Definition (1) With $A, B, \ldots, C, \ldots, Z \subseteq W$, the notation $\Xi : A \times B \times \ldots \times C \times \ldots \to Z$ indicates that for each $a \in A, b \in B, \ldots, c \in C, \ldots$ the set $\Xi(a, b, \ldots, c, \ldots) \subseteq Z$ is uniquely determined by $a, b, c, \ldots$.

(2) A set $\Theta(u)$ defined for all $u \in U$ for which $\{\Theta(u') \setminus \Theta(u'')\} \cup \{\Theta(u'') \setminus \Theta(u')\}$ is void for all $u', u'' \in U$ is said to be independent of u in U

(3α) The notation $\Xi : R\sim \to J$ indicates that $\Xi : R \to J$ and that $\Xi(a)$ is not independent of a in R ~~(β) the~~

(β) The notation $\Xi : A \times B \times \ldots \times C\sim \times D\sim \times \ldots \to Z$ indicates that $\Xi : A \times B \times \ldots \times C \times D \times \ldots \to Z$ and that for each set $S = \{a, b, \ldots, d, \ldots\}$ with $a \in A, b \in B, \ldots, d \in D, \ldots$ $\Xi(a, b, \ldots, c, d, \ldots)$ is not independent of C, and that similar dependence holds with regard to the arguments in all sets accompanied by a tilde in the above notation.

(4) With $A, B, \ldots, C, D, \ldots, Z \subseteq W$ the notation $\Xi : A \times B \times \ldots \times C. \times D. \times \ldots \to Z$ indicates ~~that $\Xi : A \times B \times \ldots \times C \times D \times \ldots \to Z$~~

(α) that $\Xi : A \times B \times \ldots \times C \times D \times \ldots \to Z$ and

(β) that for each set $S = \{a, b, \ldots, c, d, \ldots\}$ with $a \in A, b \in B, \ldots, c \in C, d \in D, \ldots$ either

(β1) $\Xi(a, b, \ldots, c', d, \ldots)$ is independent of c' in C or

(β2) with $x \in \Xi(a, b, \ldots, c, d, \ldots)$ there is only one $c' \in C$ (namely $c' = c$) for which $x \in \Xi(a, b, \ldots, c', d, \ldots)$ ~~(which~~

(which of the two conditions (β1, β2) holds may depend upon S)

and

(β3) that condition (β1) or (β2) holds not only with regard to the argument in C but also independently with regard to the arguments in all sets accompanied by a full stop in the above notation

(5) With $A, B, \ldots, C, D, \ldots, E, F, \ldots ; Z \subseteq W$ the notation
$\Xi : A \times B \times \cdots \times C : \times D : \times \ldots \times E : \times F : \times \ldots \to Z$ indicates
α) that $\Xi : A \times B \times \ldots \times C \times D \times \ldots \times E \times F \times \ldots \to Z$ and
β) that each set $S = \{a, b, \ldots, c, d, \ldots, e, f, \ldots\}$ with $a \in A$, $b \in B, \ldots, c \in C, d \in D, \ldots, e \in E, f \in F, \ldots$ establishes a partition of the system of sets $C, D, \ldots, E, F, \ldots$ accompanied by a colon in the above notation; with regard to the argument ranges $E, F, \ldots$, $\Xi(a, b, \ldots, c, d, \ldots, e', f, \ldots)$ is independent of e' in E, $\Xi(a, b, \ldots, c, d, \ldots, e, f', \ldots)$ is independent of f' in $F, \ldots$; with regard to the argument ranges $C, D, \ldots$ the conditions that if $x \in \Xi(a, b, \ldots, c, d, \ldots, e, f, \ldots)$ there is only one $c' \in C$ (namely $c' = c$), only one $d' \in D$ (namely $d' = d$), ... for which $x \in \Xi(a, b, \ldots, c', d', \ldots, e, f)$ hold simultaneously. "(The partition of the system $C, D, \ldots, E, F, \ldots$ into two systems $C, D, \ldots$ and $E, F, \ldots$ may depend upon S.)"

(6α) The notation $\Xi : R \mid\to J$ $\langle \Xi : R \to\| J \rangle$ indicates that $\Xi : R \to J$ and that $\Xi(a)$ is nonvoid for at least one $\langle$for all$\rangle$ $a \in R$.

(β) The notation $\Xi : A \times B \times \ldots \times C| \times D| \times E| \times \ldots \to Z$ indicates ~~that $\Xi : A \times B \times$~~
(β1) that $\Xi : A \times B \times \ldots \times C \times D \times E \times \ldots \to Z$ and
(β2) that for each set $\{a, b, \ldots, c, e, \ldots\}$ with $a \in A$, $b \in B$,

$\ldots, c\in C, e\in E, \ldots$, $\Xi(a, b, \ldots, c, d, e, \ldots)$ is nonvoid for at least one $d\in D$

~~and that similar properties~~ conditions hold with regard ~~to all argument ranges $C, D, E, \ldots$~~

to the arguments in all sets accompanied by a bar in the above notation.

(γ) The notation $\Xi : A\times B\times\ldots\times C\times\ldots \to\!| Z$ indicates that $\Xi : A\times B\times\ldots\times C\times\ldots \to Z$ and that $\Xi(a, b, \ldots, c, \ldots)$ is nonvoid for all $a\in A$, $b\in B, \ldots, c\in C, \ldots$.

