Bürmann series over a field

Introduction

This paper is directed towards the transformation of series expansions.

In order to abbreviate exposition in this and later sections, certain conventions are introduced.

Notation. (1) Angular brackets $\langle \ldots \rangle$ are used in a way made obvious by the context to combine two statements into one.

(2a) $(i:j;k)$ indicates that the integer i should take the values $i=j,\ldots,k$. (b) If the lower limit is unity, it and the subsequent semi-colon are omitted; thus $(i:k)$ is an abbreviation for $i=1,\ldots,k$. (c) If the upper limit is unbounded, it is omitted; thus $(i:)$ is an abbreviation for $i=1,2,\ldots$. (d) If the lower and upper limits take unbounded negative and positive values respectively, both are omitted; thus $(i:;)$ is an abbreviation for $i=\ldots,-1,0,1,\ldots$ (e) Duplicated descriptions are indicated by separating the variables concerned by a comma; thus $(i,j:)$ is an abbreviation for $i=1,2,\ldots;$ $j=1,2,\ldots$. (f) Conjoint descriptions are separated by a vertical bar; thus $(i:m|j:i)$ is an abbreviation for $i=1,\ldots,m$

$j=1,\dots,i$. (8) Statements accompanied by null index allocations are to be ignored; thus the allocation $l_{k,k}=0$ $(k:m)$ is not performed when $m=0$.

The general results derived are illustrated by application to a problem concerning the transformation of asymptotic relationships.

Notation. Let $n>0$ be a fixed finite integer.

(1) With w a fixed point in $\mathbb{C}$, the finite part of the complex plane, $\Omega(w)$ an open set of points in $\mathbb{C}\setminus\{w\}$ with limit point at w, $\bar{\Omega}(w)\equiv\{w\}\cup\Omega(w)$, and p, q mappings of $\bar{\Omega}(w)$ into $\mathbb{C}$,

$$p(z)\simeq_{(w,\Omega)} q(z)$$

means that $p(z)-q(z)=o\{(z-w)^n\}$ as z tends through $\Omega(w)$ to w.

(2) Let M be a nonvoid set of points in $\mathbb{C}$. For each $w\in M$, let $\Omega(w)$, $\bar{\Omega}(w)$ be as in (1) and $p(w,\cdot), q(w,\cdot)$: $\bar{\Omega}(w)\to\mathbb{C}$.

$$p(w,z)\simeq_{[M,\Omega]} q(w,z)$$

means that for each $w\in M$, $p(w,z)\simeq_{(w,\Omega)} q(w,z)$. $\bar{\Omega}=\bigcup_{w\in M}\bar{\Omega}(w)$ and $\Omega'=\bigcup_{w\in M}\Omega'(w)$ where $\{\Omega'(w): z-w$ for all $z\in\bar{\Omega}(w)\}$.

The main problem considered has a simple form as follows: let $a_j, f_j \in \mathbb{C}$ $(j:n)$ with $a_1 \neq 0$ and, with $w \in \mathbb{C}$ fixed, let $a, f: \bar{\Omega}(w) \to \mathbb{C}$ be such that

$$a(z) \underset{(w,\Omega)}{\simeq} a(w) + \sum_{j=1}^{n} a_j (z-w)^j$$

$$f(z) \underset{(w,\Omega)}{\simeq} f(w) + \sum_{j=1}^{n} f_j (z-w)^j$$

Determine $g_i \in \mathbb{C}$ for which

$$f(z) \underset{(w,\Omega)}{\simeq} f(w) + \sum_{i=1}^{n} g_i \{a(z)-a(w)\}^i$$

In the general form of the problem $a_j, f_j : M \to \mathbb{C}$ ~~ ~~ $(j:n)$ are mappings with $a_1(w) \neq 0$ for all $w \in M$; for each $w \in M$, $a(w,\cdot): \bar{\Omega}(w) \to \mathbb{C}$ is such that

$$a(w,z) \underset{[M,\Omega]}{\simeq} a(w,w) + \sum_{j=1}^{n} a_j(w)(z-w)^j$$

the mapping $f: \bar{\Omega} \to \mathbb{C}$ satisfies the relationship

$$f(z) \underset{[M,\Omega]}{\simeq} f(w) + \sum_{j=1}^{n} f_j(w)(z-w)^j$$

and $g_i: M \to \mathbb{C}$ for which

$$f(z) \underset{[M,\Omega]}{\simeq} f(w) + \sum_{i=1}^{n} g_i(w)\{a(w,z) - a(w,w)\}^i$$

are to be determined.

As a matter of detail it is remarked that once the $g_i(w)$ have been obtained, relationship ( ) carries with it the additional information that, for $(m:n)$ and each $w \in M$

$$f(z) - f(w) - \sum_{i=1}^{m-1} g_i(w)\{a(w,z) - a(w,w)\}^i \sim g_m(w)\{a(w,z) - a(w,w)\}^m$$

$$\sim g_m(w)\, a_1(w)^m (z-w)^m$$

as $z(\in \Omega(w)) \to w$.

The problem has, in particular, three variants

(1) In the first variant, a satisfies the condition $a(w, w+z) - a(w,w) = a(w', w'+z) - a(w', w')$ for all $w, w' \in M$ and $z \in \Omega'$. $a(w,z) - a(w,w)$ then has the form $a'(z-w)$ where $a' : \Omega' \to \mathbb{C}$ and $a'_j \in \mathbb{C}$ $(j:n)$ with $a'_1 \neq 0$ exist such that

$$a'(z) \simeq_{(0,\Omega')} \sum_{j=1}^{n} a'_j z^j$$

Now the problem is that of determining the $g_i(w)$ in relationship with the factor $\{a(w,z) - a(w,w)\}^i$ replaced by $a'(z-w)^i$. Formula ( ) is then a transformation of relationship ( ) in which the term $(z-w)$ in the latter is replaced by a function $a'(z-w)$ of $z-w$.

(2) In the second variant, the mapping $a(w,\cdot)$ in relationship ( ) is independent of w. The factor $\{a(w,z) - a(w,w)\}^i$ in relationship ( ) is now to be replaced by $\{a(z) - a(w)\}^i$, where $a : \bar{\Omega} \to \mathbb{C}$. Formula ( ) is then a mechanism for

changing the independent variable in relationship ( ) from z to $a(z)$.

This variant has as a special case a well known problem of the theory of functions of a complex variable. Let M consist of the single point w, and let the functions a, f be analytic at w, with $\mathcal{D}a(w) = \frac{da(w)}{dw} \neq 0$. The difference $a(z) - a(w)$ satisfies a relationship of the form ( ) as $z \to w$ for all values of $\arg(z-w)$, where the $a_j(w)$ are Taylor series coefficients and $\Omega(w)$ is a suitable open domain containing w. Mutatis mutandis, the same can be said of relationship ( ) with regard to the function f. Since $\mathcal{D}a(w) \neq 0$, there exists a contour Δ enclosing w, lying within a region over which a, f are analytic and such that for all z within Δ and ζ in and upon Δ the equation $a(\zeta) = a(z)$ has only the single and simple root $\zeta = z$. Accordingly, with z, w within Δ,

$$f(z) = \frac{1}{2\pi i} \int_\Delta \frac{f(\zeta)\mathcal{D}a(\zeta)}{a(\zeta) - a(z)} d\zeta$$

and

$$f(z) = f(w) + \sum_{j=1}^{n} g_j(w)\{a(z) - a(w)\}^j + R_n(w,z)$$

where

$$g_j(w) = \frac{1}{2\pi i}\int_{\Delta} \frac{f(\zeta)\,\partial a(\zeta)}{\{a(\zeta)-a(w)\}^{j+1}\{a(\zeta)-a(w)\}}\,d\zeta$$

and

$$R_n(w,z) = \frac{1}{2\pi i}\int_{\Delta} \frac{f(\zeta)\,\partial a(\zeta)}{\{a(\zeta)-a(w)\}^{n+1}\{a(\zeta)-a(w)\}}\,d\zeta\,\{a(z)-a(w)\}^{n+1}$$

In this case, an asymptotic relationship of the form ( ) holds as $z \to w$ for all values of $\arg(z-w)$: the series of which the first n terms are displayed in formula ( ) converges for all sufficiently small values of $|z-w|$. To obtain a convergence region, it is remarked that for $\zeta \in \Delta$, $\min|a(\zeta)-a(w)| = \delta > 0$. For all z such that $|a(z)-a(w)| < \delta$ the series in question converges.

Setting

$$\beta(w,\zeta) = \frac{\zeta - w}{a(\zeta)-a(w)}$$

and integrating by parts in formula ( ), Bürmann's form

$$g_j(w) = \left[\frac{\partial^{j-1}}{\partial\zeta^{j-1}}\{\beta(w,\zeta)\}^j\,\partial f(\zeta)\right]_{\zeta=w}$$

for the coefficients $g_j(w)$ is obtained.

