Interpolation theory

The following notes concern interpolation in a field by the use of polynomials and rational functions, the interpolatory argument values being assumed to be discrete. The notes serve both as a basis for subsequent more general theory concerning interpolation in the presence of confluent arguments and as a framework for the theory of the transformation of Schweins' series.

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I Notations, definitions and classes of matrices and mappings

In this preliminary section notations and definitions, together with conventions adopted in their use, are presented. Classes of matrices and mappings arising from the calculus of finite differences are also described. Since these notes are confined to classical interpolation theory, the mappings encountered are of fixed type, with the single exception of sequence difference mappings which must unavoidably be presented in terms of mappings of variable type. It is also convenient to define operations upon mappings in terms of previously determined operations over target domains.

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A. Definitions, notations and conventions

1] Prescribed mathematical systems

i) $\overline{\mathbb{N}}$, $\mathbb{N}$ and $\tilde{\mathbb{N}}$ are the integer sequences $0,1,\ldots$, $1,2,\ldots$ and $\ldots,-1,0,1,\ldots$ respectively. The integers of all three sequences possess ordering properties; those of the first two are also used for the purpose of measuring.

ii) $\mathbb{K}$ is a prescribed field which, in certain applications of the theory, is assumed to possess relevant required properties (e.g., with $i \in \mathbb{N}$ given, that of possessing at least i distinct members).

2] Sequences

In the following, where relevant, $\mathbb{S}$ and $\mathbb{T}$ are prescribed sets.

i) $\overline{\mathbb{N}}$ – based sequences

α) A finite $\overline{\mathbb{N}}$ – based sequence (from now on – a sequence) is a mapping whose source set has the form $\{0,1,\ldots,i\}$ where $i \in \overline{\mathbb{N}}$ depends upon the sequence in question and all integers from 0 to i feature in the set.

β) The notation $\Xi \in \mathrm{seq}$ indicates that Ξ is such a mapping

γ) With $\Xi \in \mathrm{seq}$ prescribed, $|\Xi| = i$, where the source set of Ξ is as just described.

δ) A sequence $\Xi \in seq$ is displayed in the form $a, b, \ldots, c$ where a in the target domain corresponds to 0 in $\overline{N}$, b to 1, ... and c to $|\Xi|$.

ε) With $\Xi \in seq$, $\{\Xi\}$ is the target set in Ξ (i.e. $\{a, b, \ldots, c\}$ in the example just given). Membership indications such as $c \in \{\Xi\}$ are abbreviated by presenting them in the form $c \in \Xi$.

bα) The function ξ occurring in a mapping in seq is called a sequential function

β) The notation $\xi \in seqf$ indicates that ξ features as a function in a prescribed mapping in seq. seqf is the set of all $\xi \in seqf$.

γ) The statement $\xi \in seqf$ having been made, $|\xi|$ is $|\Xi|$ where Ξ is the prescribed sequence in which ξ features. Ξ is then displayed as $\xi(0), \xi(1), \ldots, \xi(|\xi|)$.

cα) The notation $\Xi \in seq(T)$ indicates that T is the target domain in $\Xi \in seq$. Ξ is then said to be a T-sequence.

β) The notation $\xi \in seqf(T)$ indicates that $\xi \in seqf$ has T as a possible target domain. $seqf(T)$ is the set of all $\xi \in seqf(T)$.

ii) Void sequences

A void $\overline{N}$-based sequence (from now on – a void sequence is a mapping whose source set is a void set in $\bar{N}$. A void T-sequence is a void sequence with target domain T. (Void sequences may arise in the subtraction of one sequence from another and may also feature in operations upon matrices and matrix mappings (both contexts are dealt with below).)

iii) Special integer sequences

aα) With $h,k \in \tilde{N}$ and $h \le k$, the $\tilde{N}$-sequence $h, h+1, \ldots, k$ is denoted by $[h,k]$.

β) When $h<k$, $(h,k]$ denotes the sequence $h+1,\ldots,k$; $[h,k)$ and (h,k) are defined similarly.

b) The symbols "h," are omitted in the foregoing when $h=0$; thus $[k]$ denotes the sequence $0,1,\ldots,k$ and so on, but (h,k) is written as $((k))$.

c) With $\Xi \in seq$ and $\xi \in seqf$, $[1\Xi 1]$ and $[1 \xi 1]$ are written simply as $[\Xi]$ and $[\xi]$ respectively

d) With $h,k \in \tilde{N}$ and $k<h$, $[h,k]$ is the void $\tilde{N}$-sequence. In analogous circumstances, the symbols $(h,k]$, $[h,k)$ and (h,k) are similarly interpreted as are $(k]$ and $[k)$ when $k=0$ and $((k))$ when $k=1$. When Ξ is a void sequence $[\Xi]$ is a void $\bar{N}$-sequence.

iv) Sequence inequalities

Let $\Omega \in seq$.

a) The notation $\Xi \le \Omega$ indicates that either α) $\Xi \in \text{seq}$ also and

β) $\{\Xi\} \subseteq \{\Omega\}$, the multiplicity of each term in Ξ being not greater than that of its counterpart in Ω, or

γ) Ξ is a void sequence whose target domain is that of Ω.

The notations $\Xi = \Omega$ and $\Xi \subset \Omega$ indicate that in the first case both $\Xi \subseteq \Omega$ and $\Omega \subseteq \Xi$, while in the second only the first of these relationships is true.

(Of the conditions $\Xi \subseteq \Omega$ and $\{\Xi\} \subseteq \{\Omega\}$ the former is the stronger. Thus with Ξ and Ω being $0, 0$ and 0 respectively: $\{\Xi\} \subseteq \{\Omega\}$ but $\Xi \nsubseteq \Omega$.)

b) The notation $\Xi \leq \Omega$ indicates that $\Xi \subseteq \Omega$ and that, if Ξ is nonvoid, the ordering of the elements of Ξ appears in preserved form in Ω (so that, ξ and ω being the sequential functions in Ξ and Ω, a strictly increasing $\bar{N}$-sequence $a, b, \ldots, c$ for which $\xi(0) = \omega(a)$, $\xi(1) = \omega(b)$, …, $\xi(|\xi|) = \omega(c)$ can be determined.

The notations $\Xi = \Omega$ and $\Xi < \Omega$ have meanings analogous to those of $\Xi = \Omega$ and $\Xi \subset \Omega$ in the preceding clause.

c) The membership relationship $\nu \in \{\Omega\}$ is written simply as $\nu \in \Omega$.

v) Sections of sequences

Let $\xi \in \mathrm{seqf}(T)$.

a) Let $\Psi \subseteq [\xi]$ have the $\bar{\aleph}$-sequence representation $\psi(0), \psi(1), \ldots, \psi(|\Psi|)$. $\xi[\Psi]$ is the T-sequence $\xi(\psi(0)), \xi(\psi(1)), \ldots, \xi(\psi(|\Psi|))$.

b) With $[h,k] \subseteq [\xi]$ and $h \leq k$, $\xi[[h,k]]$ is written simply as $\xi[h,k]$. $\xi(h,k]$, $\xi[h,k)$ and $\xi(h,k)$ are similarly defined

c) When $h=0$, the abbreviated notations $\xi[k]$, $\xi(k]$ and $\xi((k))$ are used in place of those given above.

d) If Ψ is a void $\bar{\aleph}$-sequence, $\xi[\Psi]$ is a void T-sequence When $k<h$, $\xi[h,k]$ is a void T-sequence; similar conventions are adopted with regard to $\xi(h,k]$, $\xi[h,k)$ and $\xi(h,k)$ and also with regard to $\xi(k]$ and $\xi[k)$ when $k=0$ and $\xi((k))$ when $k \leq 1$.

vi) Sequence subtypes

a) With $j \in \bar{\aleph}$, the notations $\alpha \in \mathrm{seqf}(T|j)$ and $\alpha \in \mathrm{seqf}(T|\geq j)$ mean that $\alpha \in \mathrm{seqf}(T)$ with $|\alpha| = j$ and $|\alpha| \geq j$ respectively. $\mathrm{seqf}(T|j)$ and $\mathrm{seqf}(T|\geq j)$ are the sets of all $\alpha \in \mathrm{seqf}(T|j)$ and $\alpha \in \mathrm{seqf}(T|\geq j)$ respectively.

b) Let $k \in \bar{\aleph}$, $h \in [k]$, $\xi \in \mathrm{seq}(\bar{\aleph}|\geq k)$ and $j = \max \xi(\nu)$ for $h \leq \nu \leq k$. The notations $\alpha \in \mathrm{seqf}(T|\xi[h,k])$ and $\alpha \in \mathrm{seqf}(T| \geq \xi[h,k])$ are interpreted as $\mathrm{seqf}(T|j)$ and $\mathrm{seqf}(T|\geq j)$

respectively. When $h=0$, the notations $\alpha\in \mathrm{seqf}(T|\xi[k])$ and $\alpha\in \mathrm{seqf}(T|\geq\xi[k])$ are used. $\mathrm{seqf}(T|\xi[h,k])$ is the set of all $\alpha\in\mathrm{seqf}(T|\xi[h,k])$. The symbols $\mathrm{seqf}(T|\geq\xi[h,k])$, $\mathrm{seqf}(T|\xi[k])$ and $\mathrm{seqf}(T|\geq\xi[k])$ have similar meanings.

vi) Sequences with distinct members

a) The notation $\xi\in\mathrm{seqf}'(T)$ indicates that $\xi\in\mathrm{seqf}(T)$ and that the sequence $\xi(0),\xi(1),\dots,\xi(|\xi|)$ consists of distinct members of T.

b) Let $k\in\bar{N}$, $h\in[k]$ and $\xi\in\mathrm{seq}(\bar{N}|\geq k)$. The notation $\alpha\in\mathrm{seqf}'(T|\xi[h,k])$ indicates that $\alpha\in\mathrm{seqf}(T|\xi[h,k])$ and that $\alpha[\xi[h,k]]$ consists of distinct members of T. When $\xi[h,k]=[h,k]$ the notation $\alpha\in\mathrm{seqf}'(T|[h,k])$ is used. When $h=0$ the preceding notations are abbreviated to the forms $\alpha\in\mathrm{seqf}'(T|\xi[k])$ and $\alpha\in\mathrm{seqf}'(T|k)$ respectively.

viii) Operations upon sequences

a) $\mathrm{rev}:\mathrm{seq}\to\mathrm{seq}$ is the sequence reversal operator. With $\Xi\in\mathrm{seq}$ displayed as $a,b,\dots,c$, $\mathrm{rev}(\Xi)$ is $c,\dots,b,a$.

b) Let T be either $\bar{N}$, N or $\tilde{N}$.

α) The sum of $\Phi,\Psi\in\mathrm{seq}(T)$ is Φ followed by Ψ (thus if Θ is the sequential function of this sum, $\Theta[\Phi]=\Phi$ and $\Theta(|\Phi|,|\Phi|+|\Psi|+1)=\Psi$).

β) The difference $\Phi-\Psi$ is constructed by removing from

Φ all terms occurring in Ψ, the ordering of the remaining members of Φ if such exist being preserved; if Ψ contain: all terms of Φ, $\Phi - \Psi$ is a void sequence.

γ) With $|\Psi| \le |\Phi|$, the product $\Phi\Psi$ is the T-sequence constructed in the following way. Let ψ and ϕ be the sequential functions in Ψ and Φ respectively and construct $\operatorname{ord}\Psi \in \operatorname{seq}(\bar{\mathbb{N}} \mid |\Psi|)$ by letting, in the $\bar{\mathbb{N}}$-sequence representation $\operatorname{ord}\psi(0), \operatorname{ord}\psi(1), \ldots, \operatorname{ord}\psi(|\Psi|)$ of $\operatorname{ord}\Psi$, $\operatorname{ord}\phi(\omega)$ be the number of members of $\{\Psi\}$ less than $\psi(\omega)$ (such comparison is available over $\bar{\mathbb{N}}$, $\mathbb{N}$ and $\tilde{\mathbb{N}}$). $\Phi\Psi$ is then $\phi[\operatorname{ord}\Psi]$.

δ) If Φ is a void T-sequence and $\Psi \in \operatorname{seq}(T)$, $\Phi + \Psi$ and $\Psi + \Phi$ are simply Ψ, $\Phi - \Psi$ and $\Psi - \Phi$ being a void T-sequence and Ψ respectively; $\Psi\Phi$ is a void T-sequence. The sum, difference and product of two void T-sequences are all void T-sequences

α) Addition, subtraction and, if $\mathbb{K}$ is totally ordered, multiplication of sequences in $\operatorname{seq}(\mathbb{K})$ are as defined above, but the sum, difference and product of Φ and Ψ are denoted by $\Phi \dotplus \Psi$, $\Phi \setminus \Psi$ and $\Phi \times \Psi$ respectively.

