Continued fraction transformations of the Euler-Maclaurin series

by

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Abstract

Results concerning the convergence of forward diagonal sequences of quotients in the Padé table are given. In particular, it is shown that if (*) $f_\nu = \int_0^\infty t^\nu d\sigma(t)$ ($\nu = 0, 1, \ldots$), σ being a bounded nondecreasing real valued function such that all moments (*) exist, and (**) $f_\nu = O\{(\chi\nu)!\,\xi^\nu\}$ ($0 < \chi \leq 2$, $0 < \xi < \infty$) then all forward diagonal sequences of Padé quotients derived from the power series $\sum_{\nu=0}^{\infty} f_\nu z^\nu$ converge uniformly over any bounded region in the z-plane not containing any point of the nonnegative real axis to (***) $f(z) = \int_0^\infty (1 - zt)^{-1} d\sigma(t)$, $f(z)$ being the le Roy or (B, χ) sum of the given series for all finite z in the sector $\frac{1}{2}\chi\pi \leq \arg(z) \leq \frac{1}{2}(4-\chi)\pi$. This result extends to all forward diagonal sequences in the Padé table a result given by F. Bernstein (Jahresber. Deutsch. Math. Verein., **28** (1919) 50–63; **29** (1920) 94) concerning one of them, and also extends a result given by Wall (Trans. Amer.

Math. Soc. $\underset{\sim}{34}$ (1932) 409–416) for the case in which $\chi=1$. It is also shown that if the $\{f_\nu\}$ and $f(z)$ are given by (*) and (***) with the lower limit of integration replaced by $-\infty$, and $f_\nu = O(\nu!\,\xi^\nu)$ $(0<\xi<\infty)$, then uniform convergence to $f(z)$ as described above holds over any bounded region not containing any point of the real axis, $f(z)$ now being the Borel sum of the given series for finite pure imaginary values of z. This result extends to all forward diagonal sequences in the Padé table a result given by Hamburger (Math. Ann., $\underset{\sim}{81}$ (1920) 31–45) concerning one of them, and also extends a result stated by Wall (loc. cit.) concerning the diagonal sequences of the Padé table coinciding with and lying above the ~~principal~~ sub-principal diagonal; it also extends a further result of Wall (loc. cit.) holding for the case in which $f_\nu = O(\xi^\nu)$ $(0<\xi<\infty)$.

The above results are applied to the delayed Euler-Maclaurin series $\sum_{\nu=0}^{\infty} b_{J+\nu} \mathcal{D}^{2J+2\nu+1} \Psi(\mu) k^{2\nu}$ ($J \geq 0$; $\mathcal{D}$ being $d/d\mu$) regarded as a ~~power~~ series expansion in ascending powers of k^2. It is shown that if $\Psi(\mu) = \Psi_1(\mu) + \Psi_2(\mu)$, where ~~$[\hat{\mu}, \tilde{\mu}] \subseteq [-\infty, \infty]$~~ an interval $[\hat{\mu}, \tilde{\mu}] \subseteq [-\infty, \infty]$ exists such that $(-\mathcal{D})^{\nu} \Psi_1(\mu) \leq 0$ for all $\mu \in (\hat{\mu}, \infty)$ and $\mathcal{D}^{\nu} \Psi_2(\mu) \geq 0$ for all $\mu \in (-\infty, \tilde{\mu})$, both for $\nu = 2J+1, 2J+2, \ldots$, then the coefficients $\{(-1)^{J+\nu} b_{J+\nu} \mathcal{D}^{2J+2\nu+1} \Psi(\mu)\}$ may be exhibited in the form given by (*) for $\{f_\nu\}$, and these coefficients also satisfy an order relationship of the form (**) with $\chi = 2$. Convergence results concerning Padé quotients obtained from the delayed Euler-Maclaurin series are now derived from the first result outlined above; the function to which convergence holds, is, in particular, the (B^2) sum over the finite real axis of the series in question (the latter result is an extension of a result due to Hardy (Divergent series, Oxford (1949) § 13.16) in which the component of the form Φ_2 above is missing from Φ. Similar convergence results for the intercalated delayed Euler-~~Maclaurin~~ Maclaurin series (obtained by inserting single zeros between the successive terms of the delayed Euler-Maclaurin series) are derived from the second of the general convergence results stated ~~above~~ above, the function to which convergence now holds being the Borel sum over the finite real k-axis of the series in question (the latter result is also an extension in a manner similar to that described above of a further result due to Hardy (loc. cit.)).

The preceding results are illustrated by application to Stirling's asymptotic expansion of the logarithm of the gamma function, and to the asymptotic expansion of the generalised Riemann zeta-function.

1. Introduction and summary

Our main aim is to give results on the continued fraction transformation of the Euler-Maclaurin series which supplement work by Hardy on linear transformations of this series. En route to our main results we extend to general forward diagonal sequences of Padé quotients results given by F. Bernstein and Hamburger for convergents of continued fractions associated with Stieltjes and Hamburger series; we also extend results given by Wall concerning such sequences of quotients. As corollaries to our main results, we describe properties of the Padé tables derived from asymptotic expansions of the logarithms of the gamma function and of the generalised Riemann zeta-function.

The special notations used in this paper are explained at appropriate points in the text.

2. Continued fraction transformations of series

In this paper the index of single summation is always ν; if the upper limit is infinity it is omitted from the summation sign; if the lower limit is also zero it is omitted: $\sum_1^n a_\nu$, $\sum_1 a_\nu$ and $\sum a_\nu$ denote $\sum_{\nu=1}^{n} a_\nu$, $\sum_{\nu=1}^{\infty} a_\nu$ and $\sum_{\nu=0}^{\infty} a_\nu$ respectively.

The approximating fraction (Naherungsbruch) [,] or Padé quotient ([ ], [ ]Ch. , [ ]Ch.) $P_{i,j}(z)$ ($i, j \geq 0$ being fixed finite integers) derived from the series

$$(1) \qquad \sum f_\nu z^\nu$$

with real or complex number coefficients and $f_0 \neq 0$, is that irreducible rational function whose numerator polynomial is of degree $\leq j$ and whose denominator polynomial $D_{i,j}(z)$ is of degree $\leq i$ with $D_{i,j}(0) = 1$, whose series expansion in ascending powers of z agrees with the series (1) for the greatest number of initial terms. The quotients $\{P_{i,j}(z)\}$ may be placed in a two dimensional array, the Padé table, in which i and j correspond to rows and column numbers respectively. For convenience in exposition, we append the quotient $P_{0,-1}(z)=0$ to the Padé table. ~~The quotients $P_{i,i+m-1}(z)$ $(i=0,1,\dots)$~~ For a fixed finite integer $m \geq 0$, the quotients $P_{i,i+m-1}(z)$ $(i=0,1,\dots)$ and $P_{i+m,i}(z)$ $(i=0,1,\dots)$ lie on forward diagonals in the Padé table.

Simple integral expressions and Stieltjes integral expressions denote ~~are understood to be~~ denote Riemann integrals and Stieltjes-Riemann integrals respectively ~~Riemann integrals; Stieltjes integral expressions denote Riemann-~~ Stieltjes integrals. With $-\infty \leq \alpha < \beta \leq \infty$, $\sigma \in B\overset{N}{M}(\alpha,\beta)$ means that the real valued function σ is bounded and nondecreasing over $[\alpha,\beta]$, with $\sigma(\beta) > \sigma(\alpha)$, and such that all moments $\int_\alpha^\beta t^\nu d\sigma(t)$ $(\nu=0,1,\dots)$ exist.

With $-\infty \le \alpha < \beta \le \infty$, $BE[\alpha,\beta]$ denotes an arbitrary bounded open domain in the complex plane to whose corresponding closed domain in the complex plane no point of the real segment $[\alpha,\beta]$ belongs. $BE_i[\alpha,\beta]$ denotes a similar open domain to whose corresponding closed domain no point of the imaginary segment $i[\alpha,\beta]$ belongs.

~~Stieltjes series~~

An expression such as z^{α}, where $-\infty<\alpha<\infty$ and z is a complex number, refers to that branch of this function which assumes positive real values when $z\in(0,\infty)$.

2.1 Stieltjes series

Theorem 1. Let

$$(2) \qquad f_\nu = \int_0^\infty t^\nu \, d\sigma(t) \qquad (\sigma \in BM^{N(\ )}[0,\infty]; \ \nu = 0,1,\dots)$$

and let $\{P_{i,j}(z)\}$ be the Padé quotients derived from the series (1). For a fixed finite integer $n \ge 0$, let the series

(3) $$\sum_{1} f_{n+\nu}^{-1/2\nu}$$

be divergent. Then the forward diagonal sequences $\{P_{i,i+m}(z)\}$ $(m=0,1,\dots,n)$, $\{P_{i+m,i}(z)\}$ $(m=1,2,\dots,n+1)$ converge, ~~as i tends to infinity,~~ uniformly over $\mathbb{E}[0,\infty]$ to

(4) $$f(z)=\int_0^\infty (1-zt)^{-1}\,d\sigma(t). \qquad \frac{d\sigma(t)}{1-zt}$$

Proof. If, in formula (2), σ is a simple step function with nonzero salti at $N<\infty$ distinct points in $[0,\infty]$, then the function $f(z)$ of formula (4) is a rational function, and [ ] if none <one>

<P> P = f(0):

of these ~~above~~ points is zero, $P_{N+i,N+j-1}(z)$ ~~= f(z)~~ <$P_{N+i-1,N+j-1}(z)$> $=f(z)$ for $i,j=0,1,\dots$. The series (3) diverges for arbitrarily large n in this case and all forward diagonal sequences of Padé quotients $\{P_{i,j}(z)\}$ derived from the series (1), being ultimately composed of copies of $f(z)$, converge to $f(z)$ as described in the theorem. We now dismiss this case, and henceforth assume that σ in formula (2) does not have the degenerate form described at the beginning of this paragraph.

The series (1) generates a nonterminating corresponding continued fraction ([ ]Ch.) whose successive convergents have the form

$$(5)\quad C_1(z) = u_1\,,\quad C_i(z) = \frac{a_1}{1-}\;\frac{a_2 z}{1-}\cdots\frac{a_i z}{1} \qquad (i=2,3,\ldots)$$

with $0 < a_i < \infty$ $(i=1,2,\ldots)$ if and only if the $\{f_\nu\}$ have a representation of the form (2) with σ nondegenerate [ ] (the $\{C_i(z)\}$ are characterised (and hence the $\{a_i\}$ are determined) by the property that if for ~~z~~ sufficiently small z, $C_i(z) = \sum_i f_\nu^{(i)} z^\nu$, then $f_\nu^{(i)} = f_\nu$ $(\nu = 0,1,\ldots,i-1)$ for $i=1,2,\ldots$; in particular $a_1 = f_0$, $a_2 = -f_1/f_0$).

If the moment problem of the form (2) associated with σ is determinate (in the sense that for the sequence

$\{f_\nu\}$ defined by formula (2) no normalised bounded nondecreasing function distinct from σ defines the same sequence) $\{C_i(z)\}$ converges uniformly over $BE[0,\infty]$ to $f(z)$ [ ]. If the series (3) with $n=0$ diverges, the above moment problem is determinate.

This and the next paragraph concern the case in which $n>0$. With $m\geq 0$ a fixed finite integer, we have from formula (2)

$$f_{m+\nu} = \int_0^\infty t^\nu d\sigma_m(t) \qquad (\nu=0,1,\ldots) \tag{6}$$

where

$$\sigma_m(t) = \int_0^t t^m d\sigma(t) \qquad (0\leq t\leq\infty)$$

$\sigma_m \in BN(0,\infty)$ also.

and σ_m is also bounded and nondecreasing over $[0,\infty]$. From the theory of the preceding paragraph, the series $\sum f_{m+\nu} z^\nu$ generates a corresponding continued fraction whose successive convergents $C_r^{(m)}(z)$ $(r=1,2,\dots)$ have forms similar to those displayed in formula (5); if the Stieltjes moment problem of the form (6) associated with σ_m is determinate, $\{C_r^{(m)}(z)\}$ converges uniformly over $B\in[0,\infty]$ to

(7) $$f_m(z) = \int_0^\infty (1-zt)^{-1} d\sigma_m(t) \qquad \frac{d\sigma_m(t)}{1-zt}$$

If the Stieltjes moment problem of the form (6) {with m replaced by n} associated with σ_n is determinate, then the same is true with regard to the moment problems associated with σ_m for $0\le m\le n$. For if σ_m in formula (6) may be replaced by distinct functions $\sigma_m', \sigma_m'' \in B\overset{N}{M}[0,\infty)$ (and if such functions exist, $d\sigma_m'(t)$ and $d\sigma_m''(t)$ differ at points other than the origin $t=0$ [ ]) then formulae of the form (6) with m replaced by n hold true with σ_n in turn replaced by the two distinct functions $\sigma_n', \sigma_n'' \in B\overset{N}{M}[0,\infty)$, where $\sigma_n'(t)=\int_0^t t^{n-m}d\sigma_m'(t)$, $\sigma_n''(t)=\int_0^t t^{n-m}d\sigma_m''(t)$ $(0\leq t\leq\infty)$, i.e. the moment problem associated with σ_n is indeterminate.

With σ nondegenerate as described

$$P_{i,i+m}(z)=\sum_0^{m-1} f_\nu z^\nu + z^m C^{(m)}_{2i+1}(z)$$

$$P_{i+1,i+m}(z)=\sum_0^{m-1} f_\nu z^\nu + z^m C^{(m)}_{2i+2}(z)$$

for $m,i=0,1,\ldots$. If, for a fixed finite integer $n\geq 0$, the series (3) diverges, the sequences $\{C^{(n)}_{2i+1}(z)\}$ and $\{C^{(n)}_{2i+2}(z)\}$ both converge uniformly over $\mathcal{D}\in[0,\infty]$ to the function $f_n(z)$ defined by formula (7) with m replaced by n (see the penultimate paragraph) and hence, in particular, the sequence $\{P_{i,i+n}(z)\}$ converges, also as described, to

$$\cancel{\sum_0^{n-1} f_\nu z^\nu} \qquad \frac{t^n d\sigma(t)}{1-zt}$$

$$\sum_0^{n-1}\int_0^\infty t^\nu d\sigma(t)\, z^\nu + z^n\int_0^\infty (1-zt)^{-1}t^n d\sigma(t) = f(z)$$

That all diagonal sequences $\{P_{i,i+m}(z)\}$ $(m=0,1,\ldots,n)$ converge in a similar manner follows from the results of the last two paragraphs. Lastly, since $P_{i+1,i}(z) = C^{(0)}_{2i+2}(z)$ ($i=0,$ $1,\ldots$), this sequence also converges as described to $f(z)$.

We now turn to the ~~remaing seq~~ remaining sequences considered in the theorem; these exist if $n>0$, which we now assume. Let $\Sigma_i g_\nu z^\nu$ be the formal power series related to (1) by means of the equation

(8) $\quad \Sigma_i f_\nu z^\nu = f_0 / \{1 - z\, \Sigma_i g_\nu z^\nu\}$;

it generates a nonterminating corresponding continued fraction whose successive convergents (vide formulae (5)) are

$$\hat{C}_1(z) = a_2,\ \hat{C}_i(z) = \frac{a_2}{1-}\,\frac{a_3 z}{1-}\cdots\frac{a_{i+1} z}{1}. \qquad (i=2,3,\ldots)$$

Since $0 < a_{i+1} < \infty$ ($i=1,2,\ldots$), a function $\hat{\sigma} \in BM^N(0,\infty)$ exists such that

(9) $\quad g_\nu = \int_0^\infty t^\nu d\hat{\sigma}(t).$ $\qquad (\nu = 0,1,\ldots)$

Multiplying relationship (8) throughout by the denominator expression on the right hand side, and equating coefficients of corresponding powers of z, we find that

$$g_r = f_0^{-1}\{f_{r+1} - \Sigma_1^r f_\nu g_{r-\nu}\}. \qquad (r=0,1,\ldots)$$

~~If $A_\nu, B_\nu > 0$ $(\nu=0,1,\ldots)$, $A_\nu = CB_\nu - D_\nu$ with $0 \leq D_\nu < \infty$ $(\nu=0,1,\ldots)$, $0 < C < \infty$, and the series $\Sigma_1 B_\nu^{-1/2\nu}$ diverges, then the series $\Sigma_1 A_\nu^{-1/2\nu}$ does likewise.~~ Since the series (3) diverges, the series $\Sigma_1 g_{n-1+\nu}^{-1/2\nu}$ also diverges, which implies in particular (see above) that only one function $\hat{\sigma} \in B\overset{N}{M}(0,\infty)$ is permissible in the representation (9). Setting

$$\hat{f}(z) = \int_0^\infty (1-zt)^{-1} d\hat{\sigma}(t),$$

we also have

$$f(z) = \frac{f_0}{1 - z\hat{f}(z)}.$$

Furthermore, denoting the Padé quotients derived from the series $\Sigma_i g_\nu z^\nu$ by $\{\hat{P}_{i,j}(z)\}$, all diagonal sequences $\{\hat{P}_{i,i+m}(z)\}$ $(m=0,1,\dots,n-1)$ converge ~~as~~ uniformly over $B\in[0,\infty]$ to $\hat{f}(z)$. But [ ]

$$P_{i+1,j}(z) = \frac{f_0}{1 - z\,\hat{P}_{j,i}(z)}. \qquad (i,j=0,1,\dots)$$

Hence all diagonal sequences $\{P_{i+m,i}(z)\}$ $(m=1,2,\dots,n)$ converge as described to $f(z)$.

The algebraic structure of the Padé table derived from a Stieltjes series (1) was first delineated by Van Vleck [ ]. The first description of the convergence behaviour of the forward diagonal sequences of quotients in such a table was given in his doctoral dissertation by Wall [ ] who based his analysis, not upon the theory of the moment problem as we have done, but upon the convergence behaviour of the series $\Sigma_1 a_\nu$ and $\Sigma_1 \alpha_{2\nu}$, whose terms are derived from the coefficients $\{a_\nu\}$ of the corresponding continued fraction whose convergents are given by formulae (5) by use of the relationships

$$a'_{2\nu-1} = \frac{\prod_{\tau=1}^{\nu-1} a_{2\tau}}{\prod_{\tau=1}^{\nu} a_{2\tau-1}}, \qquad a'_{2\nu} = \frac{\prod_{\tau=1}^{\nu} a_{2\tau-1}}{\prod_{\tau=1}^{\nu} a_{2\tau}} \qquad (\nu = 1, 2, \dots)$$

and upon that of related series. This convergence behaviour ~~may~~ is as follows: for some fixed integer $n \geq -1$, a) for $m = 0, 1, \dots, n$, the diagonal sequence $\{P_{i,i+m}(z)\}$ converge uniformly over $BE[0,\infty]$ to the function $f(z)$ of formula (4) ~~for $m = 0, 1, \dots, n$~~; thereafter b) for $m = n+1, n+2, \dots$ each of the sequences $\{P_{i,i+m}(z)\}$ ~~$(m = n, n+1, \dots)$~~ converges uniformly over $BE[0,\infty]$, but for any fixed value of z in the convergence domain, no two neighbouring limits have equal values; similarly, below the principal diagonal, for some fixed integer $n' \geq -1$, a') the diagonal sequences $\{P_{i+m,i}(z)\}$ $(m = 1, 2, \dots, n'+1)$ converge uniformly over $BE[0,\infty]$

$n' \geq -2$

to $f(z)$ and b') for $m = n'+\cancel{1}^{2}, n'+\cancel{2}^{3}, \ldots$ each diagonal sequence converges uniformly over $B \in [0,\infty]$ but to a limit distinct from those of its neighbours as described in b) above. It may occur that one or both of the bands of diagonal sequences described under b) and b') above are ~~b) above is missing;~~ ~~similarly it may occur that the band of diagonal sequences described under b') is missing~~. At the other extreme, it may occur that one or both bands of diagonal sequences described under a) and a') are missing.

In Theorem 1 we made use of Carleman's criterion, involving the divergence of the series (2), to ensure that the moment problem associated with the function σ_n of formula (6) with $m=n$ is determinate. In the light of present knowledge this criterion is a sufficient condition for determinacy of a moment problem; i.e., we have not assumed simply that the moment problem associated with σ_n is determinate,

but have imposed the stronger condition (which implies this determinacy) that the series (3) diverges. From this stronger condition we have deduced that diagonal sequences of Padé quotients lying in symmetric bands (the axis of symmetry lying between the sequences $\{P_{i,i}(z)\}$ and $\{P_{i+1,i}(z)\}$) converge to the function $f(z)$ of formula (4). (Of course, it may occur that the series (3) diverges for all finite n, in which case all forward diagonal sequences
i.e.?
of Padé quotients in question converge to $f(z)$). As far as the author is aware, no example for which the series (3) converges and the moment problem associated with the function σ_n is determinate has been exhibited. If it ultimately transpires that Carleman's condition is also necessary, then Wall's account of the general convergence behaviour of the forward diagonal sequences of Padé quotients derived from a Stieltjes series must

be completed by the remark that the band of sequences listed under a′) above is at least as extensive as that listed under a), i.e. that $n' \geq n$.

We now refer to le Roy's method of extracting a limit function from a power series of the form (1). It is assumed that for a fixed $\chi \in (0,\infty)$ and all z belonging to a prescribed set M, a) the series $\sum f_\nu (vz)^\nu/(\chi\nu)!$ ($(\chi\nu)!$ means $\Gamma(\chi\nu+1)$) converges for small values of v, b) the function $\hat{f}_\chi(zv)$ defined by analytic continuation of the sum of this series is regular in v for all $v \in (0,\infty)$ and c) the integral

$$(10) \qquad S_\chi(z) = \chi^{-1}\int_0^\infty \exp(-v^{1/\chi})\, v^{(1/\chi)-1} \hat{f}_\chi(zv)\,dv$$

exists; following Hardy ([ ]§), the series (1) is then said to be summable (B', χ) to S_χ over M. The particular case of the above method for which $\chi = 1$ was studied in detail by Borel [ ] (Hardy uses the nomenclature (B^*) for $(B', 1)$).

Theorem 2. Let the coefficients of the series (1) have a representation of the form (2), and let $\{P_{i,j}(z)\}$ be the Padé quotients derived from this series. For fixed $\chi \in (0, 2]$, $\kappa \in [0, \infty)$, $\xi \in (0, \infty)$, let $f_\nu = O((\chi\nu + \kappa)!\,\xi^\nu)$.

$$f_\nu = O((\chi\nu + \kappa)!\,\xi^\nu) \qquad \binom{11}{10}$$

(i) All forward diagonal sequences of quotients $\{P_{i,j}(z)\}$ converge uniformly over $z \in [0, \infty]$ to the function $f(z)$ of formula (4).

(ii) $f(z)$ is the (B, χ) sum of the series (1) over ~~all points in~~ the finite part of the closed sector

all points?

$$\tfrac{1}{2}\chi\pi \le \arg(z) \le \tfrac{1}{2}(4 - \chi)\pi. \qquad \binom{12}{9}$$

Proof. Subject to the conditions imposed upon the order of magnitude of the $\{f_\nu\}$, the series (3) diverges for all finite n; the first result of the present theorem concerning

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convergence of all diagonal sequences of Padé quotients as described to the function $f(z)$ of formula (4) follows from Theorem 1.

The $\{f_\nu\}$ satisfy an order relationship similar to that given, with $K=0$ and ξ replaced by some larger number in the range $(\xi, 0)$: we may assume that

$|f_\nu| \le L(\chi\nu)!\,\xi^{-\nu}$ $\qquad L\in(0,\infty), \chi\in(0,2], \xi\in(0,\infty)$

(~~12~~ 13) $$f_\nu = O((\chi\nu)!\,\xi^{-\nu}). \qquad (\chi\in(0,2], \xi\in(0,\infty))$$

We are spared further details of analysis by knowing that F. Bernstein has shown that subject to the ~~simpler order~~ inequalities relationship (~~12~~ 13), the series (1) whose coefficients are expressible in the form (2) is (B', χ) summable over the finite part of the closed sector (~~10~~ ~~11~~ 12) to the function $f(z)$ of formula (4) (this was shown in [ ] for $0<\chi<2$, and later remarked ~~that~~ [ ] that the same result can be obtained from contemporary work of Hamburger [ ] when $\chi=2$. Hamburger's work concerned the $(B',1)$ sum of series of the form (1) for which

$$(14) \qquad f_\nu = \int_{-\infty}^{\infty} t^\nu \, d\sigma(t), \qquad (\sigma \in BM(-\infty,\infty);\ \nu=0,1,\dots)$$

where the $\{f_\nu\}$ are subject to the order relationship $f_\nu = O(\nu! \, \xi^\nu)$ $(\xi \in (0,\infty))$; the limiting form in question is obtained by taking $d\sigma$ in (14) to be symmetric about the origin and introducing appropriate variable changes).

Convergence of the forward diagonal sequences considered in the above theorem was established for the case in which $\chi=2$ in formula (11): with regard to convergence, reference to a smaller value of χ in the interval $(0,2]$ becomes superfluous. However, the range of values of z for which the series considered are (B',χ) summable does depend upon χ; this dependence corresponds to the facts of the case. The function σ concerned is that having a saltus of magnitude unity at the point $t \in (0,\infty)$ and no other points of increase, so that $c_\nu = t^\nu$ $(\nu = 0, 1, \ldots)$ and $f_\chi(zv) = \sum (tvz)^\nu / (\chi\nu)!$. When $\chi = 2$, the integral of the form (10) becomes

$$S_2(z) = \frac{1}{2}\int_0^\infty \exp(-v^{\frac{1}{2}})v^{-\frac{1}{2}} \cosh\{(tvz)^{\frac{1}{2}}\}dv$$

$$= \int_0^\infty e^{-u} \cosh\{(tz)^{\frac{1}{2}}u\}du. \tag{15}$$

For any finite $z \notin (-\infty, 0]$, a $t \in (0,\infty)$ can be found (i.e. a Stieltjes series satisfying the conditions of Theorem 2 can be constructed) such that the integral (15) fails to exist; $(B', 2)$ summability of the complete ensemble of series considered in Theorem 2 holds only for nonpositive finite real values of the argument. The series just considered as an example is also $(B', 1)$ or (B^*) summable. Nor

$\hat{f}_1(zv) = \exp(tvz)$, and

$$S_1(z) = \int_0^\infty e^{-v(1-tz)}\,dv.$$

This integral exists for all $t \in (0,\infty)$ and for all finite z for which $Re(z) \leq 0$. Naturally, special Stieltjes series exist that are (B', χ) summable over a wider range of values than that given in Theorem 2. For example, we may take $c_\nu = (2\nu)!$ $(\nu = 0, 1, \ldots)$ when $f_2(zv) = (1 - z\nu)^{-1}$, and the series in question is $(B', 2)$ summable for all finite $z \notin (0, \infty]$.

