Convergence and truncation error bounds for associated continued fractions

We propose to give a convergence theorem for the functions $P_{2r}\{\psi(\mu)\}$ (or, equivalently, [redacted] $\omega_{2r}^{(0)\prime}$) ($r = \mathbb{I}$), but before so doing derive the [redacted] convergence result upon which the proof of the theorem is based.

Theorem. Let $[\alpha, \beta]$ be a fixed interval of the finite [redacted] real axis, and $\xi(s)$ be a bounded nondecreasing [redacted] real valued function for $\alpha \leq s \leq \beta$, and not a step function with a finite number of salti. Set

[redacted] $$F(\lambda) = \int_\alpha^\beta (\lambda - s)^{-1} d\xi(s)$$

for all complex $\lambda \notin [\alpha, \beta]$, $t_\nu = \int_\alpha^\beta s^\nu d\xi(s)$ ($\nu = \mathbb{J}$) and let the r^{th} convergent of the [redacted] continued fraction associated with the series $\sum_{\nu=0}^{\infty} t_\nu \lambda^{-\nu-1}$ be

$$C_r(\lambda) = \frac{t_0}{\lambda - w_1 -} \; \frac{v_2}{\lambda - w_2 -} \cdots \frac{v_r}{\lambda - w_r} \qquad (r = \mathbb{I}_1)$$

Then, for $\lambda \notin [\alpha, \beta]$,

$$|F(\lambda) - C_r(\lambda)| \leq \text{[redacted]} \frac{(\beta - \alpha)\, t_0}{\kappa(\theta)|\lambda - \alpha||\lambda - \beta|} \Omega(\lambda; \alpha, \beta)^{2r-1} \quad (r = \mathbb{I}_1)$$

with $\theta = |\arg(\lambda - \beta)/(\lambda - \alpha)|$,

where $\kappa(\theta) = \cos\frac{1}{2}\theta$ $(0 \leq \theta \leq \frac{1}{2}\pi)$, and $\kappa(\theta) = \cos\frac{1}{2}\theta \cot\frac{1}{2}\theta$ $(\frac{1}{2}\pi \leq \theta < \pi)$ and [redacted]

$$\Omega(\lambda;\alpha,\beta) = \left| \frac{(\lambda-\alpha)^{1/2} - (\lambda-\beta)^{1/2}}{(\lambda-\alpha)^{1/2} + (\lambda-\beta)^{1/2}} \right|$$

[redacted]

If, in addition, either $\alpha \geq 0$ or $\beta \leq 0$, then the series $\sum_{\nu=0}^{\infty} t_\nu \lambda^{-\nu-1}$ also generates a corresponding continued

[redacted]

$$[redacted] (\lambda) = \frac{u_1}{\lambda-} \frac{u_2}{1-} \cdots \frac{u_{2r-2}}{1-} \frac{u_{2r-1}}{\lambda}$$

$$C_{2r}(\lambda) = \frac{u_1}{\lambda-} \frac{u_2}{1-} \cdots \frac{u_{2r-1}}{\lambda-} \frac{u_{2r}}{1}$$

[redacted]

fraction

$$\frac{u_1}{\lambda-} \frac{u_2}{1-} \cdots \frac{u_{2r-1}}{\lambda-} \frac{u_{2r}}{1-} \cdots$$

whose successive convergents of even order $\hat{C}_{2r}(\lambda)$ reduce to the form ( ) and for which, consequently, the inequality ( ) is available, and whose successive convergent of odd order

$$\hat{C}_{2r+1}(\lambda) = \frac{u_1}{\lambda-} \frac{u_2}{1-} \cdots \frac{u_{2r+1}}{\lambda} \qquad (10J)$$

satisfy the inequality

$$|F(\lambda)-\hat{C}_{2r+1}(\lambda)| \le \frac{(\beta-\alpha)\,|t_1|}{\kappa(\lambda)\,|\lambda-\alpha|\,|\lambda-\beta|\,|\lambda|}\,\Omega(\lambda;\alpha,\beta)^{2r-1}. \quad (r \ge 1)$$

where now $\lambda \in [\alpha,\beta]$ is complex and nonzero.