(7) 10 pages on

(8) With $A, B, \ldots, Z\subseteq W$, $\Xi : A\times B\times\ldots \to Z$ and $A'\subseteq A$, $B'\subseteq B, \ldots$, $\Xi(A', B', \ldots)$ is the void set if either A' or B' or $\ldots$ is void and is otherwise the set of all $x\in Z$ for which $x\in \Xi(a, b, \ldots)$ for at least one argument distribution $a\in A', b\in B', \ldots$.

(9) The notation $\Xi : A\times B\times\ldots \to\!| Z$ means that Ξ is surjective in the sense that $\Xi : A\times B\times\ldots \to Z$ and $\Xi(A, B, \ldots) = Z$

(10) With $\Xi : R\to J$ and $\Xi(a)$ void for all $a\in R$, $\Xi^{-1} : J\to R$ is defined by taking $\Xi^{-1}(b)$ to be void for all $b\in J$; when Ξ is not identically void as described, $\Xi^{-1} : \Xi(R)\to R$ is the mapping defined by taking, for each $b\in \Xi(R)$, $\Xi^{-1}(b)$ to be the set of all a for which $b\in\Xi(a)$.

(11) The notation $\Xi : A\times A\times B{:}\times B{:}\times B{:} \to Z$ is written as $\Xi : A^2\times (B{:})^3 \to Z$ and other notations are abbreviated in the same way.

(5) The notation $\Xi : A \times B \times .. \times C.\!\sim \times D.\!\sim \times \dots \to Z$ indicates that in conjunction $\Xi : A \times B \times .. \times C. \times D. \times \dots \to Z$ and $\Xi : A \times B \times \dots \times C\!\sim \times D\!\sim \times \dots \to Z$; further notations are compounded in the same way.

(12) 9 pages ~~do~~

According to ~~the d~~ clauses $\overset{(2,4)}{\cancel{(2)}}$ above, with R a set containing one point alone, ~~$\Xi : R$~~ and $\Xi : R \to J$, $\Xi : R. \to J$ automatically

With A the integer set $\{0, 1, \dots\}$ the mapping Ξ defined by setting $\Xi(a, b, c) = a(b+c)$ is described by the notation $\Xi : A \times (A.)^2 \to A$. When $a = 0$, $\Xi(a, b, c)$ is independent of both b and c, and when $a \neq 0$ and $x = a(b+c)$ the equations $x = a(b' + c)$ and $x = a(b + c')$ have unique solutions in A.

With C the integer set $\{0, \dots, 9\}$ the mapping Ξ defined by setting $\Xi(c, d) = (c/10) + (d/100) + [0, 1)/100$ is described by the notation $\Xi : (C:)^2 \to [0, 1)$. (Ξ is also surjective in the sense of (8) above).

With B, C the integer sets $\{0, 1, 2\}$ and $\{0, 1, 2, 3\}$ the mapping defined by setting $\Xi(c, d) = \{(3-d)c/10\} + \{(2-c)d/100\} + [0, 1)/100$ is not of the form described in (5) above (when $x \in \Xi(0, 0)$, there are two distinct pairs c', d' for which $x \in \Xi(c', d')$, namely $0, 0$ and $2, 3$). If $\Xi(a, b, \dots, c, d, \dots, e, f, \dots)$ is unconditionally independent of

$e, f, \ldots$ for all $a \in A$, $b \in B, \ldots, c \in C$, $d \in D, \ldots$ (for example, if $e, f, \ldots$ do not appear in a formula defining Ξ) and has property ($\beta 2$) with regard to $c, d, \ldots$ for all $a \in A$, $b \in B, \ldots$ then Ξ is, in particular, of the form described in (3) above.

i, ii, iii, iv + proof pt. (1) of (iv) foll. ~~3~~ 5 pages: proof pt. (2) of (iv)

Under the conditions of part (2), select $S = \{a, b, \ldots, d, \ldots\}$ with $a \in A$, $b \in B, \ldots, d \in D, \ldots$, $c \in C$ for which $\Xi(a, b, \ldots, c, d, \ldots)$ is nonvoid and $c' \in C$ for which $c \neq c'$. If $\Xi(a, b, \ldots, c, d, \ldots)$ and $\Xi(a, b, \ldots, c', d, \ldots)$ have a member x in common then, using the condition $\Xi' : A' \times B' \times \ldots \times C'.\sim \times D'.\sim \times \ldots \to Z$ as above, $c = c'$: with $c \neq c'$, $\Xi(a, b, \ldots, c, d, \ldots)$ and $\Xi(a, b, \ldots, c', d, \ldots)$ have no members in common. The possibility that $\Xi(a, b, \ldots, c, d, \ldots)$ is void for all $c \in C$ is excluded by condition (β). For all sets S, $\Xi(a, b, \ldots, c, d, \ldots)$ is not independent of C: $\Xi : A \times B \times \ldots \times C\sim \times D \times \ldots \to Z$ and, from the preceding proof, $\Xi : A \times B \times \ldots \times C.\sim \times D \times \ldots \to Z$. Similarly $\Xi : A \times B \times \ldots \times C \times D.\sim \times \ldots \to Z$ and, from ( ) again, $\Xi : A \times B \times \ldots \times C.\sim \times D.\sim \times \ldots \to Z$