F. Bernstein demonstrated the quoted result in the process of proving the theorem that the convergents $\{C_{2i}(z)\}$ of the continued fraction corresponding to the series (1) whose coefficients have a representation of the ~~stated~~ form (2) and ~~are subject to the order relationship ( )~~ satisfy the inequalities (13) converge uniformly over $B \in [0, \infty]$ to the analytic continuation of the (B', χ) sum of tp 21

Theorem 2 extends Wall's result in three senses: the assumption b) above is rendered superfluous; a less stringent order relationship upon the $\{b_i\}$ is assumed; Theorem 2 concerns the general (B', x) sum for $0 < x \leq 2$ of the series (1) over the finite part of the ~~sector~~ closed sector (12)

this series. The $\{C_{2i}(z)\}$ are the Padé quotients $\{P_{i,i-1}(z)\}$ derived from the series (1). Hence Theorem 2 is an extension to all forward diagonal sequences of Padé quotients of F. Bernstein's

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theorem which concerns one of them. Wall [ ] has shown that a) if the $\{f_\nu\}$ have a representation of the form (2) and are subject to the order relationship ~~$\Longleftrightarrow$~~ $f_\nu = O(\nu! \xi^\nu)$

for $m=0,1,\ldots$ $\{P_{i,i+m-1}(z)\}$

($\xi \in (0,\infty)$) then all forward diagonal sequences of Padé quotients derived from the series (1) ~~$\{P_{i,i+m-1}(z)\}$ (i increasing) for $m=0,1,\ldots$~~

function defined by analytic continuation of

converge to the (B′,1) or Borel sum of this series, and b) if the $\{g_\nu\}$ of formula (5) are subject to a similar order relationship, then the further diagonal sequences of Padé quotients $\{P_{i+m,i}(z)\}$ ($m=2,3,\ldots$) converge to the (B′,1) sum of the series (1). ~~As we have seen, assumption b) is superfluous: Theorem 2 ... also ...~~

~~extends Wall's result in two further directions: firstly, a less stringent retat order relationship upon the~~ $\{f_\nu\}$ ~~is assumed and secondly, Theorem 2 concerns the general~~ (B', x) ~~sum for~~ $0 < x \leq 2$ ~~of the series ( ).~~

In his investigations concerning the Euler-Maclaurin series (which we define below) Hardy remarks that under certain conditions this series is (B^2) summable (i.e. summable by a repetition of Borel's method. He does not define (B^2) summability explicitly; nevertheless, what he says is correct if this method of summation is defined as follows: if, for all z belonging to a prescribed set M a) the series $\sum f_\nu (vwz)^\nu/(\nu!)^2$ converges for small values of v and w, b) the function $\hat{f}(vwz)$ obtained by analytic continuation of the function so defined is regular in v and

w for all $v, w \in (0,\infty)$ and c) the double integral

$$S(z) = \int_0^\infty \int_0^\infty e^{-v-w} \hat{f}(vwz)\,dv\,dw$$

exists, then the series (1) is summable (B^2) to S over M.

As is easily verified, it is possible to replace the series in two variables ~~v~~ and w and ~~double~~ the double integral with respect to these variables by a series and integral involving one variable, and in this way derive an equivalent but more concise formulation of (B^2) summability: if, for all z belonging to a prescribed set M a) the series

(16) $$\sum f_\nu (zu)^\nu/(\nu!)^2$$

converges for small values of u, b) the function $\hat{\tilde{f}}(zu)$ ~~so defined~~ obtained by analytic continuation of the function so defined is regular for all $u \in (0,\infty)$ and

c) the integral

(17) $$S(z) = 2^{\varsigma}\int_0^\infty K_0(2u^{\frac{1}{2}})\hat{\tilde{f}}(zu)\,du$$

exists (K_0 being a modified Bessel function of the second kind) then the series (1) is summable (B^2) to S over M.

For the sake of completeness, we give a result concerning the (B^2) summability of Stieltjes series.

Theorem 3. Let the coefficients of the series (1) have a representation of the form (2) and be subject to the order relationship (11) as stated. Then this series is summable (B^2) to the function f of formula (4) over $(-\infty, 0]$.

Proof. For a fixed $z \in (-\infty, 0]$, the series (16) in the present case converges for sufficiently small u to the function

$$\hat{\tilde{f}}(zu) = \int_0^\infty J_0\{2i(zut)^{\frac{1}{2}}\}\,d\sigma(t)$$

(J_0 being a Bessel function of the first kind). Since $J_0\{2i(zut)^{\frac{1}{2}}\}$ is uniformly bounded for the z in question and all $u, t \in [0,\infty]$, $\hat{f}(zu)$ is bounded for all $u \in [0,\infty]$. The integral (17) exists in this case and, using a standard result in the theory of Bessel functions, may be shown to be equal to the function $f(z)$ of formula (4). The functions $S(z)$ ~~of~~ formula (17) and $f(z)$ of formula (4) are equivalent for all $z \in (-\infty, 0]$.

As for (B,2) summability, the restriction of (B^2) summability to nonpositive real values of z in the above theorem corresponds to the facts of the case. ~~One example~~ Again taking σ to have a saltus of magnitude unity at the point $t \in (0,\infty)$ and no other points of increase, we have $\hat{f}(zu) = J_0\{2i(zut)^{\frac{1}{2}}\}$. For large values of $(zut)^{\frac{1}{2}}$

$$J_0\{2i(zut)^{\frac{1}{2}}\} = O\left[(zut)^{-\frac{1}{4}} \cosh\{2(zut)^{\frac{1}{2}}\}\right],$$

and for large u $K_0\{2u^{\frac{1}{2}}\} = O\{u^{-\frac{1}{4}} \exp(-2u^{\frac{1}{2}})\}$. Thus for any ~~finite~~ $z \in (-\infty, 0]$ a t can be ~~found~~ chosen (i.e. a series to which Theorem 3 relates can be constructed) such that the integrand in formula (17) increases in magnitude as u increases, and the integral (17) fails to exist.

a less stringent order relationship

~~extension of Wall's result in that it concerns the general (B', χ) sum for $0<\chi\leq 2$ of the series ( ).~~

2.2 Hamburger series

Theorem ~~3~~ 4. Let the coefficients of the series (1) be given by formula (14), and let $\{P_{i,j}(z)\}$ be the Padé quotients derived from this series. For a fixed finite integer $n \geq 0$, let the series

$$\sum_{\nu=1}^{\infty} f_{2n+2\nu}^{-1/2\nu} \tag{18}$$

be divergent. Then the forward diagonal sequences $\{P_{i,i+m-1}(z)\}$ $(m=0,1,\dots,2n+1)$ converge uniformly over $B \in [-\infty,\infty]$ to

$$f(z)=\int_{-\infty}^{\infty}(1-zt)^{-1}d\sigma(t) \qquad \frac{d\sigma(t)}{1-zt} \tag{19}$$

Proof. The case in which σ in formula (14) is a simple step function with a finite number of salti can be dismissed ~~sw~~ peremptorily: the resulting Padé table has the same structure as that described at the commencement of the proof of Theorem 1. We assume henceforth that σ does not have this degenerate structure.

The series (1) generates a nonterminating associated continued fraction ([ ]Ch.) whose successive convergents have the form

$$(20)\quad \tilde{C}_1(z) = \frac{b_1}{1-c_1 z}, \quad \tilde{C}_i(z) = \frac{b_1}{1-c_1 z-}\,\frac{b_2 z^2}{1-c_2 z-}\cdots\frac{b_i z^2}{1-c_i z} \quad (i=2,3,\ldots)$$

with $0<b_i<\infty$ $(i=1,2,\ldots)$ if and only if the $\{f_\nu\}$ have a representation of the form (14) with σ nondegenerate [,] (if for sufficiently small z, $\tilde{C}_i(z) = \sum f_\nu^{(i)} z^\nu$, then $f_\nu^{(i)} = f_\nu$ $(\nu = 0,1,\ldots,2i-1)$ for $i=1,2,\ldots$; in particular $b_1 = f_0$, $c_1 = f_1/f_0$).

For a fixed finite nonreal value of z, the extended convergents

$$\tilde{C}_1(z,t) = \frac{b_1}{1-(c_1+t)z}, \quad \tilde{C}_i(z,t) = \frac{b_1}{1-c_1 z-}\,\frac{b_2 z^2}{1-c_2 z-}\cdots\frac{b_i z^2}{1-(c_i+t)z} \quad (i=2,3,\ldots)$$

describe, as t varies in the range $-\infty \le t \le \infty$, circles

$\mathcal{C}_i(z)$ in the complex plane; the circles $\{\mathcal{C}_i(z)\}$ form a nested sequence: $\mathcal{C}_{i+1}(z)$ lies inside $\mathcal{C}_i(z)$ touching the latter at $\tilde{C}_{i+1}(z,\infty) = \tilde{C}_i(z,0) = \tilde{\tilde{C}}_i(z)$ $(i=1,2,\ldots)$; all $\{\mathcal{C}_i(z)\}$ contain the value of the function $f(z)$ given by formula (13). If the moment problem of the form (14) associated with the function σ is determinate, the above associated continued fraction converges completely ([, ,][ ] Ch.) to the value of $f(z)$: the radii of the circles $\{\mathcal{C}_i(z)\}$ tend to zero.

With $m \geq 0$ a fixed finite integer, we have

(34)
$$f_{2m+\nu} = \int_{-\infty}^{\infty} t^{\nu} d\sigma_{2m}(t) \qquad (\nu = 0, 1, \ldots)$$

where $\sigma_{2m}(t) = \int_{-\infty}^{t} t^{2m} d\sigma(t)$ $(-\infty \leq t \leq \infty)$ and $\sigma_{2m} \in BM(-\infty,\infty)$.
The above theory can be applied to the nonterminating continued fraction whose convergents have the form

$$\overset{2}{(27)}\quad C_1^{(2m)}(z)=\frac{b_1^{(2m)}}{1-c_1^{(2m)}z},\quad C_i^{(2m)}(z)=\frac{b_1^{(2m)}}{1-c_1^{(2m)}z-}\,\frac{b_2^{(2m)}z^2}{1-c_2^{(2m)}z-}\cdots\frac{b_i^{(2m)}z^2}{1-c_i^{(2m)}z}\quad (i=2,3,\ldots)$$

with $0<b_i^{(2m)}<\infty$ $(i=1,2,\ldots)$, associated with the series $\sum f_{2m+\nu}z^{\nu}$. If the moment problem of the form $\overset{1}{(27)}$ associated with the function σ_{2m} is determinate then, for a fixed finite nonreal value of z, this continued fraction converges completely to the value of

$(1-zt)^{-1}$?

$$\overset{3}{(27)}\quad f_{2m}(z)=\int_{-\infty}^{\infty}\frac{d\sigma_{2m}(t)}{1-zt}$$

the geometric relationships described above are preserved

If both the points on the circles associated with the convergents $\overset{2}{(27)}$ and the limit function $\overset{3}{(27)}$ are subjected to the same linear transformation, the geometric relationships described above are preserved ~~convergence is preserved~~:

Convergence ... preserved

for a fixed finite nonreal value of z there exists a sequence of nested circles $\{\mathscr{C}_i^{(2m)}(z)\}$ described by the extended convergents z, t

c

$$C_1^{(2m)}(z) = \sum_0^{2m-1} f_\nu z^\nu + \frac{b_1^{(2m)} z^{2m}}{1 - b_1^{(2m)} z}$$

$$C_i^{(2m)}(z) = \sum_0^{2m-1} f_\nu z^\nu + \frac{b_1^{(2m)} z^{2m}}{1 - c_1^{(2m)} z -} \; \frac{b_2^{(2m)} z^2}{1 - c_2^{(2m)} z -} \cdots \frac{b_i^{(2m)} z^2}{1 - (c_i^{(2m)} + t) z} \quad (i = 2, 3, \ldots)$$

as t varies in the range $-\infty \leq t \leq \infty$, which enclose the value of

$$\sum_0^{2m-1} f_\nu z^\nu + z^{2m} f_{2m}(z) = f(z).$$

With $\{P_{i,j}(z)\}$ being the Padé quotients generated by the series (1), $P_{i,i+2m-1}(z) = C_i^{(2m)}(z; 0)$. $\mathcal{C}_{i+1}^{(2m)}(z)$ lies inside $\mathcal{C}_i^{(2m)}(z)$, touching the latter at $P_{i,i+2m-1}(z)$ ($i; r = 1, 2, \ldots$)

If the moment problem of the form (21) associated with the function σ_{2m} is determinate, the radii of the circles $\{\mathcal{C}_i^{(2m)}(z)\}$ tend to zero and, in particular, when z is finite and nonreal, the diagonal sequence $\{P_{i,i+2m-1}(z)\}$ converges, ~~for a fixed finite nonreal value of~~ z, to $f(z)$.

Denote the value of the Hankel determinant of order $r+1$

$(r+1)^{th}$ $(\tau+1)^{th}$, $f_{\tau}(\tau=0,1,\dots,r)$

ord. +1 whose τ^{th} row contains the elements $f_{m+\tau+\nu}$ $(\nu=0,1,\dots,r)$ by $H_{m,r}$ $(m,r=0,1,\dots)$, and set $H_{m,-1}=1$ $(m=0,1,\dots)$. If, in formula (14), σ is nondegenerate $H_{2m,r}>0$ $(m,r=0,1,\dots)$ [ ]. For fixed finite integers $m\geq 0$, $r\geq 1$, it can occur that $H_{2m+1,r-1}=0$

$\neq 0$ (when [ ] $H_{2m+1,r-2}\neq 0$, $H_{2m+1,r}\neq 0$). As may easily be deduced from theory given in [ ], $P_{r-1,2m+i-1}(z)=C_i^{(2m)}(z,t)$, where

$$t=-\frac{H_{2m+1,i-1}\,H_{2m,i-2}}{H_{2m,i-1}\,H_{2m+1,i-2}}$$

(this value of t always being well determined; in particular, $t=0$ when $H_{2m+1,i-1}=0$ and $t=\infty$ when $H_{2m+1,i-2}=0$). Thus $P_{i-1,2m+i-1}(z)$ lies on $\mathcal{C}_i^{(2m)}(z)$ ($P_{i-1,2m+i-1}(z)=P_{i,2m+i-1}(z)$ when $H_{2m+1,i-1}=0$ and $P_{i-1,2m+i-1}(z)=P_{i-1,2m+i-2}(z)$ when $H_{2m+1,i-2}=0$). In short, when the moment problem of the form

$(2\overset{1}{4})$ associated with the function σ_{2m} is determinate, not only do the quotients $P_{i,i+2m-1}(z)$ $(i=0,1,\ldots)$ converge to $f(z)$ for finite nonreal z, but the quotients $P_{i,i+2m}(z)$ $(i=0,1,\ldots)$ also do so also. Over any domain $B \in [-\infty,\infty]$, the $\{P_{i,i+2m-1}(z)\}$ and $\{P_{i,i+2m}(z)\}$ in question may be shown to be uniformly bounded, and uniform convergence of these sequences over such a domain follows from the Stieltjes-Vitali theorem.

If, for a finite integer $n \geq 0$, the series (18) diverges, the moment problem of the form analogous to $(2\overset{1}{4})$ associated with the function σ_{2n} is determinate [ ]. If $n > 0$, for reasons similar to those given in the proof of Theorem 1, the moment problems associated with the functions σ_{2m} $(m=0,1,\ldots,n)$ are also determinate. The result of the theorem follows immediately.

described!

Wall ~~two~~ [ ] has also ~~given an analysis of~~ described the convergence behaviour of the diagonal sequences of Padé quotients derived from the series (1) whose coefficients are given by formula (14): there are two bands of sequences $\{P_{i,i+m}(z)\}$ $(m=0,1,\dots,2n)$ and $\{P_{i+m+1,i}(z)\}$ $(m=0,1\dots,2n')$ $(n,n'\geq -1$ finite integers) converging uniformly over $B\in[-\infty,\infty]$ to the function $f(z)$ of formula (19); the remaining diagonal sequences, although behaving in a regular manner ~~systematically~~ (details are given in [ ]) do not converge in this way ~~manner~~. As in the Stieltjes case, one or both of the above two bands may be missing; alternatively the remaining sequences may be missing.

It is to be regretted that we have been unable to derive a convergence result concerning the diagonal sequences of the form $\{P_{i+m+1,i}(z)\}$ $(m>0)$ based upon the divergence of the series (18). In the following theorem, in which the coefficients $\{f_\nu\}$ are subjected to numerical constraints, this defect is partly remedied.

Theorem ~~4~~ 5. Let the coefficients $\{f_\nu\}$ of the series (1) have a representation of the form (14), and let $\{P_{i,j}(z)\}$ be the Padé quotients derived from this series. For fixed $K\in[0,\infty)$, $\xi\in(0,\infty)$, let

$$f_\nu = O((\nu+K)!\,\xi^\nu). \tag{24}$$

(i) All forward diagonal sequences of quotients $\{P_{i,j}(z)\}$ converge uniformly over $\mathcal{B}\in[-\infty,\infty]$ to the function $f(z)$ of formula (13).

(ii) $f(z)$ is the Borel sum of the series (1) over the finite part of the imaginary axis.

Proof. The series (18) now diverges for all finite n, and hence all diagonal sequences $\{P_{i,i+m-1}(z)\}$ $(m=0,1,\ldots)$ converge uniformly over $\mathcal{B}\in[-\infty,\infty]$ to the function $f(z)$ of formula (13).

We dismiss the case in which σ in formula (17) is a step function with a finite number of salti as in the proof of Theorem 3: we now assume that σ is nondegenerate. Let $\sum h_\nu z^\nu$ be the series related to (1) by means of the formal equation

$$\sum f_\nu z^\nu = \frac{f_0}{1 - c_1 z - z^2 \sum h_\nu z^\nu} \tag{$\tilde{24}$}$$

where $c_1 = f_1/f_0$. It generates a nonterminating associated continued fraction whose successive convergents (vide formula (20)) are

$$\tilde{C}_1(z) = \frac{b_2}{1-c_2 z},\ \tilde{C}_i(z) = \frac{b_2}{1-c_2 z-}\ \frac{b_3 z^2}{1-c_3 z-} \cdots \frac{b_{i+1} z^2}{1-c_{i+1} z}. \quad (i=2,3,\ldots)$$

Since $0 < b_{i+1} < \infty$ $(i=1,2,\ldots)$, a function $\tilde{\sigma} \in BM(-\infty,\infty)$ exists such that

$$h_\nu = \int_{-\infty}^{\infty} t^\nu d\tilde{\sigma}(t). \quad (\nu = 0,1,\ldots)$$

It follows from relationship ($\tilde{24}$) that

$$h_r = \frac{1}{f_0}\left\{ f_{r+2} - \frac{f_1}{f_0} f_{r+1} - \Sigma_1^r f_\nu h_{r-\nu} \right\}. \qquad (r=0,1,\dots)$$

As in the proof of Theorem 2, the order relationship (24) may be replaced without loss of generality by the inequalities

$$(26) \qquad |f_\nu| \leq K \nu! \, \xi^\nu, \qquad (\nu = 0,1,\dots)$$

where $L, \xi \in (0,\infty)$. Using these inequalities, we find by direct computation that

$$(27) \qquad |h_\nu| < 2(\nu+2)! \left(\frac{K}{f_0}\right)^{\nu+1} \xi^{\nu+2}$$

for $\nu = 0,1,2$. We now assume that inequality (26) also holds for $\nu = 0,1,\dots,r-1$, where $r \geq 3$, and, since $K/f_0 \geq 1$, have

$$\begin{aligned} |h_r| &< \frac{K}{f_0}\left\{ (r+2)!\, \xi^{r+2} + \frac{K}{f_0}(r+1)!\, \xi^{r+2} \right. \\ &\qquad \left. + 2\Sigma_1^r \nu!\, \xi^\nu (r-\nu+2)! \left(\frac{K}{f_0}\right)^{r-\nu+1} \xi^{r-\nu+2} \right\} \\ &< \left(\frac{K}{f_0}\right)^{r+1} \xi^{r+2} \left\{ (r+2)! + (r+1)! + 2\Sigma_1^r \nu!(r-\nu+2)! \right\} \\ &< 2(r+2)! \left(\frac{K}{f_0}\right)^{r+1} \xi^{r+2}. \end{aligned}$$

Hence, by induction, inequality (27) holds for $\nu = 0,1,\dots$

Denote the Padé quotients derived from the series $\sum h_\nu z^\nu$ by $\{\tilde{P}_{i,j}(z)\}$. It follows from inequalities (24), as in the first step of this proof, that all diagonal sequences of quotients $\{\tilde{P}_{i,i+m}(z)\}$ $(m=0,1,\ldots)$, in particular, converge uniformly over $B\in[-\infty,\infty]$ to the function

$$\tilde{f}(z)=\int_{-\infty}^{\infty}\frac{d\tilde{\alpha}(t)}{1-zt}.$$

But, on the one hand,

$$P_{i+m,i}(z)=\frac{f_0}{1-c_1 z-z^2\tilde{P}_{i,i+m-2}(z)} \qquad (i=0,1,\ldots;\, m=2,3,\ldots)$$

and, on the other,

$$f(z)=\frac{f_0}{1-c_1 z-z^2\tilde{f}(z)}.$$

Hence all diagonal sequences $\{P_{i+m,i}(z)\}$ $(m=2,3,\ldots)$ also converge uniformly over $B\in[-\infty,\infty]$ to $f(z)$.

It was shown by Hamburger [ ] that if the

$\{f_\nu\}$ have a representation of the form (14) and satisfy a relationship of the form (26) (i.e., the inequalities) (or, equivalently, as we have remarked, the order relationship (24)), then the function $f(z)$ of formula (13) is the Borel sum of the series (1) ~~for all finite pure imaginary values of z~~ over the finite part of the imaginary axis. This concludes the proof of the theorem.

In the above theorem, we have made use of the strict definition of (B^*) summability given by Hardy, and have shown the Hamburger series concerned to be (B^*) summable over the finite part of the imaginary axis only. As in the case of Theorem 2 concerning (B', χ) summability of certain Stieltjes series, the above restriction corresponds to the facts of the case. One function σ concerned is that having two saltus of magnitude unity at the points $\pm t$ $(t \in (0, \infty))$ and no other points of increase, so that $c_{2\nu} = 2t^{2\nu}$, $c_{2\nu+1} = 0$ $(\nu = 0, 1, \ldots)$ and $\hat{f}_1(zv) = e^{tzv} + e^{-tzv}$. The appropriate integral of the form (10) is

$$(28) \qquad S_1(z) = \int_0^{\infty} e^{-v}\{e^{tzv} + e^{-tzv}\}dv.$$

not?

For any finite nonreal value of z which is ~~not~~ also nonimaginary, a $t \in (0, \infty)$ can be found (i.e. a Hamburger series satisfying the conditions of Theorem 5 can be constructed) such that the integral (28) fails to exist. Naturally there are also Hamburger series which are (B^*) summable for values of z not restricted as in Theorem 5 (for example, that for which $c_{2\nu} = (2\nu)!$, $c_{2\nu+1} = 0$ $(\nu = 0, 1, \ldots)$, when $\hat{f}_1(zv) = (1 - z^2v^2)^{-1}$).

It is however known that Hamburger series are generated by a function whose singularities are confined to the real axis. For this reason, Hamburger [ ] was able to make use of an extension of (B^*) summability in which, in our notation, the integral (10) with $\chi = 1$ is replaced by

$$S_\phi(z) = \int_0^{\phi\infty} \hat{f}_1(zv)e^{-v}dv$$

where $\phi = \exp[i\{\theta \pm \frac{1}{2}\pi - \arg(z)\}]$, the sign attached to $\frac{1}{2}\pi$ being that of $\mathrm{Im}(z)$; he showed that the series considered in Theorem ~~4~~ 5 are (B^*) summable in this extended sense for all finite nonreal values of z. We point out that the Stieltjes series considered in Theorem 2 are also (B', χ) summable in a similarly extended sense, obtained by rotating the path of integration in formula (10), for all finite $z \notin (0, \infty]$.

Hamburger demonstrated the result quoted above in the process of proving that the convergents $\{\overset{\leftrightarrow}{C}_i(z)\}$ of the continued fraction associated with the series (1) whose coefficients have a representation of the form (14) and are subject to the inequalities (26) converge uniformly over BE$[-\infty,\infty]$ to the extended Borel sum of this series. The $\{\acute{\overset{\infty}{C}}_i(z)\}$ are the Padé quotients $\{P_{i,i-1}(z)\}$ derived from the series (1). Thus Theorem ~~4~~ 5 is an extension, to all forward diagonal sequences of Padé quotients, of Hamburger's result which concerns one of them. Theorem ~~4~~ 5 is also an extension of a result due to Wall ([ ] Th. 3): if the $\{f_\nu\}$ have a representation of the form (14) and both the series (1) and its reciprocal are Borel summable, then all forward diagonal sequences of Padé quotients converge uniformly over BE$[-\infty,\infty]$ to the Borel sum of (1). As we have seen in the proof of Theorem 5, the assumption concerning the reciprocal series can be discarded. Theorem 5 ~~is~~ also extends a further theorem of Wall ([ ] Th. 1): if the $\{f_\nu\}$ have a representation of the form (14), and the series (1) has a nonzero radius of convergence (i.e. $f_\nu = O(\frac{1}{\delta^\nu})$ for some $\delta\in(0,\infty)$) then all forward diagonal sequences of Padé quotients derived from this series ~~(1)~~ converge uniformly over $z\in[-\infty,\infty]$ to the function $f(z)$ of formula (10). As we have shown, a further term ~~of the~~ $(\nu+K)!$ $(K\in[0,\infty))$ may be inserted with impunity in this order relationship.

3. The Euler-Maclaurin series

To fix the notations used in this section, we set $b_\nu = B_{2\nu+2}/(2\nu+2)!$ $(\nu = 0, 1, \ldots)$, the $\{B_{2\nu+2}\}$ being Bernoulli numbers, so that $b_0 = 1/12$, $b_1 = -1/720, \ldots$, and for $|u| < 2\pi$,

$$\Sigma_1\, b_\nu u^{2\nu} = \frac{1}{u^2}\left\{\frac{u}{e^u - 1} - 1 + \tfrac{1}{2}u\right\}. \tag{25}$$

For the purposes of this exposition, we define the odd order periodic Bernoulli functions by means of their Fourier expansions

$$\beta_{2\tau+1}(y) = \frac{(-1)^\tau}{2^{2\tau}\pi^{2\tau+1}} \Sigma_1 \frac{\sin(2\pi\nu y)}{\nu^{2\tau+1}}. \qquad (\tau = 0, 1, \ldots) \tag{26}$$

$\mathcal{D}$ is the differential operator with respect to the indicated variable: $\mathcal{D}^2\Psi(\mu)$ is $d^2\Psi(\mu)/d\mu^2$.

With all derivatives of the function $\psi(\mu)$ assumed to exist for a prescribed finite complex value of μ and k taken to be a finite complex number,

$\underline{\Psi} = \psi$ (31) $$\sum b_\nu \mathfrak{D}^{2\nu+1}\psi(\mu)k^{2\nu}$$

is the Euler-Maclaurin series. Normally, the behaviour of this series is examined for the case in which $k=1$. Since the more general series (31) may be obtained from this special case by modifying the function ψ, no generality is lost by confining attention to this special case. However, we are particularly interested in the behaviour of the series (31) for varying values of k, and, rather than encumber the theory with

91. a varying function, retain the variable k in this series. ~~Whens~~ When $\psi(\mu)=e^{-\mu v}$, for example, v being a nonzero complex number the series (31) converges ~~in~~ over the open disc $|k|<2\pi/|v|$, and diverges over the complement with respect to the finite part of the k-plane of the corresponding closed disc.