β) The sum, difference and product of $\Phi, \Psi \in \operatorname{seq}(\mathbb{K})$ are also defined in terms of operations over $\mathbb{K}$. Let Φ and Ψ be $\phi(0), \phi(1), \dots, \phi(|\Phi|)$ and $\psi(0), \psi(1), \dots, \psi(|\Psi|)$ respectively, and set $j = \min\{|\Phi|, |\Psi|\}$. $\Phi + \Psi$ is $\phi(0)+\psi(0), \phi(1)+\psi(1), \dots, \phi(j)+\psi(j)$. $\Phi - \Psi$ and $\Phi \dot{\times} \Psi$ are defined termwise in the same way.

γ) The sum, difference and product involving at least one void $\mathbb{K}$-sequence are all void $\mathbb{K}$-sequences.

ix) Double sequences

A finite $\overline{\mathbb{N}}$-based double sequence (from now on — a double sequence) is a sequence of sequences, and is a mapping of the form $\xi : \mathfrak{S} \times [\kappa(\mathfrak{S})] \to T$, where $\mathfrak{S}$ is a source set of the form $\{[i]\}$, $i \in \overline{\mathbb{N}}$ being prescribed, and $\kappa : \mathfrak{S} \to \overline{\mathbb{N}}$. For each $\chi \in \mathfrak{S}$, the successive members $\xi(\chi|\omega)$ with $\omega \in [\kappa(\chi)]$ constitute a T-sequence $\xi(\chi)$ for which $|\xi(\chi)| = \kappa(\chi)$.

x) Implicit sequences

A finite $\overline{\mathbb{N}}$-based implicit sequence (from now on — an implicit sequence) is a mapping whose target set is $\overline{\mathbb{N}}$. The function ϕ occurring in such a mapping is called an implicit sequential function, the notation $\phi \in \mathrm{imp\ seq\,f}$ indicates that ϕ features as an implicit function occurring in a prescribed implicit sequence and imp seqf is the set of all such ϕ. The notation $\phi \in$ imp seqf($\mathbb{S}$) indicates that $\phi \in$ imp seqf has $\mathbb{S}$ as its source domain; imp seqf($\mathbb{S}$) is the set of all such ϕ.

xii) Inverses of sequences

a) Let $k \in \bar{\mathbb{N}}'$, $h \in [k]$ and $\xi \in$ seq$(\bar{\mathbb{N}}|\geq k)$ be prescribed. The mapping inv: seqf$'(T|\xi[h,k]) \to$ imp seqf$(\{\xi[h,k]\})$ is defined in the following way. Select $\theta \in$ seqf$'(T|\xi[h,k])$ and let $\theta(0), \theta(1), \ldots, \theta(|\theta|)$ represent the sequence (4) in which θ functions as a sequential function. Denote inv θ: $\theta(\xi[h,k]) \to \bar{\mathbb{N}}'$ by ϕ. Then $\phi(\theta(i)) = i$ for each $i \in \xi[h,k]$.

b) The mapping inv: seqf$'(T) \to$ imp seqf(T) is defined as a special case of the preceding in which $h=0$ and $\xi = [k]$.

c) Inverses of double sequences are defined by extension of the above.

3] Matrices

i) Matrix types

a) $[\mathbb{K}]$ is the set of all proper matrices (from now on — matrices) with elements in $\mathbb{K}$.

The elements of a member of $[\mathbb{K}]$ are indicated by use of a suffix—superscript notation, the suffix and superscript functioning as row and column indexes respectively (thus the elements of $A \in [\mathbb{K}]$ may be denoted by A^{ν}_{τ}, τ and ν indicating row and column numbers respectively). Ordering begins at zero (thus, in the example just given, all elements in the first row of A have suffix $\tau = 0$, all elements in the first column have superscript $\nu = 0$).

b) $\mathcal{L}[\mathbb{K}]$ is the set of all lower triangular members of $[\mathbb{K}]$ (i.e. matrices A in $[\mathbb{K}]$ for which $A^{\nu}_{\tau} = 0$ for all elements whose column suffix ν exceeds the row suffix τ).

c) $\mathcal{UL}[\mathbb{K}]$ is the set of all unit lower triangular members of $\mathcal{L}[\mathbb{K}]$ (now $A^{\tau}_{\tau} = 1$ for all diagonal members A^{τ}_{τ} of $A \in \mathcal{L}[\mathbb{K}]$

d) $\mathcal{U}[\mathbb{K}]$ is the set of all upper triangular members of $[\mathbb{K}]$ (now $A^{\nu}_{\tau} = 0$ when $\tau > \nu$).

e) $\mathcal{UU}[\mathbb{K}]$ is the set of all unit upper triangular member of $\mathcal{U}[\mathbb{K}]$ (again a condition of the form $A^{\nu}_{\nu} = 1$ holds).

f) $\mathrm{diag}[\mathbb{K}]$ is the intersection of $\mathcal{L}[\mathbb{K}]$ and $\mathcal{U}[\mathbb{K}]$ (now $A^{\nu}_{\tau} = 0$ when $\tau \neq \nu$).

g) $\mathcal{ULD}[\mathbb{K}]$ is the set of all unit lower diagonal matrices in $\mathcal{L}[\mathbb{K}]$ (now conditions of the form (α) $A^{\nu}_{\tau} = 0$ if either $\nu > \tau$ or $\tau > \nu + 1$ and (β) $A^{\tau}_{\tau} = 1$ hold).

h) $\mathcal{UUD}[\mathbb{K}]$ is the set of all unit upper diagonal matrices in $\mathcal{U}[\mathbb{K}]$ (now the relevant conditions have the form (α) $A_\tau^\nu = 0$ if either $\tau > \nu$ or $\nu > \tau+1$ and (β) $A_\nu^\nu = 1$).

(In most cases in which the structure of a lower triangular matrix A is prescribed, the formula expressing A_τ^ν with $\nu \le \tau$ may also be evaluated for $\tau < \nu$ and, by chance, then yields the value zero. In such a case, the symbol "$\mathcal{L}$" in the declaration $A \in \mathcal{L}[\mathbb{K}]$ is a comment upon the structure of A; the operative significance of the symbol may be dispensed with. A similar remark may be made concerning matrices A in $\mathcal{U}[\mathbb{K}]$. In the very few case in which lower and upper triangular matrices have these forms by declaration and not as a consequence of the nature of the elements A_τ^ν, a suitable statement drawing attention to the anomality is made.)

ii) Matrix subtypes

a) With $h, k \in \bar{\mathbb{N}}$ prescribed, $[\mathbb{K}|h,k]$ is the set of all matrices in $[\mathbb{K}]$ with $h+1$ rows and $k+1$ columns. The subsystems $\mathcal{L}[\mathbb{K}|h,k], \ldots, \mathcal{UUD}[\mathbb{K}|h,k]$ are similarly defined.

b) $[\mathbb{K}|h,h]$ is written simply as $[\mathbb{K}|h]$; correspondingly abbreviated notations $\mathcal{L}[\mathbb{K}|h], \ldots, \mathcal{UUL}[\mathbb{K}|h]$ are also adopted.

iii) Rows and columns

a) $\text{row}[\mathbb{K}]$ and $\text{col}[\mathbb{K}]$ are the sets of all row vectors and column vectors with elements in $\mathbb{K}$ respectively.

b) With $h \in \bar{\mathbb{N}}$ prescribed, $\text{row}[\mathbb{K}|h]$ is the set of all vectors in $\text{row}[\mathbb{K}]$ with $h+1$ elements. $\text{col}[\mathbb{K}|h]$ is similarly defined.

iv) Void matrices

~~iii)~~ Void matrices (i.e. matrices with zero number of rows or columns) and void row and column vectors also feature in theory to be given later. They may arise in the processes of row or column removal from given matrices or similar reductions of vectors and occur as submatrices in compound matrix expressions which nevertheless, represent proper matrices. Void matrices do not feature independently in matrix sums, differences or products; they are a recourse of convenience. (Instead of giving one compound expression which involves proper matrices when, for example $i, j \in \bar{\mathbb{N}}$ are such that $i \le j$, and another reduced expression to accommodate the case in which $j < i$, the first expression is used consistently; removal of the void submatrices, when they arise, yields the relevant reduced expression.

v) Zero and unit matrices

a) With $h, k \in \bar{\mathbb{N}}$ prescribed, $O_{[h]}^{[k]}$ is the zero matrix in $[\mathbb{K}|h,k]$. $O_{[h]}^{[h]}$ is written as $O[h]$. $O_{[h]}$ and $O^{[k]}$ are the zero vectors in col $[\mathbb{K}|h]$ and row $[\mathbb{K}|k]$ respectively.

b) With $i \in \bar{\mathbb{N}}$ prescribed, $I[i]$ is the unit matrix in diag $[\mathbb{K}|i]$. $\mathcal{L}I[i]$ and $\mathcal{U}I[i]$ are the members of $\mathcal{L}[\mathbb{K}|i]$ and $\mathcal{U}[\mathbb{K}|i]$ respectively whose elements not defined identically to be zero are unity (thus for the elements A_τ^ν of $A = \mathcal{L}I[i]$, $A_\tau^\nu = 0$ when $\tau < \nu$ and $A_\tau^\nu = 1$ when $\nu \leq \tau$). $I^{[i]}$ and $I_{[i]}$ are the members of row $[\mathbb{K}|i]$ and col $[\mathbb{K}|i]$ respectively whose elements are all unity.

c) Where convenience dictates, the zero matrix in $[\mathbb{K}|h-1,k]$ is written either as $O_{[h)}^{[k]}$ or $O_{(h]}^{[k]}$, and that in $[\mathbb{K}|h-2,k]$ as $O_{(h)}^{[k]}$, the selection of these alternatives being made to preserve conformity with adjacent formulae in which similar contracted formulations feature. The notations $O_{[h]}^{[k)}$, $O_{[h]}^{(k]}$, ... $I[i)$, $I(i]$, ... have corresponding meanings.

d) In addition to the proper matrices and vectors defined above, void versions are also encountered as constituents of compound matrix expressions (defined below). (When $h=0$ or $k \in \bar{\mathbb{N}}$, for example, $O_{[h)}^{[k]}$ is a void matrix.)

vi) Submatrices

a) With $h, k \in \bar{N}$ and $A \in [K|h,k]$ prescribed and $\Xi \subseteq [h]$, A_{Ξ} is formed by row selection from A: it is, when Ξ is nonvoid, that member B of $[K||\Xi|, k]$ for which

$$B_{\tau} = A_{\xi(\tau)}$$

as τ ranges through $[\Xi]$ where ξ is the sequential function of Ξ. When Ξ is void A_{Ξ} is a void matrix. Correspondingly, with $\Psi \subseteq [k]$, A^{Ψ} is formed by column selection: it is, when Ψ is nonvoid, that member C of $[K|h, |\Psi|]$ for which

$$C^{\nu} = A^{\psi(\nu)}$$

as ν ranges through $[\Psi]$, where ψ is the sequential function of Ψ. When Ψ is void, A^{Ψ} is a void matrix. The above notations are combined: A^{Ψ}_{Ξ} is in $[K||\Xi|, |\Psi|]$ when $\Xi \subseteq [h]$ and $\Psi \subseteq [k]$ are nonvoid

b) With $A \in [K]$ and $\Xi \in seq(\bar{N})$ prescribed, $A_{-\Xi} \in [K]$ is formed by removing from A those rows A_{τ} whose index τ feature in the sequence Ξ (the row dimension of $A_{-\Xi}$ is determined partly by the row dimension of A and partly by the number of distinct members of

Ξ occurring in the range covered by this row dimension).
Correspondingly, with $\Phi \in \mathrm{seq}(\bar{N})$ prescribed, $A^{-\Phi} \in [K]$ is obtained by column removal from A. $A^{-\Phi}_{-\Xi} \in [K]$ is defined by carrying out both row and column removal. (It may occur that submatrices formed in the way just described are void.)

vii) Compound matrices

a) The formation of compound matrices by adjunction of columns is indicated by the use of a multiplication sign. Thus with $h, k(0), \ldots, k(2) \in \bar{N}$ and $A \in [K|h, k(0)]$, $B \in [K|h, k(1)]$, $C \in [K|h, k(2)]$ the compound matrix

$$A \times B \times C$$

in $[K|h, k(0) + k(1) + k(2) + 2]$ contains in order the successive columns of A followed by those of B followed by those of C

b) Row adjunction is indicated by the use of a fraction slash. Thus, with suitably declared D, E, F the successive rows of

$$D/E/F$$

are those of D followed by those of E followed by those of F.

c) The above conventions feature in conjunction, auxiliary use of square brackets being made for clarity. Thus, with

$h \in \bar{N}$, $k \in \bar{N}$, the compound matrix

$$I[h+k] \times [O^{[k]}_{[h]} / I[k]]$$

is in $[\mathbb{K} | h+k, h+2k]$.

d) The convention that void constituent submatrices may occur in compound matrix constructions is adopted. (Thus, in the above example, it is also permitted that $k=0$; the compound matrix then reduces to $I[h]$.) In all cases in which this convention is adopted, the resulting compound matrix is nevertheless nonvoid.

viii) Operations upon matrices

Operations upon matrices in $[\mathbb{K}]$ are indicated in the conventional way. Thus the ordered product of suitably defined A and B is written as AB. Powers of matrices are, however, expressed with the help of braces: the square and inverse of suitably defined A are written as $\{A\}^2$ and $\{A\}^{-1}$ respectively.

ix) Determinants

Determinants of square matrices are indicated by the use of vertical bars. Thus, $|A|$ is the determinant of the suitably defined matrix A. To avoid possible confusion square brackets are introduced when dealing with determinar of compound matrices. Thus the determinant of $A \times B$, A and B being suitably defined, is expressed as $|[A \times B]|$.