(3) The third variant is arrived at by a continuation of the above discussion. Let $\mathbb{D}$ be an open domain in $\mathbb{C}$, with boundary $\mathcal{C}$. Let the functions A, f be analytic over $\mathbb{D}$, with A & mapping $\mathbb{D}$ injectively into the unit open disc and $\mathcal{C}$ into the unit circle. For any $w \in \mathbb{D}$, a representation of $f(z)$ in powers of $A(z)-A(w)$ which converges for sufficiently small values of $|z-w|$ may be obtained as above. However it may not be true that, for all $z \in \mathbb{D}$, $|A(z)-A(w)| < |A(\zeta)-A(w)|$ for all $\zeta \in \mathcal{C}$, and the representation of $f(z)$ may not converge for all $z \in \mathbb{D}$. Nevertheless it is possible to obtain such a representation. Set

$$a(w,z) = w + \frac{A(z)-A(w)}{1-A(z)\overline{A(w)}}$$

so that $a(w,w) = w$. Now, for $z \in \mathbb{D}$, $a(w,z) - a(w,w)$ is the image of $A(z)$, a point of the unit open disc, with respect to a transformation which maps this disc into itself: $|a(w,z) - a(w,w)| < 1$. For $\zeta \in \mathcal{C}$, $|a(w,\zeta) - a(w,w)| = 1$. By suitably locating the contour Δ within $\mathbb{D}$ and near $\mathcal{C}$ when $\mathbb{D}$ is bounded, and in part near $\mathcal{C}$ and in part sufficiently remote from w when $\mathbb{D}$ is unbounded, uniform convergence of the representation of f, for a fixed $w \in \mathbb{D}$, over any bounded region contained in $\mathbb{D}$ may be demonstrated. Replacing $a(\zeta)-a(w)$ by $a(w,\zeta)-w$ in the definition ( ), formula ( ) yields an extension of Bürmann's form for the coefficients $\varrho_j(w)$ in this case. The relevant remainder term $R_n(w,z)$ is given by a similarly modified version of formula ( ).

If the coefficients $\varrho_i(w)$ are to be obtained by other methods, access to the coefficients $a_j(w)$ in ~~needed,~~ relationship ( ) is needed. They may be derived recursively from the Taylor series coefficients $A_\nu(w)$ of the function A at the point w by use of the formula

$$a_j(w) = \{1 - A(w)\overline{A(w)}\}^{-1}\{A_j(w) + \overline{A(w)} \sum_{\nu=1}^{j-1} A_\nu(w) a_{j-\nu}(w)\}$$

By suitably defining a process of differentiation, the coefficients $\varrho_j(w)$ in relationship ( ) deriving from the general problem stated above may also be expressed in Bürmann's form. However, this general version is vitiated by the same disadvantage that afflicts its original form: the first few coefficients $g_i(w)$ may be written out with ease, but it is not immediately apparent that the general term may be expressed with the same facility. A transformation of variable in an asymptotic or convergent representation is evidently a useful resource; that it is not often resorted to is a consequence of the difficulty in obtaining the general term in the modified representation. But there exists a very simple algebraic process for obtaining the coefficients in question.

Transformations of polynomials

Although the problem considered was introduced in terms of complex variables, it is in essence a problem concerning the transformation of polynomials with coefficients over a field, and will be treated as such.

Definition. Let $n>0$ be a fixed finite integer and a be a prescribed sequence of members a_i $(i:n)$ of a field $\mathbb{K}$, with $a_1 \neq 0$.

(1) The sequence b of numbers b_i $(i:n)$ determined by use of the formulae $b_1 = 1/a_1$,

$$b_i = -\left\{\sum_{k=1}^{i-1} b_k a_{i-k+1}\right\} b_1 \qquad (i:2;n)$$

is the sequence reciprocal to a; it is denoted by $R(a)$.

(2) Obtain numbers $a_{i,j}$ $(i:n|j:i)$ by use of the formulae $a_{i,1} = a_i$ $(i:n)$,

$$a_{i,j} = \sum_{k=1}^{i-j+1} a_k a_{i-k,j-1} \qquad (j:2;n\,|\,i:j;n)$$

The sequence c of numbers c_i $(i:n)$ determined by use of the formulae $c_1 = 1/a_1$,

$$c_i = -\left\{\sum_{k=1}^{i-1} c_k a_{i,k}\right\} / a_{i,i} \qquad (i:2;n)$$

is the sequence inverse to a; it is denoted by $I(a)$.

For use in the proof of the following theorem and later, certain notations concerning lower triangular matrices (i.e. $n \times n$ matrices with elements $a_{i,j}$ for which $a_{i,j} = 0$ $(i:n-1|j:i+1;n)$ by definition are introduced).

Notation . (1) With a_i $\langle a_{i,j} \rangle$ as in Definition , $C\{a\}$ $\langle K\{a\} \rangle$ is the lower triangular matrix having a_{i-j+1} $\langle a_{i,j} \rangle$ as the j^{th} element of the i^{th} row $(i:n|j:i)$.

(2) L being a prescribed lower triangular matrix of order $\geq n-m$ with elements $l_{i,j}$, $V_m[L]$ with $0 \leq m \leq n$ is that lower triangular matrix having elements $l'_{k,k} = 1$ $(k:m)$, $l'_{i,j} = 0$ $(i:2;n \mid j:\min(i-1,m))$ and $l'_{i,j} = l_{i-m,j-m}$ $(i:m+1;n \mid j:m+1;i)$

(3) E_n is the $n \times n$ unit matrix.

(4) In matrix products denoted by the product sign $\prod$, factors occur in the order of the indices; thus $\prod_{\nu=1}^{2} A_\nu = A_1 A_2$. The empty product takes the value E_n.

(5) The n^{th} order vector forming the j^{th} column of the matrix A is denoted by $A^{(j)}$ $(j:n)$

(6) Sequences of members of $\mathbb{K}$ occur in matrix equations as successive members of column vectors; thus in the equation $Ag = f$, g and f are column vectors with elements g_i, f_i $(i:n)$ respectively.

It is remarked that $V_m[L]$ is obtained from L by pushing the latter down its principal diagonal a distance of m elements, adding unit principal diagonal elements in the m such positions vacated and zero elements in the further positions of the first m columns, and possibly truncating the displaced version of L to form, in total, an n^{th} order lower triangular matrix. The V_0-form of L is L itself. In the V_n-form of L, L has been pushed out of sight: $V_n[L] = E_n$.

An algebra of truncated polynomials is now defined.

Definition. Let $\mathbb{K}$ be a field and $n > 0$ be a fixed finite integer. (1) With $p_i \in \mathbb{K}$ $(i:n)$, $p[x]$ is the truncated polynomial $[\sum_{i=1}^{n} p_i x^i]$, x being a free variable. (2) The zero polynomial $\langle \text{for } (i:n)\, x^i \rangle$ is that $p[x]$ for which $p_k = 0$ $(k:n)$ $\langle p_k = 0$ $(k:n\ (k \neq i))$; $p_i = 1 \rangle$. (3) With $p'[x]$ defined as for $p[x]$, $p[x] = p'[x]$ means that $p_i = p'_i$ $(i:n)$. (4) Addition and subtraction between two polynomials are defined in terms of these operations in $\mathbb{K}$ upon coefficients of corresponding powers of x. (5) The coefficients q_i in the product of $p[x]$ and $p'[x]$ are $q_i = \sum_{k=1}^{i-1} p_k p'_{i-k}$ $(i:n)$. In particular, positive powers $p[x]^j$ $(j:)$ of $p[x]$ are defined in this way. Multiplication of $p'[x]$ by $a \in \mathbb{K}$ is defined by setting $p[x] = a\,p'[x]$ where $p_i = a\,p'_i$ $(i:n)$. (6) Composition is defined by using the above rules in the determination of the expression $p[q[x]] =$

$[\sum_{j=1}^{n} p_j q[x]^j]$. (7) The algebra of truncated polynomials just described is denoted by $\mathbb{K}_n^{(t)}[x]$.

The polynomials of $\mathbb{K}_n^{(t)}[x]$ are truncated during multiplication. With the operations of addition, subtraction and multiplication as defined above, $\mathbb{K}_n^{(t)}[x]$ is a commutative ring which is nilpotent in the sense that $p[x]^{n+1}$ is zero for all $p[x] \in \mathbb{K}_n^{(t)}[x]$; indeed, all products of more than n factors are zero.

Theorem. Let $a[x] = [\sum_{i=1}^{n} a_i x^i]$ with $a_1 \neq 0$ and $f[x] = [\sum_{i=1}^{n} f_i x^i]$. Then

$$f[x] = [\sum_{j=1}^{n} g_j a[x]^j]$$

where $f = K\{a\} g$ and $g = K\{I(a)\} f$. This decomposition of $f[x]$ is unique with respect to g in the sense that, with $a[x]$ as described, the coefficients g_j occurring in the above decomposition are uniquely determined by the f_i.

Proof. For $(j:n)$ the coefficient of x^i in the polynomial $a[x]^j$ is equal to the element $a_{i,j}$ of the matrix $K\{a\}$ for $(i:n)$; in particular, this coefficient is zero for $(i:j-1)$. The coefficient of x^i in the polynomial on the right hand side of relationship ( ) is $\sum_{j=1}^{i} a_{i,j} \underset{0}{g}_j$, that on the left being f_i $(i:n)$. Hence $f = K\{a\}\underset{0}{g}$ as stated. The diagonal elements of the lower triangular matrix $K\{\underset{0}{a}\}$ are in order $(a_1)^k$ $(k:n)$ and $K\{a\}$ is therefore nonsingular: $\underset{0}{g}$ is uniquely determined by f. It remains to be shown that the inverse of $K\{a\}$ is $K\{I(a)\}$.

