,j}/a_{r+1,r+1} \; (j:r), \quad x^{(r+1,r+1)}(A;1) = 1$$

and, for $(m:2;r+1)$

$$e^{(r-m+2)}(A;m) = -x^{(r-m+3,1)}(A;m-1)/x^{(r-m+2,1)}(A;m-1)$$

if $r < m+1$

$$x^{(r-m+2,j)}(A;m) = x^{(r-m+3,j+1)}(A;m-1) + e^{(r-m+2)}(A;m)\,x^{(r-m+2,j+1)}$$

$$(j:r-m+1)$$

and finally, if $m < n$, $x^{(r-m+2,r-m+2)}(A;m) = 1$. During the implementation of the above set of formulae, the required numbers $d^{(r+1)}(A;1)$, $e^{(r-m+2)}(A;m)$ $(m:2;r+1)$ together with the auxiliary numbers $x^{(r-m+2,j)}(A;m)$ $(m:r+1|j:r-m+2)$ are determined, use being made of the members of the latter set together with $x^{(r-m+1,j)}(A;m)$ $(m:r|j:r-m+1)$. After completion of the stage described in detail, the numbers $\{x^{(r-m+1,j)}(A;m)\}$ together with similar sets of numbers used in previous stages, are no longer required for use.

The problem converse to that considered above, namely that of expressing the elements of a lower triangular matrix A in terms of those occurring in it's band matrix decomposition, may be solved by use of sums now to be defined.

Notation With $e_k(r) \in K$ $(r:2;n|k:r-1)$ prescribed and integers m,i,j in the ranges $1 \leq m \leq n-1$, $2 \leq i \leq n-m+1$, $1 \leq j < i$ form all distinct strictly increasing sequences of integers $\tau(\nu)$ $(\nu:i-j)$ for which $m+1 \leq \tau(\nu) \leq m+i-1$; form corresponding sequences of integers $\mu(\nu) = m+j+\nu-\tau(k)$ $(\nu:i-j)$. Set

$$\Xi_{i,j}\{e_k(r),m\} = \sum \prod_{\nu=1}^{i-j} e_{\mu(\nu)}(\tau(\nu))$$

where summation is extended over all corresponding sequences of integers $\mu(\nu)$ and $\tau(\nu)$. Also set $\Xi_{i,i}\{e_k(r),m\} = 1$ $(m:n|i:n-m+1)$.

For the purposes of clarification, consider the case in which $m=1$, $i=4$, $j=2$. The sequences of integers $\tau(\nu)$ that can be formed are $(2,3)$, $(2,4)$ and $(3,4)$; the sequences of integers $\mu(\nu)$ corresponding to these are $(2,2)$, $(2,1)$ and $(1,1)$. Thus

$$\Xi_{4,2}\{e_k(r),1\} = e_2(2)e_2(3) + e_2(2)e_1(4) + e_1(3)e_1(4)$$

Clearly

$$\Xi_{i,1}\{e_k(r),m\} = \prod_{\nu=m+1}^{m+i-1} e_1(\nu) \qquad (m:n-1 \mid i:n-m+2)$$

$$\Xi_{i,1}\{e_k(r);m+1\}e_1(m+1) = \Xi_{i+1,1}\{e_k(r),m\} \quad (m:n-2 \mid i:n-m-1)$$

The sum ( ) contains certain terms for which $\tau(\nu)=m+1$. When this occurs, $\nu=1$, since the $\tau(\nu)$ form a strictly increasing sequence in the range $m+1 \leq \tau(\nu) \leq m+i-1$. When $\tau(\nu)=m+1$, $\nu=1$ the accompanying value of $\mu(\nu)$ takes the value j. Accordingly, the sum ( ) contains products of two types: those containing the sum $e_j(m+1)$ as first factor, and those not containing this factor at all. The cofactor $\Xi'_{i,j}$ derived from those products containing this term is obtained in the following way: form all distinct strictly increasing sequences of integers $\tau(\nu+1)$ $(\nu:i-j-1)$ for which $m+2 \leq \tau(\nu+1) \leq m+i-1$; form corresponding sequences of integers $\mu(\nu+1) = m+j+\nu-\tau(\nu+1)+1$ $(\nu:i-j-1)$. Then