4] Mappings

i) Sets of mapping functions

$\{\mathbb{S} \to \mathbb{T}\}$ is the complete set of functions occurring in mappings with source and target domains $\mathbb{S}$ and $\mathbb{T}$ respectively (i.e functions A occurring in mappings of the form $A : \mathbb{S} \to \mathbb{T}$). $\{\bar{\mathbb{N}} \to \bar{\mathbb{N}}\}$, $\{\bar{\mathbb{N}} \to \mathbb{K}\}$, $\{\mathbb{K} \to \mathrm{seq}(\mathbb{K}|i)\}$ with $i \in \bar{\mathbb{N}}$ prescribed and $\{\mathbb{K} \to [\mathbb{K}|h,k]\}$ with $h,k \in \bar{\mathbb{N}}$ prescribed are some examples of such sets to be encountered later.

ii) Function values

a) With $\mathbb{S}$ and $\mathbb{T}$ selected from $\bar{\mathbb{N}}$, $\mathbb{N}$, $\tilde{\mathbb{N}}$ and $\mathbb{K}$, the value in the target domain $\mathbb{T}$ assumed by $\phi : \mathbb{S} \to \mathbb{T}$ corresponding to i in the source domain $\mathbb{S}$ is denoted by $\phi(i)$.

b) Let $i \in \bar{\mathbb{N}}$ be fixed and $\xi : \mathbb{S} \to \mathrm{seqf}(\mathbb{T}|i)$. The value in the target domain $\mathrm{seqf}(\mathbb{T}|i)$ corresponding to z in the source domain $\mathbb{S}$ is the sequential function $\psi(z|)$ occurring in the mapping $\psi(z|) : [i] \to \mathbb{T}$; the term with index $\omega \in [i]$ yielded by this sequential function is denoted by $\psi(z|\omega)$. Again when $\omega \in [i]$, $\psi(\omega) : \mathbb{S} \to \mathbb{T}$ is the mapping defined by the values of the term $\psi(z|\omega)$ with index ω as z ranges through $\mathbb{S}$.

c) Let $h,k \in \bar{\mathbb{N}}$. The matrix value in the target domain

$[\mathbb{K}|h,k]$ assumed by $A: \mathbb{S} \to [\mathbb{K}|h,k]$ corresponding to z in the source domain $\mathbb{S}$ is denoted by $A(z)$. With $z \in [h]$ prescribed, $A_z: \mathbb{S} \to \text{row}[\mathbb{K}|k]$ is the row mapping with index z determined by A; with $\nu \in [k]$, $A^\nu: \mathbb{S} \to \text{col}[\mathbb{K}|h]$ is a corresponding column mapping; with z, ν as specified, $A_z^\nu: \mathbb{S} \to \mathbb{K}$ is a component mapping. The values in $\text{row}[\mathbb{K}|k]$, $\text{col}[\mathbb{K}|h]$ and $\mathbb{K}$ corresponding to $z \in \mathbb{S}$ are denoted by $A_z(z)$, $A^\nu(z)$ and $A_z^\nu(z)$ respectively. Similar conventions with regard to other matrix mappings are adopted.

iii) Equivalence and inequality

a) Equivalence of two mappings over a common source subdomain is indicated by enclosing the subdomain within triangular brackets. Thus with $\mathbb{S}', \mathbb{S}''$ two nondisjoint source domains, $\theta: \mathbb{S}' \to \mathbb{K}$, $\phi: \mathbb{S}'' \to \mathbb{K}$ and $\mathbb{B} \subseteq \mathbb{S}' \cap \mathbb{S}''$, the notation

$$\theta = \phi \quad \langle \mathbb{B} \rangle$$

indicates that $\theta(z) = \phi(z)$ for each $z \in \mathbb{B}$.

With $i \in \overline{\mathbb{N}}$, equivalence between two mappings $\theta: \mathbb{S}' \to \text{seq}(\mathbb{K}|i)$ and $\phi: \mathbb{S}'' \to \text{seq}(\mathbb{K}|i)$ is indicated in the same way as is, with $h, k \in \overline{\mathbb{N}}$, equivalence between two mappings $\theta: \mathbb{S}' \to [\mathbb{K}|h,k]$ and $\phi: \mathbb{S}'' \to [\mathbb{K}|h,k]$.

b) Inequalities between suitable mappings is treated in a similar manner. Thus, with $\mathbb{S}'$, $\mathbb{S}''$ and $\mathbb{B}$ as above, $i \in \overline{\mathbb{N}}$ and $\theta : \mathbb{S}' \to seq(T|i)$, $\phi : \mathbb{S}'' \to seq(T|i)$, the notation

$$\theta \leq \phi \quad \langle \mathbb{B} \rangle$$

indicates that $\theta(z) \leq \phi(z)$ for each $z \in \mathbb{B}$.

iv) Structural operations

Structural operations upon mappings are effected pointwise: at each argument value in the source domain, operations defined in the target domain are carried out upon function values.

Thus with $i \in \overline{\mathbb{N}}$, $\phi : \mathbb{K} \to seq(\mathbb{K}|i)$ and $\Xi \subseteq [i]$, $\phi[\Xi] : \mathbb{K} \to seq(T||\Xi|)$ is the mapping $\psi : \mathbb{K} \to seq(T||\Xi|)$ for which $\psi(z|\omega) = \phi(z, \xi(\omega))$ for each $z \in \mathbb{K}$ and $\omega \in [\Xi]$, ξ being the sequential function in Ξ.

Again, with $h, k \in \overline{\mathbb{N}}$, $A : \mathbb{K} \to [\mathbb{K}|h,k]$ and $\Xi \subseteq [h]$, $A_{\Xi} : \mathbb{K} \to [\mathbb{K}||\Xi|,k]$ is the mapping $B : \mathbb{K} \to [\mathbb{K}||\Xi|,k]$ for which $B = A_{\Xi} \langle \mathbb{K} \rangle$.

Compound matrix mappings of fixed type are dealt with in the same way.

v) Sequential and algebraic operations

Sums, differences, ~~and~~ products and, where well defined quotients of simple mappings are defined pointwise over the intersection of constituent source domains. Thus, with $\mathfrak{S}', \mathfrak{S}''$ two non-disjoint source domains the product of $B: \mathfrak{S}' \to \mathbb{K}$ and $C: \mathfrak{S}'' \to \mathbb{K}$ is the mapping $A: \{\mathfrak{S}' \cap \mathfrak{S}''\} \to \mathbb{K}$ for which $A = BC \langle \mathfrak{S}' \cap \mathfrak{S}'' \rangle$.

Operations upon sequence and matrix mappings of fixed type are treated in the same way.

(It should be noted that the difference of the fixed type sequence mappings $\theta, \phi: \mathfrak{S} \to \text{seq}(T|i)$, where $i \in \overline{\mathbb{N}}$, is of variable type: the cancellation in the formation of $\theta(z) - \phi(z)$, and hence the number of terms in this difference sequence, may vary with z over $\mathfrak{S}$.)

vi) Domains of nonsingularity

With $i \in \overline{\mathbb{N}}$ and the mapping $A: \mathfrak{S} \to [\mathbb{K}|i]$ prescribed, $N\mathfrak{S}(A)$ is the subset consisting of all $z \in \mathfrak{S}$ for which $A(z)$ is nonsingular.

5] Allocation

i) The allocation of constants and sequences

The asymmetric allocation relationship ":=" is used for the purposes of definition. Thus the numbers $b, c \in \mathbb{K}$ being prescribed, the number $a \in \mathbb{K}$ may be defined by setting

$$a := bc$$

Again, with $\phi, \psi \in seq(\bar{N})$ prescribed, $\theta \in seq(\bar{N})$ may be defined by setting

$$\theta := \phi + \psi.$$

ii) Assignment statements

The relationship ":=" is used within triangular brackets to indicate the range of running variables in assignment statements. Thus, with $\kappa \in seq(\bar{N})$, the notation $a(\nu) \in K \langle \nu := \kappa \rangle$ means that $|\kappa|+1$ numbers $a(\nu)$ are prescribed. Nested use of this notation is made: with $\kappa, \lambda \in seq(\bar{N})$, the notation $b(\tau, \nu) \in K \langle \tau := \kappa, \nu := \lambda \rangle$ has a similar meaning. In the case in which $\kappa = \lambda$, the notation just given is contracted to the form $b(\tau, \nu) \in K \langle \tau, \nu := \kappa \rangle$. (The allocation in the first of the examples just given may be presented in the slightly more cumbersome form $a\{\nu(\omega)\} \in K \langle \omega := [\kappa] \rangle$; alternative forms of the allocations in the subsequent examples may be given.)

iii) Matrix allocations

Conjoint use of the above conventions is made in fixed type matrix allocations.

a) With $\xi, \zeta \in seq(\bar{N})$ and $A(\tau, \nu) \in K \langle \tau := \xi, \nu := \zeta \rangle$ prescribed, a matrix $A \in [K \,||\xi|, |\zeta|]$ may be defined by setting

$$A := [A(\tau,\nu)]_{\tau:=\mathfrak{z}}^{\nu:=\mathfrak{b}}$$

In A, $A_\chi^\omega = A(\mathfrak{z}(\chi), \mathfrak{b}(\omega))$ $\langle \chi := |\mathfrak{z}|, \omega := |\mathfrak{b}| \rangle$. In the case in which $\mathfrak{z} = \mathfrak{b}$, A is defined by setting

$$A := [A(\tau,\nu)] \langle \tau, \nu := \mathfrak{z} \rangle$$

b) With $h, k \in \overline{\mathbb{N}}$ and $A(\tau,\nu) \in \mathbb{K}$ $\langle \tau := [h], \nu := [\min(\tau, k)] \rangle$ prescribed, a matrix $A \in \mathcal{L}[\mathbb{K}|h,k]$ is defined by use of the notation

$$A := \mathcal{L}[A(\tau,\nu)]_{\tau:=[h]}^{\nu:=[k]}$$

the requisite zero elements $A_\tau^\nu = 0$ with $\nu > \tau$ being automatically provided by the specification of A. When $h=k$ the notation

$$A := \mathcal{L}[A(\tau,\nu)] \langle \tau, \nu := [h] \rangle$$

is used.

Similar considerations relate to the definition of matrices in $\mathcal{UL}[\mathbb{K}|h,k]$, $\mathcal{U}[\mathbb{K}|h,k]$ and $\mathcal{UU}[\mathbb{K}|h,k]$.

Integer intervals that are open at the upper limit also feature in the allocations just described. Thus, with $h \in \overline{\mathbb{N}}$, $k \in \mathbb{N}$ and $A(\tau,\nu) \in \mathbb{K}$ $\langle \tau := [h], \nu := [\min(\tau, k-1)] \rangle$ prescribed, a matrix $A \in \mathcal{L}[\mathbb{K}|h,k-1]$ is defined by use of the notation

$$A := \mathcal{L}[A(\tau,\nu)]_{\tau:=[h]}^{\nu:=[k)}$$

c) With $h, k \in \overline{\mathbb{N}}$ and $d(\chi) \in \mathbb{K}$ $\langle \chi := [i] \rangle$ prescribed, where $i = \min(h,k)$, a diagonal matrix $D \in \mathrm{diag}[\mathbb{K}|h,k]$ for whic

$D_x^x = d(x)$ $\langle x := [i] \rangle$ is defined by setting

$$D := \mathrm{diag}\,[d(x)]\ \langle x := [i] \rangle$$

With $h, k \in \overline{N}$, $i = \min(h, k+1)$ and $e(x) \in K$ $\langle x := [i] \rangle$ prescribed, a unit lower diagonal matrix $E \in \mathcal{ULD}[K|h,k]$ for which $E_x^{x-1} = e(x)$ $\langle x := (i] \rangle$ is defined by setting

$$E := \mathcal{ULD}[e(x)]\ \langle x := (i] \rangle$$

With $h, k \in \overline{N}$, $i = \min(h+1, k)$ and $f(x) \in K$ $\langle x := (i] \rangle$ prescribed, a unit upper diagonal matrix $F \in \mathcal{UUD}[K|h,k]$ for which $F_{x-1}^{x} = f(x)$ $\langle x := (i] \rangle$ is defined by setting

$$F := \mathcal{UUD}[f(x)]\ \langle x := (i] \rangle$$

(When $i = 0$ in either of the last two examples, the prescription of the $e(x)$ or $f(x)$ is to be ignored; E reduces to a row vector with one nonzero element whose value is unity and F to a column vector of similar form.)

d) With $\kappa \in \mathrm{seq}(\overline{N})$ and $c(\omega) \in K$ $\langle \omega := \kappa \rangle$ prescribed, a column vector $A \in \mathrm{col}[K||\kappa|]$ is defined by setting

$$A := \mathrm{col}[c(\tau)]_{\tau := \kappa}$$

Similarly, a row vector $B \in \mathrm{row}[K||\kappa|]$ is defined by setting

$$B := \mathrm{row}[c(\omega)]^{\omega := \kappa}$$

iv) Mapping allocations

Mappings are allocated by extended use of the convention described above. Thus the mappings $b, c : \mathfrak{S} \to \mathbb{K}$ being prescribed, a product mapping $a : \mathfrak{S} \to \mathbb{K}$ for which $a = bc \langle \mathbb{K}$ may be defined as in clause (i). Sequence mappings are treated in the same way. Under conditions analogous to those of subclause (iiia), with mappings $A(z, \nu) : \mathfrak{S} \to \mathbb{K}$ $\langle z := \mathfrak{z}, \nu := \mathfrak{b} \rangle$ prescribed, a fixed type mapping $A : \mathfrak{S} \to [\mathbb{K} \| |\mathfrak{z}|, |\mathfrak{b}|]$ may be defined as in that subclause and so on.