$K\{a\}$ may be expressed as a product of displaced versions of the matrix $C\{a\}$. Firstly, from formula ( ), the relationship

$$V_{r-1}[C\{a\}]V_r[K\{a\}] = V_{r-1}[K\{a\}]$$

holds for $r=1$, and hence for $(r:n)$, so that

$$K\{a\} = \left\{\prod_{k=0}^{r-1} V_k[C\{a\}]\right\} V_r[K\{a\}] \qquad (r:n-1)$$

$$= \prod_{k=0}^{n-1} V_k[C\{a\}]$$

Since $a_1 \neq 0$, $I(a)$ exists. As may be verified directly for $j=1$, and by use of formula ( ) for subsequent values of j,

$$[K(a)C\{I(a)\}]^{(j)} = \left\{\prod_{k=0}^{j-2} V_k[C\{a\}]\right\} E_n^{(j)} = \left\{\prod_{k=1}^{j-1} V_k[C\{a\}]\right\}^{(j)}$$

$$= \left\{\prod_{k=1}^{n} V_k[C\{a\}]\right\}^{(j)} = \left\{V_1\left[\prod_{k=0}^{n-1} V_k[C\{a\}]\right]\right\}^{(j)} \qquad (j:n)$$

Therefore

$$V_{r-1}[K\{a\}]V_{r-1}[C\{I(a)\}] = V_r[K\{a\}]$$

when $r=1$ and hence for $(r:n)$. Thus

$$K\{a\}\prod_{k=0}^{r-1} V_k[C\{I(a)\}] = V_r[K\{a\}] \qquad (r:n-1)$$

and

$$K\{a\}\prod_{k=0}^{n-1} V_k[C\{a\}] = E_n$$

Use of formula ( ) with a replaced by $I(a)$ reveals that

$$K\{a\}K\{I(a)\} = E_n$$

and, since $f = K\{a\}g$, $g = K\{I(a)\}f$.

It should be remarked that although the decomposition ( ) is unique with respect to the coefficients g_j, it is not so with respect to the polynomial $a[x]$. For example, in $\mathbb{K}_2^{(t)}[x]$, $[x]^2 \equiv a[x]^2$ where $a[x] = [x + a_2x^2]$ for any $a_2 \in \mathbb{K}$. With the g_j obtained for one decomposition, many polynomials $a[x]$ may feature in the decomposition ( ).

As a corollary to the result ( ) derived in the above proof, it is remarked that the members of $I(\underline{I}(a))$

are the successive elements in the first column of the matrix inverse to $K\{I(a)\}$. They form, from formula ( ), $K\{a\}^{(1)}$. Hence $I(I(a))=a$. Corresponding results concerning the sequence $b=R(a)$ are easier to derive. The diagonal elements of $C\{a\}$ are all a_1, those of $C\{b\}$ being $b_1 = 1/a_1$. Replacing i, k by $i-j+1$, $k-j+1$ formula ( ) may be rewritten as

$$\sum_{k=j}^{i} a_{i-k+1} b_{k-j+1} = 0 \qquad (i:2;n \mid j:i-1)$$

Hence $C\{a\}C\{R(a)\}=E_n$. The members a_j $(j:2;n)$ of a may be recovered from those b_j $(j:n)$ of b by rearrangement of relationship ( ); the rearranged form is precisely that which defines the members of $R(b)$. Hence $R(R(a))=a$.

Theorem has as a special case that in which $f_1=1$, $f_j=0$ $(j:2;n)$. The coefficients in the polynomial $c[x]$ inverse to $a[x]$ (with $a_1 \neq 0$) for which $c[a[x]] \equiv [x]$ are then given by $c = K\{I(a)\}E_n^{(1)} = K\{I(a)\}^{(1)} = I(a)$. Since $a = I(c)$, it then follows that $a[c[x]] \equiv [x]$. More generally, the coefficients in the polynomial $c^{(j)}[x]$

for which $c^{(i)}[a[x]] \equiv [x^i]$ are the elements $c_{i,j}$ $(i:j;n)$ of the matrix $K\{I(a)\}$ $(j:n)$.

Assuming the members of a to be available, the set of equations $K\{a\}g = f$ is constructed and solved recursively. Initially $g_1 = f_1/a_1$. Thereafter, for $(i:2;n)$ with $a_{k,j}$ $(k:i-1|j:k)$ and g_j $(j:i-1)$ already obtained, set $a_{i,1} = a_i$ and use relationship ( ) with $(j:2;i)$ to determine the coefficients $a_{i,j}$, after which

$$g_i = \{f_i - \sum_{j=1}^{i-1} a_{i,j} g_j\}/a_{i,i}$$

If the members of $c = \underline{I}(a)$ are available, it is slightly easier to use the formula $g = K\{c\}f$ directly. For $(i:n)$ set $c_{i,1} = c_i$, evaluate $c_{i,j}$ $(j:2;i)$ by use of a modified version of formula ( ), and thereafter set

$$g_i = \sum_{j=1}^{i} c_{i,j} f_j.$$

With $w \in \mathbb{C}$ fixed, functions p of a complex variable which satisfy relationships of the form

$$p(z) \underset{(w,\Omega)}{\simeq} \sum_{j=1}^{n} p_j (z-w)^j$$

form an algebraic system which is epimorphic to

$\mathbb{C}_n^{(t)}[x]$. The correspondence ϕ for which $p[x] = \phi\{p(z)\}$ is established by comparing the coefficients p_j in relationship ( ) with those in the representation of $p[x]$. The kernel of the epimorphism is that set of functions $p(z)$ for which $p(z) = o\{(z-w)^n\}$ as $z(\in \Omega) \to w$. Accordingly, the problem of determining the coefficients g_j in relationship ( ) from those a_i, f_i occurring in relationships (,) for a fixed $w \in \mathbb{C}$ is solved by use of formulae (,) as described above. With the function a satisfying relationship ( ) specified, the determination of the coefficients g_j is unique. Naturally, if ~~$a'(z) \simeq a$~~ $a'(z) \simeq_{(w,\Omega)} a(z)$, a in relationship ( ) may be replaced by a', the g_j remaining unaltered. As the remark following the proof of Theorem indicates, however, it may occur that for fixed f and g_j, the asymptotic relationship ( ) is satisfied by a variety of functions a that are not asymptotically equivalent in the above sense.

Definition clearly permits an extension in which the coefficients p_i in $p[x]$ are mappings of $\mathbb{M} \subseteq \mathbb{K}$ into $\mathbb{K}$; the polynomials, $p[\mathbb{M}, x]$ in an extended notation, then become mappings of $\mathbb{K}_n^{(t)}[x]$ into itself. This extension permits a solution of the problem described in the introduction: relationships (,) must be reformulated with $a_{i,j}, c_{i,j}, f_i, g_j$ replaced by function values $a_{i,j}(w), \ldots, g_j(w)$ and used for each $w \in \mathbb{M}$.

The algebra $\mathbb{K}_n^{(t)}[x]$ lacks polynomials containing terms independent of x. Theorem concerns the replacement of the variable x in such polynomials by $a[x]$, where $a[x]$ is a polynomial of the same form. The described applications of the theorem concern functions $f(z)-f(w)$ asymptotically equivalent to polynomials lacking a term independent of $z-w$, and functions $a(w,z)-a(w,w)$ of the same kind. The descriptions can be reformulated in terms, in particular, of functions $f(z)$ asymptotically equivalent to polynomials possessing terms independent of $z-w$. Such descriptions derive from a more general theory concerning truncated polynomials of an algebra $\mathbb{K}_n[x]$ containing terms independent of x.

Definition . Let $\mathbb{K}$ be a field and $n>0$ be a fixed finite integer. (1) With $p_i \in \mathbb{K}$ $(i: 0; n)$, $p[x]$ is the truncated polynomial $[\sum_{i=0}^{n} p_i x^i]$, x being a free variable. (2) The zero polynomial, powers of x (now with $x^0 = 1 \in \mathbb{K}$), ~~addition and subtraction,~~ equivalence, addition and subtraction, and multiplication by members of $\mathbb{K}$ are all defined, mutatis mutandis, as in Definition . (3) $b \in \mathbb{K}$ is incorporated into the system of polynomials by letting it be represented by $p[x]$ with $p_0 = b$, $p_i = 0$ $(i: n)$. (4) The coefficients q_i in the product of $p[x]$ and similarly defined $p'[x]$ are now $q_i = \sum_{k=0}^{i} p_k p'_{i-k}$ for $(i: 0; n)$; with this modification, and $q[x]^0 \equiv 1$, composition is also as in Definition .

(5) The algebra of truncated polynomials just described is denoted by $\mathbb{K}_n[x]$. (6) For $(k:0;n)$, $\mathcal{D}_0^k$ is a mapping $\mathbb{K}_n[x] \to \mathbb{K}$ defined by setting $\mathcal{D}_0^k p[x] = k!\, p_k$ $(k:0;n)$.

$\mathbb{K}_n[x]$ contains a subalgebra $\mathbb{K}_0[x]$ isomorphic to the $\mathbb{K}$. $\mathbb{K}_n^{(t)}[x]$ is isomorphic to $\mathbb{K}_n[x]/\mathbb{K}_0[x]$, the torsive part of $\mathbb{K}_n[x]$. Letting n become unbounded, and deleting the suffix in this case, $\mathbb{K}_n[x]$ becomes $\mathbb{K}[x]$, the algebraic extension of $\mathbb{K}$. To avoid possible confusion, it is remarked that a mapping $\mathcal{D}: \mathbb{K}_n[x] \to \mathbb{K}_n[x]$ may be defined by setting $\mathcal{D}p[x] = p'[x]$, where $p_i' = (i+1)p_{i+1}$ $(i:0;n-1)$, $p_n' = 0$. However, this mapping is not a derivation: for example, with $p[x] = [x]$ in $\mathbb{K}_1[x]$, $p[x]^2 \equiv 0$ and $\mathcal{D}p[x]^2 \equiv 0$ also; but $2p[x]\mathcal{D}p[x] \equiv [2x]$.

For certain sequences f_j $(j:n)$ and a_j $(j:n)$ considered in Theorem , the sum of n terms in relationship ( ) terminates earlier in the sense that it may be replaced by one of $m < n$ terms. A corresponding result holds with regard to the transformation of polynomial in $\mathbb{K}_n[x]$. The following theorem indicates how polynomials in $\mathbb{K}_n[x]$ are transformed, and also gives the special result mentioned.