$$\Xi'_{i,j} = \Sigma \prod_{\nu=1}^{i-j-1} e_{\mu(\nu+1)}(\tau(\nu+1))$$

The above description of $\Xi'_{i,j}$ is that of $\Xi_{i-1,j}\{e_k(r);m+1\}$. In a similar way it is shown that the component independent of $e_j(m+1)$ in $\Xi_{i,j}\{e_k(r);m\}$ is $\Xi_{i-1,j-1}\{e_k(r);m+1\}$. Replacing i by $i+1$ in the preceding argument, the relationship

$$\Xi_{i,j}\{e_k(r);m+1\}e_j(m+1) + \Xi_{i,j-1}\{e_k(r);m+1\} = \Xi_{i+1,j}\{e_k(r);m\}$$
$$(m:2;n-2|i:n-m|j:i-1)$$

is derived.

Setting $x_{i,j}(m) = \Xi_{i,j}\{e_k(r);m\}$ $(m:n-1|i:n-m+1|j:i)$ $x_{i,i}(r)=1$ $(r:n|i:n-r+1)$, $X_m = V_{m-1}[L_{n-m+1}\{x_{i,j}\}]$ and $D_m = V_{m-2}[\beta_{n-m+2}\{1|e_k(m)\}]$ $(m:2;n)$ the above relationships lead to a matrix formula similar to ( ) relating to the normalisation (b) above. In particular, if the $e_k(r)$ are taken to be the $e_k(A;r)$ of formula ( ), $X_m = X_m(A)$ $(m:n)$, the $X_m(A)$ being those matrices described in connection with that normalisation. Furthermore, for the relevant diagonal matrix $D_1(A)$, $X_1(A)D_1(A) = A$. Hence

$$a_{i,j} = a_{j,j}\,\Xi_{1_{i,j}}\{e_k(A;r);1\} \qquad (i:n|j:i)$$

for the normalisation (b). In the cases of the other normalisations considered, the $a_{i,j}$ are also expressible by the use of similar formulae involving the sums $\Xi_{i,j}$.

Relationship ( ) connects sums of the form $\Xi_{i,j}\{e_k(r);m\}$ and $\Xi_{1_{i,j}}\{e_k(r);m+1\}$. A relationship connecting modified versions of these sums with equal values of the parameter m also exists. Denote by $E_\mu \Xi_{i,j}\{e_k(r);m\}$ that sum obtained from ( ) by replacing each suffix $\mu(\nu)$ by $\mu(\nu)+1$. Then

$$E_\mu \Xi_{1_{i,j}}\{e_k(r);m\} + e_1(i+m)\,\Xi_{1_{i,j+1}}\{e_k(r);m\} = \Xi_{1_{i+1,j+1}}\{e_k(r);m\}$$
$$(m:n-2|i:n-m-1|j:i-1)$$

Certain matrices A have band matrix decompositions that may be derived directly from formulae defining the $a_{i,j}$.

Notation. Set

$$R^{(m)}_{\mu,k}(u)\langle S^{(m)}_{\mu,k}(u)\rangle = \sum_1 \prod_{\nu=1}^{m} u\{\tau(\nu)\}$$

where $\{\tau(1),\dots,\tau(m)\}$ ranges over all distinct sets of m integers $\langle m$ distinct integers$\rangle$ taken from the set $\mu,\dots,k$

The two formulae

$$R^{(2)}_{1,2}(u) = u(1)^2 + u(1)u(2) + u(2)^2, \quad S^{(2)}_{1,2}(u) = u(1)u(2)$$

illustrate the above notations. Evidently $R^{(m)}_{\mu,\mu}(u) = u(\mu)^m$,

$S^{(m)}_{\mu,k}(u) = 0$ when $m > k-\mu+1$, and

$$S^{(m)}_{\mu,\mu+m-1}(u) = \prod_{\nu=0}^{m-1} u(\mu+\nu)$$

It is easily verified, that, $u(\nu)$ $(\nu:n)$ being prescribed

$$u(\mu)R^{(m)}_{\mu,k}(u) + R^{(m+1)}_{\mu+1,k}(u) = R^{(m+1)}_{\mu,k}(u) \qquad (\mu:n-1|k:\mu+1;n|m:n-1)$$