6] Sums and products

i) Sums and products of constants

With the $\bar{\mathbb{N}}$-sequence Ξ displayed as $\xi(0), \xi(1), \ldots, \xi(|\Xi|)$ and $a(\omega) \in \mathbb{K}$ $\langle \omega := \Xi \rangle$ prescribed, the sum $A \in \mathbb{K}$ for which

$$A_\xi = a(\xi(|\Xi|)) + \ldots + a(\xi(1)) + a(\xi(0))$$

is denoted by

$$A := \sum a(\omega) \quad \langle \omega := \Xi \rangle$$

and the product $B \in \mathbb{K}$ for which

$$B = a(\xi(|\Xi|)) \ldots a(\xi(1)) a(\xi(0))$$

by

$$B := \prod a(\omega) \quad \langle \omega := \Xi \rangle$$

ii) Sums and products of matrices

Similar notations are used with regard to well defined matrix sums and products with the additional stipulation that in the latter case the factors corresponding to the lower values of the running suffix occur upon the right. Thus the matrix product B indicated by use of the above notation has the ordering indicated in the preceding relationship. Products may be reordered by suitable change of the running variable under the product sign. Thus with $k \in \bar{N}$ and $h \in [k]$, and $a(\omega) \in [K]$ $\langle \omega := [h,k] \rangle$ being suitably defined matrices, the product $B' \in [K]$ for which

$$B' = a(h) \ldots a(k-1)\,a(k)$$

may be defined by setting

$$B' := \prod a(k-\omega) \quad \langle \omega := [k-h] \rangle$$

iii) Empty sums and products

Empty sums of constants (for example, those as defined in clause (i) above with Ξ taken to be a void $\bar{N}$ sequence) are given the value zero and empty products of constants the value unity. Empty matrix sums of terms of equal dimension are given the value of the ~~matrix~~ zero matrix of dimensions equal to those of the terms. Empty products of square matrices are given the corresponding unit matrix value.

iv) Mapping products

Products of mappings over identical source domains are also defined by use of the above notations. Thus, with the constants

$a(\omega)$ of clause (i) replaced by mappings $a(\omega): \mathbb{S} \to \mathbb{K} \ \langle \omega := \Xi \rangle$, the function in the mapping $A: \mathbb{S} \to \mathbb{K}$ for which

$$A = a(\tfrac{\xi}{\mathbb{S}}(|\Xi|)) + \ldots + a(\tfrac{\xi}{\mathbb{S}}(1)) + a(\tfrac{\xi}{\mathbb{S}}(0)) \quad \langle \mathbb{S} \rangle$$

is denoted as in that clause. Products of simple mappings and sums and products of matrix mappings are dealt with in the same way.

7] Structural extension

i) Members of a target domain as constant mappings

$\mathbb{D}$ being a prescribed source domain, members of a target domain $\mathbb{T}$ are treated as constant functions in $\{\mathbb{D} \to \mathbb{T}\}$ and are represented by the same symbol. (Thus $0 \in \overline{\mathbb{N}}$ is treated as the function $0: \mathbb{N} \to \overline{\mathbb{N}}$ for which $0(i) = 0$ for each $i \in \mathbb{N}$.)

$h, k \in \overline{\mathbb{N}}$ being prescribed the product of a matrix $\Phi \in [\mathbb{K}|h]$ and a matrix mapping $A: \mathbb{K} \to [\mathbb{K}|h,k]$ is subsumed within operations upon matrix mappings by treating Φ as a constant function in $\{\mathbb{K} \to [\mathbb{K}|h]\}$. A similar convention is observed regarding post multiplicative factors and components of sums and differences

ii) Members of a target domain as single member sequences

Components of structures are, in ways which depend upon the structures in question, regarded as structures of simple form

Integers in $\bar{N}$ are taken to be single member sequences in $seq(\bar{N})$. Thus 2 is $f \in seq(\bar{N}|0)$ where $f(0) := 2$. (In this way, with $h, k \in \bar{N}$, $k \geq 2$ and $A \in [\mathbb{K}|h,k]$, A^2 is the column of A with column superscript 2. Alternatively, with $k \geq 1$, A^{-1} is the submatrix obtained by removing the column with superscript 1 from A (for suitably declared A, $\{A\}^2$ and $\{A\}^{-1}$ are the square and inverse of A). Integers in N and $\tilde{N}$ and members of $\mathbb{K}$ are regarded as being single member sequences in $seq(N)$, $seq(\tilde{N})$ and $seq(\mathbb{K})$ respectively.

iii) Scalars as diagonal matrices

In expressions involving the ordered product of a member z of $\mathbb{K}$ and a matrix $A \in [\mathbb{K}|h,k]$, $h, k \in \bar{N}$ being prescribed, z is regarded as being Z in $diag[\mathbb{K}|h]$ having diagonal elements $Z_\omega^\omega = z \langle \omega := [h] \rangle$. Where z occurs as a post factor it is regarded as being in $diag[\mathbb{K}|k]$.

(This convention is observed conjointly with that described in (i). Thus $z \in \mathbb{K}$ occurring as a premultiplying factor of a mapping $A: \mathbb{K} \to [\mathbb{K}|h,k]$ may be regarded as a constant diagonal matrix in $diag[\mathbb{K}|h]$ and then treated as a constant function in $\{\mathbb{K} \to [\mathbb{K}|h]\}$.

8] Extensions of source and target domain types

A mapping $e : \bar{N} \to \bar{N}$ may immediately be extended to the form $e : seq(\bar{N}) \to seq(\bar{N})$ simply by defining $e(\sigma)$ to be the sequence $e(\sigma(0)), e(\sigma(1)), \ldots, e(\sigma(i))$ corresponding to $\sigma \in seq(\bar{N}|i)$ where $i \in \bar{N}$. Various extensions of other mappings may be devised; some finding subsequent use are now described.

The first extensions considered have, apart from a slight modification of the target domain, the primitive form of that just described: the arguments in the extended source domain are sequences; the length of the function value sequence in the target domain is that of the argument sequence. In the second extensions, selection of a subsequence of the argument sequence is possible, the source domain being modified so as to include reference to integers which specify the position and length of the subsequence selected.

i) Sequential extension of both source and target domains

Let $N' := N$, $B \in dom(\mathbb{K}, \bar{N})$ and $T \in set(\mathbb{K}, \bar{N})$

α) The mapping $e : \bar{N} \to \{B \to T\}$ is extended to the form $e : seq(\bar{N}) \to seq(B \to T)$ by setting, with $\sigma \in seq(\bar{N}|i)$,

$$e(\sigma) := \{e\{\sigma(\omega)\} \langle \omega := [i] \rangle\} \ \langle B \rangle$$

β) In particular, the mapping $e : \bar{N} \to T$ is extended to the form $e : seq(\bar{N}) \to seq(T)$ by omitting reference to B in the above declaration

bα) The mapping $v: \bar{\mathbb{N}} \times \mathbb{N}' \to \{B \to T\}$ is extended to the form $v: \mathrm{seq}^2(\bar{\mathbb{N}}) \to \mathrm{seq}^2(B \to T)$ by setting, with $\chi \in \mathrm{seq}(\mathbb{N}|k), \sigma \in \mathrm{seq}(\bar{\mathbb{N}}|i)$:

$$\sigma(\chi, \sigma) := \{e\{\chi(\tau), \sigma(\omega)\} \langle \tau := [k], \omega := [i] \rangle\} \quad \langle B \rangle$$

β) The mapping $v: \bar{\mathbb{N}} \times \mathbb{N}' \to T$ is extended to the form $v: \mathrm{seq}^2(\bar{\mathbb{N}}) \to \mathrm{seq}^2(T)$ by omitting reference to $\mathbb{B}$ in the above declaration

ii) Extension of a source domain to sequence segment form

Let $\mathbb{N}' := \bar{\mathbb{N}}$, $\mathbb{B} \in \mathrm{dom}(\mathbb{K}, \bar{\mathbb{N}})$ and $\mathbb{T} \in \mathrm{set}(\mathbb{K}, \bar{\mathbb{N}})$. Define $\mathbb{D} \in \mathrm{dom}(\mathbb{K}, \bar{\mathbb{N}})$ by setting $\mathbb{D} := \mathrm{seq}(\mathbb{K} | \geq \mathbb{N}') \times \bar{\mathbb{N}} \times \mathbb{N}'$.

aα) The mapping $g: \mathbb{K} \to \{\mathbb{B} \to \mathbb{T}\}$ is extended to the form $g: \mathbb{D} \to \{\mathbb{B} \to \mathrm{seq}(\mathbb{T}|\mathbb{N}')\}$ by setting

$$g(\bar{\beta}) := g(\beta \| n, i) := \{g(\beta_{n+\omega}) \langle \omega := [i] \rangle\} \quad \langle \mathbb{B} \rangle$$

β) The mapping $g: \mathbb{K} \to \mathbb{T}$ is extended to the form $g: \mathbb{D} \to \mathrm{seq}(\mathbb{T}|\mathbb{N}')$ by omitting reference to $\mathbb{B}$ in the above declaration.

b) Let $\mathbb{N}'' := \bar{\mathbb{N}}$.

α) The mapping $w: \mathbb{K} \times \mathbb{N}'' \to \{\mathbb{B} \to \mathrm{seq}(\mathbb{T}|\bar{\mathbb{N}}'')\}$ is extended to the form $w: \mathbb{D} \times \mathbb{N}'' \to \{\mathbb{B} \to \mathrm{seq}^2(\mathbb{T}|\mathbb{N}', \mathbb{N}'')\}$ by setting

$$w(\bar{\beta}, k) := w(\beta \| n, [i, k]) := w(\omega | \beta_{n+\tau}) \langle \mathbb{B}; \tau := [k], \omega := [i] \rangle$$

β) The mapping $w: \mathbb{K} \times \mathbb{N}'' \to \mathrm{seq}(\mathbb{T}|\mathbb{N}'')$ is extended to the form $w: \mathbb{D} \times \mathbb{N}'' \to \mathrm{seq}^2(\mathbb{T}|\mathbb{N}', \mathbb{N}'')$ by omitting reference to $\mathbb{B}$ in the above declaration.

9] Type conversion

Row and column vectors and diagonal matrices over a prescribed field $\mathbb{K}$ take their respective places in a framework of operations of linear algebra. Successive elements of such vectors and matrices form sequences in $seq(\mathbb{K})$. It is convenient to have available type conversion operators which pour subsequences of $\mathbb{K}$-sequences into appropriate linear algebraic moulds.

i) Sequence to row, column and diagonal matrix conversion operators

~~α)~~ Let $\mathbb{N}' := \bar{\mathbb{N}}$ and $\mathbb{B} \in dom(\mathbb{K}, \bar{\mathbb{N}})$

aα) The mapping $row : seq(\mathbb{B} \to \mathbb{T} | \geq \bar{\mathbb{N}}' + \bar{\mathbb{N}}) \times \mathbb{N}' \times \bar{\mathbb{N}} \to row[\mathbb{B} \to \mathbb{T} | \bar{\mathbb{N}}$ is defined by setting

$$row[(e \| m)]^{[i]} := row[e_{m+i}]^{D:=[i]} \qquad \langle \mathbb{B} \rangle$$

β) The mapping $row := seq(\mathbb{K} | \geq \mathbb{N}' + \bar{\mathbb{N}}) \times \mathbb{N}' \times \bar{\mathbb{N}} \to row[\mathbb{T} | \bar{\mathbb{N}}]$ is defined as a special case of the above, omitting reference to $\mathbb{B}$ in the above declaration.

