The Hamburger – Pick – Nevanlinna problem

The Hamburger - Pick - Nevanlinna problem may be described in the following way. Initially let p, q, r be finite positive integers.

0) Let a_ν ($\nu : 0, 2p-2$) be finite real numbers

1) Let x_k ($k : q$) be distinct real argument values, $\tau(1,k)$ ($k : q$) be corresponding finite positive integers, and $b_{k,\nu}$ ($k : q \mid \nu : 0, 2\tau(1,k)-1$) sets of ~~coeffict~~ finite real ~~valued coefficients~~ ~~numbers~~.

2) Let λ_k ($k : r$) be distinct argument values in $\mathbb{L}$, $\tau(2,k)$ ($k : r$) be corresponding finite positive integers, and $c_{k,\nu}$ ~~sets~~ ($k : r \mid \nu : 0, \tau(2,k)-1$) sets of finite complex ~~numbers~~ valued coefficients.

The problem in hand is that of determining a function $\mathbb{G}$ which

α0) satisfies the asymptotic relationship

$$\lambda^{2p-1}\left\{\mathbb{G}(\lambda) - \sum_{\nu=0}^{2p-2} a_\nu \lambda^{-\nu-1}\right\} = O(1)$$

as λ tends to infinity in a sector of the form $\pi < \gamma \leq \arg(\lambda) \leq \delta < 2\pi$

α1) for $k : q$, satisfies the asymptotic relationship

$$(\lambda - x_k)^{-2T(1,k)} \left\{ G(\lambda) - \sum_{\nu=0}^{2T(1,k)-1} b_{k,\nu} (\lambda - x_k)^{\nu} \right\} = O(1)$$

as λ tends to x_k over any open set contained in $\mathbb{L}$ with limit point at x_k

$\alpha 2)$ for $k:r$, satisfies the interpolation conditions

$$\frac{\partial^{\tau} G(\lambda_k)}{\tau!} = c_{k,\tau}$$

for $\tau : 0, T(2,k) - 1$, and

$\beta)$ maps $\mathbb{L}$ into $\mathbb{D}$.

With ~~$T(1) = \sum_{\tau=1}$~~

$$T(1) = \sum_{k=1}^{q} T(1,k) \qquad T(2) = \sum_{k=1}^{r} T(2,k)$$

$n = p + T(1) + T(2)$ is the order of the problem, which is denoted by $P[n \,|\, p:a;\, q:x, b;\, r:\lambda, c;\, T]$ or, where the context permits, simply by $P[n]$. The variants of the above problem in which conditions ~~(α0) [(α1)] {(α2)}~~ (α0), (α1) or (α2) are omitted ~~are is~~ are indicated by setting $p=0$, ~~[$q=0$] {$r=0$}~~ $q=0$ or $r=0$ in use of the preceding extended symbol.

The proof of the principal theorem of this section is largely based upon results, due to M. Riesz, ~~from~~ in the theory of linear functionals. Given a set F ~~ϕ_ν (ν:m)~~ of ~~real valued~~ functions ϕ_ν (ν:m), real valued and linearly independent ~~defined~~ over a real interval $[\alpha,\beta]$, and ~~linearly real valued and linearly independent~~ and a corresponding sequence of real numbers α_ν (ν:m), a linear functional $Q\{\psi,[\alpha,\beta]\}$ whose argument is the set of ~~function~~ values taken by the real valued function ψ ~~as its~~ over $[\alpha,\beta]$, is identified by setting

$$Q\{\phi_\nu^{(t)}[\alpha,\beta]\} = \alpha_\nu$$

for ν:m. Q is defined over the aggregate E of functions of the form

$$\phi(t) = \sum_{\nu=1}^{m} u_\nu \phi_\nu(t)$$

the u_ν being ~~fi~~ real and finite by setting

$$Q\{\phi(t),[\alpha,\beta]\} = \sum_{\nu=1}^{m} u_\nu Q\{\phi_\nu(t),[\alpha,\beta]\}.$$

Q is said to be ~~positive~~ nonnegative definite ~~over if for every~~ with respect to F over ~~the interval~~ $[\alpha,\beta]$ if for every $\psi \in E$ for which $\psi(t) \geq 0$ for all $t \in [\alpha,\beta]$, $Q\{\psi(t),[\alpha,\beta]\} \geq 0$; it is said to be positive definite with respect to F over $[\alpha,\beta]$ if for every such ψ, not identically zero, $Q\{\psi(t),[\alpha,\beta]\} > 0$.

~~A functional Q may equally~~

The above linear functional Q may be identified from a set $\mathbb{F}'$ of ~~complex valued functions ψ_ν~~ func functions $\{\psi_\nu\}$ $(\nu:m)$ that are ~~two~~ complex valued and linearly independent over a real interval $[\alpha, \beta]$ and a corresponding sequence of complex numbers $\{e_\nu\}$ by ~~taking~~ replacing m by $2m$ in the above set $\mathbb{F}$ and taking the ~~$\{\psi_\nu\}$ above~~ to be $\{\operatorname{Re}\psi_\nu, \operatorname{Im}\psi_\nu\}$ $(\nu:m)$ and ~~the~~ $\{d_\nu\}$ to be ~~a suitably a corresponding~~ ~~set~~ $\{\operatorname{Re}\{d_\nu\}, \operatorname{Im}(d_\nu)\}$ $(\nu:m)$. Q is then defined over the aggregate ~~$\bar{E}$ of~~ real linear hull $\overline{\mathbb{E}}$ of the set $\{\psi_\nu\}$, i.e. the aggregate of functions of the form

$$\psi(t) = \sum_{\nu=1}^{m} v_\nu \operatorname{Re}\psi_\nu(t) + \sum_{\nu=1}^{m} v_\nu' \operatorname{Im}\psi_\nu(t)$$

the v_ν, v_ν' being real and finite, by use of a formula similar to ( ). The sequence of complex numbers e_ν $(\nu:m)$ is said to be nonnegative (positive) definite with respect to the complex valued functions ψ_ν $(\nu:m)$ over the interval $[\alpha, \beta]$ if the linear functional Q described above is nonnegative (positive) definite with respect to $\mathbb{F}$ (the set of functions $\{\operatorname{Re}\psi_\nu, \operatorname{Im}\psi_\nu\}$) over $[\alpha, \beta]$. The linear functional Q defined over $\overline{\mathbb{E}}$ as above may be defined over ~~$\mathbb{E}$~~ the complex linear hull of $\mathbb{F}'$, i.e. the aggregate of functions given by expression ( ) with ϕ, ϕ_ν replaced by ψ, ψ_ν, and the u_ν now being finite and complex, simply by setting

$$Q\{\psi(t), [\alpha,\beta]\} = Q\{\mathrm{Re}\,\psi(t), [\alpha,\beta]\} + i Q\{\mathrm{Im}\,\psi(t), [\alpha,\beta]\}.$$

Use will be made of three results from the theory of linear functionals

(i) Let the sequence of complex numbers e_ν $(\nu:m)$ be positive definite with respect to the ~~continuous~~ complex valued functions ψ_ν $(\nu:m)$ over $[-\infty,\infty]$, the ψ_ν being continuous over this interval, and let the real linear hull $\bar{E}$ of the set $\{\psi_\nu\}$ contain a function Φ for which $\Phi(t)>0$ for all $t \in [-\infty,\infty]$. ~~Then the sequence e_ν is positive defin there~~ Then a finite interval $[\alpha,\beta]$ exists such that the sequence e_ν $(\nu:m)$ is positive definite with respect to the functions ψ_ν $(\nu:m)$ over $[\alpha,\beta]$.

(ii) Let the real linear hull $\bar{E}$ of the set $\{\psi_\nu\}$ of complex valued functions ψ_ν $(\nu:m)$, continuous over a finite interval $[\alpha,\beta]$, contain a function Φ for which $\Phi(t)>0$ for all $t\in[\alpha,\beta]$. Then the complex numbers e_ν $(\nu:m)$ have representations of the form

$$e_\nu = \int_\alpha^\beta \psi_\nu(t)\,d\sigma(t)$$

for $\nu:m$,

where $\sigma \in B[\alpha,\beta]$, if and only if the sequence $\{e_\nu\ (\nu:m)\}$ is nonnegative definite with respect to the functions $\psi_\nu\ (\nu:m)$ over $[\alpha,\beta]$.

(iii) Let the sequence of complex numbers $e_\nu\ (\nu:m)$ be ~~positive~~ nonnegative definite with respect to the complex valued functions $\psi_\nu\ (\nu:m)$ over $[-\infty,\infty]$, the ψ_ν being continuous over this interval, and let the real linear hull $\mathbb{E}$ of the set $\{\psi_\nu\}$ contain a function Φ for which $\inf \Phi(t) > 0$ for $t \in [-\infty,\infty]$ and such that all limits

$$\lim_{t \to \pm\infty} \frac{\psi_\nu(t)}{\Phi(t)} = c_{\nu,\Phi}$$

exist and are finite. A constant $A \geq 0$ and a function $\sigma \in B[-\infty,\infty]$ ~~[illegible]~~ exist such that

$$e_\nu = A c_{\nu,\Phi} + \int_{-\infty}^{\infty} \psi_\nu(t)\,d\sigma(t)$$

for $\nu:m$.

In ~~the application~~ ~~subsequent~~ later ~~use~~ application of the second of the above results, use will be made of the facts that, in the application considered, many functions σ may feature in formula ( ), and that the ~~function~~ behaviour of the function σ ~~may~~ ~~be~~ in the neighbourhoods of a finite number of points on the real axis, ~~thus~~ may be constrained. For this reason the proof of the second result ~~and~~ is now briefly sketched. If Q is a positive definite linear functional identified over ~~with respect to~~ a ~~the~~ set $\mathbb{F}$ of real valued function ϕ_ν $(\nu:m)$ ~~over the~~ and $\mathbb{F}$ with respect to the interval $[\alpha,\beta]$ is such that ~~there are two func~~ & the finite aggregate E of functions of the form ( ) contains two functions ϕ', ϕ'' for which

$\phi'(t) < \phi''(t)$ for all $t \in [\alpha,\beta]$, ~~then the a positive definite linear functional Q identified with respect to E may~~ then, ~~a positive definite linear functional $Q^{(1)}$ over~~ taking $\phi^{(1)}(t)$ to be any function for which $\phi'(t) \le \phi^{(1)}(t) \le \phi''(t)$ for all $t \in [\alpha,\beta]$, a positive definite linear functional $Q^{(1)}$ identified over $\mathbb{F}^{(1)} = \mathbb{F} \cup \phi^{(1)}$ can be constructed. The identifications of $Q^{(1)}$ ~~interms~~ in terms of the ϕ_ν are those of the form ( ) identifying Q. Take ~~Taking $d' = \sup Q\{\phi', [\alpha,\beta]\}$~~

$$d' = \sup Q\{\phi', [\alpha,\beta]\}, \quad d'' = \inf Q\{\phi'', [\alpha,\beta]\}$$

for all ϕ', ϕ'' satisfying relationship ( ), $d^{(1)}$ in the range ~~such~~ $d' \le d^{(1)} \le d''$, and the identification $Q^{(1)}\{\phi^{(1)}, [\alpha,\beta]\} = d^{(1)}$. ~~$Q^{(1)}$ to positive~~ ~~and~~ By considering the conditions under which $\phi(t) + c\phi^{(1)}(t) \ge 0$ when $\phi \in E$, and

defining appropriate inequalities [illegible]
definite. The extension of $\mathbb{F}$ to a set containing unboundedly many members of and the construction of a positive definite functional identified over this set are carried out by ~~[illegible]~~ continuing the above process. If $\phi^{(1)} \notin E$, $d' < d''$ in the above; ~~the~~ the selection of $d^{(1)}$ is open ~~to choice, and~~ to choice; ~~there is~~ ~~ip~~ ipso facto there is freedom in the construction of $Q^{(1)}$ and indeed of all subsequent positive definite linear functionals. A clear case to which the above theory has application is that in which E contains a function Φ for which $\Phi(t) > 0$ for all $t \in [\alpha, \beta]$: $\phi'(t), \phi''(t)$ can then be taken to be $-\Phi(t), \Phi(t)$.

Let γ be a fixed interior point of $[\alpha, \beta]$ and define the function $\zeta_x(t)$ ($x \in [\alpha, \beta]$) by setting, ~~ζ_x~~ when $x \geq \gamma$, $\zeta_x(t) = 1$ for $t \in [\gamma, x]$ and $\zeta_x(t) = 0$ for $t \in [\alpha, \beta] \setminus [\gamma, x]$ and, when $x < \gamma$, $\zeta_x(t) = -1$ for $t \in [x, \gamma]$ and $\zeta_x(t) = 0$ for $t \in [\alpha, \beta] \setminus [x, \gamma]$. Under the conditions of (ii) a positive ~~definite~~ (nonnegative) definite functional Q, identified from a set $\mathbb{F}$ of functions $\{Re\,\psi_\nu, Im\,\psi_\nu\}$ over $[\alpha, \beta]$ exists; furthermore the real linear hull $\mathbb{E}$ of the set $\{\psi_\nu\}$ contains a function Φ for which

$\Phi(t) > 0$ for all $t \in [\alpha, \beta]$. Accordingly, ~~for all $x \in E$~~ there exists a $C_x \in (0, \infty)$ such that $-C_x\Phi(t) < \xi_x(t) < C_x\Phi(t)$ for each $x \in [\alpha, \beta]$. Let $t(k)$ be a denumerable set of points everywhere dense in $[\alpha, \beta]$. F may be extended to a set containing all $\{\xi_{t(k)}(t)\}$ and a positive ~~de~~ ⟨nonnegative⟩ functional Q' identified over this set may be constructed. Define the function σ over $[\alpha, \beta]$ by setting $\sigma(t(k)) = Q'\{\xi_{t(k)}\}$ and $\sigma(t) = \sup \sigma(t(k))$ for $t(k) < t$, when $t \neq t(k)$. Since $\xi_{t(k)}(t) - \xi_{t(j)}(t) \geq 0$, ~~and this diff~~ for $t(j) < t(k)$, and this difference is not identically zero for all ~~$t\in$~~ $t \in [\alpha, \beta]$, ~~$\sigma(t(k))$~~ $\sigma(t') > \sigma(t'')\langle\sigma(t') \geq \sigma(t'')\rangle$ ~~when Q is positive when $t' > t''$ if Q is positive ⟨nonnegative⟩ definite~~ when $t' > t''$, i.e. σ is an increasing ⟨nondecreasing⟩ function if Q is positive ⟨nonnegative⟩ definite. For any $\varepsilon > 0$ and $\psi \in \bar{E}$, a subsequence s_ν $(\nu : m)$ of the $\{t_\nu\}$, with $\alpha = s_1 < s_2 < \dots < s_m = \beta$ can be found such that

$$-\varepsilon\Theta(t) < \psi(t) - \sum_{\nu=1}^{m-1} \psi(s_\nu)\{\xi_{s(\nu+1)}(t) - \xi_{s(\nu)}(t)\} < \varepsilon\Theta(t)$$

or

$$-\varepsilon Q\{\Theta\} < Q(\psi) - \sum_{\nu=1}^{m-1} \psi(s_\nu)[\sigma\{s(\nu+1)\} - \sigma\{s(\nu)\}] < \varepsilon Q(\Theta)$$

Increasing the length of the subsequence with ~~Letting~~ $\max\{s(\nu+1) - s(\nu)\}$ tending to zero, it may be deduced that $Q\{\psi\} = \int_\alpha^\beta \psi(t)\, d\sigma(t)$. Taking ψ to be $\mathrm{Re}\,\psi_\nu$, $\mathrm{I}\psi_\nu$ in turn, relationship ( ) is obtained.

The results (i, ii) can be ~~contained~~ compounded to show that if a sequence of numbers e_ν $(\nu:m)$ ~~which~~ is positive definite with respect to a set of complex valued functions ψ_ν $(\nu:m)$ over $[-\infty,\infty]$, the ψ_ν being continuous over this interval, and ~~such that~~ their real linear hull $\bar{E}$ ~~contains a~~ of this set contains a function θ for which $\theta(t)>0$ for all $t\in(-\infty,\infty)$, then the e_ν have a representation of the ~~form~~ Stieltjes integral form ( ). The proof of the result (i) breaks down if the assumption ~~that~~ of positive definiteness of the e_ν ~~should be positive definite~~ with respect to the ψ_ν is replaced by the less stringent ~~assumption~~ alternative of nonnegative definiteness. Simple examples may be constructed to show that the above compound result is false under the assumption of nonnegative definiteness of the e_ν with respect to the ψ_ν. To derive a result in the direction of the above compound result under the assumption of nonnegative definiteness, ~~the~~ lemma (iii) must be used. At this stage it is desirable both to show how the alternative result falls short of the complete representation ( ), and how this shortcoming may possibly be removed. The set of functions $\{\psi_\nu\}$ of lemma (iii) constains a subset $\hat{\psi}_\kappa$ $(\kappa:m')$ such that

$$\lim_{t \to \pm\infty} \frac{\hat{\psi}_\chi(t)}{\hat{\psi}_\tau(t)} = C_{\chi\tau}$$

for $\chi, \tau : m'$, the $C_{\chi,\tau}$ being nonzero and finite, while for all ψ_ν not belonging to the set $\{\hat{\psi}_\tau\}$ the limit obtained by replacing $\hat{\psi}_\chi$ by ψ_ν in relationship ( ) is zero. The representation of Θ ~~[illegible]~~ as a linear combination of the functions $\{\mathrm{Re}\,\psi_\nu, \mathrm{Im}\,\psi_\nu\}$ contains at least one member of the set $\{\mathrm{Re}\,\hat{\psi}_\tau, \mathrm{Im}\,\hat{\psi}_\tau\}$ (otherwise the limits $c_{\nu,\Theta}$ of formula ( ) is infinite when ψ_ν is one of the $\hat{\psi}_\tau$). Accordingly the limits $c_{\nu,\Theta}$ fall into two classes; they are zero when ψ_ν is not one of the $\{\hat{\psi}_\tau\}$ and nonzero when it is. For the ~~ψ_ν~~ ψ_ν relating to the first class, ~~formula~~ the representation ( ) with $[\alpha,\beta] = [-\infty,\infty]$ holds; if $A \neq 0$, formula ( ) must be replaced by ( ) for the ψ_ν of the second class. If the set $\{\psi_\nu\}$ may be extended by the introduction of a function $\psi^{(1)}$ such that one of the relationships

$$\lim_{t \to \pm\infty} \frac{\psi_\nu(t)}{\mathrm{Re}\,\psi^{(1)}(t)} = 0$$

holds, a further constant $c^{(1)}$ may be introduced such that there exists ~~a~~ ~~[illegible]~~ the extended set of constants and also the representation of the function Θ relating to the extended set of functions (?) is ~~[illegible]~~ nonnegative definite with respect to the extended the extended set of functions contains the function $\mathrm{Re}\,\psi^{(1)}$ or $\mathrm{Im}\,\psi^{(1)}$ for which relationship ( ) ~~holds~~ obtains, then all limits ( ) for the ψ_ν belonging to the original unextended set $\mathbb{F}$ are zero; accordingly, the representation ~~formula~~ ( ) with $[\alpha,\beta] = [-\infty,\infty]$ holds for all ~~these of those~~ such ψ_ν (it may ~~does~~ not hold for the introduced function $\psi^{(1)}$, but this is of little direct interest). It is clear that the appropriate function $\psi^{(1)}$ cannot be introduced by extension of Riesz' artifice as described in the above proof (such a function is dominated by $\overline{\mathbb{E}}$ in the sense that ~~for~~ $\phi', \phi'' \in \overline{\mathbb{E}}$ exist such that $\phi'(t) \leq \mathrm{Re}\,\psi^{(1)}(t), \mathrm{Im}\,\psi^{(1)}(t) \leq \phi''(t)$: there are $\psi_\nu \in \mathbb{F}$ for which relationship ( ) does not hold).

Nevertheless, in the ~~very~~ special case considered in the following theorem, an alternative ~~appropriate~~ extension is possible.

$H(n \mid p, g; q, x; r, \lambda; T, G) \quad H(n)$

(i) $p, q, r > 0$: $H(n)$ of order $n \times n$

a) $H^{(0,0)}$ $p \times p$ array whose $(k,j)^{th}$ element is $H^{(0,0)}_{k,j} = g_{k+j-2}$ $(k,j = 1, \ldots, p)$

b) obtain numbers $\{\Delta^{r} h_k\}, \{\Delta^{r} h_j\}$ from the corresponding numbers $\{\Delta^{r} G_k\}, \{\Delta^{r} G_j\}$ by setting

$$h_{1,k} = G_{1,k} - g_{-2}\lambda_k - g_{-1} - \cancel{\sum_{k=1} (c_{1,k}\lambda_k - c} \quad \cancel{c_{1,-2,k}\lambda_k - c_{1,-1}}$$

$x_\nu \quad \lambda_\nu$

$$\sum_{k=1}^{T(1)} (c_{1,k,-2}\, x_k - c_{1,k,-1}) \ ?$$

$$\Delta h_{1,k} = \Delta G_{1,k} - g_{-2}$$

Define reduced version ~~$P[n \mid p, a; q, b; r, c; T)$~~

$\square$ $\approx$ $P[n \mid p: a; q: x, b; r: \lambda, c; T)$

b) $H^{(1,0)}$ is the $T(1) \times p$ array consisting of q blocks ~~$T(1,\nu) \times p$~~

$\underset{\sim}{H}^{(1,0)}_{\kappa}$ ~~$(\kappa = 1, \ldots, q)$~~ $(\kappa; q)$, ~~of $T(1,\nu) \times p$~~ the $(k,j)^{th}$ element of

$\underset{\sim}{H}^{(1,0)}_{\kappa}$ being

$$H^{(1,0)}_{\kappa; k,j} = x_{\kappa}^{j-k-1} \sum_{\nu=0}^{j-k-1} a_{\nu}\, x_{\kappa}^{-\nu} \binom{j-\nu-2}{k-1}$$
$$- \sum_{\nu}^{k-1} \quad q\, x_{\kappa}^{j-k+\nu} \binom{j-1}{k-\nu-1} b_{\kappa,k,\nu}$$

$$H^{(1,0)}_{K;k,j} = x_K^{j-k-1} \sum_{\tau=0}^{j-k-1} a_\tau x_K^{-\tau} \binom{j-\tau-2}{k-1} - \sum_{\tau=\max(0,k-j)}^{k-1} x_K^{j-k+\tau} \binom{j-1}{k-\tau-1} b_{K,\tau}$$

for $k: T(1,K)$ | ~~and~~ $j:p$. ~~with $j:q$~~

c) ~~Let~~ $H^{(1,1)}$ is ~~be the~~ $T(1)\times T(1)$ ~~block~~ matrix ~~array consisting of~~ composed of q^2 blocks, the array $H^{(1,1)}_{K,J}$ of $T(K)\times T(J)$ elements occurring in the K^{th} row and J^{th} column $(K,J:q)$ and the $(k,j)^{th}$ element of $H^{(1,1)}_{K,J}$ being, when $K=J$,

$$H^{(1,1)}_{K,J;k,j} = -b_{K,j+k-1}$$

for $k,j: T(K)$ and when $K \neq J$

$$H^{(1,1)}_{K,J;k,j} = (-1)^{k-1}\sum_{z=0}^{j-1}\binom{k+j-z-2}{k-1}(x_K - x_J)^{z-j-k+1} b_{J;z} + (-1)^{j-1}\sum_{z=0}^{k-1}\binom{k+j-z-2}{j-1}(x_J - x_K)^{z-j-k+1} b_{K,z}$$

for $k: T(K)$ and $j: T(1,J)$.

d) $H^{(2,0)}$ is the $T(2)\times p$ ~~block~~ matrix ~~consisting of~~ consisting of a column ~~composed~~ of r blocks $H^{(2,0)}_K$ $(K:r)$, the $(k,j)^{th}$ element $H^{(2,0)}_{K;k,j}$ of $H^{(2,0)}_K$ being given by formula ( ) with $H^{(1,0)}_{K;k,j}$, x_K, $b_{K,\tau}$ being replaced by $H^{(2,0)}_{K;k,j}$, λ_K, $c_{K,\tau}$ respectively, for $k: T(2,K)$ | $j:p$.

e) $H^{(2,1)}$ is the $T(2)\times T(1)$ matrix composed of ~~qr~~ blocks, the array $H^{(2,1)}_{K,J}$ of $T(1,K)\times T(2,J)$ elements occurring in the K^{th} row and J^{th} column $(K:p|J:q)$, ~~and~~ the $(k,j)^{th}$ element of $H^{(2,1)}_{K,J}$ being given by formula ( ) with $H^{(1,1)}_{K,J;k,j}$, x_J, $b_{J,\tau}$ being replaced by $H^{(2,1)}_{K,J;k,j}$, $\bar{\lambda}_J$, $\bar{c}_{J,\tau}$ respectively for $k:T(2,K)|j:T(1,J)$

f) $H^{(2,2)}$ is the $T(2)\times T(2)$ matrix composed of r^2 blocks, the array $H^{(2,2)}_{K,J}$ of $T(2,K)\times T(2,J)$ elements occurring in the K^{th} row and J^{th} column $(K,J:r)$, ~~and~~ the $(k,j)^{th}$ element of $H^{(2,2)}_{K,J}$ being given by formula ( ) with $H^{(1,1)}_{K,J;k,j}$, x_K, x_J, $b_{K,\tau}$, $b_{J,\tau}$ being replaced by $H^{(2,2)}_{K,J;k,j}$, λ_K, $\bar{\lambda}_J$, $c_{K,\tau}$, $\bar{c}_{J,\tau}$ respectively for $k:T(2,K)|j:T(2,J)$.

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The $n\times n$ Hermitian matrix, denoted by $H(n|p:a;q:x;b;r;\lambda,c;T)$ or, if the context permits, simply by $H(n)$, is, when $p,q,r>0$, composed of nine blocks arranged in three columns and three rows; the block $H^{(\tau,\upsilon)}$ occurs in the $(\upsilon+1)^{th}$ position of the $(\tau+1)^{th}$ row $(\tau,\upsilon:0,2)$. These blocks are constructed from the data to $P[n]$ in the following way

When $p=0$ $(q=0)$ $[r=0]$ the first (second) [third] row and column of blocks are to be deleted from the above description of $H(n)$.

Denoting the elements of $H(n)$ by $H_{k,j}(n)$ $(k,j:n)$, ~~the~~ $H(m)$ is, the sub matrix whose elements are $H_{k,j}(n)$ $(k,j:m)$ for for $m:n$, ~~$m:n$.~~

~~A~~ Λ, h being complex numbers a bordered version $H(\Lambda,h|n+1)$ of order $(n+1)\times(n+1)$ of $H(n)$ is, when $p,q,r>0$, defined ~~in the following way:~~ by adjoining elements

(i)

$$H(\Lambda,h|n+1)_{n+1,j} = \overset{h_j^{(0)}}{=} \Lambda^{j-1}\left\{\sum_{\nu=0}^{j-2} a_\nu \Lambda^{-\nu-1} - h\right\}$$

and $H(\Lambda,h|n+1)_{k,n+1} = \bar{h}_k^{(0)}$ in positions j and k ~~$(j:p\ \&\ k:p$~~ ~~$j:p$ and $k:p$~~ of the $(n+1)^{\text{th}}$ row and column respectively ~~$(j:p\ \&\ k:p$~~ $(j,k:p)$

(ii)

$$H(\Lambda,h|n+1)_{n+1,m(J,j)} = \overset{h^{(1)}_{m(J,j)}}{=} (\Lambda-x_J)^{-j}\left\{\sum_{z=0}^{j-1}(\Lambda-x_J)^z b_z - h\right\}$$

and $H(\Lambda;h|n+1)_{m(K,k),n+1} = \bar{h}^{(1)}_{m(K,k)}$ in positions ~~j~~ $m(J,j) = p + \sum_{z=0}^{J-1} T(1,z) + j$ and $m(K,k)$ of the $(n+1)^{\text{th}}$ row and column respectively $(J:q \mid j:0,T(1,J)-1;$
$K:q;\ k:0, T(1,K)-1)$

(iii) $H(\Lambda, h|n+1)_{n+1,\, m(2,J,j)}$ given by formula ( ) with

~~x_J, b_z replac~~ $m(1,J,j), x_J, b_z, h^{(1)}_{m(1,J,j)}$ replaced by $m(2,J,j)$, $\bar{\lambda}_J, \bar{c}_z, h^{(2)}_{m(2,J,j)}$ respectively, and $H(\Lambda, h|n+1)_{m(2,K,k),\, n+1} = \bar{h}^{(2)}_{m(2,K,k)}$ in positions $m(2,J,j) = \cancel{m(1,q,T(q)) + \sum_{z=0}^{J-1} T(2.}\; p + T(1) + \sum_{z=0}^{J-1}{}' T(2,z) + j$ and $m(2,K,k)$ of the $(n+1)^{th}$ row and column respectively $(J: r;\ j: 0, T(2,J)-1;\ K: r;\ k: 0, T(2,K)-1)$

(iv)

$$H(\Lambda, h|n+1)_{n+1,\, n+1} = \cancel{(h -}\ (\bar{\Lambda} - \Pi)^{-1}(h - \bar{h})$$

in position $(n+1, n+1)$.

When $p = 0$ & $(q = 0)$ $[r = 0]$ references to the elements of (i) ((ii)) [(iii)] are to be deleted from the above description of $H(\Lambda, h|n+1)$.

Theorem. (A i) If ~~$H(n|p, a, q, x, r, \lambda, T, b)$~~

$H(n \mid p:a; q:x, b; r:\lambda, c; T)$ is positive definite, $P[n \mid p:a; q:x, b; r:\lambda, c; T]$ is soluble; ~~indeed~~ ~~$P(n)$ possesses contains~~ for any $n' \geq n$, $P(n)$ ~~possesses~~ contains nondenumerably many rational functions whose denominator polynomials are of degree n'; $P(n)$ also contains ~~nondenumerably~~ nondenumerably many nonrational functions.

(ii) If $H(n)$ is positive semi-definite and either

a) $p = 0$ or

b) $p > 0$, and for the least m such that $H(n')$ is singular $(n': m, n)$

α) when $m > p$ the matrix obtained by deleting the p^{th} row and column from $H(m)$ is nonsingular, or

β) when $m \leq p$, $m > 1$ and the matrix obtained by deleting rows and columns $m, m+1, \ldots, p-1$ from $H(p)$ is singular

then $P[n]$ is ~~soluble, but~~ degenerate; its degenerate solution is its only solution

(iii) If $H(n)$ is neither positive definite nor positive semi-definite, or if it is positive semi-definite and neither of conditions (ii a, b) obtain, then $P[n]$ is insoluble.

B. Let $H(n)$ be positive definite and $\Delta \in \mathbb{L} \setminus \{\lambda_1, \ldots, \lambda_r\}$ be fixed. The equation

$$\det\{H(n \mid \Delta, h)\} = 0$$

(~~the boundary of a closed~~

determines ~~a circle the boundary~~ a circle $\mathscr{C}_n(\Delta)$ ~~in the complex h plane. Let $\bar{D}_n(\Delta)$~~, {the boundary of a closed disc $\bar{D}_n(\Delta)$}, in the complex h-plane

(i) If $p = 0$, $\mathbb{P}(n) \to \bar{D}_n(\Delta)$.

(ii) If $p > 0$, there is a point ω on $\mathscr{C}_n(\Delta)$ such that $\mathbb{P}(n) \to \bar{D}_n(\Delta) \setminus \omega$.

Proof. The results of part A will first be proved for the case in which $p, q, r > 0$; the ~~necessary~~ modifications to the ~~arguments~~ reasoning involved that must be introduced to deal with the cases in which p, q or r are zero will be indicated subsequently: until further notice it is assumed that $p, q, r > 0$.

Set

$$b_{p,q}(t) = \frac{\prod_{k=1}^{q} (t - x_k)^{2T(2,k) - 1?}}{(1+t^2)^{p + T([illegible]) - 1}}$$

The following set $\mathbb{F}$ of functions $\phi_{0,\nu}, \phi_{1,k,z}, \phi_{2,j,z}$ ~~$(\nu:0,2p-2), \phi_{1,k,z} = (k:q \mid z:0,2T(1,k)-1), \phi_{2,j,z}$ $(j:r \mid z:0,T(2,k)-1)$~~

is introduced, ~~the~~ $\{\phi_{0,\nu}, \phi_{1,k,z}\}$ being real valued, ~~the~~ $\{\phi_{2,k,z}\}$ being complex valued:

$$\phi_{0,\nu}(t) = b_{p,q}(t)\,t^{\nu}$$

for $\nu:0,2p-2$,

$$\phi_{1,k,z}(t) = -\,b_{p,q}(t)(t-x_k)^{-z-1}$$

for $k:q \mid z:0,2T(1,k)-1$ and

$$\phi_{2,j,z}(t) = -\,b_{p,q}(t)(t-\lambda_j)^{-z-1}$$

for $j:r \mid z:T(2,j)-1$. The linear functional Q on $\mathbb{F}$ is determined by means of the following identifications:

$$\cancel{Q\{\phi_{0,\nu}(t)\} = a_\nu}$$

$$Q\{\phi_{0,\nu}(t)\} = a_\nu\,,\quad Q\{\phi_{1,k,z}(t)\} = b_{k,z}\quad Q\{\phi_{2,k,z}\} = c_{k,z}$$

for the relevant ν, k, j and z. ~~From~~

From formula ( )

$$H^{(0,0)}_{k,j} = Q\{b_{p,q}(t)\, t^{k+j-2}\}$$

for $k,j:r$. Expanding $t^{j-1} = (t - x_K + x_K)^{j-1}$ in powers of $t - x_K$, and using some elementary relationships involving binomial coefficients, it is easily verified that

$$H^{(1,0)}_{K;k,j} = Q\{b_{p,q}(t)\, t^{j-1}(t-x_K)^{-k}\}$$

for $K:q \mid k: T(1,K) \mid j:p$. Again, since

$$Q\{b_{p,q}(t)(t-x_K)^{-1}(t-x_J)^{-j}\} =$$
$$-\sum_{z=0}^{j-1}(x_K - x_J)^{-z-1} Q\{b_{p,q}(t)(t-x_J)^{z-j}\}$$
$$+(-1)^j (x_J - x_K)^{-j} Q\{b_{p,q}(t)(t-x_K)^{-1}\}$$

When $J \neq K$, the formula

$$H^{(1,1)}_{K,J;k,j} = Q\{b_{p,q}(t)(t-x_K)^{-k}(t-x_J)^{-j}\}$$

is correct when $k=1$ for $K,J:q\,(K \neq J) \mid j:T(1,J)$. That this formula is also correct when $j=1$ for $K,J:q\,(K\neq J) \mid k:T(1,K)$ is verified in the same way. When $K=J$ the elements $H^{(1,1)}_{K,J;k,j}$ satisfy the difference equation

$$H^{(1,1)}_{K,J;k,j+1} - H^{(1,1)}_{K,J;k+1,j} = (x_J - x_K)\, H^{(1,1)}_{K,J;k+1,j+1}$$

for all index values for which all terms displayed are defined; the values of the expressions upon the right hand side of equation ( ) do the same: ~~equation~~ formula ( ) is valid for $K, J : q\ (K \neq J) \mid k : T(1,K) \mid j : T(1,J)$. That formula ( ) ~~is~~ also ~~valid~~ holds when $K = J$ for $K : q \mid k, j : T(1,J)$ may be verified directly.

