

On stability functions

by

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1. Introduction and notations

This paper is concerned with functions of the form

$$(1) \quad f(z) = 1 + \frac{\alpha z}{1 - \frac{1}{2}\alpha z + z^2 s(z)}$$

where $0 < \alpha < \infty$, and

$$(2) \quad s(z) = \int_0^\infty \frac{d\psi(t)}{1 + z^2 t}$$

where $\psi(t)$ is a nondecreasing function of bounded variation for $0 \leq t < \infty$ such that all moments

$$(3) \quad c_\nu = \int_0^\infty t^\nu d\psi(t)$$

for $\nu = 0, 1, \dots$ exist

A function of this form will be called an F-function.

If the context permits, the notation $f \in F$ or, where convenience dictates, $f(z) \in F$, will be used. The function

s in the representation (1) plays a significant role in the theory of the function f . A function of the form (2) with ψ as described will be called an S -function; again the notations $s \in S$ or $s(z) \in S$ will be used.

The mapping properties of F - and S -functions, in particular, will be investigated. To avoid all possible misunderstanding it is stated at the outset that the statement that a function maps one open domain R into another R' means that the image of every point of R lies in R' and that, with one possible exception, every point of R' is the image of at least one point of R . Mappings

of closed domains are defined in the same way.

Frequent reference will be made to the finite part of the open left half-plane (i.e. $\{z : \operatorname{Re}(z) < 0, |z| < \infty\}$) which is denoted by L , to the open right half-plane, denoted by R , to the open unit disc (i.e. $\{f : |f| < 1\}$) denoted by D , and to the finite part of the open upper half-plane (i.e. $\{\lambda : \operatorname{Im}(\lambda) > 0, |\lambda| < \infty\}$) denoted by U . In particular, the closed infinite left half-plane is denoted by \bar{L} , and the closed unit disc by \bar{D} . $\Delta(\alpha, \beta)$ denotes the sector $\{z : \alpha + \delta \leq \arg(z) \leq \beta - \delta'\}$ where $\delta, \delta' > 0$ and it is assumed that $\alpha + \delta < \beta - \delta'$. The notation $z \rightarrow z_0 \in \Delta(\alpha, \beta)$ is used to indicate that z tends to z_0 over an open set in the sector $\Delta(\alpha, \beta)$.

with limit point z_0 (z_0 is always 0 or ∞). Reference is also made to nondecreasing functions of bounded variation over a prescribed interval $[\alpha, \beta]$; the notation $\psi \in \mathcal{B}[\alpha, \beta]$ is used to indicate that ψ is such a function.

with limit point z_0 (z_0 is always 0 or ∞).

F-functions have been characterised in the following way: the function $f(z)$: a) is real for finite negative values of z with $f(z) > 0$ for all sufficiently small negative values of z and b) is asymptotically represented as $z \rightarrow 0 \in \Delta\left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$ by a series of the form

$$f(4) \quad f(z) \sim \sum_{j=0}^{\infty} c_j z^j$$

which c) generates an associated continued fraction (see [15] §25) whose convergents

$$(5) \quad C_{2r}(z) = v_0 + \frac{v_1 z}{1+w_1 z +} \frac{v_2 z^2}{1+w_2 z +} \dots \frac{v_r z^2}{1+w_r z +}$$

for $r=1,2,\dots$ (the definition of $C_1(z)$ being obvious) map $\bar{\mathbb{L}}$ into $\bar{\mathbb{D}}$, if and only if f is an F-function.

Omitting numerous details (they are given in [26]) the above characterisation was established in the following way. $C_{2r}(z)$ is defined by the property that if, for sufficiently small z ,

$$(6) \quad C_{2r}(z) = \sum_{j=0}^{\infty} f_j^{(2r)} z^j$$

then $f_j^{(2r)} = f_j$ ($j = 0, \dots, 2r$) for all r for which the convergents (5) are defined (either the continued fraction associated with the series (4) is non-terminating, in which case the preceding relationships hold for $r = 1, 2, \dots$ or (4) is the expansion of a rational function, its associated continued fraction terminates with some $C_{2r}(z)$ equivalent to this function, and for this r , $f_j^{(2r)} = f_j$ ($j = 0, 1, \dots$)). If $C_{2r}(z)$ maps

\bar{L} into \bar{D} , it maps the imaginary axis into the unit circle; it is also real for real z : thus $C_{2r}(z)C_{2r}(-z)=1$ for pure imaginary values of z , and hence generally. The series (6) obey the same relationship and, letting r increase indefinitely or assume its terminating value, the series (4) itself, denoted by $f(z)$, satisfies the formal equation $f(z)f(-z)=1$. Series with this property generate associated continued fractions ^{with convergent} of the special form

$$(7) \quad C_{2r}(z) = 1 + \frac{az}{1 - \frac{1}{2}az +} \frac{v_2 z^2}{1 +} \cdots \frac{v_r z^2}{1 +} \cdots$$

$C_{2r}(z)$ has the form $q_r(z)/p_r(z)$, where $p_r(z)$ is an r^{th} degree polynomial with $p_r(0)=1$, and q_r and p_r cannot vanish for a common value of their

arguments. p_r has real coefficients, and $p_r(z) = 0$ only when $\operatorname{Re}(z) > 0$ if and only if $0 < a < \infty$ and $0 < v_i < \infty$ $i = 2, \dots, r$ (should $C_{2r}(z)$ be defined for $r > 1$). The continued fraction

$$\cfrac{v_2}{1+} \cfrac{v_3 z^2}{1+} \cdots \cfrac{v_r z^2}{1+} \cdots$$

(see [15] § 23)

with $0 < v_i < \infty$ corresponds to the series $\sum_{j=0}^{\infty} c_j z^{2j}$

if and only if the c_j have the form (3) with ψ as described. This series is the asymptotic expansion as $z \rightarrow 0 \in \Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$ of the function $s(z)$ of formula (2). Hence the above characterisation of the function f of formula (1) has finally been obtained.

One example of an F-function is

$$(8) e^z - 1 + \frac{z}{1 - \frac{1}{2}z + z^2 \sum_{j=1}^{\infty} \frac{\frac{1}{2}(2\pi)^{-2}}{1 + z^2 (2\pi)^{-2}}}$$

in which ψ in formula (2) is a step-function with saltus of magnitude $\frac{1}{2}(2\pi)^{-2}$ at the point $t = (2\pi)^{-2}$ ($j = 1, 2, \dots$) and no other points of increase in the range $(0, \infty)$.

The study of F-functions was motivated by the following considerations: With \tilde{A} a bounded linear operator, the solution of the differential equation

$$(9) \quad \frac{dy(t)}{dt} = \tilde{A}y(t)$$

with $y(0)$ prescribed, satisfies the relationship

$$(10) \quad \tilde{y}(nh+h) = \exp(\tilde{A}h)\tilde{y}(nh)$$

for $n=0,1,\dots$. If $0 < h < \infty$ and the eigenvalues of \tilde{A} lie in L , those of $\exp(\tilde{A}h)$ lie in D . $\|\tilde{y}(nh)\|$ remains bounded, and, indeed, decreases to zero as n increases indefinitely. An approximation $\tilde{y}^*(t)$ to the solution of equation (5) may be obtained by use of a Taylor series method based upon use of an approximate identity

$$(11) \quad \sum_{i=0}^m a_i \frac{d^i y^*(t+h)}{dt^i} = \sum_{i=0}^n b_i \frac{d^i y^*(t)}{dt^i}.$$

Setting $m=n=r$ and taking the a_i and b_i to be the denominator and numerator coefficients of powers of z in $C_{2r}(hz)$, where $C_{2r}(z)$, with $r \geq 1$ fixed, is a convergent of the continued fraction associated with

the exponential series, and setting $t=nh$, the special form of the approximate identity (11) applied to equation (10) yields the relationship

$$(12) \quad \underset{n}{y^*(nh+h)} = C_{2r}(\underset{n}{Ah}) \underset{n}{y^*(nh)}$$

for $n=0, 1, \dots$. As a consequence of the mapping properties of $C_{2r}(z)$ described above, the eigenvalues of $C_{2r}(\underset{n}{Ah})$ lie in \mathbb{D} , and the remarks concerning the behaviour of $\|\underset{n}{y(nh)}\|$ apply with equal force to $\|\underset{n}{y^*(nh)}\|$: the exact and approximate solutions of equation (9) behave in the same way. The practical details of the way in which relationship (12) is implemented are not of immediate concern; any ~~technique~~ method for the approximate solution of

equation (9) based upon use of recursion (12) is stable.

e^z is not the only F-function; $z + (1+z^2)^{1/2}$ is another, and, $F_1(\gamma+1; 2\gamma+1; z)/F_1(\gamma; 2\gamma+1; z)$ with $-\frac{1}{2} < \gamma < \infty$ yet another. Many of these functions satisfy nonlinear differential equations, just as e^z of Riccati form, just as e^z satisfies a simple linear differential equation. The above theory thus opens up the possibility of constructing stable schemes for the approximate solution of certain nonlinear differential equations.

F-functions possess interesting properties further to those outlined above. For example, for certain such functions the equation $f(z) = C_{2r}(z)$

holds for a system of points upon the imaginary axis which interlaces with the system obtained by replacing r by $r+1$. This subclass of F-functions contains the symmetric members of another class of functions occurring in the theory of smoothing functions. These matters are dealt with later.

2. A characterisation

Functions of a prescribed class may often be characterised in various ways. It is clear from the remarks concerning the approximate solution of differential equations given in the preceding section that, given this motivation, the characterisation of F-functions outlined at the beginning of that

section is reasonable. Nevertheless, a characterisation of functions in terms of a representational apparatus such as a continued fraction is likely to be, in contexts other than those relating to the use of rational functions, infelicitous. For this reason, it is proposed to give a characterisation which is independent of continued fraction theory.

Theorem 1. The function f

- a) is analytic over L
- b) maps L into D
- c) is real for negative real values of its argument with $f(z) > 0$ for all sufficiently small negative real values of z

d) is represented asymptotically by a series of the form
 $(^4)$ as $z \rightarrow 0 \in \Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$
if and only if f is an F-function.

Proof. Use will be made of the following result: the function
w A) is analytic over L , B) maps L into itself, C) has
an asymptotic representation of the form

$$(^8) \quad w(z) \sim \sum_{j=0}^{\infty} w_j z^{j-1}$$

as $z \rightarrow 0 \in \Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$ and D) is real for negative real values
of z if and only if w is represented over L by the
expression

$$(^4) \quad w(z) = w_0 z^{-1} + z s(z)$$

where $s \in S$. This result has a counterpart, obtained by
mere verbal changes, in the theory of functions which
map \mathbb{R} into itself and assume real values for positive
real argument values (the positive real functions). This
counterpart is of considerable independent interest,
and its proof is therefore given in a separate
appendix (Appendix 1).

The main result of the theorem may now be derived. Firstly, if
 f is an F-function with the representation (1), then it possesses the
properties (a-d). For f may be expressed as

$$f(z) = \frac{w(z) + \frac{1}{2}\alpha}{w(z) - \frac{1}{2}\alpha}$$

$$w_0 = 1.$$

where w has the form (14) with $A=1$. w maps L into itself and, when $0 < \alpha < \infty$, f maps L into D . The function f of formula (5) also satisfies requirement c). Since the moments (3) exist, w generates an asymptotic series of the form (18), and f in turn has the asymptotic representation of d).

The origin lies on the boundary of L . If f maps L into D , the limiting value taken by $f(z)$ as z tends to zero lies on the unit circle. If, in addition, f assumes positive real

values for all small negative real values of z , $f(z)$ tends to the only positive real value on the unit circle, namely unity; thus $f_0 = 1$ in the representation (4). Since $|f(z)| < 1$ for negative real values of z , $0 < f_1 < \infty$ in (4). With $a = f_1$, the transformation

$$(19) \quad w(z) = \frac{1}{2}a \frac{f(z)+1}{f(z)-1}$$

maps \mathbb{D} in the $f(z)$ -plane into \mathbb{L} in the $w(z)$ -plane. Since $f(z)$ is real for negative real values of z , the same is true of $w(z)$. Again, since f has the representation (18) with, in particular, $w_0 = 1$.

Thus from the second auxiliary result derived above, w has the representation (14) with $A = 1$ and $\theta \in S$. Rearrangement of formulae (14, 19) then

yields the representation (1) as described.

The theory of continued fractions is not used in the above characterisation, and it is of interest to point out how this is made possible. In the proof of the first characterisation, the behaviour of the convergents of the continued fraction associated with the asymptotic series for f for argument values lying on the imaginary axis (where f itself may be undefined) is used to deduce that these convergents are generated by an algebraic form having a certain structure. The mapping properties of the convergents over L are then used to show that certain coefficients occurring in continued fraction formulae

are real and positive, and hence that the associated continued fraction itself is generated by an expression involving a Stieltjes transform, namely that given by formula (1). In the above characterisation it is deduced more directly from the mapping properties of f itself over \mathbb{L} that this function is generated by an expression of the form (1) as described. The asymptotic series generated by the function f of the above theorem still possesses an associated continued fraction whose convergents possess the mapping properties described in the first section. These properties, however, although important in applications, are now

relegated to an ancillary role, and no longer play an essential part in characterisation.

It is, of course, possible that a function f should have the mapping properties of a-c) without generating the asymptotic series of ~~formula~~(1); such a function has the form (1) with $0 < a < 0$ and $s \in B(-\infty, \infty)$ in formula (2) a function for which all moments (3) do not exist.

Upon the basis of the proof of the above theorem, it is possible to demonstrate the existence of functions with mapping properties less specific than those of F -functions. For example, when $0 < C < 1$, the function

By making use of those aspects of the theory of continued fractions involved in the first characterisation of \mathcal{F} , it is possible to derive a number of F-functions from a given F-function. The result that if $f \in \mathcal{F}$ and $0 < b < \infty$, then

$$g(z) = \frac{(1+bz)f(z)+1}{f(z)+1-bz}$$

is an IF-function is a simple example of such a derivation.

Further F-functions may be derived by use of the second characterisation of this section. The function

$$\tilde{S}(z) = z \int_0^{\infty} (1+z^2 t)^{-1} d\sigma(t)$$

where $\sigma \in \mathcal{B}(0, \infty)$ with $y = \sigma(\infty) - \sigma(0) > 0$, maps \mathbb{L} into itself and is real for real z ; furthermore $\lim_{z \rightarrow \infty} z^{-1} S(z) = y$ as $z \rightarrow 0 \in \Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$ (see clause () of Theorem of Appendix 1 below).

It follows from use of the second characterisation that if S is as just described and $f \in \mathcal{F}$, then $f \{ S(z) \}$ is also an F-function.

Upon the basis of the proof of the above theorem, it is possible to demonstrate the existence of functions with mapping properties less specific than those of F-functions. For example, when $0 < c < 1$, the function

$$(20) \quad f(z) = 1 + \frac{az}{1 - \frac{1}{2}Caz + z^2 s(z)}$$

with $0 < a < \infty$ and $s \in S$, maps L into a region containing \bar{D} in its interior except for the point $f(z) = 1$ which lies on the boundary of this region; when $1 < C < \infty$, the above function maps L into a region lying within D .

The auxiliary results derived in the proof of Theorem 1 may naturally be presented in terms of positive functions, i.e. functions w which map the finite part of the open right half-plane into itself; such a formulation may be convenient in other contexts (for a recent paper dealing with the use of positive functions in connection with

3. Multiplicative properties

The new characterisation established in the preceding section yields, as an immediate bonus, a simple proof that F-functions are closed with respect to multiplication. The proof of this multiplicative property may be approached from three directions. The first is by way of the original characterisation described in the first section. However, a simple relationship between the convergents of the continued fractions associated with ~~two power series~~
and their the product of two power series is unknown: the first approach is not viable. The second involves the representation of formulae (^{1,2}). This method of proof is relatively difficult but possible. Since it exhibits features of independent interest, this proof is given in Appendix 2. The third approach is by way of the characterisation of the preceding ~~preceding~~ section, and in this case the proof is trivial.

Theorem 2. The product of two F-functions is another F-function.
Proof. Each of the two functions possesses properties (a-d) of Theorem 1. The product of two functions possessing them also possesses them. Such a function is an F-function.

Meromorphic F-functions

The function

$$(1) \quad s(z) = L + \sum_{i=1}^m U_i (1+z^2 u_i)^{-1}$$

where $L \geq 0$, $0 < u_i$, $U_i < \infty$ ($i=1, \dots, m$) has a representation

of the form (2) with $\epsilon(u_i + \delta) - \epsilon(u_i - \delta) = U_i$ ($i=1, \dots, m$),

and having no other points of variation over $[0, \infty]$.

Subject to the condition that all moments $\sum_{i=1}^m U_i u_i^j$

($j=0, 1, \dots$) exist, functions of this form may be inserted

into formula (1), with $0 < a < \infty$, to define a sub-class

of F-functions.

Analytic continuation of a function f of this sub-class from L across the imaginary axis is, in particular, possible; the function thus obtained is identical with that defined for argument values with positive real part by the defining formula (1). When $m < \infty$, the F-functions concerned are rational. When $m = \infty$, two classes of F-functions in particular may be distinguished,

if the u_j form an unbounded increasing sequence, the resulting F-function is meromorphic in any open domain not containing the origin, and has an essential singularity at the origin; if the u_j form a decreasing sequence with limit zero, the F-function concerned is meromorphic in any finite region of the complex plane (it may even be entire as (see formula (5)) the example $a = \frac{1}{2}, u_j = (2j\pi)^{-2}, U_j = \frac{1}{2}u_j$ ($j=1, 2, \dots$) for which $f(z) = e^z$ shows). An F-function is of the second class described if and only if it may be represented for all finite $z \notin \pm i[u^{-\frac{1}{2}}, \infty]$ by means of a continued fraction whose convergents have the form (7) with $0 < a < \infty, 0 < a_j < \infty$ ($j=1, 2, \dots$) and $\lim a_j = 0$ as j tends to infinity. For example, the function $f(z) = {}_1F_1(\gamma+1; 2\gamma+1; z) / {}_1F_1(\gamma; 2\gamma+1; z)$ ($-\frac{1}{2} < \gamma < \infty$) is a member of this type of this class [24]. As will be seen below, the same is true of certain functions occurring in the theory of smoothing operations.

The product of a quotient of two suitable linear

functions and α function of the subclass being considered also belongs to this subclass; in the representation of the form () of the product F-function, the function α s is also of the form (). If the multiplicand F-function is rational, the rational function () occurring in the product F-function is of higher degree than that occurring ~~than that occurring~~ in the multiplicand function. If the constant U occurring in the the multiplicand F-function is zero, a nonzero constant U is created in the similar representation of the product F-function, but the numbers of terms in the sums in the two versions of formula () are the same for both multiplicand and product F-functions; if the constant U occurring in the multiplicand F-function is nonzero, it is promoted in the product F-function to a term of the form occurring in the sum in formula (): the sum related to the product F-function contains one more term than that related to the multiplicand F-function however, the corresponding constant U occurring in the

product F-function is reduced to zero. If a rational t-function is multiplied by a succession of suitable fractional linear functions, the constants corresponding to U in the successive product F-functions are alternately created and promoted. Subject to certain conditions, the numbers u_i occurring in formula () in the representation of the product F-function interlace those occurring in the representation of the multiplicand F-function. Remarks similar to those above may be made concerning the multiplication of meromorphic F-functions by a suitable quotient of two quadratic functions. These matters are now dealt with.

Theorem 5. Let the F-function f have the representation () where $0 < a < \infty$ and s is defined by formula () with

$$0 \leq U < \infty \text{ and } 0 < U_i, u_i < \infty \quad (i = 1, \dots, m)$$

A) Let $0 < A < \infty$, $g(z) = (1 - \frac{1}{2}Az)^{-1}(1 + \frac{1}{2}Az)$ and $h(z) = g(z)f(z)$.

i) h is also an F-function; it has a representation of the form () where a is to be replaced by Aa and s by

$$(5) \quad s_2(z) = V + \sum_{j=1}^n V(1+\xi v_j z^2)^{-1}$$

where $0 < v_j < \infty$ ($j=1, \dots, n$).

(ii) Let $m < \infty$ and $u_1 > u_2 > \dots > u_m$.

a) If $V > 0$, then $V \neq 0$, ~~and~~ $n = m+1$ and

$$(6) \quad v_j > u_j > v_{j+1}$$

for $j=1, \dots, m$.

b) If $V = 0$, then $V > 0$, $n = m$ and relationships (7)

hold for $j=1, \dots, m-1$.

(iii) Let $m = \infty$ and $u_1 > u_2 > \dots$. Then $n = \infty$ in formula

(7) and relationships () hold for $j=1, 2, \dots$

(iv) Let $m = \infty$ and $u_1 < u_2 < \dots$. Then $n = \infty$ in formula

(8) and

$$u_j < v_j < u_{j+1}$$

for $j=1, 2, \dots$

3) Let $0 < \alpha, \beta < \infty$, $P(z) = 1 + 2\alpha z + (\alpha^2 + \beta^2)z^2$, $g(z) =$

$P(z)/P(-z)$ and $h(z) = g(z)f(z)$.

i) The function h is also an F-function; it has a representation of the form (1) where α is to be replaced by $\alpha + 4\alpha$ and s by s_2 , s_2 being given by formula

(⁵⁶) with $m=m+1$ and $V=0$ if $U=0$, $V>0$ otherwise.

(ii) Let $m<\infty$ and $u_1 > u_2 > \dots > u_m$. Relationships (⁵⁷) hold for $j=1, \dots, m$.