γ) $row[(e \| 0)]^{[i]}$ is written as $row[e]^{[i]}$ and in this way defines a mapping $row : seq(\mathbb{B} \to \mathbb{K} | \geq \bar{\mathbb{N}}) \times \bar{\mathbb{N}} \to row[\mathbb{B} \to \mathbb{T} | \bar{\mathbb{N}}]$ and a further mapping as adumbrated in (β)

b) Arithmetic operations take place in sequence expressions occurring in place of $(e \| m)$ in the above.

α) Thus $row[(e \| m) - (g \| n)]^{[i]}$, subject to suitable preliminary declaration of e, g, m and n, represents $\mathrm{row}[v]^{[i]}$, where

$v_\nu := e_{m+\nu} - g_{n+\nu} \langle \nu := [i] \rangle$

β) In a similar way, $\mathrm{row}[(e\|m) \dot{\times} (g\|n)]^{[i]}$ represents

$\mathrm{row}[w]^{[i]}$, where $w_\nu := e_{m+\nu} g_{n+\nu} \langle \nu := [i] \rangle$.

γ) Contraction in the use of these conventions also takes place: $\mathrm{row}[e \dot{\times} (g\|n)]^{[i]}$ and $\mathrm{row}[e \dot{\times} g]^{[i]}$ have meanings as described in (aγ).

c) The mappings $\mathrm{col} : \mathrm{seq}(B \to T | \geq N' + \bar{N}) \times N' \times \bar{N} \to \mathrm{col}[B \to T | \bar{N}]$ and $\mathrm{diag} : \mathrm{seq}(B \to T | \geq N' + \bar{N}) \times N' \times \bar{N} \to \mathrm{diag}[B \to T | \bar{N}]$ are defined by setting

$$\mathrm{col}[(e\|m)]_{[i]} := \mathrm{col}[e_{m+\tau}]_{\tau := [i]}$$

and

$$\mathrm{diag}[(e\|m), [i]] := \mathrm{diag}[e_{m+\omega}] \langle \omega := [i] \rangle$$

respectively. The contractions and conventions described in (aγ) and (b) above hold.

ii) Row and column interchange and row and column to diagonal matrix conversion operators

The sequences referred to in the preceding clause may also be extracted from row, column and diagonal mapping. Thus $\mathrm{diag}[(e\|m), [i]]$ in subclause (ic) is also defined, as given, when $e \in \mathrm{row}[B \to K | \geq m i]$.

(It is naturally possible to define conversion operators which translate row and column vectors and diagonal matrices into sequences, and to define the type conversion considered above in terms of joint action of two conversion operators.)

iii) Function sequence to matrix conversion operators

For a fixed value of $\alpha \in \mathbb{K}$, the function v occurring in the mapping $v : \mathbb{K} \to \mathrm{seq}(\mathbb{B} \to T)$ assumes a value $v(\alpha) \in \mathrm{seq}(\mathbb{B} \to T)$ whose components have the form $v(\omega|\alpha) : \mathbb{B} \to T \langle \omega := [v(\alpha)] \rangle$. If $|v(\alpha)|$ is sufficiently large these components may be set into successive positions of a prescribed vector in $\mathrm{row}[\mathbb{B} \to T]$. By letting α assume the values of the successive constituents of a subsequence $\alpha[m, m+i]$ of $\alpha \in \mathrm{seq}$ the sequence generated by $\alpha \in \mathrm{seqf}(\mathbb{K})$, a matrix mapping in $[\mathbb{B} \to T]$ is defined. A type conversion operator which sets fixed initial subsequen of the sequence mappings $v(\alpha_{m+z}) \in \mathrm{seq}(\mathbb{B} \to T) \langle z := [i] \rangle$ into successive row positions of a matrix mapping in $[\mathbb{B} \to T]$ is described below.

Let $\mathbb{N}'' := \mathbb{N}' := \bar{\mathbb{N}}$ and $\mathbb{B} \in \mathrm{dom}(\mathbb{K}\ddagger, \bar{\mathbb{N}})$.

a) The mapping $V : \{\mathbb{K} \to \mathrm{seq}(\mathbb{B} \to \mathbb{K} | \geq \mathbb{N}'') \times \mathrm{seqf}(\mathbb{K} | \geq \mathbb{N}' + \mathbb{N}'') \times \bar{\mathbb{N}} \times \mathbb{N}' \times \mathbb{N}'' \to \{\mathbb{B} \to [\mathbb{K} | \mathbb{N}', \mathbb{N}'']\}$ is defined by setting

$$V[v; \alpha \| m, [k, i]] := [v(\nu | \alpha_{m+z})]_{z := [k]}^{\nu := [i]} \quad \langle \mathbb{B} \rangle$$

b) The mapping $V: \{\mathbb{K} \to seq(\mathbb{K}|\geq \mathbb{N}'')\} \times seqf(\mathbb{K}|\geq \mathbb{N}'+\mathbb{N}'')$ $\times \bar{\mathbb{N}} \times \mathbb{N}' \times \mathbb{N}'' \to [\mathbb{K}|\mathbb{N}', \mathbb{N}'']\}$ is defined by omitting reference to $\mathbb{B}$ in the above declaration.

cα) The arithmetic operations of addition and subtraction take place in sequence function expressions occurring in place of v in the above. Thus $V[w - v; \alpha\|m, [k,i]]$, subject to suitable preliminary declaration of $w, \ldots, i$, represents $V[x; \alpha\|m, [k,i]]$ where, for each $z \in \mathbb{K}$, the mapping $x(\nu|z): \mathbb{B} \to \mathbb{K}$ is given by $x(\nu|z) := w(\nu|z) - v(\nu|z) \langle \nu := [i] \rangle$. If w and v are both in $\{\mathbb{K} \to seq(\mathbb{K}|\geq i$ x is in $\{\mathbb{K} \to seq(\mathbb{K}|i)\}$ and $V[x; \alpha\|m, [k,i]]$ in $[\mathbb{K}|k;i$

β) Pre and post multiplication of v by sequence functions are treated in the following way. The mapping V: $seq(\mathbb{B} \to \mathbb{K}|\geq \mathbb{N}''' + \mathbb{N}') \times \mathbb{N}''' \times \{\mathbb{K} \to seq\{\mathbb{K} \to seq(\mathbb{B} \to \mathbb{K}|\geq \mathbb{N}'')$ $\times seqf(\mathbb{K}|\geq \mathbb{N}'+\mathbb{N}'') \times \bar{\mathbb{N}} \times \mathbb{N}' \times \mathbb{N}'' \to \{\mathbb{B} \to [\mathbb{K}|\mathbb{N}', \mathbb{N}'']\}$, ~~is~~ ~~defin~~ where $\mathbb{N}''' := \bar{\mathbb{N}}$, is defined by setting

$$V[(e\|r)\dot{\times} v; \alpha\|m, [k,i]] := [e_{r+z} v(\nu|\alpha_{m+z})]_{z:=[k]}^{\nu:=[i]} \langle \mathbb{B} \rangle$$

The mapping $V: \{\mathbb{K} \to seq(\mathbb{B} \to \mathbb{K}|\geq \mathbb{N}'')\} \times seq(\mathbb{B} \to \mathbb{K}|\geq$ $\mathbb{N}''' + \mathbb{N}'') \times \mathbb{N}''' \times seqf(\mathbb{K}|\geq \mathbb{N}'+\mathbb{N}'') \times \bar{\mathbb{N}} \times \mathbb{N}' \times \mathbb{N}'' \to \{\mathbb{B} \to$ $[\mathbb{K}|\mathbb{N}', \mathbb{N}'']\}$ is defined by setting

$$V[v\dot{\times}(g\|n); \alpha\|m, [k,i]] := [v(\nu|\alpha_{m+z}) g_{n+\nu}]_{z:=[k]}^{\nu:=[i]}$$

γ) The conventions concerning arithmetic operations described in (α,β) above are used in conjunction. Thus $V[(e\|r)\times v - w;$ $\alpha\|m, [k,i]]$ is defined in terms of component mappings $e_{r+z}\, v(\nu|\alpha_{m+z}) - w(\nu|\alpha_{m+z}) : B \to K$ $<z := [k], \nu := [i]>$

δ) As in clause (b) above, $V[(e\|\phi)\times v; \alpha\|m, [k,i]]$ is $[K|k,i]$ when e is in $seq(K|\geq N''' + N')$ and v in $\{K \to seq(K|\geq N'')$. Similar remarks concern the notation relating to postmultiplication of v.

ε) $V[(e\|0)\times v; \alpha\|m, [k,i]]$ is written as $V[e\times v; \alpha\|m, [k,i]]$ this contraction defining a further mapping. The above notation concerning postmultiplication is contracted in a similar way.

10] Delimiters and parentheses

Delimiters other than the conventional comma, semicolon and colon are used. Thus, with $\mathbb{D}$ a prescribed domain, the notation $L(\bar{\alpha}/\bar{\beta})$ is used to draw attention to the fact that, with $\mathbb{T}$ a suitable target domain, $L: \mathbb{D}\times\mathbb{D} \to \mathbb{T}$ transforms according to the law

$$L(\bar{\alpha}/\bar{\beta})\,L(\bar{\beta}/\bar{\gamma}) = L(\bar{\alpha}/\bar{\gamma})$$

for all $\bar{\alpha},\bar{\beta},\bar{\gamma}$ in $\mathbb{D}$.

The choice of parentheses is also extended. Thus the notations $U/\bar{\alpha})$ and $V(\bar{\beta}/$ are adopted to emphasize the transformation properties

$$L(\bar{\alpha}/\bar{\beta})U/\bar{\alpha}) = U/\bar{\beta})$$

$$L(\bar{\alpha}/\bar{\beta})V(\bar{\beta}/ = V(\bar{\alpha}/$$

Further liberties are taken in the use of delimiters and parentheses. Thus, for example, symbols of the form $Q[f;\alpha\|m,j]^{[i]}$ and $A[f;\alpha\|m,j,(k,i]]$ are used.

B. Special matrix mappings

1] Annihilatory and permanent matrix mappings

In subsequent work two classes of matrix mappings that transform other matrix mappings are encountered. The transforming matrix mappings are those for which all row sums except one are zero (the exceptional row sum taking the value unity) and those for which all row sums are unity. Some properties of such matrix mappings are stated.

Let $i \in \bar{N}$ and $K' \leq K$.

a) A mapping $A: K' \to [K|i]$ for which

$$AI_{[i]} = \hat{I}[i]^{\circ} \qquad \langle K' \rangle$$

is said to be annihilatory over K'. $\mathrm{ann}(K'|i)$ is the complete system of such mappings and $\mathrm{ann}'(K'|i)$ the complete subsystem of all such mappings that are nonsingular over K'.

bγ) A mapping $B: \mathbb{K}' \to [\mathbb{K}|i]$ for which

$$BI_{[i]} = I_{[i]} \qquad \langle \mathbb{K}' \rangle$$

is said to be permanent over $\mathbb{K}'$; perm$(\mathbb{K}'|i)$ and perm$'(\mathbb{K}'|i$ are the complete system of such mappings and the complete subsystem of such mappings that are nonsingular over $\mathbb{K}'$ respectively.

i) Properties of annihilatory and permanent matrix mappings

Let $i \in \bar{N}$ and $\mathbb{K}' \subseteq \mathbb{K}$

aα) The mappings of ann$(\mathbb{K}'|i)$ are preserved during the formation of an arithmetic mean: with $A, A' \in$ ann$(\mathbb{K}'|i)$ and $a, a': \mathbb{K}' \to \mathbb{K}$ such that $a + a' = 1 \ \langle \mathbb{K}' \rangle$, $aA + a'A' \in$ ann$(\mathbb{K}'|i$

The same result holds for perm$(\mathbb{K}'|i)$.

β) Let $X: \mathbb{K}' \to [\mathbb{K}|i]$ have the form $[I[i]_0 / X']$ where $X': \mathbb{K}' \to [\mathbb{K}|i-1, i]$. If $A \in$ ann$(\mathbb{K}'|i)$, $XA \in$ ann$(\mathbb{K}'|i)$ also

ann$(\mathbb{K}'|i)$ is closed with respect to the multiplicative operator $\bar{x}$ defined by setting $A \bar{x} A' := A \mathcal{L} I[i] A'$ over $\mathbb{K}'$: if $A, A' \in$ ann$(\mathbb{K}'|i)$ then $A \bar{x} A' \in$ ann$(\mathbb{K}'|i)$ also. The same result holds for ann$'(\mathbb{K}'|i)$.