The formulae

$$H^{(2,0)}_{K;k,j} = Q\{ b_{p,q}(t)\, t^{j-1} (t - \lambda_K)^{-k} \}$$

~~for $K : r \mid k : T(2,K) \mid j : p$,~~

$$H^{(2,1)}_{K,J;k,j} = Q\{ b_{p,q}(t)\, (t - \lambda_K)^{-k} (t - x_J)^{-j} \}$$

~~for $K : r \mid J : q \mid k : T(1,K) \mid j : T(2,J)$ and~~

$$H^{(2,2)}_{K,J;k,j} = Q\{ b_{p,q}(t)\, (t - \lambda_K)^{-k} (t - \bar{\lambda}_J)^{-j} \}$$

~~for $K, J : r \mid k : T(2,K) \mid j : T(2,J)$ may all be verified in the same way~~

valid for all suffix values for which the entries in the left hand sides are defined may all be verified in the same way

With $\{\eta_\nu, \eta_{1,j,\nu}, \eta_{2,k,\nu}\}$ sets of complex numbers let

$$Y_1(t) = (t-i)^{1-p}\left\{\prod_{j=1}^{q}(t-x_j)^{T(1,j)}\right\}\left[\sum_{\nu=1}^{p}\eta_\nu t^{\nu-1} - \sum_{j=1}^{q}\sum_{\nu=1}^{T(1,j)}\eta_{1,j,\nu}(t-x_j)^{-\nu} - \sum_{k=1}^{r}\sum_{\nu=1}^{T(2,k)}\eta_{2,k,\nu}(t-\lambda_k)^{-\nu}\right]$$

Obtain two new sums by replacing $\{\nu; 1,j,\nu; 2,k,\nu\}$ by $\{k; 1,K,k; 2,K,k\}$ and $\{j; 1,J,j; 2,J,j\}$ respectively, and form the product $Y(t)\bar{Y}(t)$ of the first and the complex conjugate of the second. Setting

$$\xi_k = \eta_k \quad \xi_{p+m(1,K)+k} = \eta_{1,K,k} \quad \xi_{p+T(2)+m(2,K)+k} = \eta_{2,K,k}$$

where $k:p$ in the first of these relationships, with $m(\chi, K) = \sum_{\nu=1}^{K-1} T(\chi,\nu)$ $(\chi:2)$, ~~$\sum_{\nu=1}^{K-1} T(1,\nu)$, $\{K:q \mid k:T(1,K)\}$ in the second~~ the index values run $K:q \mid k:T(1,K)$ in the second relationship and $K:r \mid k:T(2,K)$ in the third, using the product formula for $Y(t)\bar{Y}(t)$ just ~~obtained~~ described, and ~~relationship~~ formulae (–), it is found that

$$Q\{|Y(t)|^2\} = \underset{\sim}{\xi}\,\underset{\sim}{H}(n)\,\underset{\sim}{\xi}^*$$

where $\underset{\sim}{\zeta}$ is the row vector with elements $\zeta_1, \dots, \zeta_n$.

Any function Φ belonging to the real linear hull of the set $\mathbb{F}$ of functions ~~defin~~ $\phi_{0,\nu}, \phi_{1,k,\pi}, \phi_{2,j,z}$ defined above has the representation

$$b_{p,q}(t) \left[\sum_{\nu=0}^{p-1} B_\nu t^\nu + \sum_{j=1}^{q} \sum_{z=1}^{T(1,j)} B_{1,j,z} (t-x_j)^{-z} + \sum_{k=1}^{r} \sum_{z=1}^{T(2,k)} \left\{ B_{2,k,z} (t-\lambda_k)^{-z} + \bar{B}_{2,k,z} (t-\bar{\lambda}_k)^{-z} \right\} \right]$$

where $\{B_\nu, B_{1,j,z}\}$ are real and $\{B_{2,k,z}\}$ are complex numbers. This Φ accordingly has the representation

$$b_{p,q}(t) \left| \prod_{k=1}^{r} (t-\lambda_k)^{T(2,k)} \right|^{-2} X(t)$$

where X is a polynomial of degree $2n-1$. If $\Phi(t) \geq 0$ for all $t \in (-\infty, \infty)$, $X(t)$ has the form $\left| \sum_{\nu=0}^{n-1} X_\nu t^\nu \right|^2$. Hence $\Phi(t) = Y(t)\bar{Y}(t)$, where Y may be expressed in the form ( )

If $H(n)$ is positive ~~semi~~ definite, the value of the expression upon the right hand side of formula ( ) with $\underset{\sim}{\zeta}$ any compatible row vector with complex elements not all zero is ~~nonneg~~ positive. It has thus been demonstrated that the condition $\Phi(t) \geq 0$ for all $t \in (-\infty, \infty)$, ~~with~~ where $\Phi \in E$, implies that $Q\{\Phi(t)\} > 0$.: the ~~sequence of~~ set of numbers $\{a_\nu, b_{j,\nu}, c_{k,\nu}\}$ is positive definite with respect to the functions $\{\phi_{0,\nu}, \phi_{1,\nu}, \phi_{2,\nu}\}$. Accordingly, from Lemma 1, ~~numerous~~ numerous corresponding intervals $[\alpha, \beta] \in (-\infty, \infty)$ and ~~a~~ function $\sigma \in B(\alpha, \beta)$ exist such that

$$a_\nu = \int_\alpha^\beta t^\nu d\sigma(t), \quad \cancel{b_\nu = \int_\alpha (t-x}$$

$$b_{j,\nu} = -\int_\alpha^\beta (t-x_j)^{-\nu-1} d\sigma(t)$$

$$c_{k,\nu} = -\int_\alpha^\beta (t-\lambda_k)^{-\nu-1} d\sigma(t)$$

for the various index values featuring in the data of P[n]. The functions

$$G(\lambda) = \int_\alpha^\beta (\lambda - t)^{-1} d\sigma(t)$$

~~is a solution to P[n].~~ are, for the σ in question, solutions to P[n]. From Lemma 1 again, nondenumerably many distinct σ with n' salti $(n' \geq n)$ and no other points of increase in the range $[\alpha, \beta]$ ~~exist~~ satisfy relationships ( ). The corresponding solutions G to P[n] are rational functions whose denominator polynomials are of degree n. Again nondenumerably many distinct σ which increase monotonically over specified subintervals of $[a, b]$ satisfy relationships ( ); they induce nonrational solutions to $P[n]$.

~~The functions~~ $\phi_{1,k,2\tau(1,j)-1}$ ~~may be expressed in the form~~ Since

$$\phi_{1,k,2\tau(1,j)-1}(t) = \cancel{\frac{\prod_{k=1}^{q}{}^{(j)}(t-x_k)^{2\tau(1,k)}}{(1+t^2)^{-\tau(1)+1}}}$$

$$= (1+t^2)^{-\tau(1)+1}\prod_{k=1}^{q}{}^{(j)}(t-x_k)^{2\tau(1,k)}$$

for $j:q$, where the symbol $\prod^{(j)}$ is used to indicate that the term involving x_j is absent from the product, it is clear that the function $-\phi_{1,k,2\tau(1,j)-1}(t)$ ~~is zero~~ vanishes when $t=x_j$, is ~~otherwise~~ positive for all $t \neq x_j$ in $(-\infty,\infty)$ and tends to zero as t tends to infinity

Again,

From formulae (,), the function $-\phi_{1,j,2\tau(1,j)-1}(t)$ vanishes at the points $t=x_k$ $(k:q)$ except at $t=x_j$, it is positive and bounded for all other $t\in(-\infty,\infty)$, and tends to zero as t tends to infinity. Again the function $\phi_{0,2p-2}(t)$ vanishes at the points $t = x_k$ $(k:q)$ without exception, is positive and bounded for all other $t \in (-\infty, \infty)$, and tends to unity as t tends to infinity. ~~Accordingly~~ ~~The~~ Hence the function

$$\Theta(t) = \phi_{0,2p-2}(t) - \sum_{j=1}^{q} \phi_{1,j,2\tau(1,j)-1}(t)$$

which belongs to the real linear hull of the set of functions $\{\phi_{0,\nu}, \phi_{1,j,\nu}, \phi_{2,k,\nu}\}$ is positive and bounded for all $t \in [-\infty, \infty]$.

The above proof that $P[n]$ is soluble if $H(n)$ is positive definite depends, ~~critically~~ in particular, upon the ~~lemma that~~ ~~fact~~ result that ~~the~~ formulae which identify ~~that~~ a positive definite functional with respect to an infinite interval also identify ~~such~~ a similar functional with respect to a finite interval. The proof of this result (see (i) above) breaks down if the assumption of positive definiteness is replaced by that of positive semi-definiteness alone. To deal with the case in which $H(n)$ is positive semi-definite, the result (iii) above ~~must be~~ is used.

and are finite. It concludes

exist $A \geq 0$ and a nondecreasing function of bounded variation over $(-\infty, \infty)$ can be found such that

$$Q\{\Theta_\nu(t)\} = A c_{\nu;\theta} + \int_{-\infty}^{\infty} \Theta_\nu(t)\, d\sigma(t)$$

It can be shown, exactly as above, that if $H(n)$ is positive semi-definite, the ~~funct~~ linear functional Q identified by means of formulae (, ,) is nonnegative definite. The ~~above~~ result iii) may be applied to this linear functional Q by taking $\{\Theta_\nu\}$ to be the complete set $\{\Theta_{k,\tau}, \psi_{k,\tau}\}$ and Θ to be the function of formula ( ). All limits of the form ( ) are then zero, except that involving $\phi_{0,2p-2}$ for which the limit in question is unity. All equations of the form (,) are satisfied except that involving t^{2p-2}, for which the alternative

$$\mathring{g}_{2p-2} = A + \int_{-\infty}^{\infty} t^{2p-2}\, d\sigma(t)$$

where $0 \leq A < \infty$, holds. Direct application of the above result (iii) ~~yields~~ leads not to the ~~conclusion~~ consequence that if $H(n)$ is positive semi-definite, $P[n]$ has a solution, but to the less ~~conclusive result~~ specific conclusion that there is an $A \in [0, \infty)$ such that the modified version $P_A[n]$ of $P[n]$ obtained by replacing

g_{2p-2} by $g_{2p-2} - A$ has a solution. This conclusion is in accordance with the facts. Consider the P[3] for which $p=2,\ q=0,\ T(2,1)=1$, $g_0 = g_1 = 1$, $g_2 = 1 + \delta^2$ where $\delta \in (-\infty, \infty)$, $\lambda_1 = 1 \mp i$, $h_1 = i$. The elements (a) of $H(3)$ are in order, row by row, $\{1, 1, i;\ 1, 1+\delta^2, i;\ -i, -i, 1\}$. For all $\delta \in (-\infty, \infty)$, $H(3)$ is positive semi-definite. The value of any N^{ε}-function which obeys asymptotic relationships of the form ( ) with the given g_0, g_1, g_2 and $\delta \neq 0$, for a fixed argument value $\lambda_1 \in \mathbb{L}$, lies on the closed disc whose boundary is defined by the continued fraction expression

$$\frac{1}{\lambda_1 - 1 -} \; \frac{\delta^2}{\lambda_1^* - t}$$

for $t \in [\infty, \infty]$, from which the exceptional point defined as the image of $t = \infty$ (i.e. $(\lambda_1 - 1)^{-1}$) is removed. When $\lambda_1 = 1 - i$, $h_1 = i$ lies at precisely this exceptional point; accordingly P[3] has no solution when $\delta \neq 0$. When $\delta = 0$, there is only one N-function obeying relationships of the form ( ) with the given g_0, g_1, g_2 namely $(\lambda - 1)^{-1}$. The value of this function when $\lambda = \lambda_1 = 1 - i$ is $h_1 = i$. ~~P[3] now has a solution. Clause ( ) describes the true state of affairs; for the given example,~~ Accordingly the modified version of P[3] obtained by replacing g_2 by $g_2 - \delta^2$ has a solution.

~~Integrals~~ ~~like~~

$$F = \int_a^b f(t)\rho(t)\,dt$$

(hpa)

~~In the note under review~~

More general convergence theory of ~~the convergence of~~ such quadrature expressions was developed by, in particular, Markoff, Stieltjes, ~~H.~~ Hamburger, R. Nevanlinna, F. Bernstein and M. Riesz. The note under review briefly sketches this development.

Quadrature, moment problem, analytic theory of continued fractions

Note to the printer:

Greek λ, ρ underlined in red

Suffixes underlined in brown: c_i

With n' as defined in clause ( )$_4$ the fact that $H(n')$ is singular while $H(n'-1)$ is ~~positive defin~~ just implies the (b) existence of a uniquely defined vector $\underset{\sim}{\omega}$ such that $\underset{\sim}{\omega} H(n'-1)$ is the ~~im~~ vector formed from the first $n'-1$ components of the n'^{th} row of $H(n')$. Letting Ω be the $(n'-1)\times(n'-1)$ ~~unit matrix~~ ~~bordered from below~~ formed by bordering the $(n'-1)\times(n'-1)$ unit matrix ~~from~~ by $\underset{\sim}{\omega}$ from below, $H(n') = \Omega H(n'-1) \Omega^*$. The matrix ~~$H^{(n',p)}$~~ $H^{(p)}(n')$ of clause ( ) is singular if and only if the p^{th} component ω_p of $\underset{\sim}{\omega}$ is zero. When $\omega_p \neq 0$, it is possible so to extend $P[n]$ by the provision of two more coefficients g_{2p-1}, g_{2p} that the Hermitian matrix associated with this extended version is also positive semi-definite: a problem of the form $P_A[n+1]$ has a solution which a fortiori is a solution to $P[n]$: $P[n]$ is soluble

Accordingly ~~there is an $A \in (\ldots)$~~ the modified version of $P[3]$ obtained by replacing g_2 by $g_2 - \delta^2$ has a solution.

Subject to suitable ~~existence~~ conditions, successive convergents of the continued fraction associated with the power series $\sum_{i=0}^{\infty} c_i \lambda^{-i-1}$ may be expressed in the form

bp6

$$C_r(\lambda) = \sum_{j=1}^{r} M_{r,j} (\lambda - t_{r,j})^{-1}$$

Christoffel ~~gave the~~ ( ) gave the first general convergence result concerning the functions $C_r(\lambda)$, namely that if ρ is bounded and positive over $[-1, 1]$ and

$$c_i = \int_{-1}^{1} t^i \rho(t)\, dt$$

then the $C_r(\lambda)$ converges for all finite λ ~~such that~~ outside the unit circle to

$$C(\lambda) = \int_{-1}^{1} (\lambda - t)^{-1} \rho(t)\, dt$$

This result ~~was~~ an ~~special case of~~ initial contribution to the theory of the convergence of quadrature expressions

$$F_r = \sum M_r f(t_{r,j})$$

to the integral

The ~~H~~ Hermitian matrix $H(3)$ described above (c)

satisfies neither of the conditions ~~of~~ (, ,) of clause ( ). If the Hermitian matrix $H(n)$ associated with the Hamburger-Pick-Nevanlinna problem $P[N]$ satisfies either ~~one~~ of these conditions, it is possible so to extend $P[N]$ by the provision of two more coefficients g_{2p-1}, g_{2p} that the Hermitian matrix $H(n+1)$ associated with this extended version $P[n+1]$ of $P[n]$ is also positive semi-definite; a problem of the form $P_A[n+1]$ has a solution which a fortiori is a solution to $P[n]$: $P[n]$ itself is soluble.

~~Of~~ The elements $H_{k,j}(n+1)$ $(k,j \colon n+1)$ of the matrix $H(n+1)$ generated | associated with the above extension of $P[n]$ are given by formulae (-). As may be observed only three of these elements, (namely $H_{p+1,p}(n+1) = H_{p,p+1}(n+1) = g_{2p-1}$ and $H_{p+1,p+1}(n+1) = g_{2p}$) depend upon ~~$g_{2p-1}, g_{2p}$~~ the coefficients g_{2p-1}, g_{2p} specifying the extension. Accordingly, setting $m(p,n') = \min(p-1, n'-1)$, the system of $n'-1$ linear equations ~~p-1 when $n' \geq p$ and $m(p,n') = n'-1$ when $n' \leq p$~~

Reversing the order of summation, and using formula ( ),
it follows that

$$\bar{\vec{z}}''_p \sum_{k=1}^{n'-1} \vec{z}_k H_{k,p}(n') = 0$$

bp c

Either $\vec{z}''_p = 0$ or equation ( ) ~~is at~~ also holds for

$j = p$.

d

$$\sum_{k=1}^{m(p,n')} H_{k,j}(n')\, w_k + \sum_{k=p+1}^{n'} H_{k,j}(n')\, w_{k-1} = H_{p+1,j}(n+1)$$

for $j: m(p,n'); p+1, n'$ is well defined (the elements $H_{p+1,p+\gamma}(n+1)$ $(\gamma=0,1)$ are missing from the $\{H_{p+1,j}(n+1)\}$ featuring in this system of equations). The numbers $\{w_k\}$ are obtained and used to set

$$g_{2p+\gamma-1} = \sum_{k=1}^{m(p,n')} w_k\, g_{p+k+\gamma-1} + \sum_{k=p+2}^{n'+1} w_{k-2}\, H_{k,p+\gamma}(n+1)$$

for $\gamma=0,1$ (Again the $H_{k,p+\gamma}(n+1)$ occurring in formula ( ) are independent of g_{2p-1}, g_{2p}; when $\gamma=0$, neither g_{2p-1} nor g_{2p} occurs on the right hand side of formula ( ); g_{2p-1} can be determined and is then used, with $\gamma=1$, to obtain g_{2p}). ~~The Hermitian matrix $H(n+1)$ constructed from the data to $P[n]$ extended by the provision of the two further coefficients $g_{2p+\gamma-1}$ $(\gamma=0,1)$ of formula ( ) is positive semi~~ The two coefficients $g_{2p+\gamma-1}$ $(\gamma=0,1)$ defined by formula ( ) are those inducing the critical extension of $P[n]$ referred to in the preceding paragraph.

for $j = m(p, n'); p+1, n'$ or such a system of relationships (1)

with the right hand sides terms ~~set equal~~ replaced by zero holds. In the latter case ~~is possible~~, there exists a nonzero $\xi' \in C^{n'-1}$ (with $\xi'_p = 0$) such that $\xi' H(n'-1) \xi'^{*} = 0$, a possibility that has been discounted. Since $H(n')$ is singular, and $H(n'-1)$ is not, $\{\xi''_k\}$ (not all zero) exist such that

$$\sum_{k=1}^{n'-1} \xi''_k H_{k,j}(n') = H_{n',j}(n')$$

for $j: n'$. ~~Accordingly. From equations (,)~~ It follows from equations (,) that $\{\xi_k\}$ not all zero exist such that

$$\sum_{k=1}^{n'-1} \xi_k H_{k,j}(n') = 0$$

for $j: m(p, n'); p+1, \ldots, n'$. Using the symmetry property of the $H_{k,j}(n')$, formula ( ) yields

$$\sum_{j=1}^{n'-1} \bar{\xi}''_j H_{k,j}(n') = H_{k,n'}(n')$$

or, setting $j = n'$ in formula ( )

$$\sum_{k=1}^{n'-1} \xi_k \sum_{k=1}^{n'-1} \bar{\xi}''_j H_{k,j}(n') = 0$$

Naturally the assertion that coefficients g_{2p-1}, g_{2p} (e)
can be obtained in the above way and then lead to an extension of $P[n]$ with positive semi-definite associated Hermitian matrix $H(n+1)$ must be justified.
With regard to the feasibility of the above construction it suffices to remark that the matrix of coefficients upon the left hand sides of the set of ~~under the condition~~ equations of which ( ) is a typical member is, by assumption, nonsingular: the $\{a_k\}$ and $\{g_{2p+r-1}\}$ can be determined.

The proof of the ~~second~~ further assertion will be carried out in stages. Firstly, it is remarked that relationships (,) are assertions concerning elements of $H(n+1)$. ~~All elements of $H(n)$, and hence of $H(n')$, are elements of $H(n+1)$. Specifically; for $j = m(p,n)$ & $k, j : p$, $H_{k,j}(n+1) = H_{k,j}(n)$; for~~
$H(n+1)$ is obtained from $H(n)$ by ~~separation~~ inserting an extra row and an extra column between the p^{th} and $(p+1)^{th}$ rows and p^{th} and $(p+1)^{th}$ columns respectively of $H(n)$.
~~The elements of the inserted row and column being~~ ~~used~~.

(2)

(bpc)

$(k=p,\dots,n')$. Thus, concerning the rows of $H(n+1)$, the statement

$$\mathrm{row}\,(p+1) = \sum_{k=1}^{m(p,n')} a_k\,\mathrm{row}\,(k) + \sum_{k=p}^{n'} a_k\,\mathrm{row}\,(k+2)$$

is correct with regard to the elements in positions $1,\dots,m(p,n')$ and $p+2,\dots,n'+2$ since the $\{a_k\}$ (see equation ( )) have been determined from this condition. The statement is also correct with regard to the elements in positions p and $p+1$, since they are respectively δ_{2p-n}, $\delta_{\cdot p}$, and these two coefficients have been determined (see formula ( )) by imposing the condition that the required relationship should be true. It will now be shown that relationship ( ) is ~~generally true~~ holds with regard to the elements in positions $n'+3,\dots,n+1$. If $n'=n-1$, nothing has to be shown; otherwise, for convenience in

~~expressed given by explicit formulae~~ so that (f)

$H_{k,j}(n+1) = H_{k,j}(n)$ $(k,j:p)$, $H_{k+1,j}(n+1) = H_{k,j}(n)$ and $H_{j,k+1}(n+1)$ $= H_{j,k}(n)$ ~~$(k:p, n \mid j:p,n)$~~ ~~$k,j:p$~~ $(k:p,n \mid j:p)$, and $H_{k+1,j+1}(n+1) = H_{k,j}(n)$ $(k,j:p;n)$. ~~The elements of the inserted~~ rows and columns inserted into $H(n)$ in order to form $H(n+1)$ ~~are given by explicit formulae taken from (, ,)~~ Thus equation ( ) may be rewritten as

$$H_{p+1,J}(n+1) = \sum_{k=1}^{m(p,n')} w_k H_{k,J}(n+1) + \sum_{k=p}^{n'-1} w_k H_{k+2,J}(n+1)$$

for $J = m(p,n'); p+2, n'+1$. This relationship also holds for $J: p, p+1$, since the elements $H_{p+1,p+q}(n+1) = \delta_{p+q-1}$ $(q=0,1)$ are determined from this condition. The term $H_{p+1,J}(n+1)$ upon the left hand side of relationship ( ) is ~~is~~ an element of the row inserted into $H(n)$ in order to form $H(n+1)$. It, and all elements of this row, are given by explicit formulae taken from (, ,). A crucial step in the proof under way is that of showing that relationship ( ) holds, not only ~~for $j: m(p,n'); p, \ldots n'+1$~~ but also for $j: n'+2, n+1$. with regard to the elements $H_{p+1,J}(n+1)$ in positions $J: m(p,n'); p, n+1$.

in consequence of the assumptions made concerning the $H'_{k,j}$, possesses a unique solution.), Subsequently, set

bpf

$$g_{2p+\gamma-1} = \sum_{k=1}^{m(p,n')} a_k g_{p+k+\gamma-1} + \sum_{k=p+2}^{n'+2} a_{k-2} H_{k,p+\gamma}$$

for $\gamma = 0,1$, where the ~~above~~ $H_{k,p+\gamma}$ are elements of $H(n+1)$ as described above (again the $H_{k,p+\gamma}$ ~~are~~ occurring in formula ( ) are independent of g_{2p-1}, g_{2p}; when $\gamma = 0$, neither g_{2p-1} nor g_{2p} occur on the right hand side of formula ( ): ~~so that~~ g_{2p-1} can be determined and is then used, ~~to determine~~ with $\gamma = 1$, to ~~determine~~ obtain g_{2p}).

Naturally the assertion that the coefficients g_{2p-1}, g_{2p} defined by formula ( ) lead to an extension of $P[n]$ with positive semidefinite associated Hermitian matrix $H(n+1)$ must be justified. To do so, it is first remarked that for ~~$j = 1, \ldots, p-1$~~ $j: m(p,n')$, $H'_{k,j} = H_{k,j}$ $(k: m(p,n'))$ while $H'_{k,j} = H_{k+2,j}$ $(k = p, \ldots, n')$; for $j = p, \ldots, n'$, $H'_{k,j} = H_{k,j+2}$ $(k: m(p,n'))$ and $H'_{k,j} = H_{k+2,j+2}$

of the inserted rows, but with regard to all elements of these rows: the $(p+1)^{th}$ row of $H(n+1)$ is expressible as a linear combination of rows ~~numbered~~ $1,\ldots,\min(p-1,n')$ and $p+2,\ldots,n'+1$

The elements of $H(n)$ form a subset of those of $H(n+1)$; ~~and~~ certain structural properties of $H(n+1)$ derive directly from those of $H(n)$. Let $H(n)$ be of rank m ~~son~~. Since $H(n)$ is Hermitian ~~and positive semi definite~~ an $n \times m$ matrix A $(m<n)$ with ~~m~~ linearly independent columns exists such that $AA^* = H(n)$. The integer n' (the n' of clause (ii b)) in the range $1 \le n' \le m+1$ exists such that rows $1,\ldots,n'-1$ of A are linearly independent while rows $1,\ldots,n'$ are linearly dependent ~~: $b \in C^{n'}$ exists such that~~ Thus, denoting the $n' \times n$ matrix formed ~~by the first n' rows of $H(n)$ by $H'(n)$~~, $\forall b \in C^{n'}$ nonzero exists such that $b^* H'(n) = 0 \in C^n$; in particular $b^* H(n') = 0$ ~~(0 in both cases represents appropriate zero vectors)~~ The case in which $n' \le p$ is not treated. If ~~$n' > p$ and~~ the p^{th} component of b^* is zero, then, denoting the cofactor of the $(p,p)^{th}$ element of $H(n')$ by $\hat{H}(n')$, and the vector in $C^{n'-1}$ obtained from b^* by deleting its p^{th} component by $\hat{b}$, ~~$\hat{b} H'(n) b$~~ $\hat{b} \hat{H}(n') \hat{b}^* = b^* H(n') b^* = 0$: $\hat{H}(n')$ is singular, contrary to the assumption of clause (ii b α). Hence the

(by δ)

equations

$$\sum_{k=1}^{n'} H'_{k,j}\, a_k = \cancel{\delta_{k+j-2}}\ \overset{m}{h}_{p+1,j}$$

~~where $j : \min(p-1, n')$ for the first right hand term, and~~

~~$h_{p+1,j}$~~ for $(j : n')$ where, with $m(p,n') = \min(p-1, n')$, the $h_{p+1,j}$ ~~the first n~~
~~where $h_{p+1,j}$~~ the $(j : n')$ are in order ~~the successive~~
~~elements of the $(p+1)^{th}$ row of the Hermitian matrix~~
$H_{p+1,j}$ $(j \neq \min(p-1, n'))$; $H_{p+1,j}$ ~~$H(n+1)$ associated with $P[n+1]$, the extended version of~~
~~$P[n]$ obtained by the adjunction of two further~~
~~elements of the $(p+1)^{th}$ row~~

elements $H_{p+1,j}$ $(j : m(p,n'))$ $H_{p+1,j+1}$ $(j = m(p,n')+1, \ldots, n')$ of the $(p+1)^{th}$ row of the Hermitian matrix $H(n+1)$ associated with $P[n+1]$, the extended version of $P[n]$ obtained by replacement of p by $p+1$ and the adjunction of two further coefficients $\delta_{2p-1}, \delta_{2p}$ in relationships of the form ( ) (~~Equations ( )~~)

There are only two elements of the $(p+1)^{th}$ row of $H(n+1)$ that involve $\delta_{2p-1}, \delta_{2p}$, namely $H_{p+1,p+q} = \delta_{2p+q-1}$ $(q = 0, 1)$ and these do not feature in the set $\{\overset{m}{h}_{p+1,j}\}$: the system of linear equations ( ) is uniquely defined and,

p^{th} component of b^* is nonzero, and may be taken to be -1. Denoting the components of b by b_k, ~~using~~ the equation $bH(n')=0$ may be expressed in (h) terms of the elements of $H(n+1)$ in the form

$$H_{p,j}(n+1) = \sum_{k=1}^{m(p,n')}{}' b_k H_{k,j}(n+1) + \sum_{k=p}^{n'}{}' b_k H_{k+1,j}(n+1)$$

holding for $j: m(p,n'): p+2, n'$. That this equation also holds when $n' \le p$ for the stated values of j (these now only run through the values $1,\ldots,n'-1$) may be deduced from the condition imposed in clause (iib,3).

by the first n' rows of $H(n)$ by $H'(n)$, nonzero $v \in C^{n'}$ exists such that $vH'(n) \in C^{n}$. Also, for any $n' \times r$ matrix $H''(n)$ formed from r columns of $H'(n)$, $vH''(n) = 0 \in C^{r}$. In particular, $vH(n') = 0 \in C^{n'}$. The case in which $n' > p$ is now treated. ~~If the p^{th} component~~ Denote the components of $\underset{\sim}{b}$ by b_k $(k: n')$. If $b_p = 0$ then, denoting ~~the cofactor of the $(p,p)^{th}$ element of $H(n')$ by $\hat{H}(n')$~~ by $\hat{H}(n')$ matrix obtained from $H(n')$ by deletion of its p^{th} row and column, and the vector in $C^{n'-1}$ obtained from b by deleting its p^{th} component by $\hat{v}$, $\hat{v}\hat{H}(n')\hat{b}^* = vH(n')b^* = 0$: $\hat{H}(n')$ is singular, contrary to the ~~assumption condition assum condition~~ assumption of

g_{2p-2} by $g_{2p-2} - A$ has a solution; and there is no reason (8)
at this point to suppose that A may be taken to be zero. It (by h)
is, however, possible so to extend $P[n]$ by the provision of
two more coefficients g_{2p-1}, g_{2p} that the ~~associated~~
Hermitian matrix associated with this extended version $P_A[n+1]$
is also positive semidefinite: ~~Now a version of formula~~
~~( ) with p replaced by p+1, and, more importantly, with~~
~~A p unchanged and A=0 A set A=0, hold. A solution to~~
~~$P_A[n+1]$ is a solution to $P[n]$; thus~~
a problem of the form $P_A[n+1]$ has a solution, which, a fortiori ~~ipso~~
~~facto~~ is a solution to $P[n]$: ~~to P[n] h a solution to P[n]~~
~~exists~~ is soluble.

The extension is carried out in the following way.
Delete the p^{th} row and column from $H(n)$, thus producing a
~~an $(n-1) \times (n-1)$~~ Hermitian matrix with elements
$H'_{k,j}$ $(k,j: n-1)$ say. If this matrix is nonsingular, set
$n' = n-1$; otherwise let n' be such that the matrix with
elements $H'_{k,j}$ $(k,j: n')$ is nonsingular, while its
successor with n' replaced by $n'+1$ is singular. ~~Det~~
Obtain numbers $\{a_k\}$ from the n' linear algebraic

clause (iii b…). Hence $v_p \neq 0$; by suitable choice of v, $v_p = -1$ may be taken. The equation ~~$vH(n')=0$~~ may be (i) expressed in terms of the elements of $H(n-1)$ in the form

$$H_{p,j}(n-1) = \sum_{k=1}^{m(p,n')} v_k H_{k,j}(n-1) + \sum_{k=p}^{n'} v_k H_{k+1,j}(n-1)$$

holding for $j: m(p,n'); p+2, n'$. Now denote by $H^{(r)}(n')$ the $n' \times n'$ matrix obtained by replacing the p^{th} column of $H(n')$ by the $(r-1)^{th}$ column of the $n' \times n$ matrix $H'(n)$. As was stated above, $vH^{(r)}(n') = 0 \in \mathbb{C}^{n'}$. Hence nonzero $w^* \in \mathbb{C}^{n'}$, depending upon r, also exists such that $H^{(r)}(n')w^* = 0 \in \mathbb{C}^{n'}$. The above proof that $v_p \neq 0$ may also be used to show that the p^{th} component of w cannot be zero, ~~w may be so chosen~~ and ~~and~~ the equation ~~$H^{(r)}(n')w^* = 0$~~ ( ) may be rewritten as

$$H_{k,r}(n-1) = \sum_{j=1}^{m(p,n')} w_j H_{k,j}(n-1) + \sum_{j=p}^{n'} w_j H_{k,j+1}(n-1)$$

holding for $k: m(p,n'); p+2, n'$. Equations ( ) with $r: m(p,n'); p+2, n'$, equation ( ) with $j: m(p,n'); p+2, n'$, and equation ( ) with a fixed value of r in the range $n'+1 \leq r \leq n+$ and $k: m(p,n'); p+2, n'$, may be used to show that equation ( ) also holds for the fixed value of r in question and hence for all r in the stated range.