Let

$$(32)\quad \hat{R}_{0,0}(\psi,\mu;k) = k^{-2}\left[\int_{\mu}^{k\infty}\psi(\mu')\,d\mu' - \tfrac{1}{2}k\sum_1\left[\psi(\mu+\omega k)+\psi\{\mu+(\omega+1)k\}\right]\right],$$

where the integral is taken over the path $\mu' = \mu + tk$ $(0 \le t \le \infty)$ so that the terms in the sum involve values of ψ taken at points on the path of integration. In certain circumstances, $\hat{R}_{0,0}(\psi,\mu;k)$ represents the sum of the series (31). When $\psi(\mu) = e^{-\nu\mu}$, the integral and series in formula (32) both converge over the finite part of the open half-plane $-\frac{1}{2}\pi - \arg(\nu) < \arg(k) < \frac{1}{2}\pi - \arg(\nu)$. Over the open semi-disc defined by the intersection of this half-plane and the open disc $|k| < 2\pi/|\nu|$, $\hat{R}_{0,0}(\psi,\mu;k)$ is the sum of the series (31); over the ~~intersection of the open half-plane, and~~ finite part of ~~this~~ above half-plane from which all points belonging to the closed disc $|k| \le 2\pi/|\nu|$ have been removed, $\hat{R}_{0,0}(\psi,\mu;k)$ defines a divergent sum of this series.

An alternative representation in closed form ~~for a~~ of the sum of the series (31) is offered by

$\mu-\mu'$ $\sim$?

$$\tilde{R}_{0,0}(\psi,\mu;k) = k^{-1}\int_{\mu}^{k\infty} \beta_1\{(\mu-\mu')/k\}\,\mathfrak{D}\psi(\mu')\,d\mu',$$

the path of integration being as for formula (32). When $\psi(\mu) = 1+e^{-\nu\mu}$, $\hat{R}_{0,0}(\psi,\mu;k)$ no longer defines a sum of the series (31); nevertheless, $\tilde{R}_{0,0}(\psi,\mu;k)$ does so in the manner described in connection with $\hat{R}_{0,0}(\psi,\mu;k)$ for the function $\psi(\mu) = e^{-\nu\mu}$. $\tilde{R}_{0,0}(\psi,\mu;k)$ may, of course, be regarded as being the remainder term of the series (31) after no terms have been summed.

The delayed form of the Euler-Maclaurin series is

$$\sum b_{J+\nu}\,\mathfrak{D}^{2J+2\nu+1}\psi(\mu)\,k^{2\nu} \tag{33}$$

where $J \geq 0$ is a fixed finite integer. Under certain conditions, the functions

$$\tilde{R}_{J,j}(\psi,\mu;k) = k^{2j-1}\int_{\mu}^{k\infty}\beta_{2J+2j+1}\{(\mu'-\mu)/k\}\,\mathfrak{D}^{2J+2j+1}\psi(\mu')\,d\mu' \qquad (j=0,1,\ldots) \tag{34}$$

(34) $$\tilde{R}_{J,j}(\psi,\mu;k) = k^{2j-1}\int_{\mu}^{k\infty} \beta_{2J+2j+1}\{(\mu'-\mu)/k\}\,\mathcal{D}^{2J+2j+1}\psi(\mu')\,d\mu' \quad (j=0,1,\ldots)$$

play the rôles of remainder terms of the series (31), in the sense

suitable

that for suitably ~~restricted~~ functions ψ,

$$\tilde{R}_{J,0}(\psi,\mu;k) = \sum_{0}^{j-1} b_{J+\nu}\,\mathcal{D}^{2J+2\nu+1}\psi(\mu)\,k^{2\nu} + \tilde{R}_{J,j}(\psi,\mu;k). \quad (j=0,1\ldots)$$

Hardy ([ ]§), using properties of the periodic Bernoulli polynomials, has shown that if $\psi(\mu)$ is real for $0\le\mu<\infty$ and, all for $j=J, J+1,\ldots$, $\mathcal{D}^{2j+1}\psi(\mu)$ tends to zero as μ tends to infinity, $\int_0^\infty |\mathcal{D}^{2j+2}\psi(\mu)|d\mu<\infty$, and $\mathcal{D}^{2j+2}\psi(\mu')$ is of fixed sign for all $\mu'\in[0,\infty)$ (this condition implies (see the footnote to p. 327 of [ ]) that, depending on the sign of $\mathcal{D}^{2j+2}\psi(\mu)$,

$2j+2$

either $(-\mathcal{D}^{4})^{j}\psi(\mu)\geq 0$ for all $\mu\in[0,\infty)$ and $j=2J+1, 2J+2,\ldots$ or $(-\mathcal{D})^{j}\psi(\mu)\le 0$ for these μ and j) then, when $\mu=0$ and $k=1$, the terms of the series (33) alternate in sign, its partial sums oscillate about $\tilde{R}_{J,0}(\psi,\mu;k)$, and this series is semi-convergent: $|\tilde{R}_{J,j}(\psi,\mu;k)| < |b_{J+j}\,\mathcal{D}^{2J+2j+1}\psi(\mu)\,k^{2j}|$ $(j=0,1,\ldots)$. Hardy ([ ] §) has also shown that without the above restrictions upon ψ, but with $\psi(\mu)$ assumed ~~regular~~ analytic in ~~for~~ the finite part of the half-plane $Re(\mu) > -\delta$ $(\delta \in (0,\infty))$ and $\psi(\mu) = O(\mu^s)$ ~~in this half-plane~~ for large μ in this region, the intercalated delayed Euler-Maclaurin series

$$(35) \qquad b_J\,\mathcal{D}^{2J+1}\psi(\mu) + 0 + b_{J+1}\,\mathcal{D}^{2J+3}\psi(\mu)k^2 + 0 + \ldots$$

with $2J+1 > s$ is, when $k = 1$, summable $(B',1)$ to $R_{J,0}(\psi,\mu;k)$ for $\mu = 0$ over the finite part of the nonnegative real axis, and further states that the series (33) itself is (B^2) summable to the same function for $\mu = 0$ over the same range of values of μ.

In conjunction with the theory of the preceding sections the above recapitulation of Hardy's results suggests that the delayed Euler-Maclaurin series should be amenable to continued fraction transformation.

Theorem 6. Let $\psi(\mu) = \psi_1(\mu) + \psi_2(\mu)$, where ψ_1 and ψ_2 are two real valued functions such that, the integer $J \geq 0$ with interval $[\hat{\mu}, \tilde{\mu}]$ $(-\infty \leq \hat{\mu} < \tilde{\mu} \leq \infty)$ being fixed,

$$(36) \qquad (-\mathcal{D})^{\nu} \psi_1(\mu') \leq 0 \qquad (\nu = 2J+1, 2J+2, \dots)$$

for all $\mu' \in (\hat{\mu}, \infty)$, and

$$(37) \qquad \mathcal{D}^{\nu} \psi_2(\mu) \geq 0 \qquad (\nu = 2J+1, 2J+2, \dots)$$

for all $\mu \in (-\infty, \tilde{\mu})$, ~~$J \geq$ being a fixed finite integer~~, then for any $\mu \in (\hat{\mu}, \tilde{\mu})$,

(i) a function $\sigma \in B\overset{N}{M}(0,\infty)$ exists such that

$$(38) \qquad D_{J+\nu} \mathcal{D}^{2J+2\nu+1} \psi(\mu) = (-1)^{J+\nu} \int_0^{\infty} t^{\nu} d\sigma(t); \qquad (\nu = 0, 1, \dots)$$

setting $\sigma_j(t) = \int_0^t t^j d\sigma(t)$ $(0 \leq t < \infty;\ j = 0, 1, \dots)$,

(ii) the functions

$$(39) \qquad R_{J,j}(\psi, \mu; k) = (-1)^{J+j} k^{2j} \int_0^{\infty} \frac{d\sigma_j(t)}{1 + k^2 t} \qquad (j = 0, 1, \dots)$$

~~where $\sigma_j(t) = \int_0^{\infty} t^j d\sigma(t)$~~ $(0 \leq t < \infty;\ j = 0, 1 \dots)$ are well defined for all finite k in both open sectors

(40) $$-\tfrac{1}{2}\pi < \arg(k) < \tfrac{1}{2}\pi, \quad \tfrac{1}{2}\pi < \arg(k) < \tfrac{3}{2}\pi;$$

(iii) if for some $\mu''\in(\mu,\infty)$ the integrals

(41) $$\int_{\cancel{0}\,\mu''}^{\infty} \mathcal{D}^{2J+2j+1}\psi(\mu')\,d\mu', \quad \left\langle \int_{-\infty}^{\cancel{0}\,\mu''} \mathcal{D}^{2J+2j+1}\psi(\mu')\,d\mu' \right\rangle \qquad (j=0,1,\ldots)$$

excist, then the functions $\tilde{R}_{J,j}(\psi,\mu;k)$ of formula (34) are well defined for all ~~to in the~~ finite k in the first <second> of the open sectors (40), and

(42) $$\tilde{R}_{J,j}(\psi,\mu;k) = R_{J,j}(\psi,\mu;k)$$

for such k;

(iv)

$$R_{J,0}(\psi,\mu;k) \sim \sum b_{J+\nu}\,\mathcal{D}^{2J+2\nu+1}\psi(\mu)\,k^{2\nu}$$

as k tends to zero in both open sectors (40);

(v) for all finite k in the closed sectors $-\tfrac{1}{4}\pi < \arg(k) < \tfrac{1}{4}\pi$, $\tfrac{3}{4}\pi < \arg(k) < \tfrac{5}{4}k$, the series (32) is semi-convergent in the sense that

(43) $$|R_{J,j}(\psi,\mu;k)| < |b_{J+j}\,\mathcal{D}^{2J+2j+1}\psi(\mu)\,k^{2j}| \qquad (j=0,1,\ldots)$$

and for finite real values of k, the values of the partial sums of this series oscillate about ~~that of~~ $R_{J,0}(\psi,\mu;k)$:

$$\cancel{(-1)^J \sum_0^{2j-1} b_{J+\nu}\,\partial^{2J+2\nu+1}\psi(\mu)k^{2\nu} < (-1)^J}$$

$$(-1)^J \sum_0^{2j-1} b_{J+\nu}\,\partial^{2J+2\nu+1}\psi(\mu)k^{2\nu} < (-1)^J R_{J,0}(\psi,\mu;k) < (-1)^J \sum_0^{2j} b_{J+\nu}\,\partial^{2J+2\nu+1}\psi(\mu)k^{2\nu},\ (j=0,1,\dots)$$

$(-1)^J R_{J,0}(\psi,\mu;k)$ being real and positive; ~~back~~

(vi) $R_{J,0}(\psi,\mu;k)$ is both the $(B',2)$ and the (B^2) sum of the

over the finite part of the nonnegative real k-axis

note that the nonnegative real k-axis

series (33) for all finite real values of k;

(vii) all forward diagonal sequences of Padé quotients derived from (33) regarded as a series in ascending powers of k^2

in the k plane

in the k-plane

converge uniformly over BE; $[-\infty,\infty]$ to $R_J(\psi,\mu;k)$.

Proof. For $|z|<4\pi^2$

$$\sum_1 b_\nu z^\nu = (2\pi^2)^{-1}\sum_1 \frac{\nu^{-2}}{1+(2\nu\pi)^{-2}z}$$

(47) $$= \int_0^{(2\pi)^{-2}} (1+uz)^{-1}\,d\sigma'(u)$$ $\frac{d\sigma'(u)}{1+uz}$

where $d\sigma'(u) = (2\nu^2\pi^2)^{-1}$ when $u=(2\nu\pi)^{-2}$ $(\nu=1,2,\dots)$ and $d\sigma'(u)=0$

elsewhere in $[0,(2\pi^2)^{-1}]$. Hence

$$(-1)^J b_{J+\nu} = (-1)^\nu \int_0^{(2\pi)^{-2}} u^\nu d\hat{\sigma}(u) \qquad (\nu=0,1,\dots) \tag{45}$$

where $\hat{\sigma}(u) = \int_0^u u^J d\sigma'(u)$ $(0 \le u \le (2\pi^2)^{-1})$.

It follows from simple extensions of a theorem of Hausdorf [,], S. Bernstein [ ] and Widder [ ] that the conditions (36) and (37) holding for all $\mu \in (\hat{\mu},\infty)$ and $\mu \in (-\infty,\tilde{\mu})$ respectively imply that

$$\begin{aligned} \partial^{2J+1}\psi_1(\mu) &= \int_0^\infty e^{-\mu v} d\sigma^{(1)}(v) \\ \partial^{2J+1}\psi_2(\mu) &= \int_0^\infty e^{\mu v} d\sigma^{(2)}(v), \end{aligned} \tag{46}$$

where $\sigma^{(1)}$ and $\sigma^{(2)}$ are both bounded nondecreasing real valued functions over $[0,\infty]$ and such that the above integrals exist at least for all stated values of μ.

Hence, for a fixed $\mu \in [\hat{\mu},\tilde{\mu}]$

$$\partial^{2J+2\nu+1}\psi(\mu) = \int_0^\infty v^\nu d\tilde{\sigma}(v), \qquad (\nu=0,1,\dots) \tag{47}$$

where

(48) $$\tilde{\sigma}(v)=\int_0^v\{\exp(-\mu v^{\frac{1}{2}})\,d\sigma^{(1)}(v^{\frac{1}{2}})+\exp(\mu v^{\frac{1}{2}})\,d\sigma^{(2)}(v^{\frac{1}{2}})\} \qquad (0\le v\le\infty)$$

($\tilde{\sigma}$ is, of course, bounded and nondecreasing over $[0,\infty]$ and all integrals (47) exist).

Combining formulae (45) and (47), we have

$$(-1)^J b_{J+\nu}\,\mathcal{D}^{2J+2\nu+1}\psi(\mu)=(-1)^\nu\int_0^{(2\pi)^{-2}}u^\nu\,d\hat{\sigma}(u)\int_0^\infty v^\nu\,d\tilde{\sigma}(v). \qquad (\nu=0,1,\dots)$$

Setting $uv=t$, we obtain formula (38), with

$$d\sigma(0)=[\hat{\sigma}\{(2\pi)^{-2}\}-\hat{\sigma}(0)]\,d\tilde{\sigma}(0)=(-1)^J b_J\,d\tilde{\sigma}(0)$$

$$d\sigma(t)=\int_0^{(2\pi)^{-2}}d\tilde{\sigma}(t/u)\,d\hat{\sigma}(u). \qquad (0<t<\infty)$$

For a fixed value of k in one of the open sectors ( ), $|(1+k^2t)^{-1}|\le M$ for some $M\in(0,\infty)$ and all $t\in[0,\infty]$. the integrals (39) exist as stated.

It is quite possible that for certain functions ψ considered in the theorem, the integral expressions (34) are not defined (for example, when $\mathcal{D}^{2J+1}\psi(\mu)=1$

and $j=0$). Furthermore, if the first set of integrals (41) exists, $\mathfrak{D}^{2J+1}\psi_2(\mu)=0$ identically; for a nonzero ~~nonconstant~~ function of the form $\mathfrak{D}^{2J+1}\psi_2(\mu)$ is also nondecreasing ~~increases at least exponentially~~ as μ tends to infinity through positive real values, and ~~if $\mathfrak{D}^{2J+1}\psi_2(\mu)$ is a nonzero constant~~ $\tilde{R}_{J,0}(\psi,\mu;h)$ does not exist. Conversely, if the second set of integrals exists, $\mathfrak{D}^{2J+1}\psi_1(\mu)=0$ identically. Nevertheless, subject to one of the auxiliary conditions stated in clause (iii), the integrals $\{\tilde{R}_{J,j}(\psi,\mu;k)\}$ do exist. We assume that the first set of integrals (41) exists, so that $\sigma^{(2)}$ may be taken to be identically zero in formula (48). Formula (34) may then be rewritten as

$$\tilde{R}_{J,j}(\psi_1,\mu;k)=k^{2j-1}\int_{\mu}^{k\infty}\beta_{2J+2j+1}\{(\mu'-\mu)/k\}\int_0^{\infty}v^{2j}e^{-v\mu'}\,d\sigma^{(1)}(v)\,d\mu'. \tag{50}$$

$|\beta_{2J+2j+1}\{(\mu'-\mu)/k\}|$ is uniformly bounded for μ' lying upon the path of integration. From the first of the auxiliary conditions of clause (iii), the double integral

$$\int_{\mu''}^{k\infty} \int_0^\infty v^{2j} e^{-v\mu'} d\sigma^{(1)}(v)\, d\mu'$$

exists for $\mu = \overset{\mu''}{0}$ and $k=1$, and is therefore absolutely convergent for all $\mu \in \overset{(\hat{\mu},\infty)}{[0,\infty)}$ and ~~for~~ finite k in the first open sector (40). Hence the double integral ( ) exists.

~~From~~ Hence formula (34) may be rewritten as

(4ϕ) $R_{r,j} =$ 30, 4ϕ, 4t, 4ϕ

Employing the notations used in formulae (-), we then obtain find that

$$\tilde{R}_{J,j}(\psi_1,\mu;k) = k^{2j}\int_0^\infty\int_0^\infty \beta_{2J+2j+1}(y)\, v^{2j} e^{-\mu v - kvy}\, d\sigma^{(1)}(v)\, dy$$

$$= \frac{(-1)^{J+j}k^{2j}}{2^{2J+2j}\pi^{2J+2j+1}}\int_0^\infty \left[\int_0^\infty \Sigma_1 \frac{\sin 2\pi\nu y}{\nu^{2J+2j+1}} e^{-kvy}\, dy\right] v^{2j} e^{-\mu v}\, d\sigma^{(1)}(v)$$

$$= \frac{(-1)^{J+j}k^{2j}}{2^{2J+2j+1}\pi^{2J+2j+2}}\int_0^\infty \left\{\Sigma_1 \frac{\nu^{-2J-2j-2}}{1+k^2(2\pi\nu)^{-2}v^2}\right\} v^{2j} e^{-\mu v}\, d\sigma^{(1)}(v)$$

$$= (-1)^{J+j}k^{2j}\int_0^\infty \left\{\int_0^{(2\pi)^{-2}} \frac{u^{J+j}\, d\sigma'(u)}{1+k^2 u \cancel{{}^{2}} v^2}\right\} v^{2j} e^{-v\mu}\, d\sigma^{(1)}(v)$$

$$= (-1)^{J+j}k^{2j}\int_0^{(2\pi)^{-2}} u^{J+j}\left\{\int_0^\infty \frac{v^j \exp(-\mu v^{\frac{1}{2}})\, d\sigma^{(1)}(v^{\frac{1}{2}})}{1+k^2 uv}\right\} d\sigma'(u)$$

$$= (-1)^{J+j}k^{2j}\int_0^{(2\pi)^{-2}}\int_0^\infty \frac{u^j v^j\, d\hat{\sigma}(u)\, d\check{\sigma}(v)}{1+k^2 uv}$$

$$= (-1)^{J+j} k^{2j} \int_0^\infty \frac{t^j d\sigma(t)}{1+k^2 t}$$
$$= R_{J,j}(\psi_1, \mu; k)$$

finite

for values of k lying in the finite part of the first open sector (40). The case in which the second set of integrals (41) exists is dealt with similarly.

With $\{f_\nu\}$ and $f(z)$ given by formulae (2; 4)

$$f(z) \sim \sum f_\nu z^\nu$$

as z tends to zero in the open sector $0 < \arg(z) < 2\pi$ [ ].

The result of clause (iv) follows immediately from this remark.

finite

Furthermore $|(1-zt)^{-1}| \leq 1$ for all finite part of z in half-plane $Re(z) \leq 0$ and all $t \in [0,\infty]$. Hence, in the notation of formula (7), $|z^m f_m(z)| \leq |f_m z^m|$ $(m=0,1,\ldots)$ for all z in the finite part of the half-plane $Re(z) \leq 0$.

~~the first result of clause (v)~~

Formula (43) follows from this result. Lastly, for $z \in (-\infty, 0]$, we have $\sum_0^{2m-1} f_\nu z^\nu < f(z) < \sum_0^{2m} f_\nu z^\nu$ $(m=0,1,\ldots)$ with

$f(z)>0$, and the second last result of clause (v) follows immediately.

From formula (45), $\mathfrak{D}_{J+\nu}=O\{(2\pi)^{-2\nu}\}$. Formula (44) implies that the function $\mathscr{D}^{2J+1}\psi_1(\mu)$, regarded as a function of the complex variable μ, is analytic in the finite part of the half-plane $\mathrm{Re}(\mu)>\hat{\mu}$; similarly, the function $\mathscr{D}^{2J+1}\psi_2(\mu)$ is analytic in the finite part of the half-plane $\mathrm{Re}(\mu)<\tilde{\mu}$. Setting $\delta=\min(\mu-\hat{\mu},\tilde{\mu}-\mu)$, we

??? have $\mathscr{D}^{2J+2\nu+1}\psi(\mu)=O\{(2\nu)!\,(\tfrac{1}{2}\delta)^{-2\nu}\}$. ~~for all $\mu\in[0,\infty)$.~~

Hence

(50) $$\mathfrak{D}_{J+\nu}\mathscr{D}^{2J+2\nu+1}\psi(\mu)=O\{(2\nu)!\,(\pi\delta)^{-2\nu}\}$$

~~for all such μ.~~ The results of Theorem 2 with $\chi=2$ and those of Theorem 3 may now be applied to prove the remaining two clauses of the theorem.

In the above theorem, the functional remainder terms ~~[illegible]~~,

$\{R_{J,j}(\nu,\mu;k)\}$ of formula (39) ~~were~~ is expressed as Stieltjes transforms, and ~~were~~ was shown to be the remainder terms for the series (33) when k belongs to the finite parts of the two open sectors (40). The further results of the theorem were expressed in terms of ~~these~~ this functions. The result of formula (42) involving the functions $\{\tilde{R}_{J,j}(\nu,\mu;k)\}$ ~~were~~ was derived merely to connect our theory with the more conventional theory of the Euler-Maclaurin series. The latter functions ~~are~~ is (see formula (49)) defined in terms of an integral whose integrand is ~~a Laplace transform~~ itself a Laplace integral. A function defined by ~~a Laplace transform such as~~ means of a Laplace integral, such as

$$\cancel{f(z)=z^{-1}\int}$$

$$f(z)=z^{-1}\int_0^{e^{i\phi}\infty} e^{-z^{-1}u}\,g(u)\,du \tag{51}$$

$(-\pi<\arg\phi\le\pi)$

is in general so defined over ~~a~~ the finite part of the half-plane $\phi+\frac{\pi}{2}<\arg(z)<\phi+\frac{3\pi}{2}$ only.

Nevertheless (see, for example, §2 of Ch. 6 of [ ]) if the function $g(u)$ in formula ($5\overset{1}{0}$) is analytic and bounded in

sector

the angle $|\arg(u)-\phi|<\xi$ $(0<\xi\leq\frac{1}{2}\pi)$, then $f(z)$ can be

$\phi+\frac{\xi}{2}\pi-\alpha<\arg(z)<\phi+\frac{\xi}{2}\pi+\alpha$

continued analytically in the sector $|\arg(z)+\phi|<\frac{1}{2}\pi+\xi$.

In particular, if in formula ($5\overset{1}{0}$) $\phi=0$ and $g(u)$ is

itself a Laplace ~~transform~~ integral transform of the form

$$(5\overset{2}{1})\qquad g(u)=\int_0^\infty e^{-tu}\,d\sigma(t)$$

with σ of bounded variation over $[0,\infty]$, then, since $g(u)$

is analytic and bounded for $|\arg(u)|<\frac{1}{2}\pi$, $f(z)$ can

$0<\arg(z)<2\pi$

be continued analytically in the sector $|\arg(z)|<\pi$. From

$\frac{1}{2i}$?

formula ($5\overset{1}{0}$) with $\phi=0$ and formula ($5\overset{2}{1}$), $f(z)$ has the form (4)

for finite $z\notin[-\infty,0]$. This Stieltjes transform automatically

provides the analytic continuation of the transform ($5\overset{1}{0}$). In

short, the function$\{R_{J,j}(\psi,\mu;k)\}$ is defined for

values of k for which the $\{\tilde{R}_{J,j}(\psi,\mu;k)\}$ is not, and naturally we preferred to use the former. These considerations arise, of course, from the special nature of the function ψ considered in the theorem. For more general functions, formula (33) is unavailable and the remainder terms $\{R_{J,j}(\psi,\mu;k)\}$ must be, and ~~are~~ is, used.

As was remarked above, Hardy has ~~given a result~~ investigated the concerning (B^2) summability of the delayed Euler-Maclaurin series. However, his result concerns only series whose (B^2) sum has a representation of the form $\tilde{\tilde{R}}_{J,0}(\psi,\mu;k)$; Hardy's result result may be applied only to those functions ψ of Theorem 6 for which one of the components ψ_1 or ψ_2 is identically zero. By expressing the sum function in the form $R_{J,0}(\psi,\mu;k)$, a result concerning the (B^2) summability of composite functions ψ whose components are not restricted in this way has been derived. In this sense clause (vii) of Theorem 6 may be regarded as being an extension of Hardy's result. A similar remark may ~~be made~~ with equal be made concerning the Borel summability of the series considered in the following theorem.

Theorem 7. Let the integer J and the functions $\phi(\mu)$ and $R_{J,0}(\nu,\mu;k)$ be as described in Theorem 6.

(i) ~~The~~ All forward diagonal sequences of Padé quotients ~~derived from ( ) regarded as a series in ascending powers of k~~

derived from the intercalated delayed Euler-Maclaurin series (35) regarded as a series in ascending powers of k in which all coefficients of odd powers of k are zero, converge uniformly over $BE_i[-\infty,\infty]$ to $R_{J,0}(\nu,\mu;k)$.

(ii) $R_{J,0}(\nu,\mu;k)$ is the Borel sum of the series (35) over the finite part of the real k-axis.

Proof. Rewrite the series (35) as $\sum_i \tilde{f}_\nu (ik)^\nu$, so that $\tilde{f}_{2\nu+1}=0$ $(\nu=0,1,\ldots)$. As was shown in the proof of Theorem 6, $\tilde{f}_{2\nu}=f_\nu$ $(\nu=0,1,\ldots)$, where the $\{f_\nu\}$ have a representation of the form (2). Thus the $\{\tilde{f}_\nu\}$ are Hamburger moments of the form

$$f_\nu=\int_{-\infty}^{\infty} t^\nu d\tilde{\sigma}(t) \qquad (\tilde{\sigma}\in BM(-\infty,\infty);\ \nu=0,1,\ldots)$$

where $d\tilde{\sigma}(0)=d\sigma(0)$, $d\tilde{\sigma}(t)=d\tilde{\sigma}(-t)=\frac{1}{2}d\sigma(t^2)$ $(0<t\leq\infty)$, the function $\tilde{\sigma}$ being anti-symmetric about the origin. In terms of this function, formula (33) with $j=0$ may be rewritten as

$$R_{J,0}(\psi,\mu;k)=(-1)^J\int_{-\infty}^{\infty}\frac{d\tilde{\sigma}(t)}{1+ikt}.$$

It follows from relationship (50) that the $\{\tilde{f}_\nu\}$ obey an order relationship of the form (34). Theorem $\overset{5}{4}$ may now be applied to derive the required results.