(The conditions $\Xi' : A \times B \times \ldots \times C. \times D. \times \ldots \to Z$ and $\Xi(a, b, \ldots, c, d, \ldots) \leq \Theta\Xi'(a, b, \ldots, c, d, \ldots)$ for all $a \in A$, $b \in B, \ldots, c \in C$, $d \in D, \ldots$ alone do not suffice to ensure that $\Theta\Xi : A \times B \times \ldots \times C. \times D. \times \ldots \to Z$. Take $R = \{m, n\}$ and ~~$\Theta'(m) = \Theta$~~ $\Xi'(m) = \Xi'(n) = Z$, so that $\Theta\Xi' : R. \to Z$. Take $\Theta\Xi(m), \Theta\Xi(n) \subset Z$ with $X = \Xi(m) \cap \Xi(n)$ nonvoid. When $x \in X$, $x \in \Xi(u)$ with $u = m$ or $u = n$.)

~~exist for which $x = c' + d'$: it is not true that $\Xi: A \times D$ ...~~

(i) Let $\Xi: A' \times B' \times .. \times C.' \times D.' \times \ldots \to Z$ and $A \subseteq A'$, $B \subseteq B'$, ..., $C \subseteq C'$, $D \subseteq D'$, ... Then $\Xi: A \times B \times .. \times C. \times D. \times .. \to Z$

$\Xi: A' \times B' \times .. \times C.' \times D.' \times .. \to Z$ if and only if $\Xi: A' \times B' \times .. \times C' \times D' \times .. \to Z$ and for each set $S = \{a, b, .., c, d, ..\}$ with $a \in A$, $b \in B$, .. either $\Xi(a, b, .., c', d, ..)$ is independent of c' in C or, with $x \in \Xi(a, b, .., c, d, ..)$ there is only one $c' \in C'$ (namely $c' = c$) for which $x = \Xi(a, b, .., c', d, ..)$, these conditions holding independently with respect to the arguments in $C, D, ...$. If $\Xi: A' \times B' \times .. \times C.' \times D.' \times .. \to Z$ and $S = \{a, b, .., c, d, ..\}$ with $a \in A \subseteq A'$, $b \in B \subseteq B'$, ... and $\Xi(a, b, .., c', d, ..)$ is independent of c' in C', then it is also independent of c in $C \subseteq C'$. If, with $x \in \Xi(a, b, .., c, d, ...)$ there is only one $c' \in C'$, namely $c' = c$ for which $x \in \Xi(a, b, .., c', d, ..)$, then, since $c \in C$, there is only one $c' \in C$ for which this condition holds.

(The corresponding result deriving from the condition $\Xi: A' \times B' \times ... \times C'_{\sim} \times D'_{\sim} ... \to Z$ is untrue. For example $\Xi(c)$ may not be independent of c in C', but independent of c in C)

(ii) $\Xi : A\times B\times\ldots\times C_\sim\times D_\sim\times\ldots\to Z$ if and only if in conjunction $\Xi : A\times B\times\ldots\times C_\sim\times D\times\ldots\to Z$, ~~and~~ $\Xi : A\times B\times\ldots\times C\times D_\sim\times\ldots\to Z,\ldots$. Corresponding results also hold for the cases in which $\Xi : A\times B\times\ldots\times C_{.\sim}\times D_{.\sim}\times\ldots\to Z$ and $\Xi : A\times B\times\ldots\times C_{.\sim}\times D_{.\sim}\times\ldots\to Z$.

$\Xi : A\times B\times\ldots\times C_{.}\times D\times\ldots\to Z$ if and only if $\Xi : A\times B\times\ldots\times C\times D\times\ldots\to Z$ and for each set $S=\{a,b,\ldots,c,d,\ldots\}$ with $a\in A, b\in B,\ldots$ either $\Xi(a,b,\ldots,c',d,\ldots)$ is independent of $c'\in C$ or, with $x\in\Xi(a,b,\ldots,c,d,\ldots)$ there is only one $c'\in C$ (namely $c'=c$) for which $x\in\Xi(a,b,\ldots,c',d,\ldots)$. Similar ~~result~~ definitions hold for $\Xi : A\times B\times\ldots\times C\times D_{.}\times\ldots\to Z,\ldots$. $\Xi : A\times B\times\ldots\times C_{.}\times D_{.}\times\ldots\to Z$ if and only if the conditions stipulating that $\Xi : A\times B\times\ldots\times C_{.}\times D\times\ldots\to Z$, $\Xi : A\times B\times\ldots\times C\times D_{.}\times\ldots$ hold in conjunction. ~~The further results are shown in the same~~ The proof of the result stated is similar and slightly simpler: the further result is shown in the same way.