The above demonstration that $P[n]$ is stable if $H(n)$ is positive definite depends critically upon ~~two~~ the quoted results from Riesz' theory of linear functionals. For the first of these — namely that a positive definite functional with respect to an infinite interval may also be presented as a positive definite functional with respect to a certain finite interval — the proof (see, for example, [ ] of ... Th) breaks down if the assumption of positive ~~semi~~ definiteness ~~alone~~ is replaced by that of positive semi-definiteness alone. ~~Thus~~ To deal with the case in which $H(n)$ is positive semi-definite, a third, less specific result [,] from Riesz' theory ~~must be~~ is used. Let Q be a ~~positive~~ nonnegative semi-definite functional defined over the linear hull E of a ~~set~~ finite set F of functions θ_ν that are real valued and continuous over $[-\infty,\infty]$, and let E contain at least one function θ such that $\inf \theta(t) < 0$ for $t \in (-\infty,\infty)$ and for all $\theta_\nu \in F$ all limits

$$\lim_{t \to \pm\infty} \frac{\theta_\nu(t)}{\theta(t)} = c_{\nu;\theta}$$

⑧

bp i

To simplify the exposition it is now assumed that $P[n]$ has the particularly simple structure for which $q = 0$, $T(2,k) = 1$ $(k:r)$. The elements of $H(n)$ and $H(n+1)$ then have a correspondingly simple form: setting

$$(f) \qquad U_{\nu,\chi} = \lambda_\nu^{\chi-2} \sum_{z=0}^{\chi-2} a_z \lambda_\nu^{-z} - \lambda_\nu^{\chi-1} c_\nu$$

$$V_{\nu,\chi} = (\bar{\lambda}_\chi - \bar{\lambda}_\nu)^{-1} (c_{\nu,0} - \bar{c}_{\chi,0})$$

$$H_{k,j}(n+1) = g\, a_{k+j-2}$$

for $k,j:p$,

$$H_{p+k+1,j}(n+1) = U_{k,j}$$

and $H_{j,p+k+1}(n+1) = \bar{H}_{p+k+1,j}(n+1)$ for $k:r \mid j:p$, and

$$H_{p+k+1,\,p+j+1}(n+1) = V_{k,j}$$

for $k,j:r$. The functions $\{U_{\nu,\chi}, V_{\nu,\chi}\}$ satisfy the relationships

$$(a) \qquad U_{\nu,\chi} = a_{\chi-2} + \lambda_\nu U_{\nu,\chi-1}$$

$$(b) \qquad \bar{\lambda}_\chi V_{\nu,\chi} = c_{\nu,0} - \bar{c}_{\chi,0} + \lambda_\nu V_{\nu,\chi}$$

In order to condense formulae describing the manipulation of multiple sums, the notation

$$[u_\nu | x_\nu, y_\nu] = \sum_{\nu=1}^{m(p,n')} u_\nu x_\nu + \sum_{\nu=1}^{n'} u_{p+\nu} y_\nu$$

is introduced; ~~in cases in which the~~ terms ~~x_ν, y_ν possess two suffixes, that governing summation is unambiguously~~ ~~suffices~~ the suffix of summation is unambiguously that of the coefficient u_ν. For the simplified version of the elements of $H(n+1)$, ~~formulae~~ the sets of formulae (,) as described become

$$(c)\quad [u_\nu | a_{\chi+\nu-2}; U_{\nu,\chi}] = a_{p+\chi-1} \quad (\chi : p-1), \quad [u_\nu | \bar{U}_{\chi,\nu}, V_{\nu,\chi}] = \bar{U}_{\chi,p+1} \quad (\chi : n'$$

$$(d)\quad [v_\nu | a_{\chi+\nu-2}; U_{\nu,\chi}] = a_{p+\chi-2} \quad (\chi : p), \quad [v_\nu | \bar{U}_{\chi,\nu}, V_{\nu,\chi}] = \bar{U}_{\chi,p} \quad (\chi : n')$$

$$(e)\quad [w_\chi | a_{\chi+\nu-2}; \bar{U}_{\chi,\nu}] = \bar{U}_{J,\nu} \quad (\nu : p), \quad [w_\chi | U_{\nu,\chi}, V_{\nu,\chi}] = V_{\nu,J} \quad (\nu : n')$$

where $J = m-p$ in formulae (,); it must be shown that for this value of J

$$[u_\nu | \bar{U}_{J,\nu}; V_{\nu,J}] = \bar{U}_{J,p+1}$$

Successively

$$[u_\nu | \bar{U}_{J,\nu}; V_{\nu,J}] = [u_\nu | [w_\chi | a_{\chi+\nu-2}; \bar{U}_{\chi,\nu}]; [w_\chi | U_{\nu,\chi}, V_{\nu,\chi}]]$$

$$= [w_\chi | [u_\nu | a_{\chi+\nu-2}; U_{\nu,\chi}]; [u_\nu | \bar{U}_{\chi,\nu}; V_{\nu,\chi}]]$$

$$= [w_\chi | a_{p+\chi-1}; \bar{U}_{\chi,p+1}]$$

(j)

$$= [w_\gamma \mid [v_\nu \mid a_{\gamma+\nu-1}; U_{\nu,\gamma+1}]; a_{p-1} + \bar{\lambda}_\gamma [v_\nu \mid \bar{U}_{\gamma,\nu}, V_{\nu,\gamma}]]$$

$$= [w_\gamma \mid [v_\nu \mid a_{\gamma+\nu-1}; a_{\gamma-1} + \lambda_\nu U_{\nu,\gamma}]; a_{p-1} + [v_\nu \mid \bar{U}_{\gamma,\nu+1} - a_{\nu-1}; c_{\nu,0} - \bar{c}_{\gamma,0} + \lambda_\nu V_{\nu,\gamma}]]$$

$$= [w_\gamma \mid 0; a_{p-1} + [v_\nu \mid \bar{U}_{\gamma,\nu+1} - a_{\nu-1}; c_{\nu,0}]]$$
$$+ [v_\nu \mid [w_\gamma \mid a_{\gamma+\nu-1}; 0]; [w_\gamma \mid a_{\gamma-1} + \lambda_\nu U_{\nu,\gamma}; -\bar{c}_{\gamma,0} + \lambda_\nu V_{\nu,\gamma}]]$$

$$= [v_\nu \mid [w_\gamma \mid a_{\gamma+\nu-1}; \bar{U}_{\gamma,\nu+1}]; \bar{U}_{J,1} + \lambda_\nu V_{\nu,J}]]$$

$$= [v_\nu \mid \bar{U}_{J,\nu+1}; \lambda_\nu V_{\nu,J} - \bar{c}_{J,0}]$$

$$= [v_\nu \mid a_{\nu-1} + \bar{\lambda}_J \bar{U}_{J,\nu}; \bar{\lambda}_J V_{\nu,J} - c_{\nu,0}]$$

$$= a_{p-1} + \bar{\lambda}_J [v_\nu \mid [w_\gamma \mid a_{\gamma+\nu-2}; \bar{U}_{\gamma,\nu}]; [w_\gamma \mid U_{\nu,\gamma}; V_{\nu,\gamma}]]$$

$$= a_{p-1} + \bar{\lambda}_J [w_\gamma \mid a_{p+\gamma-2}; \bar{U}_{\gamma,p}]]$$

$$= \cancel{a_{p-1} + \bar{\lambda}_J \bar{U}_{J,p}} \; \bar{U}_{J,p+1}$$

The first of the above relationships follows by use of ~~Ref~~ formulae (^{e}), the second by rearrangement of double sums, the third ~~second of~~ from formulae (^{e}), the fourth from the first of formulae (^{d}),

$$H_{p,j}(n'+1) = \sum_{k=1}^{m(p,n')} b_k H_{k,j}(n'+1) + \sum_{k=p}^{n'-1} b_k H_{k+2,j}(n'+1) \quad \text{(bp j)}$$

for $j: n'+1$, a relationship which may also ~~holds~~ be shown to hold

It will now be shown that relationship ( ) ~~holds~~ also holds for $j = n'+2, \ldots, n+1$. When $n' = n$, nothing has to be shown. Otherwise, as remarked above $H(n')$ is when $n' < p$. When $n+1 < m \leq n+2$, it may be shown that the ~~first~~ column vector ~~of~~ formed from the first n' elements of the $(m-1)$th column of $H(n)$ is expressible as a linear combination of columns $1;m(p,n');p+1,n$ of $H(n')$; alternatively

$$H_{k,m}(n+1) = \sum_{j=1}^{m(p,n')} c_j H_{k,j}(n+1) + \sum_{j=p}^{n'-1} c_j H_{k,j+2}(n+1)$$

for ~~$k: n'+1$~~ $k: m(p,n'); p+2, n'+1$, where the $\{c_j\}$ depend upon m $(m: n'+1; n+1)$

~~t, u,~~ ~~and ( ) with and~~ use of the complex conjugate version of

$(^a)$ with ν, χ interchanged and ν then taken to be $p+1$, ~~the~~ (x)

~~fifth from~~ and subsequent use of the second of formulae $(^d)$,

the fifth from use of ~~and~~ appropriate versions of formulae $(^a)$ ~~of~~ and formula

$(^b)$, the sixth by rearrangement, the seventh by use of

formulae $(^f)$ ~~of~~ and the first of $(^e)$ ~~with~~ both with $\chi = 1$,

~~and also of the first of formulae $(^e)$ with $\nu = 1$~~

use ~~and~~ of the complex conjugate version of formula $(^f)$ with

ν, χ interchanged and ν then taken to be 1 and then ~~use~~ of

the first of formulae $(^e)$ with $\nu = 1$, and use of the

second of formulae $(^e)$, the eighth by use of ~~formula~~

the first of formulae $(^e)$ with ν replaced by $\nu+1$ and

of the complex conjugate version of formula $(^f)$ with $\chi = 1$,

the ninth by use of appropriate versions for formulae

$(^a, {}^b)$, the tenth by use of the first of formulae

$(^d)$ with $\chi = 1$ and of formulae $(^e)$, the eleventh

by ~~use of formulae~~ rearrangement and use of formulae $(^d)$,

and the twelfth by use of the first of formulae $(^e)$

with $\nu = p$ and the complex conjugate version of formula ( )

with $\chi = p+1$, $\nu = J$.

be justified. The matrix of coefficients upon the left hand side of equation ( ) ~~has been assumed~~ is nonsingular by assumption: the $\{a_k\}$ and $g_{2p+\chi-1}$ can be (by k) determined. For $j: m(p,n')$, $H_{k,j}(n''+1) = H_{k,j}(n')$ $(k: m(p,n'))$ while $\cancel{H_{k,j}(n'+1) = H_{k+2,j}}$ $H_{k+4,j}(n''+1) = H_{k,j}(n')$ $(j: p, n')$; for $j: p, n'$ $H_{k,j+1}(n''+1) = H_{k,j}(n')$ $(k: m(p,n'))$ and $H_{k+1,j+1}(n''+1) = H_{k,j}(n')$ $(k: p, n')$. Thus ~~formula~~ equation ( ) may be rewritten

as

$$H_{p+1,j}(n''+1) = \sum_{k=1}^{m(p,n')} a_k H_{k,j}(n'') + \sum_{k=p}^{n'-1} a_k H_{k+2,j}(n''+1)$$

for $j: m(p,n'); p+2, n'+1$. This relationship also holds for ~~$j: p, m(p,n')+2, \ldots$~~ $j: p, p+1$, since the elements $H_{p+1,p+\chi}(n''+1) = g_{2p+\chi-1}$ $(\chi = 0, 1)$ ~~have been~~ are ~~been~~ determined from this condition. $H(n')$ ~~was~~ is expressible in the form $\Omega H(n'-1)\Omega^*$ where $\Omega^* = (I, \omega^*)$ and, ~~as described above~~ when $n' \geq p$ the p^{th} component of ω is nonzero. ~~and When~~ The p^{th} row of $H(n')$ is thus expressible as a linear combination of the ~~the~~ remaining rows of $H(n')$ ~~or, in terms~~ alternatively

It is possible, by the use of sums more complex ~~general~~ (l) than ( ) and relationships more involved than ( ), to show that relationship ( ) holds for all ~~values of J in~~ elements of the $(p+1)^{th}$ row of $H(n+1)$ in the general case in which ~~[illegible]~~ the elements of $H(n+1)$ are given by expressions of the form (, ,) in their full generality. This more complete result may, however, be obtained from the special result already available.

Having proved that relationship ( ) holds with regard to all elements ~~$H(n+1)$~~ $H_{p+1,J}(n+1)$ of the $(p+1)^{th}$ row of $H(n+1)$, it is a comparatively simple matter to show that $H(n+1)$ is positive semi-definite. Let C be the $(n+1)^{th}$ order unit matrix ~~ambe~~ modified by the addition of elements $-a_k$ in position ~~k~~ $(k: \cancel{1}\, p \ldots \cancel{and} \mid p+2, m)$ of the $(p+1)^{th}$ row. ~~$CH(n+1)$~~ The matrix $CH(n+1)C^*$ is ~~$H(n+1)$~~. ~~with the~~ obtained from $H(n+1)$ by replacing all elements of the $(p+1)^{th}$ row and column by zero elements. The remaining elements of $CH(n+1)C^*$ are successively thus

$$\mathring{g}_{2p+\gamma-1} = \sum_{k=1}^{m(p,n')} a_k \mathring{g}_{p+k+\gamma-1} + \sum_{k=p+2}^{n'+1} a_{k-2} H_{k,p+\gamma}(n+1)$$

for $\gamma=0,1$. (Again the $H_{k,p+\gamma}(n+1)$ occurring in formula ( ) are independent of $\mathring{g}_{2p-1}, \mathring{g}_{2p}$; when $\gamma=0$, neither $\mathring{g}_{2p-1}$ nor $\mathring{g}_{2p}$ occurs on the right hand side of formula ( ); g_{2p-1} can be determined and is then used, with $\gamma=1$, to obtain g_{2p}.)

Naturally the assertions that ~~numbers~~ coefficients $\mathring{g}_{2p-1}, g_{2p}$ can be obtained in the above way and then lead to an extension of $P[n]$ with positive semi-definite associated Hermitian matrix $H(n+1)$ must ~~be justified. Firstly, the possibility that the matrix~~ of coefficients on the left hand sides of equations ( ) ~~must be~~ is singular must be discounted. Suppose that this matrix is singular, ~~Either a system of~~ so that $\{\xi'_k\}$, not all zero, exist such that either

$$\sum_{k=1}^{m(p,n)} \xi'_k H_{k,j}(n') + \sum_{k=p+1}^{n'-1} \xi'_k H_{k,j}(n') = H_{n',j}(n')$$

of $H(n)$. C is nonsingular ~~corresponding to each $\xi\in C^{n+1}$, $\xi'\in C^{n+1}$ exists such that~~ With $\xi\in C^{n+1}$, let ~~$\xi C = \xi$~~ $\xi^{*} = \xi' C$ obtain ξ' from the equation $\xi = \xi' C$ and let $\hat{\xi}\in C^{n}$ be obtained from ξ' by ~~supp~~ deleting the $(p+1)^{th}$ element of the latter vector. ~~Since~~ ξ then $\xi H(n+1)\xi^{*} = \hat{\xi} H(n)\hat{\xi}^{*} \geq 0$. ~~It is less~~ Furthermore, by ~~taking~~ selecting nonzero $\hat{\xi}\in C^{n}$ ~~to be such~~ for which $\hat{\xi} H(n)\hat{\xi}^{*} = 0$, ~~and~~ assigning the $(p+1)^{th}$ element of ξ' arbitrarily, $\xi = \xi' C$ is a nonzero element of C^{n+1} for which the condition $\xi H(n+1)\xi^{*} = 0$ is attained: $H(n+1)$ is positive semi-definite, and for reasons stated at the commencement of the proof of the result under consideration, $P[n]$ is soluble.

The conditions of $P[m]$ are less restrictive than those of $P[n]$: ~~any~~ the solution ~~G of formula ( )~~ to $P[n]$ is a fortiori $G(\lambda) = \int (\lambda - t)^{-1} b_{p,q}(t)\, d\sigma(t)$ a solution to $P[m]$. ~~Taking the solution G of formula ( ) to be a solution to~~ The $\{a_\nu, b_{j,\nu}, c_{k,\nu}\}$ featuring in the data to $P[m]$ are given by formulae ~~( )~~ similar to ( ) with $[\alpha, \beta]$ replaced by $[-\infty, \infty]$ and $d\beta(t)$ by $b_{p,q}(t)$

The extension is carried out in the following way.

~~Denote the elements of $H(n)$ by $H_{k,j}(n)$ $(k,j:n)$~~ and

let $H(m)$ be the matrix with elements $H_{k,j}(n)$ $(k,j:m)$ for $m:n$. Let n' be such that $H(n'-1)$ is positive definite ~~while $H(n')$ is singular (n' is uniquely defined).~~ ~~Set $m(p,n') = \min(p-1,n')$~~. ~~With $\delta_{2p-1}, \delta_{2p}$ two further coefficients,~~ Let $P[n+1]$ be the extended version of $P[n]$ obtained by replacement of p by $p+1$ and the adjunction of two further coefficients $\delta_{2p-1}, \delta_{2p}$ in relationships of the form ( ), and let $H_{k,j}(n+1)$ $(k,j:n+1)$ be its elements. Set $m(p,n') = \min(p-1,n')$. Obtain numbers $\{a_k\}$ from the $n'-1$ linear algebraic equations

$$\sum_{k=1}^{m(p,n')} H_{k,j}(n') a_k + \sum_{k=p+1}^{n'} H_{k-1,j}(n') a_{k-1} = H_{p+1,j}(n+1)$$

for $j: m(p,n'); p+1, n'$. (It should be remarked that only two members of the set $\{H_{p+1,j}(n+1)\}$ involve $\delta_{2p-1}, \delta_{2p}$ namely $H_{p+1,p+\chi}(n+1) = \delta_{2p+\chi-1}$ $(\chi=0,1)$, and these are missing from the above equations.)

Subsequently set

The conditions of $P[m]$ are less restrictive than those of $P[n]$: a solution

$$G(\lambda) = \int_{-\infty}^{\infty} (\lambda - t)^{-1} b_{p,q}(t) d\gamma(t)$$

where $\gamma \in B[-\infty,\infty]$, to $P[n]$ is a fortiori a solution to $P[m]$. The $\{a_\nu, b_{j,\nu}, c_{k,\nu}\}$ featuring in the data to $P[m]$ are given by formulae similar to (-) with $[\alpha, \beta]$, $d\beta(t)$ replaced by $[-\infty,\infty]$, $b_{p,q}(t)d\gamma(t)$. ~~The elements of H(m)~~ Using these formulae, the elements of $H(m)$ are expressible as Stieltjes integrals with respect to γ over $[-\infty,\infty]$. Since $H(m)$ is positive semi-definite, there is a nonzero $\xi \in \mathbb{C}^m$ such that $\xi H(m) \xi^* = 0$. Obtain numbers $\{\eta_k$, ~~$\xi_{p+m(1,k)+k}, \xi$~~ $\eta_{1,k,k}$, $\eta_{2,k,k}\}$ by ~~use of a~~ from the components of ξ by use of a system of relationships of the form ( ). As in the consideration of the case in which $H(n)$ is positive definite, it may then be shown that for the $\{\eta_k, \eta_{1,k,k}, \eta_{2,k,k}\}$ in question

$$\int_{-\infty}^{\infty} |\hat{Y}(t)|^2 \, d\sigma(t) = 0$$

where $\hat{Y}$ has the form ( ) with $p, q, r, T(1,j), T(2,k)$ replaced by $p', \ldots, T'(2,k)$ ~~p', q', r' numbers~~ p', q', ~~r'~~ which depend upon the number of $\{x_j, \xi, \lambda_k\}$ featuring in the data to $P[m]$ (~~possibly~~ one possible $r' = 0$ and even also $q' = 0$). $\hat{Y}$ may alternatively be expressed in the form

$$\left\{(t-i)^{1-p'} \prod_{k=1}^{r'} (t-\lambda_k)^{-T'(2,k)}\right\} Z(t)$$

where $Z(t)$ is a polynomial of degree m. It follows from equation ( ) that σ is a step function with salti at the zeros $\{t_\nu\}$ of Z and no other points of increase in the range $[-\infty, \infty]$. Thus $G(\lambda)$ has the form

$$G(\lambda) = \sum_{\nu=1}^{m} M_\nu (\lambda - t_\nu)^{-1}$$

where $M_\nu \geq 0$ $(\nu : m)$.

It is ~~now~~ supposed that one of the M_ν, M_2 say, is zero. Taking $\{p', r', T'(2,k)\}$ now to be numbers appropriate to $P[m-1]$, and $Z(t)$ to be ~~the product~~

$\prod_{\nu=1}^{m}{}^{(2)}(t-t_\nu)$, the function $\hat{Y}(t)$ of expression ( ) is constructed ~~formed~~, and is then decomposed in the form

( ) where again $p, q, r, T(1,j), T(2,k), x_j, \lambda_k$ are replaced by appropriate numbers deriving from the data to $P[m-1]$. ~~The function $\hat{Y}(t)$ under consideration also satisfies equation ( ).~~

From the coefficients $\{\eta_k, \eta_{1,k,k}, \eta_{2,k,k}\}$ occurring in this decomposition obtain $m-1$ numbers ξ_ν by use of relationships of the form ( ). ~~from the numbers $\{\eta_k, \eta_{1,k,k}, \eta_{2,k,k}$ occurring~~ Then

$$\int_{-\infty}^{\infty} |\hat{Y}(t)|^2 \, d\gamma(t) = \xi H(m-1)\xi^*$$

The function $\hat{Y}(t)$ under consideration also satisfies equation ( ), since γ is a step function having salti only where $\hat{Y}(t)$ vanishes. Accordingly, there is a nonzero $\xi \in C^{m-1}$ such that $\xi H(m-1)\xi^* = 0$. But it has been assumed that $H(m-1)$ is positive definite. From this contradiction it follows that no M_ν in the representation ( ) of a solution to $P[n]$ is zero.

It is now supposed that there are two distinct solutions of the form ( ) to P[n], and hence of P[m]. Let them have coefficients M'_ν, M''_ν; the numbers t_ν, being fixed, are the same in both cases. With x finite, any function of the form ( ) with coefficients $xM'_\nu + (1-x)M''_\nu$ satisfies the algebraic relationships of the form (, ,) relating to P[m]. For all x such that $xM'_\nu + (1-x)M''_\nu \geq 0$ $(\nu : m)$ the function in question maps $\mathbb{L}$ into $\mathbb{W}$, and hence is a solution to P[m]. It has been shown that $M'_\nu, M''_\nu > 0$ $(\nu : m)$. There is a real x such that one of the $xM'_\nu + (1-x)M''_\nu$ is zero, while the remainder are nonnegative. ~~Thus a solut~~ This x induces a solution of the form ( ) to P[m] in which one of the M_ν is zero. It has been shown that P[m] possesses no such solution. Accordingly the solution to P[n] of the form ( ) is unique.

In summary it has been demonstrated that under the conditions of clause (Aii) with $p, q, r > c$ P[n] is degenerate; its degenerate solution, a rational function with denominator and numerator polynomials of degrees m and $m-1$ respectively, is its only solution.

Having shown that various conditions suffice to ensure ~~that P[n] is solub the existence of a solut~~ the existence of a solution to $P[n]$, it is now to be shown that these conditions are necessary. Suppose that the function G of formula ( ), with $[a, b] \subseteq [-\infty, \infty]$ is a solution to $P[n]$. Formulae similar to (–) ~~express~~ with $Q\{\ldots\}$ replaced by $\int_{-\infty}^{\infty} \ldots d\sigma(t)$ express the elements of $H(n)$ in terms of σ. With ξ_{ν} $(\nu : n)$ any sequence of finite complex numbers, not all zero, derive numbers $\{\eta_k, \eta_{1,k,k}, \eta_{2,k,k}\}$ by use of relationships ( ). As in the above proof of sufficiency it may be shown that formula ( ), with the above reinterpretation of Q, holds. The value of the expression upon the right ~~t~~ hand side of this relationship is nonnegative. If $H(n)$ is neither positive definite nor nonnegative definite, a solution to $P[n]$ does not exist

The Hermitian matrix $H(n+1|\Lambda, h)$ is simply the matrix of the form $H(n+1)$ derived from the extended version $P[n+1]$ of $P[n]$ which is obtained by the adjunction of a further argument and function value pair $\lambda_{r+1} = \Lambda$, $c_{r+1,0} = h$. There is a region in the complex h-plane for which $\det\{H(n+1|\Lambda, h)\} < 0$. This region, in particular, contains all points of $\mathbb{L}$, since for no: the problem $P[n+1]$ which requires its solution to take a value h in $\mathbb{L}$ when its argument value Λ also lies there is insoluble. Equally there is a region in the h-plane for which $\det\{H(n+1|\Lambda, h)\} > 0$. This region, in particular, contains the points h representing the values of all nonrational solutions to $P[n]$. ~~that are either rational functions with denominator~~ The equation $\det\{H(n+1|\Lambda, h)\} = 0$ represents the curve in the complex h-plane seperating the two above regions. By expansion of the determinant in question, the equation may be written in the form

$$\Upsilon|h|^2 + \Xi\bar{h} + \bar{\Xi}h + \Omega = 0$$

where ~~A~~ Υ, Ξ and Ω are independent of h, Υ and Ω being real. This equation determines a circle $\mathcal{C}(\Delta)$ in the complex h-plane. The values, for the fixed argument value Δ, of all ~~every~~ solutions to $P[n]$ that are either rational functions with denominator polynomial of degree ~~n~~ greater than n or are nonrational lie ~~on~~ in $\mathbb{D}(\Delta)$, the open disc bounded by $\mathcal{C}(\Delta)$. Every point of $\mathbb{D}(\Delta)$ represents the value of such a solution.

The values of h lying on $\mathcal{C}_n(\Delta)$, ~~i.e.~~ namely those values inducing the condition $\det\{H(n+1|\Delta,h)\}_0^n = 0$ may be investigated ~~with the~~ by use of clause (Aii v). ~~H[n+1] has a solution, i.e. h is~~ In the notation of that clause $m=n$. $P[n+1]$ has a solution, i.e. h is the value of a solution to $P[n]$ expressible as a rational function with denominator polynomial of degree n, if and only if the matrix $H'\{n+1|\Lambda,h\}$ derived by suppressing the p^{th} row and column from $H\{n+1|\Lambda,h\}$ is nonsingular. The locus of points ~~$H\{n+1|\Lambda,h\}$~~ h for which $\det\{H'\{n+1|\Lambda,h\}\}=0$ is a circle $\mathcal{C}_n'(\Lambda)$ bounding an ~~disc~~ inclusion disc for the solutions of $P'[n]$, the problem obtained ~~by suppressi~~ from $P[n]$ by suppressing reference to a_{2p-2} and, if $p>1$, a_{2p-3}. The conditions of $P'[n]$ are less restrictive than those of $P[n]$. $\mathcal{C}_n'(\Lambda)$ either lies entirely outside $\mathcal{C}_n(\Lambda)$, or does so except at a point of common tangency, or is identical with $\mathcal{C}_n(\Lambda)$. If the last case were to occur, $P[n]$ would have no solutions in the form of a rational function with denominator polynomial of degree n, a conclusion which ~~is easily~~ may be shown by construction to be false. The numbers w_k are derived from the system of equations

$$\sum_{k=1}^{p-1} H_{k,j}(n)w_k + \sum_{k=p}^{n-1} H_{k+1,j}(n)w_k = \tilde{H}_{n+1,j}(n)-1$$

for $j: p-1; p+1, n$ where $\tilde{H}^{m}_{n+1,j}(n+1)$ is the j^{th} element in the $(n+1)^{th}$ row of the Hermitian matrix $H'(n+1)$ associated with that extension $\tilde{P}^{*}[n+1]$ of $P[n]$ derived ~~by~~ from the adjunction of two further coefficients a_{2p-1}, a_{2p} (the $\tilde{H}^{m}_{n+1,j}(n+1)$ concerned do not involve these coefficients. Subsequently ~~we~~ fix the a_{2p-1}, a_{2p} featuring in $\tilde{P}[n+1]$ by setting

$$\cancel{a_{2p+\chi-1} = \sum_{k=1}^{p-1} H_{k,p+\chi}{}^{(n)} u_k + \sum_{k=p}^{n-1} H_{k+1,p+\chi-1}{}^{(n)} u_k}$$

$$\cancel{p-1+p+1-2\cdot1}\quad a_{2p+\chi-1} = \sum_{k=1}^{p-1} a_{k+p+\chi-2} u_k + \sum_{k=p}^{n-1} \tilde{H}_{k+1,p+\chi}(n+1) u_k$$

for $\chi = 0, 1$. $\tilde{H}(n+1)$ is positive semi-definite but, as is easily demonstrated, the ~~...~~ elements $\tilde{H}_{k,j}(n+1)$ $(k, j : n)$ form a nonsingular matrix. The matrix obtained by suppressing the $(p+1)^{th}$ row and column from $\tilde{H}(n+1)$ ~~is $H(n)$~~, being $H(n)$, is nonsingular. The results of clauses (Aii, iii)$^{!}$ may be applied: $\tilde{P}'[n+1]$ has a unique solution in the form of a rational function with denominator of degree n. This solution is also one of the solutions to $P[n]$. Hence $\mathcal{C}'_m(\Lambda)$ either lies entirely outside $\mathcal{C}_n(\Lambda)$, or does so at a point of common tangency.

The possible solutions to the Hamburger-Pick-Nevanlinna problem ~~as~~ ~~formulated~~ presented above are N-functions which, behaving in a very special way, at ~~with~~ ~~at~~ ~~regard to~~ ~~their behaviour at~~ the specified boundary points of $\mathbb{L}$ ~~(the x_k and infinity)~~ behave in a very special way: If $q>0$ it is required that the solutions should tend to ~~a~~ finite limits on the real axis at the points x_k^ν; if $p>0$ it is required that they should tend to zero at infinity. It is ~~clearly possible~~ feasible so to formulate an interpolation problem that possible solutions ~~possess singularities~~ may tend to infinity at these boundary points. ~~Nevertheless many N functions exhibiting singular behaviour it is possible to refer this~~ Many such extended versions of the interpolation problem ~~directly~~ may directly be referred to the special problem ~~stated~~ dealt with in detail above.

Many N-functions exhibiting singular behaviour at specified ~~points on the~~ boundary points of $\mathbb{L}$ may be expressed ~~decomposed into the~~ as the sum of ~~a~~ two components: simple functions ~~of permissible forms~~ exhibiting the singular behaviour in question, and an N-function tending to ~~a~~ finite limits at the boundary points. Extended interpolation problems ~~involving~~ whose

~~possible~~ solutions ~~functions~~ are required to exhibit the given singular behaviour, ~~of the type just referred to~~ at specified boundary points of $\mathbb{L}$ may thus be treated by removing the simple $\mathbb{N}$-function from the problem by suitably modifying the conditions of the form (, ,); ~~and subsequently determining, if this be possible, a solution ni~~ the resulting problem has the form described above and its possible solutions have the regular behaviour at the boundary points required of the second component of the solution to the extended problem. The ~~way in which this modification takes~~ ~~can be done is most simply illustrated~~ modification of an extended problem is most simply illustrated by supposing that in just one ~~of the~~ of the conditions of the form ( ), that concerning the point x_j, an additional term $b_{j,-1}(\lambda-x_j)^{-1}$ features in the sum ~~(2)~~ concerned. A possible solution to the extended problem has the form $G(\lambda) = b_{j,-1}(\lambda-x_j)^{-1} + \hat{G}(\lambda)$, where $\hat{G}$ satisfies a relationship of the form ( ) with g_z replaced by $\hat{g}_z = g_z - b_{j,-1}x_j^{z}$, and relationships ~~of the~~ of the form ( ) with $b_{k,z}$ replaced by $\hat{b}_{k,z} = b_{k,z} + b_{j,-1}(x_j - x_k)^{z-1}$, ~~and of the form ( ) with $c_{k,z}$ or the $b_{j,z}$~~ when $k \neq j$ and $\hat{b}_{j,z} = b_{j,z}$, and of the form ( ) with ~~all $\hat{c}_{k,z}$ modif~~ $\hat{c}_{k,z} = c_{k,z} + b_{j,-1}(x_j - \lambda_k)^{z-}$

The extended interpolation problem involving the additional term $b_{j,-1}(\lambda-x_j)^{-1}$ is soluble ~~has a solution~~ if and only if (a) $0 \le b_{j,-1} < \infty$ and (b) the Hamburger-Pick-Nevanlinna problem involving the coefficients $\hat{g}_z, \hat{b}_{k,z}, \hat{c}_{k,z}$ has a solution. Evidently, a number of terms of the form $b_{j,-1}(\lambda-x_j)^{-1}$ can be removed from an extended interpolation problem in this ~~same~~ way; ~~as can also~~ the same holds for terms of the form $b_{j,-1}\ln\{(\lambda-\alpha)^{-1}(\lambda-x_j)$ $(-\infty<\alpha<\infty)$ and certain algebraic expressions. ~~The terms of the form~~ Additional terms of the form $a_{-2}\lambda+a_{-1}$ can also be removed from the sum in relationship ( ) concerning the behaviour of $G(\lambda)$ as λ tends to infinity.

In the above analyses ~~concerning it is required, in particular, that x_k~~ concerning the behaviour of the function G at the point x_k, it is required that ~~the~~ x_k should be a regular point of the component function $\hat{G}$, i.e. that $\hat{G}(\lambda)$ should tend to a finite limit as λ tends to x_k as described. The presence of the finite coefficient $b_{k,0}$ in the sum ( ) ensures that this condition is satisfied by all solutions of the reduced interpolation problem.

It is conceivable, however, that ~~by~~ an extended interpolation problem may contain a condition concerning singular behaviour at a boundary point without accompaniment of a series of the form ( ). ~~to ensure that the regularity of the component $\hat{G}$ reduced component~~. For example, a condition of the form $\lim(\lambda - x_j)G(\lambda) = b_{j,-1}$ as λ tends to x_j over an open set in $\mathbb{L}$ with limit point at x_j, ~~may be impo~~ where x_j is not one of the x_k featuring in relationship ( ), may be imposed: now the condition ensuring regularity of the ~~reduced~~ component $\hat{G}$ function $\hat{G}$ at x_k is absent. To deal with cases of this sort, it is stated, in anticipation of results later to be given, that solubility of ~~the~~ a ~~Hamburger~~ Pick-Nevanlinna ~~Hamb~~ problem $P[n]$ is of two types. For the first, $P[n]$ is degenerate; its degenerate solution, a rational function, is its only solution. In this case the degenerate solution can be inspected; if it is regular at the point x_k, the extended problem from which $P[n]$ is derived is soluble; otherwise not.