The asymptotic and semi-convergent properties of by 69

Our account of the Euler-Maclaurin series was first motivated by an appeal to the function $\hat{R}_{0,0}(\psi,\mu;k)$ of formula (32) which involves the difference between an integral over a semi-infinite range and an infinite sum derived from a function ψ; we then proceeded rapidly to functions ψ for which neither ~~sum nor~~ integral nor sum are defined. Nevertheless, it is possible to impose conditions upon ψ which suffice to ensure that both integral and sum exist. In this way we are lead to a variant of our theory in which the function $R_{0,0}(\psi,\mu;k)$ features the intercalated delayed Euler-Maclaurin series (35) derived from the function ψ of the above theorem may easily be deduced from those of the series (31) given in Theorem 6.

For the sake of completeness, we remark that there is a simple relationship between the Padé quotients discussed in Theorems 6 and 7. Denote the Padé quotients derived from any series $\sum f_\nu k^\nu$ ($f_0 \neq 0$) by $\{P_{i,j}(k)\}$, and those from the series $\sum \tilde{f}_\nu k^\nu$, where $\tilde{f}_{2\nu} = f_\nu$, $\tilde{f}_{2\nu+1} = 0$ ($\nu = 0, 1, \ldots$) by $\{\tilde{P}_{i,j}(k)\}$. Then $\tilde{P}_{2i,2j}(k) = \tilde{P}_{2i+1,2j}(k) = \tilde{P}_{2i,2j+1}(k) = \tilde{P}_{2i+1,2j+1}(k) = P_{i,j}(k^2)$ ($i, j = 0, 1, \ldots$); these are the relationships subsisting between the two sets of Padé quotients of Theorems 6 and 7.

Theorem 8. The real valued function ψ being as described in each of the following clauses, set $P_0(\mu) = \dfrac{-\mu D\psi(\mu)}{\psi(\mu)}$, $P_{n+1}(\mu) = \ln^n[\mu\{P_n(\mu) - 1\}]$ ($n = 0, 1, \ldots$), where $\ln^n(a)$ denotes

$\ln\{\ldots \ln(a)\}$, the logarithm being taken n times.

(i) If for some $\hat{\mu} \in (-\infty, \infty)$, $(-\partial)^{\nu}\psi(\mu) \leq 0$ ($\nu = 0, 1, \ldots$) for all $\mu \in (\hat{\mu}, \infty)$, and for some finite integer $n \geq 0$, $\lim_{\mu=\infty} P_n(\mu) > 1$, then the results of Theorems 6 and 7 hold for $J=0$ and $\tilde{\mu} = \infty$ (the first variant of clause (iii) of Theorem 6 now being concerned) and, for $\mu \in (\hat{\mu}, \infty)$ and finite k in the first of the sectors (40)

$$(53) \qquad R_{0,0}(\psi, \mu; k) = \hat{R}_{0,0}(\psi, \mu; k)$$

where $\hat{R}_{0,0}(\psi, \mu; k)$ is defined by formula (32).

(ii) If for some $\tilde{\mu} \in (-\infty, \infty)$, $\partial^{\nu}\psi(\mu) \geq 0$ ~~for all $\mu \in ($~~ ($\nu = 0, 1, \ldots$) for all $\mu \in (-\infty, \tilde{\mu})$, and for some finite integer $n \geq 0$, $\lim_{\mu=-\infty} P_n(\mu) > 1$, then the results of Theorems 6 and 7 hold for $J=0$ and $\hat{\mu} = -\infty$ (the second variant of clause (iii) of Theorem 6 now being concerned) and formula (53) now holds for $\mu \in (-\infty, \tilde{\mu})$

and finite k in the second of the sectors (40).

Proof. Under the conditions of clause (i), the function ψ has a representation of the form

$$\psi(\mu) = -\int_0^\infty e^{-v\mu} d\sigma''(v)$$

where the real valued function σ' is bounded and nondecreasing over $[0,\infty]$ and the integral exists for all $\mu \in (\hat{\mu},\infty)$. As in the proof of Theorem 6, we have

$$(54)\quad R_{0,0}(\psi,\mu;k) = k^{-2}\int_0^\infty \left\{\frac{kv}{e^{kv}-1} - 1 + \tfrac{1}{2}kv\right\} v^{-1} e^{-v\mu} d\sigma''(v)$$

for all finite k in both open sectors (32).

The function $\psi(\mu)$ is analytic for all finite μ in the half-plane $Re(\mu) > \hat{\mu}$. The auxiliary conditions imposed upon ψ suffice to ensure (see, for example, [ ]§) that in formula (32) the integral exists and the infinite sum converges when $k=1$; that they continue to do so for all finite k in the first of the open sectors (40) is easily demonstrated. Furthermore

$$\int_\mu^{k\infty} \psi(\mu')\,d\mu' = -\int_0^\infty v^{-1} e^{-v\mu}\,d\sigma''(v)$$

and

$$-\frac{1}{2}k\sum_1\left[\psi(\mu+\nu k)+\psi\{\mu+(\nu+1)k\}\right] = k\int_0^\infty\left\{\frac{1}{e^{kv}-1}+\frac{1}{2}\right\}e^{-v\mu}\,d\sigma'''(v)$$

for these values of k. Formula (53) follows immediately.

The second clause of the theorem is proved in the same way.

For the sake of completeness we remark that formulae of the form (54) may also be derived for $R_{0,0}(\nu_1,\mu;k)$ under the conditions of Theorem 6 when $J=0$: one replaces $v^{-1}d\sigma'(v)$ by $-\cancel{v}\,d\sigma^{(1)}(v)$. ~~Under the same conditions the complete function $R_{0,0}(\nu,\mu;k)$ may be expressed in a similar way.~~ ($R_{0,0}(\nu_2,\mu;k)$ may also be expressed in the same way). When $J\neq 0$ an integral expression involving the function

$$-\left\{\frac{kv}{e^{kv}-1}-1+\tfrac{1}{2}kv\right\}-\sum_{0}^{J-1} b_\nu k^{2\nu}v^{2\nu}\Big\}(kv)^{-2J-2}$$

may also be derived.

Assuming the integral $\int_\mu^{k\infty}\psi(\mu')\,d\mu'$ to exist, we have $\Delta\int_\mu^{k\infty}\psi(\mu')\,d\mu' = -\psi(\mu)$, and the integral may be expressed as $-\Delta^{-1}\psi(\mu)$. This integral and the Euler-Maclaurin series (3.1) may be incorporated in an extended series $\sum b'_\nu \Delta^{2\nu-1}\psi(\mu)k^{2\nu}$, where $b'_0=1$, $b'_\nu=-b_{\nu+1}$ ($\nu=0,1,\dots$), and the suggestion naturally prompts itself that it may be possible to exhibit the numbers $\{b'_\nu \Delta^{2\nu-1}\psi(\mu)\}$ as Stieltjes moments of the form $\{(-1)^\nu f_\nu\}$, where the $\{f_\nu\}$ have a representation of the form (2). This appears in general to be difficult, and in certain cases may be proved to be impossible. Set $\psi(\mu)=e^{-\mu}$, so that when $\mu=0$, $b_\nu \Delta^{2\nu+1}\psi(\mu) = -b_\nu$ ($\nu=0,1,\dots$). The numbers $\{(-1)^\nu b_\nu\}$ are Stieltjes moments generated by the ~~distribution~~ function σ' of formula (44). It is known from theory due to Stieltjes [ ] that a sequence of Stieltjes moments $\{f_\nu\}$ permits an extension, in the sense that ~~with $f'_\nu = f_{\nu+1}$ ($\nu=0,1,\dots$)~~ a number f'_0 exists such that with $f'_\nu = f_{\nu+1}$ ($\nu = 0,1\dots$) the numbers $\{f'_\nu\}$ are also Stieltjes moments, if and only if the series $\sum f'_\nu z^\nu$ generates a corresponding continued fraction of the form

$$\frac{1}{a'_1 -}\;\frac{z}{a'_2 -}\;\frac{z}{a'_3 -}\;\cdots$$

$a'_\nu > 0$!

for which $\sum_1 a'_{2\nu} < \infty$. Using formula (29) and a contracted version of Lagrange's continued fraction for the exponential function ([ ] § 28), we find that the series $\sum (-1)^\nu f_\nu z^\nu$ generates the corresponding continued fraction

$$\frac{1}{12 -}\;\frac{z}{5 -}\;\frac{z}{28 -}\;\frac{z}{9 -}\;\frac{z}{44 -}\;\frac{z}{13 -}\;\cdots$$

Since the series $\sum_1 (4\nu+1)$ diverges, no extension of the sequence $\{(-1)^\nu b_\nu\}$ is possible, and the Euler-Maclaurin series in this case may ~~be~~ not be extended, by the adjunction of an integral or indeed any other function, in such a way that the coefficients of its extensional are related to Stieltjes moments as described.

Theorems 6 – 8 concern functions ψ of a very limited class; nevertheless this class ~~includes~~ contains the two functions generating the most extensively studied Euler-Maclaurin series.

Taking $\psi(\mu) = \ln(\mu)$ and $J=0$ in ~~form~~ Theorem 6 (so that $\overset{=0}{\hat{\mu}'}, \tilde{\mu} = \infty$) we have $\mathfrak{D}^{2\nu+1}\psi(\mu) = (2\nu)!\,\mu^{2\nu+1}$ $(\nu = 0, 1, \ldots)$

$\mu^{-2\nu-1}$

and the sum function of the Euler-Maclaurin series in question is

$$k^{-2}\int_0^\infty \left\{\frac{kv}{e^{kv}-1}-1+\frac{1}{2}kv\right\}v^{-2}e^{-v\mu}\,dv.$$

Setting $\lambda=\mu/k$, and using Binet's first integral expression for $\ln\{\Gamma(\lambda)\}$ (see [ ]Ch), we derive the formula

$$\lambda\left[\ln\{\Gamma(\lambda)\}-(\lambda-\tfrac{1}{2})\ln(\lambda)+\lambda-\tfrac{1}{2}\ln(2\pi)\right]\sim\sum \frac{B_{2\nu+2}}{(2\nu+1)(2\nu+2)}\lambda^{-2\nu}. \quad (54)$$

It follows from Theorem {6} that the numbers $\left\{\frac{(-1)^\nu B_{2\nu+2}}{(2\nu+1)(2\nu+2)}\right\}$ have a representation of the form (2) (this is known: see, for example, [ ]Ch), that formula (54) is valid as λ tends to infinity in the open sector $-\frac{1}{2}\pi<\arg(\lambda)<\frac{1}{2}\pi$ (formula (54) is, of course, equivalent to Stirling's asymptotic representation of the logarithm of the gamma function), that for $\lambda\in(0,\infty)$ for $-\frac{1}{4}\pi\leq\arg(\lambda)\leq\frac{1}{4}\pi$ the series in formula (54) is semi-convergent in the sense that if

$$\lambda\left[\ln\{\Gamma(\lambda)\}-(\lambda-\tfrac{1}{2})\ln(\lambda)+\lambda-\tfrac{1}{2}\ln(2\pi)\right]=\sum_0^{n-1}\frac{B_{2\nu+2}}{(2\nu+1)(2\nu+2)}\lambda^{-2\nu}+R_n(\lambda) \quad (n=0,1$$

then $|R_n(\lambda)| \leqslant \left|\frac{B_{2n+2}}{(2n+1)(2n+2)}\lambda^{-2n}\right|$ $(n=0,1,\ldots)$ (this simple result is not given in any of the standard ~~for~~ reference works (for example, [ ] vol 1, Ch.), although the author can hardly believe that it is new), that for $\lambda \in (0,\infty)$ the partial sums of the series upon the right hand side of relationship (54) oscillate about the value of the expression upon the left hand side (this is well known: see, for example, [ ] Ch), that all forward diagonal sequences of Padé quotients generated by the series in relationship (54), regarded as a series in ascending powers of λ^{-2} converge uniformly over any domain $BE_i[-\infty,\infty]$ lying in the right half-plane to the given function, and that the series in relationship (54) is both $(\mathcal{B}',2)$ and $(\mathcal{B}^2)$ summable to the given function over the finite part of the ~~non~~ positive real ~~axis~~ λ-axis; from Theorem 7 it follows that ~~the~~ all forward diagonal sequences of Padé quotients derived from the intercalated version of the series in relationship (54), regarded now as a series in ascending powers of λ^{-1}, converge uniformly over any domain ~~B~~ $[-\infty,\infty]$ as described above to the given function, and that the intercalated series is Borel summable to the given function over the finite part of the positive real axis. Results valid for λ in the left half-plane $Re(\lambda) < 0$ may be obtained by reversing the sign of λ in the generating function upon the left hand side of relationship (54).

??? $F(\lambda)$ singular at pts in $Re(\lambda)<0$

?

Taking $\psi(\mu) = -\mu^{-s}$ $(s \in (-1,\infty); \mu \in (0,\infty))$ and $J=0$

No mention was made in Theorem of analytic continuation of the function $R_{J,0}(\psi,\mu;k)$ across the imaginary axis in the k-plane, and of the possibility that the series ( ) might represent the function so obtained asymptotically over a sector of magnitude greater than π in the k-plane. As we shall shortly see, there exist functions ψ to which Theorem applies for which the above possibilities do not exist; conversely conditions upon ψ may be imposed such that they do.

Theorem . Let the integer J, the interval $[\hat{\mu},\check{\mu}]$ and the functions ψ_1, ψ_2, ψ and $R_{J,0}(\psi,\mu;k)$ be as defined in Theorem .

(i) In the representations ( ) which may be given for the functions $2^{2J+1}\psi_1(\mu)$ and $2^{2J+1}\psi_2(\mu)$, let either $\sigma^{(1)}$ or $\sigma^{(2)}$ be nonanalytic over an interval of the form $[V,\infty]$. Then the imaginary axis in the k-plane is a natural boundary of the function $R_{J,0}(\psi,\mu;k)$ and analytic continuation across it is impossible.

(ii) In the representations ( ), let $\frac{d\sigma^{(1)}(v)}{dv}=\omega^{(1)}(v)$, $\frac{d\sigma^{(2)}(v)}{dv}=\omega^{(2)}(v)$, where $\omega^{(1)}$ and $\omega^{(2)}$ are analytic in the neighbourhood of all points of the nonnegative real axis, and such that $\omega^{(1)}(v)=O(e^{\hat{\mu}v}v^{x})$ for some $x\in[0,\infty)$ and $\omega^{(2)}(v)=O(e^{-\tilde{\mu}v}v^{y})$ for some $y\in[0,\infty)$, both for all v in the right half-plane $Re(v)>0$.

a) The function $R_{J,0}(\psi,\mu;k)$ may be continued analytically across both the positive and negative imaginary axes, in the k-plane.

b) In the plane which is an extension of the half-plane over which the function $R_{J,0}(\psi,\mu;k)$ is defined by formula ( ), the possible singularities of this function are confined to that part of the real axis not lying in the half-plane of definition. The series ~~represents~~ ( ) represents $R_{J,0}(\psi,\mu;k)$ asymptotically as k tends to zero in any open sector not containing that part of the real axis upon which the singularities of this function lie.

c) In the Riemann surface which may be constructed for the function $R_{J,0}(\psi,\mu;k)$, all singularities of this function are confined to the real axes belonging to the constituent sheets. The series ( ) represents each function defined over each separate sheet asymptotically as k tends to zero in the two open sectors $\Delta(0,\pi)$

Proof. If either of the functions $\sigma^{(1)}$ or $\sigma^{(2)}$ is nonanalytic throughout an interval such as (A^2, B^2) $(0<A<B<\infty)$ of the real axis then, from formula ( ), $\tilde{\sigma}$ is nonanalytic over the interval (A,B). Formula ( ) may be expressed in the form $\sigma(t) = \sum_1 \sigma_\nu(t)$, where

$$\sigma_\nu(t) = 2(2\nu\pi)^{-2J-2}\,\tilde{\sigma}(4\nu^2\pi^2 t). \quad (\nu=0,1,\ldots;\ 0\leq t<\infty)$$

Subject to the above conditions, σ is nonanalytic over the succession of intervals $(4\nu^2\pi^2)[A,B)$ $(\nu=1,2,\ldots)$ (i.e. certainly over some interval of the form $[0,\epsilon]$). If $B=\infty$, σ is nonanalytic over the entire nonnegative real axis. The imaginary axis in the k-plane is a natural boundary of the function $R_{J,0}(\psi,\mu;k)$ defined by formula ( ).

With $\omega^{(1)}$ and $\omega^{(2)}$ as defined in clause (ii), we find that

$$R_{J,0}(\nu,\mu;k)=\int_0^\infty \frac{\omega(t)}{1+k^2t}\,dt$$

where

$$\omega(t)=\sum\nolimits_1 (2\nu\pi)^{-2J-1}\{\exp(-2\mu\nu\pi t^{\frac{1}{2}})\omega^{(1)}(2\nu\pi t^{\frac{1}{2}})$$
$$+\exp(2\mu\nu\pi t^{\frac{1}{2}})\omega^{(2)}(2\nu\pi t^{\frac{1}{2}})\}t^{-\frac{1}{2}}.$$

It follows from the order conditions imposed upon $\omega^{(1)}$ and $\omega^{(2)}$ that, since $\mu\in(\hat{\mu},\hat{\hat{\mu}})$, the series ( ) defines $\omega(t)$ for all $|\arg(t)|<\pi$ as an analytic function for all finite nonzero t in this sector. Thus $R_{J,0}(\nu,\mu;k)$ can be continued analytically across the imaginary k-axis and the further properties of this function, both in the half-plane containing the half-plane over which formula ( ) is taken to define $R_{J,0}(\nu,\mu;k)$ and over the Riemann surface which may be constructed for this function, are as described in clauses (ii) and (iii). Also ω obeys the order condition, easily obtained from Theorem , which suffices to ensure that the series ( ) represents both $R_{J,0}(\psi,\mu;k)$ and the functions defined over the various sheets of the Riemann surface that may be constructed for this function as described in clauses (ii) and (iii).

As a point of detail, we mention that by taking $\omega^{(2)}$ to be identically zero, $t^{-\frac{1}{2}}\omega^{(1)}(t^{\frac{1}{2}})$ to be an even function of t, and setting $\mu=0$, ω becomes an even function. In this case, the same function is obtained by analytic continuation of $R_{J,0}(\psi,\mu;k)$ over both the negative and positive imaginary axes. However, if μ is changed, ω ceases to be an even function, and two different functions are obtained by analytic continuation.

This indicates that the structure of the Riemann surface associated with $R_{J,0}(\nu,\mu;k)$ can be radically affected by the value of μ.

in Theorem 6 (so that $\hat{\mu}=0$, $\check{\mu}=\infty$ again) we have $\partial^{2\nu+1}\phi(\mu)=$ $\frac{\Gamma(s+2\nu+1)}{\Gamma(s)}\mu^{-s-2\nu-1}$ ($\nu=0,1,\ldots$). As in the first example above, we set $\lambda=\mu/k$ and derive the asymptotic expansion

$$(55)\quad \lambda^{s+1}\int_0^\infty\left\{\frac{v}{e^v-1}-1+\tfrac{1}{2}v\right\}e^{-\lambda v}v^{s-2}dv \sim \sum \frac{\Gamma(s+2\nu+1)}{\Gamma(2\nu+3)}B_{2\nu+2}\lambda^{-2\nu}$$

valid as λ tends to infinity in the open sector $-\frac{1}{2}\pi<\arg(\lambda)<\frac{1}{2}\pi$. To connect this example with more familiar theory, we assume that $s\in(1,\infty)$, so that clause (i) of Theorem 8 with $n=0$ becomes operative, and introduce the extended

$\zeta(s,\lambda)$ ζ-function defined by the series $\zeta(\lambda,s)=\sum(\lambda+\nu)^{-s}$

$\zeta(s,0)$ so that $\zeta(0,s)=\zeta(s)$ ($\mathrm{Re}(s)>1$) and $\zeta(\lambda,r)=\frac{2(-1)^r}{r!}\partial^r\ln\{\Gamma(\lambda)\}$ ($r=2,3,\ldots$). We then have

$$(56)\quad \lambda\Gamma(s-1)\left\{\lambda^s(s-1)\zeta(\lambda,s)-\lambda-s+\tfrac{3}{2}\right\}\sim\sum\frac{\Gamma(s+2\nu+1)}{\Gamma(2\nu+3)}B_{2\nu+2}\lambda^{-2\nu}$$

as λ tends to infinity in the open sector $-\frac{1}{2}\pi<\arg(\lambda)<\frac{1}{2}\pi$.

The properties of the asymptotic series upon the right hand side of relationship (55) for $s \in (-1, \infty)$ and upon that of relationship (56) for $s \in (1, \infty)$ with respect to their respective generating functions, and those of the Padé quotients derived from these series, may be derived from Theorem and listed, just as we did earlier for the series in formula (54).

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formula
λ – $\ln \Gamma(\lambda)$ –

Formula ( ) is, of course, equivalent to Stirling's asymptotic representation of the logarithm of the gamma function. The behaviour of this series, and that of the function which it represents, may be discussed in extenso upon the basis of the theory given above for the Euler-Maclaurin series, or more directly, as in § .

Taking $\psi(\mu) = -\mu^{-s}$ ($s \in (-1,\infty)$; $\mu \in (0,\infty)$) and $J=0$ in Theorem (so that $\hat{\mu}=0, \tilde{\mu}=\infty$ again) we have $\mathfrak{D}^{2\nu+1}\psi(\mu) = \dfrac{\Gamma(s+2\nu+1)}{\Gamma(s)}\mu^{-s-2\nu-1}$ ($\nu=0,1,\dots$). As in the first example above, we set $\lambda=\mu/k$ and derive the asymptotic expansion

$$\lambda^{s+1}\int_0^\infty \left\{\frac{v}{e^v-1} - 1 + \tfrac{1}{2}v\right\} e^{-\lambda v} v^{s-2}\,dv = \sum \frac{\Gamma(s+2\nu+1)}{\Gamma(2\nu+3)} B_{2\nu+2}\,\lambda^{-2\nu}$$

When $s \in (1,\infty)$, we obtain

$$\Gamma(s-1)\left\{\lambda^{s-1}(s-1)\,\zeta(\lambda;s) - \qquad\qquad \right\} \sim \sum \frac{\Gamma(s+2\nu+1)}{\Gamma(2\nu+3)} B_{2\nu+2}\,\lambda^{-2\nu-2};$$

this formula has already been discussed in §.

Analytic Continuation

Series of the form ( ) possess certain properties of symmetry which both make it possible to obtain approximating fractions of general order by use of a uniform process and greatly facilitate the further study of such fractions.

In many applications of the the theory of functions, the formulae turn out to be simpler if they are presented in terms of the series

$$\sum f_\nu \lambda^{-\nu-1}$$

and we deal with such series in this section. (Since $z\sum f_\nu z^\nu = \sum f_\nu \lambda^{-\nu-1}$ and $\lambda \sum f_\nu \lambda^{-\nu-1} = \sum f_\nu z^\nu$ when $z = \lambda^{-1}$, results formulated in terms of either series can easily be presented in terms of the other.)

Series of the form ( ) are naturally associated with functions defined by the formula

$$F(\lambda;\alpha,\beta)=\int_{\alpha}^{\beta}\frac{d\sigma(t)}{\lambda-t}$$

($[\alpha,\beta]\subseteq[-\infty,\infty]$) for $\lambda\notin[\alpha,\beta]$. In particular, if the coefficients $\{f_\nu\}$ have a representation of the form

$$f_\nu=\int_{\alpha}^{\beta}t^\nu d\sigma(t) \qquad (\sigma\in BN[\alpha,\beta];\ \nu=0,1,\ldots)$$

and $[\alpha,\beta]\subset(-\infty,\infty)$, then $F(\lambda;\alpha,\beta)=\sum f_\nu\lambda^{-\nu-1}$ for $|\lambda|>\max(|\alpha|,|\beta|)$. If the coefficients $\{f_\nu\}$ have a representation of the form ( ), then [ ]

$$F(\lambda;0,\infty)\sim\sum f_\nu\lambda^{-\nu-1}$$

as λ tends to infinity in $\Delta(0,2\pi)$. If they have a representation of the form ( ), then [ ] a relationship similar to ( ) with $F(\lambda;0,\infty)$ replaced by $F(\lambda;-\infty,\infty)$ holds for λ tending to infinity in the two open sectors $\Delta(0,\pi)$ and $\Delta(\pi,2\pi)$ (if, in addition, σ is constant over an interval of the form $[\beta,\infty]$ $\langle[-\infty,\alpha]\rangle$, then a relationship analogous to ( ) also holds for $\arg(\lambda)=0$ $\langle\arg(\lambda)=\pi\rangle$. When $0\notin(\alpha,\beta)$, the series ( ) possesses semi-convergence and inclusion properties with respect to the function $F(\lambda;\alpha,\beta)$ similar to those described at the commencement of § concerning the series ( ) and the function f of formula ( ).

In his classic memoir, Hamburger points out (see the footnote to p. 268 of part I of [ ]) that if σ is nonanalytic over $(-\infty,\infty)$, then formula ( ) with $\alpha=-\infty$, $\beta=\infty$ defines one function ~~the upper half over~~ in the upper half-plane, another in the lower half-plane, and that the real axis is a natural barrier of both functions, in the sense that analytic continuation of either function accross the real axis is impossible; however, if σ is analytic over $(-\infty,\infty)$, the two functions defined as above, although distinct, can be continued analytically across the real axis; he gives

$$F(\lambda)=\int_{-\infty}^{\infty}\frac{e^{-t^2}}{\lambda-t}dt$$

as an example of a function of the second type.

We wish to investigate the analytic continuation of functions of the form ( ); we go considerably beyond Hamburger's remark and, in particular, delineate the structure of the Riemann surface over which a function of the form ( ) is defined as a single valued function in certain simple cases which nevertheless clearly indicate what occurs in the general case. Our principal reason for carrying out this investigation is that the asymptotic relationship between functions of the form ( ) and series of the form ( ) can persist over a far wider range of $\arg(\lambda)$ than the Stieltjes and Hamburger theories imply. (This phenomenon occurs in connection with certain classical examples of functions of the form ( ); we are thus concerned to remove an apparent discrepancy between the theories of the asymptotic series generated by these examples and the Stieltjes and Hamburger theories.)$_4$

We are also directed by a subsidiary purpose which may in a wider context be more important$_5$. As is well known$_6$ (see, for example, [ ] Ch. 6, Th. 2.1) if $f(u)$ is regular and bounded over $\Delta(-\alpha,\alpha)$ $(\alpha \leq \frac{\pi}{2})$ then the function represented for $Re(\lambda)>0$ by the Laplace integral

$$F(\lambda)=\int_0^\infty e^{-\lambda u} f(u)\,du$$

can be continued analytically in the sector $\Delta(-\frac{\pi}{2}-\alpha, \frac{\pi}{2}+\alpha)$.