(iii) If $\Xi : A\times B\times\ldots\times C{:}\times D{:}\times\ldots\times E{:}\times F{:}\times\ldots\to Z$ then in conjunction $\Xi : A\times B\times\ldots\times C{:}\times D\times\ldots\to Z$, $\Xi : A\times B\times\ldots C\times D{:}\times\ldots\times E\times D\times\ldots\to Z,\ldots$

If only one argument set is accompanied by a colon, the colon has the same significance as a full stop. For example the notations $\Xi : A\times B\times\ldots\times C : \times D\times\ldots$ $\to Z$ and $\Xi : A\times B\times\ldots\times C_{.}\times D\times\ldots\times E\times F\times\ldots\to Z$

~~[illegible]~~ ~~have the same meaning~~ are synonymous. The notation ~~is~~ $\Xi: A\times B\times\ldots\times C{:}\times D{:}\times\ldots\times E{:}\times F{:}\times\ldots\to Z$ indicates that $\Xi: A\times B\times\ldots\times C\times D\times\ldots\times E\times F\times\ldots\to Z$ and that each set $S=\{a,b,\ldots,c,d,\ldots,e,f,\ldots\}$ with $a\in A$, $b\in B$ partitions the system $C,D,\ldots,E,F,\ldots$ into two subsystems: for $E,F,\ldots$ $\Xi(a,b,\ldots,c,d,\ldots,e',f,\ldots)$ is independent of $e'\in E$ (i.e. $\Xi: A\times B\times\ldots\times C\times D\times\ldots E.\times F\times\ldots$) etc.; for $C,D,\ldots$ with $x\in\Xi(a,b,\ldots,c,d,\ldots,e,f,\ldots)$ there is only one c' (namely $c'=c$) & only one d' (namely $d'=d$),... for which $x\in\Xi(a,b,\ldots,c',d',\ldots,e,f,\ldots)$: in conjunction $\Xi: A\times B\times\ldots\times C.\times D\times\ldots\times E\times F\times\ldots\to Z$, $\Xi: A\times B\times\ldots\times C\times D.\times\ldots\times E\times F\times\ldots\to Z$, ...

(The converse of the above result is false. Let $C\equiv D\equiv \cancel{A}Z\equiv A$ an additive Abelian group, ~~With~~ and let ~~$x=c+d$~~ $x=c+d$. With d fixed there is only one c', namely $c'=c$, for which $x=c'+d$, and with c fixed there is only one d', namely $d'=d$, for which $x=c+d'$; i.e. $\Xi: C{:}\times D\to Z$ and $\Xi: C\times D{:}\to Z$. It is not true that in conjunction only one c' and only one d' exist for which $x=c'+d'$: it is not true that $\Xi: C{:}\times D{:}\to Z$.)

(xiv) Let $\Xi: A \times B \times \ldots \times C \times D \times \ldots \to Z$ and $\Xi': A' \times B' \times \ldots \times C'_\nu \times D'_\nu \times \ldots \to Z$ where $A \subseteq A'$, $B \subseteq B'$, ..., $C \subseteq C'$, $D \subseteq D'$, ~~and $\Xi(a, b, \ldots, c, d, \ldots) \subseteq \Xi'(a, b, \ldots, c, d, \ldots)$ for all $a \in A$, $b \in B$, ..., $c \in C$, $d \in D$, ...~~ and $\Xi \subseteq_{(A, B, \ldots, C, D, \ldots)} \Xi'$

~~(1) If $\Xi': A' \times B' \times \ldots \times C'_\nu \times D'_\nu \times \ldots \to Z$ then~~

(1) $\Xi: A \times B \times \ldots \times C_\nu \times D_\nu \times \ldots \to Z$

(2) If ~~(2) If $\Xi': A' \times B' \times \ldots \times C'_\nu \times D'_\nu \times \ldots \to Z$ and $\Xi(a, b, \ldots, c, d, \ldots)$ is nonvoid for all $a \in A$, $b \in B$, ..., $c \in C$, $d \in$ then $\Xi: A \times B \times \ldots \times C_\nu \times D_\nu \times \ldots \times E_\nu \times D_\nu \ldots \to Z$.~~

(α) C contains two members c, c' with $c \neq c'$ and similarly for D, ... and ~~(β) for each set $S = \{a, b, \ldots, d, \ldots\}$~~

(β) $\Xi: A \times B \times \ldots \times C_1 \times D_1 \times \ldots \to Z$ ~~with $a \in A$, $b \in B$, ..., $d \in D$, ... $\Xi(a, b, \ldots, c, d, \ldots)$ is nonvoid for at least one $c \in C$, and similarly for sets of the form $\{a, b, \ldots, c, \ldots\}$ and arguments $d \in D$, ...~~

then $\Xi: A \times B \times \ldots \times C_\nu \times D_\nu \times \ldots \to Z$

If, for all sets S as defined in (2β) $\Xi(a, b, \ldots, c, d, \ldots)$ is independent of c in C, then $\Xi: A \times B \times \ldots \times C_\nu \times D \times \ldots \to Z$.