(iii) Mutatis mutandis, the results of clauses (Aiii, iv) hold

Proof. The function g of part A) is of the form (') with $a=A$ and $s(z)=0$. Thus, from Theorem 2, h is also an F-function. By inspecting the coefficients of z in the asymptotic expansions of $h(z)$ and $g(z)f(z)$ it is easily established that a in formula () for h is as described in clause (i). The function s_2 replacing s in this formula has the representation

$$(\text{**}) \quad s_2(z) = \frac{B + \Theta s(z)}{1 + \phi z^2 s(z)}$$

where $B = \frac{1}{4}aA$, $\Theta = \frac{\phi}{4}(a+A)^{-1}$, $\phi = 1 - \Theta$. Under the conditions of clause (ii), s_2 is a rational function which, since it has a representation of the form (2), is also of the form (**). That V and n are as stated in clause (Aiiia) is easily verified by use of formula (**) .

s_z , regarded as a function of z^2 , has simple poles
 (i.e. at $z^2 = -v_j^{-1}$ ($j=1, \dots, n$)) where the denominator
 $1 + \phi z^2 s(z)$ in formula (57) has simple zeros. Subject to
 the ordering imposed in clause (Aii) upon the v_i ,
 the function $1 + \phi z^2 s(z)$ behaves as follows: when $z^2 > 0$,
 it is positive; as z^2 decreases from zero, a zero
 at $z^2 = -1/v_1$, where $-1/v_1 > -1/u_1$, is first encountered;
 a simple pole at $z^2 = -1/u_1$ is then
 encountered; a further zero at $z^2 = -1/v_2$ in the range
 $(-1/u_2, -1/u_1)$ is encountered, and so on. When z^2
 is slightly less than $-1/u_m$, the terms not involving
 U are positive; they decrease in magnitude as z^2
 decreases; if $U > 0$ (as is the case in clause (Aia))
 the term Uz^2 , which is negative, increases in
 magnitude, i.e. a last zero at $z^2 = -1/v_{m+1}$ where
 $-1/v_{m+1} < -1/u_m$ is also encountered. Relationships
 (57) as stated have been established. The further
 results of part (A) may be demonstrated in the

same way. Under the conditions of part (B), formula (55) is to be replaced by

$$S_2(z) = \frac{3 + (\alpha + \beta) \{ \phi + z^2 s(z) \} + \Theta s(z)}{1 + z^2 \{ \Theta(\alpha + \beta) + \phi s(z) \}}$$

but apart from this detail the proof is as before.

The primary purpose of the above theorem is to describe the structures of certain rational functions, and to establish inequalities concerning coefficients u_i and v_i . As a point of detail, it is remarked that subject to further conditions, relationships between U and V when $m = \infty$ in formula (56) can be derived: if $U > 0$, then $V = 0$; when $U = 0, V = 0$ when $\sum_{i=1}^{\infty} U_i u_i^{-2}$ diverges and $V > 0$ otherwise.

A recursive application of the above theorem is considered below with regard to functions occurring in the theory of smoothing operations.

The points at which an F-function and its associated approximating fractions assume equal values

The continued fraction associated with the exponential series has convergents of the form $(^7)$ with $a=1$ and $a_r = \{4(4r^2 - 1)\}^{-1}$ ($r=1, 2, \dots$). The equation

$a_1 = \{4(1^2 - 1)\}^{-1}$
a=1 and $v_r = \{4(2r-3)(2r-1)\}^{-1}$ ($r=2, 3, \dots$). The equation

$e^z = C_0(z)$, i.e. $e^z = 1$, holds at a system of points

$z = \pm iy(0; j)$, where $y(0; j) = 2j\pi$ ($j=1, 2, \dots$), upon the imaginary axis. $e^z = C_2(z)$ also for a system of

values of z lying upon the imaginary axis; replacing

z by iy , this equation becomes $\tan(\frac{1}{2}y) = \frac{1}{2}y$,

and is satisfied for $z = \pm iy(1; j)$ where $y(0; j) < y(1; j)$

$< y(0; j+1)$ ($j=1, 2, \dots$). $e^z = C_4(z)$, i.e. with $z = iy$,

$\tan(\frac{1}{2}y) = (1 - \frac{y^2}{12})^{-1}(\frac{1}{2}y)$ at a further system of points

$z = \pm iy(2; j)$ where $y(2; j) < y(1; j) < y(2; j+1)$ ($j=1, 2, \dots$)

and so on. Interlacing systems of points at which

$f(z) = C_{2r}(z)$ hold which separate each other upon the imaginary axis may be associated with a

class of functions F-functions.

Theorem 6. Let the function f have a representation of the form (1) with $0 < \alpha < \infty$ and

$$(59) \quad s(z) = M + \sum_{j=1}^k M_j (1+t_j z^2)^{-1}$$

where $M \geq 0$, $0 < M_j, t_j < \infty$ ($j=1, \dots, k$). Denote the successive convergents of the continued fraction associated with the asymptotic series for f by C_{2r} . The nonzero values of z for which

$$(60) \quad f(z) = C_{2r}(z)$$

(when this equation is not satisfied identically) lie on the imaginary axis; they are given by

$$(61) \quad z = \pm i y(r;j)$$

where $y(0;j) = t_j^{-\frac{1}{2}}$ ($j=1, \dots, k$) and the further

$y(r, j)$ are, in the special cases considered, as described in the following clauses.

(i) Let $k < \infty$ and $t_1 > t_2 > \dots > t_k$. Set $\gamma = 1$ if $M = 0$ and $\gamma = 1$ otherwise.

The systems of numbers $(^{61})$ are defined for $r = 0, \dots, 2k - 2\gamma$. The members of the systems are defined for $j = 1, \dots, k - n$ when $r = 2n$ is even ($n = 0, \dots, k - \gamma$) and for $j = 1, \dots, k - n - \gamma$ when $r = 2n + 1$ is odd ($n = 0, \dots, k - \gamma - 1$). They satisfy the relationships $y(2n; j) < y(2n+1; j) < y(2n; j+1)$ ($j = 1, \dots, k - n - 1$) with $y(2n, k - n) < y(2n+1, k - n)$ if $M > 0$, and $y(2n+1; j) < y(2n+2; j) < y(2n+1; j+1)$ ($j = 1, \dots, k - n - \gamma - 1$) with $y(2n+1, k - n - 1) < y(2n+2, k - n - 1)$ if $M = 0$. Equation $(^{60})$ is not satisfied for any nonzero value of z when $r = 2k - \gamma$, and it is satisfied identically when $r = 2k - \gamma + 2$.

(ii) Let $k = \infty$, $t_1 > t_2 > \dots$ and $\sum_{j=1}^{\infty} M_j < \infty$. The systems of numbers $(^{61})$ are defined for $r = 0, 1, \dots$, and $y(r, j) < y(r+1, j) < y(r, j+1)$ ($r = 0, 1, \dots$; $j = 1, 2, \dots$).

(iii) Let $k = \infty$, $t_1 < t_2 < \dots$ and $\sum_{j=1}^{\infty} M_j t_j^n < \infty$ ($n = 0, 1, \dots$).

The systems of numbers (61) are defined for
 $r=0,1,\dots$ and $y(r;j) > y(r+1;j) > y(r;j+1)$ ($r=0,1,\dots$;
 $j=1,2,\dots$).

Proof. The function (59) has a representation of the form (2) in which $\sigma(0+) - \epsilon(0) = M$ and $\sigma(t_j+) - \sigma(t_{j-1}) = M_j$ ($j=1,\dots,k$), σ having no other points of increase over the range $(0,\infty)$. Subject to the conditions of clause (iii) all moments (3) exist; in particular the first exists. Thus (see the remarks concerning formula (72) below) $s(z)$ may also be represented in the form

$$(62) \quad s(z) = a_1 \left\{ 1 + z^2 \sigma^{(2)}(z) \right\}^{-1}$$

\Rightarrow where (since $M=0$ in the case being treated) $a_1 =$

$\sum_{j=1}^k M_j$. As is easily verified, $s^{(2)}$ has a representation of the form

$$s^{(2)}(z) = M(2; 0) + \sum_{j=1}^{k-1} M(2; j) \{1 + z^2 t(2; j)\}^{-1}$$

where $M(2; 0) > 0$. From relationship (62), $s^{(2)}(z)$ regarded as a function of z^2 has poles (i.e. at $z^2 = -t_j^{-1}$ ($j = 1, \dots, k$)) where $1 + z^2 s^{(2)}(\frac{z}{\sqrt{}})$ has zeros. Let the $t(2; j)$ be ordered by the relationships $t(2; 1) > t(2; 2) > \dots > t(2; k-1)$. When $z^2 \geq 0$, $s^{(2)}(z) > 0$; i.e., $1 + z^2 s^{(2)}(z) \neq 0$ when $z^2 \geq 0$. As z^2 decreases through from zero through negative values and approaches $-t(2; 1)^{-1}$, $s^{(2)}(z)$ is positive and becomes arbitrarily large. Since z^2 is negative $1 + z^2 s^{(2)}(z)$ has a zero at $z^2 = -t_1^{-1}$ in the range $(-t(2; 1)^{-1}, 0)$.

This function has further zeros which separate its poles, i.e. $-t(2;j)^{-1} < -t_j^{-1} < -t(2;j-1)^{-1}$ ($j=2, \dots, k-1$) and one further zero: $-t_k^{-1} < t(2;k-1)^{-1}$. In turn $s^{(3)}(z)$ has a representation of the form

$$(63) \quad s^{(3)}(z) = a_2 \left[1 + z^2 \sum_{j=1}^{k-1} M(3;j) \{ 1 + t(3;j)z^2 \}^{-1} \right]^{-1}$$

and the further system of inequalities $-t(3;1)^{-1} < -t(2;1)^{-1} < 0$, $-t(3;j)^{-1} < -t(2;j)^{-1} < -t(3;j-1)^{-1}$ ($j=2, \dots, k-1$) may be derived. The sum in relationship (63) has the same form as $s(z)$ with k replaced by $k-1$, and may be treated as above.

It is clear from formula (1) that $f(z)=1$, or $f(z)=C_0(z)$, where $s(z)$ has poles, i.e. where $z^2 - t_j^{-1} = 0$ ($j=1, \dots, k$) or for $z=\pm iy(0;j)$

$(j=1, \dots, k)$ as defined in the theorem. From relationship

(62)

$$f(z) = 1 + \frac{az}{1 - \frac{1}{2}az + \frac{a_1 z^2}{1 + z^2 s^{(2)}(z)}}$$

$f(z) = 1 + az(1 - \frac{1}{2}az)^{-1}$, or $f(z) = C_2(z)$, where $s^{(2)}(z)$

has poles, i.e. where $z^2 = -t(2; j)^{-1}$ ($j=1, \dots, k-1$)

or for $z = \pm iy(1; j)$ ($j=1, \dots, k-1$) also as defined.

The further points at which $f(z) = C_{2r}(z)$ for $r > 1$

may be treated in the same way. The stated

inequalities concerning the $y(r; j)$ may be derived

from those obtained for the $t(r; j)$ using the

relationship $y(r; j) = t(r+1; j)^{-\frac{1}{2}}$.

The remaining results of the theorem may be derived in detail as above.

Systems of mutually separating points $y(r;j)$ such as those discussed in the above theorem may behave in a variety of ways. The results of that theorem say little concerning the location of the $y(r;j)$ for fixed r and large j . Nevertheless, results in this direction can be derived.

Theorem 7. Let f, s, t_j, C_{2r} and $y(r;j)$ be as described in the introduction to Theorem 6, so that
in particular $f(z) = C_{2r}(z)$ for $z = \pm iy(r;j)$ ($j=1,2,\dots$)
with $k=\infty$ in formula (59), so that each system
of numbers $y(r;j)$ ($j=1,2,\dots$) contains ^{ing} unboundedly many ~~numbers~~ members.

(i) Let $t_1 > t_2 > \dots$ so that $y(r;j) < y(r;j+1)$ ($r=0,$

$1, \dots ; j=1,2,\dots$). Then, for a fixed $r > 0$, at least one member of the sequence $\{y(r,j)\}$ lies in the interval $[y(0;m), y(0;m+1)]$ ($m \geq 0$) and there exists a positive integer J such that $y(0;j) < y(r;n+j) < y(0;n+j)$ for $j = J, J+1, \dots, n \geq 0$ being fixed.

ii) Let $t_1 < t_2 < \dots$, so that $y(r;j) > y(r;j+1)$ ($r = 0, 1, \dots ; j = 1, 2, \dots$). With the stated interval and inequalities replaced by $[y(0;m+1), y(0;m)]$ and $y(0;j+1) < y(r;n+j) < y(0;j)$ respectively, the results of the preceding clause hold.

Proof. The relationship $f(z) = C_{2r}(z)$ holds for nonzero values of z when and only when $s(z) = C'_{2r-2}(z)$.
Equation (60)

where

$$s(z) = M' + \sum_{j=1}^{\infty} M_j (1+t_j z^2)^{-1}$$

and $C_2'(z) = \alpha_1$,

$$\Rightarrow C_{2r}'(z) = \frac{\alpha_1}{1} \frac{\alpha_2 z^2}{1} \dots \frac{\alpha_r z^2}{1}$$

for $r=2, 3, \dots$. Under the conditions of clause (i), $s(z) > 0$

for all $z^2 > -t_1^{-1}$, but in each of the ranges $[-t_{j+1}^{-1},$

$-t_j^{-1}]$ varies monotonically from ∞ to $-\infty$,

attaining these values at the respective endpoints.

Thus $C_{2r-2}'(z) = s(z)$ for at least one value of z^2

in each of these intervals. For large $-z^2$, $C_{2r-2}'(z)$

remains finite and varies monotonically over, in

particular, successive intervals of the form just

described ($C_{4r-2}'(z) \sim k_r^{1/2}$ and $C_{4r}'(z) \sim k_r^{1/2} z^{-2}$ for

$|z^2|$, k_r and k'_r being positive constants). There is
 only one value of $-z^2$ lying in the corresponding
 open interval, for which $C'_{2r-2}(z) = S(z)$. The above
 results may be presented in terms of values not
 of $-z^2$ but of z ; they are as formulated in clause
 (i). The results of clause (ii) are obtained in the same
 way.

Subject to the conditions of clause (ii), the t_j
 have a limit point β^{-2} , where $0 \leq \beta < \infty$. The
 $y(0, j)$, forming a decreasing sequence, then have
 β as a limit point, and for fixed $\delta > 0$, all
 $y(r, j)$ with sufficiently large j lie in the interval
 $[\beta, \beta + \delta]$. Since δ may be taken arbitrarily small,

the limiting values of the $y(r;j)$ may be estimated with precision. Under the conditions of clause (i), the gaps between the $y(0;j)$ may be arbitrarily large, and no such precise estimation of location is possible. Nevertheless, in certain cases precise location in the latter case is possible.

Theorem 8. Let the F-function f have the representation (1) with, in particular, s defined by formula (5) with $k=\infty$, $M \geq 0$, $0 < M_j, t_j < \infty$ ($j=1, 2, \dots$); let $f(iy) \sim e^{i\beta(y+\gamma)}$ where $0 < \beta < \infty$, $0 \leq \gamma < 2\pi$, as $|y|$ tends to infinity. Denote the successive convergents of the continued fraction associated with the ascending power series for $f(z)$ by $C_{2r}(z)$. Let

$\delta > 0$ be arbitrarily small and fixed.

There exists a positive integer J , dependent only upon δ and r , such that in each of the intervals $(\phi^{-1}(2j\pi - \frac{\pi}{r}) - \delta, \phi^{-1}(2j\pi - \frac{\pi}{r}) + \delta)$ ($\pm j = J, J+1, \dots$) only one value of y can be found for which $f(iy) = C_{4r}(iy)$, and in each of the intervals $(\phi^{-1}((2j-1)\pi - \frac{\pi}{r}) - \delta, \phi^{-1}((2j-1)\pi - \frac{\pi}{r}) + \delta)$ ($\pm j = J, J+1, \dots$) only one value of y can be found for which $f(iy) = C_{4r+2}(iy)$; apart from these values there are no others ~~left~~ for which $f(z) = C_{4r}(z)$ and $f(z) = C_{4r+2}(z)$ ($|z| > \phi^{-1}((2J-1)\pi - \frac{\pi}{r}) - \delta$) respectively.

Proof. The convergents $C_{2r}(z)$ in question have the form $\phi_r(z) / \phi_r(-z)$, where $\phi_r(z)$ is an r^{th}

degree polynomial in z with nonzero coefficient of z^r . As $|z|$ tends to infinity $C_r(z) \sim 1$ and $C_{4r+2}(z) \sim -1$.

Hardy [8,9] has investigated the values of z for which $\phi(z) \exp\{G(z)\} = \psi(z)$, ϕ, G and ψ being polynomials. A considerably simplified version of his analysis leads to the result that if $C(z) \sim 1$ for large $|z|$ and $\delta > 0$ is fixed and arbitrarily small, then there exists a positive integer J depending upon δ such that all roots of the equation $e^{iz} = C(z)$ with $|z| > 2J\pi - \delta$ lie on discs of the form $|z - 2j\pi| \leq \delta$ ($\pm j = J, J+1, \dots$) and only one root lies on each disc.

By use of a linear transformation of z , this result may be extended to the equation $e^{i(\delta z + \beta)} = C(z)$. The extended form may be applied to equations of the form $f(iz) = C_{2r}(iz)$ as considered above. In such cases, however, it is known that those values of y for which such an equation are satisfied are real: the above discs may be replaced by diameters, i.e. by segments of the imaginary axis as described in the theorem.

For simplicity in exposition the results of this section have been presented in terms of F -functions that are meromorphic over a bounded domain or over a domain bounded away from the origin. Nevertheless, the result of clause (ii) of Theorem , for example, may slightly be extended by taking s to be a sum of two functions: the first has the form $(^2)$ with $\sigma(t) = \sigma(0)$ for $0 < t < \alpha$, all moments $(^3)$ existing, with $\alpha > t_1$; the second

has the form () as described. For $\alpha^{\frac{1}{2}} < |z| < \infty$,
 $f(z)$ is meromorphic, as is true for the unextended
version of clause (ii) of Theorem 6, and the result of
that clause still holds. Clause (i) of Theorem ⁷
and ^{also} Theorem 8 may be extended in the same way.

Variation diminishing functions

The transformation of a sequence of real numbers x_i ($i=0, 1, \dots$) by use of a relationship of the form

$$(64) \quad y_j = \sum_{i=0}^n a_{m-i} x_i$$

for $j=0, 1, \dots$ is said to be variation diminishing if the number of changes of sign of the y_j is less than or equal to the number of changes of sign of the x_i . Transformations of this type underly the theory of many smoothing operations, and also occur in the numerical solution of certain partial differential equations by iterative methods (see [19] for a historical account and numerous references).

Successive studies by Polya, Schoenberg and Edrei produced the result that the transformation (64) with $a_0 = 1$ is variation diminishing if and only if, for sufficiently small z ,

$$(65) \quad \sum_{k=0}^{\infty} a_k z^k = e^{\gamma z} \left\{ \prod_{i=1}^m (1 + \alpha_i z) \right\} \left\{ \prod_{j=1}^n (1 - \beta_j z) \right\}^{-1}$$

with $\gamma \geq 0$; $\alpha_i > 0$ ($i=1, \dots, m$); $\beta_j > 0$ ($j=1, \dots, n$);

$$0 \leq m \neq n \leq \infty \text{ and } \gamma + \sum_{i=1}^m \alpha_i + \sum_{j=1}^n \beta_j < \infty.$$

It has already been shown [25] that functions
of the form (65) have a representation of the form
 \Leftrightarrow if and only if $m=n$, $\alpha_i = \beta_i$ ($i=1, \dots, n$).

The original demonstration involved properties of
the convergents of the continued fractions associated
with series of the form (65), but now becomes
a trivial consequence of the result of Theorem 1
(the function upon the right hand side of
formula (65) satisfies requirements (a-d)^a if
and only if the above conditions are satisfied).

It is clear that in the representation of
the form (1) of the function

$$(66) \quad E_m(\gamma; z) = e^{\gamma z} \prod_{i=1}^m \{(1+\alpha_i)(1-\alpha_i z)^{-1}\}$$

$\alpha = \gamma + 2 \sum_{i=1}^m \alpha_i$ and, since this function is meromorphic throughout the finite part of the complex plane, that $s(z)$ is a meromorphic function of the form

$$(67) \quad s_k(z) = \sum_{j=1}^k M_j (1+z^2 t_j)^{-1}$$

where $M_j > 0$, $0 < t_j < \infty$ ($j=1, \dots, k$) the t_j being distinct, and $\sum_{j=1}^k M_j < \infty$. If either $\gamma > 0$ or $n = \infty$, then $k = \infty$ in formula (67).

It should perhaps be remarked that not every rational F-function may be expressed in the form $E_m(0, z)$; for example, such a function with denominator $1 - \frac{1}{2}az + Mz^2$ ($0 < a, M < \infty$) and

$16M^2/a^2$ is not of the form $E_2(0; z)$. Similarly, not every function of the form involving a sum $S_{\infty}(z)$ may be expressed in the form $E_m(\gamma; z)$ where $m=\infty$ or $\gamma>0$.

For a fixed value of γ , the infinite sequence $\alpha_i > 0$ ($i=1, 2, \dots$) determines a corresponding sequence of functions $E_m(\gamma; z)$; when $\gamma=0$, these functions are rational, and otherwise but otherwise have more general meromorphic structure. Successive functions of this sequence have related structures and further properties which may be established upon the basis of previously derived theory.

Theorem 9. Let $0 \leq \gamma < \infty$, $0 < \alpha_i < \infty$ ($i=1, 2, \dots$) and

define $E_m(\gamma, z)$ by means of formula (66).

A). Let $\gamma=0$

i) The function $E_m(\gamma; z)$ has a representation of

✓ the form ⁽¹⁾ in which, in particular, s is to be

replaced by s_m , where for $k=1, 2, \dots$

$$s_{2k}(z) = M(2k, 0) + \sum_{j=1}^{k-1} M(2k, j) \{1 + t(2k, j)z^2\}^{-1}$$

$$s_{2k+1}(z) = \sum_{j=1}^k M(2k+1, j) \{1 + t(2k+1, j)z^2\}^{-1}$$

all numbers represented by means of the symbols

M and t being finite positive real numbers.

Furthermore, the inequalities

$$t(2k+1, j) > t(2k, j) > t(2k+1, j+1) \quad (j=1, \dots, k-1)$$

$$t(2k+2, j) > t(2k+1, j) > t(2k+2, j+1) \quad (j=1, \dots, k-1)$$

$$t(2k+2, k) > t(2k+1, k)$$

hold for $k=2, 3, \dots$.