If $\int_{-\infty}^{0} |d\sigma(t)| < \infty$, then the function

$$f(u) = \int_{-\infty}^{0} e^{tu} d\sigma(t)$$

satisfies the above conditions with $\alpha = \frac{\pi}{2}$. But in this case $F(\lambda) = F(\lambda; -\infty, 0)$. The Stieltjes integral expression ( ) for this example automatically provides the function defined by analytic continuation of the Laplace integral expression ( ) over the sector $\Delta(-\pi, \pi)$. In the theory ~~that~~ which is about to be described, a little more is provided, namely, in certain cases, a complete description of the Riemann surface of the function obtained by analytic continuation of the expression ( ).

The following notations are used: $\omega \in A(\alpha', \beta')$ means that ω is an ~~analytic~~ function of a complex variable, analytic in an open domain containing all points of the open interval

(α', β'); with $\mathfrak{S}$ a prescribed set of open intervals of the real axis, $\omega \in A(\mathfrak{S})$ means that a simply connected open domain $\mathbb{D}$ exists for which all points of all ~~intervals~~ ~~of~~ constituent intervals (α', β') of $\mathfrak{S}$ are interior points, with ~~$w \in A(\alpha', \beta')$~~ ω analytic over $\mathbb{D}$. $H_+ \langle H_- \rangle$ is the finite part of the half-plane $\mathrm{Im}(\lambda) > 0 \langle \mathrm{Im}(\lambda) < 0 \rangle$. $F_+ \langle F_- \rangle$ is the function defined by formula ( ) over $H_+ \langle H_- \rangle$. $\mathfrak{R}\{\omega\}$ is the Riemann surface over which ω is defined as a single valued function; the Riemann surfaces of other functions are denoted in the same way. That sheet of $\mathfrak{R}\{F_+\} \langle \mathfrak{R}\{F_-\} \rangle$ which contains the half-plane $H_+ \langle H_- \rangle$ over which $F_+ \langle F_- \rangle$ is directly defined by formula ( ) is called the principal sheet of $\mathfrak{R}\{F_+\}$ $\langle \mathfrak{R}\{F_-\} \rangle$.

Theorem . In each of the following parts of the theorem, σ is a complex valued function, finite over finite interval in $[\alpha, \beta]$, where $[\alpha, \beta]$ is prescribed, for which $\sigma(t) = O(t)$ if $\beta = \infty$ and $\sigma(t) = O(t)$ if $\alpha = -\infty$ for large positive real t in both cases, and $\int_{\alpha-}^{\beta-} (1+|t|)^{-1} d\sigma(t)$ is assumed to exist if either of these conditions holds. $\omega, \hat{\omega}, \ldots$ are functions of complex variable assumed to be devoid of natural barriers throughout their domains of definition.

A_3^*. Let $\omega \in A(\alpha', \beta')$, where $[\alpha', \beta'] \subseteq [\alpha, \beta]$, and let $\frac{d\sigma(t)}{dt} = \omega(t)$ for all $t \in (\alpha', \beta')$.

i) The function $F_+ \langle F_- \rangle$ may be continued analytically from $H_+ \langle H_- \rangle$ across the real segment (α', β') into $H_- \langle H_+ \rangle$ and also denoting the function obtained by analytic continuation in this way by $F_+ \langle F_- \rangle$, we have

$$F_+(\lambda;\alpha,\beta) = F_-(\lambda;\alpha,\beta) - 2\pi i \omega(\lambda)$$

for all λ sufficiently near (α',β').

B. Let $\alpha = -\infty$, $\beta = \infty$. Let $\mathcal{S}$ be a non-empty set of nonintersecting open intervals contained in $(-\infty,\infty)$. Let $\hat{\omega}$ be single or many valued, denote by $\omega^{(0)}$ the function defined over $\mathcal{S}$ in one of the sheets of $\mathcal{R}\{\omega\}$, and let $\omega^{(0)} \in A(\mathcal{S})$ with $\frac{d\sigma}{dt}(t) = \omega^{(0)}(t)$ over $\mathcal{S}$; let σ be non-analytic over $[-\infty,\infty] \setminus \mathcal{S}$. Denote, as above, the function defined directly by formula ( ) over $H_+ \langle H_- \rangle$ and obtained by analytic continuation of this function by $F_+ \langle F_- \rangle$.

Cut each sheet of $\mathcal{R}\{\hat{\omega}\}$ along $[-\infty,\infty] \setminus \mathcal{S}$ and detach that part of $\mathcal{R}\{\hat{\omega}\}$ which has become disconnected from the intervals over which $\omega^{(0)}$ is defined. If the half-plane H_+ partly bounded by these intervals contains branch lines of $\hat{\omega}$, cut this sheet of $R\{\hat{\omega}\}$, and thereby certain other sheets, along these lines. Again detach as above. Heal the cuts in H_+ and in the correspondingly afflicted sheets of $R\{\hat{\omega}\}$ that have not been detached (without reconnecting the sheets). Denote what remains of $R\{\hat{\omega}\}$ after the above operations have been performed upon it by $R\{\omega_+\}$.

(i) $R\{F_+\}$ and $R\{\omega_+\}$ are isomorphic

(ii) $R\{F_-\}$ is isomorphic to the surface $R\{\omega_-\}$ constructed in a similar way.

(iii) If $R\{\omega_+\}$ and $R\{\omega_-\}$ are ~~both~~ devoid of branch points in both H_+ and H_-, $R\{F_+\}$ and $R\{F_-\}$ are disjoint simple sheets cut along $[-\infty,\infty]\setminus S$

C. Let all intervals belonging to the two nonvoid sets

$\mathcal{S}_1$ and $\mathcal{S}_2$ of open intervals be non-intersecting. Let $\hat{\omega}_1$ and $\hat{\omega}_2$ be two distinct (possibly many valued) functions, neither identically zero. Let the function $\omega_1^{(0)}$ $\langle \omega_2^{(0)} \rangle$ be that defined over one of the sheets of $\mathcal{R}\{\hat{\omega}_1\}$ $\langle \mathcal{R}\{\hat{\omega}_2\} \rangle$ with $\omega_1^{(0)} \in A(\mathcal{S}_1)$ $\langle \omega_2^{(0)} \in A(\mathcal{S}_2) \rangle$, with $\frac{d\sigma(t)}{dt} = \omega_1^{(0)}(t)$ over $\mathcal{S}_1$ $\langle \frac{d\sigma(t)}{dt} = \omega_2^{(0)}(t)$ over $\mathcal{S}_2 \rangle$. Let σ be nonanalytic over $[-\infty, \infty] \setminus \{\mathcal{S}_1 \cup \mathcal{S}_2\}$.

Cut all sheets of $\mathcal{R}\{\hat{\omega}_1\}$ along $[-\infty, \infty] \setminus \{\mathcal{S}_1 \cup \mathcal{S}_2\}$. Detach from $\mathcal{R}\{\hat{\omega}_1\}$ that part no longer connected to the intervals of $\mathcal{S}_1$ over which $\omega_1^{(0)}$ is defined. Let ω_1 be the function defined over what remains, namely $\mathcal{R}\{\omega_1\}$ of $\mathcal{R}\{\hat{\omega}_1\}$. Define ω_2 similarly. If ω_1 and ω_2 have any branch points, let them be real and separate $\mathcal{S}_1$ ~~and $\mathcal{S}_2$~~ from $\mathcal{S}_2$. Let the successive sheets of $R\{w_1\}$ and $R\{w_2\}$ be so disposed that passage from H_+ to H_- in a sheet takes place across the intervals of S_1, and passage from one sheet to its neighbour is effected across the intervals of S_2.

Commencing at some point in the half-plane H_+ partly bounded by the intervals of S_1 and S_2 over which $w_1^{(0)}$ and $w_2^{(0)}$ have been defined, let λ traverse a circular path, first crossing a selected interval of S_1 and returning from H_- to its original position in H_+ via a selected interval of S_2. As λ repeatedly traverses the above path, a path in $R\{w_1\}$ is described; let the functions defined upon the selected segment of S_1 in the successive sheets of $R\{w_1\}$ encountered in this way be $w_1^{(\nu)}$ $(\nu=0,1,\ldots)$ (so that if w_1 is N–

valued $\omega_1^{(\nu N+r)} \equiv \omega_1^{(r)}$ ($\nu=0,1,\dots,\ r=0,1,\dots,N-1$), and let $\omega_2^{(\omega)}$ ($\omega=0,1,\dots$) be the functions similarly encountered upon the selected ~~segment of~~ interval of $\overset{+}{S}_2$.

(i) a) Let a finite integer $N \geq 1$ exist such that

$$\sum_1^N \{\omega_1^{(\omega)}(\lambda) - \omega_2^{(\omega)}(\lambda)\} = 0$$

identically. Then, for the smallest such N, F_+ is an N-valued function and $\mathcal{R}\{F_+\}$ has an irreducible N-sheeted form.

b) If no integer N satisfying the requirements of the preceding clause exists, $\mathcal{R}\{F_+\}$ consists of an infinite system of sheets

c) In both of the above cases, the sheets of $\mathcal{R}\{F_+\}$ may be so disposed that passage between H_+ and H_- in the same sheet is effected across the intervals of

S_1, passage from one sheet to its neighbour being effected across those of S_2, and those intervals contained in the complement of $[-\infty,\infty]\setminus\{S_1 \cup S_2\}$ are natural barriers in each sheet.

ii) Let $\mathcal{R}\{F_+\}$ be as described in subclause ic)

a) As λ repeatedly traverses the above circular path, a path upon $\mathcal{R}\{F_+\}$ is also described. Denote the functions defined upon H_+ and H_- and the intervals of S_1 encountered in this way by $F_+^{(n)}$ $(n=0,1,\dots)$. Then for $\lambda \in H_+$

$$F_+^{(n)}(\lambda;-\infty,\infty) = F_+^{(0)}(\lambda;-\infty,\infty) + 2\pi i \sum_1^n \{\omega_1^{(\nu)}(\lambda) - \omega_2^{(\nu)}(\lambda)\} \quad (n=0,1,\dots)$$

and for $\lambda \in H_-$

$$F_+^{(n)}(\lambda;-\infty,\infty) = F_+^{(0)}(\lambda;-\infty,\infty) + 2\pi i \{\sum_0^n \omega_1^{(\nu)}(\lambda) - \sum_1^n \omega_2^{(\nu)}(\lambda)\}$$
$$(n=0,1,\dots)$$

(under the conditions of clause ia) it is to be understood that $F_+^{(\tau)} \equiv F_+^{(\omega)}$ when $\tau \equiv \omega \bmod (N)$).

b) Denoting the functions encountered upon the selected intervals of S_1 and S_2 in $R\{\omega_1\}$ and $R\{\omega_2\}$ respectively as λ traverses the above circuit in a contrary direction by $\omega_1^{(\nu)}, \omega_2^{(-\nu)}$ $(\nu = 0, 1, \dots)$, formulae analogous to those given above may be derived for the functions $\{F_+^{(n)}\}$ defined for $n < 0$ in a similar way.

(iii) a) The structure of $R\{F_-\}$ is similar to that of $R\{F_+\}$ and formulae analogous to (,) may be given for functions of the form $\{F_-^{(n)}\}$.

b) If a finite integer $m > 0$ exists such that

$$\Sigma_0^m \omega_1^{(\nu)}(\lambda) = \Sigma_1^m \omega_2^{(\nu)}(\lambda)$$

then $R\{F_+\}$ and $R\{F_-\}$ are obtained from each other by displacement in the sense that with m being the smallest integer for which a relationship of the form ( ) is obeyed, $F_{+}^{(m+r)} \equiv F_{-}^{(r)}$ $(r = \dots, -1, 0, 1, \dots)$ (if relationship ( ) is also satisfied, then the above minimum value of m satisfies the relationship $m \leq N$).

c) If no integer m satisfying the requirements of the preceding clause exists, $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ are disjoint.

D. Let $\mathcal{S}'$ be a (possibly empty) set of open intervals contained in (α, β) over each of which σ is constant.

Let $\mathcal{S}_1$ be the set composed of $(-\infty, \alpha)$ (if $\alpha > -\infty$), the intervals of $\mathcal{S}'$, and (β, ∞) (if $\beta < \infty$), and let $\mathcal{S}_1$ be nonvoid (so that either $\alpha > -\infty$, or $\beta < \infty$, or σ is constant over some ~~int~~ open interval in (α, β)).

(i)a) The function defined by formula ( ) in both half-planes H_+ and H_- may be obtained from each other by analytic continuation across the intervals of S_1, and the single function now defined by formula ( ) is also defined by the same formula at points belonging to these intervals.

b) Let σ be nonanalytic at all points of $[\alpha,\beta]\setminus S_1$. Then the function $F(\lambda;\alpha,\beta)$ is single valued and analytic everywhere in the finite complex plane from which the points of $[\alpha,\beta]\setminus S_1$ have been removed; if $[\alpha,\beta]\setminus S_1$ contains intervals, then these segments of the real axis are natural barriers of $F(\lambda;\alpha,\beta)$.

(ii) Let S_2 be a non-empty set of non-intersecting open intervals contained in $(-\infty,\infty)$, none of which intersects a member of S_1. Let $\hat{\omega}$, ω and $\{\omega^{(i)}\}$ be as defined for $\hat{\omega}_2$, ω_2 and $\{\omega_2^{(i)}\}$ in part c) (the possible branch points of ω_2 was separating S_1 from S_2). Let $\frac{d\sigma(t)}{dt} = 0$ over S_1 and $\frac{d\sigma(t)}{dt} = \omega^{(0)}(t)$ over S_2, and σ be non-analytic over $[\alpha,\beta] \setminus \{S_1 \cup S_2\}$. Denote the function defined directly by formula ( ) $H_+ \cup H_- \cup S_1$ and obtained by analytic continuation of this function by F.

a) Let a finite integer $N \geq 1$ exist such that

$$\sum_1^N \omega^{(\nu)}(\lambda) = 0$$

identically. Then, for the smallest such N, F is an N-valued function and $\mathcal{R}\{F\}$ has an irreducible N-sheeted form.

b) If no N as above exists, $\mathcal{R}\{F\}$ consists of an infinite system of sheets.

c) $\mathcal{R}\{F\}$ may be given the same structure as that described for $\mathcal{R}\{F_+\}$ in clause C(c), and in the notation of part c), $R\{F_+\}$ and $R\{F_-\}$ are, in the present case, the same, being $R\{F\}$.

d) Starting from any position in $H_+ \cup H_- \cup S_1$, let λ repeatedly traverse the circular path described in part c), and let the functions defined over $H_+ \cup H_- \cup S_1$ in the successive sheets of $R\{F\}$ and encountered in this way be $F^{(n)}$ $(n=0,1,\dots)$. Then

$$F^{(n)}(\lambda;\alpha,\beta) = F^{(0)}(\lambda;\alpha,\beta) - 2\pi i \sum_1^N \omega^{(\nu)}(\lambda) \quad (n=0,1,\dots)$$

Again, a similar relationship involving functions $\{\omega^{(\nu)}\}$ for $\nu \leq 0$ may be given for the functions $\{F^{(n)}\}$ with $n<0$.

E. Let the integral $\int_\alpha^\beta t^y |d\sigma(t)|$ exist for arbitrarily large $y \in (0,\infty)$. Define the moments $\{f_\nu\}$ by means of formula ( ). With ω as further to be prescribed below, let a system θ of non-intersecting open intervals belonging to $(0, 2\pi)$ exist such that

$$\lim_{\lambda=\infty} \lambda^{y} \omega(y) = 0$$

for arbitrarily large positive real y for all $\arg(\lambda)$ belonging to the intervals of Θ over all sheets of $R\{\omega\}$.

(i) Let $\alpha = -\infty$, $\beta = \infty$, and let S, σ, ω and F_+ be defined as in part B.

a) Over the principal sheet of $R\{F_+\}$

$$F_+(\lambda; -\infty, \infty) \sim \sum f_\nu \lambda^{-\nu-1}$$

for $0 < \arg(\lambda) < \pi$ and

b) relationship ( ) also holds for all $\arg(\lambda) \neq \pi$ belonging to the intervals of Θ.

c) If $\infty \langle -\infty \rangle$ is an end point of one of the intervals of S then relationship ( ) also holds when $\arg(\lambda) = 0 \ \langle \arg(\lambda) = \pi \rangle$.

Over the remaining sheets of $\mathcal{R}\{F_+\}$ the above result b) holds; the conditional result of c) also holds if relationship ( ) is valid for $\arg(\lambda)=0$ $\langle \arg(\lambda)=\pi \rangle$.

(ii) By imposing conditions similar to ( ) upon the functions ω_1 and ω_2 of clause ( ), results similar to a) and b) above holds for the function $F_+^{(1)}$ of that clause. A further result analogous to c) holds for the remaining functions $\{F_+^{(n)}\}$ if ∞ $\langle -\infty \rangle$ is ~~the~~ an end point of one of the intervals of either S_1 or S_2.

(iii) The results of clause (i) with the condition $0<\arg(\lambda)<\pi$ replaced by $\pi<\arg(\lambda)<2\pi$ in a) for the function F_-; similar modifications of the results of clause (ii) are also valid.

(iv) Let $[\alpha,\beta]$, S_1, S_2, ω, θ and F be as described in clause Dii).

a) The results ia,b) above with $F_+(\lambda;-\infty,\infty)$ replaced by $F(\lambda;\alpha,\beta)$ hold over the principal sheet of $\mathcal{R}\{F\}$.

b) If either $\infty \langle -\infty \rangle$ is one of the end points of an interval of S_1, or $\infty \langle -\infty \rangle$ is one of the end points of an interval of S_2, and relationship ( ) is valid for $\arg(\lambda)=0$ $\langle \arg(\lambda)=\pi \rangle$, then relationship ( ) with $F_+(\lambda;-\infty,\infty)$ replaced by $F(\lambda;\alpha,\beta)$ also holds for $\arg(\lambda)=0$ $\langle \arg(\lambda)=\pi \rangle$.

c) Over the remaining sheets of $\mathcal{R}\{F\}$ the modified result ib) holds; if $\infty \langle -\infty \rangle$ is an end point of an interval of either S_1 or S_2 and relationship ( ) is valid for $\arg(\lambda)=0$ $\langle \arg(\lambda)=\pi \rangle$ then the modified relationship ( ) also holds for $\arg(\lambda)=0$ $\langle \arg(\lambda)=\pi \rangle$ over the remaining sheets of $\mathcal{R}\{F\}$.

Proof. Subject to the conditions imposed upon σ at the commencement of the theorem

$$\lim_{\beta=\infty}\Big\{(\lambda+1)\int_\alpha^\beta \frac{\lambda-1-2t}{(1+t)^2(\lambda-t)^2}\,d\sigma(t)+\{(\lambda-\beta)^{-1}+(1+\beta)^{-1}\}\sigma(\beta)-(\lambda^{-1}+1)\sigma(\alpha)\Big]$$

is finite for all $\lambda\in H_+$ when $\alpha\in[0,\infty)$. Integrating by parts, we find that this limit is

$$\lim_{\beta=\infty}\Big\{\int_\alpha^\beta(\lambda-t)^{-1}d\sigma(t)+\int_\alpha^\beta(1+t)^{-1}d\sigma(t)\Big\}.$$

Hence formula ( ) represents a well defined function over H_+ when $\alpha\in[0,\infty)$. That this is also true when $\lambda\in H_-$ is clear, and that it is true when $\alpha=-\infty$ is demonstrated in the same way. The case in which $[\alpha,\beta]\subset(-\infty,\infty)$ presents no difficulty. We may henceforth assume that formula ( ) represents two functions that are well defined over H_+ and H_- respectively.

Under the conditions of part A, we select an interval $[\alpha'', \beta''] \subset (\alpha', \beta')$ and express $F(\lambda; \alpha, \beta)$ as the sum of three integrals over the ranges $[\alpha, \alpha'']$, $[\alpha'', \beta'']$, $[\beta'', \beta]$. For the second of these, we have

$$F(\lambda; \alpha'', \beta'') = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{\omega(u)}{\lambda - u} \ln\left\{\frac{\alpha'' - u}{\beta'' - u}\right\} du$$

where $\mathscr{C}$ is a simple contour enclosing the points of $[\alpha'', \beta'']$ and lying within a domain $\mathfrak{D}$ over which ω is analytic, the principal branch of the logarithm is implied, and $\lambda \in H_+$ lies outside $\mathscr{C}$. When $\lambda \in H_+$ lies inside $\mathscr{C}$, a further term $\omega(\lambda) \ln\left\{\frac{\beta'' - \lambda}{\alpha'' - \lambda}\right\}$ must be added to the right hand side of formula ( ). λ may now be allowed to cross (α'', β'') and pass into H_-, still within $\mathscr{C}$. The expression $\ln\left\{\frac{\beta'' - \lambda}{\alpha'' - \lambda}\right\}$ now acquires a component $-2\pi i$ if $\ln$ is still to denote the principal branch of this function in the formula holding when λ has reached its last position. When $\lambda \in H_-$ lies outside $\mathcal{C}$ a formula similar to ( ) holds, and an embellished version also holds when $\lambda \in H_-$ lies inside $\mathcal{C}$. The two functions $F(\lambda;\alpha,\alpha'')$ and $F(\lambda;\beta'',\beta)$ are single-valued and analytic over the domain $\mathbb{D}$. A comparison of the two formulae which hold for $F(\lambda;\alpha'',\beta'')$ as obtained by analytic continuation into H_-, and as directly defined there, yields formula ( ). (The above is, of course, no more than a simple derivation (which we could use because ω has been assumed analytic over $\mathbb{D}$) of a result analogous to those given by Harnack [ ], Privalov [ ] and Plemelj [ ] concerning Cauchy type integrals defined over less ~~restric~~ restricted contours of integration).

Formula ( ) retains its validity for all λ for which the constituents of this formula are analytic, and serves as the basis for further analytic continuation of $F_+(\lambda;\alpha,\beta)$ and $F_-(\lambda;\alpha,\beta)$. The value of $F_+(\lambda;\alpha,\beta)$ for λ in $H_+ < H_- >$ on any of the sheets of $\mathcal{R}\{F_+\}$ is expressible as the sum of the value of $F_+(\lambda;\alpha,\beta) < F_-(\lambda;\alpha,\beta) >$ in H_+ $< H_- >$ upon its principal sheet and further terms involving the values of ω upon the sheets of its Riemann surface. ~~Under the~~

Under the conditions of part B), the segments $(-\infty,\alpha']$ and $[\beta',\infty)$ of the real axis are natural barriers of F_+ and F_- upon the principal sheet of $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ respectively, and therefore of these functions upon all sheets of $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$. The result of clauses Bi, ii) follow immediately. When $\mathcal{R}\{\omega_+\}$

and $\mathcal{R}\{w_-\}$ are devoid of branches emanating from points of both H_+ and H_-, they have no branches at all, and are simple sheets cut along $[-\infty,\infty]\setminus S$, and the same is true of $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$. Since ω is not identically zero, there is no domain throughout which F_+ and F_- take equal values, and $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ are disjoint.

Under the conditions of part c), analytic continuation of $F_+(\lambda;-\infty,\infty)$ and $F_-(\lambda;-\infty,\infty)$ may take place across intervals belonging to the two sets S_1 and S_2. If the branch points of ω_1 and ω_2 separate the intervals of these two sets, the functions $\{\omega_1^{(\nu)}\}$ and $\{\omega_2^{(\nu)}(\lambda)\}$ encountered as λ traverses the circuit described in clause (ii) are the same for all intervals of $\overset{\circ}{S}_1$ and $\overset{\circ}{S}_2$ chosen to contain the crossing points between H_+ and H_- (we remark that if $\hat{\omega}_1$ has any branch points lying to one side of all intervals of $\overset{\circ}{S}_1$ and $\overset{\circ}{S}_2$, then ω_1 no longer possesses these branch points, for only the sheet of $R\{\hat{\omega}_1\}$ containing the interval over which $\omega_1^{(0)}$ is defined remains after the process of cutting and detachment described in part C has been carried out; the same remark may be made with regard to ω_2).

After λ has made its first crossing of the selected interval of $\overset{\circ}{S}_{1\phi}$ when traversing the circuit described in clause Cii), F_+ remains upon its principal branch, and we have

$$F_+^{(0)}(\lambda;-\infty,\infty)=F_-^{(0)}(\lambda;-\infty,\infty)+2\pi i\,\omega_1^{(0)}(\lambda).$$

The behaviour of $F_-^{(0)}$ as λ makes its first crossing of the selected interval of S_2 may be examined by methods similar to those used to prove formula ( ). Since w_2 has a branch which includes this interval, we must decompose $\mathcal{C}$ in formulae such as ( ) into two parts lying in H_- in the principal sheet of $\mathcal{R}\{w_2\}$ and H_+ in a neighbouring sheet of this surface respectively; but this detail is easily attended to. w_1 also has a branch line including the above interval. In H_+ of the sheet which is adjacent to the principal sheet of $\mathcal{R}\{F_+\}$, we have

$$F_+^{(1)}(\lambda;-\infty,\infty)=F_+^{(0)}(\lambda;-\infty,\infty)-2\pi i w_2^{(1)}(\lambda)+2\pi i w_1^{(1)}(\lambda).$$

After λ makes its second crossing into H_-, we have

$$F_+^{(1)}(\lambda;-\infty,\infty)=F_-^{(0)}(\lambda;-\infty,\infty)+2\pi i\{w_1^{(0)}(\lambda)+w_1^{(1)}(\lambda)-w_2^{(1)}(\lambda)\},$$

and so on; we derive formulae of the form (,). If a relationship of the form ( ) holds, then $F_+^{(\omega)}$ and $F_+^{(N)}$ are identical. The result of clause (i) follows immediately. Those of clause (ii) have already been derived.

For the functions $\{F_-^{(n)}\}$ of clause (iiia), we have for $\lambda \in H_+$

$$F_-^{(n)}(\lambda;-\infty,\infty) = F_-^{(0)}(\lambda;-\infty,\infty) + 2\pi i\left\{\sum_1^{n-1}\omega_1^{(\omega)}(\lambda) - \sum_1^{n}\omega_2^{(\omega)}(\lambda)\right\}$$
$$(n=0,1,\ldots)$$

and for $\lambda \in H_-$

$$F_-^{(n)}(\lambda;-\infty,\infty) = F_-^{(0)}(\lambda;-\infty,\infty) + 2\pi i\left\{\sum_0^{n-1}\omega_1^{(\omega)}(\lambda) - \sum_1^{n}\omega_2^{(\omega)}(\lambda)\right\}$$
$$(n=0,1,\ldots).$$

$\mathcal{R}\{F_-\}$ is N-sheeted if and only if

$$\sum_0^{N-1}\omega_1^{(\omega)}(\lambda) = \sum_1^{N}\omega_2^{(\omega)}(\lambda)$$

for some finite integer $N \geqslant 0$.

With regard to the result of clause (iiib) (which follows immediately from formula ( )) we remark that if $m > N$ in equation ( ), subtraction of an appropriate number of sets of equations of the form ( ) reduces equation ( ) to a form in which $m \leq n$. We remark that $\mathcal{R}\{F_+\}$ can be N-sheeted while $\mathcal{R}\{F_-\}$ consists of an infinite number of sheets, and conversely (except, of course, when a relationship of the form ( ) holds; then neither or both Riemann surfaces are N-sheeted). If $\sum_1^N \omega_1^{(\omega)}(\lambda) = \sum_1^N \omega_2^{(\omega)}(\lambda) = 0$, $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ are both N-sheeted; even in these circumstances, $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ can still be disjoint.