In the second case $P[n]$ possesses a continuum of solutions. ~~Members of this continuum may~~ ~~These~~.

for which firstly explicit constructions in the form ( ) ~~are available~~ and, secondly, representations in continued fraction form involving an arbitrary N-function are available. There is sufficient liberty available in the construction of the function σ in the representation ( ) to ensure that the solution is regular at the point x_j; the arbitrary N-function may also be so chosen that ~~re~~ the required regularity occurs. ~~In the~~ when the derived problem $P[n]$ is nondegenerate, the extended problem ~~has a solution~~ is soluble.

Conditions ~~as~~ of the form $G(\lambda) \sim a_{-2}\lambda$ and $G(\lambda) \sim a_{-2}\lambda + a_{-1}$ as λ tends to infinity ~~in an~~ appropriately ~~such~~ occurring in an extended interpolation problem ~~was~~ and unaccompanied by a regularising condition of the form ( ) may be treated in the same way. If the problem obtained by elimination of the additional condition is degenerate, its solution may be inspected; if this solution tends to a finite limit as λ tends to infinity in the case of the first of the above conditions, or if the solution tends to zero in the second case, the extended interpolation problem is soluble; otherwise not. If the derived interpolation problem $P[n]$

is nondegenerate, the arbitrary N-function occurring in the general representation of its solutions may be so chosen ~~that a solution~~ as to define a subclass of solutions behaving in the required manner as ~~t~~ their argument tends to infinity: in this case, the extended problem from which P[n] is derived is soluble.

Throughout the above proof it has been assumed that $p, q, r > 0$. It remains to consider the necessary modifications and possible simplifications of the proof that arise when these conditions upon p, q and r are not satisfied. The case in which $p > 0$ and either ~~p~~ $q = 0$ or $r = 0$ may be dismissed by the remark that ~~the above proof then merely becomes simpler;~~ various functions are not defined and sundry sums and double sums fall away from the formulae ~~the structure of the above proof is retained; the details are~~ the above proof retains its ~~structure~~ outline but becomes simpler in appearance.

The assumption $p > 0$ imposes a condition upon possible solutions to $P[n]$, namely that they should tend to zero as their argument tends appropriately to ~~zero~~ infinity. The solutions to $P[n]$ constructed in the above proof have a special form which reflects this behaviour; ~~When~~ they are members of a ~~cl~~ subclass of

N-functions. When $p=0$, $P[n]$ may possess solutions that are not members of this subclass; in particular when $H(n)$ is positive semi-definite, it may occur that the only solution to $P[n]$ is not of the special form described. Accordingly, ~~forms expressions for~~ when dealing with the case in which $p=0$, possible solutions must be expressed in ~~the~~ a form appropriate to general N-functions.

As a preliminary to the consideration of the case in which $p=0$, $r>0$, it is remarked that solubility of $P[n]$ is unaffected by the replacement of ~~the~~ all x_j, λ_k by Ax_j+B, $A\lambda_k+B$ where $0<A<\infty$, $-\infty<B<\infty$. It is also unaffected by the replacement of ~~all~~ $b_{j,0}$ by $Cb_{j,0}+D$, of $b_{j,\rho}$ $(\rho>0)$ by $Cb_{j,\rho}$, where $0<C<\infty$, $-\infty<D<\infty$, and ~~by~~ ~~the~~ similar replacements of $c_{k,0}$, $c_{k,\rho}$. Such replacements ~~carry~~ ~~these~~ may be used to transform the number pair λ_1, $c_{1,0}$ into $-i, i$: for the purposes of investigating the solubility of $P[n]$ there is no loss of generality in ~~assuming~~ restricting attention to ~~considering~~ a normalised version of $P[n]$ in which ~~that~~ $\lambda_1=-i$, $c_{1,0}=i$. ~~The set F of functions of~~

~~functions:~~

With $b_{p,q}$ the function defined by formula ( ), set $b_q(t) = b_{1,q}(t)$ and take $\mathbb{F}$ to be the set of functions

$$\phi_{1,j,0}(t) = \cancel{(x_j - t)^{-1}}\, b_q(t)(x_j - t)^{-1}(1 + x_j t)$$

$$\phi_{1,j,z}(t) = -b_q(t)(t - x_j)^{-z-1}$$

for $j:q$, and $z: 2T(1,j)-1$ in the second formula, and

$$\phi_{2,k,0}(t) = b_q(t)(\lambda_k - t)^{-1}(1 + \lambda_k t)$$

$$\phi_{2,k,z}(t) = -b_q(t)(t - \lambda_k)^{-z-1}$$

for $k:r$ and $z: T(2,k)-1$. The linear functional Q on $\mathbb{F}$ is determined by use of the ~~second and third~~ last two of formulae ( ). In the verification of formula ( ) with $b_{p,q}$ replaced by b_q, for the special case in which $k=1$, formula ( ) must be replaced by

$$Q\{b_q(t)(t - x_K)^{-1}(t - x_J)^{-j}\} =$$
$$-\sum_{z=0}^{j-2}(x_K - x_J)^{-z-1} Q\{b_q(t)(t - x_J)^{z-j}\}$$
$$+(-1)^j (x_J - x_K)^{-j} Q\{(x_J - t)^{-1}(1 + x_J t) - (x_K - t)^{-1}(1 + x_K t)\}$$

but otherwise the ~~general~~ proof of the new version of formula ( ), ~~is as before~~ and the proofs of the new version of formulae (,) are as before. With $Y(t)$ redefined by replacing $(t-i)^{1-p}$ by $\{t-i\}$ and discarding the sum involving η_k, and deleting the first of relationships ( ), formula ( ) is obtained and used to show that the set of numbers ~~$\{b_{j,\nu}$~~ $\{b_{j,\nu}, c_{k,\nu}\}$ is positive definite with respect to the functions $\{\phi_{1,\nu}, \phi_{2,\nu}\}$.

~~In the case being considered~~

Subject to the current assumptions, the function $\mathrm{Im}\{\phi_{2,1,0}(t)\} = b_q(t)$ vanishes at the points $\{x_j\}$, is otherwise positive over the range $-\infty \leq t \leq \infty$, and tends to unity as t tends to infinity. Accordingly the function

$$\Theta(t) = \mathrm{Im}\{\phi_{2,1,0}(t)\} - \sum_{j=1}^{q} \phi_{1,j,2T(1,j)-1}(t)$$

which belongs to the real linear hull of the set of functions, $\{\phi_{1,j,\nu}, \phi_{2,k,\nu}\}$, is positive and bounded for all $t\in[-\infty,\infty]$. As in the main proof, ~~the existence~~ it is deduced that if $H(n)$ is positive definite, there exists a system of solutions of the normalised version of $P[n]$ having the form ( ) with the interval of integration taken to be $[\alpha,\beta]\subset(-\infty,\infty)$, $B=0$ and $\gamma\in B[\alpha,\beta]$

$$\cancel{G(\lambda)=B+\int_{\alpha}^{\beta}(\lambda-t)^{-1}(1+\lambda t)\,d\gamma(t)}$$

~~with $B=0$ and $\gamma\in B[\alpha,\beta]\in(-\infty,\infty)$ where $[\alpha,\beta]\subset(-\infty,\infty)$ is established~~. The corresponding ~~system of~~ solutions of the unnormalised version of $P[n]$ for which the condition $b_1=-i$, $\xi_{2,1,0}=i$ is not assumed ~~have the~~ are derived by means of suitable linear transformations of argument and function values. These solutions have the modified form ( ) with $A=0$, $-\infty<B<\infty$, and they are analytic at the point at infinity. ~~and, lacking~~ They are ~~all~~ members of ~~the full class of solut~~ a subclass of solutions to $P[n]$ which is sufficiently extensive to support the statements made in clause (Ai).

The relevant arguments outlined above ~~may be used~~ suffice to show that if the Hermitian matrix

$H(n)$ associated with the normalised version of $P[n]$ is positive definite, the linear functional Q identified above is nonnegative definite. The function Θ used in the subsequent application of the result (iii) stated at the commencement of this section is that defined by formula ( ). Then $\lim \Theta(t)^{-1}\phi_{1,j,0}(t) = -x_j$ as $t \to \pm\infty$ $(j:q)$; the similar limits involving $\phi_{1,j,1}$, $\phi_{2,k,0}$ and, where appropriate, $\phi_{2,k,1}$ are -1, λ_k and -1 respectively; all remaining limits formed from the functions $\{\phi_{1,j,\nu}, \phi_{2,k,\nu}\}$ are zero. Application of the result (iii) then leads to the conclusion that the normalised version of $P[n]$ has a solution of the form ( ) with $B=0$; the corresponding solution of the unnormalised version of $P[n]$ has the general form ( ) with unrestricted B. The form of this solution is investigated, as in the main proof, by taking $\{\xi_\nu\}$ ~~(2:n)~~ to be the components of that nonzero

$\underset{\sim}{\xi} \in \mathbb{C}^n$ for which $\xi H(m) \xi^* = 0$, deriving coefficients $\{\gamma_{1,k,k}, \gamma_{2,k,k}\}$ by use of the second and third of relationships ( ) with $p=0$, and showing that $\xi H(n) \xi^*$ ~~ha~~ is equal to

$$\cancel{A \left| \sum_{j=1}^{n} \xi_j \right|^2 + \int_{-\infty}^{\infty} \left| \sum_{j=1}^{q} \sum_{\nu=1}^{2\tau(1,j)} \gamma_{1,j,\nu} (t - z_j)^{-\nu} \right|}$$

$$\cancel{+ \sum} \quad A|\eta|^2$$

$$\cancel{A \left| \sum_{j=1}^{q} \gamma_{1,j,1} + \sum_{k=1}^{r} \gamma_{2,j,1} \xi \right|} + \int_{-\infty}^{\infty} |Y(t)|^2 \, d\gamma(t)$$

where $\eta = \sum_{j=1}^{q} \gamma_{1,j,1} + \sum_{k=1}^{r} \gamma_{2,k,1}$ and ~~where~~ Y ~~is~~ again has the form ( ) with $(t-i)^{1-p}$ replaced by $t-i$ and the sum involving ~~the~~ γ_k discarded. ~~As at~~ Y is expressed as the quotient of two polynomials and it is deduced that γ can only be a step function with saltus at the zeros of the numerator polynomial. The degenerate solution accordingly has the form

$$G(\lambda) = A\lambda + B + \sum_{\nu=1}^{m'-1} M_\nu (\lambda - b_\nu)^{-1} (1 + b_\nu \lambda)$$

Two cases are possible: either $\eta \neq 0$, $A = 0$ and $m' = m$ ~~$M_\nu > 0$ ($\nu = \ldots$)~~ or $\eta = 0$, $A \neq 0$ and $m' = m-1$. As in the main proof it is shown that $M_j > 0\ (j : m'-1)$ in both cases and that the degenerate solution ( ) is the only solution of P[n] when H(n) is positive semi-definite.

The construction of an inclusion disc for the values, corresponding to a fixed argument value $\Lambda \in \mathbb{H}$, of all solutions to P[n] is as described in the main proof. It has only to be remarked that in the case under consideration there is no exceptional point on the boundary point that does not represent a solution value.

Having dealt with the case in which $p = 0,\ q, r > 0$, that ~~that in which the argument values featuring in P[n] lie~~ ~~the case~~ in which $p = q = 0,\ r > 0$ is dismissed ~~entirely in H, i.e.~~ by remarking that the proof of the first case, with considerably simpler formulae, can be taken over in entirety.

It remains to consider the case in which $p = 0,\ q > 0,\ r = 0$; the argument values featuring in P[n] now lie solely on the finite part of the real axis. Now the artifice of normalising P[n] and subsequent use of the function $\phi_{2,1,0}$ in the definition of $\Theta(t)$ are no longer possible.

~~[illegible]~~

Certain conventions and notations that are adhered to throughout this section and, in some cases, later in the paper are now stated.

(over leaf)

$\mathbb{L}$ and $\mathbb{U}$ are the finite parts of the open lower and upper half-planes respectively (i.e. for $\mathbb{L}$: $\{\lambda : \operatorname{Im}(\lambda) < 0, |\lambda| < \infty\}$). ~~$R(r)$, $R^{(0)}(r)$, $R^{(1)}(r)$ are the sets of all irreducible rational functions expressible as quotients of two r^{th} degree polynomials, as~~ respectively

$R(r)$, $R^{(0)}(r)$ and $R^{(1)}(r)$ are respectively the sets of all irreducible rational functions with denominator and numerator polynomials of ~~[illegible]~~ degrees r and r, $r-1$ and r, and r and $r-1$ respectively $(r = 1, 2, \ldots)$ the coefficients of highest powers of the variable in these polynomials being, in all cases, nonzero. ~~Subject to suitable existence conditions~~, the successive convergent $\{C_r\}$ of the terminating continued fraction

(over leaf) 2

$$b_0 + \frac{a_1}{b_1 +} \cdots \frac{a_n}{b_n}$$

~~may be determined by setting $D_{-1}=0$, $D_0=1$, $N_{-1}=1$~~
and ~~and~~ $N_0 = b_0$ and evaluating

$$D_r = b_r D_{r-1} + a_r D_{r-2}$$

~~for $r=1,\dots,n$ and~~ ~~D_r~~ and N_r similarly for $r=1,\dots,n$ when $C_r = D_r^{-1} N_r$ $(r=0,\dots,n)$. A reference to the denominator and numerator of the convergent C_r is a reference to ~~the num~~ D_r and N_r ~~con~~ determined in this way, and as such is unambiguous. The symbol ∂ denotes the value of the derivative of a function at the indicated argument value; thus $\partial G(-\lambda_1)$ is the value of $dG(\lambda)/d\lambda$ when $\lambda = -\lambda_1$. ∂ is also used in conjunction with complex constants to denote a complex number; ~~value,~~ thus $\partial^2 G_1$ is a complex number. Compound expressions involving the use of ∂ in both senses are to be interpreted with the help of Leibniz' rule; thus $\partial\{G(\lambda_1)G_1\}$ means $G_1 \partial G(\lambda_1) + G(\lambda_1)\partial G_1$. $x \in B[\alpha,\beta]$, where $-\infty \le \alpha < \beta \le \infty$ means that

α is a nondecreasing real-valued function of bounded variation over $[\alpha, \beta]$.

With $\mathbb{M}$ a set of points, $\mathbb{F}$ a set of functions which regarded as functions of λ have ~~as~~ a representation $F(\lambda, t)$ where $t \in \mathbb{M}$, $F(\lambda, \mathbb{M}) \equiv \mathbb{F}$ means that there is a one to one correspondence between the members ~~of~~ of $\mathbb{F}$ and $\mathbb{M}$ established by use of the representation F. If $\mathbb{M}$ has only one point, t' say, and $\mathbb{F}$ has only one member, the notation $F(\lambda, t')$ ~~$\equiv \mathbb{F}$ is used. With Δ a fixed complex number,~~ $M(\Delta)$ a set of points and $\mathbb{F}$ a set of functions, $M(\Delta)$ is said to be an inclusion set, or inclusion domain if $M(\Delta)$ is a domain in the complex plane, if $F(\Delta) \in M(\Delta)$ for every $F \in \mathbb{F}$ ~~and every p~~

$\equiv \mathbb{F}$ is used. A function is said to map one ~~domain~~ open domain into another, $\mathbb{D}'$, if the image of every point of $\mathbb{D}$ lies in $\mathbb{D}'$ and, with one possible exception, every point of $\mathbb{D}'$ is the image of one or more points of $\mathbb{D}$. With Λ a fixed complex number, $M(\Lambda)$ a set of points and $\mathbb{F}$ a set of functions, ~~for $\mathbb{F}$ at the point Λ~~ $M(\Lambda)$ is said to be an inclusion set (or inclusion domain if $M(\Lambda)$ is a domain in the complex plane) for $\mathbb{F}$ at the point Λ if $F(\Lambda) \in M(\Lambda)$ for every $F \in \mathbb{F}$ and every point of $M(\Lambda)$ represents the value of at least one function of $\mathbb{F}$ for the argument value Λ; this fact is indicated by use of the notation $\mathbb{F} \to M(\Lambda)$. If, in the preceding, each point of $M(\Lambda)$ represents the value of only one function of $\mathbb{F}$, $M(\Lambda)$ is said to be a univalent inclusion ~~domain~~ set for $\mathbb{F}$ at the point Λ, and to denote this fact the notation $\mathbb{F} \leftrightarrow M(\Lambda)$ is used. In conjunction with this notation, the same symbol is used to denote a circular arc and the points of this arc. The notation $\mathbb{F} \leftrightarrow M(\Lambda)$ is retained when F and M each possess only one member.

A function G maps $\mathbb{L}$ into $\mathbb{U}$ if and only if ([ ], see also [ ] Ch.) it is a nonconstant function of the form

$$G(\lambda) = A\lambda + B + \int_{-\infty}^{\infty} \frac{1+t\lambda}{\lambda - t}\, d\sigma(t)$$

where $-\infty < A \le 0$, $-\infty < B < \infty$ and $\sigma \in B[-\infty,\infty]$; the class of such functions is denoted by $\mathbb{N}$. ~~$\mathbb{N}'$ is the subclass of $\mathbb{N}$-functions G for which $G(\lambda) = o(1)$ as λ tends to infinity in a nonvoid sector of the form $0 < \alpha \le \arg(\lambda) \le \beta < \pi$. $\mathbb{N}''$ is the subclass of $\mathbb{N}$-~~ functions G for which $1/G(\lambda) = o(1)$ as λ behaves in the above manner. $\overline{\mathbb{N}}$ is $\mathbb{N}$ closed by the adjunction of all functions G such that $G(\lambda) = t \in (-\infty, \infty]$; ~~that are identically equal to a constant $t \in (-\infty,\infty]$;~~ $\overline{\mathbb{N}}''$ is $\mathbb{N}''$ closed by the adjunction of the single function $G(\lambda) = \infty$.

The Hamburger-Pick-Nevanlinna problem may be described in the following way. Let p, q be finite positive integers, g_ν ($\nu = -2, -1, \dots, 2p-3$) be finite real numbers, λ_k ($k = 1, \dots, q$) be distinct argument values in $\mathbb{L}$ and $\tau(k)$ ($k = 1, \dots, q$) be corresponding finite positive integers; let $\Delta^{\tau} G_k$ ($\tau = 0,$

..., $T(k)-1$) be prescribed complex numbers corresponding to the argument value λ_k ($k=1,..,q$). The problem in hand is that of determining a function G which (α1) satisfies the asymptotic relationship

$$\lambda^{2p-1}\left\{G(\lambda) - \sum_{\nu=-2}^{2p-3} g_\nu \lambda^{-\nu-1}\right\} = O(1)$$

as λ tends to infinity as described above, (α2) satisfies the interpolation conditions

$$\Delta^r G(\lambda_k) = \Delta^r g_k$$

for $r=0,..,T(k)-1$; $k=1,..,q$ and (α3) maps $\mathbb{L}$ into

~~(iii). The two variants of the above problem in which~~

~~first conditions (α1) and secondly~~

(iii). $n = p + \sum_{k=1}^{q} T(k)$ is the order of the problem, which is denoted by $P[n|p,q,T:g;\lambda,G]$, or where the context permits, simply by $P[n]$. (iv). The two variants of the above problem in which first conditions (α1) and secondly (α2) are omitted are indicated by setting $p=0$ and $q=0$ respectively ~~when using~~ in use of the preceding extended symbol. The diminished version of the above problem, in defined ~~only when~~ $p>0$, ~~is obtained~~ by replacing condition ( ) by the less stringent alternative

$$\lambda^{2p-2}\left\{G(\lambda)-\sum_{\nu=-2}^{2p-4} g_\nu \lambda^{-\nu-1}\right\}=O(1)$$

with λ tending to infinity as described; it is denoted by $\underset{\sim}{P}[n]'$; when $p=0$, $\underset{\sim}{P}[n]'$ is taken to be $\underset{\sim}{P}[n]$. The doubly diminished version ~~again~~ is, ~~defined only when~~ $p>0$, ~~is~~ simply $P[n-1|p-1,q,T;g;\lambda,G]$; it is denoted by $\underset{\sim}{P}[n]''$. The sets of solutions of $\underset{\sim}{P}[n]$, ~~and, when defined, of~~ $\underset{\sim}{P}[n]'$, $\underset{\sim}{P}[n]''$ are denoted by $\mathbb{P}[n]$, $\mathbb{P}[n]'$, $\mathbb{P}[n]''$. $\underset{\sim}{P}[n]$ is said to be nondegenerate if it is soluble but ~~does~~ does not have, as ~~its~~ a ~~possible~~ solution, ~~when p=0~~ a rational function whose denominator is of degree n when $p=0$ and $n-1$ when $p>0$.

The integer sequence $i(\nu)$ $(\nu=1,..,n)$ is said to be compatible with $\underset{\sim}{P}[n]$ if it contains p zeroes and $T(k)$ copies of k $(k=1,..,q)$, these integers ~~being placed occurring~~ being placed in an order open to choice but fixed. Such a sequence induces a corresponding sequence of subordinate problems from $\underset{\sim}{P}[n]$ defined in the following way. With ~~in the fixed~~ $1 \le r \le n$ let $p^{(r)}$ be the number of times zero occurs in the list $(i(1),\dots,i(r))$, let $j(r,k)$ $(k=1,\dots,q^{(r)})$ be the further distinct entries to be found in this list, let $T^{(r)}(k)$ be the number of times $j(r,k)$ occurs in the list $(k=1,\dots,q^{(r)})$ and let $\Delta^{\tau} G^{(r)}_{j(r,k)}$ $(k=1,\dots,q^{(r)}, \tau=0,\dots,T^{(r)}(k)-1)$ be complex numbers corresponding to the argument value $\lambda^{(r)}_{k} = \lambda_{j(r,k)}$ and appropriately extracted from the complete set $\{\Delta^{\tau} G_k\}$. The rth member of the sequence of subordinate problems is $\underset{\sim}{P}[r \mid p^{(r)}, q^{(r)}, T^{(r)} : g ; \lambda^{(r)}, G^{(r)}]$ ~~$(r=1,\dots,n)$~~ and is denoted by $\underset{\sim}{P}[r]$ $(r=1,\dots,n)$. The diminished and doubly diminished versions of $\underset{\sim}{P}[r]$, defined when $p^{(r)} > 0$, ~~are defined~~ together with the sets of solutions to these problems, are denoted by notations analogous to those given above.

Matrix criteria

In this section Akhiezer's treatment of the simple Pick-Nevanlinna problem is extended to the diminished Hamburger-Pick-Nevanlinna problem; conditions that are necessary and sufficient for the solubility of the latter problem are established and, assuming this problem to be nondegenerate, inclusion discs for its solutions are located.

As a preliminary, a Hermitian matrix denoted by $H(n|p,q,\tau:g;\lambda,\mathfrak{G})$ or, if the context permits, simply by $H(n)$, is constructed from the data to $\underset{\sim}{P}[n|p,q,\tau:g;\lambda,\mathfrak{G}]$ in the following way.

(i) If $p>1, q>0$, $H(n)$ is of order $(n-1)\times(n-1)$

set $r = p-1$;

a) let $\underset{\sim}{H}_{0,0}$ be the $r \times r$ Hankel array whose $(k,j)^{th}$ element is $H_{0,0}^{(k,j)} = g_{k+j-2}$ $(k,j=1,\ldots,r)$

b) obtain numbers $\{\Delta^{\tau} h_k\}$ from the corresponding numbers $\{\Delta^{\tau} G_k\}$ by use of the relationships

$$h_k = G_k - g_{-2}\lambda_k - g_{-1}, \quad \Delta h_k = \Delta G_k - g_{-2}$$

(the latter if $T(k)>1$) and $\Delta^{\tau} h_k = \Delta^{\tau} G_k$ $(\tau = 2,\ldots,T(k)-1)$

all for $k=1,\ldots,q$;

c) let $\underset{\sim}{H}_{K,0}$ be the array containing $T(K)$ rows and r columns whose $(k,j)^{th}$ element is

$$H_{K,0}^{(k,j)} = \lambda_K^{j-k-1} \sum_{\tau=0}^{j-k-1} g_{\tau} \lambda_K^{-\tau} \binom{j-\tau-2}{k-1} - \sum_{\tau=\max(0,k-j)}^{k-1} \lambda_K^{j-k+\tau} \binom{j-1}{k-\tau-1} \frac{\Delta^{\tau} h_K}{\tau!}$$

for $k=1,\ldots,T(K)$; $j=1,\ldots,r$ with $K=1,\ldots,q$;

d) let $\underset{\sim}{H}_{1,J;K,J}$ be the array containing $T(K)$ rows and $T(J)$ columns whose $(k,j)^{th}$ element is

$$H^{(k,j)}_{1,1;K,J} = (-1)^{k-1} \sum_{z=0}^{j-1} \binom{k+j-z-2}{k-1} (\lambda_K - \bar{\lambda}_J)^{z-j-k+1} \frac{\overline{\Delta^z h_J}}{z!}$$

$$+ (-1)^{j-1} \sum_{z=0}^{k-1} \binom{k+j-z-2}{j-1} (\bar{\lambda}_J - \lambda_K)^{z-j-k+1} \frac{\Delta^z h_K}{z!}$$

for $k = 1, \dots, T(K)$; $j = 1, \dots, T(J)$ with $K, J = 1, \dots, q$ and with $m = \sum_{k=1}^{q} T(k)$, let $\underset{\sim}{H}_{1,1}$ be the $m \times m$ block matrix composed of q^2 blocks, the array $H_{1,1;J,K}$ occurring in the K^{th} row and J^{th} column ($K, J = 1, \dots, q$

$\underset{\sim}{H}(n)$ is compounded from blocks arranged as follows: the first ~~row~~ column contains $\underset{\sim}{H}_{0,0}$ followed by $\underset{\sim}{H}_{K,0}$ ($K = 1, \dots, q$) in order; the first row contains ~~$\bar{\underset{\sim}{H}}^T_{0,0}$~~ $\underset{\sim}{H}_{0,0}$ again followed by $\bar{\underset{\sim}{H}}^T_{J,0}$ ($J = 1, \dots, q$) in order; the remaining $m \times m$ interior block is $\underset{\sim}{H}_{1,1}$. $\underset{\sim}{H}(n)$ accordingly has the following appearance

$$\begin{pmatrix} \underset{\sim}{H}_{0,0} & \{\underset{\sim}{\bar{H}}{}^{T}_{0,J}\} \\ \{\underset{\sim}{H}_{K,0}\} & \underset{\sim}{H}_{1,1} \end{pmatrix}$$

(ii) If $p=1, q>0$, $\underset{\sim}{H}(n)$ is again of order $(n-1)\times(n-1)$. Obtain numbers $\{\Delta^{\tau} h_k\}$ as in (b) above, setting $g_{-1}=0$ in the first of relationships ( ). $\underset{\sim}{H}(n)$ is the block $\underset{\sim}{H}_{1,1}$ defined in (d) above.

(iii) If $p=0, q>0$, $\underset{\sim}{H}(n)$ now of order $n\times n$, is the block $\underset{\sim}{H}_{1,1}$ obtained as in (d) above by taking the $\{\Delta^{\tau} h_k\}$ to be simply $\{\Delta^{\tau} G_k\}$.

(iv) If $p>0, q=0$, $H(n)$ again of order $(n-1)\times(n-1)$, is simply the Hankel array $\underset{\sim}{H}_{0,0}$ of (ia).

An extended version $\underset{\sim}{H}(n|\Lambda, h)$ of $\underset{\sim}{H}(n)$, Λ and h being complex numbers, is also defined.

(i) If $p>1, q>0$, $\underset{\sim}{H}(n|\Lambda, h)$, of order $n \times n$, is obtained by bordering $\underset{\sim}{H}(n)$ by the elements

a)

$$h^{(j)} = \Lambda^{j-1}\left\{\sum_{\nu=0}^{j-2} g_\nu \Lambda^{-\nu-1} - h\right\}$$

and $\bar{h}^{(j)}$ in positions $j=1,\dots,r$ of the n^{th} row and n^{th} column respectively,

b)

$$h_J^{(j)} = (\Lambda - \bar{\lambda}_J)^{-j}\left\{\sum_{z=0}^{j-1} (\Lambda - \bar{\lambda}_J)^z \frac{\overline{\Delta^z h_J}}{z!} - h\right\}$$

and $\bar{h}_J^{(j)}$ in position $p + \sum_{z=1}^{J-1} T(z) + j$ ($J=1,\dots,q$; $j=0,\dots,T(J)-1$) of the n^{th} row and n^{th} column respectively and

c) by the single element

$$\Delta(\Lambda, h) = (\Lambda - \bar{\Lambda})^{-1}(\bar{h} - h)$$

in position (n,n).

$\underset{\sim}{H}(n|\Lambda, h)$ accordingly has the following appearance.

$$\begin{pmatrix} H_{0,0} & \{\bar{H}^T_{0,J}\} & \{h^{(k)}\} \\ \{H_{K,0}\} & H_{1,1} & \{h^{(k)}_K\} \\ \{h^{(j)}\} & \{h^{(j)}_J\} & \Delta(\Lambda, h) \end{pmatrix}$$

(ii) If $p=1, q>0$, obtain $\underset{\sim}{H}(n|\Lambda,h)$ simply by bordering $\underset{\sim}{H}_{1,1}$ as in (a, c, d), $\{\Delta^{\tau} h_k\}$ being defined as in (ii) above.

(iii) If $p=0, q>0$, obtain $\underset{\sim}{H}(n|\Lambda,h)$ as in the preceding clause, $\Delta^{\tau} h_k$ being simply $\Delta^{\tau} G_k$.

(iv) If $p>0, q=0$, obtain $\underset{\sim}{H}(n|\Lambda,h)$ by bordering $\underset{\sim}{H}_{0,0}$ as in (a, b, d) above.

Theorem. (i) $\underset{\sim}{P}[n|p,q,T:g;\lambda,G]'$ is soluble ⟨nondegenerate⟩ if and only if ~~$H(n|p,q,T;g,\lambda,G)$~~ (a) $g_{-2} \leq 0$ when $p>0$ and (b) $H(n|p,q,T;g;\lambda,G)$ is positive semi-definite ⟨positive definite⟩.

(ii) If $\underset{\sim}{P}[n]'$ is degenerate, its degenerate solution is its only solution.

(iii) Let $P[n]'$ be nondegenerate and $\Lambda \in \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_q\}$ be fixed. The equation

$$\det\{\underset{\sim}{H}(n\,|\,\underline{\Lambda},h)\}=0$$

determines a circle $\mathcal{C}'$ in the complex h-plane. If $p>1$, let $\mathcal{C}_n(\underline{\Lambda})$ be the circle given symbolically by the equation

$$\mathcal{C}_n(\underline{\Lambda})=g_{-2}\Lambda+g_{-1}+\mathcal{C}'$$

if $p=1$, let $\mathcal{C}_n(\underline{\Lambda})=g_{-2}\Lambda+\mathcal{C}'$, and if $p=0$ let $\mathcal{C}_n(\underline{\Lambda})=\mathcal{C}'$. Let $\bar{D}_n(\underline{\Lambda})$ be the closed disc whose boundary is $\mathcal{C}_n(\underline{\Lambda})$.

$$\mathbb{P}(n)'\to\bar{\mathbb{D}}_n(\underline{\Lambda}).$$

Proof. Assuming that $p>1$, $q>0$, attention is first directed to a reduced version of $\underset{\sim}{P}[n]'$, namely the problem of determining a function $\hat{G}$ which (α'1) satisfies the asymptotic relationship

$$\lim \lambda^{2r}\{\hat{G}(\lambda)-\sum_{\nu=0}^{2r-2} g_\nu\lambda^{-\nu-1}\}=O(1)$$

as λ tends to infinity in a sector of the form $\pi+\delta<\arg(\lambda)<2\pi-\delta$ $(0<\delta<\frac{1}{2}\pi)$ where $r=p-1$, (α'2) satisfies the interpolation conditions

$$\lambda^\tau \hat{G}(\lambda_k) = \lambda^\tau h_k$$

for $\tau = 0, \dots, T(k)-1$ $(k = 1, \dots, q)$ and (β) maps $\mathbb{L}$ into $\mathbb{D}$.

The following set $\mathbb{F}$ of real valued functions $\{\phi_{k,\tau}, \psi_{k,\tau}\}$ defined over $(-\infty, \infty)$ is introduced:

$$\phi_{0,\tau}(t) + i\psi_{0,\tau}(t) = (t-i)^{\tau+1-r}$$

for $\tau = 0, \dots, r-1$ and with $\lambda_k = x_k + iy_k$,

$$b_r(t) = (1+t^2)^{1-r}, \quad B_k(t) = \{(x_k - t)^2 + y_k^2\}^{-1}$$

$$\phi_{k,\tau}(t) + i\psi_{k,\tau}(t) = -b_r(t)\{B_k(t)(t - x_k + iy_k)\}^{\tau+1}$$

for $k = 1, \dots, q$; $\tau = 0, \dots, T(k)-1$. In particular, $\phi_{0,r-1}(t) = 1$, $\psi_{0,r-1}(t) = 0$ and the remaining $\{\phi_{0,\tau}, \psi_{0,\tau}\}$ have the form

$$\phi_{0,\tau}(t) = b_r(t)\sum_{\nu=0}^{r+\tau-1} a_{\tau,\nu} t^\nu, \quad \psi_{0,\tau}(t) = b_r(t)\sum_{\nu=0}^{r+\tau-1} b_{\tau,\nu} t^\nu$$

where the $\{a_{\tau,\nu}, b_{\tau,\nu}\}$ are real numbers.

The linear hull $\mathbb{E}$ of $\mathbb{F}$ is formed, and the linear functional Q on $\mathbb{E}$ is determined by means of the following identifications:

$$Q\{\phi_{0,\tau}(t)\} = \sum_{\nu=0}^{r+\tau-1} a_{\tau,\nu}\, g_\nu \,, \quad Q\{\psi_{0,\tau}(t)\} = \sum_{\nu=0}^{r+\tau-1} b_{\tau,\nu}\, g_\nu$$

for $\tau = 0,\dots,r-1$ and

$$Q\{\phi_{k,\tau}(t)\} = (\tau!\,2)^{-1}\left(\Delta^\tau h_k + \overline{\Delta^\tau h_k}\right), \quad Q\{\psi_{k,\tau}(t)\} = (\tau!\,2i)^{-1}\left(\Delta^\tau h_k - \overline{\Delta^\tau h_k}\right)$$

both for $k = 1,\dots,q;\ \tau = 0,\dots,T(k)-1$. It follows from the definition ( ) and formulae (,) that

$$Q\{b_r(t)\, t^\nu\} = g_\nu$$

for $\nu = 0,\dots,2r-2$. It is also clear from formulae ( ) that

$$Q\{b_r(t)\,(t-\lambda_k)^{-\tau-1}\} = -(\tau!)^{-1}\Delta^\tau h_k$$

for $k = 1,\dots,q;\ \tau = 0,\dots,T(k)-1$.