(ii) Denote the successive convergents of the continued fraction associated with the series expansion of the function $E_m(0; z)$ in ascending powers of z by $C_{m,2r}(z)$. For $k=0,1,\dots; \chi=0,1$ the equation $E_{2k+\chi}(0; z) = C_{2k+\chi,2r}(z)$ is satisfied at points $z = \pm y(2k+\chi, r; j)$ defined for $r=0,\dots,2k-2$ with $j=1,\dots,k-n+\chi-1$ when $r=2n$ is even ($n=0,\dots,k-1$) and with $j=1,\dots,k-n-1$ when $r=2n+1$ is odd ($n=0,\dots,k-2$). These numbers satisfy the inequalities

$$y(2k+\chi, 2n; j) < y(2k+\chi, 2n+1; j) < y(2k+\chi, 2n; j+1)$$

for $j=1,\dots,k-n+\chi-2$ with $y(2k, 2n; k-n-1) < y(2k, 2n+1; k-n-1)$ if $\chi=0$ and

$$y(2k+\chi, 2n+1; j) < y(2k+\chi, 2n+2; j) < y(2k+\chi, 2n+1; j+1)$$

for $j=1,\dots,k-n-2$ with $y(2k+1, 2n+1; k-n-1) < y(2k+1, 2n+2; k-n-1)$

if $\chi=1$. The equation $E_{2k+\chi}(0; z) = C_{2k+\chi,2r}(z)$ is not satisfied for any nonzero value of z when $r=2k+\chi-2$ or $r=2k+\chi-1$; it is satisfied identically when $r=2k+\chi$.

3) Let $\gamma > 0$

i). The function $E_m(\gamma, z)$ has a representation of the form (1) in which s is to be replaced by

$$s_m(z) = \sum_{j=1}^{\infty} M(m, j) \{1 + z^2 t(m, j)\}^{-1}$$

with $0 < M(m, j), t(m, j) < \infty$ ($j = 1, 2, \dots$). In particular

$$M(0, j) = \frac{1}{2}(j\pi\gamma)^{-2}, \quad t(0, j) = (2j\pi\gamma)^{-2} \quad (j = 1, 2, \dots) \text{ and}$$

$$t(m+1, j) > t(m, j) > t(m, j+1)$$

for $m = 0, 1, \dots ; j = 1, 2, \dots$.

ii) Let the numbers $y(m, r; j)$ be as described in clause (Ia). They are defined and satisfy the inequalities $y(m, r; j) < y(m, r+1; j) < y(m, r; j+1)$ for $r = 0, 1, \dots ; j = 1, 2, \dots$. For fixed m and r , at least one $y(m, r; j)$ lies in the interval $[y(m, 0; k), y(m, 0; k+1)]$ ($k = 1, 2, \dots$).

b) Let $\delta > 0$ be fixed. An integer J , depending only upon r and δ , exists such that in each of the intervals $(2j\pi\gamma^{-1} - \delta, 2j\pi\gamma^{-1} + \delta)$ only one number pair y, y' can be found for which $E_{2k}(\gamma, iy) = C_{4r}(iy)$, $E_{2k+1}(\gamma, iy') = C_{4r+2}(iy')$ and in each of the intervals $((2j-1)\pi\gamma^{-1} - \delta, (2j-1)\pi\gamma^{-1} + \delta)$ only one further pair v, v' can be found for which $E_{2k}(\gamma, iv) = C_{4r+2}(iv)$, $E_{2k+1}(\gamma, iv') = C_{4r}(iv')$ ($\pm j = \{J, J+1, \dots\}$). Apart from the values, there are no others of z outside the circle $|z| = (2J-1)\pi\gamma^{-1} - \delta$ for which the equation $E_{2k}(\gamma, z) = C_{4r}(z)$, and others derived from those given above, can hold.

Proof. The results of clauses A_i) and B_i) are

simple corollaries to Theorem 5; those of clause (Aii) and that of the first part of clause (Bia) follow from Theorem 6; the second part of clause (Bia) follows from Theorem 7; clause (Bib) results from a simple application of Theorem 8.

Appendix

At various points in the main body of the above text, alternative use might have been made of results concerning the derivation of one \mathcal{S} -function from another. These results may be subsumed within a more general theory of the derivation of functions of the form

$$(68) \quad F(\lambda) = \int_{-\infty}^{\infty} (\lambda - t)^{-1} ds(t)$$

where $s \in \mathbb{R}(-\infty, \infty)$, from others of the same form. This

domain the convergents $C_{4r}(z)$ and $C_{4r+2}(z)$ ($r=0, 1, \dots$) converge uniformly to two differing F-functions with a common value of α in formula (1), but two extremal solutions of the moment problem replacing σ in formula (2).

5. Interpolation

All F-functions are analytic over L ; some cease to be analytic at the origin. Nevertheless, every F-function has a limiting value as its argument tends to zero through, in particular, negative real values, and the same is true of all derivatives of F-functions; in terms of the asymptotic series (4) for an F-function $f(z)$,

$$\lim \frac{d^D f(z)}{dz^D} = z! \phi_D$$

as z tends to zero as described, for $D=0, 1, \dots$. The convergent $C_{2p}(z)$ of the continued fraction associated with this asymptotic series is derived from the coefficients with subscripts up to and including that of order $2p$ of the series; i.e. from the corresponding suitably defined derivatives of the generating F -function at the origin. This convergent is an approximating function for its generating function in the sense that it also maps L into D and has derivatives at the origin equal in value to those of the generating function up to and including that of order $2p$.

It is possible to construct a rational F-function whose derivative values agree with those of a generating F-function up to prescribed orders not only at the origin but at a prescribed sequence of points in L; furthermore, this rational function may be derived by the use of purely algebraic methods of rational function interpolation.

Theorem. Let f be an F-function and $p > 0, T \geq 0$ be two finite integers. Let z_j ($j=1, \dots, T$) be a prescribed set of points in L and $T(j) \geq 1$ ($j=1, \dots, T$) be a prescribed sequence of integers. A) A rational F-function g exists such that

$$2^z g(z) = 2^z f(z)$$

$\alpha)$ as $z \rightarrow 0 \in \Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$ for $z = 0, \dots, 2p$ and also

$\beta)$ when $z = z_j$ ($z = 0, \dots, T(j)-1; j=1, \dots, T$).

B) Let the argument values $\{z_j\}$ be specified in further detail as follows: they are distinct; they are members of two subsets, the first of which contains ~~the~~

✓ elements z_j with $\operatorname{Im}(z_j) \neq 0$ ($j=1, \dots, q$) and no complex conjugate pair, the second containing elements for which $\operatorname{Im}(z_j) = 0$ ($j=q+1, \dots, J$) (either set may be empty, i.e. 0 and J are possible values of q).
Set $n = p + 2 \sum_{j=1}^q T(j) + \sum_{j=q+1}^J T(j)$.

- (i) a) If f is the irreducible quotient of two polynomials of degree $n' < n$, set $g(z) = f(z)$.
- b) Otherwise, an F-function g , which may be expressed as the quotient of two polynomials of degree n , exists such that conditions $(A\alpha, \beta)$ hold.
- (ii) In both of the preceding cases (ia, b), g is the rational F-function of lowest degree to satisfy conditions $(A\alpha, \beta)$.
- (iii) The function g above also satisfies relationship () at the complex conjugate points $z = \bar{z}_j$, with $z = 0, \dots, T(j)-1$ for $j=1, \dots, q$.
- c) Let the conditions of part B) obtain, with f not an irreducible quotient of two polynomials of degree $n' < n$; let $(^4)$ be the asymptotic representation

f as $z \rightarrow 0 \in \Delta\left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$.

(i) Let function and derivative values of f at points $-zeR$, where required, be obtained by use and differentiation of the equation $f(z)f(-z)=1$. Let C be the algebraic interpolation problem of determining the quotient g of two n^{th} degree polynomials, whose derivatives $\Delta^q g(z)$ assume prescribed numerical values at the following points: at $z=0$ for $z=0, \dots, 2p$; at $z=z_j, \bar{z}_j, -z_j, -\bar{z}_j$ in each case with $z=0, \dots, T(j)$ for $j=1, \dots, q$; at $z=z_j, -z_j$ in both cases with $z=0, \dots, T(j)-1$ for $j=q+1, \dots, T$; let the successive derivative values at the origin be $z! f_z$ ($z=0, \dots, 2p$) while at the remaining points of the above set, let the prescribed numerical values be those of the function \tilde{f} .

The interpolation problem C has a unique solution if it is the rational function of referred to in clause (ii) above.

(ii) The rational function g of the preceding clause

has the form $g(z) = d(z)/d(z)$ where

$$d(z) = \sum_{j=0}^n d_j z^j$$

The coefficients $\{d_j\}$ are unique solutions to the following sets of linear algebraic equations with real coefficients:

$$\sum_{j=0}^2 f_{2z-2j+1} d_{2j} - \sum_{j=0}^{2-1} f_{2z-2j} d_{2j+1} - 2d_{2z+1} = 0$$

for $z=0, \dots, p-1$; with $n=2k+\gamma$ where $\gamma=0$ or 1

$$\sum_{j=0}^k [Re, Im \{ \Delta^2 (1-f(z_j)) z_j^{2j} \}] a_{2j}$$
$$+ \sum_{j=0}^{k+\gamma-1} [Re, Im \{ \Delta^2 (1+f(z_j)) z_j^{2j+1} \}] a_{2j+1} = 0$$

with $z=0, \dots, T(j)-1$ for $j=1, \dots, q$ (these equations are to be interpreted as two sets involving the symbols Re, Im alone) and a further set of two equations Re and Im from () by removal of the symbols Re and Im again with $z=0, \dots, T(j)-1$ but for $j=q+1, \dots$

lastly the single equation $d_0 = 1$.

(iii) The rational function g of the preceding

clauses may also be constructed by means of the following algebraic process which involves five principal stages:

0) The constant a in the representation () of

f and the coefficients s_k ($k=0, \dots, p-2$) in the asymptotic representation () of the associated

S -function s are obtained from f_{2j+1} ($j=0, \dots, p-1$)

by use of the formulae $a=f_1$ and, if $p>1$,

$$s_k = \left\{ \frac{1}{2} f_1 f_{2k+2} - f_{2k+3} - \sum_{j=0}^{k-1} s_j f_{2k-2j+1} \right\} / f_1$$

for $k=1, \dots, p-1$

1) If $p>1$ the coefficients a_k ($k=1, \dots, p-1$) in the associated continued fraction () and those

$\{b_{p+\gamma-2,j}\}, \{\pi_{p+\gamma-2,j}\}$ in the numerator and

denominator polynomials of the convergents

$$\frac{b_{p+\gamma-2}(z)}{\pi_{p+\gamma-2}(z)} = \frac{a_1}{1+} \frac{a_2 z^2}{1+} \frac{a_{p+\gamma-2} z^2}{1+}$$

for $\gamma=0, 1$ are determined by use of the following

orthogonalisation process which concerns sequences of coefficients $\bar{u}_{2r,d}$ ($d=0, \dots, r$), $\bar{b}_{2r,d}$ ($d=0, \dots, r-1$), $\bar{u}_{2r+1,d}$ ($d=0, \dots, r$), $\bar{b}_{2r+1,d}$ ($d=0, \dots, r$) and scalar products

$$\underline{\Phi}_{2r-1} = \sum_{d=0}^{r-1} \bar{u}_{2r-2,d} s_{2r-d-2}, \underline{\Phi}_{2r} = \sum_{d=0}^{r-1} \bar{u}_{2r-1,d} s_{2r-d-1}$$

derived from them. Initially

$$\bar{u}_{0,0} = \bar{u}_{1,0} = 1, \bar{b}_{1,0} = s_0$$

and then, for $r=1, 2, \dots$

$$a_{2r} = -\underline{\Phi}_{2r} / \underline{\Phi}_{2r-1}, \bar{u}_{2r,0} = 1, \bar{u}_{2r,r} = u_{2r} \bar{u}_{2r-2,r-1}, \bar{b}_{2r,0} = s_0$$

$$\bar{u}_{2r,d} = \bar{u}_{2r-1,d} + a_{2r} \bar{u}_{2r-2,d-1}, \bar{b}_{2r,d} = b_{2r-1,d} + a_{2r} b_{2r-2,d-1}$$

for $d=1, \dots, r-1$, and

$$a_{2r+1} = -\underline{\Phi}_{2r+1} / \underline{\Phi}_{2r}, \bar{u}_{2r+1,0} = 1, \bar{b}_{2r+1,0} = s_0, \bar{b}_{2r+1,r} = a_{2r+1} \bar{b}_{2r,r}$$

$$(*) \bar{u}_{2r+1,d} = \bar{u}_{2r,d} + a_{2r} \bar{u}_{2r-1,d}, \bar{b}_{2r+1,d} = b_{2r,d} + a_{2r} b_{2r-1,d}$$

the first of relationships (*) for $d=1, \dots, r$ and the second for $d=1, \dots, r-1$, the above process being terminated with the formation of the coefficients

$\{\bar{u}_{2r,d}\}$, $\{\bar{b}_{2r,d}\}$ if $p=2r+1$ is odd, and of $\{\bar{u}_{2r+1,d}\}$

$\{\bar{b}_{2r+1,d}\}$ if $p=2r+2$ is even.

2a) Function and derivative values of $s^{(1)}(z) = z s(z)$, corresponding to those prescribed for f , where s is the s -function occurring in formula (), are derived by use of the formulae

$$\Delta^z s^{(1)}(z) = (1 - f(z_j))^{-1} \left[z! z^{-z-1} \left\{ \sum_{\nu=0}^z z_j^\nu \Delta^\nu f(z_j) / \nu! \right\} - \right]$$

$$- \frac{1}{2} a \Delta^z f(z_j) - \frac{1}{2} a \delta_{0,z} - \sum_{\nu=0}^z \binom{z}{\nu} \Delta^\nu f(z_j) \Delta^{z-\nu} s^{(1)}(z_j)$$

for $j=1, \dots, J; z=0, \dots, T(j)-1$ (here and in the following subclause $\delta_{\mu,\infty}=1$ if $\mu=\infty$ and $\delta_{\mu,\infty}=0$ otherwise).

b) If $p > 1$, function and derivative values of $s^{(p)}(z)$ corresponding to those obtained above are derived by use of the formula

$$\Delta^z s^{(k+1)}(z_j) = \{z_j s^{(k)}(z_j)\}^{-1} \left\{ a_k (\delta_{1,z} + \delta_{0,z} z_j) - \Delta^z s_k(z_j) \right.$$

$$\left. - z_j \sum_{\nu=0}^{z-1} \binom{z}{\nu} \Delta^\nu s^{(k+1)}(z_j) \Delta^{z-\nu} s^{(k)}(z_j) \right.$$

$$\left. - \sum_{\nu=0}^{z-1} \binom{z-1}{\nu} \Delta^\nu s^{(k+1)}(z_j) \Delta^{z-1-\nu} s^{(k)}(z_j) \right\}$$

with $k=1, \dots, p-1$ for $j=1, \dots, J; z=0, \dots, T(j)-1$.

3a) Function and derivative values of $h(\lambda) = s^{(p)}(z)$ with respect to λ , where $\lambda = -iz^{-1}$, corresponding to the function and derivative values of $s^{(p)}$ with respect to z obtained above, are derived as follows: with $T' = z$ obtained above, are derived as follows: with $T' = \max\{T(j)\} - 1$ for $1 \leq j \leq T$, numbers $y_{j'}^{(z)}$ for $z = 1, \dots, T'$ and $j = 0, \dots, z-1$ one derived from the initial values $y_0^{(1)} = -i$ by means of the recursion

$$y_0^{(z)} = -iz y_0^{(z-1)}, \quad y_{z-1}^{(z)} = -i y_{z-2}^{(z-1)}$$

$$y_{j'}^{(z)} = -i \left\{ (z+j) y_{j'}^{(z-1)} + y_{j'-1}^{(z-1)} \right\} \quad (j=1, \dots, z-2)$$

for $z = 2, \dots, T$. Then, with $\lambda_j = -iz_j^{-1}$, $h(\lambda_j) = s^{(p)}(z_j)$ and

$$\sum_{j=0}^{T-1} h(\lambda_j) = \sum_{j=0}^{T-1} y_{j'}^{(z)} z_j^{z+2+1} \sum_{j=0}^{z+1} s^{(p)}(z_j)$$

for $j = 1, \dots, T; z = 1, \dots, T(j) - 1$ are determined.

3b) The coefficients $P_{0,R,j}^{(0)}$ ($j=0, \dots, N$) and $P_{1,R,j}^{(0)}$ ($j=0, \dots, N+k-1$) where $2 \sum_{j=1}^q T(j) + \sum_{j=q+1}^T T(j) = 2N+k$ (N an integer, $k=0$ or 1) in the denominator and numerator polynomials of the first extremal solution of the symmetric Pick-Nevanlinna problem deriving from

the argument and derivative value pairs $\{\lambda_j, 2^e h(\lambda_j)\}$ obtained above are derived by means of the following process which involves sequences of coefficients $\{P_{\mu, r, \nu}^{(\omega)}\}$ ($\mu, \omega = 0, 1$) of lengths later to be specified, and values of derivatives of ~~residual~~ residual functions defined by the formula

$$\rho(x, j, z, \omega, r, l, \varepsilon) = \\ \frac{x!}{z!} \Lambda_j^{x+\varepsilon} \sum_{\nu=0}^l P_{0, r-1, \nu}^{(\omega)} \Lambda_j^{z\nu} \sum_{k=0}^{\min(z, x+2\omega+\varepsilon)} \binom{x+2\omega+\varepsilon}{k} \frac{2^{z-k} h(\lambda_j) \Lambda_j^{-h}}{(z-k)!} \\ - \Lambda_j^{-\varepsilon-z+1} \sum_{D=\max(0, \left[\frac{1}{2}(z+\varepsilon-1)\right])}^{l+\varepsilon+\omega-1} P_{1, r-1, D}^{(\omega)} \frac{(2\omega-\varepsilon+1)!}{(2\omega-\varepsilon-z+1)!} \Lambda_j^{zD}$$

During execution of the process, the argument value suffix j , the derivative order z and the polynomial order r are related by the equation $r = \sum_{k=1}^{j-1} T(k) + z + 1$; as j and z increase through the ranges $j = 1, \dots, J$; $z = 0, \dots, T(j)-1$, r increases by steps of unity from 1 to $R = \sum_{j=1}^J T(j)$.

Initially $P_{0,0,0}^{(0)} = P_{1,0,0}^{(1)} = 1$, $P_{1,0,0}^{(0)} = P_{0,0,0}^{(1)} = 0$.

The argument values λ_j with nonzero imaginary parts are attended to by setting, for $j=1, \dots, q; z=0, \dots, T(j)-1$

$$\rho_x^{(\omega)} = \rho(x, j, z, \omega, r, r-\omega-1, \omega)$$

for $\omega=0, 1; x=0, 1, 2$ and, using the auxiliary variable

$$a_1 = \text{Im}(\bar{\rho}_2^{(0)} \rho_1^{(1)}), a_2 = \text{Im}(\bar{\rho}_0^{(0)} \rho_2^{(0)}), a_3 = \text{Im}(\bar{\rho}_1^{(0)} \rho_0^{(1)})$$

$$b_1 = \text{Im}(\bar{\rho}_0^{(0)} \rho_1^{(1)}), b_2 = \text{Im}(\bar{\rho}_2^{(1)} \rho_0^{(0)})$$

determining the constants

$$\alpha_r = a_1/b_1, \beta_r = a_2/b_1, \gamma_r = b_2/(a_3 - b_2)$$

and thereafter the coefficients

$$P_{0,r,0}^{(0)} = -\alpha_r P_{0,r-1,0}^{(0)}, P_{0,r,r}^{(0)} = 1$$

$$P_{1,r,0}^{(0)} = -\alpha_r P_{1,r-1,0}^{(0)} - \beta_r P_{1,r-1,0}^{(1)}, P_{1,r,r-1}^{(0)} = P_{1,r-1,r-2}^{(0)} - \beta_r P_{1,r-1,r}^{(1)}$$

$$P_{0,r,0}^{(1)} = \gamma_r (P_{0,r-1,0}^{(1)} - P_{0,r-1,0}^{(0)}), P_{0,r,r-1}^{(1)} = 1$$

$$P_{1,r,0}^{(1)} = \gamma_r P_{1,r-1,0}^{(1)}, P_{1,r,r}^{(1)} = (1 + \gamma_r) P_{1,r-1,r-1}^{(1)} - \gamma_r P_{1,r-1,r-1}^{(0)}$$

and, if $r > 1$,

$$P_{0,r,D}^{(0)} = P_{0,r-1,D-1}^{(0)} - \alpha_r P_{0,r-1,D}^{(0)} - \beta_r P_{0,r-1,D-1}^{(1)}$$

$$P_{1,r,D}^{(1)} = (1+\gamma_r) P_{1,r-1,D-1}^{(1)} + \gamma_r (P_{1,r-1,D}^{(1)} - P_{1,r-1,D-1}^{(0)})$$

for $D=1, \dots, r-1$ and, if $r > 2$

$$P_{1,r,D}^{(0)} = P_{1,r-1,D-1}^{(0)} - \alpha_r P_{1,r-1,D}^{(0)} - \beta_r P_{1,r-1,D}^{(1)}$$

$$P_{0,r,D}^{(1)} = (1+\gamma_r) P_{0,r-1,D-1}^{(1)} + \gamma_r (P_{0,r-1,D}^{(1)} - P_{0,r-1,D}^{(0)})$$

for $D=1, \dots, r-2$

Subsequently, the pure real argument values λ_j^q
 are dealt with by setting $K = \sum_{j=1}^q T(j)$, $k = \left\lceil \frac{1}{2}(r-K) \right\rceil$
 $K' = \left[\frac{1}{2}(r-K-1) \right]$ and, using the auxiliary variables,

$$\rho_{xy}^{(0)} = \rho(x, j, 0, r, K+k, k-k')$$

$$\rho_{xy}^{(1)} = \rho(x, j, r, 1, r, K+k-1, 1+r-k)$$

both for $x=0, 1$ determining the constants

$$\Theta_r = \rho_1^{(0)} / \rho_0^{(1)}, \quad \phi_r = \rho_0^{(0)} / (\rho_0^{(0)} - \rho_1^{(1)})$$

and thereafter, if $r-K+2k$ the coefficients

$$P_{0,r,K+k-1}^{(0)}, \quad P_{1,r,0}^{(0)} = -\Theta_r P_{1,r-1,0}^{(1)}$$

$$P_{0,r,0}^{(1)} = (1-\phi_r) P_{0,r-1,0}^{(0)}, \quad P_{1,r,K+k}^{(1)} = \phi_r P_{1,r-1,K+k}^{(1)}$$

and, if $K+k > 0$,

$$P_{0,r,D}^{(0)} = P_{0,r-1,D}^{(0)} - \Theta_r P_{0,r-1,D}^{(1)}$$

$$P_{1,r,D}^{(1)} = (1-\phi_r) P_{1,r-1,D}^{(0)} + \phi_r P_{1,r-1,D}^{(1)}$$

both for $D=0, \dots, K+k-1$ and

$$P_{1,r,D}^{(0)} = P_{1,r-1,D-1}^{(0)} - \Theta_r P_{1,r-1,D}^{(1)}$$

$$P_{0,r,D}^{(1)} = (1-\phi_r) P_{0,r-1,D}^{(0)} + \phi_r P_{0,r-1,D-1}^{(1)}$$

both for $D=1, \dots, K+k$, while if $r-K=2k$

$$P_{0,r,0}^{(0)} = -\Theta_r P_{0,r-1,0}^{(1)}, P_{0,r,K+k-1}^{(0)}, P_{0,r,K+k-1}^{(1)} = 1$$

$$P_{1,r,0}^{(1)} = (1-\phi_r) P_{1,r-1,0}^{(0)}, P_{1,r,K+k}^{(1)} = \phi_r P_{1,r-1,K+k-1}^{(1)}$$

and, if $K+k > 1$,

$$P_{0,r,D}^{(0)} = P_{0,r-1,D-1}^{(0)} - \Theta_r P_{0,r-1,D}^{(1)}$$

$$P_{1,r,D}^{(1)} = (1-\phi_r) P_{1,r-1,D}^{(0)} + \phi_r P_{1,r-1,D-1}^{(1)}$$

both for $D=1, \dots, K+k-1$ and, if $K+k > 0$,

$$P_{1,r,D}^{(0)} = P_{1,r-1,D}^{(0)} - \Theta_r P_{1,r-1,D}^{(1)}$$

for $D=0, \dots, K+k-1$ and if $K+k > 1$

$$P_{0,r,D}^{(1)} = (1-\phi_r) P_{0,r-1,D}^{(0)} + \phi_r P_{0,r-1,D}^{(1)}$$

for $D=0, \dots, K+k-2$.