The preliminary results of clause Di) are easily verified. Those of the remainder of part D may be derived independently, as above, or by suitably modifying the above argument after setting $\hat{\omega}_1(\lambda)=0$, $\hat{\omega}_2(\lambda)=\omega(\lambda)$. Since F_- may now be obtained from F_+ by analytic continuation, $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ are identical.

When $[\alpha,\beta]\subset(-\infty,\infty)$ and σ is as described at the commencement of the theorem, all moments ( ) exist; otherwise the existence of the integral ~~refererred~~ referred to in part E ensures the existence of all of these moments

Under the conditions of clause Ei), σ may be decomposed into four constituents σ', ..., the positive and negative parts of the real and imaginary parts of σ for each of which $\sigma'\in BN(-\infty,\infty)$ or $-\sigma'\in BN(-\infty,\infty)$. The function $F_+(\lambda;-\infty,\infty)$ defined directly by formula

( ) and the series ( ) may each be decomposed into four constituents for which a relationship of the form ( ) holds as described; these four relationships may be reassembled to yield a single relationship involving the function $F_+(\lambda;-\infty,\infty)$ and the series ( ); we have derived the result of clause Eia). The same may be done for the function $F_-(\lambda;-\infty,\infty)$, the ~~same may be done for the function I~~ same series being involved. We now have two asymptotic relationships holding for $\lambda \in \Delta(0,\pi)$ and $\lambda \in (\pi, 2\pi)$ respectively.

Since $F_+(\lambda;-\infty,\infty)$ for $\lambda \in H_+$ in the principal sheet of $\mathcal{R}\{F_+\}$ is given by formula ( ), and for $\lambda \in H_-$ in the principal sheet of $\mathcal{R}\{F_+\}$ is given by formula ( ), the result of clause Eib) follows immediately. If the stated condition upon σ holds when $\beta=\infty$, and ∞ is also an endpoint of one of the intervals of $\mathfrak{S}$ (so that $\omega(t)=\frac{d\sigma(t)}{dt}$ is analytic over a domain containing all points of a segment of the form $[\alpha',\infty)$ in its interior) then relationship ( ) holds for $\arg(\lambda)=0$ over the principal sheet of $\mathcal{R}\{\omega\}$. Relationship ( ) also holds when $\arg(\lambda)=0$. The further result of clause Eic) is demonstrated in the same ~~way. Concerning the remaining sheets of $\mathcal{R}\{F_+\}$ we remark firstly that an~~

way. Over the remaining sheets of $\mathcal{R}\{F_+\}$, F_+ is expressible as the sum of the value of F_+ defined upon the principal sheet of $\mathcal{R}\{F_+\}$ and a sum of functions deriving from the various sheets of $\mathcal{R}\{\omega\}$. Over sectors for which relationship ( ) holds, the asymptotic relationship ( ) also holds. However, an independent proof of relationship ( ) over a sector ( ) for which condition ( ) does not hold is no longer available, and the result of the form Eia) over sheets other than the principal sheet of $\mathcal{R}\{F_+\}$ is therefore also unavailable.

The remaining results of part E are demonstrated in a similar fashion.

For an illustration of the results of the above theorem, we may return to Hamburger's example of formula ( ) $\omega(t) = e^{-t^2}$ is analytic over the finite part of the complex plane, $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ in this case reduce to distinc single sheets, F_+ and F_- being analytic over $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ respectively, and we have $F_+(\bar{\lambda}) = \bar{F}_-(\lambda)$ for all finite λ. The asymptotic series of the form ( ) which is known to represent $F_+(\lambda; -\infty, \infty)$ as defined directly by formula ( ) over $\Delta(0,\pi)$ actually represents this function as we have defined it by analytic continuation over $\Delta(-\frac{\pi}{4}, \frac{5\pi}{4})$; $F_-(\lambda; -\infty, \infty)$ is represented by the same series ~~for~~ over $\Delta(-\frac{5\pi}{4}, \frac{\pi}{4})$.

As an adjunct to the result of clause B iii) we remark that if $\mathcal{R}\{\omega_+\}$ and $\mathcal{R}\{\omega_-\}$ possess branches emanating from points in H_+ and H_-, it can occur that $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ are derived from each other by displacement (i.e. these surfaces are not disjoint). Consider the function $\hat{\omega}(t) = \{(i-t)^{\alpha}(i+t)^{\beta}\}^{\frac{1}{2\pi i}}$. $\mathcal{R}\{\hat{\omega}\}$

possesses a system of sheets with branches along the segment $i[1,\infty]$ of the imaginary axis and deriving from the term involving $i-t$; a further subsystem of sheets each having branches along the segment $i[-\infty,-1]$ is associated with each of the sheets of the first system (the numbers of sheets in the first system and each subsystem depend upon α and β). $R\{\omega_+\}$ consists of one principal half-plane H_+ (which contains no branch cuts) joined across the intervals of S to a half-plane H_- which is connected to a number of half-planes H_- across branch-lines coincident with $i[-\infty,-i]$; each further half-plane H_- is joined across the intervals of S to a half-plane H_+ which is connected to a system of planes across branc lines coincident with $i[1,\infty]$. $\mathcal{R}\{\omega_-\}$ consists of one principal half-plane H_- devoid of branch cuts and further systems of sheets determined in a manner analogous to the above. Denote by $F_+^{(0)}$ $\langle F_-^{(0)} \rangle$ that branch of F_+ $\langle F_- \rangle$ defined upon the principal half-plane of $\mathcal{R}\{F_+\}$ $\langle \mathcal{R}\{F_-\} \rangle$. For $\lambda = \lambda' \in H_- \setminus i[-\infty,-i]$ $(|\lambda' + i| < 1)$, H_- being directly connected to the principal half-plane H_+ of $\mathcal{R}\{F_+\}$, we have

$$F_+^{(0)}(\lambda;-\infty,\infty) = F_-^{(0)}(\lambda;-\infty,\infty) + 2\pi i\omega(\lambda).$$

Let λ describe a circle with centre at $-i$ ν times in a clockwise direction, returning to λ'. Denote by $F_+^{(1)}$ that function which $F_+^{(0)}$ has now become. $F_-^{(0)}$, being defined over the principal half-plane H_- of $\mathcal{R}\{F_-\}$ remains unchanged. We have, with $\lambda = \lambda'$,

$$F_+^{(1)}(\lambda;-\infty,\infty) = F_-^{(0)}(\lambda;-\infty,\infty) + 2\pi i e^{\beta\nu}\omega(\lambda).$$

Let λ move from λ', across one of the intervals of $\mathcal{S}$ to $\lambda'' \in H_+ \setminus i[1,\infty]$ $(|\lambda''-i|<1)$. With $\lambda=\lambda''$, we have

$$F_+^{(1)}(\lambda;-\infty,\infty) = F_+^{(0)}(\lambda;-\infty,\infty) + 2\pi i e^{\beta\nu}\omega(\lambda) - 2\pi i\omega(\lambda).$$

Let λ now describe a circle, with centre at i, τ times in an anticlockwise direction and denote by $F_+^{(2)}$ that function which $F_+^{(1)}$ has now become. $F_+^{(0)}$ remains unchanged. With $\lambda=\lambda''$, we have

$$F_+^{(2)}(\lambda;-\infty,\infty) = F_+^{(0)}(\lambda;-\infty,\infty) + 2\pi i e^{\alpha\tau}(e^{\beta\nu}-1)\omega(\lambda).$$

Let λ return from λ'' across one of the intervals of $\mathcal{S}$ to λ'. With $\lambda=\lambda'$, we have

$$F_+^{(2)}(\lambda;-\infty,\infty) = F_-^{(0)}(\lambda;-\infty,\infty) + 2\pi i\{1+e^{\alpha\tau}(e^{\beta\nu}-1)\}\omega(\lambda).$$

By suitably choosing α, β, ν and τ we may arrange that $1+e^{\alpha\tau}(e^{\beta\nu}-1)=0$, and then $F_+^{(2)}$ and $F_-^{(0)}$ are identical. In this case $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ are derived from each other by displacement and are not disjoint.

We remark parenthetically that the intervals of $\mathcal{S}_1$ and $\mathcal{S}_2$ of classes) and) may be separated by a single point. For example, over an interval (α', β') ($[\alpha', \beta'] \subset [\alpha, \beta]$) we may have $\frac{d\sigma(t)}{dt} = \omega(|t-\gamma|)$ where $\gamma \in A(\alpha', \beta')$ and $\omega \in A(0, \max\{(\gamma-\alpha'), (\beta'-\gamma)\})$ with σ nonanalytic over $[\alpha, \beta] \setminus [\alpha', \beta']$. Now $\mathcal{S}_1 \langle \mathcal{S}_2 \rangle$ contains the single interval $(\alpha', \gamma) \langle (\gamma, \beta') \rangle$. We find that for the function defined by analytic continuation across the interval (α', γ) the function $\omega(\lambda)$ in relationship ( ) is to be replaced by $\omega'(\lambda-\gamma) - (\lambda-\gamma)\omega''(\lambda-\gamma)$, while for that defined by analytic continuation across the interval (γ, β'), $\omega(\lambda)$ should be replaced by $\omega'(\lambda-\gamma) + (\lambda-\gamma)\omega''(\lambda-\gamma)$, where

$\omega(t) = \omega'(t) + t\,\omega''(t)$, ω' and ω'' both being even functions, (if ω'' is identically zero, then $\frac{d\sigma(t)}{dt}$ is analytic over the whole interval (α', β'), and we are no longer concerned with analytic continuation across two separated intervals).

In the above theorem, we confined our attention to the case in which $\frac{d\sigma(t)}{dt}$ was equal to no more than two distinct analytic functions over two sets of intervals in (α, β). Naturally, the case in which $\frac{d\sigma(t)}{dt}$ is equal to a denumerably infinite number of analytic functions over a corresponding number of sets of intervals may also be considered. However, $\Re\{F_+\}$ and $\Re\{F_-\}$, or $\Re\{F\}$, as is appropriate, then have far more complex structures; the simple descriptions of these functions then no longer suffice. One has, in particular, to introduce a system of enumeration of the various paths which lead from one part of these surfaces to another. This is perhaps an appropriate juncture at which to remark that if σ is real valued over (α,β), $F_+(\bar{\lambda};\alpha,\beta) = \bar{F}_-(\lambda;\alpha,\beta)$ for $\lambda \in H_+$. In this case, when $\alpha = -\infty$, $\beta = \infty$ and σ increases everywhere over $(-\infty,\infty)$, $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$ may be taken to be mirror images of each other in the real axis bounding the half-plane H_+ over which F_+ is directly defined by formula ( ), and an analogue of the relationship between F_+ and F_- given above holds for all corresponding points on these surfaces. When $[\alpha,\beta] \neq [-\infty,\infty]$, or σ is constant over some interval in $(-\infty,\infty)$, $\mathcal{R}\{F\}$ is symmetric about the real axis in its principal sheet and again an analogue of the above relationship holds at points symmetrically distributed with respect to this axis.

Theorem   may easily be applied to integral expressions more general than ( ); for example, to the functions F_+ and F_- jointly defined over ~~H_+ and H_- respectively~~ H_{i+} and H_{i-} respectively by the formula

$$\hat{F}(\lambda) = \int_0^\infty \frac{\omega(t)}{\lambda^2 + t^2} dt$$

where ω is analytic and not identically zero over $(0,\infty)$. This may be treated as $-F(-\lambda^2; 0,\infty)$ and suitable distortions of $R\{F\}$ and modifications of the results of clause   ) of Theorem introduced; alternatively, assuming the integrals to exist independently, we have

$$\hat{F}(\lambda) = \frac{i}{2\lambda}\left\{ \int_{-\infty}^0 \frac{\omega(-t)}{i\lambda - t} dt + \int_0^\infty \frac{\omega(t)}{i\lambda - t} dt \right\}$$

(so that $\hat{F}(\lambda)$ has the same form as $\frac{i}{2\lambda} F(i\lambda; -\infty, \infty)$). $\hat{F}_+(\lambda)$ may be continued analytically across the positive imaginary axis, and the analogue of relationship ( ) is

$$\hat{F}_+(\lambda) = \hat{F}_-(\lambda) - \pi \frac{\omega(i\lambda)}{\lambda};$$

continuation is also possible across the negative ~~real~~ imaginary axis, and the associated relationship is

$$\hat{F}_+(\lambda) = \hat{F}_-(\lambda) - \frac{\pi}{\lambda}\omega(-i\lambda).$$

If ω is not an even function, the two functions thus obtained by continuation are distinct: F_+ is many valued.

The theory supporting the proof of Theorem may also be extended to integrals of the form ( ) in which the expression $(\lambda - t)^{-1}$ is replaced by a rational function $P(\lambda, t)$ which may easily be decomposed in the form

$$P(\lambda,t) = \sum_{i=1}^{N} \sum_{j=1}^{n(i)} \frac{p_{i,j}(\lambda)}{\{q_{i}(\lambda)-t\}^{j}} + \sum_{\nu=0}^{M} Q_{\nu}(\lambda) t^{\nu}$$

where the $\{p_{i,j}\}$, $\{q_i\}$ and $\{Q_\nu\}$ are algebraic functions, the $\{q_i\}$ being nonzero. With (α', β'), σ and ω as described in part A of Theorem , the new function, $\tilde{\tilde{F}}_+(\lambda; P; \alpha, \beta)$ say, may be continued analytically across a curve in the λ-plane defined by the formula $q_i(\lambda) = t$ $(\alpha' < t < \beta')$, and the analogue of formula ( ) is

$$\tilde{\tilde{F}}_+(\lambda; P; \alpha, \beta) = \tilde{\tilde{F}}_-(\lambda; P; \alpha, \beta) + 2\pi i\, p_{i,1}(\lambda)\, \omega\{q_i(\lambda)\}$$

where $\tilde{\tilde{F}}_-$ is defined directly by the modified version of formula ( ). In the example of the preceding paragraph, in which $P(\lambda; t) = (\lambda^2 + t^2)^{-1}$, we may take $q_1(\lambda) = -i\lambda$, $p_{1,1}(\lambda) = -i/2\lambda$, $q_2(\lambda) = i\lambda$, $p_{2,1}(\lambda) = i/2\lambda$; there are two curves involved, the finite parts of the negative and positive imaginary axes respectively.

There is a curious relationship between the domains over which functions defined by the analytic continuation of certain functions of the form $F(\lambda;0,\infty)$ possess asymptotic expansions and those over which these expansions are (B'_*,χ) summable. Taking $\frac{d\sigma(t)}{dt} = \omega(t) = \exp(-t^{\frac{1}{\chi}})$ $(0<\chi\leq 2)$, ω satisfies a relationship of the form ( ) over the open sector $\Delta(-\frac{1}{2}\chi\pi, \frac{1}{2}\chi\pi)$: the various functions defined over successive sheets of $R\{F\}$ are asymptotically represented by the same asymptotic expansion of the form ( ) over this open sector in each sheet. It is precisely where these various functions (other than $F(\lambda;0,\infty)$ itself) are no longer represented by the series ( ) namely over the closed sector $\Delta(\frac{1}{2}\chi\pi, \frac{1}{2}(4-\chi)\pi)$ (substitute $\lambda = z^{-1}$ in clause (ii) of Theorem) that the series ( ) is (B',χ) summable to $F(\lambda;0,\infty)$.

For the sake of completeness, we mention that, assuming ~~ω to be analytic in~~ ~~we~~ that $\omega \in A(\alpha,\beta)$ $[\alpha,\beta] \neq [-\infty,\infty]$, with $\frac{d\sigma(t)}{dt} = \omega(t)$ over this interval, the function $F(\lambda;\alpha,\beta)$ of formula ( ) can be expressed as a contour integral. When $[\alpha,\beta] \subset (-\infty,\infty)$, it is obtained by replacing α'',β'' by α,β in the last term in formula ( ); when $\alpha \in (-\infty,\infty)$, $\beta = \infty$, we have

$$F(\lambda;\alpha,\infty) = -\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\omega(u)}{\lambda - u} \ln(\alpha - u)\,du$$

and a similar formula, obtained by replacing $-\frac{1}{2\pi i}$ by $\frac{1}{2\pi i}$ and $\alpha - u$ by $\beta - u$, for $F(\lambda;-\infty,\beta)$ when $\beta \in (-\infty,\infty)$. In each case the contour of integration encloses all points of $[\alpha,\beta]$, and λ lies outside the contour.

Although Theorem deals ~~with~~ in detail with cases in which the function σ in formula ( ) is equal to either one analytic function over a given set of ~~to~~ open intervals in $(-\infty, \infty)$, or to one of two such functions defined over two sets of disjoint intervals, and is nonanalytic elsewhere over $[-\infty, \infty]$, it is clear that the principles involved in the analytic continuation of the function F_+, for example, hold with regard to functions σ of more general structure. With σ being equal to one of a number of analytic functions defined over a corresponding number of sets of open intervals in $(-\infty, \infty)$, the functions obtained by analytic continuation ~~over~~ of F_+ have, over H_+, the form

$$F_+^{(n)}(\lambda; -\infty, \infty) = F_+^{(i)}(\lambda; -\infty, \infty) + 2\pi i \Omega(\lambda)$$

where $\Omega(\lambda)$ is a sum of functions derived from those ~~over~~ to which σ has been assumed equal over certain intervals, and depends upon the route taken during the process of analytic continuation; n depends upon the enumeration of the sheets of $\mathcal{R}\{F_+\}$; various illustrations of simple forms that Ω can take are provided in Theorem .. .

It is, as part E) of Theorem indicates, quite possible that $F_+^{(i)}(\lambda;-\infty,\infty)$ and $F_+^{(n)}(\lambda;-\infty,\infty)$ both have the same asymptotic expansion over a fixed sector in $\arg(\lambda)$. Various methods exist for extracting from an asymptotic expansion a unique member of the class of functions which generate it. Among these, the two most important are a method arising from theory due to Watson [ ] and F. Nevanlinna [ ] which expresses the unique function as a Laplace transform, and a method arising from theory due to Hamburger [ ] and R. Nevanlinna [ ] which expresses the function in question as a Stieltjes transform. We wish to discuss, with reference to a simple example, why it is that these methods extract the function $F_+^{(0)}(\lambda;-\infty,\infty)$ and not any related function with differing superscript — why it is, in other words, that these methods land on the principal sheet of $\mathcal{R}\{F^+\}$. The example with which we are concerned is defined over H_+ by the formula

$$F_+(\lambda) = \int_{-\infty}^{\infty} \frac{e^{-|t|^\alpha}}{\lambda - t}\,dt$$

where $\alpha \in (-\infty,\infty)$. When α is an even integer, we are dealing with a function of the type considered in part B of Theorem , and $\mathcal{R}\{F_+^{\phi}\}$ is a single sheet. When this is not so, we are dealing with a function of the type considered in part C of Theorem ; we may take S_1 and S_2 to consist of the single intervals $(-\infty, 0)$ and $(0, \infty)$ respectively, with $\omega_1(t) = \exp(-(-t)^\alpha)$ $(-\infty < t \leq 0)$, $\omega_2(t) = \exp(-t^\alpha)$ $(0 \leq t < \infty)$.

Briefly to describe the Watson-F. Nevanlinna method, we suppose that the function $F(\lambda)$ is regular in $\mathbb{D}(a,b;c)$, the union of the concave sector $(a-\frac{1}{2})\pi \leq \arg(\lambda) \leq (b+\frac{1}{2})\pi$ $(a<b\}, |\lambda| < \infty)$ and the disc $|\lambda| < c \in (0,\infty)$, and that with $\{f_\nu\}$ a prescribed sequence of finite complex numbers and

$$R_n(\lambda) = F(\lambda) - \sum_0^{n-1} f_\nu \lambda^{-\nu-1} \qquad (n=0,1,\dots)$$

we have, for some fixed $\xi \in (0,\infty)$, $f_\nu = O(\nu! \xi^{-\nu})$ for large ν and, uniformly for λ in $\mathbb{D}(a,b;c)$, $R_n(\lambda) = O(n! \xi^{-n} \lambda^{-n-1})$ for large n. Under these conditions

$F(\lambda)$ is uniquely determined in this convex sector; when $\lambda \in \Delta(a\pi, b\pi)$, the function $\tilde{F}(\lambda; u)$ defined for small u by the series $\sum f_\nu u^\nu/(\nu!\lambda^{\nu+1})$ is regular for $u \in (0,\infty)$ and $F(\lambda) = \int_0^\infty e^{-u}\tilde{F}(\lambda, u)du$. We remark that although, from the above, $F(\lambda)$ satisfies asymptotic relationships of the form ( ) over the constituent concave sector of $\mathbb{D}(a, b; c)$, this function is represented by an integral of Laplace form over a convex sector whose included angle is less by π than that of the concave sector. (As Hardy illustrates by means of an example [ ] § , this discrepancy should occur).

The numbers $\{f_\nu\}$ with which the function F of formula ( ) are associated in the above theory are the moments

$$f_{2\nu} = \int_{-\infty}^{\infty} t^{2\nu} e^{-|t|^\alpha} dt = \frac{2}{\alpha}\Gamma\left(\frac{2\nu+1}{\alpha}\right), \; f_{2\nu+1} = 0 \quad (\nu = 0, 1, \ldots)$$

and if we are to apply the above theory, we must take $\alpha \in [1,\infty)$. Since nonvanishing constituents of the form Ω in the appropriate version of formula ( ) do not tend to ~~zero~~ zero uniformly over a concave sector, it follows from parts B. and C of Theorem , that of all the functions defined over parts of sheets of $R\{F_+\}$ which may replace F in formula ( ), the only one to induce a remainder term $R_n(\lambda)$ with the requisite behaviour over a concave sector is $F^{(0)}$ continued (if α is not an even integer) to $F^{(-1)}$ over $(0,\infty)$ over the sheets over which these functions are defined (in Theorem we arranged that transition from one sheet of $R\{F_+\}$ to a neighbour should take place across the intervals of S_2, in the case being considered the single interval

$(0,\infty)$; the concave sector in question is $\Delta(-\frac{\pi}{2\alpha}, \frac{(2\alpha+1)\pi}{2\alpha})$ Thus $F_+^{(6)}$ is the only function determined over a sector in H_+ by the Watson-F. Nevanlinna method in this case. The sector, $\Delta(\frac{(\alpha-1)\pi}{2\alpha}, \frac{(\alpha+1)\pi}{2\alpha})$, over which $F_+^{(6)}$ is defined as an integral of Laplace type, depends upon α. We, of course, know that $F_+^{(6)}$ is regular over H_+, independent of α, and as we shall see, subject to certain conditions the Hamburger-~~R. Nevanlinna~~ R. Nevanlinna method extracts from the asymptotic series generated by $F_+^{(6)}$ a function having this property. However, we remark that the Watson-F. Nevanlinna method still produces a corresponding result if, for example, $e^{-|t|^\alpha}$ is replaced by $-e^{-|t|^\alpha}$ for $t \in (0,\infty)$ in formula ( ), thus producing an example to which the Hamburger–R. Nevanlinna method does not refer.

Using another notation, we have adverted to the Hamburger–R. Nevanlinna theory earlier (see the proof of Theorem). We are concerned with a function F which a) is regular over H_+ with $\operatorname{Im} F(\lambda) \leq 0$ for all $\lambda \in H_+$, and b) has an asymptotic expansion of the form ( ) valid for some fixed value of ~~$\arg(\lambda) \in \Delta(0,\pi)$~~ $\arg(\lambda) = \phi_*$ $(0 < \phi < \pi)$. The continued fraction associated with the series ( ) can be constructed, and its successive convergents determine, for a fixed value of $\lambda \in H_+$, a sequence of nested circles within which the value of $F(\lambda)$ must lie. If c) the Hamburger moment problem of the form ( ) is determinate, the diameters of the successive circles become arbitrarily small, and the value of the unique function satisfying conditions a) and b) is determined. This function has the representation ( ), σ being the normalised solution of the moment problem in question.

If the functions $F_+^{(n)}(\lambda)$ and $F_+^{(0)}(\lambda)$ have the same asymptotic expansion of the form ( ) for some $\arg(\lambda) = \phi$ $(0 < \phi < \pi)$ and $\mathrm{Im}\{F_+^{(0)}(\lambda)\} \leq 0$ and $\mathrm{Im}\{F_+^{(0)}(\lambda)\} + 2\pi\,\mathrm{Re}\{\Omega(\lambda)\} \leq 0$ over H_+, and condition c) above is satisfied, then $F_+^{(n)}$ and $F_+^{(0)}$ both have the same representation of the form ( ) (so that Ω is identically zero), and $\mathcal{R}\{F_+\}$ may so be constructed that the sheets over which $F_+^{(0)}$ and $F_+^{(n)}$ are defined coincide: the Hamburger-R.Nevanlinna method lands on the principal sheet of $\mathcal{R}\{F_+\}$. This occurs in the case of the function ( ) when $\alpha \in [1, \infty)$.

The case in which conditions a) and b) above hold, but c) does not, has been investigated in the celebrated memoir of R. Nevanlinna [ ]. In this case, many functions $F(\lambda; -\infty, \infty)$ of the form ( ) with $\sigma \in BN(-\infty, \infty)$ generate the same asymptotic expansion of the form ( ), valid over $\Delta(0, \pi)$. (With reference to the example of formula ( ), this occurs when $\sigma \in (0,1)$). In terms of certain entire functions P, Q, U and V derived from the coefficients of the continued fraction associated with the series ( ), R. Nevanlinna exhibits the totality of functions satisfying conditions a) and b) above in the form

$$F_\phi(\lambda) = \frac{P(\lambda) - \phi(\lambda) U(\lambda)}{Q(\lambda) - \phi(\lambda) V(\lambda)}$$

where ϕ ranges over all functions satisfying condition a) alone.

Functions of the form ( ) as described are of considerable interest in other contexts, and naturally they find application in the theory developed in this section. It is clear, for example, that P, Q, U and V having been determined, ϕ having been chosen, and $F_+^{(\phi)}$ taken to be F_ϕ as defined over H_+, $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{\phi\}$ are isomorphic. We wish to construct an example for which the functions defined over certain sheets of $\mathcal{R}\{F_+\}$ have the form ( ), each of these functions having the same asymptotic expansion over $\Delta(0,\pi)$ in its sheet. Select an arbitrary sequence $\{f_\nu\}$ associated with an indeterminate moment problem (the sequence ( ) with $\alpha \in (0,1)$, for example). Fix a finite sequence S_1 of disjoint open intervals in $(-\infty,\infty)$ and take S_2 to be the complement in

$(-\infty,\infty)$ of the sets of closed intervals corresponding to those of S_1. Define $\phi^{(0)}$ over $H_+ \cup H_- \cup S_2$ by means of the formula

$$\phi^{(0)}(\lambda) = \int_{S_1} \frac{\kappa\,dt}{\lambda - t}. \qquad (0<\kappa<\infty)$$

We are dealing with a function of the form considered in part D of Theorem , S_1 and S_2 having been interchanged, with $w(t)=\kappa$ over S_1. Define $F^{(0)}$ over H_+ by means of formula ( ) with F_ϕ and ϕ replaced by $F^{(0)}$ and $\phi^{(0)}$ respectively. Let λ, starting in H_+, repeatedly traverse a circular path, crossing from H_+ to H_- via an interval of S_1 and returning via one of S_2. Denote the functions encountered upon the successive sheets of $\mathcal{R}\{\phi\}$ $\langle\mathcal{R}\{F_+\}\rangle$ (these surfaces are isomorphic) by $\phi^{(n)}$ $\langle F_+^{(n)}\rangle$ $(n=0,1,\dots)$. We have

$$\phi^{(n)}(\lambda) = \phi^{(0)}(\lambda) - 2n\pi\kappa i \qquad (n=0,1,\dots)$$

and

$$F_n(\lambda) = \frac{P(\lambda) - \phi^{(n)}(\lambda)U(\lambda)}{Q(\lambda) - \phi^{(n)}(\lambda)V(\lambda)}. \qquad (n=0,1,\dots)$$

The functions $\phi^{(n)}(\lambda)$ $(n=0,1\dots)$ satisfy the condition a) above, and we have constructed our example. The asymptotic series generated by all functions $F_+^{(n)}$ $(n=0,1,\dots)$ over $\Delta(0,\pi)$ has as coefficients the moment sequence $\{f_\nu\}$ originally chosen.