Proof. We start with a result of Gragg [ ]: for a fixed finite complex value of z for which $|\arg(1+z)|<\pi$ the function [redacted]

$$g(z) = \int_0^1 \frac{d\xi'(s')}{1+zs'},$$

[redacted]

where $\xi'(s')$ behaves [redacted] for $0 \le s' \le 1$ as does $\xi(s)$ for $\alpha \le s \le \beta$, generates a π-fraction whose [redacted] $4r$th convergent

[redacted]

$$w_{2r,2}(z) = \frac{\pi_0}{1+z-}\,\frac{z}{1+}\cdots [redacted] \frac{\pi_{2r-1}}{1+z-}\,\frac{z}{1}$$

is a boundary point of a convex [redacted] domain [redacted] containing [redacted] the value of $f(z)$ and whose diameter is less than

$K(z)\left|\{1-(1+z)^{1/2}\}/\{1+(1+z)^{1/2}\}\right|^{2r-1}$, where

$$K(z) = \max\{1, [redacted] \tan\tfrac{1}{2}\theta\}\,\frac{\pi_0}{\mathrm{Re}\sqrt{1+z}}\left|\frac{z}{\sqrt{1+z}}\right|$$

[redacted] $\theta = \arg(1+z)$ and $\pi_0 = \xi'(1)-\xi'(0)$. Setting [redacted] $z = -\lambda'^{-1}$, $\lambda'^{-1}g(z)$ becomes

$$F'(\lambda') = \int_0^1 \frac{d\xi'(s')}{\lambda'-s'} [redacted]$$

(the even part is taken twice)

$+\lambda^{-1} W_{2r+2}(z)$ contracts to an equivalent form

[redacted]

$$C_r'(\lambda') = \frac{v_1'}{\lambda - w_1' -} \frac{v_2'}{\lambda' - w_2' -} \cdots \frac{v_r'}{\lambda' - w_r'}$$

$\sum_{\nu=0}^{\infty} t_\nu' \lambda'^{-\nu-1}$ where $t_\nu' = \int_0^1 s^\nu d\psi'(s')$ $(\nu \geq 0)$

$C_r(\lambda')$ now being the rth convergent of the continued fraction associated with the series [redacted] Now $C_r(\lambda')$ is a boundary point of a convex domain containing the value of $F'(\lambda')$ and whose diameter is less than $W(\lambda') | \{\lambda'^{1/2} - (\lambda'-1)^{1/2}\} / \{\lambda'^{1/2} + (\lambda'-1)^{1/2}\}|^{2r+1}$

where

$$W'(\lambda') = \max\{1, \tan \tfrac{1}{2}\theta\} \text{ [redacted] } \pi_0 \,/\, |\lambda'(\lambda'-1)| \cos\tfrac{1}{2}\theta$$

[redacted] $\theta = |\arg$ [redacted] $\{(\lambda'-1)/\lambda'\}|$ [redacted] $< \pi$ and π_0 is as before. We now use [redacted] a result due to Stieltjes (L J [redacted] [redacted] see [ ] §): let [redacted] a series $\sum_{\nu=0}^{\infty} t_\nu \lambda^{-\nu-1}$ generate an associated continued fraction whose rth convergent is $C_r'(\lambda')$ $(r \in J_1)$, [redacted] let $\sum_{\nu=0}^{\infty} t_\nu \lambda^{-\nu-1}$ be the series obtained by setting $\lambda' = (\lambda - B)/A$ [redacted] (A, B being finite complex numbers with $A \neq 0$) in the above series, expanding each term, and regrouping, then the series $\sum_{\nu=0}^{\infty} t_\nu \lambda^{-\nu-1}$ also generates an associated continued fraction for whose rth convergent $C_r(\lambda)$ we have $C_r'(\lambda') = C_r(\lambda)$.

In formula ( ), we now set $s^{\circ} = \alpha + (\beta-\alpha)s'$, $\xi^{\circ}(s^{\circ}) = (\beta-\alpha)\,\xi'(s')$ $(0 \le s' \le 1)$, $\lambda = (\beta-\alpha)\lambda' - \alpha$, so that the [redacted] $F'(\lambda')$ [redacted] now becomes the function $F(\lambda)$ of formula ( ), and $t_0 = (\beta-\alpha)\tau_0$. [redacted] From the result just stated, the convergent $C_r'(\lambda')$ of formula ( ) becomes the $C_r(\lambda)$ of formula ( ), and, [redacted] finally, the result obtained for the diameter of the convex domain enclosing $F'(\lambda')-C_r'(\lambda')$ [redacted] leads to the inequality for $|F(\lambda)-C_r(\lambda)|$ stated in the theorem.