4) The coefficients $\{d_{2k}\}$ occurring in the required interpolating rational F-function are obtained from the number a derived in step 0), the coefficients $\{T_{p+\gamma-2, k}\}$ and $\{b_{p+\gamma-2, k}\}$ ($\gamma=0, 1$) derived in step 1) and the coefficients $\{P_{0,R,2}^{(0)}\}$, $\{P_{1,R,2}^{(0)}\}$ derived in step 3) by use of the formulae

$$d_{2k} = (-1)^k \sum_{j=\max(0, k-N)}^{\min(k, r+k-1)} (-1)^j T_{p-1, j} P_{0,R,N+j-k}^{(0)}$$

$$+ \sum_{j=\max(0, k-N-\gamma-1)}^{\min(k-2, r-1)} (-1)^j T_{p-2, j} P_{1,R,N+\gamma+j-k-3}^{(0)}$$

$$- \sum_{j=\max(0, k-N-1)}^{\min(k-1, r-1)} (-1)^j b_{p-1, j} P_{0,R,N+j-k+1}^{(0)}$$

$$+ \sum_{j=\max(0, k-N-3)}^{\min(k-2, r+k-2)} (-1)^j b_{p-2, j} P_{1,R,N+\gamma+j-k-3}^{(0)}$$

for $k=0, \dots, [\frac{1}{2}n]$, and

$$d_{2k+1} = (-1)^{k+1} \frac{1}{2} a \left[\sum_{j=\max(0, k-N)}^{\min(k, r+k-1)} (-1)^j T_{p-1, j} P_{0,R,N+j}^{(0)} \right.$$

$$\left. - \sum_{j=\max(0, k-N-\gamma)}^{\min(k-1, r-1)} (-1)^j T_{p-2, j} P_{1,R,N+\gamma+j}^{(0)} \right]$$

for $k=0, \dots, [\frac{1}{2}(n-1)]$, where $p=2r+t$, $R=2N+y$ with t and y each being 0 or 1.

Proof. The results of part 3) of the theorem state, in particular, that a rational F-function g exists such that its derivative values to specified orders agree with those of a given F-function at the origin, at a system of points on the negative part of the real axis, and at a system of complex conjugate point in \mathbb{L} . Into such a point and order system any less restricted point and order system as prescribed in part A) may be embedded: the results of part A) are implied by those of part 3) whose proof is now embarked upon.

f either has a representation of the form

$$(2b) \quad f(z) = 1 + \frac{az}{1 - \frac{1}{2}az +} \frac{a_1 z^2}{1 +} \dots \frac{a_{p-1} z^2}{1 +} \frac{z^2 s'(z)}{1}$$

where $s' \in S$, or is a rational function having a similar terminating representation in which the terms involving s' and a subsequence a_{p-1}, a_{p-2}, \dots are

missing. In the latter case $g(z) = f(z)$ is a solution to the interpolation problem of clause Bb); no rational F-function of lower degree generates an ascending power series agreeing with that of $f(z)$ as far as the term involving z^{2p} ; $g(z)$ is real for real values of z and the required interpolation properties of $f(z)$ and the required interpolation properties of clause (Biii) automatically hold. All results of part B) have been established for the degenerate case in question, which is now disregarded.

The function

$$(7) \quad g(z) = 1 + \frac{az}{1 - \frac{1}{2}az^2} \frac{a_1 z^2}{1 + \dots} \frac{a_{p-1} z^{2(p-1)}}{1 + \dots} \frac{z^{2p} s''(z)}{1}$$

where s'' is a rational S-function, is analytic at the origin and has there a power series expansion of the form (7) agreeing as far as the term involving z^{2p} with the asymptotic expansion of f at the origin. The interpolation conditions at the origin are thus satisfied by such a function g , and by no other F-function having different

coefficients preceding the term involving s'' . The F -function $f(z)$ is a rational function of z and $s'(z)$; $g(z)$ is the same function of z and $s''(z)$. Values of derivatives of s' at the points $\{z_j\}$ and of orders $z=0, \dots, T(j)-1$ corresponding to those of f occurring in the interpolation problem of part B) may be obtained by use of this rational function. If s'' satisfies the interpolation conditions

$$\sum z^j s''(z) = \sum z^j s'(z)$$

with $z=0, \dots, T(j)-1$ for $j=1, \dots, J$, g satisfies the required interpolation conditions. The interpolation problem of part B) has thus been reduced to that of determining a rational S -function s'' which satisfies conditions () as described, where s' is a prescribable S -function. The latter interpolation problem may be transformed, by means of a change of variable, into one for which much existing theory is available. Setting $\tau = (iz)^{-1}$, the function $-zs'(z)$ may be exhibited as

in the form

$$h(\lambda) = \int_{-\infty}^{\infty} (t-\lambda)^{-1} d\sigma'(t)$$

where $\sigma' \in SB(-\infty, \infty)$. If g is also to be an F-function, $-zs''(z)$ may be exhibited in a similar form

$$G(\lambda) = \int_{-\infty}^{\infty} (t-\lambda)^{-1} d\sigma''(t)$$

where $\sigma'' \in SB(-\infty, \infty)$ also; if g is rational, G is rational, and σ'' is a step function with a finite number of salti. The interpolation problem being considered has now been transformed into that of determining a function G of the form () with σ'' as described, which satisfies the conditions

$$\Delta^z G(\lambda_j) = \Delta^z h_j$$

with $z=0, \dots, T(j)-1$ for $j=1, \dots, J$ where $\lambda_j = (jT_j)^{-1}$ and $\Delta^z h_j = \Delta^z h(\lambda_j)$.

For this interpolation problem the following results are available (they are of considerable independent interest and are derived in a wide context in Appendix 3): if a function G which

has a representation of the form $\epsilon'' \in \mathcal{B}(-\infty, \infty)$ and also
b) satisfies conditions () with $z=0, \dots, \bar{T}(j)-1$ and

$\Lambda_j \in \mathbb{W}$ for $j=1, \dots, J$ can be found then either A)
G is a rational function expressible as the quotient

of two polynomials of degrees less than ~~m-2~~
 $m = 2 \sum_{j=1}^q T(j) + \sum_{j=q+1}^J T(j)$ and in this case G is

the only function having the required form and
satisfying the stated interpolation conditions or B)

there is a unique rational function G both
expressible in the form () with $\epsilon'' \in \mathcal{SB}(-\infty, \infty)$
and as the quotient of two polynomials, the
denominator of degree m and the numerator
of degree $m-1$, which satisfies the interpolation
conditions (); in this case G is the unique
solution of the algebraic interpolation problem
derived by imposing conditions () as stated,
and the further conditions

$$\Delta^z G(-\bar{\Lambda}_j) = -\overline{\Delta^z h_j}$$

with $z=0, \dots, \bar{T}(j)-1$ for $j=1, \dots, q$, and also

two further complete sets of conditions obtained from the above simply by replacing 1_j and $\Omega^2 h_j$ by their complex conjugates consistently, the algebraic problem being normalised by setting the coefficient of the m^{th} power of the variable in the denominator equal to unity.

With regard to the interpolation problem in hand, it is first remarked that a solution satisfying conditions a, b) certainly exists: it is the function h providing the data. If h is a rational function as described under A) above, the \mathfrak{g} G may be taken to be h ; indeed this is the only ~~easy~~ possible choice. The functions s' and s'' are identical, and ipso facto the same is also true of f and g . In this case, as is easily demonstrated, f is a rational function of deficient degree as described in clause Bi. Since g is unique, the result of clause Bii) will

and, since g is real for real argument values,
that of clause Biii) also holds.

The above degenerate case is now disregarded.
The algebraic conditions which determine G as
a function of λ in terms of the derivatives of
~~h~~ with respect to λ may be replaced by
conditions which determine s' as a function
of z in terms of the derivatives of s' with
respect to z . f and g are the same rational
functions of s' and s'' ; the conditions just
referred to may be reformulated as conditions which
partly determine g in terms of the derivatives
of f at points $z_j \in L$. The remaining conditions
which complete the determination of g are those at
the origin: the first $2p-1$ derivatives of f and g
agree there. The algebraic interpolation problem
C described in clause C) is obtained.
If P, Q are two polynomials and, for a fixed

value of z_j , $\Delta^2 \{ Q(z_j) / P(z_j) \} = \Delta^2 f(z_j)$ for $z=0, \dots, T(j)-1$, then equally $\Delta^2 Q(z_j) = \Delta^2 \{ P(z_j) f(z_j) \}$ for $z=0, \dots, T(j)-1$. The algebraic conditions of the interpolation problem considered in the preceding paragraph may be presented in terms of sets of linear equations involving the coefficients in the denominator and numerator of g . This algebraic problem is, however, of a special form. The data values relating to the points z_j for which $\text{Im}(z_j) \neq 0$ are complex valued. However the coefficients in the rational function representation of g are real numbers. By decomposing the conditions imposed upon g at the points z_j in question into real and imaginary parts, an equivalent set of algebraic conditions involving real arithmetic alone is obtained. Since g satisfies the equation $g(z)g(-z) = 1$, it is expressible as the quotient $d(-z)/d(z)$ of

two polynomials of the form (). Accordingly, the conditions imposed upon \mathcal{S} at the origin may be expressed in the form (), and the further algebraic conditions may be reduced to the form () and the further form relating to pure real values of z ; described in clause (Cii).

It remains to justify the steps of the algorithm described in clause (Ciii).

o) Since $f_0 = 1$ in expansion (), $a = f_1$. Rearrange formula (), the relationship

$$\left\{ 1 - \frac{1}{2} f_1 z + z^2 s(z) \right\} \{f(z) - 1\} = f_1 z$$

is obtained. Replacing s and f by their asymptotic expansions, and equating coefficients of z^{2k+3} ($k = \dots, p-1$) on both sides of this formula, recursion (is obtained.

i) With π the function occurring in formulae a system of orthogonal polynomials

$$\pi_k^{(r)}(x) = \sum_{v=0}^k \pi_{k,v}^{(r)} x^v$$

with $k=0, 1, \dots$ exists for $r=0, 1, \dots$. Under current assumptions, namely that in the continued fraction () a_1, \dots, a_{p-1} are nonzero, the orthogonal polynomials $\pi_k^{(r)} (k=0, \dots, p-1)$ are well defined for $r=0, 1, \dots$. These polynomials may be determined from the conditions

$$\int_0^\infty t^{n+r} \pi_k^{(r)}(t) d\sigma(t) = 0$$

for $\gamma=0, \dots, r-1$ and $\pi_{k,k}^{(r)} = 1$. They satisfy relationships of the form

$$\pi_k^{(r)}(x) = x \pi_{k-1}^{(r+1)}(x) - a_{2k}^{(r)} \pi_{k-1}^{(r)}(x), \quad \pi_{k-1}^{(r+1)}(x) = \pi_{k-1}^{(r)}(x) - a_{2k+1}^{(r)} \pi_{k-1}^{(r)}$$

where the $a_k^{(r)}$ are real numbers. Multiplying the first of relationships throughout by x^{r+k-1} and using formula () with $\gamma=k-1$, the result

$$a_{2k}^{(r)} = \sum_{j=0}^{k-1} \pi_{k-1,j}^{(r+1)} s_{r+k+j} / \sum_{j=0}^{k-1} \pi_{k-1,j}^{(r)} s_{r+k+j-1}$$

is obtained. A similar formula may be obtained for $a_{2k+1}^{(r)}$ from the second of relationships (). The formulae and relationships () serve as the ba-

of a recursive process for the construction of the polynomials $\{\pi_k^{(r)}, \pi_k^{(r+1)}\}$ for a fixed value of r : $a_{2k}^{(r)}$ is determined from formula (), $\pi_k^{(r)}$ is constructed by the use of the first of relationships (), $a_{2k+1}^{(r)}$ is determined from the counterpart to formula (), $\pi_k^{(r+1)}$ is constructed by use of the second of formulae (), formula () is used with k replaced by $k+1$ and so on.

The polynomials $t\pi_{k-1}^{(r+1)}(t), \pi_k^{(r)}(t)$ in order ($k=1, 2, \dots$) are the denominators of the successive convergents of the continued fraction

$$\frac{a_1^{(r)}}{t-} \frac{a_2^{(r)}}{1-} \dots \frac{a_{2k-1}^{(r)}}{t-} \frac{a_{2k}^{(r)}}{1-} \dots$$

The numerator polynomials $\{b_k^{(r)}\}$ of the above convergents satisfy relationships similar to (). The orthogonalisation algorithm described above is thus a process for determining the coefficients in the continued fraction () and

those in its convergents from the coefficients of its generating series $\sum_{r=0}^{\infty} s_{r,t} t^{-r-1}$.

In the above, r may be taken to be zero, and the substitutions $t = -z^{-2}$, $\pi_{2k}(z) = (-t)^k \pi_k^{(1)}(-t)$, $\pi_{2k+1}(z) = (-t)^{k+1} \pi_k^{(1)}(-t')$ may be made. The continued fraction () evolves to the form exhibited in formula (). Its convergents are

$b_k(z)/\pi_k(z)$ ~~$t = 0, 1, 2, \dots$~~ ($k=1, 2, \dots$) where

$$\pi_{2k}(z) = \sum_{j=0}^k \pi_{2k,j} z^{2j}, \quad \pi_{2k+1}(z) = \sum_{j=0}^k \pi_{2k+1,j} z^{2j}$$

$$b_{2k}(z) = \sum_{j=0}^{k-1} b_{2k,j} z^{2j}, \quad b_{2k+1}(z) = \sum_{j=0}^k b_{2k+1,j} z^{2j}$$

The orthogonalisation process and recursions involving numerator polynomials may be given in terms of the coefficients $\{\pi_{k,j}, b_{k,j}\}$, the superscript $r=0$ may be discarded from the symbol $a_k^{(r)}$, and the resulting algorithm is that given in stage 1).

2a) The functions f and $s^{(1)}$ satisfy the relationship

$$z^{-1} + \frac{1}{2}\alpha + s^{(1)}(z) = z^{-1}f(z) + \frac{1}{2}\alpha f(z) + f(z)s^{(1)}(z)$$

differentiation of which, and subsequent rearrangement, yields formula ().

2b) The functions

$$s^{(k)}(z) = \frac{a_k z}{1+} \frac{a_{k+1} z^2}{1+} \dots \frac{a_{p-1} z^2}{1+} \frac{z^2 s'(z)}{1}$$

$k=1, \dots, p-1$ and $s^{(p)}(z) = z s'(z)$ are related by

the formula

$$s^{(k)}(z) \{1 + z s^{(k+1)}(z)\} = a_k z$$

$k=1, \dots, p-1$. Differentiation of this formula yields recursion ().

3a) If $\lambda = (iz)^{-1}$, Δ_λ and Δ_z denote differentiation with respect to λ and z respectively, then $\Delta_\lambda =$

$-iz^2 \Delta_z$ and more generally

$$\Delta_\lambda^2 = \sum_{D=0}^{z-1} H_D^{(z)}(z) \Delta_z^{D+1}$$

for $z=1, 2, \dots$, where the $\{H_{\nu}^{(z)}\}$ are polynomials in z . Applying the operator \mathcal{D}_{λ} on the left hand side of this relationship, and $-iz^2 \Delta_z$ on the right, and thus deriving a differential recursion for the $\{H_{\nu}^{(z)}\}$ it is shown both that these polynomials have the form $H_{\nu}^{(z)}(z) = \gamma_{\nu}^{(z)} z^{2\nu+1}$ ($\nu=0, \dots, z-1$) and that the coefficients $\{\gamma_{\nu}^{(z)}\}$ satisfy the recurrence relationship stated in stage 3a). The derivatives of h with respect to λ at the points λ_j are thus as given by formula ().

3b) The algebraic problem of determining a rational function, expressible both in the form () with $\sigma' \in SB(-\infty)$ and as a quotient of two polynomials with denominator and numerator of degrees equal to m and $m-1$ respectively, which satisfies the interpolation conditions () as described, may be solved by means of a recursive process. This process, in a completely general form permitting treatment of the points

$\{\lambda_j\}$ and corresponding successive derivative values $\{2^r h(\lambda_j)\}$ in any order, is described and justified in Appendix 3 (see clause () of Theorem). In the present case, in which the points $\{\lambda_j\}$ are dealt with in consecutive order, and each set of derivatives $\{2^r h(\lambda_j)\}$ are completely taken into account before a new argument value λ_j is dealt with, the process assumes a simplified form.

Four sequences of polynomials $P_{\mu,r}^{(\omega)}$ ($r=0,\dots,R$) for $\omega, \mu=0,1$ are constructed, use being made of associated residual functions

$$P_{\gamma,r}^{(\omega)} = 2^r [\Delta_j^r \{ P_{0,r-1}^{(\omega)}(\lambda_j) H(\lambda_j) - P_{1,r-1}^{(\omega)}(\lambda_j) \}]$$

With $P_{0,r-1}^{(\omega)}, P_{1,r-1}^{(\omega)}$ polynomials of the form

$$P_{0,r-1}^{(\omega)}(\lambda) = \sum_{D=0}^l P_{0,r-1,D}^{(\omega)} \lambda^{2D+E}, \quad P_{1,r-1}^{(\omega)}(\lambda) = \sum_{D=0}^{l+2+\omega-1} P_{1,r-1,D}^{(\omega)} \lambda^{2D}$$

the residual function () is simply () expressed in terms of the coefficients $\{P_{0,r-1,D}^{(\omega)}, P_{1,r-1,D}^{(\omega)}\}$. Initially

$P_{\mu,\omega}^{(\omega)}(\lambda) = 1 - l\omega - \mu l$ for $\omega, \mu = 0, 1$. During the process, the polynomial order r , the argument value suffix j , and the derivative order τ are related and increase in value as described in the theorem.

During treatment of the argument values λ_j ($j=1, \dots, q$) with nonzero imaginary part, the polynomial

$P_{\mu,r}^{(\omega)}$ have the form

$$P_{\mu,r}^{(\omega)}(\lambda) = \sum_{\nu=0}^{r-l\omega-\mu l} P_{\mu,r,\nu}^{(\omega)} \lambda^{2\nu + l\omega - \mu l}$$

for $\omega, \mu = 0, 1$; accordingly the substitutions $l = r - \omega - 1$, $\varepsilon = \omega$ must be made in the residual formula (). With $\alpha_r, \dots, \gamma_r$ defined by formulae (,), successive polynomials are constructed by use of the recursions

$$P_{\mu,r}^{(0)}(\lambda) = (\lambda^2 - \alpha_r) P_{\mu,r-1}^{(0)}(\lambda) - \beta_r \lambda P_{\mu,r-1}^{(1)}(\lambda)$$

$$P_{\mu,r}^{(1)}(\lambda) = \{ (1 + \gamma_r) \lambda^2 + \gamma_r \} P_{\mu,r-1}^{(1)}(\lambda) - \gamma_r \lambda P_{\mu,r-1}^{(0)}(\lambda)$$

for $\mu = 0, 1$. Formulae (,) are merely versions of these two relationships written out in explicit algebraic form in terms of the coefficients of the

polynomials $P_{\mu,r}^{(\omega)}$. At this point, a set of four polynomials of the form () with r replaced by $K = \sum_{j=1}^q T(j)$ has been produced.

During treatment of the argument values λ_j ($j = q+1, \dots, J$) with zero imaginary part λ_j ($j = q+1, \dots, J$) with zero imaginary part, with zero imaginary polynomials $P_{\mu,r}^{(\omega)}$ with r increasing from $K+1$ by steps of unity to $\sum_{j=q+1}^J T(j)$ are produced. With $k = [\frac{1}{2}(r-K)]$, $\gamma = [\frac{1}{2}(r-K-1)]$ these polynomials have the form

$$P_{\omega,r}^{(\omega)}(\lambda) = \sum_{j=0}^{K+k} P_{\omega,r,j}^{(\omega)} \lambda^{2j+1+\gamma-k}$$

$$P_{1-\omega,r}^{(\omega)}(\lambda) = \sum_{j=0}^{K+\gamma} P_{1-\omega,r,j}^{(\omega)} \lambda^{2j+k-\gamma}$$

both for $\omega = 0, 1$; accordingly, the substitutions $l = K + \gamma$, $\varepsilon = k - \gamma$ when $\omega = 0$ and $l = K + k - 1$, $\varepsilon = 1 + \gamma - k$ when $\omega = 1$ must be made in the residual formula (). With Θ_r, ϕ_r defined by formulae (), successive polynomials are constructed by use of the recursions

$$P_{\mu,r,m}^{(0)}(\lambda) = \lambda P_{\mu,r-1}^{(0)}(\lambda) - \Theta_r P_{\mu,r-1}^{(1)}(\lambda)$$

$$P_{\mu,r}^{(1)}(\lambda) = (1 - \phi_r) P_{\mu,r-1}^{(0)}(\lambda) + \phi_r \lambda P_{\mu,r-1}^{(1)}(\lambda)$$

for $\mu=0,1$. Again formulae () are versions of these two relationships written out in explicit algebraic form in terms of coefficients.