Since $\mathrm{Im}\{\phi^{(n)}(\lambda)\}$ tends to zero as $\mathrm{Im}(\lambda)$ tends to infinity in H_+, it is clear that the condition a) above is violated by all functions of the form ( ) with negative values of n; the corresponding functions $F_+^{(n)}(\lambda)$ $(n=-1,-2,\dots)$ are not R. Nevanlinna functions. Nevertheless, they still generate the same asymptotic series as do the functions $F_+^{(n)}(\lambda)$ $(n=0,1,\dots)$. We have

$$F_+^{(n)}(\lambda) = \frac{U(\lambda)}{V(\lambda)} + \frac{\frac{P(\lambda)}{Q(\lambda)} - \frac{U(\lambda)}{V(\lambda)}}{\frac{Q(\lambda)}{V(\lambda)} - \phi^{(n)}(\lambda)}$$

The ~~functions~~ quotients $\frac{U(\lambda)}{V(\lambda)}$ and $\frac{P(\lambda)}{Q(\lambda)}$ both generate the same asymptotic series ( ) over $\Delta(0,\pi)$ (its coefficients are those chosen for the construction of P, Q, U and V). For large λ in $\Delta(0,\pi)$, $\ln Q(\lambda) \sim a\lambda^2$ and $\ln V(\lambda) \sim b\lambda$ for some $a, b \in (0,\infty)$ ([ ] §§13,16); $\phi^{(n)}(\lambda) \sim -2n\pi ki$ $(n=-1,2,\dots)$ of course. Hence, as stated, all $\{F_+^{(n)}\}$ generate the same asymptotic series over $\Delta(0,\pi)$. We preferred to use R. Nevanlinna's representation as a basis for the construction of the above example and the investigation of its asymptotic properties, rather than appeal to Theorem for this purpose. Nevertheless, for the sake of completeness we remark that each of the functions $F_+^{(n)}$ $(n=0,1,$

has a representation of the form ( ) in which σ has positive saltus at a denumerably infinite number of points in S_2 and no other points of increase in S_2, and $\sigma(t) = \frac{d\omega^{(0)}(t)}{dt}$ over S_1, where $\omega^{(0)}$ is defined upon one of the sheets of a surface $R\{\omega\}$ each of which has branch cuts along S_1. For the other functions defined over successive sheets of $R\{\omega\}$, we have

$$\frac{P(\lambda) - \{\phi^{(0)}(\lambda) - 2n\pi\kappa i\}U(\lambda)}{Q(\lambda) - \{\phi^{(0)}(\lambda) - 2n\pi\kappa i\}V(\lambda)} = F_+^{(0)}(\lambda) - 2\pi i \sum_0^{n-1} \omega^{(0)}(\lambda).$$

$$(n = 0, 1, \ldots)$$

Each function $\omega^{(n)}(\lambda)$ $(n > 0)$ possesses two sets of denumerably infinitely many poles in S_2, at $a_i^{(n,1)}$ $\langle a_i^{(n,2)} \rangle$ with residue $b_i^{(n,1)} \langle b_i^{(n,2)} \rangle$ $(i = 1, 2, \ldots)$ say; $\omega^{(0)}($ possesses the first set alone. We have $a_i^{(n,2)} = a^{(n-1,1)}$ $b_i^{(n,2)} = -b_i^{(n-1,1)}$ $(i = 1, 2, \ldots)$ for $n = 1, 2, \ldots$.

Although the matter is not in the main line of our interests, we mention that, subject to certain conditions, formula ( ), interpreted as a Cauchy principal value, serves directly to define a function of the real variable λ and, by implication, defines over the complex plane the function obtained by analytic continuation of this function of a real variable. Subject to the conditions of part B of Theorem , we denote by $F_{\pm}$ the function defined directly by formula ( ) for λ belonging to some interval of $\mathcal{S}$, and have

$$F_{\pm}(\lambda;-\infty,\infty) = F_{+}(\lambda;-\infty,\infty) - \pi i\,\omega(\lambda)$$

$$F_{\pm}(\lambda;-\infty,\infty) = F_{-}(\lambda;-\infty,\infty) + \pi i\,\omega(\lambda),$$

relationships which continue to hold over $H_{+} \cup H_{-} \cup \mathcal{S}$; under the conditions of clause (iii), $R\{F_{\pm}\}$ is a single sheet distinct from both $\mathcal{R}\{F_+\}$ and $\mathcal{R}\{F_-\}$. When the conditions of part C hold, we have two functions, $F_{1\pm}$ and $F_{2\pm}$, defined directly over $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, to consider. The formula analogous to both of formulae ( ) for the first of these functions is

$$F_{1\pm}^{(n)}(\lambda;-\infty,\infty) = F_{1\pm}^{(0)}(\lambda;-\infty,\infty) + \pi i\{\omega_1^{(0)}(\lambda) - \omega_1^{(n)}(\lambda)\}$$
$$+ 2\pi i \sum_1^n \{\omega_1^{(\nu)}(\lambda) - \omega_2^{(\nu)}(\lambda)\} \quad (n=0,1,\dots)$$

and the nature of $\mathcal{R}\{F_{1\pm}\}$ and of its counterpart $\mathcal{R}\{F_{2\pm}\}$ can be discussed as in the proof of part C.

The functions considered in the preceding paragraph may also generate asymptotic series. Under the conditions of clause Ei) of Theorem , for example, relationship ( ) holds for all $\arg(\lambda) \neq \pi$ belonging to the intervals of Θ over all sheets of $\mathcal{R}\{F_\pm\}$, and a further subsidiary result of the form given in clause Eic also holds. Thus, reverting to Hamburger's example of formula ( ), the series which represents F_+ in this case over $\Delta(-\frac{1}{4}\pi, \frac{5\pi}{4})$, and F_- over $\Delta(-\frac{5\pi}{4}, \frac{\pi}{4})$, also represents $F_\pm$ over the sectors $\Delta(-\frac{\pi}{4}, \frac{\pi}{4})$ and $\Delta(\frac{3\pi}{4}, \frac{5\pi}{4})$.

Summation formulae and a class of functions

Under appropriate conditions an integral transform expresses the sum of a series in closed form. For example, taking a_ν $(\nu = 0, 1, \dots)$ to be a sequence of finite complex numbers, setting $da(u) = a_\nu$ when $u = \nu$ $(\nu = 0, 1, \dots)$ and $da(u)$ elsewhere for $0 \le u < \infty$, and $e^{-x} = z$, the Laplace transform is, subject to convergence, $\sum_{\nu=0}^{\infty} a_\nu z^\nu$. As we have seen in §§ , a Stieltjes transform represents, again under suitable conditions, a generating function of an asymptotic series. A Stieltjes transform may also sum the partial fraction decomposition of the logarithmic derivative of an entire function: taking

$$\phi(\lambda) = A x^k \exp\left\{\sum_{\nu=0}^{p} a_\nu \lambda^\nu\right\} \prod_{\tau=1}^{\infty} \left[\left(1 - \frac{\lambda}{t_\tau}\right) \exp\left\{\sum_{\nu=1}^{p} \frac{\lambda^\nu}{\nu t_\tau^\nu}\right\}\right]$$

with the $\{t_\tau\}$ real, we have

$$(-\lambda)^{-\rho}\left\{\frac{d}{d\lambda}\ln\{\phi(\lambda)\}-\frac{k}{\lambda}-\sum_{\nu=1}^{p-1}\nu a_\nu\lambda^\nu\right\}=\sum_{r=1}^{\infty}(t_r)^{-\rho}(\lambda-t_r)^{-1}$$

$$=\int_{-\infty}^{\infty}\frac{d\sigma(t)}{\lambda-t}$$

where $d\sigma(t)=(t_r)^{-\sigma}$ when $t=t_r$ $(r=1,2,\ldots)$ and $d\sigma(t)$ is zero elsewhere in $(-\infty,\infty)$. (This decomposition and summation with $k=p=0$ and $\rho=1$, and the $\{t_r\}$ symmetrically distributed about the origin, occurs in the classical work of Grommer [ ]).

We have shown in § that Stieltjes transforms of the form ( ) can be expressed as contour integrals. Taking ω in formula ( ) to be a suitable meromorphic function, and expanding the contour, we find that the Stieltjes transform represents the sum of a series in closed form. We deal with functions that are meromorphic in the finite part of the complex plane, and with simple poles $a_\nu \notin [0,\infty]$ $(\nu = 1, 2, \ldots)$ only, with corresponding residues b_ν $(\nu = 1, 2, \ldots)$ and with $0 < |a_1| \leq |a_2| \leq |a_3| \leq \ldots$. It is supposed that there exists a sequence of closed contours $\{\mathcal{C}_n\}$, $\mathcal{C}_n$ including a_ν $(\nu = 1, 2, \ldots, n)$ but excluding a_ν $(\nu = n+1, n+2, \ldots)$, that the minimum distance R_n of $\mathcal{C}_n$ to the origin tends to infinity, and that the length of $\mathcal{C}_n$ is $O(R_n)$

When, for t on $\mathcal{C}_n$, $\omega(t) = O(R_n / \ln(R_n))$, we have

$$\int_0^\infty \frac{\omega(t)}{\lambda^2 + t^2}\,dt = \frac{1}{2\lambda}\Big[\pi\{\omega(0) + \omega(i\lambda) - \omega(-i\lambda)\} - i\ln(\lambda)\{\omega(i\lambda) - \omega(-i\lambda)\}\Big]$$

$$+ \frac{\pi}{2\lambda}\sum_1 \frac{b_\nu^2}{a_\nu(a_\nu^2 + b_\nu^2)} - \sum_1 \frac{b_\nu}{\lambda^2 + a_\nu^2}\ln(a_\nu)$$

When, for t on $\mathcal{C}_n$, $\omega(t) = O(R_n)$, and $a_{2\nu} = -a_{2\nu-1} = a'_\nu$, $b_{2\nu} = -b_{2\nu-1} = b'_\nu$ $(\nu = 1, 2, \ldots)$, we have

$$\int_0^\infty \frac{\omega(t)}{\lambda^2 + t^2}\,dt = \frac{1}{2\lambda}\Big[\pi\{\omega(0) + \omega(i\lambda) - \omega(-i\lambda)\} - i\ln(\lambda)\{\omega(i\lambda) - \omega(-i\lambda)\}\Big] + \pi i \sum_1 \frac{b'_\nu}{\lambda^2 + a'^2_\nu}$$

When, for t on ℓ_n, $\omega(t)=O(R_n^2/m(R_n))$, we have

$$\int_0^\infty \frac{\omega(t)}{\lambda^2+t^2}dt - \int_0^\infty \frac{\omega(t)}{1+t^2}dt = \tfrac{1}{2}\pi(\lambda^{-1}-1)\omega(0) - \omega'(0)\, m(\lambda)$$
$$-\sum_1 b_\nu \left[\frac{\frac{1}{2}\pi a_\nu^2(1-\lambda)(a_\nu^2-\lambda)+a_\nu^2\, m(-a_\nu)(1-\lambda^2)+m(\lambda)\lambda^2(1+a_\nu^2)}{a_\nu^2(1+a_\nu^2)(\lambda^2+a_\nu^2)}\right]$$

When, for t on ℓ_n, $\omega(t)=O(R_n^2)$, and $\{a_\nu'\}, \{b_\nu'\}$ are as described in connection with formula ( ), we have

$$\int_0^\infty \frac{\omega(t)}{\lambda^2+t^2}dt - \int_0^\infty \frac{\omega(t)}{1+t^2}dt = \tfrac{1}{2}\pi\{i\omega'(0)-\omega(i)\}$$
$$+\frac{\pi}{2\lambda}\{\omega(i\lambda)-i\lambda\omega'(0)\}-\omega'(0)\, m(\lambda)-i\pi(1-\lambda^2)\sum_1 \frac{b_\nu'}{(1+a_\nu'^2)(\lambda^2+a_\nu'^2)}.$$

With ω a suitable function, the structure of $\mathcal{R}\{F\}$ is made evident by inspection of the expressions upon the right hand sides of the above formulae. In particular, if in formulae (,) ω is not an even function, then $\mathcal{R}\{F\}$ clearly has an infinite number of sheets arising from the nonvanishing factor accompanying $m(\lambda)$.

A systematic study of the convergence properties of the above summation formulae and their extensions, and of the asymptotic series and approximating fractions with which they may be associated, leads to the introduction of a certain class of functions. This class, which includes the logarithmic derivative of the Γ-function, the logarithm of the Γ-function, and the Hurwitz and Riemann zeta-functions, is of considerable interest in itself. We prefer to delineate the numerous interconnections of the theory of this new class of functions elsewhere, and upon this occasion focus attention only upon those aspects of the theory leading to asymptotic series and the approximating fractions generated by these series

Theorem . a) Let W be a complex valued function a real variable, finite over $[0,\infty)$, with $W(0)=0$, and

b) let $W(v)=W_0+W_1v+W'(v)$ for $v\in(0,\infty)$, where W_0 and W_1 are complex numbers, and $W'(v)=O(v^{x})$ for small v, for arbitrarily large $x\in(0,\infty)$, and

c) let $W(v)=W_0+W_2v+W''(v)$, where W_2 is a complex number and $\int_0^\infty \left|\frac{W''(v)}{v}\right|^2 dv<\infty$.

Set

$$w(W;0)=0\,,\quad w(W,\omega)=\int_0^\infty \frac{\sin(\omega v)}{v}dW(v)$$

$$\omega(W;s,t)=\frac{t^s}{\Gamma(s)}\int_0^\infty v^{s-1}e^{-vt}dW(v)$$

$$F_+(W;s,\lambda)=\int_0^\infty \frac{\omega(W;s,t)}{\lambda^2+t^2}dt$$

$$P(W;t)=\int_0^\infty e^{-vt}dW(v)$$

and

$$B_{2\nu+2}(W)=(-1)^{\nu}(2\nu+2)!\int_0^\infty v^{-2\nu-2}dW'(v)\quad(\nu=0,1,\dots)$$

A. Define $\psi(W;\lambda)$ for all finite $\lambda \notin (-\infty,0]$ by

$$\psi(W;\lambda) = -\gamma(W) + \int_0^\infty \{(1+u)^{-1} - (\lambda+u)^{-1}\}\, dw(W;u)$$

where

$$\gamma(W) = \cancel{\tfrac{\pi}{2}} \tfrac{\pi}{2} W_1 + F_+(W';1,1).$$

~~Then~~ Then

$$W_0 \ln(\lambda) - \tfrac{\pi}{2} W_1 \lambda^{-1} - \psi(W;\lambda) \sim \sum \frac{B_{2\nu+2}(W)}{(2\nu+2)} \lambda^{-2\nu-2}$$

as λ tends to infinity in $\Delta(-\pi,\pi)$

~~B) Let $\ell(W;\lambda)$~~

B. Define $\ell(W;\lambda)$ for all finite $\lambda \notin (-\infty,0]$ by

$$\ell(W;\lambda) = \gamma(W)(1-\lambda) + \int_0^\infty \left\{\frac{\lambda-1}{1+u} - \ln\left(\frac{\lambda+u}{1+u}\right)\right\} dw(W;u)$$

Then

$$\ell(W;\lambda) + \tfrac{\pi}{2} W_1 \ln(\lambda) - W_0\{\ln(\lambda) - \lambda + 1\} + C(W) \sim \sum \frac{B_{2\nu+2}(W)}{(2\nu+1)(2\nu+2)} \lambda^{-2\nu-1}$$

as λ tends to infinity in $\Delta(-\pi,\pi)$, where

$$C(W) = \int_0^\infty \arctan(t)\, P(W';t)\, dt$$

C. Define $\zeta(W;s,\lambda)$ for all finite $\lambda \notin (-\infty,0]$ ~~for~~ and

$\mathrm{Re}(s) > 1$ by

$$\zeta(W; s, \lambda) = \int_0^\infty \frac{dw(W; u)}{(\lambda + u)^s}$$

(i)a) For $\lambda \in H_+$, $\zeta(W; s, \lambda)$ is defined for all finite $s \neq 1$ by

$$\zeta(W; s, \lambda) = W_0 (s-1)^{-1} \lambda^{1-s} + \frac{\pi}{2} W_1 \lambda^{-s} + \int_0^\infty (\lambda^2 + y^2)^{-\frac{1}{2}s} \sin\{s \arctan(\tfrac{y}{\lambda})\} P(W'; y)\, dy$$

~~b) $\zeta(W; s, \lambda)$ is defined for finite nonzero im~~

b) The function obtained by analytic continuation of the function $\zeta(W; s, \lambda)$ defined by formula ( ) across the positive imaginary axis is given for $\lambda \in H_- \cap H_{i+}$ by

$$\zeta(W; s, \lambda) = W_0 (s-1)^{-1} \lambda^{1-s} + \frac{\pi}{2} W_1 \lambda^{-s} + \int_0^\infty (\lambda^2 + y^2)^{-\frac{1}{2}s} \sin\{s \arctan(\tfrac{y}{\lambda})\} P(W'; y)\, dy - \pi \lambda^{\cancel{\phi}-s} \omega(W; s, -i\lambda),$$

that obtained by analytic continuation across the negative imaginary axis is given for $\lambda \in H_- \cap H_{i-}$ by replacing the last term in this formula by $+\pi\lambda^{-s}\omega(W; s, i\lambda)$.

$\xi(W; s, \lambda)$ is defined for finite nonzero imaginary values of λ and all finite $s \neq 1$ by taking the integral in formula ( ) to be a Cauchy principal value.

(ii) With $\xi(W; s, \lambda)$ defined for all finite $\lambda \notin (-\infty, 0]$ by appeal to the relevant subclause of (i) above, we have, for all finite $s \neq 1$

$$\lambda^{s-1}\xi(W; s, \lambda) - W_0(s-1) - \frac{\pi}{2} W_1 \lambda^{-1} \sim \sum_\nu \frac{(s)_{2\nu} B_{2\nu+2}(W)}{(2\nu+2)!} \lambda^{-2\nu-2}$$

as λ tends to infinity in $\Delta(-\pi, \pi)$, where $(s)_{2\nu} = s(s+1)\ldots(s+2\nu)$ $(\nu = 0, 1, \ldots)$

D. Let W' be real valued and nondecreasing over $[0, \infty]$.

(ia) The series in relationship ( ) is semi-convergent for $|\arg(\lambda)| \leq \frac{1}{4}\pi$, in the sense that for finite nonzero values of λ in this sector

$$\left| W_0 \mathfrak{m}(\lambda) - \frac{\pi}{2} N_1 \lambda^{-1} - \psi(W;\lambda) - \sum_{\nu=0}^{n-1} \frac{B_{2\nu+2}(W)}{(2\nu+2)} \lambda^{-2\nu-2} \right| < \left| \frac{B_{2n+2}(W)}{(2n+2)} \lambda^{-2n-2} \right|$$

$(n = 0, 1 \ldots)$

b) This series also possesses an inclusion property: for real values of W_0 and W_1, and positive real values of λ

$$\sum_{\nu=0}^{2n-1} \frac{B_{2\nu+2}(W)}{(2\nu+2)} \lambda^{-2\nu-2} < W_0 \mathfrak{m}(\lambda) - \frac{\pi}{2} N_1 \lambda^{-1} - \psi(W;\lambda) < \sum_{\nu=0}^{2n} \frac{B_{2\nu+2}(W)}{(2\nu+2)} \lambda^{-2\nu-2}$$

$(n = 0, 1 \ldots)$

(ii) Let $W'(v) = 0$ for all $v \in [0, \delta]$ for some $\delta \in (0, \infty)$.

a) The series in relationship ( ) is both $(B, 2)$ and B^2 summable to the function upon the left hand side of this relationship over the finite part of the real λ-axis.

b) ~~Aff~~ All forward diagonal sequences of Padé quotients generated by the series $\sum \frac{B_{2\nu+2}(W)}{(2\nu+2)} z^{\nu}$ converge uniformly to $z\left[W_0 \mathfrak{m}(z^{\frac{1}{2}}) - \frac{\pi}{2} W_1 z^{-\frac{1}{2}} - \psi(W; z^{\frac{1}{2}})\right]$ over any domain $\mathcal{B} E_i[-\infty, \infty]$ lying in H_+

$$I_1(V,u) = \int_0^V \frac{\sin(uv)}{v^2}\left(1-\frac{v}{V}\right)W''(v)\,dv$$

$$I_2(V,u) = \int_0^V \frac{\cos(uv)}{v}\left(1-\frac{v}{V}\right)W''(v)\,dv$$

From conditions b) and c)

$$I_1(V,u) = \int_0^V \frac{\sin(uv)}{v^2}\left(1-\frac{v}{V}\right)W'(v)\,dv + (W_1-W_2)\int_0^V \frac{\sin(uv)}{v}\left(1-\frac{v}{V}\right)dv$$

Since $\frac{W''(v)}{v} \in L^2(0,\infty)$, it follows from condition b) upon the behaviour of $W'(v)$ for small v that $\frac{W'(v)}{v^2} \in L^2(0,\infty)$ also. Hence, from a result of Plancherel [ ], the limit as V tends to infinity of the value of the first integral upon the right hand side of formula ( ) exists for almost all $u \in [0,\infty)$. The similar limit of the second integral is $\frac{\pi}{2}(W_1-W_2)$. Again, from Plancherel's result, $I_2(V,u)$ tends to a limit as V tends to infinity for almost all $u \in [0,\infty)$, and in conclusion, $I(V;u)$ does the same. But

$$I(V,u) = \int_0^V \frac{\sin(uv)}{v}\left(1-\frac{v}{V}\right)dW''(v) - \frac{1}{V}\int_0^V \frac{\sin(uv)}{v}\,W''(v)\,dv.$$

(iii) Mutatis mutandis, the results of clauses (i) and (ii) above also hold with regard to the series occurring in relationship ( ).

(iv) Results similar to the above also hold with regard to the series occurring in relationship ( ) when $s \in (0,\infty)$.

Proof. Concerning the functions defined at the commencement of the theorem, we remark that when W satisfies conditions a–c), the constants W_0, W_1, W_2 and the functions W' and W'' are uniquely determined.

The function $w(W;u)$ is defined by formula ( ) for almost all $u \in [0,\infty)$ and has the form

$$w(W;u) = \frac{\pi}{2} W_2 + W_0 u + w(W'';u)$$

where $\frac{w(W'';u)}{u} \in L^2(0,\infty)$. To prove this assertion, we consider the function $I(V;u) = I_1(V,u) - u I_2(V,u)$, when

From a further result of Plancherel [ ],

$$\int_0^V \frac{\sin(uv)}{v} W''(v)\,dv = o(\ln(V))$$

for large $V \in (0,\infty)$ for almost all $u \in [0,\infty)$ when $\frac{W''(v)}{v} \in L^2(0,\infty)$. Hence, as stated $\lim_{V=\infty} I(V,u) = w(W'';u)$ exists for almost all $u \in [0,\infty)$.

We now consider the decomposition

$$\int_0^V \frac{\sin(uv)}{v} dW''(v) = \int_0^V \frac{\sin(uv)}{v^2} W'(v)\,dv + (W_1 - W_2)\int_0^V \frac{\sin(uv)}{v}\,dv - u\int_0^V \frac{\cos(uv)}{v} W'(v)\,dv$$

Since $\frac{W'(v)}{v^2}$ and $\frac{W''(v)}{v}$ are ~~$L^2(0,\infty)$~~ functions of the class $L^2(0,\infty)$, the first and third integrals upon the right hand side of the above equation converge in mean, as V tends to infinity. to $L^2(0,\infty)$ functions; the second tends to $\frac{\pi}{2}(W_1 - W_2)$. We have shown, however, that the integral on the left hand side of the above equation converges, not only in mean, but almost everywhere, to $w(W'';u)$.

Thus $w(W''; u)$ is the sum of two $L^2(0,\infty)$ functions and a constant. By splitting the range of integration in formula ( ) into the two segments $[0,1]$, $[1,V]$, it is easily shown that $\lim_{V=\infty} I_1(V,u) = O(u)$ for small u, and hence that $w(W''; u) = O(u)$ also. In conclusion $\frac{w(W'';u)}{u} \in L^2(0,\infty)$ as stated.

$\omega(W'; s, t)$ can be written in the form

$$\omega(W'; s, t) = \frac{t^s}{\Gamma(s)} \left[\left\{ \int_0^{3\delta} + \int_{3\delta}^{\infty} \right\} v^{s-1} e^{-vt} dW'(v) \right]$$

where $\delta \in (0,\infty)$. From condition b), an $M \in (0,\infty)$ exists such that $|dW(v)| < M v^y dv$ for all $v \in [0, 2\delta]$, where $y \in (0,\infty)$ is arbitrarily large. Hence

$$\left| \frac{t^s}{\Gamma(s)} \int_0^{3\delta} v^{s-1} e^{-vt} dW'(v) \right| < \left| \frac{t^s}{\Gamma(s)} \int_0^{\infty} \left| v^{s-1} e^{-vt} dW'(v) \right| \right|$$

and, for all finite s, the first component upon the right hand side of relationship ( ) is $O(t^{-y})$ for large t

when $Re(t)>0$. Furthermore

$$\frac{t^s}{\Gamma(s)}\int_{3\delta}^{\infty} v^{s-1}e^{-vt}\,dW'(v) = \frac{t^s}{\Gamma(s)}\,e^{-2\delta t}\int_{\delta}^{\infty}(v+2\delta)^{s-1}e^{-tv}\,dW'(v+2\delta),$$

For all finite s, the second component upon the right hand side of relationship ( ) is $O(e^{-\delta t})$ for large t when $Re(t)>0$. Hence $\omega(W';s,t)=O(t^{-y})$ for large t, when $Re(t)>0$, where $y\in(0,\infty)$ is arbitrarily large. For small $t\in(0,\infty)$, the first component of $\omega(W';s,t)$ arising from the decomposition ( ) is visibly $O(t^s)$ for all finite s. The second component is $O(1)$ for $Re(s)>0$. Thus $\omega(W';s,t)=O(1)$ for small $t\in(0,\infty)$ and $Re(s)>0$. From what has been deduced concerning $\omega(W';s,t)$, $F_+(W';s,t)$ is defined by formula ( ) as an analytic function of λ over H_+ for $Re(s)>0$. Since $P(W';t)=t^{-1}\omega(W';1,t)$, $P(W';t)=O(t^{-y})$ for large $t\in(0,\infty)$, where $y\in(0,\infty)$ is arbitrarily large, and $P(W';t)=O(t^{-1})$

for small $t \in (0,\infty)$.