From formula ( )

$$H_{0,0}^{(k,j)} = Q\{b_r(t)\, t^{k+j-2}\}$$

for $k,j = 1,\dots,r$. Expanding $t^{j-1} = (t-\lambda_k+\lambda_k)^{j-1}$

in powers of $t-\lambda_K$, and using some elementary relationships involving binomial coefficients, it is easily verified that

$$H_{K,0}^{(k,j)} = Q\{b_r(t)\, t^{j-1}(t-\lambda_K)^{-k}\}$$

for $K=1,\dots,q$; $k=1,\dots,T(K)$; $j=1,\dots,r$. Again, since

$$Q\{b_r(t)(t-\lambda_K)^{-1}(t-\bar{\lambda}_J)^{-j}\} =$$

$$-\sum_{z=0}^{j-1}(\lambda_K-\bar{\lambda}_J)^{-z-1}Q\{b_r(t)(t-\bar{\lambda}_J)^{z-j}\}$$

$$+(-1)^j(\bar{\lambda}_J-\lambda_K)^{-j}Q\{b_r(t)(t-\lambda_K)^{-1}\}$$

the formula

$$H_{1,1;K,J}^{(k,j)} = Q\{(t-i)^{1-r}(t-\lambda_K)^{-k}(t+i)^{1-r}(t-\bar{\lambda}_J)^{-j}\}$$

is correct when $k=1$ for $K,J=1,\dots,q$; $j=1,\dots,T(J)$. That this formula is also correct when $j=1$ for $K,J=1,\dots,q$; $k=1,\dots,T(K)$ is verified in the same way. The elements $H_{1,1;K,J}^{(k,j)}$ satisfy the difference equation

$$H^{(k,j+1)}_{1,1;K,J} - H^{(k+1,j+1)}_{1,1;K,J} = (\bar{\lambda}_J - \lambda_K) H^{(k+1,j+1)}_{1,1;K,J}$$

for all index values for which all ~~the~~ terms displayed are defined; the values of the expressions upon the right hand side of equation ( ) do the same: equation ( ) is valid for $K,J=1,\ldots,q$; $k=1,\ldots,T(k)$; $j=1,\ldots,T(j)$.

With η_ν $(\nu=1,\ldots,r)$ and $\eta_{k,\nu}$ $(k=1,\ldots,q;\nu=1,\ldots,T(k))$ sets of complex numbers, and

$$Y(t)=(t-i)^{1-r}\left\{\sum_{\nu=1}^{r}\eta_\nu t^{\nu-1}-\sum_{k=1}^{q}\sum_{\nu=1}^{T(k)}\eta_{k,\nu}(t-\lambda_k)^{-\nu}\right\}$$

it is clear that by replacing the index sets $\{\nu;k,\nu\}$ by $\{k;K,k\}$, $\{j;J,j\}$, forming the complex conjugate of the second expression and subsequently the product of the two expressions obtained in this way, the relationship

$$Y(t)\overline{Y(t)} = b_r(t)\Big\{\sum_{k=1}^{r}\sum_{j=1}^{r} t^{k+j-2}\eta_k\bar{\eta}_j$$

$$-\sum_{k=1}^{r}\sum_{J=1}^{q}\sum_{j=1}^{T(J)} t^{k-1}(t-\bar{\lambda}_J)^{-j}\eta_k\bar{\eta}_{J,j} - \sum_{K=1}^{q}\sum_{k=1}^{T(K)}\sum_{j=1}^{r}(t-\lambda_K)^{-k}t^{j-1}\eta_{K,k}\bar{\eta}_j$$

$$+\sum_{K=1}^{q}\sum_{J=1}^{q}\sum_{k=1}^{T(K)}\sum_{j=1}^{T(J)}(t-\lambda_K)^{-k}(t-\bar{\lambda}_J)^{-j}\eta_{K,k}\bar{\eta}_{J,j}\Big\}$$

is obtained. Setting

$$\xi_k = \eta_k\,, \qquad \xi_{r+\chi_r} = \eta_{k,\tau}$$

where $k=1,\dots,r$ in the first of these relationships and, with $\chi_r = \sum_{\nu=1}^{k-1} T(\nu)+\tau$, $k=1,\dots,q$; $\tau = 1,\dots,T(k)$ in the second, it follows from relationships (, ,) that

$$Q\{|Y(t)|^2\} = \underset{\sim}{\xi}\,\underset{\sim}{H}(n)\,\underset{\sim}{\xi}^T$$

where $\underset{\sim}{\xi}$ is the row vector with elements $\xi_1,\dots,\xi_{n-1}$

Any real $\Phi \in E$ has the representation

$$\sum_{\tau=0}^{r-1}\{B_{0,\tau}(t-i)^{1-r} + \bar{B}_{0,\tau}(t+i)^{1-r}\}t^{\tau}$$

$$+\sum_{k=1}^{q}\sum_{\tau=0}^{T(k)-1}\{B_{k,\tau}(t-i)^{1-r}(t-\lambda_k)^{-\tau-1} + \bar{B}_{k,\tau}(t+i)^{1-r}(t-\bar{\lambda}_k)^{-\tau-1}\}$$

where the $B_{k,r}$ are complex numbers. This Φ accordingly has the representation

$$|(t-i)^{8r-1}\prod_{k=1}^{q}(t-\lambda_k)^{\tau(k)}|^{-2}X(t)$$

where X is a polynomial of degree $2n-2$. If $\Phi(t)\geq 0$ for all $t\in(-\infty,\infty)$, $X(t)$ has the form $|\sum_{\nu=0}^{n-1}x_\nu t^\nu|^2$. Hence $\Phi(t)=Y(t)\overline{Y(t)}$, where Y may be expressed in the form ( ).

If $\underset{\sim}{H}(n)$ is positive semi-definite, the value of the expression upon the right hand side of formula ( ) with $\underset{\sim}{\xi}$ any compatible row vector with complex elements, not all zero, is nonnegative. It has thus been demonstrated that the condition $\Phi(t)\geq 0$ for all $t\in(-\infty,\infty)$ with $\Phi\in\mathbb{E}$ real implies that $Q\{\Phi(t)\}\geq 0$: the functional Q determined by means of the identifications (,) is nonnegative.

Use is now made of a result due to Riesz [,

]: let Q be a nonnegative functional defined over the linear hull $\mathbb{E}$ of a ~~set~~ finite set $\mathbb{F}$ of functions Θ_ν that are real valued and continuous over $(-\infty,\infty)$, and let $\mathbb{E}$ contain at least one function Θ such that $\inf \Theta(t) > 0$ for $t \in (-\infty,\infty)$ and for all $\Theta_\nu \in \mathbb{F}$ all limits

$$\lim_{t\to\pm\infty} \frac{\Theta_\nu(t)}{\Theta(t)} = c_{\nu;\Theta}$$

exist and are finite. A constant $A \geq 0$ and a nondecreasing function σ of bounded variation over $(-\infty,\infty)$ can be found such that

$$Q\{\Theta_\nu(t)\} = A c_{\nu;\Theta} + \int_{-\infty}^{\infty} \Theta_\nu(t)\, d\sigma(t).$$

In the above result, let the set $\{\Theta_\nu\}$ be $\{\phi_{k,r}\, \psi_{k,r}\}$ and take $\Theta_{0,r-1}(t) = 1$ to be $\Theta(t)$. All limits of the form ( ) (except that for which

$\theta_\nu = \Theta$) are then zero. A function $\sigma \in B(-\infty,\infty)$ exists such that, with $d\beta(t) = b_r(t)d\sigma(t)$ for $t \in (-\infty,\infty)$, so that $\beta \in B(-\infty,\infty)$ also,

$$\int_{-\infty}^{\infty} t^\nu d\beta(t) = g_\nu$$

for $\nu = 0,\dots,2r-2$ and

$$-\tau! \int_{-\infty}^{\infty} (t-\lambda_k)^{-\tau-1} d\beta(t) = \lambda^{\tau} h_k$$

for $k=1,\dots,q;\ \tau=0,\dots,T(k)-1$. Set

$$\hat{G}(\lambda) = \int_{-\infty}^{\infty} (\lambda - t)^{-1} d\beta(t).$$

Since the moments ( ) exist, $\hat{G}$ satisfies the asymptotic relationship ( ) ([,], [ ]g.. [ ]). $\hat{G}$ also satisfies the interpolation conditions ( ) of the reduced interpolation problem $\hat{P}[n]'$.

Accordingly the function

$$G(\lambda) = g_{-2}\lambda + g_{-1} + \hat{G}(\lambda)$$

satisfies conditions (,) pertaining to

$P[n]'$. If $g_{-2} \le 0$, $\mathcal{G}$ is also an N-function and hence maps $\mathbb{L}$ into $\mathbb{U}$: it has been demonstrated that (at least when $p>1, q>0$) if $\underset{\sim}{H}(n)$ is positive semi-definite, $P[n]'$ is soluble.

To prove the necessity of the stated conditions, it is assumed that the N-function ( ) satisfies conditions (,). The constant A in expression ( ) is uniquely determined by the condition $\lim \{\mathcal{G}(\lambda) - A\lambda\} = O(1)$ as λ tends to infinity in the sector $\pi - \delta < \arg(\lambda) < 2\pi - \delta$ $(0 < \delta < \frac{1}{2}\pi)$; accordingly $A = g_{-2}$ and since $A \le 0$, $g_{-2} \le 0$. Since $\mathcal{G}$ satisfies the asymptotic relationships ( ), ρ in formula ( ) is such that condition ( ) applies, and $g_{-1} = B - \int_{-\infty}^{\infty} t d\rho(t)$. The function $\hat{\mathcal{G}}(\lambda) = \mathcal{G}(\lambda) - g_{-2}\lambda - g_{-1}$ has the representation ( )

$$\hat{\mathcal{G}}(\lambda) = \int_{-\infty}^{\infty} \frac{(1+t^2)}{\lambda - t} d\rho(t)$$

and is a solution to the reduced interpolation problem ~~$\hat{P}[n]'$; formulae (,) hold as described. With ξ_ν ($\nu = 1$,~~

$\hat{\underset{\sim}{P}}[n]'$s. Formulae (,) connect the function β and the data $\{g_\nu ; \Delta^\tau h_k\}$ of $\hat{\underset{\sim}{P}}[n]'$. The elements of $\underset{\sim}{H}(n)$ may be expressed in terms of β by use of formulae similar to (, ,) with $Q\{\dots\}$ replaced by $\int_{-\infty}^{\infty} \dots d\beta(t)$. With ξ_ν $(\nu = 1, \dots, n-1)$ any sequence of finite complex numbers, not all zero, derive numbers $\{\eta_\nu, \eta_{k,\nu}\}$ by use of relationships ( ). As in the above proof of sufficiency, it may be shown that

$$\underset{\sim}{\xi}\underset{\sim}{H}(n)\underset{\sim}{\bar{\xi}}^{T} = \int_{-\infty}^{\infty} \left| \sum_{\nu=1}^{r} \eta_\nu t^{\nu-1} + \sum_{k=1}^{q} \sum_{\nu=1}^{\tau(k)} \eta_{k,\nu} (t-\lambda_k)^{-\nu} \right|^2 \mathfrak{p}_r(t) d\beta(t)$$

The value of the expression upon the right hand side of this relationship is nonnegative: it has been demonstrate that (again when $p>1, q>0$) if $P[n]'$ is soluble, $H(n)$ is semidefinite.

Having proved that, when $p>1, q>0$, $P[n]'$ is soluble if and only if $g_{-\nu} \leq 0$ and $\underset{\sim}{H}(n)$ is positive semi-definite, it remains to indicate the modifications to the above proof that must be introduced to deal with other values of p and q. When $p=1$, the asymptotic relationship ( ) is absent from the reduced interpolation problem $\hat{\underset{\sim}{P}}[n]'$: the functions $\{\phi_{0,\tau}, \psi_{0,\tau}\}$ are not included in the set $\mathbb{F}$. Furthermore it may not be assumed of any function ~~that satisfies~~ which satisfies the conditions of $P[n]'$ that in the representation ( ) β satisfies ~~the~~ condition ( ). Accordingly the functions $\phi_{k,0}, \psi_{k,0}$ are defined to be

$$\cancel{\phi_{k,0}(t) = \frac{(x_k - t)(1+t^2)}{}}$$

$$\phi_{k,0}(t) = B_k(t)(x_k - t)(1+t^2) + t, \quad \psi_{k,0}(t) = -B_k(t) y_k (1+t^2)$$

so that

$$\phi_{k,0}(t) + i\psi_{k,0}(t) = (\lambda_k - t)^{-1}(1+\lambda_k t)$$

The remaining functions $\phi_{k,\tau}, \psi_{k,\tau}$ are defined by formula ( ) with $r=0$ and $k=1,\dots,q$; $\tau=1,\dots,T(k)-1$. The linear functional Q on $\mathbb{E}$ is defined by formulae ( ) alone, and now

$$Q\{(\lambda_k - t)^{-1}(1+\lambda_k t)\} = h_k$$

for $k=1,\dots,q$ while formula ( ) with $r=0$ holds for $k=1,\dots,q$; $z=1,\dots,T(k)-1$. In the verification of formula ( ) for $k=1$, formula ( ) must be replaced by

$$Q\{(1+t^2)(t-\lambda_k)^{-1}(t-\bar{\lambda}_J)^{-j}\} =$$
$$-\sum_{z=0}^{j-2}(\lambda_k-\bar{\lambda}_J)^{-z-1}Q\{(1+t^2)(t-\bar{\lambda}_J)^{z-j}\}$$
$$+(-1)^j(\bar{\lambda}_J-\lambda_k)^{-j}Q\{(\bar{\lambda}_J-t)^{-1}(1+\bar{\lambda}_J t)-(\lambda_k-t)^{-1}(1+\lambda_k t)\}$$

but otherwise the general proof of formula ( ) is as before, as is the subsequent working, now with $r=0$. Riesz' result, quoted above, must be subjected to slight modification (obtained simply be replacing θ by $-\theta$ in the original formulation) namely that if the reference function θ satisfies the condition that $\sup\theta(t)<0$ (in place of $\inf\theta(t)>0$) for $t\in(-\infty,\infty)$, then the stated result with $A \leq 0$ (in place of $A \geq 0$) holds. In the application of the modified version of Riesz' result, any one of the functions $\{\psi_{k,0}\}$ may be taken to be θ. Taking $\psi_{1,0}$ to be the chosen function

$$\lim \frac{\phi_{k,0}(t)}{\psi_{1,0}(t)} = \frac{x_k}{y_1}, \lim \frac{\psi_{k,0}(t)}{\psi_{1,0}(t)} = \frac{y_k}{y_1}, \lim \frac{\phi_{k,1}(t)}{\psi_{1,0}(t)} = \frac{1}{y_1}$$

as $t \to \pm\infty$. All other similar limits formed from the functions $\{\phi_{k,0}, \psi_{k,0}\}$ are zero. In particular, for the functional Q being considered,

$$Q\{\phi_{k,0}(t)\} = Ax_k + \int_{-\infty}^{\infty} \phi_{k,0}(t)d\sigma(t), Q\{\psi_{k,0}(t)\} = Ay_k + \int_{-\infty}^{\infty} \psi_{k,0}(t)d\sigma(t)$$

The subsequent analysis is as for the case in which $p > 1$. The case in which $p = 0$ is dealt with in a similar way.

When $q = 0$, the interpolation conditions (,) are absent from the problem $P[n]'$ and its reduced counterpart $\hat{P}[n]'$. Now the elements of the submatrices $\underset{\sim}{H}_{k,0}$ and $\underset{\sim}{H}_{1,1;k,J}$ are absent from the formation of $\underset{\sim}{H}(n)$ and the subsequent analysis, but in principle the proof proceeds as for the case in which $p > 1$.

The above analysis concerns the problem of solubility only. To deal with the question of degeneracy, it is assumed both that $H(n)$ is positive semi-definite and that a nonzero vector $\underset{\sim}{\xi}$ can be found such that

$$\underset{\sim}{\xi} H(n) \bar{\xi}^{T} = 0.$$

$P[n]'$ generates a complete system of subordinate interpolation problems in which $p, q, \{T(k)\}$ are replaced by $p' \le p$, $q' \le q$, $\{T'(k) \le T(k)\}$ with $n' = p' + \sum_{k=1}^{q'} T'(k) < n$, the coefficients $\{g_\nu\}$ and derivative values $\{\Delta^{\tau} G_k\}$, to the extent that they occur in the subordinate problem, being those that occur in $\underset{\sim}{P}[n]'$. (The complete system is that generated by all integer sequences $\{i(r)\}$ compatible with $P[n]'$, as described in the introduction to this section.) The conditions imposed upon $\boldsymbol{H}(n)$ above imply that $P[n]'$ has a solution. The conditions of the subordinate interpolation problems are less restrictive than those of $P[n]'$: the subordinate problems all have solutions: for all compatible nonzero $\boldsymbol{\xi}$, $\boldsymbol{\xi}\boldsymbol{H}(n')\bar{\boldsymbol{\xi}}^T \geq 0$ for all $H(n')$ constructed from the data for the subordinate problems. There is an n' for which nonzero $\boldsymbol{\xi}$ exists such that $\boldsymbol{\xi}H(n')\bar{\boldsymbol{\xi}}^T = 0$ while, for all compatible nonzero $\boldsymbol{\xi}$, $\boldsymbol{\xi}H(n'')\bar{\boldsymbol{\xi}}^T > 0$ for all $n'' < n'$. There may, of course, be several subordinate problems, all of degree n', inducing these two conditions; one of them is chosen for examination.

Assume that $p > 1$, $q > 0$, and let the function

$\hat{g}$, having the form ( ), satisfy the reduced version $\hat{\underset{\sim}{P}}[n]'$ of $\underset{\sim}{P}[n]'$ and, automatically, the reduced version of the subordinate problem in question, and derive coefficients $\{\eta_\nu, \eta_{k,\nu}\}$ from the $\{\xi_\nu\}$ for which $\underset{\sim}{\xi}\underset{\sim}{H}(n')\underset{\sim}{\xi}^* = 0$ by use of relationships similar to ( ). A version of equation ( ), with n replaced by n', holds. The expression contained between bars may be replaced by

$$\left\{(t-i)^{1-r'} \prod_{k=1}^{K'} (t-\lambda_k')^{-\tau'(k)}\right\} Z(t)$$

where $r' = p'-1$, the $\{\lambda_k'\}$ form an appropriate subset of the $\{\lambda_k\}$, and $Z(t)$ is a polynomial of degree $n'-1$. The value of the term on the left hand side of the modified version of equation ( ) is zero, and the same is therefore true of that upon the right hand side: σ is a step function with salti at the zeros of Z and no other points of increase over the range $[-\infty,\infty]$ $\hat{G}$ is ~~then~~ a uniquely defined quotient of two polynomials, the denominator being of degree n'. This rational function satisfies ~~the reduced version~~ ~~of~~ $\hat{P}[n]'$ and, in so doing, the reduced versions of all subordinate problems of degree n'. Since it is uniquely defined it is, in particular, the only solution to be obtained from consideration of all subordinate problems of degree n' for which a condition of the form $\xi H(n')\xi^* = 0$ holds. If a rational function with denominator of degree less than n' were to satisfy $\hat{P}[n]'$, it would follow that $\xi H(n'')\xi^* = 0$ for some $n'' < n'$ and suitable choice of nonzero ξ, a possibility that has been excluded. Accordingly it has been shown that if $H(n)$ is positive semi-definite and a condition of the form ( ) holds, $P[n]'$

has a degenerate solution which is it's ~~deg~~ only solution. (It has also been shown how the degree of this rational function solution may be determined from the structures of the matrices $\underset{\sim}{H}(n')$ deriving from the data to the complete system of problems ~~de~~ subordinate to $P[n]'$.) If $\underset{\sim}{H}(n)$ is positive ~~semi~~ definite, no equation of the form ( ) with nonzero $\underset{\sim}{\xi}$ holds: no solution of $P[n]'$ can, via a relationship of the form ( ), induce an equation of the form ( ): $P[n]'$ is soluble but has no degenerate solution. All results of clause (i, ii) have now been demonstrated.

In all cases relating to permissible values of p and q, the Hermitian matrix $\underset{\sim}{H}(n|\Lambda, h)$ is simply the matrix of the form $H(n+1)$ ~~obtained~~ derived from the extended version of $\underset{\sim}{\hat{P}}[n]'$ obtained by the adjunction of a further argument and function value pair $\lambda_{q+1} = \Lambda$, $h_{q+1} = h$. The equation

$\det\{H(n|\Lambda, h)\} = 0$ has, by expansion of this ~~the~~ determinant, the form

$$\Upsilon|h|^2 + \Xi\bar{h} + \bar{\Xi}h + \Omega = 0$$

where Υ, Ξ and Ω are independent of h, Υ and Ω being real. This equation defines a circle $\mathscr{C}'$ in the complex h-plane. ~~; it is the locus of all points~~ For values of h outside this circle $\det\{\underset{\sim}{H}(n|\Lambda, h)\} > 0$ or $\det\{\underset{\sim}{H}(n|\Lambda, h)\} < 0$ consistently. There is certainly one point h' outside $\mathscr{C}'$ for which $\mathrm{Im}(h') < 0$. If $\det\{\underset{\sim}{H}(n|\Lambda, h')\} > 0$, there exists a function satisfying the conditions of the extended interpolation problem with $h = h'$: in particular, this function both maps $\mathbb{L}$ into $\mathbb{D}$ and assumes a value, h', with negative real part when its argument lies in $\mathbb{L}$. Since this is impossible, $\det\{\underset{\sim}{H}(n|\Lambda, h)\} < 0$ for all h lying outside $\mathscr{C}'$. Referred to the coordinates $\{\mathrm{Re}(h), \mathrm{Im}(h), \Psi\}$, the ~~formula~~ equation $\Psi = \det\{\underset{\sim}{H}(n|\Lambda, h)\}$ represents a quadric surface. For values of $\mathrm{Re}(h)$,

$\operatorname{Im}(h)$ lying outside $\mathscr{C}'$, Ψ is negative; for such values on $\mathscr{C}'$, Ψ is zero: when h lies inside $\mathscr{C}'$, $\det\{H(n|\Delta,h\}>0$

Taking $\hat{G}$ to be any solution to $\hat{P}[n]'$, the extended problem obtained by supplying the extra argument-function value pair $(\Delta, \hat{G}(\Delta))$ has a solution, namely $\hat{G}$ itself. Thus $\det\{H(n|\Delta, \hat{G}(\Delta))\} \geq 0$: the value of $\hat{G}$, and indeed of all solutions to $\hat{P}[n]'$ for the argument value Δ, lie on $\bar{\mathbb{D}}'(\Delta)$, the ~~disc~~ closed disc bounded by $\mathscr{C}'$. Conversely, for any $h \in \bar{\mathbb{D}}'(\Delta)$, $\det\{H(n|\Delta,h)\} \geq 0$: the extended problem obtained by adjoining the pair (Δ, h) has a solution $\hat{G}$ say. $\hat{G}$ is also a solution of $\hat{P}[n]'$ itself: h is the value, for the fixed argument value Δ in question of a solution to $\hat{P}[n]'$. Thus $\hat{P}[n]' \to \bar{D}_n'(n)$.

Using relationship ( ) or ~~one of~~ its ~~second~~ appropriate counterpart, it follows that $P[n]' \to \bar{D}_n(\Delta)$.

The result of clause (i) of the above theorem in the case in which $q=0$ concerns the connection between the existence of an N'-function which generates an asymptotic series whose initial coefficients satisfy a relationship of the form ( ) and the sign of the (real symmetric) Hankel determinant formed from these coefficients; the result for the case in which $p=0$ and $T(k)=1$ $(k=1,\dots,q)$ concerns the connection between the existence of a solution to the simple Pick-Nevanlinna problem and the sign of the determinant ~~formed~~ of the Hermitian matrix with elements of the form $(\lambda_k-\bar{\lambda}_J)^{-1}(\bar{h}_J-h_k)$. Both of these special results are classical; the first was established by Hamburger [ ] and the second was given by Pick [ ] and Nevanlinna [ ] in their original memoirs. The general result of clause (i) shows, in particular, how the Hankel matrix and the Pick-Nevanlinna Hermitian matrix may be combined to permit a unified treatment of the Hankel and simple Pick-Nevanlinna problems and, more generally, how a generalisation of the second problem may be subsumed within this unified treatment.

The existence of inclusion discs for the values of all solutions to the Hamburger problem and the simple Pick-Nevanlinna problem is also known {[, , ,]}; the result of clause (iii) is accordingly an extension of the work referred to.

The results of the above theorem are exclusively concerned with the diminished version of the Hamburger-Pick-Nevanlinna problem. It is possible to extend the method of proof to the treatment of the Hamburger-Pick-Nevanlinna problem itself, to examine the structure of the rational function solutions to this problem and its diminished form, and to describe the relative positions of the inclusion discs deriving from a sequence of subordinate problems. These matters are, however, more conveniently dealt with by the use of methods described in the following section, in which explicit expressions for the general solutions to the interpolation problems concerned are described.

Theorem Let p, K be finite positive integers, g_ν ($\nu=-2,-1,\ldots,$ 3? $2p-2$) be finite real numbers, λ_k ($k=1,\ldots,K$) be distinct argument values in $\mathbb{D}$ and $T(k)$ ($k=1,\ldots,K$) be corresponding finite positive integers; let $\partial^\tau H_k$ ($\tau=0,\ldots,T(k)-1$) be prescribed complex numbers complex numbers corresponding to the argument value λ_k ($k=1,\ldots,K$). Set $m=\sum_{k=1}^{K} T(k)$ and $n=m+p$. ?

Let Δ_ν ($\nu=0,\ldots,n-1$) be a sequence whose members ? consist of $p+1$ infinities and $T(k)$ copies of λ_k ($k=1,\ldots,K$); ~~these members may be~~ are placed in an order ~~open to choice but~~

Let $\{i(\nu)\}$ be an argument suffix sequence compatible ~~fixed at the outset.~~ with the Hamburger–Pick–Nevanlinna problem $P[n|p,q,T:g;\lambda,\mathfrak{G}]$.

A) The following scheme for the construction of numbers $\{g_\tau^{(\nu)}, c_{k,\tau}^{(\nu)}\}$ is proposed. Initially $g_\tau^{(0)}=g_\tau$ ($\nu=-2,-1,\ldots,2p-3$)

$c_{k,\tau}^{(0)}=\partial^\tau H_k/\tau!$ ($k=1,\ldots,\overset{q}{K}$; $\tau=0,\ldots,T(k)-1$). ~~Let $p(\nu)$ be~~

For $\nu=1,\ldots,n$ let $\hat{p}(\nu)$ be the number of times zero ~~infinity~~ occurs in the list $\{\Delta_{i(\nu)},\Delta_{i(\nu+1)},\ldots,\Delta_{i(n)}\}$ and let $\hat{T}(\nu,k)$ ~~(k,ν)~~ be the number of times λ_k occurs in it ~~the list~~ ($k=1,\ldots,q$).

For $\nu=1,\ldots,n$

(i) if $\Delta_{i(\nu)}=\infty$, ~~set $g_{-2}^{(\nu+1)}=$~~

a) if ~~$p(\nu+1)>0$~~, set $g_{-2}^{(\check{\nu}+2)} = -1/g_0^{(\check{\nu})}$ and

$$g_{r}^{(\nu+1)} = g_{-2}^{(\nu+1)} \sum_{j=1}^{\nu+2} g_{r-j}^{(\nu+1)} g_j^{(\nu)}$$

b) if $\overset{\wedge\,(\nu+1,k)}{T}(k,\nu+1)>0$ set $c_{k,0}^{(\check{\nu}+1)} = \{ g_{-2}^{(\check{\nu})}\lambda_k + g_{-1}^{(\check{\nu})} - c_{k,0}^{(\check{\nu})} \}^{-1}$; if $T^{(\nu+1,k)}(k,\nu+1)>1$ set $c_{k,1}^{(\check{\nu}+1)} = c_{k,0}^{(\check{\nu}+1)^2}(c_{k,0}^{(\check{\nu})} - g_{-2}^{(\check{\nu})})$; if $T^{(\nu+1,k)}(k,\nu+1)>2$

~~b) if $T(k,\nu+1)>0$ set $\psi_{\nu,k} = \{g_{-2}^{(\nu)}\lambda_k + g_{-1}^{(\nu)} - c_{k,0}^{(\nu)}\}^{-1}$ and~~

~~$c_{k,0}^{(\nu+1)} =$~~

set $c_{k,z}^{(\check{\nu}+1)} = c_{k,0}^{(\nu+1)} \{ \sum_{j=1}^{z} c_{k,z-j}^{(\check{\nu}+1)} c_{k,j}^{(\check{\nu})} - g_{-2}^{(\check{\nu})} c_{k,z-1}^{(\check{\nu}+1)} \}$ for $z=2$, $\ldots, T^{(\nu+1,k)}(k,\nu+1)-1$, all for $k=1,\ldots,q$.

ii) if ~~$\Lambda_{\nu} = \lambda_k$~~ $i(\nu)=k$

a) if $\hat{p}(\check{\nu}+1)>0$ set

$$g_{-2}^{(\check{\nu}+1)} = \{ \mathrm{Im}(c_{k,0}^{(\check{\nu})} - \mathrm{Im}(\lambda_k) g_{-2}^{(\check{\nu})}) \}^{-1} \mathrm{Im}(\lambda_k) g_{-2}^{(\check{\nu})}$$

$$g_{-1}^{(\check{\nu}+1)} = (g_{-2}^{(\check{\nu}+1)}+1)\left[\mathrm{Im}(c_{k,0}^{(\check{\nu})})^{-1} \{ \mathrm{Im}(\lambda_k) g_{-1}^{(\check{\nu})} - \mathrm{Im}(\lambda_k \bar{c}_{k,0}^{(\check{\nu})})(g_{-2}^{(\check{\nu})}+1) \} - 2\mathrm{Re}(\lambda_k) \right]$$

and if $p(\check{\nu}+1)>1$ set

$$g_0^{(\check{\nu}+1)} = (g_{-2}^{(\check{\nu}+1)}+1)\left[|\lambda_k|^2 - \mathrm{Im}(c_{k,0}^{(\check{\nu})})^{-1} \left[\{ \mathrm{Im}(\lambda_k \bar{c}_{k,0}^{(\check{\nu})}) - \mathrm{Im}(\lambda_k) g_{-1}^{(\check{\nu})} \} g_{-1}^{(\check{\nu}+1)} + (g_{-2}^{(\check{\nu}+1)}+1) g_0^{(\check{\nu})} \right] \right]$$

$$g_j^{(\check{\nu}+1)} = \mathrm{Im}(c_{k,0}^{(\check{\nu})})^{-1} (g_{-2}^{(\check{\nu}+1)}+1) \left[\mathrm{Im}(\lambda_k) \{ (g_{-2}^{(\check{\nu}+1)}+1) g_j^{(\check{\nu})} + \sum_{\tau=1}^{j} g_{j-\tau}^{(\check{\nu})} g_{\tau-2}^{(\check{\nu}+1)} \} - \{ \mathrm{Im}(\lambda_k \bar{c}_{k,0}^{(\check{\nu})}) - \mathrm{Im}(\lambda_k) g_{-1}^{(\check{\nu})} \} g_{j-1}^{(\check{\nu}+1)} \right]$$

for $j = 1,\dots, 2\check p(\check p+1)-1$

b) ~~if $T(k,p+1)>0$~~ set $\check\theta^{(p)}_k = \{\mathrm{Im}(\check c^{(p)}_{k,0}) - \mathrm{Im}(\lambda_k)\check c^{(p)}_{k,1}\}^{-1}$ and

$\check c^{(p+1)}_{k,0} = \check\theta^{(p)}_k\{\lambda_k \mathrm{Im}(\lambda_k)\check c^{(p)}_{k,1} - \bar\lambda_k \mathrm{Im}(\check c^{(p)}_{k,0})\}$; if $\hat T^{r+1,k}(\cancel{k,p+1})>1$ set

$\check c^{(p+1)}_{k,1} = \check\theta^{(p)}_k\left[\mathrm{Im}(\check c^{(p)}_{k,0}) + \mathrm{Im}(\lambda_k)\check c^{(p)}_k\{\lambda_k + \check c^{(p+1)}_{k,0}\}\right]$;

if $\hat T^{r+1,k}(\cancel{k,p+1})>2$ set

$$\check c^{(p+1)}_{k,z} = \check\theta^{(p)}_k \mathrm{Im}(\lambda_k)\left[\lambda_k \check c^{(p)}_{k,z+1} + \sum_{j=0}^{z-1} \check c^{(p+1)}_{k,z-j-1}\check c^{(p)}_{k,j}\right].$$

for $z = 2,\dots, \hat T^{r+1,k}(\cancel{k,p+1})-1$.

β) for $j = 1,\dots, \cancel{k}\ (j\neq k)$ if $\hat T^{r+1,j}(\cancel{j,p+1})>0$ set $\cancel{\phi_k}$

$$\check\phi^{(p)}_j = \{\mathrm{Im}(\lambda_k)\check c^{(p)}_{j,0} - \mathrm{Im}(\lambda_k\bar{\check c}^{(p)}_{k,0}) - \mathrm{Im}(\check c^{(p)}_{k,0})\lambda_j\}^{-1}$$

$$\check c^{(p+1)}_{j,0} = \check\phi^{(p)}_j\left[|\lambda_k|^2\mathrm{Im}(\bar{\check c}^{(p)}_{k,0}) + \{\mathrm{Im}(\lambda_k\check c^{(p)}_{k,0}) - \mathrm{Im}(\lambda_k)\check c^{(p)}_{j,0}\}\lambda_j\right]$$

if $\hat T^{(r+1,j)}(\cancel{j,p+1})>1$ set

$$\check c^{(p+1)}_{j,1} = \check\phi^{(p)}_j\left[\mathrm{Im}(\lambda_k\check c^{(p)}_{k,0}) - \mathrm{Im}(\lambda_k)\{\check c^{(p+1)}_{j,0}\check c^{(p)}_{j,1} + \lambda_j\check c^{(p)}_{j,1} + \check c^{(p)}_{j,0}\}\right.$$
$$\left. + \mathrm{Im}(\check c^{(p)}_{k,0})\check c^{(p+1)}_{j,0}\right]$$

if $\hat T^{(r+1,j)}(\cancel{j,p+1})>2$ set

$$\check c^{(p+1)}_{j,z} = \check\phi^{(p)}_j\left[\mathrm{Im}(\check c^{(p)}_{k,0})\check c^{(p+1)}_{j,z-1} - \mathrm{Im}(\lambda_k)\{\lambda_j\check c^{(p)}_{j,z} + \check c^{(p)}_{j,z-1} + \sum_{\nu=1}^{z}\check c^{(p+1)}_{j,z-\nu}\check c^{(p)}_{j,\nu}\}\right.$$

for $z = 2,\dots,\hat T^{r+1,k}(\cancel{k,p+1})-1$

The ~~interpolation problem (α,β,γ) has a solution which is not the quotient of two polynomials of degrees $\le n$~~ $P[n]$ is nondefective if ~~and~~ and only if, in the ~~implementation~~ execution of the above ~~scheme $g_{-2}^{(\nu+1)} > 0$ and $\operatorname{Im}(c_{k,0}^{(\nu+1)}) > 0$~~

(a) $g_{-2} \le 0$ if $i(\nu)=0$, or $\operatorname{Im}(c_{k,0}^{(\nu)}) > 0$ if $i(\nu)=k \ne 0$ and

scheme (b) $g_{-2}^{(\nu+1)} < 0$ at each implementation of step (ia) and (c) $\operatorname{Im}(c_{k,0}^{(\nu+1)}) > 0$ at each implementation of step (iib).