The end product of this stage is a rational function

$$W_R(\lambda) = \frac{\sum_{j=0}^{N+\gamma-1} P_{1,R,j}^{(0)} \lambda^{2D-\gamma+1}}{\sum_{j=0}^N P_{0,R,j}^{(0)} \lambda^{2D+\gamma}}$$

where $R = \sum_{j=1}^J T(j)$ and $2 \sum_{j=1}^q T(j) + \sum_{j=q+1}^J T(j) = 2N+\gamma$

where N, γ are integers with $\gamma=0$ or 1.

4) Replacing λ by $(iz)^{-1}$, the above function $W_R(\lambda)$ yields the term $z^2 s''(z)$ in expansion (). Accordingly

$$z^2 s''(z) = Q(z)/P(z)$$

where

$$Q(z) = \sum_{j=0}^{N+\gamma-1} P_{1,R,N+j-1}^{(0)} (-1)^j z^{2(D+1-\gamma)}$$

$$P(z) = \sum_{j=0}^N P_{0,R,N-j}^{(0)} (-1)^j z^{2j}$$

In terms of these polynomials, the tail of expansion ()

$$\frac{u_1 z^2}{1+} \cdots \frac{u_{p-1} z^2}{1+} \frac{Q(z)}{P(z)} = \frac{z^2 \{ P(z) b_{p-1}(z) + Q(z) \bar{b}_{p-2}(z) \}}{P(z) \bar{b}_{p-1}(z) + Q(z) \bar{b}_{p-2}(z)}$$

where $\bar{b}_{p-2}, \dots, \bar{b}_{p-1}$ are the polynomials whose coefficients were obtained in stage 1). Accordingly the denominator of g is

$$(1 - \frac{1}{2}az) \{ P(z) \bar{b}_{p-1}(z) + Q(z) \bar{b}_{p-2}(z) \} + z^2 \{ P(z) b_{p-1}(z) + Q(z) b_{p-2}(z) \}$$

The coefficients in this polynomial may be obtained from those of $P, Q, \bar{b}_{p-2}, \dots, \bar{b}_{p-1}$ and are as defined in stage 4.

As a point of detail concerning clause (iii) of the above theorem, it is remarked that the coefficients a and $\{a_k\}$ in the associated continued fraction expansions of certain F-functions are known explicitly (this is true, for example, for the expone function). In such a case stage 0) can be dispensed with; the determination of the $\{a_k\}$ by use of the numbers () can be omitted (it is still necessary to determine the coefficients $\{\bar{b}_{p+\gamma-2}, \dots, \bar{b}_{p+1}\}$,

($j=0,1$) using the known values of the $\{a_k\}$). Use of known values of a and $\{a_k\}$ not only reduces the amount of computational labour involved, but may also improve the accuracy of the process. The orthogonalisation process described in stage 1) is equivalent to the Gauss-Banachiewicz decomposition of the Hankel matrix whose $(i,j)^{\text{th}}$ element is s_{i+j-1} ($i,j = 1, \dots, [\frac{1}{2}p]$). It can occur that this matrix is poorly conditioned; for example, this is certainly so when $s_{ij} = (\lambda + M)^{-1}$ ($\lambda = 0, 1, \dots$), $M > 0$ being large, the matrix in question being a segment of the Hilbert matrix. The use of closed expressions for the $\{a_k\}$ avoids loss of accuracy in such cases. It is further remarked that in the cases being considered, the set of equations described in clause (Cii) is poorly conditioned, but for the method of this clause recourse to closed expressions for auxiliary numbers is

unavailable. The process of clause (Ciii), although apparently far more complicated, is to be preferred in such poorly conditioned cases.

During implementation of a stage of the algorithm of clause () of Theorem, the coefficients of the polynomials of the form $P_{\mu, R}^{(1)}(\lambda)$ ($\mu=0, 1$) are determined merely to assist the construction of the coefficients of the polynomials of the form $P_{\mu, R}^{(0)}, P_{\mu, R}^{(1)} (\mu=0, 1)$ in a subsequent stage. At the final stage the first coefficients of $P_{\mu, R}^{(1)}$ are no longer required, and, as is stated in the theorem, the construction of this final set of coefficients may be dispensed with. The rational function formed from the two polynomials in question has the form

$$W_R^{(1)}(\lambda) = \frac{\sum_{d=0}^N P_{1, R, d}^{(1)} \lambda^{2d+\gamma}}{\sum_{d=0}^{N+\gamma-1} P_{0, R, d}^{(1)} \lambda^{2d-\gamma+1}}$$

when $m=2N+\gamma$ with $\gamma=0$ or 1 . It is of interest to point out that if this function is used in place of the function $W_R(\lambda)$ of formula () during stage (), the function g , so derived is also an F-function, it

the quotient of two polynomials of degree $n-1$; it satisfies conditions A α) with p replaced by p^{-1} and A β). This function g is, in short, the F-function constructed as described in the theorem with p replaced by p^{-1} .

6. Exponential fitting

The method for the numerical solution of differential equations offered by the use of formula (1) is particularly simple. The connection between the special variant of it described and the stability function $C_{2r}(z)$ is direct; that the behaviour of this function over the complex plane determines the stability of the above special method when applied to an equation of the form (9) is evident. The stability behaviour of all methods for the numerical solution of initial value problems when applied to an equation of the form (9), may be presented in terms of an associated stability function.

function of polynomial or rational form (for details of the derivation of the stability function from the numerical method, see Ch 2 of the recent important book [21] by van der Houwen).

Equations of the form (5) constitute a very restricted class; it is evidently desirable to exploit the stability function in the treatment of more general equations by the use of an associated numerical method. One such method of exploitation concerns the equation

$$(51) \quad \frac{dy(x)}{dx} = f(y(x))$$

where $y, f \in \mathbb{R}^n$.

The linearised version of this equation, obt

by expanding the components of \tilde{f} in powers of those of $y(x)$, is

$$(52) \quad \frac{dy(x)}{dx} = \tilde{f}(0) + \tilde{J}y(x)$$

where the $(i,j)^{\text{th}}$ element of the Jacobian matrix \tilde{J} is $\partial f_i(y(x)) / \partial y_j(x)$ ($i, j = 1, \dots, n$). Assuming \tilde{J} to be nonsingular, it follows that for the solution to

equation (52)

$$(53) \quad y(x+h) = e^{\tilde{J}h} \{ y(x) + \tilde{J}^{-1}f(0) \} - \tilde{J}^{-1}f(0)$$

Making the further assumptions that the eigenvalues

$\lambda_{j,k}$ of \tilde{J} are distinct, and are disposed in

in clusters containing $r(j)$ approximately

equal members ($j = 1, \dots, m$), so that $\lambda_{j,k} = \Lambda$

($j = 1, \dots, m; k = 1, \dots, r(j)$) approximately, and

function associated with the method. For the function $y^*(x+h)$, a formula similar to (54) with $e^{\Lambda(j)h}$ replaced by $\tilde{R}(\Lambda(j)h)$ also holds approximately.

The series expansions of $y(x+h)$ and $y^*(x+h)$ in ascending powers of h agree as far as the term in h^{N+1} if $\frac{d^p R(z)}{dz^p} = 1$ ($p=0, \dots, N$) when $z=0$.

A further, non-local, condition of agreement between $y(x+h)$ and $y^*(x+h)$ is achieved by

arranging that $\tilde{R}(\Lambda(j)h) = e^{\Lambda(j)h}$ ($j=1, \dots, K$): the

value of $y(x+h)$ given by formula (54), and that

$y^*(x+h)$ given by the counterpart to this formula

as described above, then agree. A slightly more

elaborate treatment leads to the condition that

derivatives of prescribed orders of \tilde{R}_n^0 should agree with those of the exponential function at the points $\Lambda(j)h$ ($j=1, \dots, K$). It is, of course, still required that the numerical method associated with \tilde{R}_n^0 should be stable in terms of the treatment of equation (5) as described above. It is, in summary, required to construct a rational function

$\tilde{R}_n(z)$ such that a) $|\tilde{R}_n(z)| \leq$

~~\tilde{R}_n a) which maps L into D and b) is such that~~

~~with $\tilde{R}_{np}(z)$ such that a) $|\tilde{R}_{np}(z)| \leq 1$ when $z \in L$,~~

b) with $\tilde{R}_{np}(z) = \sum_{j=0}^{\infty} f_j^{(n)} z^j$, $f_j^{(n)} = (j!)^{-1}$ ($j=0$,

and c)

$$\frac{d^i \tilde{R}_n(z)}{dz^i} = e^z \quad \tilde{g}(z) = e^z$$

$z = 0, \dots, T(j)^{-1}$

for $j = 1, \dots, K$; $z = z_j$; $i = 0, \dots, T(j)$ where the z_j are distinct points in \mathbb{L} and the T_j are finite integers.

Solutions to this problem in very simple cases

have been proposed by Liniger and Wilbrough [11]

who derive three functions. They are, together with equivalent continued fraction representations,

as follows:

$$L_1(\alpha_1; z) = \frac{1 + (1 + \alpha_1)z}{1 + \alpha_1 z}$$

$$= 1 + \frac{z}{1 +} \frac{\alpha_1 z}{1} \quad 1 + \frac{z}{1 + \alpha_1 z}$$

$$L_2(\alpha_1, \alpha_2; z) = \frac{1 + (1 + \alpha_1)z + (\frac{1}{2} + \alpha_1 + \alpha_2)z^2}{1 + \alpha_1 z + \alpha_2 z^2}$$

$$= 1 + \frac{z}{1 - \frac{1}{2}z +} \frac{\{\alpha_2 + \frac{1}{2}(\frac{1}{2} + \alpha_1)\}z^2}{1 +} \frac{(\frac{1}{2} + \alpha_1)z}{1}$$

$$L_3(\alpha_1, z) = \frac{1 + (1 + \alpha_1)z + (\frac{1}{3} + \frac{1}{2}\alpha_1)z^2}{1 + \alpha_1 z - (\frac{1}{6} + \frac{1}{2}\alpha_1)z^2}$$

$$= 1 + \frac{z}{1 - \frac{1}{2}z + \frac{1}{12}z^2} \frac{\frac{1}{12}z^2}{1 +} \frac{(\frac{1}{2} + \alpha_1)z}{1}$$

Formulae may be and have been given for the determination of α_1 such that $L_1(\alpha_1; z_1) = e^{z_1}$. When $z_1 \in (-\infty, 0)$, then $\alpha_1 \in (-\infty, -\frac{1}{2})$ and L_1 satisfies requirement a) and also b) with $N = 1$. Similarly

α_1 and α_2 may be determined such that

$$L_2(\alpha_1, \alpha_2; z) = e^z \text{ with } z = z_1, \bar{z}_1. \text{ For all } z, L_2$$

~~in an infinite domain in the left half plane~~

bounded

~~unbounded in the right half plane~~

in an unbounded domain in L^\vee bounded on the right by a curve with asymptotic behaviour

$x = e^{-y}$ where $z = x+iy$, L_2 satisfies requirement a);
 does not completely satisfy
 it satisfies requirement b) with $\frac{N}{P} = 2$. Lastly,
 may be determined such that $L_3(\alpha_1; z_1) = e^{z_1}$. When
 $z_1 \in (-\infty, 0)$, L_3 satisfies requirement a) and also
 b) with $\frac{N}{P} = 3$.

The extent to which requirement b) is met
 by the above functions L_1 , L_2 and L_3 is made
 clear from a comparison of their continued fraction
 representations and that of the associated cont.
 fraction (22) —

(for e^z . The function $L_1(z)$ is, when $\alpha_1 \in (-\infty,$
 the simplest function of the form (20) as de-

that it maps an unbounded region containing the nonzero part of the imaginary axis in its interior into \mathbb{D} is an elementary corollary to the theory stated in connection with functions of that form. Subject to the stated fitting conditions, none of the above functions is an F-function: they map a domain either more extensive or less extensive than \mathbb{L} into \mathbb{D} . It is also difficult to discern a general principle underlying the construction of L_1, L_2 and L_3 which would also underly construction of general functions of the same type which also satisfy conditions a, b, c) above. The process of exponential fitting is:

practice, extremely important. In what is perhaps the most powerful collection of computer software for the numerical solution of ordinary differential equations in existence [10] the functions L_1, L_2 and L_3 above are extensively made use of. It is eminently desirable that a method for the construction of general rational functions mapping \mathbb{L} into \mathbb{D} and having prescribed orders of contact with the exponential function at the origin and other specified points in \mathbb{L} should be made available. This is precisely the service offered by § of this paper. In Theorem 4, the F -function producing the interpolation data is taken to be

exponential function; the results of that theorem then show that the required function is obtained simply by means of rational function interpolation.

7. Meromorphic F-functions

The function

$$(ss) \quad s(z) = M + \sum_{i=1}^m M_i \frac{u}{(1+z^2 k_i)^{-1}}$$

where $M \geq 0$, $0 < k_i, M_i < \infty$ ($i=1, \dots, k$) has a representation of the form (2) with $s(k_i+0) - s(k_i-0)$
 $= M_i$ ($i=1, \dots, k$), $s(t)$ having no other points of variation

in the range $0 \leq t \leq \infty$. Subject to the condition that

all moments $\sum_{i=1}^m M_i t_i^j$ ($j=0, 1, \dots$) exist, functions

this form may be inserted into formula (1) with $a < \infty$, to define a sub-class of F-functions:

Properties of the corresponding continued fraction
An F-function generates a corresponding continued fraction
whose r^{th} convergent is

$$(1) C_r(z) = 1 + \frac{c_1 z}{1 +} \frac{c_2 z}{1 +} \dots \frac{c_r z}{1}$$

The series expansion of $C_r(z)$ in ascending powers of z
agrees with the asymptotic expansion of $f(z)$ as far as
the term involving z^r ($r=1, 2, \dots$). The

corresponding continued fraction terminates with
the convergent $C_2(z)$ if the function ϵ in formula
(1) is constant; it terminates with the convergent
 $C_{2n+2}(z)$ ($1 \leq n < \infty$) if ϵ has one salto at the
origin, $n-1$ saltos in the range $(0, \infty)$, and no
other points of increase in this range; it
terminates with the convergent $C_{2n+4}(z)$ if ϵ
has n saltos in the range $(0, \infty)$ and no other
points of increase; otherwise it is nonterminating.

In the notation of formulae (1, 2), $\hat{u}_1 = a$, $\hat{u}_2 = -\frac{1}{2}a$

and $\hat{u}_{2r} = -\hat{u}_{2r-1}^2$ with $u_{2r-1} > 0$ ($r \geq 1$) for those
coefficients that are defined. For example, the

The convergents $C_{2r}(z)$ of the above corre-

coefficients that are defined. For example, the corresponding continued fraction generated by the exponential series is

$$(2) \quad 1 + \frac{z}{1 - \frac{\frac{z^2}{2}}{1 + \frac{\frac{z^2}{2^2}}{1 - \frac{\frac{z^2}{6}}{1 + \dots}}}} = 1 + \frac{z}{1 - \frac{\frac{z^2}{2}}{1 + \frac{\frac{z^2}{2^2}}{1 - \frac{\frac{z^2}{6}}{1 + \frac{\frac{z^2}{2(2r-1)}}{1 - \frac{\frac{z^2}{2(2r+1)}}{1 + \dots}}}}}} \dots$$

The convergents $C_{2r}(z)$ of the corresponding continued fraction generated by an F-function may be expressed in the contracted form (). As mentioned above, they map \mathbb{L} into $\bar{\mathbb{D}}$. The odd order convergents $C_{2r+1}(z)$ do not have this property: evidently this remark holds for the function $C_1 = 1 + az$ which maps the disc $|z + a^{-1}| \leq a^{-1}$ into $\bar{\mathbb{D}}$, i.e. a domain lying entirely within \mathbb{L} except for the boundary point at the origin, into $\bar{\mathbb{D}}$. The function $C_1(-z)^{-1} = (1 - az)^{-1}$ maps the complement of the open disc $|z - a^{-1}| < a^{-1}$ into $\bar{\mathbb{D}}$, i.e. $C_1(-z)^{-1}$ maps a region containing \mathbb{L} and all nonzero points of the imaginary axis in its interior into a region contained in \mathbb{D} . The remarks above remarks hold for all functions $C_{2r+1}(z)$ and $C_{2r+1}(-z)^{-1}$ that are defined.

Theorem For those odd order convergents $C_{2r+1}(z)$

the continued fraction corresponding to the asymptotic power series expansion of an F-function that are defined:

- (i) the function $C_{2r+1}(z)$ maps a closed domain contained entirely within \mathbb{L} , except for the single boundary point at the origin, into $\bar{\mathbb{D}}$ ($|C_{2r+1}(z)| > 1$ for all nonzero $z \in \mathbb{L}$); this domain is symmetric with respect to the real axis;
- (ii) the function $C_{2r+1}(-z)^{-1}$ maps $\bar{\mathbb{L}}$ into a closed domain contained entirely within \mathbb{D} except for the single point $1+i0$ which is the image of $z=0$; this domain is symmetric with respect to the real axis, and the line parallel to the imaginary axis and passing through the point $1+i0$ is a tangent to its boundary.

Proof. Clause (ii) of the above theorem is dealt with first. The result of this clause is true of the convergents of the form $C_1(z)$ for all F-functions. Assume that it is true of all convergents of the form $C_{2r-1}(z)$ for some fixed r . Since $C_{2r+1}(z)$, for any given F-function generation, convergent of this order, may

be expressed in the form

$$C_{2r+1}(z) = 1 + \frac{az}{1-} \frac{\frac{1}{2}az}{1+} \frac{\overset{c}{\mu_3} z}{1-} \frac{\overset{c}{\mu_3} z}{1+} \dots \frac{\overset{c}{\mu_{2r-1}} z}{1-} \frac{\overset{c}{\mu_{2r-1}} z}{1+} \frac{\overset{c}{\mu_{2r+1}} z}{1}$$

it follows that

$$(23) \quad C_{2r+1}(-z)^{-1} = \frac{1}{1-} \frac{az}{1+} \frac{\frac{1}{2}az}{1+\frac{1}{2}\{D(z)-1\}}$$

where

$$D(z) = 1 - \frac{2\overset{c}{\mu_3} z}{1+} \frac{\overset{c}{\mu_3} z}{1-} \dots \frac{\overset{c}{\mu_{2r-1}} z}{1+} \frac{\overset{c}{\mu_{2r-1}} z}{1-} \frac{\overset{c}{\mu_{2r+1}} z}{1}$$

Formula (23) reduces to

$$(24) \quad C_{2r+1}(-z)^{-1} = \frac{D(z) + 1 + az}{(1 - az) D(z) + 1}$$

$D(z)$ is of the form $C'_{2r-1}(-z)$, where C'_{2r-1} is

the $(2r-1)^{th}$ order convergent of a continued

fraction corresponding to the asymptotic series generated by an F-function (for which, as a p-

of detail, $a = \frac{c}{2\pi\beta}$ in formula (1)). When z takes a finite nonzero value in $\bar{\mathbb{L}}$, $az = x + iy$ is such that $x \leq 0$, and $D(z)^{-1} = \alpha + i\beta$ is such that $\alpha^2 + \beta^2 <$

i.e. $\alpha < 1$. From formula (24), $|C_{2r+1}(-z)^{-1}| < 1$ if

$$|1 + (1 - x - iy)(\alpha + i\beta)| < |\alpha + i\beta + 1 + x + iy|.$$

Subject to the stated inequalities, this inequality

also holds, as is easily verified. Thus $|C_{2r+1}(-z)^{-1}|$

1 when $\operatorname{Re}(z) \leq 0, z \neq 0$ for all values of r for which

C_{2r+1} is defined. Since $C_{2r+1}(0) = 1$, the point 1

upon the boundary of the region into which $\bar{\mathbb{L}}$ is

mapped by means of the function $C_{2r+1}(-z)^{-1}$.

$C_{2r+1}(z)$ is real for real values of z , this region

symmetric with respect to the real axis. The f

symmetric with respect to the real axis. The real and imaginary components of $C_{2r+1}(-iy)^{-1}$ are rational functions of y , bounded and indeed differentiable for $-\infty < y < \infty$.

The image of the imaginary axis under the transformation $C_{2r+1}(-z)^{-1}$ is a closed continuous curve with, in particular, a continuously turning tangent. The points of the curve lie on \bar{D} except for $1+i0$, the image of $y=0$. The curve is symmetric with respect to the real axis. Hence the line parallel to the imaginary axis through the point $1+i0$ is a tangent to the curve.

Turning to clause (i), it is remarked that C_{2r+1} is a rational function, and hence maps some closed domain into \bar{D} . Since $|C_{2r+1}(-z)^{-1}| < 1$ for all $z \in L$; it follows that for all nonzero z for which $\operatorname{Re}(z) \geq 0$, $|C_{2r+1}(z)| > 1$. The closed domain just referred to lies entirely in L , except for the single boundary point at the origin. Since $C_{2r+1}(z)$ is symmetric for real values of z , the domain is symmetric with respect to the real axis.

$C_{2r+1}(-z)^{-1}$ is analytic at the origin; the part of the contour upon which $|C_{2r+1}(-z)^{-1}| = 1$ lying in the neighbourhood of the origin is free of cusps.

The images of the point at infinity under the transformations offered by the successive convergents of a continued fraction corresponding to the asymptotic series generated by an F-function vary. With regard to the even order convergents $\lim C_{4r+2}(z)^{-1}$ and $\lim C_{4r+4}(z) = -1$ as $\{z\} \rightarrow \infty$; for the odd order convergents $\lim C_{4r+1}(z) = -\infty$ and $\lim C_{4r+3}(z) = \infty$ as $z \rightarrow \infty$; for the reciprocals of the latter $\lim C_{2r+1}(z)$ as $z \rightarrow \infty$ (these results hold for $r=0, 1, \dots$ for such convergents as are defined)

The mapping properties described above do not depend upon the convergence behaviour of the corresponding continued fraction whose convergents are given by formula (21). If the Stieltjes moment problem associated with the function σ of formula (2) is determinate, the sequence of convergents (21) converges uniformly to its generating function over any bounded domain lying in L ; this is a fortiori true of the sequence $C_{2r}(z)$ ($r=0,1,\dots$) and is also true of the sequence $C_r(-z)^{-1}$ ($r=0,1,\dots$). If the moment problem in question is indeterminate, the sequence of convergents $C_{2r}(z)$ ($r=0,1,\dots$), in particular, diverges by oscillation, over the above

domain the convergents $C_{4r}(z)$ and $C_{4r+2}(z)$ ($r=0,1,\dots$) converge uniformly to two differing F-functions with a common value of α in formula (1), but two extremal solutions of the moment problem replacing σ in formula (2).