$$u(k+1)R^{(m)}_{\mu,k+1}(u) + R^{(m+1)}_{\mu,k}(u) = R^{(m+1)}_{\mu,k+1}(u) \quad (\mu:n-1|k:\mu;n-1|m:n-1)$$

$$\mu(u)S^{(m)}_{\mu+1,k}(u) + S^{(m+1)}_{\mu+1,k}(u) = S^{(m+1)}_{\mu,k}(u) \qquad (\mu:n-2|k:\mu+2;n|m:k-\mu-1)$$

$$\mu(k+1)S^{(m)}_{\mu,k}(u) + S^{(m+1)}_{\mu,k}(u) = S^{(m+1)}_{\mu,k+1}(u) \quad (\mu:n-2|k:\mu+1;n-1|m:k-\mu)$$

$v(k)$ $(k:n)$ being available, set

$$e_k(A;r) = -\frac{u(k)}{v(k+r-1)} \qquad (r:2;n|k:n-r+1)$$

$$x_{i,i}(A;r) = 1 \;\; (r:n|i:n-r+2), \quad x_{i,j}(A;r) = \frac{(-1)^{i-j}R^{(i-j)}_{1,j}(u)}{\prod_{k=j+1}^{i} v(k+r-1)} \quad (r:n-1|i:2;n-r+1|j:i-1)$$

As is easily verified by use of relationships (,)

$$x_{i-1,1}(A;r+1)e_1(A;r+1) = x_{i,1}(A;r) \qquad (i:2;n-r+1)$$

$$x_{i-1,j}(A;r+1) + x_{i-1,j+1}(A;r+1)e_{j+1}(A;r+1) = x_{i,j+1}(A;r)$$

$$(r:n-1 \mid i:2;n-r+1 \mid j:i-1)$$

Defining the band matrices $D_r(A)$ by means of formula ( ), and setting $X_r(A) = V_{r-1}[L_{n-r+1}\{x_{i,j}(A;r)\}]$ $(r:n)$, the above relationships lead to formula ( ) for the normalisation (b) connecting the matrices of these two systems. Accordingly, setting $D_1(A) = E_n$, it follows that the matrices $D_r(A)$ occurring in the factorisation, according to the normalisation of (b), of the matrix A with ~~diagonal~~ elements $a_{j,j} = 1$ $(j:n)$ and

$$a_{i,j} = \frac{(-1)^{i-j} R^{(i-j)}_{1,j}(u)}{\prod_{\nu=j+1}^{i} v(\nu)} \qquad (i:2;n \mid j:i-1)$$

are the $D_r(A)$ just defined; in particular the $D_r(A)$ with $r>1$ have off diagonal elements defined by formula ( ).

Again, set

$$e^{(k)}(A;r) = \frac{u(r-1)}{v(k+r-1)} \qquad (r:2;n \mid k:n-r+1)$$

$$x^{(i,i)}(A;r) = 1 \ (r:n \mid i:n-r+2), \quad x^{(i,j)}(A;r) = \frac{(-1)^{i-j} R^{(i-j)}_{r,j+r-1}(u)}{\prod_{k=j+1}^{i} v(k+r-1)} \quad (r:n-1 \mid i:2;n-r+1 \mid j:i-1)$$

Then

$$e^{(i)}(A;r+1)\,x^{(i,1)}(A;r)+x^{(i+1,1)}(A;r)=0 \qquad (r:n-1\,|\,i:n-r+1)$$

$$x^{(i,j)}(A;r+1)=e^{(i)}(A;r+1)\,x^{(i,j+1)}(A;r)+x^{(i+1,j+1)}(A;r)$$

$$(r:n-2\,|\,i:2;n-r\,|\,j:i-1)$$

Setting $D_{\S}^{(1)}(A)=E_n$, defining band matrices $D^{(r)}(A)$ by means of formula ( ), and letting $X^{(r)}(A)=V_{r-1}[L_{n-r+1}\{x^{(i,j)}(A;r)\}]$ $(r:n)$ it follows that the matrices $D^{(r)}(A)$ occurring in the factorisation, according to the normalisation of (b), of the inverse of the matrix A with elements ( ) are the $D^{(r)}(A)$ just defined; the $D^{(r)}(A)$ with $r>1$ have off-diagonal elements defined by formula ( ).