B) Assume that the above conditions concerning the numbers $\{g_{-2}^{(\nu+1)}, c_{k,0}^{(\nu+1)}\}$ hold. For $\nu = 0, \dots, n-1$

a) if $i(\nu)=0$ ~~$\Delta_\nu = \infty$~~, set $W_\nu(\lambda) = g_{-2}^{(\nu)}\lambda + g_{-1}^{(\nu)}$, $V_{\nu+1}(\lambda) = 1$, $\xi_{\nu+1} = 0$

b) if $i(\nu) = k \ne 0$ ~~$\Delta_\nu = \lambda_k$~~, set

$$W_\nu(\lambda) = b_0^{(\nu)} + b_1^{(\nu)}\lambda$$
$$U_{\nu+1}(\lambda) = b_1^{(\nu)}(\lambda - \lambda_k)(\lambda - \bar\lambda_k)$$

~~$W_\nu(\lambda) = \operatorname{Im}(\lambda_k)^{-1}\{\lambda \operatorname{Im}(c_{k,0}^{(\nu)}) + \operatorname{Im}(\lambda_k \bar c_{k,0}^{(\nu)})\}$~~

~~$V_{\nu+1}(\lambda) = \operatorname{Im}(\lambda_k)^{-1}\operatorname{Im}(c_{k,0}^{(\nu)})(\lambda-\lambda_k)(\lambda-\bar\lambda_k)$~~

and $\xi_{\nu+1} = 1$. where $b_0^{(\nu)} = \operatorname{Im}(\lambda_k)^{-1}\operatorname{Im}(\lambda_k \bar c_{k,0}^{(\nu)})$, $b_1^{(\nu)} = \operatorname{Im}(\lambda_k)^{-1}\operatorname{Im}(c_{k,0}^{(\nu)})$

~~All solutions to~~

(i) The general expression of a solution to ~~the interpolation problem (α,β,γ) is~~ $P(n)$

$$\mathcal{G}(\lambda) = \overset{\vee}{W}_0(\lambda) + \frac{\overset{\cup}{V}_1(\lambda)}{\xi_1\lambda + \overset{\vee}{W}_1(\lambda) +} \cdots \frac{\overset{\cup}{V}_{n-1}(\lambda)}{\xi_{n-1}\lambda + \mathcal{G}_{n}(\lambda) V_{n-1}(\lambda) +} \frac{\Delta_n(\lambda)}{\xi_n\lambda + \mathcal{G}^{(n+1)}(\lambda)}$$

~~where $\mathcal{G}_n \in \bar{N}$ if $\Delta_n \neq \infty$ and $\mathcal{G}_n \in \bar{N}$ if $\Delta_n = \infty$.~~

where $\mathcal{G}_n \in N$, where $\mathcal{G}^{(n+1)} \in \bar{N}$ if $p = 0$ and otherwise $\mathcal{G}^{(n+1)} \in \dot{N}$ if $i(n) \neq 0$ and $\mathcal{G}^{(n+1)} \in \dot{N}''$ if $i(n) = 0$.

(ii) All solutions to the interpolation problem (α, β, γ) expressible as quotients of two n^{th} degree polynomials may be represented in the form

$$W_0(\lambda) - \frac{V_1(\lambda)}{\xi_1\lambda + W_1(\lambda) -} \cdots \frac{V_n(\lambda)}{\xi_n\lambda + t^{(n)}}$$

where $-\infty \leq t^{(n)} \leq \infty$. Distinct values of $t^{(n)}$ (taking $\pm\infty$ to be equivalent) are associated with distinct rational functions.

c) Subject to the conditions of part B) above, expression ( ) represents as $t^{(n)}$ varies over the range $[-\infty, \infty]$ a circle (the case $n = 1$, $\Delta_0 = \infty$ is excepted) ~~with $n > 1$,~~

c) Assume again that the conditions of part B) hold, and let $\lambda \in \mathbb{U}$ be fixed. (i) As $t^{(n)}$ varies over the range $[-\infty, \infty]$ expression ( ) represents, ~~if $n > 1$, a circle in a circle in $\mathbb{U}$. The~~ circle $\mathcal{C}^{(n)}$ in $\mathbb{U}$. $\mathcal{C}^{(n)}$ is the boundary

(ii) The rational functions of λ represented by the expression

$$C_n(\lambda, t_n) = V_0(\lambda) + \frac{U_1(\lambda)}{\xi_1\lambda + V_1(\lambda) +} \cdots \frac{U_{n-1}(\lambda)}{\xi_{n-1}\lambda + V_{n-1}(\lambda) +} \frac{U_n(\lambda)}{\xi_n\lambda + t_n}$$

with $-\infty < t_n \le \infty$ are ~~elements of~~ solutions to $\mathbf{P}(n)$, only $\mathbf{P}(n)'$ or even only $\mathbf{P}(n)''$. In the following $t_n^{(0)} = \infty$, and $t_n^{(1)}$ is a finite real number uniquely determined in each clause; in all cases $t_n^{(1)} = 0$ if $i(n) = 0$.

a) If $p = 0$, $C_n(\cdot, (-\infty, \infty) \setminus t_n^{(1)}) = P(n) \cap R(n)$ and $C_n(\cdot, t_n^{(\omega)}) = P(n) \cap R^{(\omega)}(n)$ $(\omega = 0, 1)$

b) If $p > 0$ and $g_{-2} \neq 0$, $C_n(\cdot, (-\infty, \infty] \setminus t_n^{(1)}) \equiv P(n)' \cap R^{(0)}(n)$, $C_n(\cdot, \infty) = P(n) \cap R^{(0)}(n)$ and $C_n(\cdot, t_n^{(1)}) = P(n)'' \cap R^{(0)}(n-1)$.

c) If $p > 0$, $g_{-2} = 0$ and $g_{-1} \neq 0$, $C_n(\cdot, (-\infty, \infty] \setminus t_n^{(1)}) \equiv P(n)' \cap R(n-1)$ and $C_n(\cdot, \infty) = P(n) \cap R(n-1)$; if $p = 1$, $C_n(\cdot, t_n^{(1)}) =$ ~~$P(n)'' \cap R^{(0)}(n-1)$ while if $p > 1$, $C_n(\cdot, t_n^{(1)}) = P(n)'' \cap R(n-2)$~~
$P(n)'' \cap \{R^{(0)}(n-1)$ if $p = 1 \mid R(n-2)$ if $p > 1\}$

d) Let $g_{-2} = g_{-1} = 0$.

~~α) If $p = 1$, $C_n(\cdot, (-\infty, \infty) \setminus t_n^{(1)}) = P(n)' \cap R(n-1)$ while if $p > 1$~~

α) $C_n(\cdot, (-\infty, \infty) \setminus t_n^{(1)}) = P(n)' \cap \{R(n-1)$ if $p = 1 \mid R^{(1)}(n-1)$ if $p > 1\}$

β) $C_n(\cdot, \infty) = P(n) \cap R^{(1)}(n-1)$

γ) $C_n(\cdot, t_n^{(1)}) = P(n)'' \cap \{R^{(\omega)}(n-1) \text{ if } p=1 \mid R^{(1)}(n-2) \text{ if } p>1\}$.

(iii) The members G of the sets of functions considered in the preceding clause are distinguished by ~~certain limiting and asymptotic properties in the following way~~ the behaviour of $G(\lambda)$ as λ tends to infinity.

a) if $p=0$ and $G \in \{P(n) \cap R(n)\}$, $\lim G(\lambda) = g'(G) \in (-\infty,\infty) \setminus \{0\}$ ~~as λ tends to infinity~~. This relationship establishes a one to one correspondence between the functions of $P(n) \cap R(n)$ and the points of $(-\infty,\infty) \setminus \{0\}$. For the unique member G of $P(n) \cap R^{(\omega)}(n)$, $\lim G(\lambda) = \omega^{-1}(1-\omega)$ $(\omega = 0,1)$.

b) if (α) $p>0$, $g_{-2}+|g_{-1}| \neq 0$ and $G \in \{P(n)' \cap R^{(0)}(n) \text{ if } g_{-2}<0 \mid R(n-1) \text{ if } g_{-2}=0\}$, or (β) $p=1$, $g_{-2}=g_{-1}=0$ and $G \in P(n)' \cap R(n-1)$ or (γ) $p>1$, $g_{-2}=g_{-1}=0$ and $G \in P(n)' \cap R^{(1)}(n-1)$ then

$$\lim \lambda^{2p-1}\left\{G(\lambda) - \sum_{\nu=-2}^{2p-4} g_\nu \lambda^{-\nu-1} - g'(G)\lambda^{-2p+2}\right\} = 0$$

~~as λ tends to infinity~~ where $g'(G) \in (-\infty,\infty)$. This relationship establishes a one-to-one correspondence between the functions of the class in question ($P(n)' \cap R^{(1)}(n)$ if $p>0$, $g_{-2}+|g_{-1}| \neq 0$, etc.)

and the points of $(-\infty,\infty)$.

In each of the above cases there is only one member of the class concerned for which relationship ( ) is satisfied with $g'(G) = g_{2p-3}$; it is the unique member of the class defined by replacing $P(n)'$ by $P(n)$.

(C) Assume again that the conditions of part B) hold, and let $\Lambda \in \mathbb{L} \setminus \{\lambda_1, \dots, \lambda_p\}$ be fixed.

(i) As t_n varies over the range $(-\infty, \infty]$, expression ( ) represents a circle $b_n^{\neq}(\Lambda)$; the boundary of open and closed discs $\mathbb{D}_n(\Lambda)$ and $\bar{\mathbb{D}}_n(\Lambda)$ lying in $\mathbb{D}$. (If $n=p=1$, $b_1(\Lambda)$ is the line parallel to the real axis through the point $ig_{-2}\operatorname{Im}(\Lambda)^{\neq}$; $\mathbb{D}_1(\Lambda)$ is the half-plane lying above $b(\Lambda)$.

$b_n(\Lambda)$ is the similarly named circle of part of Theorem .

(ii) Let $C_n^{(\omega)}(\Lambda)$ be the point on $b_n(\Lambda)$ represented by $C_n(\Lambda, t_n^{(\omega)})$ $(\omega=0,1)$, $b_n^{(1)}(\Lambda)$ be the arc obtained by removing $C_n^{(1)}(\Lambda)$ from $b_n(\Lambda)$, and $b_n^{(0,1)}(\Lambda)$ be the two arcs obtained by removing the points $C_n^{(\omega)}(\Delta)$ $(\omega=0,1)$.

a) If $p=0$, $P(n)\to\bar{D}_n(\Delta)$, while if $p>0$, $P(n)'\to$ $\bar{D}_n(\Delta)\setminus C_n^{(1)}(\Delta)$ and $P(n)\to D_n(\Delta)\cup C_n^{(0)}(\Delta)$

b) Inclusion results may be obtained directly from the results of clause (Bii) by replacing $G_n(\cdot,(-\infty,\infty]\setminus t_n^{(1)})$, $G_n(\cdot,(-\infty,\infty)\setminus t_n^{(1)})$, $C(\cdot,t_n^{(\omega)})$ $(\omega=0,1)$ by $\mathcal{C}_n^{(1)}(\Delta)$, $\mathcal{C}_n^{(0,1)}(\Delta)$, $C_n^{(\omega)}(\Delta)$ respectively, and appropriately changing the notation. Thus clause (Biib) yields the result that if $p>0$ and $g_{-2}\neq 0$, $P(n)'\cap R^{(0)}(n)\to\mathcal{C}^{(1)}(\Delta)$, $P(n)\cap R^{(0)}(n)\Leftrightarrow$ $C_n^{(0)}(\Delta)$ and $P(n)''\cap R^{(0)}(n-1)\Leftrightarrow C_n^{(1)}(\Delta)$

(iii) Circles $\mathcal{C}_r$ and discs D_r, $\bar{D}_r$ $(r=1,\dots,n-1)$ may be defined in analogy with those described above

a) If $i(r)\neq 0$ $(r>1)$, $\mathcal{C}_r(\Delta)\subset D_{r-1}(\Delta)$

b) If $g_{-2}\neq 0$, $i(r)=0$ and $r>1$, $\mathcal{C}_r(\Delta)$ lies within $\mathcal{C}_{r-1}(\Delta)$, touching the latter at the point

$$C_r(\Lambda, t_r^{(1)}) = C_{r-1}(\Lambda, t_{r-1}^{(1)})$$

$$= V_0(\lambda) - \frac{U_1(\Lambda)}{\xi_1\Lambda + V_1(\Lambda) -} \cdots \frac{U_{p-2}(\Lambda)}{\xi_{p-2}\Lambda + V_{p-2}(\Lambda)}$$

c) If $g_{-2} = 0$, $i(r) = 0$, $r > 0$ and $i(r') \neq 0$ ($r' = 2, \ldots, r$), then $b_r(\Lambda)$ is $b_{r-1}(\Lambda)$; but if $i(r') = 0$ for some r' in the range $2 \leq r' \leq r$, the result of (b) above holds.

Proof. The following proof of part A) ~~to be given~~ is based upon the recursive construction of a sequence of ~~N~~ functions by ~~means~~ use of a process which steadily exhausts the data. It is shown that all functions of this sequence are N-functions if and only if the conditions stated in part A) hold; the ~~result of part~~ required result then follows directly.

Assume firstly that $\cancel{\Delta_1 = \infty}$ $i(1)=0$. If a function $G^{(1)}$ satisfies an asymptotic relationship of the form ( ), ~~with~~ then a) if ~~$G^{(1)}$ is an~~ $G^{(1)} \in N$, $g_{-2} \leq 0$ and b) if $g_{-2} > 0$, $G^{(1)}$ is not an N-function. Hence, with regard to the problem ~~it is possible~~ ~~in hand, $g_{-2} = g^{(1)}_{-2} \geq 0$, the possibility that~~ $N(n)$ has a solution ~~remains~~ if and only if, with regard to the single coefficient $g^{(1)}_{-2}$, $g^{(1)}_{-2} \leq 0$. ~~If~~ Since $G^{(1)}$ satisfies a relationship of the form ( ), ~~σ is such that condition ( )~~ ~~in the formula ( )~~ σ in the representation of the form ( ) of $G^{(1)}$ is such that condition ( ) holds: $G^{(1)}$ may be decomposed in the form

$$G^{(1)}(\lambda) = g_{-2}\lambda + g_{-1} + G'(\lambda)$$

where $G' \in \bar{N}'$. If G' is a constant, G is a rational function

of the type expressly excluded from consideration in part A)s.
Hence $G' \in N'$ and, ~~replacing G, g_{-2}, g_{-1} by~~ adding a superscript to G, g_{-2}, g_{-1}, $G = G^{(1)}$ has a representation of the form

$$G^{(r)}(\lambda) = g_{-2}^{(r)}\lambda + g_{-1}^{(r)} - G^{(r+1)}(\lambda)^{-1}$$

with $r=1$,
~~where $G_2 \in N''$.~~ $R(n)$ may be ~~form~~ reformulated as an extended Pick-Nevanlinna problem involving the function $G^{(2)}$: ~~the~~ Asymptotic relationships ~~involving coeff~~ of the form ( ) involving $G^{(2)}$ and coefficients $\{g_\nu^{(2)}\}$ may be obtained from those involving $G = G^{(1)}$ and $\{g_\nu = g_\nu^{(1)}\}$ by use of formula ( ). The values of the $\{g_\nu^{(2)}\}$ may be extracted from ~~these relation~~ new relationships; they are as given by formulae ( ) with $r=1$. The derivative values $\{\Delta^2 G_k^{(2)}\}$ to be assumed by $G^{(2)}$ at the points $\{\lambda_k\}$ may also be obtained from relationship ( ). The values of $\{c_k^{(2)} = (2!)^{-1}\Delta^2 G_k^{(2)}\}$ are as given by formulae ( ) with $r=1$. It is to be noted that in this case the coefficients $\{g_\nu^{(2)}\}$ are in number two less than the $\{g_\nu^{(1)}\}$. The order of the derived ~~extended~~ Hamburger-Pick-Nevanlinna problem is one less than that of the original.

The contrary assumption that ~~$\Delta_1 = \Delta_k$~~ $i(1) = k \neq 0$ is now made.

In this case ~~an application of~~ use is made of ~~the~~ an inequality of Schwarz ~~inequality, of which fluent use was made by Nevanlinna in his original investigations, will be exploited. For all z such that~~ (Let f be analytic with $|f(z)|<1$ over the unit open disc, with $f(0)=0$; then $f(z)<|z|$ for all $|z|<1$).

Set

$$F(\lambda) = \{G(\lambda) - \bar{G}_k\}^{-1}\{G(\lambda) - G_k\}, \quad Z(\lambda) = (\lambda - \bar{\lambda}_k)^{-1}(\lambda - \lambda_k)$$

$$Z(\lambda)^{-1}F(\lambda) = \{G^{(2)}(\lambda) - \bar{\lambda}_k\}^{-1}\{G^{(2)}(\lambda) - \lambda_k\}$$

~~so that, with If $G \in \mathbb{N}$, $|F(\lambda)| <$ it follows from the above inequality that $|F(\lambda)| < |Z(\lambda)|$ for all $\lambda \in \mathbb{U}$~~

If G_k is to be the value assumed by an $\mathbb{N}$-function when its argument value is in $\mathbb{U}$, $\mathrm{Im}(G_k) > 0$. If $G \in \mathbb{N}$, it follows from the above inequality that $|F(\lambda)| < |Z(\lambda)|$ ~~if this ineq the latter inequality~~ for all $\lambda \in \mathbb{U}$; if the latter inequality holds, $G^{(2)} \in \bar{\mathbb{N}}$. Conversely, as ~~As~~ is easily demonstrated, if $G^{(2)} \in \bar{\mathbb{N}}$ then $G \in \mathbb{N}$: $G \in \mathbb{N}$ if and only if $G^{(2)} \in \bar{\mathbb{N}}$.

~~Setting $c_k^{(1)} = G_k$ at G~~ Replacing G, G_k by $G^{(1)}, c_{k,0}^{(1)}$

relationships (,) yield

$$\mathcal{G}^{(r)}(\lambda) = \operatorname{Im}(\lambda_k)^{-1}\operatorname{Im}(\mathcal{G}^{(r)}_{k,0})\lambda + \operatorname{Im}(\lambda_k)^{-1}\operatorname{Im}(\lambda_k\bar{\mathcal{G}}^{(r)}_{k,0}) - \frac{\operatorname{Im}(\lambda_k)^{-1}\operatorname{Im}(c^{(r)}_{k,0})(\lambda-\lambda_k)(\lambda-\bar{\lambda}_k)}{\lambda + \mathcal{G}^{(r+1)}(\lambda)}$$

with $r=1$.

If $\mathcal{G}^{(2)}$ is a constant, then $\mathcal{G}^{(1)} = \mathcal{G}$ is a rational function of the type excluded from consideration in part (A): $\mathcal{G}^{(2)}$ in relationship ( ) is an N-function. Again $P(n)$ may be reformulated in terms of ~~the function~~ $\mathcal{G}^{(2)}$. The Asymptotic relationships involving $\mathcal{G}^{(2)}$ and coefficients $\{g^{(2)}_{\nu}\}$ are obtained from formulae (,), and the $\{g^{(2)}_{\nu}\}$ are extracted from them by means of formulae (,). The derivative values $\{\partial^{z}\mathcal{G}^{(2)}_{kj}\}$ to be assumed by $\mathcal{G}^{(2)}$ at the points $\{\lambda_j\}$ may also be obtained from relationship ( ). The values of $\{c^{(2)}_j = (z!)^{-1}\partial^{z}\mathcal{G}^{(2)}_j\}$ are, ~~as given~~ when $j=k$, given by formulae (,) ~~with r=1~~ and when $j \neq k$ by formulae (,) with $r=1$ in both cases. It is to be noted that when $\lambda = \lambda_k$, the value of ~~$\mathcal{G}^{(2)}$ is~~ $\mathcal{G}^{(2)}(\lambda)$ is to be obtained from relationship ( ) in the form of the quotient of two derivatives; in consequence the $\{c^{(2)}_k\}$ are one less in number than the $\{c^{(1)}_k\}$. Again the degree of the derived ~~extended~~ Hamburger Pick-Nevanlinna problem is one less than that of the original.

The above construction of ~~extended~~ ~~extended~~ Hamburger Pick-Nevanlinna problems is continued until the data is exhausted. ~~At the r^{th} stage $G = G^{(1)}$ may be is expressed in terms of $G^{(r+1)}$ by means of a version of formula ( ) with n replaced by r.~~ Comparing terms in formulae (,) with the formulae for ~~V~~ V_1, V_r given in part (B), and eliminating $G^{(2)}, \dots G^{(r)}$ from r versions of formulae (,), it can be seen that at the r^{th} stage of the reduction process, G ~~$= G^{(r)}$~~ is expressed in terms of $G^{(r+1)}$ by means of a version of formula ( ) with n replaced by r. ~~The possibility~~ ~~If at~~ ~~the~~ ~~The possibility that $G^{(r+1)}$ is a constant~~ ~~function~~ ~~at any stage can be discounted~~ If $G^{(r+1)}$ were to be ~~is~~ a constant, G would be a rational function of the type excluded from consideration in part A): the possibility that $G^{(r+1)}$ is a constant function at any stage can be discounted. At each stage a condition of the form $g^{(r)}_{-2} \leq 0$ or $\mathrm{Im}(c_0^{(r)}) > 0$ is ~~[illegible]~~ derived: the result of part (A) has been obtained. That of part (B) has already been obtained in passing as a consequence of the remark concerning the expression of $\mathbb{G}$ in terms of $\mathbb{G}^{(r+1)}$.

Clauses (B_{iii}) are concerned ~~by the~~ with the functions represented by expression ( ) when $\mathbb{G}^{(r+1)}$ is replaced by a real number, which lies on the boundary of the value-domain of the ~~values taken by~~ the functions $\{\mathbb{G}^{(r+1)}\}$ when their arguments lie in $\mathbb{L}$. For $r=1,\dots,n$ denote by $\hat{P}(n-r+1)$ the Hamburger Pick-Nevanlinna problem deriving from the coefficients $\delta_\nu^{(n-r+1)}$ $(\nu=-2,\dots,2\hat{p}(n-r+1)-3)$ and derivative values $\tau!\,c_{k,\tau}^{(n-r+1)}$ $\tau=0,\dots,\hat{\imath}(n-r+1,k)$ for those values of k contained in the list $i(n-r+1),\dots,i(n)$, so that $\hat{P}(n)$ is $P(n)$, and ~~denotes by C_{n-r+1}~~ set

$$\hat{C}_{n-r+1}(\lambda,t_n) = V_{n-r}(\lambda) - \frac{U_{n-r+1}(\lambda)}{\xi_{n-r+1}\lambda + V_{n-r+1}(\lambda)} \cdots \frac{U_{n-1}(\lambda)}{\xi_{n-1}\lambda+V_{n-1}(\lambda)-}\,\frac{U_n(\lambda)}{\xi_n\lambda+t_n}$$

~~so that $\hat{C}_n(\lambda) = C$~~ so that $\hat{C}_n(\lambda,t_n) = C_n(\lambda,t_n)$. +4 pages Under the conditions of clause (Biia), $\xi_r=1$ for $r=1,\dots,n$ and, with $i(n)=k$

$$\hat{C}_1(\lambda,t_n) = b_0^{(n)} + b_1^{(n)}\lambda - \frac{b_1^{(n)}(\lambda-\lambda_k)(\lambda-\bar{\lambda}_k)}{\lambda+t_n}$$

~~$= \{$~~

$$\hat{C}_1(\lambda, t_n) = \frac{\left[b_0^{(n)} + b_1^{(n)}\{t_n + 2\mathrm{Re}(\lambda_k)\} \right]\lambda + b_0^{(n)} t_n - |\lambda_k|^2 b_1^{(n)}}{\lambda + t_n}$$

It is evidently true to say of the function $\hat{C}_r(\lambda, t_n)$ that for all $t_n \in (-\infty, \infty]$, $\hat{C}_r(\lambda, t_n) \in \hat{P}(r)$, when $r = 1$ (that for all $t_n \in (-\infty, \infty) \setminus \hat{t}_r^{(1)}$ (where $\hat{t}_r^{(1)}$ ~~is such as to make the coefficient of~~ λ in the ~~$t_r^{(1)} = -(b_1^{(n)})^{-1}\{b_0^{(n)}$~~ numerator of expression ( ) zero) $\hat{C}_r(\cdot, t_n) \in R(r)$) (and, in view of the above remarks, that there are no other members of $\hat{P}(r) \cap R(r)$) ~~that $\hat{C}_r(\cdot, t_n^{(0)}) \in R$~~ that with $\hat{t}_n^{(\omega)} = \infty$, $\hat{C}_r(\cdot, \hat{t}_n^{(\omega)}) \in R^{(\omega)}(r)$ for $\omega = 0, 1$ (again uniquely); ~~furthermore formula ( )~~ and that ~~one to one cores~~ the function $g_r'(t_n) = \lim \hat{C}_r(\lambda, t_n)$ as λ tends to infinity ~~is a univalent~~ establishes a univalent mapping of the real segment $(-\infty, \infty]$ ~~into~~ onto itself (the point at ~~∞ is its~~ infinity is its own image). It is now assumed that the foregoing statements are correct when r is replaced by $r-1$, where $2 \leq r < n$. $\hat{C}_{r-1}(\lambda, t_n)$ and $\hat{C}_r(\lambda, t_n)$ are connected by means of the formula

$$\hat{C}_r(\lambda, t_n) = b_0^{(n-r)} + b_1^{(n-r)}\lambda - \frac{b_1^{(n-r)}(\lambda - \lambda_j)(\lambda - \bar{\lambda}_j)}{\lambda + \hat{C}_{r-1}(\lambda, t_n)}$$

Where λ_j is $(n-r+1) = j$. By ~~methods~~ the use of methods employed in the proof of part (A) and clause (Bi), it can be shown that if $\hat{C}_{r-1}(\cdot, t_n) \in \hat{P}(r-1)$ for all $t_n \in (-\infty, \infty]$, then $\hat{C}_r(\cdot, t_n) \in \hat{P}(r)$ for these values of t_n. ~~Reduction~~ Transformation of the expression upon the right hand side of formula ( ) to a simple fraction, ~~it~~ indicates that for all $t_n \in (-\infty, \infty) \setminus \hat{t}_r^{(1)}$ (where $\hat{t}_r^{(1)}$ is now that value of t_n for which $g'_{r-1}(t_n) = -(b_1^{(n-r)})^{-1} b_0^{(n-r)} 2\mathrm{Re}(\lambda_j)$) $\hat{C}_r(\cdot, t_n) \in R(r)$ and that $\hat{C}_r(\cdot, t_n) \in R^{(\omega)}(r)$ $(\omega = 0, 1)$, ~~these statements unless the in the reduced transformed expression possesses a common polynomial factor~~ (these are in each case the true forms of C_r, since reduction of ~~the denominator and numerator~~ rational functions cannot occur) ~~possess a common polynomial factor.~~ If ~~this latter eventuality were to occur, however, it would then follow that, P(n) is defective~~ since $C_r(\cdot, t_n) \in \hat{P}_*(r)$ ~~for r = 1,...,n, P(n) is defective; this possibility has been discounted.~~ The further statements concerning $C_r(\cdot, t_n)$ made above can ~~may easily~~ be verified; the results of clause (Biia, Biiia) have been proved.

The case in which $p>0$ is now considered. It is now possible that $i(n)=0$, ~~Also~~ and indeed when $p>1$ that

$$\cancel{\hat{C}_1(\lambda, t_n) = g_{-2}^{(n)}\lambda + g_{-1}^{(n)} - t_n^{-1}}$$

~~It is now t~~ ~~As a preliminary, it is assumed that $g_{-2}\neq 0$, in which case (see formula ( )), $g_{-2}^{(r)}\neq 0$ $(r=1,\ldots,n)$.~~

~~It is now~~ ~~Whether $\hat{C}_1$ has the form ( ) or ( ), it is now true that when $r=1$ that for all $t_n\in(-\infty,\infty]\setminus t_n^{(1)}$, $\hat{C}_r(\cdot, t_n)\in\hat{P}(r)'$ where $t_n^{(1)}=0$~~

$i(n-1)=0$ also, ~~in wh~~ ~~Also~~ and then

$$\hat{C}_2(\lambda, t_n) = g_{-2}^{(n-1)}\lambda + g_{-1}^{(n-1)} - \frac{1}{g_{-2}^{(n)}\lambda + g_{-1}^{(n)} - t_n^{-1}}$$

As a preliminary it is assumed that $g_{-2}\neq 0$, in which case (see formula ( )), $g_{-2}^{(r)}\neq 0$ $(r=1,\ldots,n)$

It is now true that when $r=2$ that for all $t_n\in(-\infty,\infty]\setminus t_n^{(1)}$, where $t_n^{(1)}=0$, $\hat{C}_r(\cdot, t_n)\in\hat{P}(r)'$, that ~~$C_2$~~ $\hat{C}_r(\cdot, t_n^{(0)}\infty)\in\hat{P}(r)$ and that $\hat{C}_r(\cdot, t_n^{(1)})\in\hat{P}(r-1)'$.

~~With~~ With regard to ~~the~~ the structure ~~of~~ of $\hat{C}_r$, $\hat{C}_r(\cdot, t_n)\in R^{(0)}(r)$ for all $t_n(-\infty\,\infty]\setminus\{t_n^{(1)}\}$ while $\hat{C}_r(\cdot, t_n^{(1)})\in R^{(0)}(r-1)$ ~~or~~

Furthermore, for all $t_n \in (-\infty,\infty] \setminus \{t_n^{(1)}\}$, $\hat{C}_p(\lambda, t_n)$ has a series expansion in inverse powers of λ of the form $\sum_{\nu=-2}^{2r} g_\nu^{(n-r+1)} \lambda^{-\nu-1} + g_r'(t_n)\lambda^{-2r+2}$ where g_r' establishes a univalent mapping of $(-\infty,\infty] \setminus \{t_n^{(1)}\}$ onto $(-\infty,\infty)$; there is one value of t_n, namely $t_n = t_m^{(0)} = \infty$, for which $g_r'(t_n) = g_{2r-3}^{(n-r+1)}$. ~~For the stated initial values, namely $i(n)=i(n-1)=0$~~ It is again verified by induction that, starting from the ~~stated~~ given initial values $i(n)=i(n-1)=0$, the stated results are true for all r in the range $2 \le r \le n$. That they are true starting from other initial values ~~and indeed all results of clauses (Bii, iii)~~ can be verified in the same way, as can all results of clauses (Bii, iii)

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[P] Concerning expressions of the form ( ), it is ~~remarked as a preliminary~~ appropriate to remark that for all $t_n \in (-\infty,\infty)$, the denominator $D_n(\lambda, t_n)$ and numerator $N_n(\lambda, t_n)$ of $C_n(\lambda, t_n)$ cannot possess a common polynomial ~~factor. Denoting by $D_{n-1}(\lambda)$, $N_{n-1}(\lambda)$ the denominator and~~ factor. ~~It is a consequence of part (A) that if $P(n)$ is nondegenerate, $P(n-1)$ is also nondegenerate the same is true~~

~~of $P(n-1)$~~ if $i(n) \neq 0$ or $i(n) = 0$ with $g_{-2}^{(n)} \neq 0$, ~~$\hat{C}_1(\lambda, t)$ is~~ $\hat{C}_1(\cdot, t_n) \in N$ for $t_n \in (-\infty, \infty)$. Accordingly, from the result of clause (Bi), $\hat{C}_n(\cdot, t_n) \in P(n-1)$. ~~If $D_n(\lambda, t_n)$, $N_n(\lambda, t_n)$ were to possess a common polynomial factor~~ Cancellation of a possible polynomial factor from $D_n(\lambda, t_n)$ $N_n(\lambda, t_n)$ would reveal that $P[n-1]$ has a solution of such rational form as to render $P[n-1]$ degenerate. However, it is a consequence of part (A) that if $P[n]$ is nondegenerate, the same is true of $P[n-1]$. Thus, with the above assumptions concerning $i(n)$, $g_{-2}^{(n)}$ and t_n, $C_n(\cdot, t_n)$ is irreducible. The cases in which $i(n) = 0 \vee g_{-2}^{(n)} = 0$, or $t_n \in \infty$, may be dismissed by appeal to the problem $P[n-2]$. Naturally the property of irreducibility is ~~shared~~ also possessed by expressions of the form ( ). This property has implications concerning the uniqueness of representations of the form ( ). All rational function solutions to $P[n]$ either have a representation of the form ( ) ~~in which $g^{(m)}$ is a rational N-function; they have a representation of the form ( ) in which n is replaced by a larger integer~~ (the circumstances in which this occurs are dealt with below) or have a representation of the form ( ) in which $G^{(n+1)}$ is a rational N-function, and this case they have a representation of the form ( ) in which n is replaced by a larger integer, the further coefficients being subjected to restrictions similar to those imposed upon the existing coefficients in ( ). The extended representation cannot reduce to ( ) by cancellation of common factors in denominator and numerator. It may occur that for certain t_n, an extended form is ~~equivalent~~ equal to ( ) (for example ~~when $t_{n+1}=0$,~~ of t_n and $G_{-2}^{(n+1)}\lambda + G_{-1}^{(n)} - t_{n+1}^{-1}$ take equivalent values when $t_n=\infty$ and $t_{n+1}=0$) but in such a case a new function of the form ( ) is not produced. Thus if it may be shown~~, for example, that~~ ~~$G_n(\cdot, t_n) \in R(n)$~~ by use of the representation ( ) that ~~all~~ $G_n(\lambda, t_n)$ is a ~~are rational functions of for all t_n in a certain set are~~ rational function of a certain form, if and only if t_n belongs to a certain real set, no other extended or reduced version of $G_n(\lambda, t_n)$ is of this form.]