When $0 \notin (\alpha,\beta)$, the series $\sum_{\nu=0}^{\infty} t_{\nu+1}\lambda^{-\nu-1}$ also generates an associated continued fraction to which the above theory may be applied, where now $F(\lambda)$ is to be replaced by

$$F^{(1)}(\lambda) = \int_{\alpha}^{\beta} (\lambda-s)^{-1}\, s\, d\xi(s),$$

to [redacted] by $|t_1|$, and $C_r(\lambda)$ by $C_r^{(1)}(\lambda)$, the rth convergent of the new associated continued fraction. Furthermore, $F(\lambda) =$ [redacted] $\lambda^{-1}\{t_0 + F^{(1)}(\lambda)\}$ and, using the notation of formula $\hat{C}_{2r+1}(\lambda) = \lambda^{-1}\{t_0 + C_{2r}^{(1)}(\lambda)\}$ for nonzero $\lambda \notin [\alpha,\beta]$. Formulae [redacted] ( ) now leads directly to the inequality ( ).

It is clear that when $\lambda \in [\alpha, \beta]$, the factor in formula ( ) independent of r is bounded, while $\Omega(\lambda; \alpha, \beta) \in (0,1)$, implying that [redacted] the convergents $C_r(\lambda)$ $(r \in \mathrm{I})$ converge to $F(\lambda)$ for the values of λ in question, a [redacted] result first demonstrated by Markoff [ ] for general functions $\xi(s)$ of the type described. He derived the follow inequalities for $|F(\lambda) - C_r(\lambda)|$: when $\mathrm{Im}(\lambda) \neq 0$

$$|F(\lambda) - C_r(\lambda)| \leq \frac{2t_0}{d(\lambda; \alpha, \beta)} \text{[redacted]} \left\{ \frac{A\bar{B} - \mathrm{Re}(A\bar{B})}{AB - \mathrm{Re}(AB)} \right\}^r \quad (r \in \mathrm{I}_1)$$

where $d(\lambda; \alpha, \beta)$ is the distance from λ to the nearest point of $[\alpha, \beta]$ (i.e. $d(\lambda; \alpha, \beta)$ is $|\lambda - \alpha|$ when $\mathrm{Re}(\lambda) < \alpha$, and $|\lambda - \beta|$ when $\mathrm{Re}(\lambda) > \beta$) and $\mathrm{Im}(\lambda)$ when $\alpha \leq \mathrm{Re}(\lambda) \leq \beta$, and $A = \lambda - \alpha$, $B = \lambda - \beta$ [redacted];

when λ is real,

$$|F(\lambda) - C_r(\lambda)| \leq \frac{t_0}{\min\{|\lambda - \alpha|, |\lambda - \beta|\}} \left(\frac{\beta - \alpha}{2\lambda - \alpha - \beta} \right)^{2r} \quad (r \in \mathrm{I}_1)$$

and, as a further refinement,

$$|F(\lambda) - C_r(\lambda)| \leq \frac{4t_0}{\min\{|\lambda - \alpha|, |\lambda - \beta|\}} \frac{(\beta - \alpha)^{2r}}{\lambda_1^{2r} + \lambda_2^{2r}}. \quad (r \in \mathrm{I}_1)$$

where $\lambda_1, \lambda_2 = 2\lambda - \alpha - \beta \pm \{(\lambda - \alpha)(\lambda - \beta)\}^{1/2}$. It is easily deduced that the geometric term in formula ( ) is smaller than any of [redacted] the geometric terms occurring in formulae (–), and hence that inequality ( ) is sharper than these three inequalities. In particular, we have

$$\Omega(\lambda;\alpha,\beta) = (\beta-\alpha)/\{|\lambda-\alpha|^{1/2}+|\lambda-\beta|^{1/2}\}^2$$

[redacted]

$$= \frac{(\beta-\alpha)}{[2\lambda-\alpha-\beta+2\{(\lambda-\alpha)(\lambda-\beta)\}^{1/2}]}$$

[redacted]

Thus the geometric term in formula ( ) is derived by discarding the square root terms in the denominator of expression ( ); [redacted]

[redacted]

in the notation of formula ( ), $\Omega(\lambda;\alpha,\beta)^{2r} = (\beta-\alpha)^{2r}/\lambda_1^{2r}$, where λ_1 is the larger of the two terms λ_1, λ_2. The ratio of the geometric progression to which the sequence of formula ( ) tends is $\Omega(\lambda;\alpha,\beta)^2$; however, the accompanying factor in formula ( ) is larger than that accompanying the powers of $\Omega(\lambda;\alpha,\beta)^2$ in inequality ( ); and formula ( ) is also valid for complex values of λ.