Assume that a set S for which $\Xi(a, b, \ldots, c, d, \ldots)$ is not independent of c in C exists. Let $x \in \Xi(a, b, \ldots, c, d, \ldots)$ and let $c' \in C$ be such that $x \in \Xi(a, b, \ldots, c', d, \ldots)$ also

Since $\Xi(a, b, \ldots, c, d, \ldots) \subseteq \Xi'(a, b, \ldots, c, d, \ldots)$, $x \in \Xi'(a, b, \ldots, c, d, \ldots)$

Similarly $x \in \Xi'(a, b, .., c', d, ..)$. The condition $\Xi' : A \times B \times .. \times C \sim \times D \sim \times .. \to Z$ implies with respect to the notation $\Xi' : A \times B \times .. \times C. \times D. \times .. \to Z$ that only clause (4β2) of Definition is operative: $x \in \Xi'(a, b, .., c, d, ..)$ and $x \in \Xi'(a, b, .., c', d, ..)$ only when $c = c'$. For each set S ~~as~~ either $\Xi(a, b, .., c, d, ...)$ is independent of c in C or, with $x \in \Xi(a, b, .., c, d, ...)$ there is only one $c' \in C$, namely $c' = c$, for which $x \in \Xi(a, b, .., c', d, ...)$

$\therefore \Xi : A \times B \times .. \times C. \times D \times ... \to Z$. Similarly

$\Xi : A \times B \times ... \times C \times D. \times .. \to Z$ and, from ( ),

$\Xi : A \times B \times ... \times C. \times D. \times ... \to Z$.

~~iv) Assume the condition of (2) to hold, and~~ select S, and $c \in C$ for which $\Xi(a, b, .., c, d, ..)$ is nonvoid. C contains at least one member c' for which $c \neq c'$. If $\Xi(a, b, .., c', d, ..)$ is void, $\Xi(a, b, .., c, d, ..)$ is not independent of c in C. Assuming $\Xi(a, b, .., c', d, ..)$ ~~to be nonvoid~~

and Assume $c, c' \in C$ for which $c \neq c'$. If $\Xi(a, b, .., c, d, ..)$ and $\Xi(a, b, .., c', d, ..)$ have a ~~point x~~ member x in common then, using the condition $\Xi' : A' \times B' \times .. C' \sim \times D \sim \times .. \to Z$ as above, $c = c'$: ~~B~~ with $c \neq c'$, $\Xi(a, b, .., c, d, ..)$ and $\Xi(a, b, .., c', d, ...)$ have no members in common. The possibility that $\Xi(a, b, .., c, d, ...)$ is void for all $c \in C$ is excluded by condition (β). For all sets S, $\Xi(a, b, .., c, d, ...)$ is not independent of ~~C~~ c in C; $\Xi : A \times B \times .. \times C \sim \times D \times .. \to Z$

(v) Let $\Xi : R \to J$ and $A \subseteq B \subseteq R$. Then $\Xi(A) \subseteq \Xi(B)$

Let $b \in \Xi(A)$. There is an $a \in A$ for which $b \in \Xi(a)$. This a is in B: $b \in \Xi(B)$.

(vi) Let $\Xi : R| \to J$. For all $a \in R$ and $b \in \Xi(R)$,

$a \in \Xi^{-1}(b)$ if and only if $b \in \Xi(a)$

If $a \in \Xi^{-1}(b)$, a is one of the members of R for which $b \in \Xi(a)$; if $a \notin \Xi^{-1}(b)$, a is not one of these numbers.

(vii)(1) Let $\Xi : R| \to J$.

$A \subseteq \Xi^{-1}\{\Xi(A)\}$ for all $A \subseteq R$ and $B \subseteq \Xi\{\Xi^{-1}(B)\}$ for all $B \subseteq \Xi(R)$

(2) Let $\Xi : R|.\sim \to J$. $A = \Xi^{-1}\{\Xi(A)\}$ for all $A \subseteq R$.

Select $a \in A$. With $b \in \Xi(R)$ let $A(b)$ be the set of all $a' \in R$ for which $b \in \Xi(a')$. $\Xi^{-1}\{\Xi(a)\}$ is $\cup A(b)$ $(b \in \Xi(a))$. For all $b \in \Xi(a)$, $a \in A(b) = \Xi^{-1}(b)$, from ( ). Hence $a \in \Xi^{-1}\{\Xi(a)\} \subseteq \Xi^{-1}\{\Xi(A)\}$ from ( ); this result holds for all $a \in A$. That $B \subseteq \Xi\{\Xi^{-1}(B)\}$ for all $B \subseteq \Xi(R)$ is shown in the same way.

If $\Xi : R. \to J$ and $\Xi(a)$ is not independent of a for all $a \in R$, there is only one a', namely $a' = a$, for which $b \in \Xi(a')$ when $b \in \Xi(a)$: $A(b) = a$ for all $b \in \Xi(a)$: $\Xi^{-1}\{\Xi(a)\} = a$. Hence $A = \Xi^{-1}\{\Xi(A)\}$ for all $A \subseteq R$.

(viii) Let $\Xi : R \rightsquigarrow J$.

(1) $\Xi^{-1} : \Xi(R) .\!\to R$ if and only if $\Xi : R \to . J$ ~~$\Xi(a)$ is a single member of $\Xi(R)$ for each $a \in R$~~

(2) If this condition is satisfied, $B = \Xi\{\Xi^{-1}(B)\}$ for all $B \subseteq \Xi(R)$.

If $\Xi^{-1}(b)$ is independent of b for all $b \in \Xi(R)$ then the set of all a for which $b \in \Xi(a)$ is independent of b for all $b \in \Xi(R)$, and then $\Xi(a) = \Xi(R)$ for all $a \in R$: $\Xi(a)$ is independent of a ~~for all $a \in$~~ in R.