We add to the list of functions defined at the commencement of the theorem one, which although not featuring in the statement of the theorem, occurs in its proof, namely

$$Q(W;z) = z^2 \int_0^\infty \frac{dW(v)}{z^2+v^2}.$$

$Q(W;z)$ is defined by this formula as an analytic function of z over H_+, and $\lim Q(W;z)$ is bounded as z tends to zero in H_+.

It follows from condition b) that all integrals ( ) exist, and we remark that if we set $B_0(W) = W_0$, $B_1(W) = -\frac{\pi}{2}W_1$, $B_{2\nu+1}(W) = 0$ $(\nu = 1,2,\dots)$, then

$$Q(W;z) \sim \sum \frac{(-1)^\nu B_\nu(W)}{\nu!} z^\nu$$

as z tends to zero in H_+. We also remark that it follows from what has been shown above concerning the behaviour of $\omega(W'; s, t)$, that all moments generated by this function over the interval $[0, \infty]$ exist, and indeed

$$\int_0^\infty t^{2\nu}\omega(W'; s, t)\,dt = \frac{(-1)^\nu (s)_{2\nu} B_{2\nu+2}(W)}{(2\nu+2)!}. \quad (\nu = 0, 1, \ldots)$$

We are now in a position to commence the proof of of the successive results given in the theorem. From formula ( )

$$x\int_0^\infty e^{-xu}\,dw(W; u) = W_0 + \frac{\pi}{2}W_1 x + Q(W'; x)$$

for all $x \in [0, \infty)$. Taking $\lambda \in H_+$, multiplying this relationship throughout by $(e^{-\lambda x} - e^{-x})/x$, and integrating, we have

$$\int_0^\infty \{(\lambda+u)^{-1} - (1+u)^{-1}\}\,dw(W; u) = W_0\, m(\lambda) + \frac{\pi}{2}W_1(\lambda^{-1} - 1)$$
$$+ \int_0^\infty e^{-\lambda x} x \int_0^\infty \frac{dW'(v)}{x^2+v^2}\,dx - \int_0^\infty e^{-x} x \int_0^\infty \frac{dW'(v)}{x^2+v^2}\,dx$$

Changing the variable of integration to $t = \lambda x/v$, the first double integral in the above formula is shown to be $F_+(W'; 1, \lambda)$; the second is $F_+(W'; 1, 1)$. By rearrangement we derive the formula

$$W_0\, m(\lambda) - \frac{\pi}{2} W_1 \lambda^{-1} - \psi(W; \lambda) = F_+(W; 1, \lambda)$$

holding for $\lambda \in H_+$.

The analytic continuation of functions of the form $F_+(W; 1, \lambda)$ across the finite part of the imaginary axis has already been discussed in § ; relationships analogous to ( ) and ( ) hold for the functions obtained. For nonzero finite pure imaginary values of λ, the values of $F_+(W; 1, \lambda)$ obtained by analytic continuation are obtained by interpreting the integral in formula ( ) as a Cauchy principal value.

A relationship of the form ( ), ~~for~~ with $F_+(\lambda;1,\lambda)$ suitably interpreted, holds for all finite λ in the sector $\Delta(-\pi,\pi)$.

As is easily verified by expanding the expression $(\lambda^2+t^2)^{-1}$ in ascending powers of t^2/λ^2 with remainder term, and using formula ( ) with $s=1$, the series upon the right hand side of relationship ( ) represents $F_+(W;1,\lambda)$ asymptotically as λ tends to infinity in H_+. That it also represents the function obtained by analytic continuation throughout the sector $\Delta(-\pi,\pi)$ follows from what has been shown concerning the behaviour of $\omega(W;1,-i\lambda)$ for large λ in H_{i+} and $\omega(W;1,i\lambda)$ in H_{i-}, and from the theory given in § . The ~~ret~~ result of part A follows directly.

As is evident from formulae (,)

$$\ell(W;\lambda) = \int_1^\lambda \psi(W;\lambda')d\lambda'$$

where $\lambda \notin [-\infty, 0]$ and the path of integration is a straight line joining the points $\lambda'=1$ and $\lambda'=\lambda$. Hence, from formula ( ),

$$\ell(W;\lambda) = W_0\{\lambda \ln(\lambda) - \lambda + 1\} - \frac{\pi}{2}W_1 \ln(\lambda) - C(W) + \int_0^\infty \arctan(t/\lambda)P(W';t)dt$$

where $C(W)$ is defined by formula ( ). Integrating by parts on the right hand side of this relationship, we have

$$\ell(W;\lambda) = W_0\{\lambda \ln(\lambda) - \lambda + 1\} - \frac{\pi}{2}W_1 \ln(\lambda) - C(W) + \lambda\int_0^\infty \frac{d\sigma(W;t)}{\lambda^2+t^2}$$

where $\sigma(W;t) = \int_0^\infty v^{-2}(1-e^{-vt})dW'(v)$. The function directly represented over H_+ by the integral expression in formula ( ) may be analytically continued over the finite part of the sector $\Delta(-\pi,\pi)$, and relationships similar to those concerning the function $F_+(W;1,\lambda)$ also hold.

We also have

$$(-1)^{\nu}\int_0^{\infty} t^{2\nu}\,d\sigma(W;t) = \frac{B_{2\nu+2}(W)}{(2\nu+1)(2\nu+2)}. \qquad (\nu=0,1,\ldots)$$

As in the proof of the preceding part of the theorem, an asymptotic expansion for the function obtained by analytic continuation of the function defined by the integral expression in formula ( ) is easily derived (its coefficients are the values of the integral expressions ( )) and the result of part B follows immediately.

Taking $\lambda \in H_+$, $Re(s)>1$, multiplying relationship ( ) throughout by $\frac{x^{s-2}e^{-\lambda x}}{\Gamma(s)}$ and integrating, we have

$$\zeta(W;s,\lambda) = N_0(s-1)^{-1}\lambda^{1-s} + \frac{\pi}{2}N_1\lambda^{-s} + \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-2}e^{-\lambda x}Q(W';x)\,dx$$

and formula ( ) follows. From what has been established above concerning the behaviour of $P(W';y)$, it follows that formula ( ) serves to define $\zeta(W;s,\lambda)$ not only for $Re(s)>1$, but for all finite $s\neq 1$.

Changing the variable of integration in formula ( ) to $t = \lambda x/v$, we have, when $\lambda \in H_+$

$$f_0(W; s, \lambda) = W_0 (s-1)^{-1} \lambda^{1-s} + \frac{\pi}{2} W_1 \lambda^{-s} + \lambda^{1-s} F_+(W'; s, \lambda).$$

Using formula ( ), we have

$$F_+(W'; s, \lambda) = \lambda^{s-1} \int_0^\infty (\lambda^2 + y^2)^{-\frac{1}{2}s} \sin\{s \arctan(\frac{y}{\lambda})\} P(W'; y) dy,$$

a representation which serves to define $F_+(W'; s, \lambda)$ for $\lambda \in H_+$ and all finite s. Hence

$$F_+(W'; s, \lambda) = \frac{\lambda^{s-1}}{2i} \left\{ \int_0^\alpha + \int_\beta^\infty \right\} \left\{ \frac{1}{(\lambda - iy)^s} - \frac{1}{(\lambda + iy)^s} \right\} P(W'; y) dy$$

$$- \frac{\lambda^{s-1}}{4\pi} \int_{\mathcal{C}} \left\{ \frac{1}{(\lambda - iu)^s} - \frac{1}{(\lambda + iy)^s} \right\} \ln \left\{ \frac{\alpha - u}{\beta - u} \right\} P(W'; y) dy$$

where $[\alpha^*, \beta] \in (0, \infty)$, $\mathcal{C}$ is a loop enclosing all points of this closed segment of the real axis, and with $\lambda \in H_+$, the points $\pm i\lambda$ lie outside $\mathcal{C}$. The behaviour of $F_+(W'; s, \lambda)$ as λ crosses the segment $i(\alpha, \beta)$ of the positive imaginary axis can now be ascertained by an extension of the methods of § . To accommodate the case in which s is not an integer, $\mathcal{C}$ is first distended to Hankel form, so that it encloses the points of the segment $[\alpha, \infty)$ of the real axis; it is then extended further to include the point $-i\lambda$ (so that formula ( ) acquires a further term), and then still further to contain the segment of $i(\alpha, \beta)$ of the imaginary axis; λ is then made to cross this segment. We obtain

$$F_+(W'; s, \lambda) = F_-(W'; s, \lambda) - \frac{\pi\omega(W'; s, -i\lambda)}{\lambda}$$

for $\lambda \in \xi H_-$, where $F_-(W'; s, \lambda)$ is the function defined over H_- by a formula analogous to ( ). Formulae (,) then lead to the first result stated in clause C(c). The further result is derived analogously. The result of clause (ib) is similar to that of clause of Theorem and is derived in the same way.

To deal with the asymptotic expansion of the function $\zeta(W;s,\lambda)$, we first consider the function

$$R_n(\lambda) = \frac{(-1)^n \lambda^{s-n} \Gamma(n+2s)}{\Gamma(s)} \int_0^\infty (\lambda^2+y^2)^{-n-\frac{1}{2}s} \sin\{(2n+s)\arctan(\tfrac{y}{\lambda})\} \int_0^\infty v^{-2n} e^{-vy} dW'(y)\,dy$$

where $\lambda \in H_+$ and n is a nonnegative integer. Using substitution of the same type as those exploited above, it is found that when $2n+\mathrm{Re}(s)>-1$

$$R_n(\lambda) = \frac{(-1)^n}{\Gamma(s)\lambda^{2n}} \int_0^\infty \frac{t^{2n}\omega(W;s,t)}{\lambda^2+t^2} dt$$

and hence

$$R_{n+1}(\lambda) - R_n(\lambda) = \frac{(s)_{2n} B_{2n+2}(W)}{(2n+2)!\,\lambda^{2n+2}}.$$

for $2n > -\mathrm{Re}(s)-1$. The function $R_n(\lambda)$ of formula ( ) is clearly an entire function of s: the same is true of the function on the right hand side of equation ( ): this equation holds for all finite s and $n=0,1,\ldots$ (this is true even when $2n+\mathrm{Re}(s)\leq -1$, although formula ( ) then no longer holds). As is easily verified, the series upon the right hand side of relationship ( ) is the asymptotic expansion of $R_0(\lambda)$ as λ tends to zero in H_+. However, $R_0(\lambda)$ as given by formula ( ), is $F_+(W;s,\lambda)$. The series upon the right hand side of relationship ( ) is also the asymptotic expansion of $F_-(W;s,\lambda)$ in the sector $\Delta(\frac{\pi}{2},\frac{3\pi}{2})$, and is also the asymptotic expansion of the function obtained by setting $n=0$ in formula ( ), and interpreting the integral as a Cauchy principal value for large pure imaginary values of λ. At the commencement of the proof, we established that $\omega(W;s,-i\lambda)$

is $O(\lambda^{-y})$ for arbitrarily large $y \in (0,\infty)$ when $\lambda \in \Delta(0,\pi)$; $\omega(W;s,i\lambda)$ behaves in a similar fashion when $\lambda \in \Delta(-\pi,0)$. From formula ( ), we conclude that the series upon the right hand side of relationship ( ) represents $F_+(W;s,\lambda)$ asymptotically in the sector $\lambda \in \Delta(-\pi,\pi)$. The result of clause (ii) follows from formula ( ). (For the sake of completeness, we remark that when s is a nonpositive integer, $F_+(W;s,\lambda)$ is a polynomial in λ^{-2}; the series in relationship ( ) then terminates; when $2n > -s-1$, formula ( ) holds, and $R_n(\lambda)=0$.

When W' is real-valued and nondecreasing and $s \in (0,\infty)$, $\omega(W';s,t)$ is positive in the range $0 \le t < \infty$, and the moments of formula ( ) are Stieltjes moments. This is true in particular when $s=1$. The results of clauses Di a,b) now follow from the remarks made at the beginning of § . When $W'(v) = 0$ for all $v \in [0, 8]$, $\frac{B_{2\nu+2}(W)}{(2\nu+2)!} = O(8^{-2\nu})$ for large ν. The results of clause Dii a,b) follow from Theorem . The further results of part D are derived in a similar way.

Having proved Theorem , we remark that the function

$$\tilde{W}(v) = 1 + \frac{1}{\pi}v + 2\left[\frac{v}{2\pi}\right] \qquad (0 < v < \infty)$$

is of particular interest. This function has decompositions of the form given under conditions b) and c) with $\tilde{W}_0 = 1$, $\tilde{W}_1 = \frac{1}{\pi}$, $\tilde{W}'(v) = 2\left[\frac{v}{2\pi}\right]$, $\tilde{W}_2'' = 1 - \frac{1}{\pi}v + 2\left[\frac{v}{2\pi}\right]$. We also have $w(\tilde{W}, u) = [u+1]$ $(0 < u < \infty)$,

$$\omega(\tilde{W}; s, t) = \frac{2^s \pi^{s-1} t^s}{\Gamma(s)} \sum_{\nu=1}^{\infty} \nu^{s-1} e^{-2\pi\nu t}$$

and $\omega(\tilde{W}; 1, t) = 2t/(e^{2\pi t} - 1)$, $Q(\tilde{W}, z) = z/(1 - e^{-z})$. The $\{B_\nu\{\tilde{W}\}\}$ defined by formulae ( ) and as at the commencement of the proof are Bernoulli numbers. $\gamma(\tilde{W})$ is Euler's constant; $\psi(\tilde{W};\lambda)$ is defined by formula ( ) as the partial fraction decomposition of $\psi(\lambda) = \frac{d}{d\lambda} \ln\{\Gamma(\lambda)\}$. Formula ( ) now yields a differentiated version of Binet's second expression [ ] for $\ln\{\Gamma(\lambda)\}$ and expansion ( ) a differentiated form of Stirling's asymptotic series for $\ln\{\Gamma(\lambda)\}$. $l(\tilde{W},\lambda)$ as defined by formula ( ), is simply $\ln\{\Gamma(\lambda)\}$; formula ( ) now yields Binet's expression referred to above directly; and expansion ( ) Stirling's series itself (the inclusion property of this particular series, as described in clause Diib), is of course well known (see [ ] §)). $\zeta(\tilde{W};s,\lambda)$ as defined by formula ( ), is Hurwitz' zeta-function $\zeta(s,\lambda) = \sum (\lambda+\nu)^{-s}$. Formula ( ) yields

Hermite's formula ( ) for $\zeta(s,\lambda)$, valid for $\lambda \in H_+$ and all finite $s \neq 1$. Formulae of the type ( ) and the asymptotic expansion ( ) for $\zeta(s,\lambda)$ are not given for example in the standard reference works ([ ] Ch.13, [ ] vol.3, Ch 17) and appear to be new. The results of clause D iii) concerning the approximating fractions generated by the above special functions are, of course, new.

The Riemann surfaces of the various functions of the form $F_+(N;s,\lambda)$ occurring in Theorem and its proof may, by exploiting the properties of the functions $w(N;s,\lambda)$ and utilising relationships of the form (,) be investigated by use of the methods developed in § . However, explicit representations of the functions obtained by analytic continuation across the imaginary axis are, in the cases being considered, already available (see, for example, formulae (,)). By examining such a representation, we may deduce, for example, that the singularities of $F_+(W;1,\lambda)$ upon the principal sheet of its Riemann surface, for example, are confined to the nonpositive real axis. If $w(W;u)$ in formula ( ) is analytic over some interval of the positive real axis, analytic continuation of $F_+(W;1,\lambda)$ across the negative real axis is possible. In the particularly simple case in which W is the function $\tilde{N}$ of formula ( ), the Riemann surface of $F_+(W;1,\lambda)$ consists of an infinite system of sheets, $F_+(W;1,\lambda)$ has a logarithmic singularity at the origin, and simple poles at $\lambda = -1, -2, \ldots$ on each of the sheets, and is analytic elsewhere. The Riemann surfaces of $F_+(\tilde{W};1,\lambda)$ and $F_-(\tilde{W};1,\lambda)$ are disjoint.

Under appropriate conditions, the function $w(W';u)$ of formula ( ) may be expressed as the limit of a sequence of Fejer-polynomials derived from a Fourier series, simply by setting $W'(v) = \sum_{\nu=1}^{[\frac{v}{2\pi}]} c_\nu$, $\{c_\nu\}$ being a sequence of complex numbers. With W_0 and W_1 prescribed, and $c_\nu = c + c'_\nu$ $(\nu = 1,2,\ldots)$, where $\sum_1 \frac{|c'_\nu|^2}{\nu^2} < \infty$, we then have

$$W_2 = W_1 + \frac{c}{2\pi}, \quad W''(v) = \sum_{\nu=1}^{[\frac{v}{2\pi}]} c'_\nu + c\left(\left[\frac{v}{2\pi}\right] - \frac{v}{2\pi}\right)$$

and

$$w(W;u) = W_0 u + \frac{\pi}{2} W_1 + \frac{1}{2\pi} \lim_{n=\infty} \sum_{\nu=1}^{n} c_\nu \left(1 - \frac{\nu}{n}\right) \frac{\sin(2\pi\nu u)}{\nu}$$

for all $u \in [0,\infty)$. When $\tilde{W}_0 = 1$, $\tilde{W}_1 = \frac{1}{\pi}$, $c = 2$, $c'_\nu = 0$ $(\nu = 1, 2, \ldots)$ we again obtain the function $w(\tilde{W},u) = [u+1]$.

Alternatively, W may be extracted from the partial fraction decomposition of the logarithmic derivative of an entire function, and $w(W;u)$ may subsequently be expressed as the limit of a sequence of Fejer-Bochner polynomials derived from the Fourier expansion of a uniformly almost periodic function. Assuming ϕ to be an entire function having a representation of the form

$$\phi(z) = cz^k e^{az} \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{z_\nu}\right) e^{\frac{z}{z_\nu}}$$

where $z_{2\nu-1} = iv_\nu$, $z_{2\nu} = -iv_\nu$, $v_\nu \in (0,\infty)$ $(\nu = 1,2,\dots)$, we have

$$\frac{z\phi'(z)}{\phi(z)} = Q(W;z)$$

where, with W defined by formula ( ), $W_0 = k$, $\frac{\pi}{2}W_1 = a$, $W'(v) = 4n(v)$, $n(v)$ being the number of zeros of $\phi(z)$ (discounting that at the origin when $k>0$) inside the circle $|z| = v$. With W_0 and W_1 prescribed, and $v_\nu = d\nu + v'_\nu$ $(d \in (0,\infty), \nu = 1,2,\dots)$, where $\sum_1 \left(\frac{v'_\nu}{d_\nu + v'_\nu}\right)^2 < \infty$, we then have

$$W_2 = W_1 + \frac{2}{d}, \quad W''(v) = 4n(v) - \frac{2}{d} v$$

and

$$w(W;u) = W_0 u + \frac{\pi}{2} W_1 + 2 \lim_{n=\infty} \sum_{\nu=1}^{n} \left(1 - \frac{v}{v_n}\right) \frac{\sin(v_\nu u)}{v_\nu}$$

for all $u \in [0,\infty)$. When $\phi(z) = e^z - 1$, we again obtain the function $w(\tilde{W}, u) = [u+1]$. The functions $\mathfrak{F}(W;s,\lambda)$ arising from the functions W considered in this paragraph are integral transforms of functions of the type represented on the left hand side of relationship ( ) when $p=2$ and considered by Grommer [ ]. The results of part of Theorem concern the approximating fractions generated not by functions of the form ( ) themselves, but by integral transforms of such functions. In this sense our work can be considered to be an extension of that of Grommer.

The functions W discussed in the preceding two paragraphs satisfy the conditions of clause ( ) of Theorem (we may take δ in the clause in question to be 2π for the functions of the penultimate paragraph, and $\delta = \min v_\nu$ $(1 \leq \nu < \infty)$ for those of the last paragraph) and the results of that clause hold for them. It is of interest to consider a function for which the conditions of clause ( ) do not hold. We take

$$W(v) = \int_0^v e^{-\beta^2/v}\,dv \qquad (0<\beta<\infty)$$

(i.e. $W_0 = W_1 = W_2$, $W'(v) = W''(v) = W(v)$) and have, in particular

$$w(W;u) = \tfrac{1}{4}\pi J_0(2\beta u^{\frac{1}{2}})$$

$$\mathfrak{f}(W;s,t) = \frac{\pi\beta^2\lambda^{1-s}}{4\Gamma(s)} G_{13}^{21}\left(\beta^2\lambda \,\middle|\, {0 \atop s-1,0,-1}\right)$$

$$(|s|<\infty,\ \lambda \notin [-\infty,0])$$

$$\omega(W;s,t) = 2\beta^s t^{\frac{1}{2}s} K_s(2\beta t^{\frac{1}{2}}) \qquad (|s|<\infty,\ 0 \leq t<\infty)$$

where G_{13}^{21} is one of Meijer's G-functions. In the notation of formula ( ), we have

$$\mathcal{B}_{2\nu+2}(W) = (-1)^{\nu}(2\nu)!(2\nu+2)!\,\beta^{-4\nu-2}. \quad (\nu = 0,1,\ldots)$$

The series

$$\sum (-1)^{\nu}(s)_{2\nu}(2\nu)!\,\beta^{-4\nu-2}\lambda^{-2\nu-2}$$

represents the function $f(W;s,t)$ of formula ( ) asymptotically for large λ in the sector $\Delta(-\pi,\pi)$ for all finite s, and for $s \in (0,\infty)$ possesses the semi-convergent and inclusion properties described in clauses ( ) of Theorem . However, the numbers $(s)_{2\nu}(2\nu)!\,\beta^{-4\nu-2}$ $(\nu=0,1,\ldots)$ are associated with an indeterminate Stieltjes moment problem: the series ( ) is neither $(B,2)$ nor (B^2) summable for real values of λ, and the approximating fractions which it generates do not converge uniformly ~~for~~ to

$\mathfrak{f}(W;s,\lambda)$ as described in clause ( ) of Theorem .

The function $L(s,\lambda;\kappa)$ studied by Lerch, which is defined for $|\kappa|<1$, $Re(s)>1$ and finite $\lambda\neq 0,-1,-2,\ldots$ by the series expansion

$$L(s,\lambda;\kappa)=\sum\frac{\kappa^{\nu}}{(\lambda+\nu)^{s}}$$

is a natural extension of the Hurwitz zeta-function. Setting $w(W;u)=\kappa^{[u]}$ $(0\leq u<\infty)$, $L(s,\lambda;\kappa)$ is the function $\mathfrak{f}(W;s,\lambda)$ given by formula ( ). Using the inversion formula

$$W(v)=\int_0^\infty\frac{\sin(uv)}{u}dw(W;u)$$

corresponding to ( ), we have

$$W(v)=v+\arctan\left\{\frac{\kappa\sin(v)}{1-\kappa\cos(v)}\right\}.$$

$L(s,\lambda;\kappa)$ has a representation of the form ( ), valid for $Re(s)>0$, in which $W_0=W_1=0$ and

$$\omega(s,t) = 1 + \frac{\kappa t^s}{\Gamma(s)} \int_0^\infty \frac{v^{s-1} e^{-vt} \{\cos(v) - \kappa\}}{1 - 2\kappa\cos(v) + \kappa^2} dv.$$

Unfortunately, the function W of formula ( ) satisfies neither of the conditions β, γ) of Theorem , and in consequence the results of clause ( ) concerning asymptotic series cannot be applied.

We may, however, introduce a function related to that considered by Lerch, which may with advantage be investigated by use of the methods supporting the proof of Theorem . This function is defined for $\Omega \in (0, \infty)$, $Re(k) > 0$, $Re(s) > 0$ by the formula

$$L'(s, \Omega, k) = \int_0^\infty \frac{e^{-\Omega x} x^{s-1}}{1 - e^{-k(1+x)}} dx$$
$$= \sum e^{-\nu k} (\Omega + \nu k)^{-s}.$$

Expanding the function $\{1 - e^{-k(1+x)}\}^{-1}$ as a series of partial fractions, we find that for $Re(k) > 1$

$$L'(s,\Omega,k)=k^{-1}\Omega^{-s}{}_2F_0(s,1;-\Omega^{-1})+\frac{1}{2}\Omega^{-s}+k\tilde{\tilde{F}}_+(s,k)$$

where, with

$$\omega(s,t)=\frac{1}{\Gamma(s)}\sum_1^\infty \omega_\nu(t)$$

$$\omega_\nu(t)=0 \quad (0\le t\le(2\nu\pi)^{-1})$$

$$\omega_\nu(t)=2e^{-\Omega t}(2\nu\pi t-1)^{s-1}e^{-2\nu\pi\Omega t}, \quad ((2\nu\pi)^{-1}\le t<\infty)$$

$\tilde{\tilde{F}}$ is defined by the formula

$$\tilde{\tilde{F}}_+(s,t)=\int_0^\infty \frac{\omega(s,t)}{1+k^2t^2}\,dt$$

The function $\omega(s,t)$ is an analytic function of t over each of the open intervals $((2\pi)^{-1},\infty)$, $(\{2(\nu+1)\pi\}^{-1},(2\nu\pi)^{-1})$ $(\nu=1,2,\ldots)$, but a different analytic function is defined over each interval. The function $L'(s,\Omega,k)$ can be continued analytically across the imaginary k-axis, and formulae similar to ( ) (with $\frac{\omega(\pm i\lambda)}{\lambda}$ replaced by $k\omega^{(\nu)}(\pm ik^{-1})$) ~~obtained,~~ hold for the functions so obtained, but a different branch of this function is obtained by continuation across each interval $i[2\nu\pi, 2(\nu+1)\pi]$ ($\nu = \ldots, -1, 0, 1, \ldots$). Subsequent analytic continuation across the negative real k-axis is also possible, and if s is not an integer, $L'(s, \Omega, k)$ has a further system of sub-branches associated with the components $(k - i2\nu\pi)^{s-}$. All functions derived by analytic continuation of the function $L'(s, \Omega, k)$ defined over H_+ in the k-plane by formula ( ) are devoid of poles throughout their domains of definition. The component function $\tilde{\tilde{F}}_+(s, k)$ defined by formula ( ) over H_+, and all functions obtained from it by analytic continuation are asymptotically represented by a series of the form

$$\sum f_\nu (-k^2)^\nu$$

valid over the sectors $|2\tau\pi + \arg(k)| < \pi$ $(\tau = 0, 1, \ldots)$. When $s \in (0, \infty)$, the $\{f_\nu\}$ have a representation of the form ( ). In this case the series ( ) is semi convergent, in a manner, analogous to that described in clause ( ) of Theorem , for $|\arg(k)| \leq \frac{1}{4}\pi$; it possesses an inclusion property similar to that of clause ( ) of Theorem when $k \in (0, \infty)$; it is both $(B, 2)$ and (B^2) summable to $\tilde{F}_+(s, k)$ over the finite part of the positive real k-axis; it generates approximating fractions to which the results Theorem with z replaced by $-k^2$ may be applied.

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