As a preliminary, it is remarked that in this case $G^{(n)}$ given by formula ( ) is a member of N, not only when $G^{(n+1)} \in N$ but when $G^{(n+1)} \in \overline{N}$, in particular when $G^{(n+1)}(\lambda) = t_n \in (-\infty, \infty]$. Thus, ~~from the proof of~~ as in the proof of part (A), it may be shown that $C_n(\lambda, t_n) \in PN(n)$.

To prove the results of part (C), first assume ~~assume first~~ that $i(1) \neq 0$ ~~$\Delta_1 \neq \infty$~~ and $\Delta \neq \lambda_k$ ~~$\lambda \neq \Delta_1$~~. Expression ( ) with $n=1$ then ~~represents~~ maps all points $t_1 \in (-\infty, \infty]$ into a circle $C_1(\Delta)$ in U. This circle contains all values of expression ( ) with $n=1$ as $G^{(2)}$ ranges over N, and since for every point of U a function $G^{(2)} \in N$ can be found which takes this value for the argument value Δ in question, every ~~D_1 is~~ point within $C_1^{(\Delta)}$ is the value of ~~some member of $PN(1)$.~~ Expression ( ), which ~~may also~~ be written as the quotient in which $D_n(\lambda, t_n)$ of formula ( ) is denominator and a similarly defined counterpart $N_n(\lambda, t_n)$ is numerator, ~~also~~ represents a line, a circle, or a point the boundary of ~~an inclusion domain~~ for the values of $PN(n)$. Since $PN(n)$ is more restrictive than $PN(1)$, $C^{(n)}$ either is a circle some member of $P(1)$. ~~(also, in the present case, every point of $\mathscr{B}_1(A)$ represents the value of some member of $P(1) \cap B$~~

Let the denominators and numerators of ~~successive~~ the convergents of order r of the terminating continued fraction ( ) be $D_r(A)$, $N_r(A)$ $(r = 0, \ldots, n-1)$ so that

$$C_n(A, t_n) = \frac{N_{n-1}(A)\, t_n + \xi_n A\, N_{n-1}(A) - \upsilon_n(A)\, N_{n-2}(A)}{D_{n-1}(A)\, t_n + \xi_n A\, D_{n-1}(A) - \upsilon_n(A)\, D_{n-2}(A)}$$

As t_n varies over the range $(-\infty, \infty]$, expression ( ) represents a line, a circle or a point, the boundary of an inclusion domain for the values of $P(n)$. Since $P(n)$ is more restrictive than $P(1)$, the representation is either that of a circle or a point, the latter occurring when or ~~reduces to a point, the latter occurring when~~

$$\cancel{D_{n-1}(\lambda)\{V_n(\lambda)N_{n-2}(\lambda) - \dot{\xi}_n\lambda N_{n-1}(\lambda)\} - N_{n-1}(\lambda)\{V_n(\lambda)D_{n-2}(\lambda) - \dot{\xi}_n D_{n-1}(\lambda)\}}$$

$$= V_n(\Lambda)\{D_{n-1}(\Lambda)N_{n-2}(\Lambda) - N_{n-1}(\Lambda)D_{n-2}(\Lambda)\}$$

~~The cross product Using an elementary result in continued fraction theory (see [ ]) the cross product enclosed in braces may be represented as a continued p~~

$$= \prod_{r=p}^{n} V_p(\lambda) = 0 \qquad (-1)^n ?$$

the transition from the first ~~second~~ to the second ~~third~~ line being secured by use of an elementary result in continued fraction theory (see [ ]). Since the last relationship is ~~the representation is that of~~ never true when $\Lambda \in ℍ \setminus \{\lambda_1, \dots \lambda_n\}$, $\mathscr{C}_n(\Lambda)$ is a circle. When $\dot{\xi}(1)=0$, $\mathscr{C}_2(\Lambda)$ is a circle, and the above proof can be started by considering it.

It follows from the representation ( ) ~~result~~ of clause (Bi) that the argument value of every element of $\mathbb{P}(n)'$ for the fixed value of Λ in question is either contained within $\mathscr{C}_n(\Lambda)$ or lies at some point upon this circle. Furthermore, every point of $\mathbb{D}_n^{(\omega)}(\Delta)$ is the value of an element of $PN(n)'$. The circle $\mathcal{C}_n^{(\nu)}(\Delta)$ of clause ( ) of Theorem — defines a ~~simila~~ ~~inclusion~~ domain with ~~the same~~ identical properties: the two circles $\mathcal{C}_n^{(\omega)}(\Delta)$ are ~~the identical~~, the same.

The results of ~~part (B)~~ clause (Bii) concern rational functions ~~(elements of $PN(n)$, $PN(n)'$)~~ obtained by inserting numerical values of t_n in expression ( ) and treating λ as a variable. Naturally these results have counter-parts obtained by giving λ a fixed value, as in ~~part (C)~~ clause (Ci), and treating t_n as a variable; they concern values, ~~of rational funct~~ ~~by~~ represented by points of $\mathcal{C}_n^{(\omega)}(\Delta)$, of rational functions, and are as ~~stated~~ occur in clauses (Cii, iii). ~~It suffices merely to remark that the points $\mathcal{W}_n^{(\omega)}(\Delta)(\omega=0,1)$ referred to are represent values of $C_n(\lambda, t_n^{(\omega)})$ $(\omega=0,1)$. The further results of clause (Ci) concerning the numerical values of elements of $PN(n)$, $PN(n)'$ follow directly from the representation ( ).~~

Two circles $\mathcal{C}^{(\nu)}(\Delta)$, $\mathcal{C}^{(\nu)}(\Delta)$ as considered in clause (Biii), ~~touch~~ have one or more points in common if and only if ~~the~~ real t_ν, t_ν can be found such that ~~the values of~~ two expressions of the form ( ) with n replaced by $\cancel{r, r+1}$ $r-1, r$ assume equal values or, equivalently, such that

$$\xi_r t_{r-1} = \xi_r V_{r-1}(\Delta) - \frac{V_{r-1}(\Delta)}{\xi_r \Delta + \xi_r}$$

~~equality the~~

This possibility arises only when $\xi_r = 0$; i.e. $i(r) = 0$;

~~limiting case $t_r = \infty$, $t_{r+1} = 0$~~ equality then occurs in the limiting case $t_{r-1} = \infty$, $t_r = 0$. ~~The result of clause (cii) follows from this remark.~~

~~and $V_{r-1}(\Delta) = g_{-2}^{(r-1)}\Delta + g_{-1}^{(r-1)}$~~

~~$V_{r-1}(\Delta)$ has the form $g_{-2}^{(r-1)}\Delta + g^{(r-1)}$~~

When $i(r) = 0$, $V_{r-1}(\Delta) = g_{-2}^{(r-1)}\Delta + g_{-1}^{(r-1)}$. If $g_{-2} \neq 0$, $g_{-2}^{(r-1)} \neq 0$ also for all $r-1$ concerned (see formula ( )). The above values of t_{r-1}, t_r are then the only two to induce equal values of expressions of the form ( ): $b^{(r-1)}(\Delta)$, $b^{(r)}(\Delta)$ touch, as described in clause (ciiib).

If $g_{-2} = 0$ and $i(r') \neq 0$ $(r' = 1, \ldots, r-1)$ then (again see formula ( )) $g_{-2}^{(r-1)} = 0$ and formula ( ) then becomes $t_{r-1} = g_{-1}^{(r-1)} - t_r^{-1}$. Now at least three distinct pairs of real numbers ~~&~~ t_{r-1}, t_r ~~can be found for which~~ induce equality between values of expressions of the form ( ): $\mathcal{C}^{(G \cup \{v\})}_{r-1}(\Delta)$ and $\mathcal{C}^{G}_{r}(\Lambda)$ are identical as described in clause (Ciiic). Subsequently, since $\mathcal{G}^{(r+1)}_{ki} \in N''$, $\delta^{(r)}_{-2i}$ ~~to~~ if it is calculated, is nonzero, and (yet again see formula ( )) all subsequent $\delta^{(r')}_{-2i}$ are nonzero. ~~The poss~~ Subsequent pairs of circles have no more than one point in common, as is also stated in clause (Ciiic)

The method of employing Schwarz' inequality in the proof of the above theorem is due to Nevanlinna, [ ]. In his work, allowing for a change of notation, the function $Z(\lambda)^{-1}F(\lambda)$ of formula ( ) is replaced by

$$Z(\lambda)^{-1}F(\lambda) - \{G^{(2)}(\lambda) - \gamma_1\}^{-1}\{G^{(2)}(\lambda) - \bar{\gamma}_1\}$$

where γ_1 is an arbitrarily chosen point of $\mathbb{U}$, and the recurrence relation for the functions $\{G^{(r)}\}$ to which use of this function leads is

$$\cancel{G^{(r)}(\lambda) = g^{(r)}_{k,0} + \mathrm{Im}(\lambda_k)^{-1}\mathrm{Im}(g^{(r)}_{k,0})(\lambda-\lambda_k)}$$

$$G^{(r)}(\lambda) = \mathrm{Im}(\lambda_k)^{-1}\Big\{\mathrm{Im}(c^{(r)}_{k,0})\lambda + \mathrm{Im}(\lambda_k \bar{c}^{(r)}_{k,0})$$

$$+ \frac{\mathrm{Im}(c^{(r)}_{k,0})(\lambda-\lambda_k)(\lambda-\bar{\lambda}_k)}{\mathrm{Im}(\gamma_r)^{-1}\mathrm{Im}(\bar{\gamma}_r\lambda_k) - \lambda - \mathrm{Im}(\gamma_r)^{-1}\mathrm{Im}(\lambda_k)G^{(r+1)}(\lambda)}$$

in place of ( ), where the $\{\gamma_r\}$ are arbitrarily chosen points in $\mathbb{U}$. This relationship expresses an N-function $G^{(r)}$ in terms of a parameter γ_r and another N-function $G^{(r+1)}(\lambda)$. Recursive use of it leads to a continued fraction expression for $G^{(1)} \in \mathbb{N}$ in terms of n parameters $\gamma_1, \ldots, \gamma_n$ and another N-function $G^{(n+1)}$. However, since $\mathrm{Im}(\lambda_k) < 0$, $G^{(r+1)} \in \mathbb{N}$ if and only if $\hat{G}^{(r+1)} \in \mathbb{N}$, where

$$\hat{G}^{(r+1)}(\lambda) - \mathrm{Im}(\gamma_r)^{-1} \mathrm{Im}(\gamma_r \lambda_k) - \mathrm{Im}(\gamma_r)^{-1} \mathrm{Im}(\lambda_k) G^{(r+1)}(\lambda)$$

~~& Im(λ_k) < 0~~.. The parameter γ_r in formula ( ) may be subsumed within ~~the~~ an N-function ~~as~~ $\hat{G}^{(r+1)}$; ~~γ_r drops out of~~ in this way γ_r drops out of formula ( ). Similarly the n parameters $\gamma_1, \ldots, \gamma_n$ drop out of the continued fraction representation of $G^{(1)}$ in terms of $G^{(n+1)}$. The simplified formulation of the theory can be reached from Nevanlinna's ~~formulation by~~ presentation by taking $\gamma_1 = \bar{\lambda}_k$ in formula ( ), which then becomes formula ( ).

Relationship ( ) and, when $g_{-1}^{(r)} \neq 0$, ( ) also, may be written in the extended forms

$$G^{(r)}(\lambda) = c_{k,0}^{(r)} + \frac{\operatorname{Im}(\lambda_k)^{-1} \operatorname{Im}(c_{k,0}^{(r)})(\lambda - \lambda_k)}{1 +} \; \frac{\lambda - \bar{\lambda}_k}{\bar{\lambda}_k - G^{(r+1)}(\lambda)}$$

$$G^{(r)}(\lambda) = g_{-2}^{(r)}\lambda + \frac{g_{-1}^{(r)}}{1 +} \; \frac{(g_{-1}^{(r)})^{-1}}{(-g_{-1}^{(r)})^{-1} + G^{(r+1)}(\lambda)}$$

These latter forms may be used, either consistently, or as ~~convenience~~ choice disposes, in conjunction with relationships (,) (use of relationship ( ) being forced when $g_{-1}^{(r)} = 0$) to form extended versions of the continued fraction expansion ( ) of $G^{(1)}(\lambda)$ in terms of $G^{(m+1)}(\lambda)$. If $i(r) = i(r+1) = \ldots = 0$ and relationship ( ) is used exclusively in succession, certain of the partial denominators in the resulting continued fraction have the form $(-g_{-1}^{(r+j-1)})^{-1} + g_{-2}^{(r+j)}\lambda$ with $j = 1, 2, \ldots$. Nevertheless it is possible so to modify the transformation leading to formula ( ) that intervening partial denominators have simpler form. Set

$$u_{r,j} = g_{-1}^{(r+j-1)} - \frac{1}{g_{-1}^{(r+j-2)} -} \cdots \frac{1}{g_{-1}^{(r+1)} -} \; \frac{1}{g_{-1}^{(r)}}$$

for $j=$ ~~, then~~

$$G^{(r)}(\lambda) = g^{(r)}_{-2}\lambda + \frac{u_{r,1}}{1+}\;\frac{u_{r,1}^{-1}}{g^{(r+1)}_{-2}\lambda+}\;\frac{u_{r,2}}{1+}\;\frac{u_{r,2}^{-1}}{g^{(r+2)}_{-2}\lambda+}\cdots\frac{u_{r,j}}{1+}\;\frac{u_{r,j}^{-1}}{(-u_{r,j})^{-1}+G^{(r+j)}(\lambda)}$$

for $j=1,2,\ldots$ The m^{th} convergent (beginning with ~~the first~~ $g^{(r)}_{-2}\lambda$ for $m=1$) of expansion ( ) possesses a series expansion in descending powers of λ which agrees with the series $-1+\nu=\overset{1}{\cancel{2}}$

$\sum_{\nu=-2}^{2j-3} g^{(r)}_{\nu}\lambda^{-\nu-1}$ as far as the term involving λ^{-m+2};

adopting terminology from similar transformations of power series into continued fractions (see [ ]), expansion ( ) corresponds to the given series. The above theory is, of course, formal: the corresponding expansion may not exist (e.g. when $g^{(r)}_{-1}=0$).

$$P^{(\omega)}_{\mu,r}(\lambda) = \sum_{\nu=0}^{d(\omega,\mu,r)} P^{(\omega)}_{\mu,r,\nu}\lambda^{\nu} \qquad P^{(0)}_{0,r,d(0,\mu,r)} = 1$$

$$d(\omega,\mu,r) = r - |\omega-\mu| \qquad r = 1,\dots,J\,;\ \omega,\mu = 0,1$$

$$d(\omega,0,r) = r - \omega - 1 \qquad r = J+1,\dots,n\,;\ \omega = 0,1$$

$K = 0$ if $g_{-2} > 0$, 1 if $g_{-2} = 0,\ g_{-1} \neq 0$, 2 if $g_{-2} = g_{-1} = 0$

$$d(0,1,r) = r - K \qquad r = J+1,\dots,n$$

if $g_{-2} > 0$ $\quad d(1,1,r) = r-1 \quad r = J+1,\dots,n$

~~if $g_{-2} \geq 0$ $g_{-1} \neq 0$ $\quad d(1,1,r) = r-1,\ r = J+1,\dots J+q+1$~~

~~$= r-2 \quad r = J+q+2,\dots,n$~~

~~if $g_{-2} = 0,\ g$~~

if $g_{-2} = 0$ $\quad d(1,1,r) = r-1$ for $r = J+1,\dots,J+q+1$

if $g_{-2} = 0,\ g_{-1} \neq 0$ $\quad d(1,1,r) = r-2$ for $r = J+q+2,\dots,n$

if $g_{-2} = g_{-1} = 0$ $\quad d(1,1,r) = r-3$ $\dots$

~~Let the problem $HPN[p,K,\{T\}:g;\lambda,G]$ be nondefective and ~~the sequence~~ $\{\Delta_r\}$ be a , $k(r)$? and let the integer sequence $\{\Delta_r\}$ generate a the ~~sequence of subordinate~~ problems $\{HPN(r)\}$ subordinate ~~with respect~~ to the nondefective problem $HPN[p,K,\{T\}:g,\lambda,G]$~~

Theorem: Let the ~~integer~~ sequence $\{k(r)\}$ induce ~~generate~~ the problems $\{HPN(r)\}$ subordinate to the nondefective problem $HPN[p,K,T:g,\lambda,G]$

A) Assume that $j(r) \neq 0$, $r = 1, \ldots, J$, $j(J+1) = 0$, $j(r) \neq 0$ $r = J+2, \ldots, J+q+1$ $(J, q > 0, J+q+1 \leq n)$ $j(J+q+2) = 0$

~~Set k(r)~~ ~~With~~

Set $K = 0$ if $g_{-2} > 0$ | 1 if $g_{-2} = 0, g_{-1} \neq 0$ | 2 if $g_{-2} = g_{-1} = 0$.

$$\cancel{P^{(\omega)}_{\mu,r}(\lambda) = \sum_{\nu \geq 0} P^{(\omega)}_{\mu,r,\nu} \lambda^\nu}$$

$$\cancel{W^{(\omega)}_r(\lambda) = P^{(\omega)}_{0,r}(\lambda)^{-1} P^{(\omega)}_{1,r}(\lambda)}$$

~~for $r = 1, \ldots, n$; $\omega = 0, 1$~~ let ~~$W^{(\omega)}_r = P(r) \cap R($~~

For $r = 1, \ldots, n$, $\omega = 0, 1$, let $W^{(\omega)}_r$ be the unique members of the following sets of functions: ~~for $\omega = 0, 1$; $r = 1, \ldots,$~~ J, $W^{(\omega)}_r = \{P(r) \cap R^{(\omega)}(r)\}$ for $(\omega = 0, 1$; $r = 1, \ldots, J)$. $W^{(0)}_r = \{P(r) \cap R^{(0)}(r)$ if $g_{-2} > 0$ | $R(r-1)$ if $g_{-2} = 0, g_{-1} \neq 0$ | $R^{(1)}(r-1)$ if $g_{-2} = g_{-1} = 0\}$ for $r = J+1, \ldots, n$;

~~$W^{(1)}_r = \{P(r)'' \cap R^{(0)}(r-1)\}$ for $r = J+1, \ldots, n$ if $g_{-2} > 0$ and for $r = J+1, \ldots, J+q+1$ if $g_{-2} = 0$, $W^{(1)}_r = \{P(r)'' \cap R(r-2)$ if $g_{-2} = 0$ $g_{-1} \neq 0$ | $R^{(1)}(r-2)$ if $g_{-2} = g_{-1} = 0\}$ for $r = J+q+2, \ldots, n$, so~~

$W^{(1)}_r = \{P(r)'' \cap R^{(1)}(r-1)\}$ for $r = J+1, \ldots, J+q+1$; $W^{(1)}_r = \{P(r)'' \cap R^{(0)}(r-1)$ ~~that~~ if $g_{-2} > 0$ | $R(r-2)$ if $g_{-2} = 0, g_{-1} \neq 0$ | $R^{(1)}(r-2)$ if $g_{-2} = g_{-1} = 0\}$ for $r = J+q+2, \ldots, n$. ~~so that thus~~

~~where~~ with

$$W^{(\omega)}_r(\lambda) = P^{(\omega)}_{0,r}(\lambda)^{-1} P^{(\omega)}_{1,r}(\lambda)$$

$$P^{(\omega)}_{\mu,r}(\lambda) = \cancel{P^{(\omega)}_{0,r}(\lambda)^{-1} P^{(\omega)}_{1,r}} \sum_{\nu=0}^{d(\omega,\mu,r)} P^{(\omega)}_{\mu,r,\nu} \lambda^\nu$$

for $(\omega = 0, 1$; $r = 1, \ldots, n)$ where the degrees $d(\omega, \mu, r)$ are as follows: $d(\omega, \mu, r) = r - |\omega - \mu|$ for $r = 1, \ldots, J$; $\omega, \mu = 0, 1$. ~~$d(\omega, 0, r) =$~~

~~replace $P(r)''$ by $P(r)^{(1)}$~~

~~$r-\omega-1$ for $r=J+1,\dots,n;\ \omega=0,1;\ d(0,1,r)=r-K$ for $r=J$~~

$d(\omega,0,r)=r-\omega-1$ for $\omega=0,1$, ~~$d(\omega,0,r)=r-\omega-1$~~ $d(0,1,r)=r-K$ and if $g-x>0$, $d(1,1,r)=r-1$ all for $r=J+1,\dots,n$.

If $g-x=0$, $d(1,1,r)=r-1$ for $r=J+1,\dots,J+q+1$. and $d(1,1,r)=r-K-1$ for $r=J+q+2,\dots,n$.

The polynomials $P^{(\omega)}_{\mu,r}$ may be constructed by means of the following algorithm. i) With $j(1)=k$, set

$$P^{(0)}_{0,1}(\lambda)=\operatorname{Im}(G_k)^{-1}\operatorname{Im}(\lambda_k),\quad P^{(0)}_{1,1}(\lambda)=\lambda+\operatorname{Im}(G_k)^{-1}\operatorname{Im}(\lambda_k G_k)$$

$$P^{(1)}_{0,1}(\lambda)=\lambda-\operatorname{Im}(G_k)^{-1}\operatorname{Im}(G_k\lambda_k),\quad P^{(1)}_{1,1}(\lambda)=-\operatorname{Im}(G_k)^{-1}\operatorname{Im}(\lambda_k)|G_k|^2$$

ii) For $r=2,\dots,J$ set

$$P^{(\omega)}_{\mu,r}(\lambda)=(\lambda+\theta^{(\omega)}_{1-\omega,r})P^{(\omega)}_{\mu,r-1}(\lambda)+\theta^{(\omega)}_{\omega,r}P^{(1-\omega)}_{\mu,r-1}(\lambda)$$

for $\omega=0,1;\ \mu=0,1$ where with $j(r)=k$, $\tau=T(r,k)$,

$$\cancel{\rho^{(\omega)}_{\psi,r,\tau}(\lambda)=\lambda^{\tau}\left[\lambda^{\psi}\{P^{(\omega)}_{0,r}(\lambda_k)G_k-P^{(\omega)}_{1,r}(\lambda_k)\}\right]_{\tau}}$$

$$\theta^{(\omega)}_{\psi,r}=\frac{\operatorname{Im}\{\bar\rho^{(\omega\psi)}_{\psi,r-1,\tau}(\lambda_k)\,\rho^{(\omega+\psi-\omega\psi)}_{1-\psi,r-1,\tau}(\lambda_k)\}}{\operatorname{Im}\{\rho^{(0)}_{0,r-1,\tau}(\lambda_k)\,\bar\rho^{(1)}_{0,r-1,\tau}(\lambda_k)\}}$$

for $\omega,\psi=0,1$.

iii) a) If $g_{-2} < 0$ set

$$\theta^{(0)}_{0,J+1} = \left(P^{(1)}_{0,J,J}\, g_{-2}\right)^{-1} \beta^{(0)}_{2,J+1}$$

$$\theta^{(0)}_{1,J+1} = \left(\beta^{(0)}_{2,J+1}\right)^{-1}\left\{\beta^{(0)}_{3,J+1} - \left(P^{(1)}_{0,J,J-1}\, g_{-2} + P^{(1)}_{0,J,J}\, g_{-1} - P^{(1)}_{1,J,J-1}\right)\theta^{(0)}_{0,J+1}\right\}$$

evaluate $P^{(0)}_{\mu,J+1}$ $(\mu=0,1)$ by use of recursion ( ), and set $P^{(1)}_{\mu,J+1}(\lambda) = P^{(0)}_{\mu,J}(\lambda)$ $(\mu=0,1)$

b) If $g_{-2} = 0$, $g_{-1} \neq 0$, set

$$P^{(0)}_{\mu,J+1}(\lambda) = g_{-1}\, P^{(0)}_{\mu,J}(\lambda) + P^{(1)}_{\mu,J}(\lambda)$$

and $P^{(1)}_{\mu,J+1}(\lambda) = P^{(1)}_{\mu,J}(\lambda)$ $(\mu=0,1)$

c) If $g_{-2} = g_{-1} = 0$ set $P^{(0)}_{\mu,J+1}(\lambda) = P^{(1)}_{\mu,J}(\lambda)$, $P^{(1)}_{\mu,J+1}(\lambda) = P^{(0)}_{\mu,J}(\lambda)$ $(\mu=0,1)$

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which makes use of differentiated residual functions evaluated at the point λ_k

$$\Leftarrow \cancel{\rho^{(\omega)}_{\gamma,r,\tau}(\lambda_k) = \Delta^{\tau}\left[\lambda_k^{\gamma}\left\{P^{(\omega)}_{0,r-1}(\lambda_k)\, g_k - P^{(\omega)}_{1,r-1}(\lambda_k)\right\}\right]}$$

and at the point at infinity (the range $0,\dots,d(\omega,1,r-1)$)

$$b^{(\omega)}_{\gamma,r} = \sum_{\nu=\max(0,\,r-\gamma-2p(r-1)+k)}^{r-\omega-2} P^{(\omega)}_{0,r-1,\nu}\, g_{2p(r-1)+2-r+\gamma-2} - P^{(\omega)}_{1,r-1,\,2p(r-1)-\gamma+1}$$

($P^{(\omega)}$ ... is taken to be zero if the last index ...)

~~iii)~~ For $r = \cancel{2}, J+2,\ldots,n$ a) make use of formulae (,) if $j(r) \neq 0$, while b) if $j(r)=0$ make use of formula ( ) ~~with~~ for $\omega = 0$ with

$$\theta^{(0)}_{\gamma,r} = \frac{b^{(\gamma)}_{0,r}\, b^{(0)}_{2,r} - b^{(\gamma)}_{1,r}\, b^{(0)}_{1,r}}{b^{(0)}_{0,r}\, b^{(1)}_{1,r} - b^{(0)}_{1,r}\, b^{(1)}_{0,r}}$$

for $\gamma = 0,1$ and set $P^{(1)}_{\mu,r}(\lambda) = P^{(1)}_{\mu,r-1}(\lambda)$ for $\mu = 0,1$.

For $r = 1,\ldots,J$ the polynomials $P^{(\omega)}_{\mu,r}$ are normalised by the condition that $P^{(\omega)}_{1-\omega,r,r} = 1$ $(\omega = 0,1)$. For $r = J+1,\ldots,n$ $\cancel{P^{(\omega)}_{0,r,d(0,0,r)} = 1}$ $P^{(0)}_{0,r,r-1} = 1$.

In the above, it is possible to take $q = 0$, in which case the descriptions of ~~the~~ $W^{(\omega)}_r$, $P^{(\omega)}_{\mu,r,d(\omega,\mu,r)}$ for $r = J+2,\ldots,J+q+1$ are ~~is~~ to be discounted. It is also possible to take $q = n-J-1$, in which case the above descriptions are to be discounted for $r = J+q+2,\ldots,n$. In both cases the ~~algorithm~~ implementation of the algorithm is as stated.

~~It is also possible to take $J=0$~~

It is also possible to take $J = n$, so that $p = 0$ in the generating problem. ~~The above~~ Now both sets of descriptions mentioned in the preceding paragraph and the normalising condition for $r = J+1,\ldots,n$ are ~~now~~ to be discounted. ~~For~~ Furthermore, stages (iii, iv) of the algorithm are never implemented.

Finally it is possible to take $J = 0$, so that $j(1) = 0$. Now the initial conditions of (i) are to be replaced by

$$P^{(0)}_{0,1}(\lambda) = 1 \qquad P^{(0)}_{1,1}(\lambda) = g_{-2}\lambda + g_{-1}$$

$$P^{(1)}_{0,1}(\lambda) = 0 \qquad P^{(1)}_{1,1}(\lambda) = 1$$

and stage (ii) ~~as described~~ alone is implemented. The descriptions of $W^{(\omega)}_r$, $P^{(\omega)}_{\mu,r}$, $d(\omega,\mu,r)$ and the normalising conditions for $r = 1, \ldots, J$ are to be discounted. If, in addition, $q = 0$, the descriptions of $W^{(\omega)}_r$, $P^{(\omega)}_{\mu,r}$, $d(\omega,\mu,r)$ for $r = J+2, \ldots, J+q+1$ are to be discounted, and if $q = n-1$ these descriptions are to be discounted for $r = J+q+2, \ldots, n$

B) ~~Let J and q be as described~~ ~~Assume that J, q > 0, J + q ≤ n and set~~

$$W_r(\omega) = \frac{(1-\omega)\, P^{(\omega)}_{1,r}(\lambda) + \omega\, P^{(1)}_{1,r}(\lambda)}{(1-\omega)\, P^{(\omega)}_{0,r}(\lambda) + \omega\, P^{(1)}_{0,r}(\lambda)}$$

$W_r(\omega, (-\infty,\infty] \setminus \{0,1\}) \equiv \{P(r) \cap R(r)\}$ for $r = 1, \dots, J$.

For further values of r, the set $\{P(r) \cap R(r)\}$ is to be replaced by others as follows: for $r = J+1, \dots, n$ by $\{P(r)' \cap R^{(\omega)}(r)$ if $g_{-2} \neq 0 \mid R(r-1)$ if $g_{-2} = 0, g_{-1} \neq 0\}$; if $g_{-2} = g_{-1} = 0$ by $P(r)' \cap R(r-1)$ for $r = J+1, \dots, J+q+1$ and by $P(r)' \cap R^{(1)}(r-1)$ for $r = J+q+2, \dots, n$.

Again it is possible to take $J = 0$ or $q = 0$ or $J+q+1 = n$ in the above, when, as at the conclusion of part (A), ~~certain~~ appropriate sets of the above results are to be discounted.

B) Assume that $J, q > 0$, $J+q < n$.

The rational function solutions to $\underset{\sim}{P}[r]$, $\underset{\sim}{P}[r]'$ may be represented in terms of the extremal solutions described above. ~~by setting~~ Set

C) For r fixed in the range $1 \leq r \leq n$ and $\underline{\Lambda} \in \mathbb{L} \setminus \{\lambda_1, \dots, \lambda_n\}$ fixed, expression ( ) with $\lambda = \underline{\Lambda}$ describes, as ω traverses

~~the real axis, a circle $\mathscr{C}_r(\Delta)$, the circle $\mathscr{C}_r(\Delta)$ of~~

~~part ( ) of Theorem and clause ( ) of Theorem~~

inclusion therein

~~with the properties described in the~~

the real axis, the circle $\mathscr{C}_r(\Delta)$ of clause ( ) of Theorem .. having the inclusion properties therein described. In terms of the values of the extremal polynomials $P^{(\omega)}_{\mu,r}$ $(\omega,\mu=0,1)$ the centre of $\mathscr{C}_r(\Delta)$ ~~may be expressed as~~ lies at the point

$$\{2\,\mathrm{Im}(\gamma\bar{\delta})\}^{-1}\{\mathrm{Im}(\alpha\bar{\delta}-\beta\bar{\gamma})+i\,\mathrm{Re}(\beta\bar{\gamma}-\alpha\bar{\delta})\}$$

and its radius is

$\frac{1}{2}$?

$$\{2|\mathrm{Im}(\gamma\bar{\delta})|\}^{-1}\left[\{\mathrm{Re}(\alpha\delta)+\mathrm{Im}(\beta\gamma)\}^2+\{\mathrm{Im}(\alpha\delta)-\mathrm{Re}(\beta\gamma)\}^2\right]$$

where $\alpha=P^{(0)}_{1,r}(\Delta)$, $\beta=P^{(1)}_{1,r}(\Delta)$, $\gamma=P^{(0)}_{0,r}(\Delta)$ and $\delta=P^{(1)}_{0,r}(\Delta)$.

Proof. It is first necessary to give a systematic account of the structure of the ~~algebraic~~ convergents $C_r(\lambda, t_r)$ defined by formula ( ) with n replaced by r. With the integers J, q defined as at the commencement of the theorem, ~~systematic~~ application of the results of clause (3ii) of Theorem ... leads to the following: (a) $C_r(\cdot,(-\infty,\infty)\setminus t_r^{(1)}) = \{P(r) \cap R(r)\}$ and $C_r(\cdot, t_r^{(\omega)}) = \{P(r) \cap R^{(\omega)}(r)\}$ $(\omega = 0,1)$ both for $r = 1,\ldots,J$; (b) $C_r(\cdot,(-\infty,\infty]\setminus t_r^{(1)}) = \{P(r)' \cap R^{(0)}(r)$ if $g_{-2}\neq 0 \mid R(r-1)$ if $g_{-2}=0$, $g_{-1}\neq 0\}$ for $r=J+1,\ldots,n$; if $g_{-2}=g_{-1}=0$, $C_r(\cdot,(-\infty,\infty)\setminus t_r^{(1)}) = \{P(r)'\cap R(r-1)\}$ for $r=J+1,\ldots,J+q+1$ and $C_r(\cdot,(-\infty,\infty)\setminus t_r^{(1)}) = P(r)'\cap R^{(1)}(r-1)$ for $r=J+q+2,\ldots,n$; (c) $C_r(\cdot,\infty) = \{P(r)\cap R^{(\omega)}(r)$ if $g_{-2}\neq 0 \mid R(r-1)$ if $g_{-2}=0$, $g_{-1}\neq 0 \mid R^{(1)}(r-1)$ if $g_{-2}=g_{-1}=0\}$ for $r=J+1,\ldots,n$; (d) $C_r(\cdot, t_r^{(1)}) = P(r)''\cap R^{(0)}(r-1)$ for $r=J+1,\ldots,n$ if $g_{-2}\neq 0$ and for $r=J+1,\ldots,J+q+1$ if $g_{-2}=0$; $C(\cdot,t_r^{(1)}) = \{P(r)''\cap R(r-2)$ if $g_{-2}=0$, $g_{-1}\neq 0 \mid R^{(1)}(r-2)$ if $g_{-2}=g_{-1}=0\}$ for $r=J+q+2,\ldots,n$.