For the unit lower triangular matrix C with elements

$$c_{i,j}=\frac{S_{1,i-1}^{(i-j)}(u)}{\prod_{\nu=j+1}^{i} v(\nu)} \qquad (i:2;n\,|\,j:i-1)$$

the formulae corresponding to (-) are

$$e_k(C;r)=\frac{u(r-1)}{v(k+r-1)},\quad x_{i,j}(C;r)=\frac{S_{r,i+r-2}^{(i-j)}(u)}{\prod_{\nu=j+1}^{i} v(\nu+r-1)};$$

$$e^{(k)}(C;r)=-\frac{u(k)}{v(k+r-1)},\quad x^{(i,j)}(C;r)=\frac{S_{1,i-1}^{(i-j)}(u)}{\prod_{\nu=j+1}^{i} v(\nu+r-1)}$$

valid for index values as for formulae (-). Again with $D_1(C) = D^{(1)}(C) = E_n$, the off-diagonal elements of the band matrices $D_r(C)$ with $r>1$ occurring in the factorisation of C according to the normalisation ( ) are given by the first of formulae ( ), and those of $D^{(r)}(C)$ with $r>1$ occurring in the similar factorisation of C^{-1} are given by the first of formulae ( ).

Comparison of formulae (,) and the first of (;) reveals, as direct computation also does, that $C = A^{-1}$.

Among the elements $x_{i,j}(A;r), \ldots, x^{(i,j)}(A;r)$ with equal values of r, the following relationships prevail:

$$v(i+r)x_{i+1,j}(A;r) = v(j+r-1)x_{i,j-1}(A;r) - u(j)x_{i,j}(A;r)$$

$$v(i+r)x^{(i+1,j)}(A;r) = v(j+r-1)x^{(i,j-1)}(A;r) - u(j+r-1)x^{(i,j)}(A;r)$$

$$v(i+r)x_{i+1,j}(C;r) = v(j+r-1)x_{i,j-1}(C;r) + u(i+r-1)x_{i,j}(C;r)$$

$$v(i+r)x^{(i+1,j)}(C;r) = v(j+r-1)x^{(i,j-1)}(C;r) + u(i)x^{(i,j)}(C;r)$$

all for $(r:n-1 \mid i:n-r \mid j:i)$.

Decomposability of a lower triangular matrix, as described in Definition , is determined by the values of the determinants $a(i,j)$ formed from arrays whose first column is always part of the first column of A. Under more stringent conditions upon A, involving determinants formed from arrays whose first columns are taken from general columns of A, a determinantal proof of a recursion similar to ( ) for the solution of the system of linear algebraic equations $Ag=f$ may be given.

Notation . A being a lower triangular $n\times n$ matrix with elements $a_{i,j} \in K$, $a(i,k;j)$ is, for $(i,k:n \,|\, j:\min(n-i+1,n-k+1))$, the j^{th} order determinant formed from the array whose τ^{th} row $(\tau:j)$ contains the elements $a_{i+\tau-1,k+\nu-1}$ $(\nu:j)$. Also $a(i,k;0)=1$ $(i:n|k:i)$.

Evidently $a(i,k;j)=0$ when $k>i$; furthermore, in terms of Notation , $a(i,1;j)=a(i,j)$. The determinants $a(i,k;j)$ satisfy the elementary identity

$$a(i,k;j-1)a(i+1,k+1;j-1)-a(i,k+1;j-1)a(i+1,k;j-1)$$
$$=a(i+1,k+1;j-2)a(i,k;j) \qquad (i,k:n-1|j:2;\min(n-i+1,n-k+1))$$