The assumption that the last condition does not hold implies that $\Xi^{-1}(b)$ is not independent of b for all $b \in \Xi(R)$ and in this case the notation $\Xi^{-1} : \Xi(R) .\!\to R$ asserts that, for all $b \in \Xi(R)$, if $a \in \Xi^{-1}(b)$ then only one $b' \in \Xi(R)$, namely $b' = b$, exists for which $a \in \Xi^{-1}(b')$.
But, from ( ), $a \in \Xi^{-1}(b)$ if and only if $b \in \Xi(a)$.
Thus $\Xi^{-1} : \Xi(R) .\!\to R$ if and only if, for all $b \in \Xi(R)$, if $b \in \Xi(a)$ then only one $b' \in \Xi(R)$, namely $b' = b$, exists for which $b' \in \Xi(a)$. Part (1) has been disposed of. Part (2) is a corollary to ( ).

result of (2) follows from its predecessor.

(ix) Let $\Xi : R \rightsquigarrow J$.

(1) Let $\Xi : R .\!\to J$. If $B \subseteq \Xi(A)$ then $\Xi^{-1}(B) \subseteq A$

(2) Let $\Xi : R \to . J$ ~~$\Xi(a)$ be a single member of $\Xi(R)$ for each $a \in R$~~ and $B \subseteq \Xi(R)$. If $A \subseteq \Xi^{-1}(B)$ then $\Xi(A) \subseteq B$

Since $B \subseteq \Xi(A)$, it follows from (, 2) that $\Xi^{-1}(B) \subseteq \Xi^{-1}\{\Xi(A)\} = A$. $\Xi^{-1}(b)$ is independent of b in $\Xi(R)$ if and only if $\Xi(a)$ is independent of a in R: under the initial assumption $\Xi^{-1}(b)$ is not independent of b in $\Xi(R)$. From ( ), $\Xi^{-1} : \Xi(R) .\!\to R$ under the assumption of (2). The

(7) The notation $\Xi : A\times B\times\ldots\times C\times\ldots \to . Z$ indicates that $\Xi : A\times B\times\ldots\times C\times\ldots \to Z$ and that for each set $a, b, \ldots, c, \ldots$ with $a\in A, b\in B, \ldots, c\in C, \ldots$ $\Xi(a, b, \ldots, c, \ldots)$ is a single member of Z.

(12α) With $\Xi' : A'\times B'\times\ldots\times C'\times\ldots \to Z$, $\Xi'' : A''\times B''\times\ldots\times C''\times\ldots \to Z$ and $A\subseteq A', A''$; $B\subseteq B', B''$; ...; $C\subseteq C', C''$; ... the notation $\Xi' \subseteq_{(A, B, \ldots, C, \ldots)} \Xi''$ indicates that

$\Xi'(a, b, \ldots, c, \ldots) \subseteq \Xi''(a, b, \ldots, c, \ldots)$ for all $a\in A, b\in B, \ldots,$ $c\in C, \ldots$

(β) With, in addition, $\Theta' : \hat{A}\times\hat{B}\times\ldots\times\hat{C}\times\ldots \to Z$, $\Theta'' : \tilde{A}\times\tilde{B}\times\ldots\times\tilde{C}\times\ldots \to Z$, ... ~~the not~~ and $A\subseteq\hat{A}, \tilde{A}$; ... $B\subseteq\hat{B}, \tilde{B}$; ..., $C\subseteq\hat{C}, \tilde{C}$; ... the notation $\{\Xi', \Theta', \ldots\} \subseteq_{(A, B, \ldots, C, \ldots)} \{\Xi'', \Theta'', \ldots\}$ indicates that $\Xi' \subseteq_{(A, B, \ldots, C, \ldots)} \Xi''$, $\Theta' \subseteq_{(A, B, \ldots, C, \ldots)} \Theta''$, ...

[n.0] ?

..0 Aggregates

Definition. Let $R, S \subseteq W$.

(1) A nonvoid aggregate σ in (R,S) is composed of

(α) a nonvoid set $I(\sigma) \subseteq R$ and

(β) a system ~~an aggregate~~ of nonvoid sets $M(\sigma,a) \subseteq S$ ~~each~~ defined for each $a \in I(\sigma)$

(2α) $O_{R,S}$, the void aggregate in (R,S), is the aggregate σ for which $I(\sigma)$ is void

(β) $1_{R,S}$ is the aggregate σ in (R,S) for which $I(\sigma) = R$ and $M(\sigma,a) = S$ for all $a \in I(\sigma)$

(3α) A is the class of aggregates

(β) $A(R)$ is the class of aggregates σ for which $I(\sigma) \subseteq R$

(γ) $A[R,S]$ is the complete set of aggregates in (R,S)

(4) Let $\mu, \rho \in A[R,S]$

(α) $\mu \wedge \rho$ is the aggregate ω in (R,S) for which

(α) $I(\omega)$ is void ~~if~~ when $I'(\omega) = I(\mu) \cap I(\rho)$ is void and

(β) is otherwise the subset of $a \in I'(\omega)$ for which $M'(\omega,a) = M(\mu,a) \cap M(\rho,a)$ is nonvoid

(β) when $I(\omega)$ is nonvoid, $M(\omega,a) = M'(\omega,a)$ for each $a \in I'(\omega)$.