Since it is [redacted] relatively easy to do so, we extend the results of Theorem to the Padé table. Let [redacted]

[redacted] $[\alpha, \beta]$, $\xi(s)$ and $\{t_\nu\}$ be as described in Theorem 2, let

$$f(z) = \int_\alpha^\beta (1 - zs)^{-1} d\xi(s)$$

and let $\{P_{i,j}(z)\}$ $(i,j \geq \mathrm{I})$ be the ensemble of Padé quotients generated by the series $\sum_{\nu=0}^{\infty} t_\nu z^\nu$. Then for $z^{-1} \notin [\alpha, \beta]$

$$|f(z) - P_{r,2m+r-1}(z)| \leq [\text{redacted}]\, w^{(2m)}(z)\, \omega\{z;\alpha,\beta\}^{r-1} \quad (r \geq \mathrm{I},\ m \geq \mathrm{I})$$

where $w^{(2m)}(z) = t_{2m} |z|^{2m+1} (\beta - \alpha) / \{|1-\alpha z| |1-\beta z| \kappa(\theta)\}$, $\theta = |\arg((1-\beta z)/(1-\alpha z))|$, $\kappa(\theta)$ is defined by formula ( ), and

$$\omega\{z;\alpha,\beta\} = \left| [\text{redacted}] \frac{(1-\alpha z)^{1/2} - (1-\beta z)^{1/2}}{(1-\alpha z)^{1/2} + (1-\beta z)^{1/2}} \right|$$

If $0 \notin (\alpha, \beta)$, then [redacted] $2m$ may be replaced by m consistently in [redacted] the above formulae. If $0 \in (\alpha, \beta)$, then an [redacted] inequality of the form ( ) holds [redacted] for all differences $|f(z) - P_{r,m+r-1}(z)|$ $(r \geq \mathrm{I},\ m \geq \mathrm{I})$ (when $\beta < 0$, [redacted] and m [redacted] $= 2m'+1$ is odd, t_{2m} must be replaced by $|t_{2m'+1}|$ in the expression for $\omega^{(2m)}(z)$. To derive [redacted] similar inequalities for the [redacted] quotients lying beneath the sub-principal diagonal of the Padé table, we remark that if $F(\lambda)$ and w_1 (we have, of course $w_1 = t_1/t_0$), are as defined by formulae (,), and

$$F(\lambda) = t_0/\{\lambda - w_1 - \tilde{F}(\lambda)\}$$

then [redacted] the function $\tilde{F}(\lambda)$ has a representation of the form ( ) with $\xi(s)$ replaced by $\tilde{\xi}(s)$, which has the same properties as $\xi(s)$. Furthermore, if the series $\sum_{\nu=0}^{\infty} \tilde{t}_\nu z^\nu$ is derived from the relationship $\sum_{\nu=0}^{\infty}$ [redacted] $t_\nu z^\nu = t_0/\{1 - w_1 z - z^2 \sum_{\nu=0}^{\infty} \tilde{t}_\nu z^\nu\}$ ($w_1 = t_1/t_0$) and [redacted] the Padé quotients generated by the series $\sum_{\nu=0}^{\infty} \tilde{t}_\nu z^\nu$ are $\tilde{P}_{i,j}(z)$ ($i,j = \mathbb{I}$), then $P_{i+2,j}(z) = t_0/\{$[redacted]$1 - w_1 z - z^2 \tilde{P}_{j,i}(z)\}$ ($i,j \geq \mathbb{I}$). We derive the inequality

$$|f(z)^{-1} - P_{2m+r+1,r}(z)^{-1}| \leq \tilde{\omega}^{(2m)}(z)\, \omega\{z;\alpha,\beta\}^{2r-1} \quad (r = \mathbb{I}_1, m = \mathbb{I})$$

where $\tilde{\omega}^{(2m)}(z) = \tilde{t}_{2m} |z|^{2m+3} (\beta - \alpha)/\{|1-\alpha z||1-\beta z||\xi(\theta)\}$.

Again, when $0 \notin (\alpha,\beta)$ an inequality of the form ( ) holds for all differences $|f(z)^{-1} - P_{2m+r+1,r}(z)^{-1}|$ ($r = \mathbb{I}_1, m = \mathbb{I}$) with $\tilde{t}_{2m}$ being replaced by $|\tilde{t}_{2m'+1}|$ when $m = 2m'+1$ is odd.

Formula ( ) leads to the [redacted] direct a posteriori error estimate

[redacted]

$P_{2m+r+1,r}(z)$ ✓ x_{-1}

$$|f(z) - P_{2m+r+1,r}(z)| \le \frac{(\beta-\alpha)(t_0 t_2 - t_1^2)\max\{|\alpha|,|\beta|\}^{2m}|z|^{2m+2}\,\omega(z;\alpha,\beta)}{t_0^2\,k(0)\,|1-\alpha z|\,|1-\beta z|\,d\{z^{-1};\alpha,\beta\}}$$

and a similar inequality for general values of $m \in J$ when $0 \notin (\alpha,\beta)$.