γ) perm$(\mathbb{K}'|i)$ and perm$'(\mathbb{K}'|i)$ are closed with respect to multiplication in $[\mathbb{K}|i]$.

bα) $A \in$ ann$'(\mathbb{K}'|i)$ if and only if $(\{A\}^{-1})^0 = I[i]$

β) $B \in$ perm$'(\mathbb{K}'|i)$ if and only if $\{B\}^{-1} \in$ perm$'(\mathbb{K}|i)$.

cα) $A \in \operatorname{ann}(\mathbb{K}'|i)$ if and only if $B \in \operatorname{perm}(\mathbb{K}'|i)$, where $B := \mathcal{L}I[i]A \quad \langle \mathbb{K}' \rangle$. This result with $\operatorname{ann}(\mathbb{K}'|i)$, $\operatorname{perm}(\mathbb{K}'|i)$ replaced by $\operatorname{ann}'(\mathbb{K}'|i)$, $\operatorname{perm}'(\mathbb{K}'|i)$ also holds.

β) Of the three conditions

(1) $A \in \operatorname{ann}(\mathbb{K}'|i)$ (2) $B \in \operatorname{perm}(\mathbb{K}'|i)$ (3) $AB \in \operatorname{ann}(\mathbb{K}'|i)$

(1,2) imply (3) and (2,3) imply (1). If A is nonsingular over $\mathbb{K}'$, conditions (1,3) imply (2).

γ) The above observations relate in equal measure to the three conditions (1,2) and

$$(3') \quad \mathcal{L}I[i]AB \in \operatorname{perm}(\mathbb{K}'|i)$$

dα) The mappings $A, C : \mathbb{K}' \to [\mathbb{K}|i]$ satisfy the relationship

$$(A \operatorname{diag}[f]B)^{0} = Af \qquad \langle \mathbb{K}' \rangle$$

for all $f : \mathbb{K}' \to \operatorname{col}[\mathbb{K}|i]$ if and only if $B^{0} = I_{[i]} \ \langle \mathbb{K}' \rangle$.

β) The mapping $A : \mathbb{K}' \to [\mathbb{K}|i]$ satisfies the relationship

$$(A \operatorname{diag}[f]\{A\}^{-1})^{0} = Af \qquad \langle \mathbb{K}' \rangle$$

for all $f : \mathbb{K}' \to [\mathbb{K}|i]$ if and only if $A \in \operatorname{ann}'(\mathbb{K}|i)$

γ) The mapping $B : \mathbb{K}' \to [\mathbb{K}|i]$ satisfies the relationship

$$(B \operatorname{diag}[f]\{B\}^{-1}\mathcal{L}I[i])^{0} = Bf \quad \langle \mathbb{K}' \rangle$$

for all $f : \mathbb{K}' \to [\mathbb{K}|i]$ if and only if $B \in \operatorname{perm}'(\mathbb{K}|i)$.

(It is possible to define a post-annihilatory mapping $A : \mathbb{K}' \to [\mathbb{K}|i]$ in terms of the relationship

$$AI_{[i]} = I[i]^{i}$$

and develop its theory as above. The operator $\overline{x}$ in $(a\beta)$ must be defined by $A\overline{x}A' := A\mathcal{U}I[i]A'$. The relationship stated in $(b\alpha)$ is now $(\{A\}^{-1})^{i} = I_{[i]}$. That in $(c\alpha)$ is $B := \mathcal{U}I[i]A$ and in $(c\gamma)$ is $\mathcal{U}I[i]AB \in perm(\mathbb{K}'|i)$. The result of $(d\alpha)$ is presented in terms of the relationships $(A\,diag[f]B)^{i} = Af\langle\mathbb{K}'\rangle$ and $B^{i} = I_{[i]}$, and the relationships of $(d\beta,\gamma)$ become $(A\,diag[f]\{A\}^{-1})^{i} = Af\langle\mathbb{K}'\rangle$ and $(B\,diag[f]\{B\}^{-1}\mathcal{U}I[i])^{i} =$

$Bf\langle\mathbb{K}'\rangle$.

The above treatment and its original counterpart may be subsumed under a general theory of annihilatory mappings based upon the relationship $AI_{[i]} = I[i]^{\nu}$ with $\nu \in [i]$.)

2] Continued products of triangular matrices

With $i, k \in \overline{\mathbb{N}}$, the elements of the product $A(k) \in [\mathbb{K}|i]$ of the $k+1$ square matrices $a(\omega) \in [\mathbb{K}|i]\ \langle \omega := [k]\rangle$ in the order $a(k)a(k-1)\ldots a(0)$ may be expressed in terms of the elements of the factor matrices by means of the formula

$$A(k)^{\nu}_{\tau} = \{\Sigma' a(k)^{\omega(k-1)}_{\tau}\{\Sigma' a(k-1)^{\omega(k-2)}_{\omega(k-1)}\{\Sigma' a(k-2)^{\omega(k-3)}_{\omega(k-2)}$$

$$\ldots\{\Sigma' a(1)^{\omega(0)}_{\omega(1)}a(0)^{\nu}_{\omega(0)}\ \langle\omega := [i]\rangle\}\ldots$$

$$\langle\omega(k-3) := [i]\rangle\}\langle\omega(k-2) := [i]\rangle\}\langle\omega(k-1) := [i]\rangle\}$$

holding for $\tau, \nu := [i]$.

Simple laws of formation of continued products of triangular matrices may also be formulated. Lower triangular matrices are considered in detail.

With $i \in \bar{N}$, the elements of the product $A(1) \in \mathcal{L}[K/i]$ of the two matrices $a(1), a(0) \in \mathcal{L}[K/i]$ in the given order may be expressed by means of the formulae

$$A(1)^{\nu}_{\tau} = \sum_{1} a(1)^{\omega}_{\tau} a(0)^{\nu}_{\omega} \quad \langle \omega := [\nu, \tau] \rangle$$

holding for $\nu := [i]$, $\tau := [\nu, i]$ and

$$A(1)^{\nu}_{\nu+\chi} = \sum_{1} a(1)^{\nu+\omega}_{\nu+\chi} a(0)^{\nu}_{\nu+\omega} \quad \langle \omega := [\chi] \rangle$$

holding for $\nu := [i]$, $\chi := [i-\nu]$

In both formulae the superscript of $a(1)$ and the suffix of $a(0)$ have the same value. In the first formula summation runs from the value of the superscript of $A(1)$ to that of the suffix of $a(1)$. In the second, summation runs from zero to the difference between the suffix of $a(1)$ and the superscript of $A(1)$. These laws are preserved in the formation of continued products of lower triangular matrices as is shown in the following theorem which is presented in terms of matrix mappings.

( ) Let $i \in \bar{N}$, $n \in N$, $K' \subseteq K$ and $a(k): K' \to \mathcal{L}[K/i]$ $\langle k := [n] \rangle$. Define $A(k): K' \to \mathcal{L}[K/i]$ $\langle k := (n] \rangle$ by setting

$$A(k) := \prod a(\kappa) \quad \langle \kappa := [k] \rangle$$

With $k := (n]$, $\nu := [i]$ in both cases, for $\tau := [\nu, i]$

$$A(k)^{\nu}_{\tau} = \Big\{ \sum a(k)^{\omega(k-1)}_{\tau} \Big\{ \sum a(k-1)^{\omega(k-2)}_{\omega(k-1)} \Big\{ \sum a(k-2)^{\omega(k-3)}_{\omega(k-2)}$$

$$\ldots \Big\{ \sum a(1)^{\omega(0)}_{\omega(1)} a(0)^{\nu}_{\omega(0)} \langle \omega(0) := [\nu, \omega(1)] \rangle \Big\} \ldots$$

$$\langle \omega(k-3) := [\nu, \omega(k-2)] \rangle \Big\} \langle \omega(k-2) := [\nu, \omega(k-1)] \rangle \Big\} \langle \omega(k-1) := [\nu, \tau] \rangle \Big\}$$

$$\langle k' \rangle$$

and for $\chi := [i - \nu]$

$$A(k)^{\nu}_{\nu+\chi} = \Big\{ \sum a(k)^{\nu+\omega(k-1)}_{\nu+\chi} \Big\{ \sum a(k-1)^{\nu+\omega(k-2)}_{\nu+\omega(k-1)} \Big\{ \sum a(k-2)^{\nu+\omega(k-3)}_{\nu+\omega(k-2)}$$

$$\ldots \Big\{ \sum a(1)^{\nu+\omega(0)}_{\nu+\omega(1)} a(0)^{\nu}_{\nu+\omega(0)} \langle \omega(0) := [\omega(1)] \rangle$$

$$\langle \omega(k-3) := [\omega(k-2)] \rangle \Big\} \langle \omega(k-2) := [\omega(k-1)] \rangle \Big\} \langle \omega(k-1) := [\chi] \rangle \Big\}$$

$$\langle k' \rangle$$

(The second result is obtained by replacing $\omega(\kappa)$ by $\nu + \omega(\kappa)$ in the first.)

3] Submatrix mappings of products of triangular matrix mappings.

$h, i, j, k, n \in \bar{N}$ being suitably prescribed, the formation of a submatrix $X^{[h,h+j]}_{[k,k+i]}$ of the matrix product $X = YZ$ with $Y, Z \in [K|n]$ requires, in general, the multiplication of $Y_{[k,k+i]}$ in $[K|i,n]$ and $Z^{[h,h+j]}$ in $[K|n,j]$. If, however, Y and Z are of triangular form, multiplication of two submatrices of reduced dimension suffices. The same considerations hold true for matrix mappings. The minimal dimension requirements are specified in the following theorem

( ) Let $i, j, k \in \bar{\mathbb{N}}$, $h \in [k+i]$, $n \geq \max(h+j, k+i)$ and $\mathbb{K}' \subseteq \mathbb{K}$. Let $x(\tau, \nu) : \mathbb{K}' \to \mathbb{K}$ $\langle \tau := [n], \nu := [\tau] \rangle$ and define the matrix mapping $X : \mathbb{K}' \to \mathcal{L}[\mathbb{K}|n]$ by setting

$$X := \mathcal{L}[x(\tau, \nu)] \quad \langle \tau := [n], \nu := [\tau] \rangle$$

Define the matrix mappings $Y, Z : \mathbb{K}' \to \mathcal{L}[\mathbb{K}|n]$ similarly.

Define the submatrix mapping $X[h, k, [i, j]] : \mathbb{K}' \to [\mathbb{K}|i, j]$ of X by setting

$$X[k, h, [i, j]] := X^{[h, h+j]}_{[k, k+i]}$$
$$:= [x(k+\tau, h+\nu)]^{\langle \nu := [j] \rangle}_{\langle \tau := [i] \rangle}$$

and define the submatrix mappings $Y[k, h, [i, k+i-h]] : \mathbb{K}' \to [\mathbb{K}|i, k+i-h]$ and $Z[h, h, [k+i-h, j]] : \mathbb{K}' \to [\mathbb{K}|k+i-h, j]$ similarly.

If $X = YZ \langle \mathbb{K}' \rangle$ then

$$X[k, h, [i, j]] = Y[k, h, [i, k+i-h]]\, Z[h, h, [k+i-h, j]] \quad \langle \mathbb{K}' \rangle$$

C. Transformation laws

$\mathbb{D}$ being a suitable domain and T and A′ being functions between which multiplication is possible, both having the source domain $\mathbb{D}\times\mathbb{D}$, T transforms A′ according to a law of the form

$$(*)\qquad T(\bar{\alpha},\bar{\beta})A'(\bar{\alpha},\bar{\beta})=B'(\bar{\alpha},\bar{\beta})$$

holding for $\bar{\alpha},\bar{\beta}:=\mathbb{D}$. Indeed $B'(\bar{\alpha},\bar{\beta})$ may simply be defined by use of the above formula to ensure that the transformation law holds. In many cases it is found, however, that T is such that the above law is satisfied for an A′ for which $A'(\bar{\alpha},\bar{\beta})$ is independent of $\bar{\alpha}$ and a B′ for which $B'(\bar{\alpha},\bar{\beta})$ is independent of $\bar{\beta}$: the transformation law then takes the form

$$T(\bar{\alpha},\bar{\beta})A(\bar{\beta})=B(\bar{\alpha})$$

holding for $\bar{\alpha},\bar{\beta}:=\mathbb{D}$, where A and B have the common source domain $\mathbb{D}$. There are even cases in which A and B are the same function: a transformation law of the form

$$T(\bar{\alpha},\bar{\beta})A(\bar{\beta})=A(\bar{\alpha})$$

holding for $\bar{\alpha},\bar{\beta}:=\mathbb{D}$ exists.

The above remarks may also be formulated in terms of a post-multiplying factor T. The special transformation laws then become

$$C(\bar{\alpha})T(\bar{\alpha},\bar{\beta}) = D(\bar{\beta})$$

and

$$C(\bar{\alpha})T(\bar{\alpha},\bar{\beta}) = C(\bar{\alpha})$$

both holding for $\bar{\alpha},\bar{\beta} := D$.