Theorem . Let $a[x] = [\sum_{i=0}^{n} a_i x^i]$ with $a_1 \neq 0$, $f[x] = [\sum_{i=0}^{n} f_i x^i]$, m be a fixed integer in the range $1 \leq m \leq n$, and $g_j' \in \mathbb{K}$ $(j:0;m)$ be prescribed.

$$f_i = \sum_{j=0}^{m} g_j' \frac{\Delta_0^i}{i!} a[x]^j \qquad (i:0;n)$$

if and only if

$$f[x] \equiv [\sum_{j=0}^{m} g_j' a[x]^j]$$

If the algorithm of Theorem is applied to the numbers f_j, a_j $(j:n)$ to obtain the numbers g_i $(i:n)$ occurring in the decomposition

$$[f[x] - f_0] \equiv [\sum_{i=1}^{n} g_i [a[x] - a_0]^i]$$

in $\mathbb{K}_n^{(t)}[x]$, then $g_i = 0$ $(i:m+1;n)$ and

$$g_i' = \sum_{j=i}^{m} \binom{j}{i} (-a_0)^{j-i} g_j \qquad (i:m)$$

The coefficient g_0' in the decomposition ( ) is given by

$$g_0' = f_0 + \sum_{j=1}^{m} (-a_0)^j g_j$$

The decomposition ( ) is unique with respect to the sequence g_j' in the sense that if also

$$f[x] = [\sum_{j=0}^{n} g_j'' a[x]^j]$$

then $g_j'' = g_j'$ $(j:0;m)$, $g_j'' = 0$ $(j:m+1;n)$.

Proof. That the coefficients f_i occurring in the polynomial expressible in the form ( ) are as given by formula ( ) is immediately demonstrated by effecting the operation Δ_0^i upon both sides of relationship ( ). That the converse result is true is demonstrated by transforming the polynomial in $\mathbb{K}_n[x]$ with coefficients ( ).

The mapping $\Delta_0 : \mathbb{K}_n[x] \to \mathbb{K}$ is a derivation. Letting

$q[x] = p[x]\, p'[x]$

$$\Delta_0^i q[x] = i!\, q_i = i! \sum_{k=0}^{i} p_k p'_{i-k} = \sum_{k=0}^{i} \binom{i}{k} \Delta_0^k p[x] \Delta_0^{i-k} p'[x] \quad (i:0;n)$$

The value of the sum

$$a_{i,j} = \sum_{\tau=0}^{j-1} \binom{i}{\tau} (-a_0)^{\tau} \frac{\Delta_0^i}{i!} a[x]^{j-\tau}$$

is evidently zero for $(j:2;n|i:j-1)$. For then

$$a_{i,j} = \frac{\Delta_0^i}{i!}\{a[x]-a_0\}^j - (-a_0)^j \frac{\Delta_0^i}{i!} a[x]^0$$

$$= \frac{\Delta_0^i}{i!}\{a[x]-a_0\}^j$$

and the first j coefficients in the polynomial $[a[x]-a_0]^j$ have zero values. It is also clear that formula ( ) represents the elements $a_{i,j}$ of the composition matrix $K\{a\}$ when $j=1$ for $(i:n)$. It is now supposed that it does so for some fixed j in the range $1 \leq j \leq n$. With j replaced by $j+1$ in formula ( ) and, using the result just derived concerning zero values of the sum ( )

$$a_{i,j+1} = \sum_{k=1}^{i-j} a_k a_{i-k,j} = \sum_{k=1}^{i-j} \frac{\Delta_0^k}{k!} a[x] \sum_{\tau=0}^{j-1} \binom{j}{\tau} (-a_0)^{\tau} \frac{\Delta_0^{i-k}}{(i-k)!} a[x]^{j-\tau}$$

$$= \sum_{\tau=0}^{j-1} \binom{j}{\tau} (-a_0)^{\tau} \Big[\sum_{k=0}^{i} \frac{\Delta_0^k}{k!} a[x] \frac{\Delta_0^{i-k}}{(i-k)!} a[x]^{j-\tau} -$$

$$- a_0 \frac{\Delta_0^i}{i!} a[x]^{j-\tau} - a_0^{j-\tau} \frac{\Delta_0^i}{i!} a[x] \Big]$$

$$= \sum_{\tau=0}^{j} \binom{j}{\tau} (-a_0)^{\tau} \frac{\Delta_0^i}{i!} a[x]^{j-\tau+1} + \sum_{\tau=1}^{j} \binom{j}{\tau-1} (-a_0)^{\tau} \frac{\Delta_0^i}{i!} a[x]^{j-\tau+1}$$

$$= \sum_{\tau=0}^{j} \binom{j+1}{\tau} (-a_0)^{\tau} \frac{\Delta_0^i}{i!} a[x]^{j-\tau+1}$$

for $(i:j+1;n)$. Hence formula ( ) holds generally for the elements of $K\{a\}$. (It should perhaps be remarked that, despite appearances, the value of the sum ( ) is independent of a_0.)

From formula ( )

$$\frac{\Delta^i}{i!} a[x]^j = \sum_{k=1}^{j} \binom{j}{k} a_0^{j-k} a_{i,k} \qquad (i:n)$$

Thus formula ( ) may be expressed as

$$f_j = \sum_{k=1}^{m} a_{j,k} \sum_{\tau=k}^{m} \binom{\tau}{k} g_\tau' a_0^{\tau-k} \qquad (j:n)$$

From formula ( ), the numbers g_i occurring in relationship ( ) are given by

dup:

$$g_i = \sum_{j=1}^{n} c_{i,j} f_j \qquad (i:n)$$

where $c_{i,j}$ is the representative element of $K\{I(a)\}$, so that $\sum_{j=1}^{n} c_{i,j} a_{j,k} = \delta_{i,j}$. Accordingly, when the f_j are given by formula ( )

$$g_i = \sum_{\tau=i}^{m} \binom{\tau}{i} a_0^{\tau-i} g_\tau' \qquad (i:n)$$

so that, in particular, $g_i = 0$ $(i:m+1;n)$. By substitution in relationship ( ) it then follows that

$$[f[x]-f_0] \equiv \left[\sum_{j=1}^{m} g_j' \sum_{i=0}^{j} \binom{j}{i} a_0^{j-i} [a[x]-a_0]^i\right]$$
$$\equiv \left[\sum_{j=0}^{m} g_j' a[x]^j - \sum_{j=0}^{m} g_j' a_0^j\right]$$

Setting

$$g_0' = f_0 - \sum_{j=1}^{m} g_j' a_0^j$$

relationship ( ) then yields the decomposition ( ). Formulae (,) then follow directly from their counterparts (,). It has been shown that polynomials $f[x]$ with coefficients given by formula ( ) have the decomposition ( ) and also, in passing, how to obtain the decomposition of any polynomial in $K_n[x]$ by use of the algorithm of Theorem .

The theorem may be stated with $m=n$ and $g_j'=0$ $(j:m+1;n)$. With this formulation, the decomposition ( ) is unique with respect to the sequence g_i $(i:n)$. Equations (,) uniquely determine the g_i' in terms of the g_i. The coefficients g_0' of formula ( ) is the only one permitting the decomposition ( ). In conclusion, the decomposition ( ) is unique in the sense stated in the theorem.

With $w \in \mathbb{C}$ fixed, functions p of a complex variable which satisfy relationships of the form

$$p(z) \simeq_{(w,\Omega)} \sum_{j=0}^{n} p_j (z-w)^j$$

form an algebraic system which is epimorphic to $\mathbb{C}_n[x]$. Furthermore, the derivatives of p at the single point w may be defined by setting $\mathcal{D}^i p(w) = i!\, p_i$ $(i:0;n)$; such derivatives satisfy a relationship of the form ( ). If $p(z) \sim p[x]$, $\mathcal{D}^i p(w) = \mathcal{D}^i_0\, p[x]$ $(i:0;n)$. Theorem asserts in particular that, with a, f two functions of the type just described and $\mathcal{D}a(w) \neq 0$,

$$\mathcal{D}^i f(w) = \sum_{j=0}^{m} g_j' \,\mathcal{D}^i \{a(w)\}^j \qquad (i:0;n)$$

if and only if

$$f(z) \simeq_{(w,\Omega)} \sum_{j=0}^{m} g_j'\, a(z)^j$$

Application of the algorithm of Theorem to the determination of the coefficients g_j in the relationship

$$f(z) \simeq_{(w,\Omega)} f(w) + \sum_{j=1}^{n} g_j \{a(z) - a(w)\}^j$$

reveals that $\overset{o}{\delta_j} = 0$ $(j: m+1; n)$; use of formula ( ) with a_0 replaced by $a(w)$ ~~reveals that~~ $\overset{o'}{\delta_j}$ yields the $\overset{o}{\delta_j}$ in relationship ( ). That $f(z) = \sum_{j=0}^{m} \overset{o'}{\delta_j}\, a(z)^j$ for all $z \in \Omega$ suffices to ensure that relationship ( ) is satisfied; if a and f are analytic at the point w and relationship ( ) holds for all values of $\arg(z-w)$, this condition is implied by relationship ( ).