Theorem . Let A be an $n\times n$ lower triangular matrix for which $a(i,k;r)\neq 0$ $(r:n|i:n-r|k:i)$, and f, g be two column vectors with elements f_i, g_i $(i:n)$ respectively. The system of linear algebraic equations $Ag = f$ may be solved in the following way: first, determine numbers $\psi(i,k;r)$ $(r:0;n-2|i:2;n-r|k:i-1)$, $\theta(i,k;r)$ $(r:0;n-3|i:2;n-r-1|k:i-1)$ and $\phi(i,k;r)$ $(r:n-1|i:n-r|k:i)$ by setting

$$\psi(i,k;0) = \frac{a_{i,k+1}}{a_{i,k}} \quad (i:2;n|k:n-1), \quad \theta(i,k;0) = \frac{a_{i,k+1}}{a_{i,k}} \quad (i:2;n-1|k:i-1)$$

initially and thereafter for $(r:n-1)$ determining

$$\phi(i,i;r) = \psi(i+1,i;r-1) \qquad (i:n-r)$$

with, if $r<n-1$,

$$\phi(i,k;r) = \psi(i+1,k;r-1) - \theta(i,k;r-1) \qquad (i:n-r|k:i-1)$$

and, again if $r<n-1$,

$$\psi(i,k;r) = \frac{\phi(i,k+1;r)\psi(i+1,k;r-1)}{\phi(i,k;r)} \qquad (i:2;n-r|k:i-1)$$

and, if $r<n-2$,

$$\theta(i,k;r) = \frac{\phi(i,k+1;r)\theta(i,k;r-1)}{\phi(i,k;r)} \qquad (i:2;n-r|k:i-1)$$

Having produced the above numbers, set

$$\phi(i,r) = \phi(i,1;r) \qquad (r:n-1|i:n-r)$$

and determine numbers $g_i^{(r)}$ $(r:n|i:n-r+1)$ by setting $g_i^{(1)} = f_i/a_{i,1}$ $(i:n)$ and using the formula

$$g_i^{(r+1)} = \frac{g_{i+1}^{(r)} - g_i^{(r)}}{\phi(i,r)} \qquad (r:n-1|i:n-r)$$

when $g_i^0 = g_1^{0(i)}$ $(i:n)$.

Proof. Under the conditions of the theorem concerning the determinants $a(i,k;r)$, the numbers

$$\psi(i,k;r) = \frac{a(i,k;r)a(i,k+1;r+1)}{a(i,k+1;r)a(i,k;r+1)}$$

$$\theta(i,k;r) = \frac{a(i+1,k;r)a(i,k+1;r+1)}{a(i,k;r+1)a(i+1,k+1;r)}$$

$$\phi(i,k;r) = \frac{a(i+1,k;r-1)a(i,k;r+1)}{a(i,k;r)a(i+1,k;r)}$$

are well determined for the index values of these numbers to which formulae (-) relate. That the initial values ( ) are consistent with formulae (,) may be verified directly. That the above determinantal expressions satisfy relationships (, ,) is easily confirmed, and that they also satisfy relationship ( ) follows from formula ( ).

In terms of Notation ,

$$\phi(i,r) = \frac{a(i+1,r-1)a(i,r+1)}{a(i,r)a(i+1,r)} \qquad (r:n-1|i:n-r)$$

Set

$$g_i^{(r)} = \frac{a(i,r|(f))}{a(i,r)} \qquad (r:n|i:n-r+1)$$

$g_r^{(1)}$ is as defined in the Theorem; relationship ( ) is a consequence of formula ( ). The successive components of the solution vector to the system of equations $R_g = f$ are expressible in the form $g_1^{(i)}$ $(i:n)$.

The determinants $a(i,k;r)$ may be placed in a three dimensional array. The quotient

$$q_i(i,k;r) = \frac{a(i+1,k;r)}{a(i,k;r)}$$

and similar quotients with respect to k and r are defined at all points of the array for which the constituents of the right hand side are well defined. The double quotient

$$q_{i,k}(i,k;r) = \frac{a(i,k;r)a(i+1,k+1;r)}{a(i+1,k;r)a(i,k+1;r)}$$

and others of its kind are defined in a similar way. In terms of such double quotients, formulae (, ,) may be expressed as

$$\psi(i,k;r) = q_{r,k}(i,k;r), \quad \Theta(i,k;r) = \frac{q_{r,k}(i,k;r)}{q_{i,k}(i,k;r)}$$