It can occur that T transforms itself according to a law of the form

$$T(\bar{\alpha},\bar{\beta})T(\bar{\beta},\bar{\gamma}) = T(\bar{\alpha},\bar{\gamma})$$

$\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ being restricted to certain domains. Subject to suitable conditions, a simple transformation law involving T and A alone (or such a law involving T and C alone) induces the above cancellation property of T.

1] Transformation systems

Let $i, j, k \in \bar{\mathbb{N}}$, $D' := D'' := D \in dom(\mathbb{K}\$, \bar{\mathbb{N}})$, $B' : D \times D \subseteq \mathbb{K}$ $\hat{B}, \tilde{B} : D \subseteq \mathbb{K}$ and $B : D' \times D'' \subseteq \mathbb{K}$ be such that

$$B(\alpha,\beta) \subseteq \hat{B}(\alpha) \cap B'(\alpha;\beta) \cap \tilde{B}(\beta) \qquad \langle \alpha,\beta := D \rangle$$

Let $T : D' \times D'' \to \{B'(D',D'') \to [\mathbb{K}|i,j]\}$

aa) With $A : D \to \{\hat{B}(D) \to [\mathbb{K}|j,k]\}$ and $B : D \to \{\tilde{B}(D) \to [\mathbb{K}|i,k]\}$, the notation $\{A,B\} \in push\{T,D,B|i,j;k\}$ indicates that

$$T(\alpha,\beta)A(\beta) = B(\alpha)$$

for $\alpha, \beta := \mathbb{D}$. A is then said to be pushed into B by T throughout $\mathbb{D}$ over $\mathbb{B}$. In the special cases in which $i=j$ and $i=j=k$ the contracted symbols push$\{T, D, B \mid i; k\}$ and push$\{T, D, B \mid i\}$ are used in the above notation.

β) If, in the preceding, $\hat{\mathbb{B}} = \tilde{\mathbb{B}}$, $i=j$ and

$$A = B \langle \mathbb{D} \rangle$$

A is said to be pushed around by T throughout D over B and the notation $A \in$ push$\{T, D, B \mid i; k\}$ is used, this notation being abbreviated by use of the symbol push$\{T, D, B \mid i\}$ in the case in which $i=k$.

bα) With $C: \mathbb{D} \to \{\hat{\mathbb{B}}(\mathbb{D}) \to [K \mid k, i]\}$ and $D: \mathbb{D} \to \{\tilde{\mathbb{B}}(\mathbb{D}) \to [K \mid k, j]\}$, the notation $\{C, D\} \in$ pull$\{T, D, B \mid i, j; k\}$ indicates that

$$C(\alpha) T(\alpha, \beta) = D(\beta)$$

for $\alpha, \beta := \mathbb{D}$. C is then said to be pulled into D by T throughout $\mathbb{D}$ over $\mathbb{B}$. The contracted symbols pull$\{T, D, B \mid i; k\}$ and pull$\{T, D, B \mid i\}$ are used in circumstances similar to those described in (aα) above.

β) If $\hat{\mathbb{B}} = \tilde{\mathbb{B}} \langle \mathbb{D} \rangle$, $i=j$ and

$$C = D \langle \mathbb{D} \rangle$$

C is said to be pulled about by T throughout $\mathbb{D}$ over $\mathbb{B}$ and the notation $C \in$ pull$\{T, D, B \mid i; k\}$, together with a contracted form when $i=k$, are used.

i) The extension of transformation systems

Multiplying relationship (*) throughout by a suitable post-factor $F'(\alpha,\beta)$, a similar transformation law involving T, a' and b', where

$$a'(\alpha,\beta)=A'(\alpha,\beta)F'(\alpha,\beta),\ b'(\alpha,\beta)-B'(\alpha,\beta)F'(\alpha,\beta)$$

is obtained. Imposing appropriate conditions upon F', an ordered pair in push$\{T,\mathbb{D},\mathbb{B}|i,j:k\}$ yields an ordered pair in a related system. Transformation involving T as a post-multiplying factor may be extended in a similar way.

( ) Let $h,i,j,k\in\bar{\mathbb{N}}$, $\mathbb{D}':=\mathbb{D}'':=\mathbb{D}\in\mathrm{dom}(\mathbb{K},\bar{\mathbb{N}})$, $\mathbb{B}':\mathbb{D}\times\mathbb{D}\leq\mathbb{K}$
$\hat{\mathbb{B}},\tilde{\mathbb{B}}:\mathbb{D}\leq\mathbb{K}$, $\mathbb{B}''\leq\mathbb{K}$ and $\mathbb{B}:\mathbb{D}'\times\mathbb{D}''\leq\mathbb{K}$ be such that

$$\hat{\mathbb{B}}(\alpha),\tilde{\mathbb{B}}(\alpha)\leq\mathbb{B}''\ \langle\alpha:-\mathbb{D}\rangle$$

and

$$\mathbb{B}(\alpha,\beta)\leq\hat{\mathbb{B}}(\alpha)\cap\mathbb{B}'(\alpha,\beta)\cap\tilde{\mathbb{B}}(\beta)\quad\langle\alpha,\beta:=\mathbb{D}\rangle$$

Let $T:\mathbb{D}'\times\mathbb{D}''\to\{\mathbb{B}'(\mathbb{D}',\mathbb{D}'')\to[\mathbb{K}|i,j]\}$

$\alpha\alpha$) Let $A:\mathbb{D}\to\{\hat{\mathbb{B}}(\mathbb{D})\to[\mathbb{K}|j,k]\}$, $B:\mathbb{D}\to\{\tilde{\mathbb{B}}(\mathbb{D})\to[\mathbb{K}|i,k]$ and $F:\mathbb{B}''\to[\mathbb{K}|k,h]$. Define $a:\mathbb{D}\to\{\hat{\mathbb{B}}(\mathbb{D})\to[\mathbb{K}|j,h]\}$ and $b:\mathbb{D}\to\{\tilde{\mathbb{B}}(\mathbb{D})\to[\mathbb{K}|i,h]\}$ by setting, for each $\alpha\in\mathbb{D}$,

$$a(\alpha):=A(\alpha)F,\quad b(\alpha):=B(\alpha)F$$

If $\{A,B\}\in$ push$\{T,\mathbb{D},\mathbb{B}|i,j:k\}$ then $\{a,b\}\in$ push$\{T,\mathbb{D},\mathbb{B}|i,j:h\}$

β) push$\{T,\mathbb{D},\mathbb{B}|i,j:k\}F\leq$ push$\{T,\mathbb{D},\mathbb{B}|i,j:h\}$.

$b\alpha$) Let $C : \mathbb{D} \to \{\hat{\mathbb{B}}(\mathbb{D}) \to [\mathbb{K}|k,i]\}$, $D : \mathbb{D} \to \{\tilde{\mathbb{B}}(\mathbb{D}) \to [\mathbb{K}|k,j]\}$; and $G : \mathbb{B}'' \to [\mathbb{K}|h,k]$. Define $c : \mathbb{D} \to \{\hat{\mathbb{B}}(\mathbb{D}) \to [\mathbb{K}|h,i]\}$ and $d : \mathbb{D} \to \{\tilde{\mathbb{B}}(\mathbb{D}) \to [\mathbb{K}|h,j]\}$ by setting, for each $\alpha \in \mathbb{D}$,

$$c(\alpha) := G\,C(\alpha)\,,\quad d(\alpha) := G\,D(\alpha).$$

If $\{C, D\} \in \text{pull}\{T, \mathbb{D}, \mathbb{B}|i,j;k\}$ then $\{c,d\} \in \text{pull}\{T, \mathbb{D}, \mathbb{B}|i,j;h\}$.

β) $G\,\text{pull}\{T, \mathbb{D}, \mathbb{B}|i,j;k\} \subseteq \text{pull}\{T, \mathbb{D}, \mathbb{B}|i,j;h\}$.

The result of subclause ($a\alpha$) of the above theorem naturally holds when $i=j$, $\hat{\mathbb{B}} = \tilde{\mathbb{B}}\langle\mathbb{D}\rangle$ and A and B are the same function. The result also concerns functions that are pushed around. Clause ($b\alpha$) also concerns functions that are pulled about.

It may occur in subclause ($a\alpha$) that $A(\alpha)$, $B(\beta)$ are constant functions over their source domains $\hat{\mathbb{B}}(\alpha)$, $\tilde{\mathbb{B}}(\beta)$ respectively — they are simply members of $[\mathbb{K}|j,k]$ and $[\mathbb{K}|i,k]$. F is in general a function with representa $F(z)$ defined for all $z \in \mathbb{B}''$. $A(\alpha)$, $B(\beta)$ then become, under transformation, no longer constants but functions $a(\alpha,z)$, $b(\beta,z)$ defined for all $z \in \mathbb{B}(\alpha,\beta)$. In this way a law involving the transformation of functional forms (e.g. polynomial forms) is derived from a law involving constants. The same considerations hold regarding clause (b).

2] Multiplicative domains

The behaviour of a function $T(\alpha,\beta)$ featuring in a mapping $T(\alpha,\beta): \mathbb{B}(\alpha,\beta) \to \mathbb{T}$, where α, β range over a domain $\mathbb{D}$, is to be considered. In particular, multiplicative properties of the form $T(\alpha,\beta)T(\beta,\gamma) = T(\alpha,\gamma)$ are to be established. The product function $T(\alpha,\beta)T(\beta,\gamma)$ is defined over the source domain $B(\alpha,\beta) \cap B(\beta,\gamma)$. Before investigating such properties it is convenient to introduce an assumption relating this domain intersection to the domain $\mathbb{B}(\alpha,\gamma)$.

Let $\mathbb{D} \in dom(\mathbb{K}, \bar{\mathbb{N}})$. A mapping $\mathbb{B}: \mathbb{D} \times \mathbb{D} \subseteq \mathbb{K}$ for which

$$\{B(\alpha,\beta) \cap B(\beta,\gamma)\} \subseteq B(\alpha,\gamma)$$

for $\alpha, \beta, \gamma := \mathbb{D}$ is said to be a multiplicative domain

$\mathcal{M}(\mathbb{D})$ is the complete system of such domains.

i) The existence of multiplicative domains

( ) Let $\mathbb{D} \in dom(\mathbb{K}, \bar{\mathbb{N}})$

a) Let $\mathbb{B}: \mathbb{D} \times \mathbb{D} \subseteq \mathbb{K}$ be seperable in the sense that $\mathbb{B}', \mathbb{B}'': \mathbb{D} \subseteq \mathbb{K}$ for which

$$B(\alpha,\beta) = \mathbb{B}'(\alpha) \cap \mathbb{B}''(\beta) \qquad \langle \alpha, \beta := \mathbb{D} \rangle$$

exist.

$B \in \mathcal{M}(\mathbb{D})$

b) Let $\mathbb{B}: \mathbb{D} \times \mathbb{D} \subseteq \mathbb{K}$ be such that $\mathbb{B}(\alpha,\beta)$ is independent of β for $\alpha :- \mathbb{D}$, so that $\mathbb{B}' : \mathbb{D} \leq \mathbb{K}$ for which $\mathbb{B}(\alpha,\beta) = \mathbb{B}'(\alpha)\langle \alpha, \beta :$

$\mathbb{B} \in \mathcal{M}(\mathbb{D})$.

c) A result analogous to the preceding holds for the case in which $\mathbb{B}(\alpha,\beta)$ is independent of α for $\beta :- \mathbb{D}$.

d) Such a result also holds for $\mathbb{B}$ independent of both α and β so that $\mathbb{K}' \leq \mathbb{K}$ for which $\mathbb{B}(\alpha,\beta) = \mathbb{K}'\langle \alpha, \beta :- \mathbb{D} \rangle$ exists.

ii) The intersection of multiplicative domains

( ) Let $\mathbb{D} \in dom(\mathbb{K}, \bar{\mathbb{N}})$ and $\mathbb{B}', \mathbb{B}'' \in \mathcal{M}(\mathbb{D})$. Define $\mathbb{B} : \mathbb{D} \leq \mathbb{K}$ by setting

$$\mathbb{B}(\alpha,\beta) := \mathbb{B}'(\alpha,\beta) \cap \mathbb{B}''(\alpha,\beta)$$

$\mathbb{B} \in \mathcal{M}(\mathbb{D})$.

$$\begin{aligned}(\mathbb{B}(\alpha,\beta) \cap \mathbb{B}(\beta,\gamma) &= \mathbb{B}'(\alpha,\beta) \cap \mathbb{B}'(\beta,\gamma) \cap \mathbb{B}''(\alpha,\beta) \cap \mathbb{B}''(\beta,\gamma) \\ &\leq \mathbb{B}'(\alpha,\gamma) \cap \mathbb{B}''(\alpha,\gamma) \\ &= \mathbb{B}(\alpha,\gamma))\end{aligned}$$

3] Mapping systems with cancellation

Let $\mathbb{D}'' := \mathbb{D}' := \mathbb{D} \in dom(\mathbb{K} \mathbin{\$} \bar{\mathbb{N}})$ and $i \in \bar{\mathbb{N}}$

The notation $T \in \mathcal{MB}(\mathbb{D}, \mathbb{B}|i)$ indicates

a) that $\mathbb{B} \in \mathcal{M}(\mathbb{D})$ and

b) that $T : \mathbb{D}' \times \mathbb{D}'' \to \{\mathbb{B}(\mathbb{D}', \mathbb{D}'') \to [\mathbb{K}|i]\}$ possesses

α) the cancellation property

$$T(\alpha/\beta)T(\beta/\gamma)=T(\alpha/\gamma) \quad \langle B(\alpha,\beta)\cap B(\beta,\gamma)\rangle$$

for ~~all~~ $\alpha,\beta,\gamma:=\mathbb{D}$ and

β) the internal reduction property

$$T(\alpha/\alpha)=I[i] \qquad \langle B(\alpha,\alpha)\rangle$$

for $\alpha:=\mathbb{D}$

A mapping $T\in \mathcal{MC}(D,B|i)$ for some triple D,B,i is said to be a mapping system with cancellation

$\S$ i) The existence of mapping systems with cancellation

( ) Let $D\in dom(\mathbb{K},\bar{\mathbb{N}})$, $B\in \mathcal{M}(\mathbb{D})$ and $i\in\bar{\mathbb{N}}$.