(β) $\mu \vee \rho$ is the aggregate λ in (R,S) for which

(a) $I(\lambda) = I(\mu) \cup I(\rho)$ and

(b) $I(\mu)$ and $I(\rho)$ not both being void, $M(\lambda, a) = M(\mu, a)$ when $a \notin I(\rho)$, $M(\lambda, a) = M(\rho, a)$ when $a \notin I(\mu)$ and $M(\lambda, a) = M(\mu, a) \cup M(\rho, a)$ when $a \in I(\mu) \cap I(\rho)$

(5α) $\sigma, \mu \in A[R, S]$ being such that

(a) $I(\sigma) \subseteq I(\mu)$ and

(b) $I(\sigma)$ being non void, $M(\sigma, a) \subseteq M(\mu, a)$ for all $a \in I(\sigma)$

μ is said to include σ; the notation $\sigma \le \mu$ indicates that σ is included by μ;

(β) the notations $\sigma = \mu$ and $\sigma < \mu$ indicate that $\sigma \le \mu$, $\mu \le \sigma$ and that $\sigma \le \mu$, $\mu \not\le \sigma$ respectively.

~~; $\mu \wedge \mu = \ldots$ $\mu \in A(R,S)$~~

~~Set $\tau = \mu \wedge \mu$. $I(\tau)$ is the subset of $a \in I(\mu) \cap I(\mu)$~~

~~( ) Let $\sigma \leq \mu$, With $\sigma \leq$~~

~~( ) $\sigma \leq \sigma$ for all~~

~~( )(1) Let $\sigma, \mu, \xi \in A(R,S)$. If $\sigma \leq \mu$ and $\mu \leq \xi$ then $\sigma \leq \xi$.~~

~~(2). $\sigma = \sigma$ for all $\sigma \in A(R,S)$. (3) Let $\sigma, \mu, \xi \in A(R,S)$. If $\sigma \leq \mu$.~~

(i)(1) $\sigma = \sigma$ for all $\sigma \in A[R,S]$. (2) ~~If~~ Let $\mu, \sigma \in A[R,S]$. ~~If~~ $\mu = \sigma$ ~~then~~ if and only if $\sigma = \mu$. (3) Let $\sigma, \mu, \xi \in A[R,S]$.

If $\sigma \leq \mu$ and $\mu \leq \xi$ then $\sigma \leq \xi$. If $\sigma = \mu$ and $\mu = \xi$ then $\sigma = \xi$. If either $\sigma = \mu$, $\mu \leq \xi$ or $\sigma \leq \mu$, $\mu = \xi$ then $\sigma \leq \xi$. If either $\sigma < \mu$, $\mu \leq \xi$ or $\sigma \leq \mu$, $\mu < \xi$ then $\sigma < \xi$.

~~(1) is a simple conseq~~ The ~~relationship~~ condition $\sigma = \mu$ is equivalent to the two conditions $\sigma \leq \mu$, $\mu \leq \sigma$; (2) follows directly.

Evidently $\sigma \leq \sigma$, ~~for all σ~~ so that $\sigma = \sigma$. If $\sigma \leq \mu$, $I(\sigma) \subseteq I(\mu)$ and $M(\sigma, a) \subseteq M(\mu, a)$ for all $a \in I(\sigma)$. The relationship $\mu \leq \xi$ implies similar conditions. Thus $I(\sigma) \subseteq I(\xi)$ and $M(\sigma, a) \subseteq M(\xi, a)$ for all $a \in I(\sigma)$. The further results of (3) are proved in the same way

(ii) 9 pp. on

(iii) $\mu \wedge \mu = \mu \vee \mu = \mu$ for all $\mu \in A[R,S]$.

Set $\tau = \mu \wedge \mu$. $I(\tau)$ is the subset of $a \in I(\mu) \cap I(\mu)$ for which $M'(\tau, a) = M(\mu, a) \cap M(\mu, a)$ is nonvoid, and then $M(\tau, a) = M'(\tau, a)$ for all $a \in I(\tau)$. $M'(\tau, a) = M(\mu, a)$ is nonvoid for all $a \in I(\mu) = I(\tau)$: $\tau = \mu$. $\mu \vee \mu$ is dealt with similarly.

(iv) Let $\mu, \rho, \xi \in A[R,S]$. ~~Then~~ "(1) $\xi \wedge (\mu \wedge \rho) = (\xi \wedge \mu) \wedge \rho$, and "(2)

$\xi \vee (\mu \vee \rho) = (\xi \vee \mu) \vee \rho$.

Set $\kappa = \mu \wedge \rho$. $I(\kappa)$ is the subset of $a \in I'(\kappa) = I(\mu) \cap I(\rho)$ for which $M'(\kappa, a) = M(\mu, a) \cap M(\rho, a)$ is nonvoid, and then $M(\kappa, a) = M'(\kappa, a)$ for all $a \in I(\kappa)$. $\omega = \xi \wedge \kappa$ is defined in the same way. $I(\omega)$ is the subset of $a \in I(\xi) \cap I(\mu) \cap I(\rho)$ for which $M'(\omega, a) = M(\xi, a) \cap M(\mu, a) \cap M(\rho, a)$ is nonvoid and $M(\omega, a) = M'(\omega, a)$ for all $a \in I(\omega)$. The same description ~~result holds with regard to~~ serves for $(\xi \wedge \mu) \wedge \rho$. The second result is proved in the same way.