5. Interpolation

All F-functions are analytic over L ; some cease to be analytic at the origin. Nevertheless, every F-function has a limiting value as its argument tends to zero through, in particular, negative real values and the same is true of all derivatives of F-functions; in terms of the asymptotic series (4) for an F-function $f(z)$,

Appendix 1. The asymptotic expansion of positive real functions.

In the proof of Theorem 1 use is made of a result concerning the asymptotic expansion of an S-function as its argument tends to zero in a sector contained in the left half-plane. The required result may be derived by mere verbal changes from a counterpart concerning functions which map the open right half-plane into itself (positive functions) and also assume real values for positive real argument values (positive real functions). Functions of the latter type have been made the subject of extensive study (for a recent treatment in connection with the stability theory of methods for the numerical solution of differential equations, see [3]) and for this reason the required result is presented in terms of them.

Theorem . (i) The function w

A) is analytic over \mathbb{R}

B) maps \mathbb{R} into itself

C) satisfies an asymptotic relationship of the form

$$^{(13)} \quad \lim \left\{ z^{-1} w(z) - w_0 z^{-2} \right\} = C$$

as $z \rightarrow 0 \in \Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ where $0 \leq w_0 < \infty$, $0 < C < \infty$, and

D) is real for positive real values of z

if and only if w is represented over \mathbb{R} by formula

(*) where s is given by formula $^{(14)}$ with $\varphi \in B(0, \infty)$

(ii) The function w satisfies conditions A, B, D) above

and also has an asymptotic representation of the

form $^{(15)}$ as $z \rightarrow 0 \in \Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ if and only if w is

represented over \mathbb{R} by formula $^{(14)}$ where $s \in S$; if

the above conditions are satisfied, then automatically

$$w_{2i+1} = 0 \quad (i=0, 1, \dots)$$

Proof. The function $N: \omega$ is analytic over \mathbb{U} and β)

maps \mathbb{U} into itself if and only if

$$N(\lambda) = w_0 \lambda + A + \int_{-\infty}^{\infty} \frac{1+t\lambda}{t-\lambda} d\beta(t)$$

where $0 \leq w_0 < \infty$, $-\infty < A < \infty$ and $\beta \in B(-\infty, \infty)$ ([13]

see [20], Ch. 2, § 2). The function W has the properties

α, β) and also 8) satisfies the condition

$$\lim_{\lambda \rightarrow \infty} \{W(\lambda) - w_0 \lambda - A\} = B$$

as $\lambda \rightarrow \infty \in \Delta(0, \pi)$, where $0 \leq w_0, C < \infty, -\infty < A < \infty$, if and only if W has a representation of the form ⁽¹⁵⁾ as described for which

$$(16) \quad \int_{-\infty}^{\infty} t^2 d\beta(t) < \infty$$

In this case W may be expressed in the form

$$(17) \quad W(\lambda) = w_0 \lambda + A' + \int_{-\infty}^{\infty} \frac{d\gamma(t)}{t - \lambda}$$

where $-\infty < A' < \infty$ and $\gamma(t) = \int_{-\infty}^t (1+t^2) d\beta(t)$ (again see [13], [20], Ch II, §2).

The function W has the properties α, β, γ and also takes pure imaginary values for positive pure imaginary values of its argument if and only if W has the representation ⁽⁷⁾ where $A' = 0$ and $\gamma \in SB(-\infty, \infty)$. This last proposition, although not a standard result that may be quoted from the literature, is easily demonstrated. W has the form ⁽¹⁷⁾ as described

For sufficiently large values of λ on the positive imaginary axis, $W'(\lambda) = W(\lambda) - w_0 \lambda$ takes pure imaginary values,

the modulus of the value of the integral expression in formula (17) becomes arbitrarily small: the value of the real component of A' is thus arbitrarily small and, since A' is real, it is also zero. The even order derivatives of $W'(\lambda)$ for λ on the positive imaginary axis are pure imaginary; those of odd order are pure real. Expanding $W'(\lambda)$ in powers of $\lambda - iy'$, where y' is real, positive, and sufficiently large, it is easily shown that $\text{Im}\{W(-x+iy)\} = \text{Im}\{W(x+iy)\}$ for all finite $y > 0$ and $-\infty < x < \infty$. Use of the Stieltjes inversion formula (see, for example, [15], § 32) to recover γ from W' then reveals that γ satisfies the symmetry condition $\gamma(t) + \gamma(-t) = 2\gamma(0)$.

The integral expression in formula (17) can be decomposed into two components over the ranges $[-\infty, 0]$ and $[0, \infty]$; the first may be transformed to the range $[0, \infty]$ and combined with the second. Setting $z = i\lambda^{-1}$ and $w(z) = -iW(\lambda)$, the result of

part (i) is obtained.

Condition $(^{18})$ implies condition $(^3)$, so that w has a representation of the form $(^4)$ as described in the first result; the same condition also implies that the moments $(^3)$ exist. The special form of δ then implies that the w_{2j+1} are zero as stated.

Appendix 2. The construction of functions belonging to certain classes

At various points in the main body of the above text, alternative use might have been made of results concerning the derivation of one S-function from another. These results may be subsumed within a more general theory of the derivation of functions of the form

$$(6) \quad F(\lambda) = \int_{-\infty}^{\lambda} (\lambda-t)^{-1} ds(t)$$

where $s \in \mathbb{B}(-\infty, \infty)$, from others of the same form. This

theory, of considerable interest in itself, does not lie central to the development of the above paper; it is treated in this appendix.

Functions of the form (68) ^{with $\sigma \in P(-\infty, \infty)$} as described will be called H-functions; the notation $F \in H$ will be used. It is an elementary consequence of classic work upon the moment problem over the interval $(-\infty, \infty)$ that if $F \in H$, $0 < v < \infty$, $-\infty < w < \infty$, and

$$F'(\lambda) = \frac{v}{\lambda - w - F(\lambda)}$$

then $F' \in H$ also. Conversely, if the moment

$$(69) \quad f_0 = \int_{-\infty}^{\infty} t^0 ds(t)$$

with $\lambda = 2$ exists, $v_1 = f_0$, $w_1 = f_1/f_0$ and

$$F''(\lambda) = \lambda - w_1 - v_1/F(\lambda)$$

then $F'' \in H$ also.

By considering the case in which $\sigma(t) + \sigma(-t) = 2\sigma(0)$ ($0 < t < \infty$), the above results may be presented in terms

of functions of the form

$$\psi \quad (70) \quad r(z) = \int_0^\infty (1+zt)^{-1} d\sigma(t)$$

where $\sigma \in \mathcal{B}(0, \infty)$. Such a function will be called an

ii) S' -function; the notation ' $r \in S'$ ' will be used. If $r \in S'$,

$0 < v < \infty$, and

$$q) \quad (71) \quad r'(z) = \{1+zh(r(z))^{-1}\}^{-v}$$

then $r' \in S'$ also. Again, if the moment (3) **

$$\psi \quad r_p = \int_0^\infty t^p d\sigma(t)$$

with $p=1$ exists, and

$$\psi \quad s_0 \quad (72) \quad r''(z) = z^{-1} \left\{ r(z)^{-1} v - 1 \right\}$$

then $r'' \in S'$ also. (Presented in terms of S -functions,
this result has been made use of in the Proof of
Theorem 6; viz. formula (62).)

Borel ([2] § 31 p.78) gives a specialized result:

if r_1, r_2 are two S' -functions with, in each case, the
relevant function σ in the representation (70) being

twice differentiable over $(0, \infty)$, then $r_1(z)r_2(z)$ is
also an S' -function. (That some restriction upon the
functions σ associated with r_1 and r_2 is necessary is

easily seen from the counter-example $r_1(z) = r_2(z) = (1+z)$
in which the restrictions ~~above~~ are not obeyed and
the result is untrue.)

There are numerous results (for example, those

Fejér [5, 6], Polya [18] and the author [22, 23]) relating moments of the form (3) or (69) of special types to moments obtainable from other generating functions; they may be used to derive H- and S'-functions from others of the same kind.

In the following, further results based upon function-theoretic considerations alone, are given.

Lemma (i) Let the functions G and H be analytic on U, with $\operatorname{Im}\{G(\lambda)\} > 0$ and $\operatorname{Im}\{H(\lambda)\} \leq 0$ there, and $G(\lambda)H(\lambda) = o\{G(\lambda) - H(\lambda)\}$ as $\lambda \rightarrow \infty \in \Delta(0, \pi)$. Then

$$F(\lambda) = \overline{\{G(\lambda) - H(\lambda)\}}^{-1} G(\lambda) H$$

$$(73) \quad F(\lambda) = \frac{G(\lambda) H(\lambda)}{G(\lambda) - H(\lambda)}$$

is an H-function.

(ii) Let G and H be as described in the preceding clause, and in addition let $G(\lambda)$ and $H(\lambda)$ be pure imaginary for pure imaginary $\lambda \in L$. Then the function

$$F(\lambda) = \frac{G(\lambda)H(\lambda)}{\lambda\{G(\lambda) - H(\lambda)\}}$$

has a representation of the form

$$\oint_{\gamma} \frac{g(z)h(z)}{z-t} dz$$

$$(7A) \quad F(\lambda) = \int_0^\infty (\lambda^2 - t)^{-1} ds(t)$$

where $s \in \mathbb{B}(0, \infty)$

(iii) Let the functions g and h be analytic over L

with $\operatorname{Re}\{g(z)\} < 0$ and $\operatorname{Re}\{h(z)\} \leq 0$ there, let $\operatorname{arg}\{g(z)h(z)\}$

~~as $z \rightarrow 0 \in \Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$~~ . Then the function

~~$\operatorname{arg}\{g(z)+h(z)\}$~~ as $z \rightarrow 0 \in \Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$ and let $g(z)$ and $h(z)$ be pure real for real $z \in L$. Then the function

$$s(z) = \frac{g(z)h(z)}{z\{g(z) + h(z)\}}$$

has a representation of the form (2) in which $\alpha \in B(0, \infty)$

(iv) Let the functions u, v and w be analytic over L ,

with $\operatorname{Re}\{u(z)\} \leq 0, \operatorname{Re}\{v(z)\} \leq 0, \operatorname{Re}\{w(z)\} < 0$ there; also let u, v and w be real for real $z \in L$ and

$$[u(z)v(z) + w(z)\{\Theta u(z) + \phi v(z)\}] = o[w(z) + \Theta v(z) + \phi u(z)]$$

as $z \rightarrow 0 \in \Delta(\frac{L\pi}{2}, \frac{3\pi}{2})$ where $\Theta, \phi \geq 0$ and $\Theta + \phi = 1$. Then

the function

$$(25) \quad s(z) = \frac{u(z)v(z) + w(z)\{\Theta u(z) + \phi v(z)\}}{z\{w(z) + \Theta v(z) + \phi u(z)\}}$$

also has a representation of the form (2) in which $\alpha \in B(0, \infty)$.

Proof. The numerator and denominator functions in expression (73) are both analytic over U and, since the imaginary part of the denominator is never zero, in this half-plane, F itself is analytic there. The imaginary part of $F(\lambda)$ may be represented as a fraction with positive denominator, and numerator equal to $|G(\lambda)|^2 \operatorname{Im}\{H(\lambda)\} - |H(\lambda)|^2 \operatorname{Im}\{G(\lambda)\}$. Subject to the stated conditions, this is negative for $\lambda \in U$: $\operatorname{Im}\{F(\lambda)\} < 0$ for all $\lambda \in U$. Furthermore, $F(\lambda)$ tends to zero as λ tends to infinity as described in clause (i). A function F with these properties may be represented in the form (68) with σ as described ($[13]$, $[20] \S 2$).
Ch II,

Subject to the further conditions upon G and

imposed in clause (ii), $s(t) + s(-t) = 2s(0)$ ($-\infty < t < \infty$) for the function s occurring in formula (68). The product of such a function and λ^{-1} may be represented in the form () as stated.

The result of clause (iii) is a restatement of that of clause (i).

Setting $u(z) = a + ib$, $v(z) = c + id$, $w(z) = x + iy$, and making use of the relationship $\theta + \phi = 1$, it is easily shown that for the function s of clause (iv),

$$|w(z) + \theta \{v(z) + \phi u(z)\}| \operatorname{Re} \{z\bar{s}(z)\} = \\ \theta a \{x^2 + (d+y)^2 + c^2\} + \phi c \{x^2 + (b+iy)^2 + a^2\} + \\ x [2ac + \theta \phi \{(a-c)^2 + (b-d)^2\}]$$

It is clear that, subject to the stated conditions, $a, c \leq 0$; $x < 0$; $\theta, \phi \geq 0$ the expression on the right hand side of equation () always takes nonpositive values, i.e. $\operatorname{Re}\{z\bar{s}(z)\}$

for all $z \in L$. Since $u(z)$, $v(z)$ and $w(z)$ are real for real $z \in L$, the same is true of $s(z)$. The result of clause (iv)

now follows from the first result of Theorem of Appendix.

Perhaps the simplest illustration of the first clause of the above theorem is that provided by the

choice of functions $G(\lambda) = \lambda$, $H(\lambda) = F(\lambda)$; if $F \in H$, the function $\{\lambda - F(\lambda)\}^{-1} \lambda F(\lambda)$ is also an H -function. Again, taking $g(z) = z$, $h(z) = z s(z^2)$, it follows from the result of clause (iii) that if $s \in S'$, $\{1 + s(z)\}^{-1} s(z)$ is also an S' -function.

Slightly more generally, S' -functions are invariant with respect to the transformation

$$\frac{a + b s(z)}{c + d s(z)}$$

with $ac > 0$, $cd > 0$, $bc > ad$.

The above lemma may also be used to show that ~~less~~ the roots of certain polynomials are all real. Considering the behaviour of

$$F(\lambda) = \sum_{i=1}^n (\lambda - \lambda_i)^{-1} M_i$$

An elementary result, obtained by considering the alternation on the real axis of poles and zeros of the function

$$F(\lambda) = \sum_{j=1}^n (\lambda - \lambda_j)^{-1} M_j$$

where the λ_j are real and distinct and the M_j are positive, and noting that $F(\lambda) = P(\lambda) \prod_{j=1}^n (\lambda - \lambda_j)^{-1}$ where

$$P(\lambda) = \sum_{j=1}^n M_j \prod_{i=1}^{n(j)} (\lambda - \lambda_i)$$

(the symbol $\prod^{(j)}$ indicates that the term $\lambda - \lambda_j$ is to be omitted from the product) is that the roots λ'_k ($k=1, \dots, n-1$) of P are real and separate the λ_j . A slightly more advanced result is obtained from clause (i) of the above theorem by taking $G(\lambda) = M_0(\lambda - \lambda_0)$, $H(\lambda) = F(\lambda)$: the roots of

$$Q(\lambda) = M_0 \prod_{j=0}^n (\lambda - \lambda_j) - P(\lambda)$$

with $M_0 > 0$ and λ_0 real ($\lambda_0 \neq \lambda'_k$ ($k=1, \dots, n-1$)) are separated by λ_0 and λ'_k ($k=1, \dots, n-1$).

An alternative proof of Theorem 2, based upon expressions of the form (75), runs as follows: let the

two functions in question have representations of the form (1) with a, s replaced by a_j, s_j ($j=0, 1$) respectively.

As is easily verified, their product also has such a representation with $a=a_0+a_1$ and

$$s(z) = \frac{A + z^2 s_0(z) s_1(z) + p s_0(z) + q_1 s_1(z)}{1 + z^2 \{ p s_1(z) + q_1 s_0(z) \}}$$

where $A = \frac{1}{4} a_0 a_1$, $p = a_0(a_0 + a_1)^{-1}$, $q_1 = 1 - p$. The component fraction of the right hand side with numerator A alone is an S-function (see the

remarks above concerning formula (71)). The

remaining fraction can be expressed in the form

(75), with $r(z) = z^{-1}$, $g(z) = z s_0(z)$, $h(z) = z s_1(z)$ and by use of clause (iv), is also an S-function.

Forming the sum of these two representations, we obtain the desired result. The above argument does not make use of the characterisation offered by Theorem 1, and ~~is perhaps~~ offers perhaps the most elementary what is simplest direct proof of the multiplicative property of F-functions.

Appendix 3. Extremal solutions of the Pick-Nevanlinna problem

The general problem of establishing the existence of a function G which maps \mathbb{D} into itself and assumes, together with its derivatives up to prescribed orders, a system of numerical values at specified argument points in \mathbb{D} was investigated by Pick and Nevanlinna (see [16, 17, 12], [1] Ch.3). In the proof of Theorem use is made of results concerning the interpolating function G in the case in which the argument points, together with the function and derivative values, are symmetrically distributed with respect to the real axis. These results are of considerable independent interest and for this reason their derivation, placed in a wider setting, is dealt with separately in this appendix.

The function G is analytic over \mathbb{D} and maps \mathbb{D} into itself if and only if it is nonconstant.

and has the form

$$G(\lambda) = A\lambda + B + \int_{-\infty}^{\infty} \frac{1+t\lambda}{t-\lambda} d\beta(t)$$

where $0 \leq A < \infty$, $-\infty < B < \infty$ and $\beta \in \mathcal{B}(-\infty, \infty)$

([13], see [20] Ch II, § 2). Such functions will be referred to as N-functions; should the context permit, the notation $G \in \mathbb{N}$ will be used to denote the fact that G is an N-function. If

$$\int_{-\infty}^{\infty} u^2 d\beta(u) < \infty,$$
 then, setting $\gamma(t) = \int_{-\infty}^t (1+u^2) d\beta(u)$

G may be expressed as

$$G(\lambda) = A\lambda + B - \int_{-\infty}^{\infty} (1+u^2) d\beta(u) + \int_{-\infty}^{\infty} (t-\lambda)^{-1} d\gamma(t).$$

This alternative representation is always permissible when β above is a step function with a finite number of salts, G then being a rational function.

With λ_k ($k=1, \dots, m$) distinct points in \mathbb{D} and h_k ($k=1, \dots, m$) corresponding complex

numbers, it is possible to find a junction gen
such that

$$() \quad G(\lambda_k) = h_k$$

for $k=1, \dots, m$ if and only if the form

$$2 \quad () \quad \tilde{\Psi}_m(\underline{\lambda}) = \underline{\lambda}^T \tilde{\Psi}_m \underline{\lambda}$$

is nonnegative, where $\underline{\lambda}$ is the column vector
with complex elements λ_k ($k=1, \dots, m$) and

$\tilde{\Psi}_m$ is the $m \times m$ Hermitian matrix whose

$(i,j)^{th}$ element is

$$\tilde{\Psi}_{i,j} = \frac{h_i - \bar{h}_j}{\lambda_i - \bar{\lambda}_j}$$

for $i, j = 1, \dots, m$. If, when $\tilde{\Psi}_m(\underline{\lambda})$ is nonnegative

and with $\lambda_{\mu(k)}$ ($k=1, \dots, r \leq m$) a subset of λ

and $h_{\mu(k)}$ ($k=1, \dots, r$) the corresponding

subset of the h_k , the form $\tilde{\Psi}_m(\underline{\lambda})$ obtained
from () by reduction of dimension and

replacement of the two full sets of λ_k and h_k by the $\lambda_{u(k)}$ and $h_{u(k)}$ is indefinite (i.e. the corresponding Hermitian matrix $\tilde{\Psi}_r$ is singular) then all forms

$\tilde{\Psi}_k(\xi)$ of dimension $k=1, \dots, m$ obtained by similar selection of subsets of the λ_k and h_k are indefinite (in particular $\tilde{\Psi}_m$ is singular) and the above interpolation problem has a unique solution; if r may not be replaced by a smaller integer, this solution is the irreducible quotient of two polynomials of degree $r-1$.

When the above Hermitian form $\tilde{\Psi}_m(\xi)$ is definite, and a new argument value λ_{m+1} distinct from λ_k ($k=1, \dots, m$) is prescribed, a closed disc upon which a new corresponding function value h_{m+1} must lie if the extended interpolation problem is to be soluble is determined. The equation determining the boundary of this disc

is $\det(\underline{I}_{m+1}) = 0$, where \underline{I}_{m+1} is the $(m+1)^{\text{th}}$ order Hermitian matrix with elements of the form (), and $\det(\underline{I}_{m+1})$ is its determinant. Written out in extenso, this equation has the form

$$(*) \sum |h|^2 - \sum h + \sum h + \Omega = 0$$

where, in particular, \sum is obtained from $\det(\underline{I}_{m+1})$ by replacing the elements in the last row by $\bar{h}_r / (\bar{\lambda}_r - \lambda_{m+1})$ ($r=1, \dots, m$) and those in the last column by $1 / (\bar{\lambda}_{m+1} - \lambda_r)$ ($r=1, \dots, m+1$). If h_{m+1} lies within the circle in the h -plane described by equation (*), \underline{I}_{m+1} is positive definite.

Equation (*) also determines a one real parameter system of rational functions satisfying the original unextended interpolation problem, each ~~quotient being~~ function being the quotient of two m^{th} degree polynomials

With h_{m+1} fixed upon the circle described by equation (*) there is a unique function of the system, which is also an N-function, which assumes the values h_k ($k=1, \dots, m+1$) when its argument assumes the value λ_k ($k=1, \dots, m+1$). This rational function has real coefficients; it also assumes the value \bar{h}_k when its argument is equal to $\bar{\lambda}_k$ ($k=1, \dots, m+1$). It is homogeneous in its numerator and denominator coefficients and therefore possesses ~~$2m+1$ degrees of freedom~~ disposable parameters. These are determined by the foregoing $2m+2$ interpolatory conditions which form an overdetermined but consistent rational interpolation problem. The entire one real parameter system of rational N-functions each of them the quotient of two m^{th} degree polynomials, which assumes the values h_k

$\dots, m)$ when their arguments take the values λ_k ($k=1, \dots, m$) is obtained by letting h_{m+1} traverse the above circle (the real parameter in question may, for example, be taken to be $\tan[\arg\{\frac{1}{2}(h_{m+1} - C)\}]$ where C is the centre of the circle).