The constant mapping $T:\mathbb{D}\times\mathbb{D}\to[\mathbb{K}|i]$ defined by setting

$$T(\alpha/\beta):=I[i]$$

is in $\mathcal{MC}(\mathbb{D},B|i)$

ii) The inversion of mapping systems with cancellation

( ) Let $\mathbb{D}\in dom(\mathbb{K},\bar{\mathbb{N}})$, $i\in\bar{\mathbb{N}}$ and $T\in\mathcal{MC}(\mathbb{D},B|i)$

For each pair $\alpha,\beta\in\mathbb{D}$,

a) $$\{B(\alpha,\beta)\cap B(\beta,\alpha)\}\subseteq NS\{T(\alpha/\beta)\}$$

and

b) $$\{T(\alpha/\beta)\}^{-1}=T(\beta/\alpha) \quad \langle B(\alpha,\beta)\cap B(\beta,\alpha)\rangle$$

iii) The extension of classes of mapping systems with cancellat

( ) Let $\mathbb{D}'':=\mathbb{D}':=\mathbb{D}\in dom(\mathbb{K},\bar{\mathbb{N}})$, $i\in\bar{\mathbb{N}}$, $B':\mathbb{D}\subseteq\mathbb{K}$ and $\mathcal{D}:\mathbb{D}\to\{B'(\mathbb{D})\to[\mathbb{K}|i]\}$.

a) Let $T \in \mathcal{M}\mathcal{C}(\mathbb{D}, \mathbb{B}|i)$. Define the mapping $\hat{\mathbb{B}} : \mathbb{D} \times \mathbb{D} \subseteq \mathbb{K}$ by setting

$$\hat{\mathbb{B}}(\alpha, \beta) := \mathbb{B}'(\alpha) \cap \mathbb{B}(\alpha, \beta) \cap NS\{\mathbb{D}(\beta)\}$$

and the mapping $\hat{T} : \mathbb{D}' \times \mathbb{D}'' \to \{\hat{\mathbb{B}}(\mathbb{D}', \mathbb{D}'') \to [\mathbb{K}|i]\}$ by setting

$$\hat{T}(\alpha/\beta) := \mathbb{D}(\alpha)\, T(\alpha, \beta)\, \{\mathbb{D}(\beta)\}^{-1} \qquad \langle \hat{\mathbb{B}}(\alpha, \beta) \rangle$$

$\hat{T} \in \mathcal{M}\mathcal{C}(\mathbb{D}, \hat{\mathbb{B}}|i)$.

b) Let $\mathbb{B} \in \mathcal{M}(\mathbb{D})$. Define the mapping $\tilde{\mathbb{B}} : \mathbb{D} \times \mathbb{D} \subseteq \mathbb{K}$ by setting

$$\tilde{\mathbb{B}}(\alpha, \beta) := \mathbb{B}'(\alpha) \cap NS\{\mathbb{D}(\beta)\}$$

and the mapping $\tilde{T} : \mathbb{D}' \times \mathbb{D}'' \to \{\tilde{\mathbb{B}}(\mathbb{D}', \mathbb{D}'') \to [\mathbb{K}|i]\}$ by setting

$$\tilde{T}(\alpha/\beta) := \mathbb{D}(\alpha)\, \mathbb{D}(\beta)^{-1} \qquad \langle \tilde{\mathbb{B}}(\alpha, \beta) \rangle$$

$\tilde{T} \in \mathcal{M}\mathcal{C}(\mathbb{D}, \tilde{\mathbb{B}}|i)$.

iv) Transformation systems and mapping systems with cancellation

( ) Let $i \in \bar{\mathbb{N}}$, $\mathbb{D}'' := \mathbb{D}' := \mathbb{D} \in \mathrm{dom}(\mathbb{K}, \bar{\mathbb{N}})$, $\mathbb{B}' : \mathbb{D} \times \mathbb{D} \subseteq \mathbb{K}$ and $T : \mathbb{D}' \times \mathbb{D}'' \to \{\mathbb{B}'(\mathbb{D}', \mathbb{D}'') \to [\mathbb{K}|i]\}$.

a) Let $A \in \mathrm{push}\{T, \mathbb{D}, \mathbb{B}'|i\}$ and $\mathbb{B} : \mathbb{D} \times \mathbb{D} \subseteq \mathbb{K}$ exist such that

α) for each $\alpha, \beta \in \mathbb{D}$

$$\mathbb{B}(\alpha, \beta) \subseteq NS\{A(\beta)\} \cap \mathbb{B}'(\alpha, \beta)$$

and

β) $\mathbb{B} \in \mathcal{M}(\mathbb{D})$.

The above conditions imply that $T \in \mathcal{M}\mathcal{C}(\mathbb{D}, \mathbb{B}|i)$.

b) Let $C \in \mathrm{pull}\{T, \mathbb{D}, B' | i\}$ and $B : \mathbb{D} \times \mathbb{D} \subseteq \mathbb{K}$ exist such that for each $\alpha, \beta \in \mathbb{D}$

$$B(\alpha,\beta) \subseteq NS\{C(\alpha)\} \cap B'(\alpha,\beta)$$

and condition (β) above holds. Again $T \in \mathcal{MC}(\mathbb{D}, B|i)$.

4] Similarity factors

With $i \in \bar{\mathbb{N}}$ and $\mathbb{K}' \subseteq \mathbb{K}$, the two mappings $F, F' : \mathbb{K}' \to [\mathbb{K}|i]$ connected by a relationship of the form

$$F = \Delta F' \{\Delta\}^{-1} \qquad \langle \mathbb{K}'' \rangle$$

where $\mathbb{K}'' \subseteq \mathbb{K}'$ and $\Delta : \mathbb{K}'' \to [\mathbb{K}|i]$ is a third mapping, are said to be similar over $\mathbb{K}''$; Δ is called a similarity factor of the ordered pair $\{F, F'\}$ over $\mathbb{K}''$.

i) The derivation of one similarity factor from another

It is possible to derive one similarity factor of $\{F, F'\}$ from another by use of conditions involving either F or F'.

( ) Let $\mathbb{K}' \subseteq \mathbb{K}$ and $F, F', \Delta_0 : \mathbb{K}' \to [\mathbb{K}|i]$ be such that

$$F\Delta_0 = \Delta_0 F' \qquad \langle \mathbb{K}' \rangle$$

so that Δ_0 is a similarity factor of $\{F, F'\}$ over $NS(\Delta_0)$. Let $\Delta_1 : NS(\Delta_0) \to [\mathbb{K}|i]$

a) If F commutes multiplicatively with Δ_1 over $NS(\Delta_0)$, i.e

$$F\Delta_1 = \Delta_1 F \qquad \langle NS(\Delta_0) \rangle$$

then

$$F\Delta_1\Delta_0 = \Delta_1\Delta_0 F' \qquad \langle NS(\Delta_0) \rangle$$

and $\Lambda_0\Delta_0$ is a similarity factor of $\{F, F'\}$ over $NS(\Lambda_0, \Lambda_1)$

b) If F' commutes multiplicatively with Δ_1 over $NS(\Lambda_0)$ then

$$F\Delta_0\Delta_1 = \Delta_0\Delta_1 F' \qquad \langle NS(\Lambda_0)\rangle$$

and $\Delta_0\Delta_1$ is a similarity factor of $\{F, F'\}$ over $NS(\Lambda_0, \Delta_1)$

cα) F commutes multiplicatively with the product $\Delta_1\{\Lambda_0\}^{-1}$ over $NS(\Lambda_0)$ if and only if F' is related to the product $\{\Lambda_0\}^{-1}\Delta_1$ in the same way.

β) If the above conditions hold

$$F\Delta_1 = \Delta_1 F' \qquad \langle NS(\Lambda_0)\rangle$$

and Δ_1 is a similarity factor of $\{F, F'\}$ over $NS(\Lambda_0, \Lambda_1)$.

dα) F commutes multiplicatively with the product $\Lambda_0\Delta_1$ over $NS(\Lambda_0)$ if and only if F' is related to the product $\Lambda_1\Delta_0$ in the same way.

β) If the above conditions hold

$$\Delta_1 F = F'\Delta_1 \qquad \langle NS(\Lambda_0)\rangle$$

and $\{\Delta_1\}^{-1}$ is a similarity factor of $\{F, F'\}$ over $NS(\Lambda_0, \Lambda_1)$.

(In special cases it may be shown that Δ_1 defined over the source domain K' satisfies the relationship of (cβ) above over K'; in this case Δ_1 is a similarity factor of $\{F, F'\}$ over $NS(\Delta_1)$. A similar remark may be made concerning subclause (dβ).)

Observations

Attempt to avoid isolated reference to sets $S, S', \ldots$ in the theory (thus $\in S$, $S \times S' \times \ldots$ etc alone)

$|\cdot|$, $[\cdot]$, $[\cdot,\cdot]$ $\{\cdot\}$ as parenthetic operators (mappings)?

A4vi): $NS(A) \cap NS(B) \cap \ldots \cap NS(C)$ as $NS(A,B,..,C)$? $NS\{\}$ or $NS($

Transfer A[10] on delimiters and parentheses to A[4] (vii) since the conventions of A[10] are used in A[8] etc.

B1] Possible introductory passage:

With $f \in seq(\mathbb{K}|i)$ prescribed, differences $\Delta^r f_m$ $\langle r := [i]$, $m := [i-r] \rangle$ may be formed by use of the recursion

$$\Delta^{r+1} f_m = \Delta^r f_{m+1} - \Delta^r f_m$$

The $\Delta^r f_m$ may be arranged in a difference table in which entries with a common value of r lie in a common column and those with a common value of m on a common forward diagonal. This process of differencing is annihilative in the sense that if f is a sequence of equal members, so that $f_m = c \in \mathbb{K}$ $\langle m := [i] \rangle$, the differences Δf_m in the first column are reduced to zero, those in the further columns remaining zero. Denoting the successive differences in the leading diagonal by $g_r = \Delta^r f_0$ $\langle r := [i]$ a matrix $M \in [\mathbb{K}|i]$ for which

$$M\,col[f]_{[i]} = col[g]_{[i]}$$

exists. M is annihilatory in the sense that all row sums except the first are zero.

With f as above, means $a^r f_m$ $\langle r:=[i], m:=[i-r]\rangle$ may be formed by use of the recursion

$$a^{r+1} f_m = \frac{1}{2}(a^r f_{m+1} + a^r f_m)$$

The $a^r f_m$ may also be arranged in a table similar to that described above. This process of mean formation is permanent in the sense that if f is a sequence of equal members c, all means $a^r f_m$ take the value c. Setting now $h_r = a^r f_0$ $\langle r:=[i]\rangle$ a matrix $A \in [\mathbb{R}|i]$ for which

$$A\,\mathrm{col}[f]_{[i]} = [h]_{[i]}$$

exists. A is permanent in the sense that all row sums are unit.

Extend definition of annihilatory matrices by use of the relationship $A(I_{[i]} \times I_{[i]} \times .. \times I_{[i]}) = I[i]^{\beta}$

C3] etc, matrix mappings, mapping systems?

C4]: express set of similarity factors of $\{F, F'\}$ as $\langle F'/F\rangle$

Then $\langle F'/F\rangle\langle F''/F'\rangle \subseteq \langle F''/F\rangle$

Displaced matrices $\mathrm{disp}^{\omega} A$?

":=:" as sign of structural equivalence, e.g.

$$[A/B]\times[C/D] :=: [A\times C]/[B\times D]$$

$$A^{\beta}\times A^{\gamma} :=: A^{\beta+\gamma} \qquad (A^{\beta})^{\gamma} :=: A^{\beta\gamma}$$

etc.