Naturally the above theory may be extended to the case in which w is allowed to vary over a set $M \subseteq \mathbb{C}$, and $p(z)$, p_j in formula ( ), the set Ω, and $a(z)$ in relationship ( ) depend upon w and assume the form $p(w,z)$, $p_j(w)$, $\Omega(w)$ and $a(w,z)$. The above theory then has, in particular, three variants in analogy with those treated in the introduction. In the first of these the terms $(z-w)^j$ occurring in relationship ( ) are replaced by powers of a function $a'(z-w)$ as described in (1) of the introduction. If then

$$f(z) \simeq_{[M,\Omega]} \sum_{j=0}^{m} \overset{o'}{\delta_j}(w)\, a'(z-w)^j$$

systematic application of the algorithm of Theorem recovers the function values $\overset{o'}{\delta_j}(w)$ in relationship ( )

from the coefficients $f_j(w)$ in the expansion of $f(z)$ in powers of $z-w$.

Function pairs f, a' which permit relationships of the form ( ) exist. The pair $a'(x)=dx$ with $d\in\mathbb{C}\setminus\{0\}$ and f such that

$$f(z)\simeq_{[M,\Omega]}\sum_{i=0}^{m}\mu_i z^i$$

serve as an example. Now

$$f(z)\simeq_{[M,\Omega]}\sum_{j=0}^{m}g_j'(w)a'(z-w)^j$$

where

$$g_j'(w)=(dw)^{-j}\sum_{i=j}^{m}\binom{i}{j}\mu_i w^i \qquad (j:0;m)$$

Again, it is possible to take $a'(x)=e^x-1$ and f such that

$$f(z)\simeq_{[M,\Omega]}\sum_{i=0}^{m}\mu_i e^{iz}$$

when relationship ( ) is satisfied with

$$g_j'(w)=\sum_{i=j}^{m}\binom{i}{j}\mu_i e^{iw} \qquad (j:0;m)$$

With regard to the second variant considered in the introduction, in which the variable z is replaced by a function $a(z)$ of z as described, the above theory has far more general application. Taking f such that

$$f(z) \simeq_{[M,\Omega]} \sum_{i=0}^{m} \mu_i a(z)^i$$

in the above sense, f satisfies the relationship

$$f(z) \simeq_{[M,\Omega]} \sum_{j=0}^{m} \delta_j^0(w)\{a(z)-a(w)\}^j$$

where

$$\delta_j^0(w) = \sum_{i=j}^{m} \binom{i}{j} \mu_i a(w)^{j-i} \qquad (j:0;m)$$

In particular, if the $a_j(w), f_j(w)$ to which the algorithm of Theorem is systematically applied are respectively the coefficients of the Taylor series expansion ~~of a(z)~~ at the point $z=w$ of $a(z)$ and of an m^{th} degree polynomial ~~with~~ in $a(z)$ with constant coefficients, the algorithm simply constructs the polynomial in $a(z)$ in question. Bürmann's series development of $f(z)$ in powers of $a(z)-a(w)$ terminates at its m^{th} term, a result which might in this special case be derived by complex variable methods. Such termination occurs at all points in $\mathbb{C}$ at which a is analytic.

In the third variant considered in the introduction, the function $a(w,z)$ is a linear fractional function, with coefficients depending upon w, of a fixed function $A(z)$ of the single variable z, it may occur that $f(z)$ is asymptotically equal to an m^{th} degree polynomial in $a(w,z)$, with $m<n$, for one or more isolated values of ~~z~~ w, but this is not so for values of z ranging over a domain in $\mathbb{C}$; the result concerning termination in the case of the second variant has no counterpart in that of the third.

A fact 7in?
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For use in the derivation of Bürmann's form for the coefficients g_i of Theorem , it is convenient to introduce an algebra of truncated power series.

Definition . Let $\mathbb{K}$ be a field and $n>0$ be a fixed finite integer. (1) With j a finite integer and $p_i \in \mathbb{K}$ $(i: j; j+n-1)$ one, at least, of the p_i being nonzero, the expression

$$\left[\sum_{i=j}^{j+n-1} p_i x^i\right]$$

represents a nonzero truncated formal power series. The expression $\left[\sum_{i=k}^{k+n-1} p_i' x^i\right]$ is equivalent to ( ) if and only if, when $j \geq k$, $p_i = 0$ $(i: k; j-1)$, $p_i = p_i'$ $(i: j: k+n-1)$, $p_i' = 0$ $(i: k+n; j+n-1)$ and similarly when $j<k$. The truncated formal power series $p[\![x]\!]$ is the class of expressions equivalent to ( ). (2) The statement

$$p[\![x]\!] = \left[\sum_{i=j}^{j+n-1} p_i x^i\right]$$

means that expression ( ) belongs to $p[\![x]\!]$. (3) A zero truncated formal power series, with properties about to be ~~defined~~ described, is also defined. (4) The nonzero member b of $\mathbb{K}$ is incorporated into the system of series by letting it be represented by expression ( ) with $j=0$, $p_0=b$, $p_i=0$ $(i:n-1)$; the zero element of $\mathbb{K}$ is incorporated as the zero series. (5) For $(j:3)$ x^j is that $p[\![x]\!]$ one of whose representations has the form ( ) with $p_j=1$, $p_i=0$ $(i:j+1;j+n-1)$. (6) $p[\![x]\!]=q[\![x]\!]$ means either that both terms in this relationship are the zero series, or that both are nonzero and, with $p[\![x]\!]$ represented as above, one of the representations of $q[\![x]\!]$ also has the form ( ). (7) The sum of two two series is, when one of them is zero, defined to be the other; otherwise, letting $p[\![x]\!]$ and $q[\![x]\!]$ represented by $\left[\sum_{i=k}^{k+n-1} q_i x^i\right]$ be the series in question, the sum $p'[\![x]\!]$ is defined as follows: if $k \geq j+n$ $\langle j \geq k+n \rangle$ $p'[\![x]\!]$ is $p[\![x]\!]\langle q[\![x]\!]\rangle$; otherwise, supposing without loss of generality that $k \geq j$, set $p'_i = p_i$ $(i:j;k-1)$, $p'_i =$

$p_i + q_i$ $(i:k;j+n-1)$; if all p'_i thus allocated are $0 \in \mathbb{K}$, $p'[\![x]\!]$ is the zero series; otherwise $p'[\![x]\!]$ is that series represented by $\left[\sum_{i=j}^{j+n-1} p'_i x^i\right]$. Subtraction is defined analogously. (8) The product of two series is, when one of them is zero, defined to be zero. Otherwise the product of $p[\![x]\!]$ and $q[\![x]\!]$ represented as above, is represented by the expression $\left[\sum_{i=j+k}^{j+k+n-1} p'_i x^i\right]$, where $p'_{j+k+\nu} = \sum_{\tau=0}^{\nu} p_{j+\tau} q_{k+\nu-\tau}$ $(\nu:0;n-1)$. Multiplication of series by members of $\mathbb{K}$ is accomplished by incorporating the latter into the system of series, as above. (9) The algebra of truncated formal power series just described is denoted by $\mathbb{K}_n[x]$. (10) The mapping $\mathcal{J}: \mathbb{K}_n[x] \to \mathbb{K}_n[x]$ is defined by taking $\mathcal{J} p[\![x]\!]$ to be the zero series when $p[\![x]\!]$ corresponds to a member of $\mathbb{K}$, and is otherwise represented by the expression $\left[\sum_{i=j-1}^{j+n-2} (i+1) p_{i+1} x^i\right]$.

Although addition over $\mathbb{K}_n[\![x]\!]$ is associative and commutative, the truncated formal power series $p[\![x]\!]$ do not, when n is bounded, form an additive

Abelian group: there are many ~~$q[\![x]\!]$ for which~~ nonzero $q[\![x]\!]$ for which $p[\![x]\!]+q[\![x]\!]=p[\![x]\!]$. However, $\mathbb{K}_{n-1}[x]$ is isomorphic to $\mathbb{K}_n[\![x]\!]$: within $\mathbb{K}_n[\![x]\!]$ the series represented by expressions of the form ( ) with $j=0$, ~~form an algebraic~~ together with the zero series, form an algebraic system isomorphic to $\mathbb{K}_{n-1}[x]$. Letting n become unbounded and omitting the suffix in this case, $\mathbb{K}_n[\![x]\!]$ becomes $\mathbb{K}[\![x]\!]$, the algebra of formal power series over $\mathbb{K}$.

Theorem . In $\mathbb{K}_n[\![x]\!]$, let $a[\![x]\!]=[\sum_{i=1}^{n} a_i x^i]$ with

and? $a_1 \neq 0$; ~~let~~ $\beta[\![x]\!]$ be the truncated formal power series satisfying the relationship

$$\beta[\![x]\!]\,a[\![x]\!]=[x]$$

In $\mathbb{K}_{n-1}[x]$, let $\beta[x]$ correspond to $\beta[\![x]\!]$, and $\mathcal{D}f[x]=[\sum_{i=0}^{n-1}(i+1)f_{i+1}x^i]$. The coefficients g_i of Theorem are given by

$$g_i=(i!)^{-1}\mathcal{D}_0^{i-1}[\beta[x]^i\mathcal{D}f[x]] \qquad (i\,n).$$

Proof. The series $\beta[\![x]\!]$ is nonzero. With $b=R(a)$, $\beta[\![x]\!]$

$=[\sum_{i=0}^{n-1}\beta_i x^i]$ where $\beta_i = b_{i+1}$ $(i:0;n-1)$. Let $\beta[\![x]\!]^i = [\sum_{\nu=0}^{n-1}\beta_\nu^{(i)}x^\nu]$ $(i:)$. In $\mathbb{K}_n[\![x]\!]$, nonzero series ($\beta[\![x]\!]$ in particular) have reciprocals: $\beta[\![x]\!]^{-1}$ exists such that $\beta[\![x]\!]\beta[\![x]\!]^{-1} = [1]$, and $a[\![x]\!] = [x]\beta[\![x]\!]^{-1}$.