Having accomplished this task, recursions involving the denominators and numerators of the $\{C_r(\lambda,t_r^{(\omega)})\}$ are derived. With $t_m\in(-\infty,\infty)$ let the denominators of the successive convergents

$$\frac{(1-\omega)a+\omega b}{(1-\omega)c+\omega d} = \frac{(1-\omega')a+\omega' x b}{(1-\omega')c+\omega' x d} \quad \omega' \rightarrow$$

$$\frac{\omega}{1-\omega} = \frac{\omega' x}{1-\omega'}, \quad \omega^{-1} = \frac{\omega' x + 1 - \omega'}{1-\omega'} \qquad \omega = \frac{1-\omega'}{\omega'(x-1)+1}$$

$$\frac{1}{\omega} = \frac{\omega' x}{1-\omega'} + 1 = \frac{\omega' x + 1 - \omega'}{1-\omega'}$$

must consider initial condition when $J=0$ $\gamma_{-2}=0 ; \gamma_{-1} \neq 0 ; \gamma_{2}=\gamma_{-1}=0$

or simply set $\langle r-2\rangle = \langle r-1\rangle = 0$ when $r=1$

coefficient of highest power of λ in $P_{0,r}^{(\omega)}$ is 1

when $J=0$ for all r if $P_{0,0}^{(\omega)}(\lambda) = 1$

~~the denominators of the~~ ~~With t_n real~~ $\in(-\infty,\infty)$ ~~for all $t_n \neq 0$ if $\frac{1}{t_n}=0$~~

~~Let successive convergents~~ of t_n terminating continued fraction ( ) be

$D_r(\lambda)$ $r=0,\ ,n-1$, $D_n(\lambda, t_n)$. Let $D_n(\lambda,\infty) = D_{n-1}(\lambda)$, ~~and $D_n(\lambda,0) = D_{n-2}(\lambda)$ if $\frac{1}{t_n}=0$~~, so that $D_n(\lambda, t_n)$ is now defined for all $t_n \in (-\infty,\infty]$.

Define $D_r(\lambda, t_r)$ for $1 \leq r \leq n$ by replacing n by r in the preceding, and define $N_r(\lambda, t_r)$ in a similar way. Let $\{t_r^{(\omega)}\}$ be the numbers ~~occurring~~ defined in clause ( ) of Theorem - . It will first be shown that finite real numbers $\alpha^{(\omega)}, \dots, \delta^{(\omega)}$ can be found such that ~~when $\Delta_r \neq \infty$~~

$$D_r(\lambda, t_r^{(\omega)}) = (\alpha_r^{(\omega)}\lambda + \beta_r^{(\omega)}) D_{r-1}(\lambda, t_{r-1}^{(\omega)}) + (\gamma_r^{(\omega)}\lambda + \delta_r^{(\omega)}) D_{r-1}(\lambda, t_{r-1})$$

a similar relationship ~~holding obtaining among~~ with the same numbers $\alpha_r^{(\omega)}, \dots, \delta_r^{(\omega)}$ connecting $N_r(\lambda, t_r^{(\omega)})$ $\dots$, $N(\lambda, t_{r-1}^{(1)})$, ~~with~~, all for $\omega=0,1$ when $\Delta_r \neq \infty$, and for $\omega=0$ when $\Delta_r=\infty$. The case in which $t_r^{(\omega)}=t_r^{(\omega)}=\infty$ is first considered: it must be shown that numbers $\alpha_r^{(\omega)}, \ldots, \delta_r^{(\omega)}$ can be found for which 51

in particular

$$D_{r-1}(\lambda) = (\alpha_r^{(\omega)}\lambda + \beta_r^{(\omega)}) D_{r-2}(\lambda) + (\gamma_r^{(\omega)}\lambda + \delta_r^{(\omega)}) D_{r-1}(\lambda, t_{r-1}^{(1)}).$$

Since

$$D_{r-1}(\lambda, t_{r-1}^{(1)}) = (\xi_{r-1}\lambda + t_{r-1}^{(1)}) D_{r-2}(\lambda) - V_{r-1}(\lambda) D_{r-3}(\lambda)$$

$$D_{r-1}(\lambda) = \{\xi_{r-1}\lambda + W_{r-1}(\lambda)\} D_{r-2}(\lambda) - V_{r-1}(\lambda) D_{r-3}(\lambda)$$

relationship ( ) may be written as

$$D_{r-1}(\lambda) = (\gamma_r^{(\omega)}\lambda + \delta_r^{(\omega)}) D_{r-1}(\lambda)$$

$$+ [\alpha_r^{(\omega)}\lambda + \beta_r^{(\omega)} + (\gamma_r^{(\omega)}\lambda + \delta_r^{(\omega)})\{t_{r-1}^{(1)} - W_{r-1}(\lambda)\}] D_{r-2}(\lambda)$$

Equating coefficients of $D_{r-1}(\lambda)$, $\gamma_r^{(\omega)} = 0$, $\delta_r^{(\omega)} = 1$. Both when $\Lambda_r = \lambda_k \neq \infty$, and when $\Lambda_r = \infty$, simple formulae for $\alpha_r^{(\omega)}, \beta_r^{(\omega)}$ may be obtained ~~$\alpha_r^{(\omega)} = \mathrm{Im}(\lambda_k)^{-1}\mathrm{Im}(c_{k,0}^{(r-1)})$, $\beta_r = \mathrm{Im}(\lambda_k)^{-1}\mathrm{Im}(\lambda_k \bar{c}_{k,0}^{(r-1)}) - t_{r-1}^{(1)}$. When~~ by equating the coefficient of $D_{r-2}(\lambda)$ in ~~when~~ formula ( ) to zero. In the latter ~~$\Lambda_r = \infty$~~ case, in particular $\alpha_r^{(\omega)} = \delta_{-2}^{(r-1)}$, $\beta_r^{(\omega)} = \delta_{-2}^{(r-1)} - t_{r-1}^{(1)}$. ~~If $r = J+1$, and only in the case, $\delta_{-2}^{(r-1)} = 0$, so that $\beta_r^{(\omega)} = 0$ also~~. A relationship ~~with~~ similar to ( ) in which $D_{r-1}(\lambda), \ldots, D_{r-1}(\lambda, t_{r-1}^{(1)})$ are replaced by $N_{r-1}(\lambda), \ldots, N_{r-1}(\lambda, t_{r-1}^{(1)})$ also holds with the stated values of $\alpha_r^{(\omega)}, \ldots, \delta_r^{(\omega)}$: ~~In short~~ The statement made concerning relationship ( ) and its counterpart has been verified for the case in which $\omega = 0$. It can be verified in a similar way when $\omega = 1$, $\Lambda_r \neq \infty$ but in this case it is necessary to appeal to the ~~limiting~~ results, ~~of clauses ( ) of Theorem~~

~~to establish that $\alpha_r^{(1)}=0$.~~
concerning the behaviour of the quotient $D_r(\lambda, t_r^{(1)})^{-1}\Delta_r(\lambda, t_r^{(1)})$, for large λ, and given in clause ( ) of Theorem —, to establish that $\alpha_r^{(1)}=0$. In short the preliminary result announced above has been obtained. As a point of detail it is mentioned that when $\Delta_r=\infty$, $\alpha_r^{(0)}=0$ if only when $g_{-2}^{(r-1)}=0$. This possibility, if it arises at all, ~~ex~~ does so only once. If $g_{-2}=0$ and $\Delta_r\neq\infty$ $(r=1,\dots,J)$ $\Delta_{J+1}=\infty$ then (see formula ( )) $g_{-2}^{(r-1)}=0$ for $r=2,\dots,J+1$. ~~In this case α~~ In this case $\alpha_{J+1}^{(0)}=0$. However, since $G^{(J+2)}\in N''$ in formula ( ), $g_{-2}^{(J+1)}$ if it is calculated (i.e. if $p>1$) is nonzero, ~~and~~ all subsequent corresponding coefficients $\{g_{-2}^{(r-1)}\}$ are nonzero, and ~~all resulting~~ this is true of all resulting coefficients $\alpha_r^{(0)}$ when $\Delta_r=\infty$. When this singular event, i.e. that $\alpha_r^{(0)}=0$ when $\Delta_r=\infty$, occurs, it can also occur that $\beta_r^{(0)}=0$ (this happens when $g_{-1}=0$). In this case the only nonzero member of the set $\alpha^{(0)},\dots,\delta^{(0)}$ ~~to~~ occurring in formula ( ) is $\delta_r^{(0)}=1$, and this relationship reduces to

$$D_r(\lambda, t_r^{(0)}) = D_{r-1}(\lambda, t_{r-1}^{(1)}).$$

With $\{\eta_r^{(0)}\}$ ~~$\{\eta_r'\}$ two~~ $(n=0,1)$ two ~~a~~ sequences of ~~finite~~ nonzero, finite, real normalising constants and $P^{(\omega)}_{0,r}(\lambda)=\eta^{(\omega)}_r D_r(\lambda, t^{(\omega)}_r)$ $P^{(\omega)}_{1,r}(\lambda)=\eta^{(\omega)}_r N_r(\lambda, t^{(\omega)}_r)$ ($\omega=0,1$; $r=1,\dots,n$). it follows from the above that finite real numbers $\alpha^{(\omega)}_r,\dots,\delta^{(\omega)}_r$ can be found such that

$$P^{(\omega)}_{\mu,r}(\lambda)=(\alpha^{(\omega)}_r\lambda+\beta^{(\omega)}_r)P^{(0)}_{\mu,r-1}(\lambda)+(\gamma^{(\omega)}_r\lambda+\delta^{(\omega)}_r)P^{(1)}_{\mu,r-1}$$

for $\mu=0,1$ and $\omega=0,1$ when $\Delta_r\neq\infty$, $\omega=0$ when $\Delta_r=\infty$

~~for $\omega=0$, $\mu=0,1$~~ When $\Delta_r\neq\infty$, $\gamma^{(0)}_r=0$ and when $\Delta_r=\infty$, $\alpha^{(0)}_r=0$ In the latter case it occurs once and only once, namely when in the notation of the theorem $r=J+1$ and $g_{-2}=0$, that $\gamma^{(1)}_r=0$ also, and then $\beta^{(1)}_r=0$ if $g_{-1}=0$. {Unless $\eta^{(\omega)}_r=1$ consistently, the nonzero values of $\alpha^{(\omega)}_r,\dots,\delta^{(\omega)}_r$ occurring in relationships (,) differ}; ~~those occurring in relationship ( ) may be obtained expressed in terms of λ_k, $\alpha^{(r-1)}_{k,0}$, $c^{(r)}_{k,1}$, $g^{(r-1)}_{-2}$, .. and the normalising~~ using ~~corresponding expressions for the $\alpha^{(\omega)}_r,\dots,\delta^{(\omega)}_r$ occurring in formula ( ) which, although not given may easily be derived, and the normalising constants $\eta^{(\omega)}_r$.~~ It has been established that the rational functions $P^{(\omega)}_{0,r}(\lambda)^{-1}P^{(\omega)}_{1,r}(\lambda)=C_r(\lambda, t^{(\omega)}_r)$ { are the only functions with their structure possessing certain interpolatory properties (see clause ( ) of theorem -) and certain limiting properties (see clause ( ) of Theorem)?. It ~~the~~ The existance of a relationship of the form ( ) connecting ~~has been established above that, apart from a~~ the polynomials $\{P^{(\omega)}_{\mu,r}\}$ has been established above; ~~common multiplicative factor, the numbers $\alpha_r^{(\omega)},\ldots,\delta_r^{(\omega)}$ occurring in relationship are ( ) are uniq~~ ~~the polynomials $\{P^{(\omega)}_{\mu,r}\}$ satisfy relationships are connected by formulae of the form ( ), and that~~, apart from a common multiplicative factor, the numbers $\alpha_r^{(\omega)},\ldots,\delta_r^{(\omega)}$ occurring in these relationships are unique. The values of these numbers may thus be determined by use of the interpolatory conditions satisfied by the $\{P^{(\omega)}_{\mu,r}\}$, ~~and~~ the common multiplicative factors being so chosen as to simplify the resulting formulae. This determination is carried out at each stage of the algorithm described in the theorem.

The ~~denominators~~ polynomials $\{P^{(\omega)}_{\mu,r}\}^{(\omega,\mu=0,1)}_{r}$, ~~in particular,~~ satisfy recursions of the form ( )$_4$ in which ~~The coefficients $\alpha_r,\dots,\delta_r$ are not these to be obtained by replacement of t_r by $t^{(\omega)}_r$ in the formulae given, since the $\{P^{(\omega)}_{0,r}\}$ and $D_r(\lambda, t^{(\omega)}_r)$ are subject to different normalisations;~~ ~~nevertheless,~~ linear functions of λ, ~~also appear.~~ ~~coefficients.~~ The coefficients in these linear functions may be determined ~~from~~ by consideration of the algebraic conditions which ~~which the functions $\{W^{(\omega)}_r\}$ are known to possess~~ ~~uniq,~~ uniquely characterise the functions $\{W^{(\omega)}_r\}$.

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To begin with, assuming that $J>0$, the quotients $P^{(\omega)}_{0,1}(\lambda)^{-1}P^{(\omega)}_{1,1}(\lambda)$ formed from the polynomials ( ) are equivalent to $C(\lambda, t^{(\omega)}_1)^4$ $(\omega=0,1)$. Assume that for some r in the range $1<r\le J$, ~~quotients~~ polynomials $P^{(\omega)}_{\mu,r-1}$ $(\omega,\mu=0,1)$ ~~with the requ. of the requisite forms ( ) and~~ such that $P^{(\omega)}_{0,r-1}(\lambda)^{-1}P^{(\omega)}_{1,r-1}(\lambda)$ is ~~an~~ the unique element of $PN(r-1)\cap R^{(\omega)}(r-1)$ $(\omega=0,1)$ ~~have been obtained. From the above it is known that real $\alpha^{(\omega)},\dots,\delta^{(\omega)}$ exist such that with~~

$$\cancel{P^{(\omega)}_{\mu,r}(\lambda) = (\alpha^{(\omega)}\lambda + \beta^{(\omega)}) P^{(0)}_{\mu,r-1}(\lambda) + (\gamma^{(\omega)}\lambda + \delta^{(\omega)}) P^{(1)}_{\mu,r-1}(\lambda)}$$

~~for $\mu = 0,1$, $P^{(\omega)}_{0,r}(\lambda)^{-1} P^{(\omega)}_{0,r}(\lambda)$ is the unique element of $PN(r) \cap R^{(\omega)}(r)$.~~

and with $P^{(\omega)}_{1-\omega,r-1,r-1} = 1$ $(\omega = 0,1)$
have been obtained. They satisfy the conditions

$$\Delta^{z}\{P^{(\omega)}_{0,r-1}(\lambda_j)^{-1} P^{(\omega)}_{1,r-1}(\lambda_j)\} = \Delta^{z} G_j \quad \cancel{(z=0,\dots,T(r-1,j)-1)}$$

for $z = 0, \dots, T(r-1,j)-1$
and in consequence the conditions

$$\Delta^{z}\{P^{(\omega)}_{1,r-1}(\lambda_j)\} = \Delta^{z}\{P^{(\omega)}_{0,r-1}(\lambda_j) G_j\}$$

for $z = 0, \dots, T(r-1,j)-1$ for every λ_j that features in the data specifications to $PN(r-1)$, and $\omega = 0,1$. From the above it is known that real $\alpha^{(\omega)}, \dots, \delta^{(\omega)}$ exist such that with $P^{(\omega)}_{\mu,r}$ defined by formula ( ),

$$\cancel{P^{(\omega)}_{\mu,r}(\lambda) = (\alpha^{(\omega)}\lambda + \beta^{(\omega)}) P^{(\omega)}_{\mu,r-1}(\lambda) + (\gamma^{(\omega)}\lambda + \delta^{(\omega)}) P^{(1)}_{\mu,r-1}}$$

for $\mu = 0,1$, $P^{(\omega)}_{0,r}(\lambda)^{-1} P^{(\omega)}_{0,r}(\lambda)$ is the unique element of $PN(r) \cap R^{(\omega)}(r)$. It is easily verified that for any complex $\alpha^{(\omega)}, \dots, \delta^{(\omega)}$ not all zero, the polynomials $P^{(\omega)}_{\mu,r}$ $(\mu = 0,1)$ satisfy relationship ( ) with $r-1$ replaced by r, for the g^{z}_{j} and λ_j given in connection with that relationship. Let $\Delta^{j(r)=k}_{\lambda_k}$. Since

$T(r,k) = T(r-1,k)+1$, it is required that ~~the~~ $P^{(\omega)}_{\mu,r}$ $(\mu=0,1)$ should satisfy one further relationship of the form ( ). Using formula ( ), the additional condition may be written as

$$\lambda^2\Big[(\alpha^{(\omega)}\lambda+\beta^{(\omega)})\{P^{(0)}_{0,r-1}(\lambda_k)G_k - P^{(0)}_{1,r-1}(\lambda_k)\} + (\gamma^{(\omega)}\lambda+\delta^{(\omega)})\{P^{(1)}_{0,r-1}(\lambda_k)G_k - P^{(1)}_{1,r-1}(\lambda_k)\}\Big] = 0$$

~~with $z = T(r,k)$.~~

The polynomials $\{P^{(\omega)}_{\mu,r}\}$ with $\omega=0$ are now considered. As was shown above, $\gamma^{(0)}=0$ ~~(this is also evident from the fact that~~ ~~The degree of $P^{(1)}_{1,r-1}$ is r, while~~ ~~formula ( ), since $P^{(0)}_{0,r}$ is~~ ~~The degree of both $P^{(0)}_{1,r}$ and $P^{(1)}_{1,r-1}$ are both of degree r). Hence, from formula ( ), $\gamma^{(0)}=0$~~. Also the coefficients ~~of both λ^r in $P^{(0)}_{0,r}$ and~~ of λ^{r-1} in $P^{(0)}_{0,r-1}$ is unity and ~~are both unity. Since $P^{(1)}_{0,r-1}$ is only of degree $r-2$,~~ Hence taking $\alpha^{(0)}=1$, it may be arranged that the coefficient of λ^r in $P^{(0)}_{0,r}$ is ~~it follows, again from formula ( ), that $\alpha^{(0)}=1$. The~~ unity. The ~~two~~ values obtained for $\alpha^{(0)}, \gamma^{(0)}$ may be set in formula ( ); ~~and~~ two conditions, obtained by taking real and imaginary parts, remain for the determination of $\beta^{(0)}, \delta^{(0)}$. With $\beta^{(0)} =$ $\quad$ $\delta^{(0)} =$ $\quad$, these two numbers with $z = T(r,k)$. The polynomials $\{P^{(\omega)}_{\mu,r}\}$ with $\omega = 0$ are now considered. As was shown above, $\gamma^{(\omega)} = 0$. $\alpha^{(\omega)}$ is nonzero, and, since equation ( ) is homogeneous in $\alpha^{(\omega)}, \ldots, \delta^{(\omega)}$, may be set equal to unity. Condition ( ) reduces to

$$\rho^{(0)}_{0,r,z}(\lambda_k)\beta^{(0)} + \rho^{(1)}_{0,r,z}(\lambda_k)\delta^{(0)} + \rho^{(0)}_{1,r,z}(\lambda_k) = 0$$

where

$$\rho^{(\omega)}_{\gamma,r,z}(\lambda_k) = \Delta^{z}[\lambda^{\gamma}\{P^{(\omega)}_{0,r}(\lambda_k)G_k - P^{(\omega)}_{1,r}(\lambda_k)\}]$$

or, equivalently, $\rho^{(\omega)}_{\gamma,r,z}(\lambda_k)$ is given by formula ( ). Two linear algebraic equations involving $\beta^{(0)}, \delta^{(0)}$ are obtained from equation ( ) by taking real and imaginary parts. With $\beta^{(0)} = \Theta^{(0)}_{1,r}$, $\delta^{(0)} = \Theta^{(0)}_{0,r}$, these two numbers are as given by formula ( ). Formula ( ) with $\omega = 0$ is formula ( ) ~~rewritten in another~~ with $\alpha^{(0)} = 1$, $\gamma^{(0)} = 1$ and $\beta^{(0)}$, $\delta^{(0)}$ ~~rewritten~~ expressed in another notation. Since the coefficient of λ^{r-1} in $P^{(0)}_{1,r-1}$ is unity and $P^{(1)}_{0,r-1}$ is only of degree $r-2$, the coefficient of λ^{r} in $P^{(0)}_{0,r}$ is unity. The determination of the polynomials $P^{(1)}_{\mu,r}$ ($\mu = 0,1$) is established in the same way. Stages (i, ii) of the algorithm have now been justified.

the transition point, indicated by the ~~first~~ least value of r for which $j(r)=0$, must now be dealt with. It is required that the series expansion of $P^{(0)}_{0,r}(\lambda)^{-1}P^{(0)}_{1,r}(\lambda)$, where $P^{(0)}_{\mu,r}$ $(\mu=0,1)$ ~~is~~ are given by formula ( ), in descending powers of λ should be of the form ~~begin with terms of the form~~ $g_{-2}\lambda+g_{-1}\ldots$. Alternatively, since $r=J+1$,

$$\{(\alpha^{(0)}\lambda+\beta^{(0)})P^{(0)}_{0,J}(\lambda)+(\gamma^{(0)}\lambda+\delta^{(0)})P^{(1)}_{0,J}(\lambda)\}\{g_{-2}\lambda+g_{-1}+O(\lambda^{-1})\}$$

$$=(\alpha^{(0)}\lambda+\beta^{(0)})P^{(0)}_{1,J}(\lambda)+(\gamma^{(0)}\lambda+\delta^{(0)})P^{(1)}_{1,J}(\lambda)$$

~~where $O(\lambda^{-1})$ represents a rational function with the indicated behaviour for large values of λ, or~~

where $O(\lambda^{-1})$ represents the sum of ~~a series~~ series expansion in inverse powers of λ, convergent for sufficiently large values of λ, which begins with a term involving $g\lambda^{-1}$. When $g_{-2}\neq 0$, It is known from the above that $\gamma^{(0)}=0$ and $\alpha^{(0)}$, known to be nonzero, may be set equal to unity. Equating coefficients of $\lambda^{J}, \lambda^{J-1}$ ~~to zero~~ in relationship ( ), two equations for $\beta^{(0)}, \delta^{(0)}$ are obtained. ~~Again~~ With $\beta^{(0)}=\theta^{(0)}_{1,J+1}$, $\delta^{(0)}=\theta^{(0)}_{0,J+1}$ their solutions ~~they~~ are as given by formulae (,). When $g_{-2}=0$, $\alpha^{(0)}=\gamma^{(0)}=1$. Equating coefficients of λ^{J-1} in relationship ( ), and using the normalisations $P^{(0)}_{1,J,J} = P^{(1)}_{0,J,J} = 1$, it is found that $\beta^{(0)} = g_{-1}\delta^{(0)}$. Setting $\delta^{(0)} = 1$, relationship ( ) is obtained. The dispositions concerning $P^{(1)}_{\mu,J+1}$ $(\mu=0,1)$ in stages (iiia, b) ~~and the special result of used in step (iiic) follow from the~~ are consequences of the relationships ~~unique characterisations of the functions $W^{(1)}_{J+1} = W^{(0)}_{J}$~~ $C_{J+1}(\lambda, t^{(1)}_{J+1}) = C(\lambda, t^{(0)}_{J})$ in the first case and ~~$W^{(1)}_{J+1} = W_{J}$~~ ${}^{(1)}C_{J+1}(\lambda, t^{(1)}_{J+1}) = {}^{(0)}C_{J}(\lambda, t^{(0)}_{J})$ in the second, ~~established above~~. The special result used in step (iiic) ~~has also been~~ is justified ~~in the same way,~~ above.

After the transition point has been passed, $p(r-1) \geq 1$ and the polynomials $P^{(0)}_{\mu,r-1}$ $(\mu=0,1)$ available for transformation at each stage satisfy a relationship of the form

$$P^{(0)}_{0,r-1}(\lambda)^{-1} P^{(0)}_{1,r-1}(\lambda) - \sum_{\nu=-2}^{2p(r-1)-3}{}' g_{\nu}\lambda^{-\nu-1} = O(\lambda^{-2p(r-1)+1})$$

where the symbol O has the meaning explained above, or alternatively, with the same convention

$$P^{(0)}_{0,r-1}(\lambda) \sum_{\nu=-2}^{2p(r-1)-3}{}' g_{\nu}\lambda^{-\nu-1} - P^{(0)}_{1,r-1}(\lambda) = O(\lambda^{r-2p(r-1)})$$

Similarly

$$P^{(1)}_{0,r-1}(\lambda)\sum_{\nu=-2}^{2p(r-1)-3} g_\nu \lambda^{-\nu-1} - P^{(1)}_{1,r-1}(\lambda) = O(\lambda^{r-2p(r-1)+1})$$

~~the term on the right hand side being replaced, when~~ ~~$p(r-1)=1$, by $O(\lambda^{r-1})$ when $p(r-1)=1$~~. The possibility that $j(r) = k \neq 0$, ~~is no~~ so that $p(r)=p(r-1)$, is now considered. The polynomials $P^{(c)}_{\mu,r}$ $(\mu=0,1)$ are to be constructed by use of formula ( ) with $\omega=0$ and $\gamma^{(c)}=0$. With this value of $\gamma^{(c)}$ ~~, as is easily verified,~~ the and finite $\alpha^{(c)}, \beta^{(c)}, \delta^{(c)}$ the polynomials $P^{(c)}_{\mu,r}$ satisfy relationship ( ) with $r-1$ replaced by r. ~~$\alpha^{(c)}$ can be taken to be unity~~ The subsequent determination of $\alpha^{(c)}, \beta^{(c)}, \delta^{(c)}$ is as described above with regard to those values of r in the range $2 \leq r \leq J$. ~~step (iva) has been validated~~. The construction of the polynomials $P^{(1)}_{\mu,r}$ $(\mu=0,1)$ is also as before: step (iva) has been validated.

When $j(r)=0$, it is required of the polynomials $P^{(c)}_{\mu,r}$ $(\mu=0,1)$ that they should satisfy a relationship of the form ( ) with $r-1$ replaced consistently by r, where $p(r)= p(r-1)+1$. It is known that $\gamma^{(c)}=0$, and $\alpha^{(c)}$ may be taken to be unity, so that the new condition may be written as

$$(\lambda+\beta^{(0)})\left\{P^{(0)}_{0,r-1}(\lambda)\sum_{\nu=-2}^{2p(r-1)-1} g_\nu\lambda^{-\nu-1} - P^{(0)}_{1,r-1}(\lambda)\right\}$$
$$+\delta^{(0)}\left\{P^{(1)}_{0,r-1}(\lambda)\sum_{\nu=-2}^{2p(r-1)-1} g_\nu\lambda^{-\nu-1} - P^{(1)}_{1,r-1}(\lambda)\right\} = O(\lambda^{r-2p(r-1)-1}),$$

As can be verified, relationships (,) ~~are~~ still hold if the upper limits of summation, $2p(r-1)-3$, are replaced by $2p(r-1)-1$. Thus relationship ( ) imposes two constraints upon $\beta^{(0)}, \delta^{(0)}$ obtained by equating to zero the ~~sums of the~~ coefficients of $\lambda^{r-2p(r-1)-3}, \lambda^{r-2p(r-1)-2}$ in the composite expression upon the left hand side. ~~The two~~ ~~res~~ With $\beta^{(0)}=\theta_{1,r}$, $\delta^{(0)}=\theta^{(0)}_{0,r}$ the solutions to the two resulting linear equations are as given by formula ( ). It is already known that the quotient $P^{(0)}_{0,r}(\lambda)^{-1}P^{(0)}_{1,r}(\lambda)$ satisfies the interpolation conditions at the points λ_j occurring in the data specifications to $P(r-1)$, which are the same as those for $P(r)$; but this fact can easily be confirmed by use of formulae of the form (,). The allocations of $P^{(1)}_{\mu,r}$ $(\mu=0,1)$ in stage (iv b) are consequences of the relationship $C_r(\lambda, t^{(1)}_r) = C_{r-1}(\lambda, t^{(0)}_{r-1})$. The justification of stage (iv) has now been completed.

Concerning the concluding statements of ~~the theorem~~ part (A), it suffices to remark that when $j(1)=0$, the transition point can be dealt with immediately by appropriately assigning the initial conditions. The normalisations $P^{(i)}_{0,r,r-1}=1$ are then preserved.

As is easily verified, the denominator polynomials $D_r(\lambda,t)$, $N_r(\lambda,t)$ recurring ~~etc the~~ in the early stages of the above proof satisfy the relationship

$$D_r(\lambda,t)=D_r(\lambda,t_r^{(1)})+(t-t_r^{(1)})D_r(\lambda,t_r^{(2)})$$

and ~~the numerator polynomial $N_r(\lambda,t)$ satisfies~~ and one similar to it, respectively. Accordingly $C_r(\lambda,t)$ is expressible as a fractional linear function of t with coefficients formed from $D_r(\lambda,t_r^{(1)}),\ldots,N_r(\lambda,t_r^{(2)})$. Introducing suitable normalising constants, and ~~an appropriate~~ ~~for~~ a new variable ω expressible as an appropriate fractional linear function with real coefficients of t, $C_r(\lambda,t)=W_r(\lambda,\omega)$ where W_r ~~(λ,ω) has the~~ is given by formula ( ). The results of part (B) are restatements of certain of those concerning G_r given at the commencement of the above proof.

~~If the function $(\alpha t'+\beta)(\gamma t'+\delta)^{-1}$ maps the real t'-axis into a circle in the complex plane, the centre and radius of this circle are as given in~~

It is known from clause ( ) of Theorem .. that for the fixed value of Λ of part C, the values of $G_r(\lambda, t)$ lies on the circle $\mathcal{C}_r^{\epsilon}(\Lambda)$ of that clause ~~as to~~ when $t \in (-\infty, \infty]$. Accordingly, ~~the values of $W_r(\lambda, \omega)$~~ since ω is a fractional linear function of t with real coefficients, the values of $W_r(\lambda, \omega)$ also lie on $\mathcal{C}_r(\Lambda)$ when $\omega \in (-\infty, \infty]$. The centre and radius of the circle in the complex plane into which the function $(\alpha t'+\beta)(\gamma t'+\delta)^{-1}$ maps the real t'-axis are as stated ~~given~~ in part (C).

In his treatment of the simple Pick-Nevanlinna problem (for which, in the notation of the above theorem, $\{[\;\;]\}$ $J = n$, ~~pass~~ ~~...~~ and $T(k) = 1$ $(k = 1, \dots, n)$) Pick ~~made the suggestion~~ ~~that~~ [ ] suggested a recursive method of solution ~~that a recursive solution is possible~~ in which two distinct members, ~~of $R(\lambda)$ ... $P(r) \cap R(r)$~~ $D'_{r-1}(\lambda)^{-1} N'_{r-1}(\lambda)$, $D''_{r-1}(\lambda)^{-1} N''_{r-1}(\lambda)$ say, of $P(r-1) \cap \{R(r-1) \cup R^{(\omega)}(r-1)\ (\omega = 0, 1)\}$ are chosen, and a system of elements of $P(r) \cap \{R(r) \cup R^{(\omega)}(r)\ (\omega = 0, 1)\}$ is ~~determined~~ obtained by imposing the condition that the quotient

$$\frac{\Delta(\lambda) N'_{r-1}(\lambda) + N''_{r-1}(\lambda)}{\Delta(\lambda) D'_{r-1}(\lambda) + D''_{r-1}(\lambda)}$$

~~where $\Delta(\lambda) = (\alpha\lambda + \beta)(\gamma\lambda + \delta)^{-1}$ is~~

~~where the coefficients $\Delta(\lambda) = (\alpha\lambda + \beta)$~~

should satisfy the additional interpolation condition which features in $P[r]$ but not in $P[r-1]$, where $\Delta(\lambda) = (\alpha\lambda + \beta)(\gamma\lambda + \delta)^{-1}$ is a fractional linear function with real coefficients $\alpha, \dots, \delta$. The additional condition, involving

complex numbers, imposes two constraints upon the $\alpha, \ldots, \delta$;
Δ is homogeneous in these constants, and hence expression ( ) represents a one real parameter family of solutions to $P[r]$s. ~~Except for the two isolated members of $R^{(\omega)}(r)$ ($\omega = 0, 1$) these solutions are members of $R(r)$~~. Two members of this family may be ~~select~~ chosen as ~~the~~ a basis for the generation of a system of solutions to $P[r+1]$, and the process continued.

~~In~~ It is shown in the above theorem ~~it is shown~~ that Pick's method can be extended to the treatment of derivative values in the Pick-Nevanlinna problem. In the algorithm described in the theorem, the two chosen members of the family ~~of rational functions~~ are the isolated members of $R^{(\omega)}(r)$ ($\omega = 0, 1$) at each stage, and only two members of the subsequent family, namely elements of $R^{(\omega)}(r)$ ($\omega = 0, 1$) are constructed. ~~It then happens that in Δ~~ It is also shown that this method of treatment may be extended to the point at infinity.

If the data specifications for the Hamburger – Pick – Nevanlinna problem considered concern ~~only~~ the point at infinity alone, the substitutions $P_{\mu,r}^{(1)}(\lambda) = P_{\mu,r-1}^{(0)}(\lambda)$ $(\mu = 0,1)$ are made consistently. In this case recursion ( ) becomes

$$P_{\mu,r}^{(0)}(\lambda) = (\lambda + \theta_{1,r}^{(0)}) P_{\mu,r-1}^{(0)}(\lambda) + \theta_{0,r}^{(0)} P_{\mu,r-2}^{(0)}(\lambda)$$

with $\mu = 0,1$, and formulae (,) alone ~~are~~ need for the construction of the numbers $\theta_{\gamma,r}^{(0)}$ $(\gamma = 0,1)$. Formula ( ) is the familiar three term recurrence relationships involving denominators and numerators of ~~the~~ ~~associated~~ continued fraction, and formulae (,) are (extensions of ~~equivalent to~~) those of an orthogonalisation process due to Tschebyscheff [9] for the construction of the continued fraction associated with the series $\sum_{r=0}^{\infty} g_r \lambda^{-r-1}$.

The above theorem brings together the methods of ~~Pick and~~ Tschebyscheff and Pick.