This entire system may also be constructed recursively as follows: at the k^{th} stage it is assumed that the entire system concerning λ_r, h_r ($r=1, \dots, k-1$) has been constructed. Two distinct quotients $Q'_{k-1}(\lambda)/P'_{k-1}(\lambda)$ and $Q''_{k-1}(\lambda)/P''_{k-1}(\lambda)$ of $(k-1)^{\text{th}}$ degree polynomials are selected from this system; a fractional linear function with real coefficients

$$\stackrel{(35)}{\Delta(\lambda)} = \frac{a\lambda+b}{c\lambda+d}$$

which maps the upper half-plane into itself (so that $ad-bc > 0$) is introduced, and its coefficient are subjected to the constraint that when $\lambda=\lambda_k$ the quotient

$$(36) \quad \frac{\Delta(\lambda) Q'_{k-1}(\lambda) + Q''_{k-1}(\lambda)}{\Delta(\lambda) P'_{k-1}(\lambda) + P''_{k-1}(\lambda)}$$

should assume the value h_k (by considering real and imaginary parts, this constraint imposes two conditions involving real numbers upon the coefficients a, \dots, d). Subject to the constraints upon the coefficients in formula (35) just mentioned (36) is the general expression for all quotients GEN of the k^{th} degree polynomials for which $G(\lambda_r) = h_r$ ($r=1, \dots, k$) (since expression (35) is homogeneous in the coefficients a, \dots, d , expression (36) represents a one real parameter family of rational functions). The above process may be initialised by taking $Q'_0(\lambda) = P''_0(\lambda) = 0$, $Q''_0(\lambda) = P'_0(\lambda) = 1$. When $k=m$, the rational functions produced are those of the preceding paragraph.

The above results may be extended to the

in which certain of the λ_k tend to identical values, and it is required that systems of successive derivatives of the interpolating N -function should assume prescribed values at common argument values. In the extended theory, certain of the difference quotients of the form $(^{(3)})$ which occur as elements of the Hermitian matrix $\hat{\Psi}_m$ are to be replaced by difference quotients involving derivatives.

It is clear from the above conspectus of results from the Pick-Nevanlinna theory, that the solution of the problem of determining a rational function which satisfies the mapping and interpolation conditions described above differs radically from that of constructing a rational function which satisfies interpolation conditions alone in at least two respects: in the solution of the first problem a combination of function-theoretic and algebraic methods is involved, while in that of the se-

algebraic methods are exclusively deployed; furthermore, each stage of the solution of the first problem results, not in the construction of a single interpolating function as is the case for the second problem, but in that of a ^{family} of functions with the required properties. Nevertheless, in the case in which the prescribed argument and function and derivative values are symmetrically distributed with respect to the imaginary axis, there are two extremal solutions, also with properties of symmetry, that can be extracted by purely algebraic methods alone: the construction of circles in the complex plane, as described above, is made redundant. This possibility arises in the following way. In the general case each interpolatory quotient G of m^{th} degree polynomials satisfies conditions of the form $G(\lambda_k) = h_k$, ~~f_{k+1}, \dots, f_m~~ $G(\bar{\lambda}_k) = \bar{h}_k$ ($k=1, \dots, m+1$)

where the last two conditions, relating to the values for $k=m+1$, are obtained by placing h_{m+1} upon a circle constructed as described above. This system of conditions is overdetermined: the last condition relating to the pair $(\bar{\lambda}_{m+1}, \bar{h}_{m+1})$ may be dropped, but it is still necessary to construct the circle. In the case in which the argument and function values (λ_k, h_k) ($k=1, \dots, m$) are symmetrically distributed with respect to the imaginary axis, it may be shown that the above circle has its centre on the imaginary axis, and that upon this circle ~~there are~~ two values of h_{m+1} may be chosen such that the resulting interpolating functions G satisfy conditions of the form $G(\bar{\lambda}) = -G(\lambda)$. It follows from this condition that for the first function the coefficient of λ^m in the numerator is zero, and for the second that it holds true in the denominator. In both cases

the known zero value of the coefficient concerned may be removed from the above system & interpolatory conditions and, in consequence, the condition involving h_{m+1} may be discarded: the circle itself may be discarded.

In the following theorem the symbol $\Delta^{\tau} H_j$ represents a numerical value which, as will transpire must be assumed by the derivatives $\Delta^{\tau} G(\lambda_j)$ of a function G at the point λ_j , the symbol Δ in the latter case being used in the sense explained at the beginning of this paper. Composite expressions involving Δ , λ_j and $\Delta^{\tau} H_j$ are to be interpreted according to Leibniz' rule: thus $\Delta\{\lambda_j; \Delta^{\tau} H_j\}$ means $\Delta^{\tau} H_j + \lambda_j \Delta^{\tau+1} H_j$.

Theorem . Let the distinct argument values λ_j ($j=1, \dots, T$) in W be members of two subsets: for the elements of the first $\operatorname{Re}(\lambda_j) \neq 0$ ($j=1, \dots, q$),

this subset containing no conjugate pair $\Lambda_j, \bar{\Lambda}_k$ for which $\Lambda_j = -\bar{\Lambda}_k$; for those of the second, $\text{Re}(\Lambda_j) > 0$ ($j = q+1, \dots, J$); either set may be empty (i.e. 0 and J are possible values of q). Let $T(j) \geq 1$ be a sequence of integers, and let $(j=1, \dots, J)$ be a sequence of prescribed derivative values corresponding to the argument value Λ_j ($j=1, \dots, J$). Set $R = \sum_{j=1}^J T(j)$ and

$$m = 2 \sum_{j=1}^q T(j) + \sum_{j=q+1}^J T(j).$$

Assume that the problem of finding a function

G which

a) satisfies the interpolation conditions

$$\Delta^z G(\Lambda_j) = \Delta^z H_j, \quad \Delta^z G(-\bar{\Lambda}_j) = -\overline{\Delta^z H_j}$$

for $z = 0, \dots, T(j)-1$ ($j = 1, \dots, q$) and

$$\Delta^z G(\Lambda_j) = \Delta^z H_j$$

for $z = 0, \dots, T(j)-1$ ($j = q+1, \dots, J$) and

b) maps \mathbb{W} into itself &
~~is~~ ~~solvble~~ has a solution.

A) If this solution is the quotient of two polynomial
 both of degrees less than m , then this rational
 function is the only solution of the above interpolation
 problem.

B) Otherwise

(i) there are two rational functions $W_R^{(\omega)}$ ($\omega=0,1$)

having the forms

$$W_R^{(0)}(\lambda) = \frac{U_0^{(R)}}{\lambda} + \sum_{j=1}^N \left\{ \frac{U_j^{(R)}}{u_j^{(R)} - \lambda} - \frac{U_j^{(R)}}{u_j^{(R)} + \lambda} \right\}$$

$$W_R^{(1)}(\lambda) = V_0^{(R)} \lambda + \sum_{j=1}^N \left\{ \frac{V_j^{(R)}}{v_j^{(R)} - \lambda} - \frac{V_j^{(R)}}{v_j^{(R)} + \lambda} \right\}$$

if $m = 2N+1$ is odd, and

$$W_R^{(0)}(\lambda) = \sum_{j=1}^{N+1} \left\{ \frac{U_j^{(R)}}{u_j^{(R)} - \lambda} - \frac{U_j^{(R)}}{u_j^{(R)} + \lambda} \right\}$$

$$W_R^{(1)}(\lambda) = V_0^{(R)} \lambda - \frac{V_1^{(R)}}{\lambda} + \sum_{j=1}^N \left\{ \frac{V_j^{(R)}}{v_j^{(R)} - \lambda} - \frac{V_j^{(R)}}{v_j^{(R)} + \lambda} \right\}$$

If $m=2N+2$ is even, which with $G(\lambda) = W_R^{(0)}(\lambda)$
 $(\omega=0,1)$ are solutions to the symmetric interpolation
 problem (α, β) .

These two functions are the only two rational
 functions expressible as quotients of two
 polynomials of degree m and also satisfying
 an equation of the form $G(-\bar{\lambda}) = -\overline{G(\lambda)}$ for
 all $\lambda \in \mathbb{D}$, that are solutions to the interpolation
 problem (α, β) .

(ii) The rational functions $W_R^{(\omega)}$ may be obtained
 directly as follows. When $m=2N+1$ is odd, the

system of $m+1$ linear equations

$$\sum_{j=0}^N \operatorname{Re} \left[2^z \left\{ H_j \Delta_j^{2j+1} \right\} \right] P_{0,R,j}^{(0)} = \sum_{j=0}^N \operatorname{Re} \left[2^z \Delta_j^{2j} \right] P_{1,R,j}^{(0)}$$

$$\sum_{j=0}^N \operatorname{Im} \left[2^z \left\{ H_j \Delta_j^{2j+1} \right\} \right] P_{0,R,j}^{(0)} = \sum_{j=0}^N \operatorname{Im} \left[2^z \Delta_j^{2j} \right] P_{1,R,j}^{(0)}$$

for $j=1, \dots, q$; $z=0, \dots, T(j)-1$ and

$$\sum_{j=0}^N \Delta^z \{ H_j \lambda_1^{2D+1} \} P_{0,R,D}^{(0)} = \sum_{j=0}^N \Delta^z \{ \lambda_j^{2D+1} \} P_{1,R,D}^{(0)}$$

for $j=q+1, \dots, T; z=0, \dots, T(j)-1$ with $P_{0,R,N}^{(0)}=1$ is solved for the $m+1$ coefficients $P_{0,R,D}^{(0)}, P_{1,R,D}^{(0)}, \dots, P_{m,R,D}^{(0)}$ and then

$$W_R^{(0)}(\lambda) = \frac{P_{1,R}^{(0)}(\lambda)}{P_{0,R}^{(0)}(\lambda)}$$

where

$$P_{\mu,R}^{(0)}(\lambda) = \sum_{j=0}^N P_{\mu,R,D}^{(0)} \lambda^{2D-\mu+1}$$

for $\mu=0,1$. Again when $m=2N+1$

$$W_R^{(1)}(\lambda) = \frac{P_{1,R}^{(1)}(\lambda)}{P_{0,R}^{(1)}(\lambda)}$$

where

$$P_{\mu,R}^{(1)}(\lambda) = \sum_{j=0}^N P_{\mu,R,D}^{(1)} \lambda^{2D+\mu}$$

for $\mu=0,1$. A further system of linear equations, analogous to those given above, with $P_{0,R,N}^{(1)}=1$, may be constructed and solved for the $m+1$ coefficients.

$$\{ P_{\mu, R, \omega}^{(1)} \} (\mu=0,1).$$

When $m = 2N+2$ is even

$$() W_R^{(\omega)}(\lambda) = \frac{P_{1,R}^{(\omega)}(\lambda)}{P_{0,R}^{(\omega)}(\lambda)}$$

for $\omega = 0,1$ where

$$P_{\mu, R}^{(\omega)}(\lambda) = \sum_{\nu=0}^{N-|\omega-\mu|+1} P_{\mu, R, \nu}^{(\omega)} \lambda^{2\nu+|\omega-\mu|}$$

for $\omega, \mu = 0,1$ and $P_{0,R,N-\omega+1}^{(\omega)} = 1$ ($\omega=0,1$) may

be taken. Systems of linear algebraic equations may be constructed and solved for the coefficients in the two rational functions () as indicated above.

(iii) The rational functions $W_R^{(\omega)}$ may also be obtained as the end result of the following recursive process. Let the sequence λ_r ($r=1, \dots, R$) be formed from sets containing $T(j)$ copies of Λ_j ($j=1, \dots, T$), these elements being arranged in any order open to choice but

fixed at the outset. Let H'_r ($r=1, \dots, R$) be an extension and rearrangement of the sequence H_j ($j=1, \dots, \sigma$) which corresponds to the sequence λ'_r ($r=1, \dots, r$) (so that pairs (λ'_r, H'_r) are preserved from (λ_j, H_j)). Let $t(r)$ be the number of times the value of λ'_r is to be found in the subsequence $\lambda'_1, \dots, \lambda'_{r-1}$ ($r=1, \dots, R$)

Four sequences of polynomials $P_{\mu, r}^{(\omega)}$ (~~λ'_r~~ ($r=0, \dots, R$) for $\omega, \mu = 0, 1$) are to be determined, use being made of values of differentiated residual functions defined by the formula

$$P_{\lambda'_r, r}^{(\omega)} = \Delta^{t(r)} \left[\lambda'_r^{\times} \{ P_{0, r-1}^{(\omega)}(\lambda'_r) H'_r - P_{1, r-1}^{(\omega)}(\lambda'_r) \} \right]$$

Initially

$$P_{0, 0}^{(0)}(\lambda) = P_{1, 0}^{(1)}(\lambda) = 1, \quad P_{1, 0}^{(0)}(\lambda) = P_{0, 0}^{(1)}(\lambda) = 0.$$

For $r=1, \dots, R$ the values $\rho_{x_r, r}^{(\omega)}$ ($\omega, x_r = 0, 1$) are determined and

a) if $\operatorname{Re}(\lambda'_r) = 0$ the numbers

$$\Theta_r = \rho_{1,r}^{(0)} / \rho_{0,r}^{(1)}, \quad \phi_r = \rho_{0,r}^{(0)} / (\rho_{0,r}^{(0)} - \rho_{1,r}^{(1)})$$

and polynomials

$$P_{\mu,r}^{(0)}(\lambda) = \lambda P_{\mu,r-1}^{(0)}(\lambda) - \Theta_r P_{\mu,r-1}^{(1)}(\lambda)$$

$$P_{\mu,r}^{(1)}(\lambda) = (1 - \phi_r) P_{\mu,r-1}^{(0)}(\lambda) + \phi_r P_{\mu,r-1}^{(1)}(\lambda)$$

both for $\mu = 0, 1$ are also determined, while

b) if $\operatorname{Re}(\lambda'_r) \neq 0$, the additional values $\rho_{2,r}^{(\omega)}$ ($\omega = 0, 1$), numbers

$$\alpha_1 = \operatorname{Im}(\bar{\rho}_{2,r}^{(0)} \rho_{1,r}^{(1)}), \quad \alpha_2 = \operatorname{Im}(\bar{\rho}_{0,r}^{(0)} \rho_{2,r}^{(0)}), \quad \alpha_3 = \operatorname{Im}(\bar{\rho}_1^{(0)} \rho_0^{(1)})$$

$$b_1 = \operatorname{Im}(\bar{\rho}_{0,r}^{(0)} \rho_{1,r}^{(1)}), \quad b_2 = \operatorname{Im}(\bar{\rho}_{2,r}^{(1)} \rho_{0,r}^{(1)})$$

and

$$\alpha_r = \alpha_1 / b_1, \quad \beta_r = \alpha_2 / b_1, \quad \gamma_r = b_2 / (\alpha_3 - b_2)$$

and polynomials

$$P_{\mu, r}^{(0)}(\lambda) = (\lambda^2 - \alpha_r) P_{\mu, r-1}^{(0)}(\lambda) - \beta_r \lambda P_{\mu, r-1}^{(1)}(\lambda)$$

$$P_{\mu, r}^{(1)}(\lambda) = \{(1 + \gamma_r)\lambda^2 + \gamma_r\} P_{\mu, r-1}^{(1)}(\lambda) - \gamma_r \lambda P_{\mu, r-1}^{(0)}(\lambda)$$

both for $\mu=0,1$ are determined.

Let the number of members of the sequence $\lambda_1, \dots, \lambda_r$ with nonzero real part be q' , and set $m(r) = q' + r$. With m replaced by $m(r)$, and the definition of N correspondingly modified, the forms of the polynomials constructed by means of the above process are as exhibited by formulae (-) of (ii).

The rational functions

$$W_r^{(\infty)}(\lambda) = \frac{P_{1,r}^{(\infty)}(\lambda)}{P_{0,r}^{(\infty)}(\lambda)}$$

($\omega=0,1; r=1, \dots, R$) formed from the polynomials produced by the above recursive process are extremal solutions of intermediate interpolation

problems. Denote the distinct members of the
 sequence $\lambda'_1, \dots, \lambda'_r$ with nonzero real part by
 λ''_j ($j=1, \dots, q'$) and those with zero real part
 by λ''_j ($j=q'+1, \dots, T'$); let the number of times
 that λ''_j occurs in this sequence be $T'(j)$ ($j=1,$
 \dots, T'). Let the function values H''_j ($j=1, \dots, T'$)
 be extracted from the sequence H_1, \dots, H_r in
 an order corresponding to the λ''_j (so that
 pairs (λ''_j, H''_j) are preserved from pairs (λ'_j, H_j) .
 The functions (\cdot) are extremal solutions of
 the interpolation problem obtained from (α, β)
 above by replacing q, J, T, Λ and H by
 q', T', T'', Λ'' and H'' respectively. Mutatis
 mutandis, they have the properties attributed
 to $W_R^{(\alpha)}$ above.

Proof. The interpolation problem (α, β) is a Pick-

Nevanlinna problem of degree m in which firstly the argument and function values are symmetrically distributed with respect to the imaginary axis, and secondly the m argument values have been allowed to coalesce to $J+q$ points Λ_j and $-\bar{\Lambda}_j$ for $j=1, \dots, q$ and Λ_j for $j=q+1, \dots, J$. It has been assumed at the outset that this problem is soluble. Either, as under the conditions of part A), the problem has a unique solution in the form of a quotient of two polynomials of degrees less than m (as will subsequently transpire, this quotient then necessarily has one of the forms () or ()) or, as under the alternative conditions of part B), there exists a class of quotients of two polynomials of degree m , each quotient satisfying the stated interpolation problem.

Part A has been dealt with; it is henceforth assumed that the conditions of part B hold. To simplify the initial exposition it is assumed that $T(j) = 1$ ($j = 1, \dots, T$) and to relate the given problem to that discussed above, the argument values are relabelled according to the scheme $\lambda_{2j-1} = \Lambda_j$, $\lambda_{2j} = -\bar{\Lambda}_j$ for $j = 1, \dots, q$ and $\lambda_{q+j} = \Lambda_j$ for $j = q+1, \dots, T$. The data values H_j are relabelled as h_j according to a similar scheme. The restricted version of the problem (α, β) under consideration has now been reduced to the form () with $k = 1, \dots, m$.

³³
 The Hermitian matrix $\tilde{\Psi}_m$ with elements is positive definite. Select a point λ_{m+1} on the positive imaginary axis and not identical with λ_k ($k = 2q+1, \dots, m$). There is a system of N-funct

every one of them an ~~function~~ irreducible quotient
 of two m^{th} degree polynomials, which assume the
 value h_k when their argument takes the value λ_k
 ($k=1, \dots, m$); individual functions of the system are
 distinguished by the value h_{m+1} assumed when their
 argument takes the value λ_{m+1} ; admissible values
 λ_{m+1} lie on the circle described by equation
 $(*)$. Denote the constant Ξ in this equation
 by $\Xi(\lambda_{m+1})$. In the determinantal formula for
 $\Xi(\lambda_{m+1})$ described above, replace λ_{2j-1} by $-\bar{\lambda}_{2j}$,
 λ_{2j} by $-\bar{\lambda}_{2j-1}$, h_{2j-1} by $-\bar{h}_{2j}$ and h_{2j} by $-\bar{h}_{2j-1}$
 $(j=1, \dots, q)$ and also λ_j by $-\bar{\lambda}_j$ and h_j by $-\bar{h}_j$
 $(j=2q+1, \dots, m)$; thereafter interchange rows
 $2j-1$ and $2j$ and also columns $2j-1$ and $2j$
 $(j=1, \dots, q)$; comparing the resultant expression
 with the original form it is found that $\Xi(-\bar{\lambda})$

$= -\overline{\Sigma(\lambda_{m+1})}$. Since λ_{m+1} is pure imaginary, $\operatorname{Re}\{\Sigma(\lambda_{m+1})\} = 0$: the centre of the circle described by equation (34) lies on the imaginary axis. There are two pure imaginary admissible values of h_{m+1} , determined by the intersection of this circle with the imaginary axis; that within shortest distance of the origin will be referred to as $h_{m+1}^{(0)}$, the other as $h_{m+1}^{(1)}$.

In the expression of an irreducible quation

$$(35) \quad G'(\lambda) = \frac{\sum_{j=0}^m Q_j \lambda^j}{\sum_{j=0}^m P_j \lambda^j}$$

which is also an N-function, the coefficient P_m either is not or is zero. In the former case, due to the homogeneous nature of expression (38), P_m may be set equal to unity and G' then has the form

$$(39) \quad G'(\lambda) = C + \sum_{j=1}^m M_j (t_j - \lambda)^{-1}$$

where $-\infty < C < \infty$, $0 < M_j < \infty$, $-\infty < t_j < \infty$ ($j=1, \dots, m$);
in the alternative case P_{m+1} may be set equal to unity and G' then has the form

$$G'(\lambda) = M\lambda + C + \sum_{j=1}^{m+1} M_j (t_j - \lambda)^{-1}$$

where $0 < M < \infty$, and C and the M_j, t_j are as above.

Since, in both of the above cases, $G'(\lambda)$ is real for

real λ , it follows that if $G'(\lambda_k) = h_k$ ($k=1, \dots, m+1$)

then $G'(\bar{\lambda}_k) = \bar{h}_k$ ($k=1, \dots, m+1$) also. If the set λ_k ($k=1, \dots, m+1$) consists of conjugate and pure imaginary points as described above, and is extended by the addition of further points $\lambda_{m+k+1} = \bar{\lambda}_k$ ($k=1, \dots, m+1$)

then $\lambda_{m+2j} = \bar{\lambda}_{2j-1} = -\lambda_{2j}$, $\lambda_{m+2j+1} = \bar{\lambda}_{2j} = -\lambda_{2j-1}$ ($j=1, \dots$)

and $\lambda_{m+j+1} = \bar{\lambda}_j = -\lambda_j$ ($j=2q+1, \dots, m+1$): introducing a suitable renumberation, the extended set is the union of two subsets λ''_k ($k=1, \dots, m+1$) and $-\lambda''_k$ ($k=1, \dots, m+1$).

The set h_k ($k=1, \dots, m+1$) may be treated in the same way: whether h_{m+1} is taken to be $h_{m+1}^{(0)}$ or $h_{m+1}^{(1)}$, the end

set is the union of two corresponding subsets h_k
 $(k=1, \dots, m+1)$ and $-h_k''$ ($k=1, \dots, m+1$).