$$\phi(i,k;r) = \frac{q_{r,r}(i,k;r-1)}{q_{r,i}(i,k;r-1)}$$

A simple illustration of the above theorem is provided by the example in which

$$a_{i,j} = (\sigma - w)^{-i} \binom{i-1}{j-1} \qquad (i:n|j:i)$$

(if $a[x] = \left[\sum_{i=1}^{n} (\sigma-w)^{-i} x^i\right]$ and $a[x]^j = \left[\sum_{i=j}^{n} a_{i,j} x^i\right]$ $(j:n)$, the $a_{i,j}$ for $(i:n|j:i)$ have the form ( )). In this case

$$\psi(i,k;r) = \frac{k-i-r}{k+r}, \quad \theta(i,k;r) = \frac{k-i}{k+r}, \quad \phi(i,k;r) = -\frac{r}{k+r-1}$$

for index values as in the theorem. Now $\delta_r^{(1)} = (\sigma-w)^i f_i$ $(i:n)$ and recursion ( ) becomes

$$\delta_i^{(r+1)} = \delta_i^{(r)} - \delta_{i+1}^{(r)} \qquad (r:n-1|i:n-r)$$

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Let A, C be $n \times n$ lower triangular matrices with elements $a_{i,j}$, $c_{i,j}$ ~~$(i:n|j:i)$~~ respectively, and Z be the $n \times n$ diagonal matrix with elements z^{k-1} $(k:n)$. Let $P_{i,j}(z)$ be the representative element of $P(z) = CZA$, so that

$$P_{i,j}(z) = \sum_{k=j}^{i} c_{i,k} a_{k,j} z^k \qquad (i:n|j:i)$$

The $P_{i,j}$ satisfy a recursion of the form

$$P_{i+1,j}(z) = \rho_{i,j} P_{i+1,j+1}(z) + \eta_{i,j} P_{i,j}(z) \qquad (i:n-1|j:i-1)$$

if and only if

$$c_{i+1,k}a_{k,j} = \rho_{i,j}c_{i+1,k}a_{k,j+1} + \eta_{i,j}c_{i,k}a_{k,j} \qquad (k:j,i+1)$$

Setting $k=j$ and $k=i+1$, this relationship yields in turn

$$\eta_{i,j} = \frac{c_{i+1,j}}{c_{i,j}} \qquad \rho_{i,j} = \frac{a_{i+1,j}}{a_{i+1,j+1}}$$

When $n=3$ and $C=A^{-1}$, relationships ( ) are satisfied. They are not satisfied for the matrices $K\{a\}, K\{I(a)\}$ when $a_i = (-1)^{i-1}/i!$ for $i=4, j=1$.

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$a_i = (-1)^{i-1}/i!$ $\quad A = K\{a\}$ $\quad u(k)=v(k)=k$

$X(A;1)$

$$\begin{matrix} 1 \\ -\frac{1}{2} & 1 \\ \frac{1}{6} & -1 & 1 \\ -\frac{1}{24} & \frac{7}{12} & -\frac{3}{2} & 1 \\ \frac{1}{120} & -\frac{1}{4} & \frac{5}{4} & -2 & 1 \\ -\frac{1}{720} & \frac{31}{360} & -\frac{3}{4} & \frac{13}{6} & -\frac{5}{2} & 1 \end{matrix}$$

$X(A;2)$

$$\begin{matrix} 1 \\ -\frac{1}{3} & 1 \\ \frac{1}{12} & -\frac{3}{4} & 1 \\ -\frac{1}{60} & \frac{7}{20} & -\frac{6}{5} & 1 \\ \frac{1}{360} & -\frac{1}{8} & \frac{5}{6} & -\frac{5}{3} & 1 \end{matrix}$$

$X(A;3)$

$$\begin{matrix} 1 \\ -\frac{1}{4} & 1 \\ \frac{1}{20} & -\frac{3}{5} & 1 \end{matrix}$$

$$e_k(A;r) = -\frac{k}{k+r-1}$$

$$a_{i,j} = -\frac{j}{i}(a_{i-1,j} - a_{i-1,j-1})$$

$c_i = 1/i \qquad C = K\{c\} \qquad u(k) = v(k) = k$

$X(C;1)$					$X(C;2)$				$X(C;3)$		
1					1				1		
$\frac{1}{2}$	1				$\frac{2}{3}$	1			$\frac{3}{4}$	1	
$\frac{1}{3}$	1	1			$\frac{1}{2}$	$\frac{5}{4}$	1		$\frac{3}{5}$	$\frac{7}{5}$	1
$\frac{1}{4}$	$\frac{11}{12}$	$\frac{3}{2}$	1		$\frac{2}{5}$	$\frac{13}{10}$	$\frac{9}{5}$	1			
$\frac{1}{5}$	$\frac{5}{6}$	$\frac{7}{4}$	2	1							