Over $\mathbb{K}_n[\![x]\!]$, the mapping ∂ functions as a derivation:

$$\partial[p[\![x]\!]q[\![x]\!]] = [p[\![x]\!]\partial q[\![x]\!] + q[\![x]\!]\partial p[\![x]\!]]$$

for all $p, q \in \mathbb{K}_n[\![x]\!]$.

With $a_{i,j}$ the representative element of $\mathbb{K}\{a\}$, consider the sum

$$z_{i,j} = \sum_{k=j}^{i}\frac{k}{j}\beta_{i-k}^{(i)}a_{k,j}$$

for $(i:n|j:i)$ When $j<i$, $z_{i,j}$ is the coefficient of x^{i-1} in a representation of

$$i^{-1}\beta[\![x]\!]^i\partial[a[\![x]\!]^j] = i^{-1}\beta[\![x]\!]^i\partial[x^j\beta[\![x]\!]^{-j}]$$

$$= i^{-1}j[\beta[\![x]\!]^{i-j}x^{j-1} - x^j\beta[\![x]\!]^{i-j-1}\partial\beta[\![x]\!]]$$

This coefficient is that of x^{-1} in a representation of

$$i^{-1}j[x^{j-i-1}\beta[\![x]\!]^{i-j} - x^{j-i}\beta[\![x]\!]^{i-j-1}\partial\beta[\![x]\!]] = \{i(j-i)\}^{-1}\partial[x^{j-i}\beta[\![x]\!]^{i-j}]$$

If a representation of a series $\mathfrak{D}p[\![x]\!]$ in $K_n[\![x]\!]$ contains a term involving x^{-1}, the coefficient in that term is zero. Hence $z_{i,j}=0$ $(i:2;n\,|\,j:i-1)$. Also $z_{i,i}=\beta_0^{(i)}a_{i,i}=(b_1a_1)^i=1$ $(i:n)$. Denoting by C the lower triangular matrix with element $i^{-1}j\beta^{(i)}_{i-j}$ in the j^{th} portion of its i^{th} row $(i:n\,|\,j:i)$, it has just been shown that $CK\{a\}=E_n$. $K\{a\}$ has, however, the unique inverse $K\{I(a)\}$: $C=K\{I(a)\}$. Letting $c_{i,j}$ be the representative element of $K\{I(a)\}$,

$$ic_{i,j}=j\beta^{(i)}_{i-j} \qquad (i:n\,|\,j:i)$$

The series $\beta[\![x]\!]^i$ $(i:n)$ belong to that part of $\mathbb{K}_n[\![x]\!]$ which is isomorphic to $\mathbb{K}_{n-1}[\![x]\!]$. Relationship ( ) refers to members of K; it holds for the coefficients $\beta_\nu^{(i)}$ of the polynomial $\beta[x]^i$ in $\mathbb{K}_{n-1}[x]$ to which $\xi_1\beta[\![x]\!]^i$ corresponds $(i:n)$.

From formulae (,)

$$g_i = i^{-1}\sum_{j=1}^{i} j\beta^{(i)}_{i-j}f_j$$

$$=(i!)^{-1}\sum_{j=0}^{i-1}\binom{i-1}{j}(i-j-1)!\,\beta^{(i)}_{i-j-1}\,j!\,(j+1)f_{j+1}$$

$$=(i!)^{-1}\sum_{j=0}^{i-1}\binom{i-1}{j}\{\mathfrak{D}_0^{\,i-j-1}\beta[x]^i\}\{\mathfrak{D}_0^{\,j}\mathfrak{D}f[x]\} \quad (i:n)$$

Use of the derivational property of the mapping $\mathfrak{Q}_0$, as expressed by formula ( ), yields the result of the theorem.

Formula ( ) expresses $K\{I(a)\}$ as a product of matrices derived from $C\{R(a)\}$. Formula ( ) relates the elements of $K\{I(a)\}$ to certain of the coefficients in certain of the powers of the truncated formal power series reciprocal to $a[\![x]\!]$ in the sense of formula ( ). The series $\beta[\![x]\!]^i$ are defined for all positive and negative integer values of i; in an array of n rows ~~and~~ of unbounded extent, the coefficient $\beta_\nu^{(i)}$ may be set in the $(\nu+1)^{th}$ row and column with index i $(i:;\nu:0;n-1)$ Relationship ( ) concerns numbers in a triangular portion of this array upon whose edges the numbers $\beta_0^{(i)}$ $(i:n)$, $\beta_{i-1}^{(i)}$ $(i:2;n-1)$ and $\beta_i^{(n)}$ $(i:n-1)$ lie.

With $w \in \mathbb{C}$ fixed and a, a_j the mapping and coefficients occurring in formula ( ), the function $\beta(z) = (z-w)/\{a(z)-a(w)\}$ is defined for $z \in \Omega$; also, if $a_1 \neq 0$

$$(z-w)\beta(z) \simeq_{(w,\Omega)} \sum_{j=1}^{n} \beta_{j-1} (z-w)^j$$

where, with $b = R(a)$, $\beta_{j-1} = b_j$ $(j:n)$. Furthermore $\beta_\nu^{(i)} \in \mathbb{C}$ exist such that

$$(z-w)\beta(z)^{i}\underset{(w,\Omega)}{\Omega}\sum_{j=1}^{n}\beta_{j-1}^{(i)}(z-w)^{j} \qquad (i:n)$$

Formula ( ) for the coefficients ϱ_i in relationship ( ) resulting from the transformation of relationships (,) is now to be interpreted in the sense of its predecessor.

If the derivatives $\frac{d^n a(z)}{dz^n}$, $\frac{d^n f(z)}{dz^n}$ are defined over Ω, and $\lim \frac{d^n a(z)}{dz^n}$ as $z(\in\Omega)\to w$ is finite with a similar relationship holding with regard to f, formula ( ) may be interpreted in the more direct sense

$$\varrho_i = (i!)^{-1} \lim \frac{d^{i-1}}{dz^{i-1}}\left\{\beta(z)^{i-1}\frac{df(z)}{dz}\right\} \qquad (i:n)$$

as $z(\in\Omega)\to w$.

The above remarks extend to the case in which the mapping a depends upon $w\in M$ and takes the form $a(w,\cdot)$, and the a_j, f_j become mappings $M\to C$, as described in the introduction. Concerning that variant of the more general case in which $a(w,z)-a(w,w)$ takes the form $a'(z-w)$ with $a'(0)=0$, where $\frac{d^n a'(t)}{dt^n}$ is defined for $t\in\Omega'$ and $\lim\frac{d^n a'(t)}{dt^n}$ as $t(\in\Omega')\to 0$ is finite and, for each $w\in M$, $\frac{d^n f(z)}{dz^n}$ is defined for over $\Omega(w)$ and $\lim\frac{d^n f(z)}{dz^n}$ as $z(\in M)\to w$ is finite, formula ( ) takes the special form

$$g_i = (i!)^{-1} \lim \frac{d^{i-1}}{dt^{i-1}} \left\{ \beta(t)^i \frac{df(w+t)}{dt} \right\}$$

as $w+t \, (\in \Omega(w)) \to w$ for each $w \in M$, where β is defined over Ω' by the formula $\beta(t) = t/a'(t)$. The further variants in which $a(w,z)$ is independent of w, and M is a domain in the complex plane over which a and f are analytic has been treated by complex variable methods in the introduction, as has the further variant in which $a(w,z)$ is a fractional linear function, depending upon w, of a further analytic function $A(z)$.

The composition matrix $K\{a\}$ is defined in terms of formula ( ) and may be constructed by its use from the sequence a. In certain special cases a simpler process is available.

Theorem. With $0 \le n(\tau) < n \; (\tau : 3)$ let

$$p[\![x]\!] = \left[\sum_{i=0}^{n-1} p_i x^i\right],\; q[\![x]\!] = \left[\sum_{i=0}^{n-1} q_i x^i\right],\; r[\![x]\!] = \left[\sum_{i=0}^{n-1} r_i x^i\right].$$

where p_0, q_0, r_0 are not all zero, $p_i = 0 \; (i : n(1)+1 ; n-1)$ and similarly for q_i, r_i. Let

$$a[\![x]\!] = \left[\sum_{i=1}^{n} a_i x^i\right]$$

with $a_1 \neq 0$ satisfy the differential equation

$$p[\![x]\!]\,\Omega\, a[\![x]\!] + q[\![x]\!]\, a[\![x]\!] + r[\![x]\!] = 0$$

The elements $a_{i,j}$ of $K\{a\}$ and $c_{i,j}$ of $K\{I(a)\}$ satisfy the relationships

$$\sum_{k=0}^{m(1,i,j)} (i-k) p_k a_{i-k,j} + j\left\{\sum_{k=0}^{m(2,i,j)} q_k a_{i-k-1,j} + \sum_{k=0}^{m(3,i,j)} r_k a_{i-k-1,j-1}\right\} = 0$$

$$(i : 2 ; n \,|\, j : 2 ; i)$$

$$\sum_{k=0}^{m(1,i,j)} p_k c_{i-1,k+j-1} + (i-1) \sum_{k=0}^{m(2,i,j)} (k+j)^{-1} q_k c_{i-1,k+j} + i \sum_{k=0}^{m(3,i,j)} (k+j)^{-1} r_k c_{i,k+j} = 0$$

$$(i : m ; n \,|\, j : m ; n)$$

respectively, where $m(\tau,i,j)=\min(n(\tau),i-j)$ for $\tau=1,3$, $m(2,i,j)=\min(n(2),i-j-1)$ and $m=\max(n(1),n(2)+1,n(3)+1)$. With $\beta[\![x]\!]$ defined by the relationship $\beta[\![x]\!]a[\![x]\!]\equiv x$, and

$$\beta[\![x]\!]^j=\left[\sum_{i=0}^{n-1}\beta_i^{(j)}x^i\right] \qquad (j:)$$

the $\beta_i^{(j)}$ satisfy the relationship

$$\sum_{k=0}^{n(1,\nu)}(\nu-k)p_k\beta_{\nu-k}^{(j)}-j\sum_{k=0}^{n(2,\nu)}p_k\beta_{\nu-k}^{(j)}-j\sum_{k=0}^{n(3,\nu)}q_k\beta_{\nu-k-1}^{(j)}-j\sum_{k=0}^{n(4,\nu)}r_k\beta_{\nu-k}^{(j+1)}=0$$

$$(j:|\nu:n-1)$$

where $n(1,\nu)=\min(n(1),\nu-1)$, $n(2,\nu)=\min(n(1),\nu)$, $n(3,\nu)=\min(n(2),\nu-1)$ and $n(4,\nu)=\min(n(3),\nu)$.