After this digression into the theory of continued fractions, we return to the convergence theory of the functions $\rho_{2\nu}\{\psi(\mu)\}$ $(\nu \in J)$.

Theorem 3. Let $[\alpha,\beta]$ be a fixed interval of the finite real axis, and $\xi(s)$ be a bounded nondecreasing real valued function for $\alpha \le s \le \beta$, and not a step function with a finite number of saltu. [redacted] Let $\mu \in (-\infty,\infty) \setminus [\alpha,\beta]$ be fixed, and

$$\psi(\mu) = \int_\alpha^\beta (\mu - s)^{-2}\, d\xi(s)$$

$$\rho\{\psi(\mu)\} = \int_\alpha^\beta (\mu - s)^{-1}\, d\xi(s).$$

[redacted], Then $\lim_{r=\infty} \rho_{2r}\{\psi(\mu)\} = \rho\{\psi(\mu)\}$, and

$$|\rho\{\psi(\mu)\} - \rho_{2r}\{\psi(\mu)\}| \leq (\beta - \alpha)|\psi(\mu)|\, p(\mu;\alpha,\beta)^{2r-1} \quad (r = \mathcal{J}_1)$$

where

$$p(\mu;\alpha,\beta) = \big|\,|\mu-\alpha|^{1/2} - |\mu-\beta|^{1/2}\big| \big/ \big|\,|\mu-\alpha|^{1/2} + |\mu-\beta|^{1/2}\big|$$

Proof. Let $\mu \leq \alpha$. Then $\mathcal{D}^{\nu}\psi(\mu)/(\nu+1)! = \int_{\alpha'}^{\beta'} s'^{\nu} d\xi'(s')$ $(\nu = \mathcal{J})$ where $\alpha' = (\beta-\mu)^{-1}$, $\beta' = (\alpha-\mu)^{-1}$ and $s' = (\mu - s)^{-1}$

$d\xi'(s') = s'^{2} d\xi(\mu + s'\xi^{-1})$ for [redacted] $\alpha' \leq s' \leq \beta'$. Hence, for $\nu = \mathcal{J}$, [redacted] $t'_{\nu} = \mathcal{D}^{\nu}\psi(\mu)/(\nu+1)!$ ~~$(\nu = \mathcal{J})$ have~~ has a representation of the form considered in Theorem 2, with $\alpha', \beta', s', \xi'$ in place of α, β, s, ξ in that theorem and, as is [redacted] easily verified, the function $F(\lambda)$ considered reduces in the present case to $\phi\{\psi(\mu)\}$ when $\lambda = 0$. The series $\sum_{\nu=0}^{\infty} t'_{\nu}\lambda^{-\nu-1}$ generates an associated continued fraction for whose convergents, using known determinantal formulae [ ], we have $\lim_{\lambda=0} C_r(\lambda) = H[t'_{\tau-1}]_{r,2}/H[t'_{\tau+1}]_{r-1}$ $(r = \mathcal{J}_1)$ (the series $\sum_{\nu=0}^{\infty} b'_{\nu}\lambda^{-\nu-1}$ from which these convergents are derived becomes meaningless continued fraction considered terminates, with $C_r(\lambda) = F(\lambda)$; inequality ( ) holds for the convergents $C_r(\lambda)$ $(r = I_1^{n-1})$.

If one of the saltus is at the origin, the corresponding continued fraction considered terminates with $C_{2n-1}(\lambda) = F(\lambda)$, otherwise with $C_{2n}(\lambda) = F(\lambda)$, and again inequality ( ) may meaningfully be applied to all but the final convergent. Again, if one of the saltus is at the origin, the Padé table associated with the function ( ) has an infinite block of quotients $P_{n+i-1,n+j-1}(z)$ $(i, j = I)$ equal to $f(z)$, otherwise an infinite block $P_{n+i,n+j-1}(z)$ $(i = I)$ equal to $f(z)$. Inequalities of the form ( ) may meaningfully be applied to quotients not belonging to the infinite block in question. Finally, for $\xi(\sigma)$ as described, the sequence $\{\rho_{2r}\{\psi(w)\}\}$ terminates with $\rho_{2n}\{\psi(w)\} = \psi\{\psi(w)\}$ and inequality ( ) holds for the functions $\rho_{2r}\{\psi(w)\}$ $(r = I_1^{n-1})$.