Attention is now directed to the case in which $m = 2N+1$ is odd. The N-function expressible in the form $(^3)$ and taking the value h_k when $\lambda = \lambda_k$ ($k=1, \dots, m$) and $h_{m+1}^{(0)}$ when $\lambda = \lambda_{m+1}$ is denoted by $G^{(0)}$. If it exists it (i.e. if $P_m \neq 0$ in the equivalent representation $(^2)$) satisfies a system of constraints $G^{(0)}(\lambda_k') = h_k''$ and $G^{(0)}(-\lambda_k') = -h_k''$ ($k=1, \dots, m+1$), and is the only N-function expressible in the form $(^3)$ to do so. It contains $2m+1$ disposable parameters, as does the equivalent homogeneous representation $(^8)$; those in the latter may be determined by use of the first $2m+1$ of the above constraints, each of which may be expressed as a linear algebraic equation in which $2m+1$ of the P_ν and Q_ν appear as unknown variables. This system of equations, if satisfied at all, is known to have one and only one solution, which automatically satisfies the last constraint $G^{(0)}(-\lambda_{m+1}'') = -h_{m+1}''$.

Since P_m in the above is nonzero, its value may be taken to be unity, and the $2m+2$ constraints described above assume the form

$$(40) \sum_{k=0}^N Q_{2k} \lambda_k^{2k} + \sum_{k=0}^N Q_{2k+1} \lambda_k^{2k+1} - \sum_{k=0}^N P_{2k} h_k \lambda_k^{2k} - \sum_{k=0}^{N-1} P_{2k+1} h_k \lambda_k^{2k+1} = h_k \lambda_k^{2N+1}$$

$$(41) \sum_{k=0}^N Q_{2k} \lambda_k^{2k} - \sum_{k=0}^N Q_{2k+1} \lambda_k^{2k+1} + \sum_{k=0}^N P_{2k} h_k \lambda_k^{2k+1} - \sum_{k=0}^{N-1} P_{2k+1} h_k \lambda_k^{2k+1} = h_k \lambda_k^{2N+1}$$

both for $k=1, \dots, m+1$ (the dashes have been dropped from the symbols λ_k^{2k}, h_k for typographical economy).

Corresponding equations from these two sets may be added together to produce $m+1$ equations involving m variables Q_{2k} and P_{2k+1} ; one of these equations may be discarded and the resulting system solved for these variables. Equations (40, 41) may also be subtracted to form a system of $m+1$ equations involving $m+1$ variables Q_{2k+1} and P_{2k} ; the right hand side entries of this system are all zero, and also ~~all zero~~, and in consequence all Q_{2k+1} and P_{2k} are zero also. With this condition the two represent

(^{3x}, ^{3y}) yield the special representation

$$(\text{42}) \quad G'(\lambda) = M_0 \lambda^{-1} + \sum_{\nu=1}^N M_\nu \left\{ (t_\nu - \lambda)^{-1} - (t_\nu + \lambda)^{-1} \right\}$$

The assumption that $P_m = 0$ and $P_{m-1} = 1$ may also be taken into account and it yields a representation of the form

$$(\text{43}) \quad G'(\lambda) = C\lambda + \sum_{\nu=1}^N M_\nu \left\{ (t_\nu - \lambda)^{-1} - (t_\nu + \lambda)^{-1} \right\}$$

When $m = 2N+2$ is even, the two representations derived

are

$$(\text{44}) \quad G'(\lambda) = \sum_{\nu=1}^{N+1} M_\nu \left\{ (t_\nu - \lambda)^{-1} - (t_\nu + \lambda)^{-1} \right\}$$

$$(\text{45}) \quad G'(\lambda) = C\lambda + M_0 \lambda^{-1} + \sum_{\nu=1}^N M_\nu \left\{ (t_\nu - \lambda)^{-1} - (t_\nu + \lambda)^{-1} \right\}$$

Expressions (⁴², ⁴³) possess a required symmetry property of the form $G(-\bar{\lambda}) = -\bar{G}(\lambda)$. It is at this stage still not known which of them assumes the

value $h_{m+1}^{(0)}$ when $\lambda = \lambda_{m+1}$; one of them does, and this

one also assumes the value h_k when $\lambda = \lambda_k$ ($k = 1, \dots, N$)

With λ_{m+1} fixed, the selection of a point $h_{m+1}^{(0)}$ which

lies at the intersection of the imaginary axis and the circle defined by equation (37) led with particular ease to the construction of a symmetric N-function (12) or (13) which assumes the value h_k when $\lambda = \lambda_k$ ($k=1, \dots, m$). Nevertheless, the possibility is not at this stage excluded that the selection of another ~~inter-~~
 interpolating function value upon this circle might also have led to the construction of a further function with, for example, the symmetry properties of expression (12) which also assumes the values h_k ($k=1, \dots, m$) as described. It will now be shown that two distinct functions of this type do not exist.

With $m=2N+1$ odd, it is supposed that two distinct functions of the form (12) with the stated interpolation properties; they have the form $Q_{\chi}(\lambda)/P_{\chi}(\lambda)$ ($\chi=1, 2$) where

$$Q_{\chi}(\lambda) = \sum_{j=0}^N Q_{\chi,j} \lambda^{2j}, \quad P_{\chi}(\lambda) = \sum_{j=0}^n P_{\chi,j} \lambda^{2j+1}$$

With λ_{m+1} fixed on the positive imaginary axis, a point h'_{m+1} lying on that part of the imaginary axis lying inside the circle of equation () is selected. The value of the two quotients $Q_{\chi}(\lambda_{m+1})/P_{\chi}(\lambda_{m+1})$ ($\chi=1,2$) lie upon the circumference of the circle; hence $Q_{\chi}(\lambda_{m+1})/P_{\chi}(\lambda_{m+1}) \neq h'_{m+1}$ ($\chi=1,2$). With $h_{m+1} = h'_{m+1}$, the Hermitian matrix $\tilde{\Psi}_{m+1}$ of the form given above is positive definite.

It is possible to determine a rational N-function G the irreducible quotient of two polynomials of degree $m+1$ such that $G''(\lambda_k) = h_k$ ($k=1, \dots, m+1$) and which maps the positive imaginary axis onto itself (it has one of the forms () or ()). All rational N-functions that one quotients of two polynomials of degree $m+1$ and assume the value h_k when their arguments take the value λ_k ($k=1, \dots, m+1$) have the form

$${}^{(46)} \quad \frac{\Delta(\lambda) Q_1(\lambda) + Q_2(\lambda)}{\Delta(\lambda) P_1(\lambda) + P_2(\lambda)}$$

where Δ is a fractional linear transformation which maps the upper half-plane into itself, and whose coefficients are subject to the further restriction that the quotient ${}^{(46)}$ should assume the value h'_{m+1} when $\lambda = \lambda_{m+1}$. The mapping property imputed to Δ implies that $\Delta(\lambda)$ is not simply a real constant. $G'(\lambda)$, in particular, has a representation of the form ${}^{(46)}$ and maps the positive imaginary axis into itself. When λ is pure imaginary, $Q_y(\lambda)$ is pure real and $P_y(\lambda)$ is pure imaginary ($y=1, 2, \dots, m$). The quotient ${}^{(46)}$ assumes pure imaginary value for all pure imaginary λ only when $\Delta(\lambda)$ is real for such λ . It is required of the fractional linear transformation $\Delta(\lambda)$ whose insertion into

expression (4⁰) produces the function $G''(\lambda)$, that it
 should map \mathbb{W} into itself and the positive imaginary
 axis into part of the real axis. Too much is
 asked; the initial assumption that there could be two
 distinct functions of the form (1²) which assume
 the value h_k when their argument takes the value
 λ_k ($k=1, \dots, m$) with $m=2N+1$ is false. Similarly it
 may be shown that two distinct functions of the
 form (1³) with the stated interpolation property do
 not exist. The case in which m is even is dismissed
 in the same way.

It has been established that, when $m=2N+1$ is
 odd, one of the functions (1², 1³) is such that $G'(\lambda_k) =$
 h_k ($k=1, \dots, m$) and also $G'(\lambda_{m+1}) = h_{m+1}^{(0)}$, and that
 this function is the only function of its form to
 satisfy the first of these interpolatory properties. What
 of these functions is that in question has yet to

to be determined. Similar doubts concerning which of
 the two functions assumes the value $h_{m+1}^{(1)}$ when $\lambda = \lambda_{m+1}$
 also exist. The point λ_{m+1} may be slightly perturbed
 from its initial fixed value. The point $h_{m+1}^{(0)}$,
 originally defined by the intersection of a circle
 with the imaginary axis, now becomes a dependent
 variable, $h_{m+1}^{(0)}(\lambda_{m+1})$ say: it is the root of smaller
 modulus of a quadratic equation function derived
 from equation (34). As is easily demonstrated,
 $h_{m+1}^{(0)}(\lambda_{m+1})$ remains bounded for large pure
 imaginary values of λ_{m+1} . As λ_{m+1} varies, new
 attempts to construct functions of the form (42-43)
 may be made, using the varying value of $h_{m+1}^{(0)}$.
 Since the original functions are the only function
 of their type to take the value h_k when the argument
 assumes the value λ_k ($k=1, \dots, m$), the same two

functions are advanced as candidates for every value of λ_{m+1} . In short, the value of $h^{(6)}(\lambda_{m+1})$ is that of one of the functions $(^{42}, ^{43})$ when $\lambda = \lambda_{m+1}$. Of these two functions, only that represented by formula $(^{42})$ is bounded for large pure imaginary values of λ . It is the only N-function having the symmetry property $G(-\bar{\lambda}) = -\bar{G}(\lambda)$, expressible as the quotient of two m^{th} degree polynomials, satisfying the interpolation conditions $()$ for $k=1, \dots, m$, and also bounded for large pure imaginary values of λ . Similarly it may be shown that there is only one N-function with all properties just described except the last, which is to be replaced by the condition that the function should be unbounded for large pure imaginary values of λ : the function is that of formula $()$. The

same two conclusions may be drawn for the case in which $m = 2N+2$ is even. Certain of the argument values λ_k may be allowed to coalesce; the conclusions still remain valid. The existence and uniqueness result concerning the extremal solutions of the symmetric interpolation problem (a, b) and stated as clause (ii) of part B) has been established

For the case considered in detail above ($m = 2N+1$ odd, and all $T(j)$ concerned in formulae equal to unity) a reduced system of linear equations may be derived from equations (1), simply by discarding the sums involving the $\{Q_{2j+1}\}$ and $\{P_{2j}\}$ all of which are known to be zero. The reduced system contains complex valued coefficients (relating to the argument values with suffix $j \leq 2q$) and is known to have

real valued solutions. Accordingly the coefficients may be decomposed into real and imaginary components; further systems of linear equations with real valued coefficients alone may be obtained. The treatment of the case in which certain of the λ_k are allowed to coalesce is simplified by use of the remark that if, for a certain λ , $\sum \{Q(\lambda)/P(\lambda)\} = H(\lambda)$ for $z=0, \dots, T$, then equally $\sum Q(\lambda) = \sum \{P(\lambda)H(\lambda)\}$ for $(z=0, \dots, T)$ for this value of λ . Equations of the form () involving derivatives of rational functions may be replaced by simpler equations involving derivatives of polynomials. Replacing the symbols $P_{2\omega+1}$ and $Q_{2\omega}$ by $P_{0,\omega}^{(0)}$ and $P_{1,\omega}^{(0)}$, these equations, in the confluent case, reduce to the forms (,). The remaining results of clause (ii) are derived in a similar manner.

To prove clause (iii) it is first supposed that $\Im=1, T(1)=1$ in the conditions of the theorem. If $\text{Re}(\lambda'_1)=0$, the rational functions produced by use of the algorithm described are

$$W_1^{(0)}(\lambda) = h'_1 \lambda'_1 / \lambda, \quad W_1^{(1)}(\lambda) = (h'_1 / \lambda'_1) \lambda$$

If $\text{Re}(\lambda'_1) \neq 0$, they reduce to

$$W_1^{(0)}(\lambda) = \frac{h'_1 \bar{h}'_1 (\lambda'^2 - \bar{\lambda}'^2) \lambda}{h'_1 \bar{\lambda}'_1 (\lambda^2 - \bar{\lambda}'^2) + \bar{h}'_1 \lambda'_1 (\lambda^2 - \bar{\lambda}'^2)}$$

$$W_1^{(1)}(\lambda) = \frac{h'_1 \bar{\lambda}'_1 (\lambda^2 - \bar{\lambda}'^2) + \bar{h}'_1 \bar{\lambda}'_1 (\lambda^2 - \bar{\lambda}'^2)}{(\lambda^2 - \bar{\lambda}'^2) \lambda}$$

In both cases the above extremal solutions $W_1^{(0)}$ ($\omega=0, 1$) are identical with those produced, for

the simple values of m and $T(1)$ concerned, as in clause (ii). Indeed all statements concerning these extremal solutions made in clause (iii) hold

It is now assumed that all results of clause (iii)
are correct, and a new interpolation argument
 $\lambda_{m+1} \in \mathbb{W}$ is considered. It is assumed that
 $\lambda_j = \bar{\lambda}_j$ ($j=1, \dots, q$) and λ_j
differs from all $\lambda_j - \bar{\lambda}_j$ ($j=q+1, \dots, J$) and is pure imaginary
($j=q+1, \dots, J$) and is pure imaginary
is pure imaginary and differs from all λ_j ($j=$
 $q+1, \dots, J$) ^{a priori} (^{ipso facto}), it also differs from $\lambda_j - \bar{\lambda}_j$
($j=1, \dots, q$); a corresponding pure imaginary
function value $h_{m+1} \in \mathbb{W}$, for which the extended
interpolation problem (α, β) is soluble, is also
introduced. Two distinct rational functions, expressed
as quotients of polynomials of degrees m , are
known to satisfy the interpolation problem (α, β)
involving m points; they are given by formulae
(α, β) or formula (α, β). The extremal solution
to the extended interpolation problem (α, β)

also exist, they have representations of the form
 $(\ , \)$ or $(\)$; they also have representations of
 the form

$$\frac{(a\lambda + b)P_{1,R}^{(0)}(\lambda) + (c\lambda + d)P_{1,R}^{(1)}(\lambda)}{(a\lambda + b)P_{0,R}^{(0)}(\lambda) + (c\lambda + d)P_{0,R}^{(1)}(\lambda)}$$

where the fractional linear transformation $(a\lambda + b)/(c\lambda + d)$
 has certain mapping and algebraic properties
 described in connection with formula $(\)$. If expression
 $(\)$ is to reduce to

$$\frac{P_{1,R+1}^{(0)}(\lambda)}{P_{1,R+1}^{(1)}(\lambda)}$$

where the polynomials $P_{\mu,R+1}^{(0)} (\mu=0,1)$ have forms,
 depending upon whether $m+1$ is odd or even,
 exhibited in clause (ii) then, from inspection of
 odd and even powers of λ in both numerator and
 denominator expressions $(\ , \)$, $b=c=0$. If the

normalisation convention that the coefficient of highest power of λ in $P_{0,R+1}^{(\omega)}(\lambda)$ should be unity is to be preserved, then $a=1$. The remaining unknown coefficient d in expression () is determined by equating the value of this expression with $\lambda=\lambda_{m+1}$ to h_{m+1} ; the value of d obtained in this way for θ_r is that of θ_r in formula () modified by replacement of the suffix r by $R+m+1$. In short, if the algorithm of clause (iii) produces the extremal solutions $W_R^{(\omega)}$ of the symmetric interpolation problem ($\omega=0,1$) of the interpolation problem extended in the manner described. That it also produces the corresponding extremal solution $W_{R+1}^{(1)}$ is verified in the same way.

If λ_{m+1} , although still unequal to λ_k ($k=1, \dots, m$), is not pure imaginary, two stages of the construction by use of fractional linear transformations as discussed in connection with formula () must be taken into account. In the resulting expression of the form (), the linear factors $a\lambda+b$ and $c\lambda+d$ must be replaced by quadratic factors. After powers of λ have been inspected and suitable coefficients in the quadratic factors equated to zero, and after the normalisation of the denominator has been taken into account, two unknown real coefficients remain. They are obtained from the single interpolating condition at the argument value $\lambda=\lambda_{m+1}$ by considering the real and imaginary parts of the resulting equation. In this way versions of the quadratic extension formulae of subclause b) are obtained.

It remains to consider the case in which the

extending argument value is equal to λ_k , one of
 ?, the $\Lambda_j, -\bar{\Lambda}_j$ ($j=1, \dots, q$) and the interpretation
 problem is extended by the provision of a further
 derivative value, for example $\mathfrak{D}^{T(j)+1} h_j$ with
 $q+1 \leq j \leq m$. The extended extremal solution
 $W_{R+1}^{(0)}$ still has the form (), and still $b=c=0$,
 $a=1$. The remaining unknown coefficient d is
 to be determined by the condition that the $(T(j)+1)$
 derivative of the rational function (), or equivalent
 condition involving $\mathfrak{D}^{T(j)+1} \{ P_{1,R+1}^{(0)}(\lambda_k) \}$ and
 $\mathfrak{D}^{T(j)+1} \{ \mathfrak{D} h_j P_{0,R+1}^{(0)}(\lambda_k) \}$ (see the derivation of
 formulae (,)). A version of formulae (,) is
 denied. The formula used to construct the
 extended extremal solution $W_{R+1}^{(1)}$ is also a ver-

of that used previously to construct extremal
 solutions of the same kind. Similar analysis deals
 with the case in which the extending argument
 value agrees with that of a member of the set
 $\Delta_j \quad (j=a+1, \dots, \bar{J} \quad j=1, \dots, q)$. In summary, if the
 algorithm of clause (iii) produces the extremal
 solutions $W_R^{(\omega)} (\omega=0,1)$ of the symmetric interpolation
 problem (α, β) , it also produces the extremal
 solutions $W_{R+1}^{(\omega)}$ of the interpolation problem extended
 in all possible ways. Since the correctness of the
 algorithm has been verified for the primitive
 case $m=1, T(1)=1$, it is correct in all cases.

That the constituent numerator and denominator
 polynomials of the recursively constructed functions
 $\{W_r^{(\omega)}\}$ are as described in clause (iii) is a by-
 product of the above method of proof. Each

national function $W_{\text{ext}}^{(\alpha)}$ may be derived as the end result of application of an algorithm, similar to that described in clause (iii), to a diminished version of the interpolation (α, β) ; the $W_r^{(\alpha)}$ are thus extremal solutions of intermediate interpolation problems, as described.

The results of the above theorem are algebraic in nature, set in his first paper by Pick [16], who established the connection between his interpolation problem and the Cauchy interpolation problem, rather than that indicated by Nevanlinna. The use of consecutive conformal mappings has been followed.

References

1. Akhiezer N. I., The classical moment problem, and some related questions in analysis, Moscow (1961)
(English translation: Oliver and Boyd, Edinburgh and London (1965))
2. Borel E., Leçons sur les séries divergentes, Gauthier-Villars, Paris (1928)
3. Dahlquist G., Positive functions and some applications to stability questions for numerical methods, in:
Recent advances in numerical analysis (Eds: de Boor C. and Gårding G.H.) Academic Press,
New York - London (1978) 1-29
4. Ehle B.L., On Padé approximations to the exponential function, J. Approx. Theory 1 (1968) 34-45

function and A-stable methods for the numerical
solution of initial value problems, Univ. of Waterloo,
Dept. Applied Analysis and Computer Science, Research
Report CSRR 2010 (1999)

5. Fejér L., Potenzreihen mit mehrfach monotoner
Koeffizientenfolge und ihre Legendre-Polynome,
Proc. Camb. Phil. Soc., 31 (1935) 307 - 316

6. Fejér L., Trigonometrische Reihen und Potenzreihen
mit mehrfach monotoner Koeffizientenfolge,
Trans. Amer. Math. Soc., 39 (1936) 18-59

7. Hamburger H., Über eine Erweiterung des
Stieltjes'schen Momentenproblems, Math. Ann., 81
(1920) 235-319; 82 (1921) 120-164; 168-187

8. Hardy G.H., On the zeroes of certain integral functions

Messenger of Math., 32 (1903) 26-45

9. Hardy G.H., The asymptotic solution of certain

transcendental equations, Quart. Jour. Math., 35

(1904) 261-282

10. Hemker P.W. (Ed.), NUMAL, a library of Algol 60

procedures in numerical mathematics, ~~Math.~~

Mathematisch Centrum, Amsterdam (1973)

11. Liniger W. and Willoughby R.A., Efficient numeric

integration of stiff systems of ordinary

differential equations, SIAM Jour. Numer. Ana

7 (1970) 47-66

12. Nevanlinna R., "Über beschränkte Funktionen."

die in gegebenen Punkten vorgeschriebene Werte

annehmen, Ann. Acad. Sci. Fenniae, A13 (1919) No. 1

13. Nevanlinna R., Asymptotische Entwickelungen

beschränkter Funktionen und das Stieltjes'schen

Momentenproblem, Ann. Acad. Sci. Fenniae, A18

(1922) No. 5

14. Nörlund N. E., Vorlesungen über Differenzenrechnung

Springer, Berlin (1937)

15. Perron O., Die Lehre von den Kettenbrüchen, vol. II,

Teubner, Stuttgart (1957)

16. Pick G., Über die Beschränkungen analytischer

Funktionen, welche durch vorgegebene

Funktionswerte bewirkt werden, Math. Ann., 77 (1916) 7-2

17. Pick G., Über beschränkte Funktionen mit vorgegebenen
Wertzuordnungen, Ann. Acad. Sci. Fennicae, A15
(1920) No. 3
18. Polya G., Application of a theorem connected with
the problem of moments, Messenger of Math., 55
(1926) 189 - 192
19. Schoenberg I.J., On smoothing operations and
their generating functions, Bull. Amer. Math. Soc.,
59 (1953) 199 - 230
20. Shohat J.A. and Tamarkin J.D., The problem of
moments, Amer. Math. Soc. Math. Surveys, 1 (1963)
21. van der Houwen P.J., Construction of integration
formulas for initial value problems, North Holland,
Amsterdam (1977)

22. Wynn P, On an extension of a result due to Polya,
Jour. reine ang. Math., 248 (1971) 127-132
23. Wynn P, Sur les suites totalement monotones, Comptes Rendus de l'Acad. Sci. (Paris) 275A (1972) 1065-7
24. Wynn P, On the zeros of certain confluent hypergeometric functions, Proc. Amer. Math. Soc., 40 (1973) 173-182
25. Wynn P, On the intersection of two classes of functions, Rev. Rom. Math. Pures Appl., 19 (1974) 949-959
26. Wynn P, Upon a class of functions connected with the approximate solution of operator equations, Ann. mat. pura appl., 104 (1975) 1-29