Proof. If, in $K_n[\![x]\!]$, $u[\![x]\!]\equiv u'[\![x]\!]$ then $u[\![x]\!]v[\![x]\!]\equiv u'[\![x]\!]v[\![x]\!]$ also for all $v[\![x]\!]\in K_n[\![x]\!]$; if, in addition $v[\![x]\!]\equiv v'[\![x]\!]$ then $u[\![x]\!]\pm v[\![x]\!]\equiv u'[\![x]\!]\pm v'[\![x]\!]$. The relationship

$$\Delta\{a[\![x]\!]^j\}\equiv j\,a[\![x]\!]\,\Delta a[\![x]\!]$$

is evidently correct when $j=1$. Writing $\Delta\{a[\![x]\!]^{j+1}\}$ as $\Delta\{a[\![x]\!]^j a[\![x]\!]\}$ and using an appropriate version of formula ( ), it may be shown by induction that relationship ( ) holds for $(j:)$. Multiplying relationship ( ) throughout by

$ja[\![x]\!]^{j-1}$ and setting $a^{(j)}[\![x]\!] = a[\![x]\!]^j$, the formula

$$p[\![x]\!]\Delta a^{(j)}[\![x]\!] + j\{q[\![x]\!]a^{(j)}[\![x]\!] + r[\![x]\!]a^{(j-1)}[\![x]\!]\} = 0$$

is obtained. With $a[\![x]\!]$ represented by a polynomial with coefficients a_i $(i:n)$ as in formula ( ), $a^{(j)}[\![x]\!]$ is, for $(j:n)$, represented by an expression involving x^i $(i:j;j+n-1)$ in which the coefficient of x^i is $a_{i,j}$ $(i:j;n)$. The coefficient of x^i in that representation of the left hand side expression in relationship ( ) derived from the representation of $a^{(j)}[\![x]\!]$ just described and the representations ( ) occurs on the left hand side of equation ( ).

A suitable version of formula ( ) applied to the relationship $\Delta\{a[\![x]\!]^{-1}a[\![x]\!]\} \equiv 0$ when $a[\![x]\!]$ is nonzero yields the result $\Delta a[\![x]\!]^{-1} \equiv -a[\![x]\!]^{-2}\Delta a[\![x]\!]$ which in turn may be used to show by induction that relationship ( ) holds, when $a[\![x]\!]$ is nonzero, for all ~~n~~ positive values of j.

With $u[\![x]\!]$ nonzero, $u[\![x]\!] \equiv v[\![x]\!]$ if and only if each of the ~~representations~~ expressions representing $u[\![x]\!]$ also represents $v[\![x]\!]$.

$$c[\![x]\!] = \left[\sum_{i=1}^{n} c_i x^i\right]$$

? $i=1$ nec.

being nonzero, $u[\![c[\![x]\!]]\!]$ is represented by an expression derived from a representation of $u[\![x]\!]$ and that occurring in formula ( ). When $u[\![x]\!] \equiv v[\![x]\!]$, the same representation is obtained for $v[\![c[\![x]\!]]\!]$; hence $u[\![c[\![x]\!]]\!] \equiv v[\![c[\![x]\!]]\!]$. Letting

$$u[\![x]\!] = \left[\sum_{i=k}^{k+n-1}{}' u_i x^i\right]$$

$$u'[\![x]\!] = \mathfrak{D}u[\![x]\!] = \left[\sum_{i=k-1}^{k+n-2}{}' (i+1)u_{i+1} x^i\right]$$

Accordingly, with $c[\![x]\!]$ nonzero

$$u[\![c[\![x]\!]]\!] \equiv \sum_{j=k}^{k+n-1}{}' u_j c[\![x]\!]^j$$

$$u'[\![c[\![x]\!]]\!] \equiv \sum_{j=k}^{k+n-1}{}' j u_j c[\![x]\!]^{j-1}$$

Using formula ( ) with a replaced by c, it follows that

$$\mathfrak{D}u[\![c[\![x]\!]]\!] = u'[\![c[\![x]\!]]\!]\,\mathfrak{D}c[\![x]\!]$$

The mapping $\mathfrak{D}$ over $\mathbb{K}_n[\![x]\!]$ is not only a derivation but obeys the rule ( ) with regard to change of independent variable.

The above rules concerning transformation of independent variable may, in particular, be applied to equation ( ) with $c[\![x]\!]$ being taken to be the series inverse to $a[\![x]\!]$, so that $a[\![c[\![x]\!]]\!] \equiv x$ and $\mathfrak{D}a[\![c[\![x]\!]]\!] = 1$. Multiplication of formula ( ) throughout by $\mathfrak{D}c[\![x]\!]$ then yields the differential equation

$$p[\![c[\![x]\!]]\!] + \{x q[\![c[\![x]\!]]\!] + r[\![c[\![x]\!]]\!]\}\vartheta c[\![x]\!] \equiv 0$$

which, if p, q or r contain nonzero nonlinear terms, is nonlinear in $c[\![x]\!]$. As in the derivation of formula ( ), equation ( ) is multiplied throughout by $c[\![x]\!]^{j-1}$; inspection of the term involving x^i in the expression representing the left hand side of the resulting relationship yields formula ( ).

The differential equation for the reciprocal series $\beta[\![x]\!]$ as defined in the theorem is

$$x p[\![x]\!]\vartheta\beta[\![x]\!] - \{p[\![x]\!] + x q[\![x]\!]\}\beta[\![x]\!] - r[\![x]\!]\beta[\![x]\!]^2 \equiv 0$$

It is treated as in the derivation of formulae (,) and formula ( ) follows.

The way in which formulae (, ,) are used to construct the numbers $a_{i,j}$, $c_{i,j}$, $\beta_\nu^{(j)}$ depends upon the structures of the polynomials p, q and r. Illustrations are provided by the following simple examples. In each of them $\mu, z \in \mathbb{K}$ are taken to be nonzero, $a_{i,i} = (\mu z)^i$ $(i:n)$, $c_{i,i} = (\mu z)^{-i}$ $(i:n)$ and $\beta_0^{(j)} = (\mu z)^{-j}$ $(j:\mathfrak{k})$.

Let $\gamma \neq -i \ (i:n-1)$ and

$$a[\![x]\!] = \left[\mu \sum_{i=1}^{n} \left\{ z^i \prod_{\nu=1}^{i-1} \left\{ \frac{\alpha+\nu}{\gamma+\nu} \right\} \right\} x^i \right]$$

so that

$$x(1-zx)\Delta a[\![x]\!] + (\gamma - 1 - \alpha z x) a[\![x]\!] - \gamma\mu z x \equiv 0$$

Now

$$a_{i,j} = \frac{z}{i+j(\gamma-1)} \left\{ (i-1+\alpha j) a_{i-1,j} + \gamma\mu j a_{i-1,j-1} \right\} \ (i:3;n \mid j:2;i-1)$$

$$c_{i,j} = -\frac{j}{i\gamma\mu z} \left[\frac{z\{j+(i-1)\alpha\}}{j} c_{i-1,j} - \frac{j-1+(i-1)(\gamma-1)}{j-1} c_{i-1,j-1} \right]$$

$(i:3;n \mid j:2;i-1)$

when $\gamma \neq 0$ and, when $\gamma = 0$, $\alpha \neq -j/i \ (i:2;n \mid j:i)$

$$c_{i,j} = \frac{(j-i-1)j}{(j-1)z(j+i\alpha)} c_{i,j-1} \qquad (i:3;n \mid j:2;i-1)$$

$$\beta_\nu^{(j+1)} = (j\gamma\mu z)^{-1} \left\{ (j\gamma - \nu)\beta_\nu^{(j)} - z(j\alpha + j - \nu + 1)\beta_{\nu-1}^{(j)} \right. \quad (j:\mid \nu:n-1)$$

when $\gamma \neq 0$ and, when $\gamma = 0$,

$$\beta_\nu^{(j)} = \frac{z(\nu - 1 - j - \alpha j)}{\nu} \beta_{\nu-1}^{(j)} \qquad (j:\mid \nu:n-1)$$

Again with γ as above, let

$$a[\![x]\!] = \left[\mu \sum_{i=1}^{n} \left\{ z^i \prod_{\nu=1}^{i-1} (\gamma+\nu)^{-1} \right\} x^i \right]$$

so that

$$x \mathcal{D} a[\![x]\!] + (\gamma - 1 - zx) a[\![x]\!] - \gamma\mu z x \equiv 0$$

The formulae corresponding to (-) are

$$a_{i,j} = \frac{jz}{i + j(\gamma - 1)} \{ a_{i-1,j} + \gamma\mu a_{i-1,j-1} \}$$

$$c_{i,j} = -\frac{j}{i\gamma\mu z} \left\{ \frac{z(i-1)}{j} c_{i-1,j} - \frac{j - 1 + (i-1)(\gamma-1)}{j-1} c_{i-1,j-1} \right\}$$

$$c_{i,j} = \frac{(j-i-1)j}{(j-1)zi} c_{i,j-1}$$

$$\beta_\nu^{(j+1)} = (j\gamma\mu z)^{-1} \{ (j\gamma - \nu)\beta_\nu^{(j)} - zj\beta_{\nu-1}^{(j)} \}$$

$$\beta_\nu^{(j)} = -\frac{zj}{\nu} \beta_{\nu-1}^{(j)}$$

the indices in each case taking the values associated with its counterpart.