$$e_k(C;r) = \frac{r-1}{k+r-1}$$

$$c_{i,j} = \frac{(i-1)}{i} c_{i-1,j} + \frac{j}{i} c_{i-1,j-1}$$

$a_i = (-1)^{i-1}/i! \qquad \beta_\nu^{(j)}: \quad \left\{\frac{x}{a(x)}\right\}^j = \sum_{\nu=0} \beta_\nu^{(j)} x^\nu$

$\beta_\nu^{(j)}$

$\nu \backslash j$	1	2	3	4	5
0	1	1	1	1	1
1	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
2	$\frac{1}{12}$	$\frac{5}{12}$	1	$\frac{11}{6}$	$\frac{35}{12}$
3	0	$\frac{1}{12}$	$\frac{3}{8}$	1	$\frac{25}{12}$
4	$-\frac{1}{720}$	$\frac{1}{240}$	$\frac{19}{240}$	$\frac{261}{720}$	1

$$\beta_\nu^{(j+1)} = \left(1 - \frac{\nu}{j}\right)\beta_\nu^{(j)} + \beta_{\nu-1}^{(j)}$$

The determinants formed from the subarrays taken from the matrix inverse to a given nonsingular lower triangular matrix A may be expressed in terms of corresponding determinants connected with A.

Theorem. Let $a_{i,j}$ $(i:n|j:i)$ be the elements of a nonsingular lower triangular matrix, and $c_{i,j}$ $(i:n|j:i)$ those of its inverse. Then

$$c(i,k;j) = \frac{(-1)^{i+k} a(j+k,k;i-k)}{\prod_{\tau=k}^{i+j-1} a_{\tau,\tau}} \qquad (i:n|k:i|j:\min(n-i+1,n-k+1))$$

Proof. The formula

$$c_{i,k} = c(i,k;1) = \frac{(-1)^{i+k} a(k+1,k;i-k)}{\prod_{\tau=k}^{i} a_{\tau,\tau}}$$

is obtained from the system of equations

$$a_{k,k} c_{k,k} = 1 \,, \quad \sum_{\nu=k}^{\mu} a_{\mu,\nu} c_{\nu,k} = 0 \qquad (\mu:k+1;i)$$

Thus the result ( ) is correct when $j=1$. A short inductive proof, based upon the use of formula ( ), suffices to show that the result is true as stated.

Expression of g_i by use of derivative $\frac{1}{\Delta a}\Delta$. Jacobi form

Transformation $\mathcal{C}_{\phi,\theta}: \sum f_i \phi^i \to \sum g_i \theta^i$. Show $\mathcal{C}_{\phi,\theta}\mathcal{C}_{\theta,\phi} = \mathcal{C}_{\phi,\psi}$

$\theta = a(\phi)$, $\psi = a'(\theta)$ $\psi = a'(a(\phi))$

Extension to $\sum_{i=1} g_i \prod_{\tau=1}^{i} a^{(\tau)}(x)$, negative powers, Newton series

Examine band matrix decomposition for $a_{i,j}$ deriving from differential equation

Partial sum transformation $\sum_{\tau=0}^{i} f_\tau x^\tau \to \sum_{\tau=0}^{i} g_\tau a(x)$. Convergence acceleration

Integral transform of transformation, as in motivation of γ-algorithm

Band matrix decomposition of $C\{a\}$ and Newton series extension

Extension of Bernoulli polynomials $\{a(xy)+1\}^{j}\beta(x)^{j} = \sum \beta_\nu^{(j)}(y)x^\nu$

Algebraic form of $\mathcal{G}$-function integral

$e^{(k)}(A;r)$ from $e_k(A;r)$; direct inversion

Open lower triangular matrix decomposition

Apply ~~to~~ decompositions to LU factorisation of square A.

Decomposition of lower triangular matrix as $V_0(L_0)V_1(L_1)\ldots$