Lastly, let

$$a[\![x]\!] = \left[\mu \sum_{i=1}^{n}{}' \left\{ z^i \prod_{\nu=1}^{i-1} (\alpha + \nu) \right\} x^i \right]$$

so that

$$z x^2 \mathcal{D} a[\![x]\!] + (z\alpha x - 1) a[\![x]\!] + \mu z x \equiv 0$$

The formulae corresponding to (, ,) are

$$a_{i,j} = \frac{z}{j} \{ (i - 1 + \alpha j) a_{i-1,j} + \mu j a_{i-1,j-1} \}$$

$$c_{i,j} = -\frac{j}{i\mu z}\left[\frac{z\{j+\alpha(i-1)\}}{j}c_{i-1,j} - \frac{(j-1)}{j-1}c_{i-1,j-1}\right]$$

$$\beta_{\nu}^{(j+1)} = (j\mu z)^{-1}\{j\beta_{\nu}^{(j)} - z(j\alpha+j-\nu+1)\beta_{\nu-1}^{(j)}\}$$

the indices again taking the values associated with the counterpart.

It is not always true that the series inverse to $a[\![x]\!]$ of formula ( ) is either a series of the same form or one of the form ( ); nevertheless, when $\gamma=1$, this is so. When $\gamma=1$, equations (,) for the series $a[\![x]\!]$ of formula ( ) and the inverse $c[\![x]\!]$ of $a[\![x]\!]$ assume the simpler forms

$$(1-zx)\Delta a[\![x]\!] - \alpha z a[\![x]\!] - \mu z = 0$$

$$(\mu z+\alpha z x)\Delta c[\![x]\!] + z c[\![x]\!] - 1 = 0$$

Equations (,) for the series $a[\![x]\!]$ of formula ( ) and its inverse $c[\![x]\!]$ assumes the forms

$$\Delta a[\![x]\!] - z a[\![x]\!] - \mu z = 0$$

$$(\mu + x) z\Delta c[\![x]\!] - 1 = 0$$

All four equations are of the same type. Let

$$\phi(\mu,z,\alpha;[\![x]\!]) = \left[\mu\sum_{i=1}^{n}\{(i!)^{-1}z^i\prod_{\nu=1}^{i-1}(\alpha+\nu)\}x^i\right]$$

$$\theta(\mu,z;[\![x]\!]) = \left[\mu\sum_{i=1}^{n}\{(i!)^{-1}z^i\}x^i\right]$$

and $\phi^{(-1)}(\mu,z,\alpha;[\![x]\!])$, $\theta^{(-1)}(\mu,z;[\![x]\!])$ be the corresponding inverse series. Then

$$\phi^{(-1)}(\mu,z,\alpha;[\![x]\!])=\phi\left(-\frac{1}{\alpha z},-\frac{\alpha}{\mu},\frac{1}{\alpha};[\![x]\!]\right) \qquad (\alpha\neq 0)$$

$$\phi^{(-1)}(\mu,z,0;[\![x]\!])=\Theta\left(-\frac{1}{z},-\frac{1}{\mu};[\![x]\!]\right)$$

$$\theta^{(-1)}(\mu,z;[\![x]\!])=\phi\left(-\frac{1}{z},-\frac{1}{\mu},0;[\![x]\!]\right)$$

When $\gamma=1$, recursions (, , ,) become

$$a_{i,j}=\frac{z(i-1+\alpha j)}{i}a_{i-1,j}+\frac{\mu z j}{i}a_{i-1,j-1}$$

$$c_{i,j}=-\frac{j+(i-1)\alpha}{i\mu}c_{i-1,j}+\frac{j}{i\mu z}c_{i-1,j-1}$$

$$a_{i,j}=\frac{jz}{i}a_{i-1,j}+\frac{\mu z j}{i}a_{i-1,j-1}$$

$$c_{i,j}=-\frac{(i-1)}{i\mu}c_{i-1,j}+\frac{j}{i\mu z}c_{i-1,j-1}$$

valid for suffix values as stated with respect to the above recursions.

Taking $\mathbb{K}$ to be the field of complex numbers and letting n increase without limit, the series of formulae (,) are generated by the functions

$$\phi(\mu,z,\alpha;x)=\alpha^{-1}\mu\{(1-zx)^{-\alpha}-1\} \qquad (\alpha\neq 0)$$

$$\phi(\mu, z, 0; x) = -\mu \ln(1-zx)$$

$$\theta(\mu, z; x) = \mu(e^{zx}-1)$$

respectively. ($\mu(e^{zx}-1) = \lim_{\alpha=0} \mu\{(1-\alpha z x)^{-1/\alpha}-1\}$.)

—o—

The Bernoulli numbers $B_\nu^{(j)}$ and polynomials $B_\nu^{(j)}(y)$ feature in the expansions

$$\frac{x^j}{(e^x-1)^j} = \sum_{\nu=0} \frac{B_\nu^{(j)}}{\nu!}x^\nu, \quad \frac{x^j e^{xy}}{(e^x-1)^j} = \sum_{\nu=0} \frac{B_\nu^{(j)}(y)}{\nu!}x^\nu$$

From these definitions, the relationships

$$B_\nu^{(j)}(0) = B_\nu^{(j)}, \quad B_\nu^{(j)}(j) = (-1)^\nu B_\nu^{(j)}, \quad B_\nu^{(j)}(y) = \sum_{\tau=0}^{\nu} \binom{\nu}{\tau} B_{\nu-\tau}^{(j)} y^\tau$$

$$B_\nu^{(j+1)}(y) = \left(1-\frac{\nu}{j}\right)B_\nu^{(j)}(y) + (y-j)\frac{\nu}{j}B_{\nu-1}^{(j)}(y)$$

$$B_\nu^{(j+1)} = \left(1-\frac{\nu}{j}\right)B_\nu^{(j)} - \nu B_{\nu-1}^{(j)}$$

$$B_j^{(j+1)}(y) = (y-j)B_{j-1}^{(j)}(y), \quad B_j^{(j+1)}(y) = \prod_{\tau=1}^{j}(y-\tau)$$

$$\frac{d}{dy}B_\nu^{(j)}(y) = \nu B_{\nu-1}^{(j)}(y), \quad B_{\nu+1}^{(j)}(y+1) - B_\nu^{(j)}(y) = \nu B_{\nu-1}^{(j-1)}(y)$$

follows.

Taking $\mathbb{K}$ to be the field of complex numbers, and letting n increase without limit, the general term in Bürmann's series with $a(x) = 1-e^{-x}$ assumes the form

$$g_i = \frac{1}{i!}\Delta_0^{i-1}\left[\left\{\frac{xe^x}{e^x-1}\right\}^i \Delta f(x)\right]$$

$$= \frac{1}{i!}\sum_{\tau=0}^{i-1}\binom{i-1}{\tau}B^{(i)}_{i-\tau-1}(i)\,\Delta_0^{\tau}\{\Delta f(x)\}$$

$$= \frac{(-1)^{i-1}}{i!}B^{(i)}_{i-1}(-\Delta_0)\Delta f(x)$$

$$= \frac{1}{i!}\prod_{\tau=1}^{i-1}(\tau+\Delta_0)\Delta f(x)$$

The numbers g_i may be determined in the following way. Set

$$f_i^{(r)} = \frac{1}{r!}\prod_{\tau=1}^{r-1}(\tau+\Delta_0)\Delta_0^{i-1}\Delta f(x) \qquad (r, i:)$$

so that $f_i^{(1)} = i!\,f_i$ $(i:)$ and determine

$$f_i^{(r+1)} = \frac{r f_i^{(r)} + f_{i+1}^{(r)}}{r+1} \qquad (r, i:)$$

then $g_i = f_1^{(i)}$ $(i:)$.

When $f(x) = e^{-kx}$ with k a nonnegative integer, $f_i^{(r)} = 0$ $(r:k+1;|i:)$ so that, if

$$f(x) = \sum_{k=0}^{m} A_k e^{-kx}$$

$f_i^{(r)} = 0$ $(r:k+1;|i:)$ and, in particular, $g_i = 0$ $(i:k+1;)$. In this case

$$\sum_{i=0}^{\infty} f_i x^i = f_0 + \sum_{i=1}^{m} g_i(1-e^{-x})^i$$

The function $c(z)=\ln(1+z)$ is inverse to $a(z)=e^z-1$. $c(e^{i\theta})=\ln\{2(1+\cos\theta)\}+i\frac{\theta}{2}$ maps the segment $-\pi\leq\theta<\pi$ onto a curve $\mathcal{C}$, symmetric about the real axis, enclosing its nonpositive part, containing the real point $\ln(2)$, the imaginary points $\pm i\frac{\pi}{3}$, and having as asymptotes the lines $z=\pm i\frac{\pi}{2}$. $\mathbb{D}$ being the open domain bounded by $\mathcal{C}$, a maps $\mathbb{D}$ bijectively onto the unit open disc.