(v) Let $\mu, \rho \in A[R,S]$. Then "(1) $\mu \wedge \rho = \rho \wedge \mu$ and "(2) $\mu \vee \rho = \rho \vee \mu$.

The conditions defining $\tau = \mu \wedge \rho$ in terms of μ and ρ are symmetric in μ and ρ: $\tau = \rho \wedge \mu$. $\mu \vee \rho$ is dealt with similarly.

(vi) Let $\mu, \rho, \xi \in A(R,S)$; then $\xi \wedge (\mu \vee \rho) = (\xi \wedge \mu) \vee (\xi \wedge \rho)$

Set $\lambda = \mu \vee \rho$ ~~$I(\lambda)$~~ so that $I(\lambda) = I(\mu) \cup I(\rho)$ and, for ~~all~~ ~~$M(\lambda,a)$~~ $a \in I(\lambda)$, $M(\lambda, a) = M(\mu, a)$ when $a \notin I(\rho)$, $M(\lambda, a) = M(\rho, a)$ when $a \notin I(\mu)$, and $M(\lambda, a) = M(\mu, a) \cup M(\rho, a)$ when $a \in I(\rho) \cap I(\mu)$. Set $\omega = \xi \wedge \lambda$. $I(\omega)$ is the subset of $a \in I'(\omega) = I(\xi) \cap \{I(\mu) \cup I(\rho)\}$ for which $M'(\omega, a) = M(\xi, a) \cap M(\lambda, a)$

is nonvoid and then $M(\omega,a) = M'(\omega,a)$ for all $a \in I(\omega)$.

$I' = \{I(\xi)\cap I(\mu)\} \cup \{I(\xi)\cap I(\rho)\}$ may be divided into three mutually exclusive subsets: I_1, all $a \notin I(\rho)$ for which $a \in I(\xi)\cap I(\mu)$; I_2; all $a \notin I(\mu)$ for which $a \in I(\xi) \cap I(\rho)$ and I_3 all $a \in I(\xi)\cap I(\mu)\cap I(\rho)$. ~~Over I_1, M'~~ Set

~~Over I_1, $M'(\omega,a) = M(\xi,a)\cap M(\mu,a)$; over I_2~~

$M_1(a) = M(\xi,a)\cap M(\mu,a)$ for $a \in I_1$, $M_2(a) = M(\xi,a)\cap M(\rho,a)$ for $a \in I_2$ and $M_3(a) = M(\xi,a) \cap \{M(\mu,a) \cup M(\rho,a)\}$ for $a \in I_3$.

$I'(\omega) = I$. ~~Over I_1,~~ For $a \in I_1$, $M'(\omega,a) = M_1(a)$; for $a \in I_2$, $M'(\omega,a) = M_2(a)$; for $a \in I_3$, $M'(\omega,a) = M_3(a)$. The subset ~~of~~ in I_1 belonging to $I(\omega)$ is the set of a for which $M_1(a)$ is nonvoid and similarly for I_2 and I_3.

Set $\kappa = \xi \wedge \mu$. $I(\kappa)$ is the subset of $a \in I'(\kappa) = I(\xi)\cap I(\mu)$ for which $M'(\kappa,a) = M(\xi,a)\cap M(\mu,a)$ is nonvoid, and then $M(\kappa,a) = M'(\kappa,a)$ for all $a \in I(\kappa)$. Set $\zeta = \xi \wedge \rho$. $I(\zeta)$, $M'(\zeta,a)$ and $M(\zeta,a)$ are similarly defined. Set $\tau = \kappa \vee \zeta$. $I(\tau) = I(\kappa) \cup I(\zeta)$ and for all $a \in I(\tau)$, $M(\tau,a) = M(\kappa,a) = M(\xi,a)\cap M(\mu,a)$ when $a \notin I(\zeta)$, $M(\tau,a) = M(\zeta,a) = M(\xi,a)\cap M(\rho,a)$ when $a \notin I(\kappa)$, and $M(\tau,a) = M(\kappa,a) \cup M(\zeta,a) = \{M(\xi,a)\cap M(\mu,a)\} \cup \{M(\xi,a)\cap M(\rho,a)\} = M(\xi,a)\cap\{M(\mu,a)\cup M(\rho,a)\}$ when $a \in I(\kappa)\cap I(\zeta)$.

$I(\tau)$ is a subset of $I'(\kappa) \cup I'(\beta) = \xi I$. When

$a \notin I(\kappa)$, $a \notin I'(\kappa) = I(\xi) \cap I(\mu)$

$M_1(a)$

$a \notin I(\kappa)$ either $a \notin I(\xi) \cap I(\mu)$ or $a \in I(\xi) \cap I(\mu)$ but $M(\xi,a) \cap M(\mu,a)$

is void || I_2 $a \notin I(\mu)$ $a \in I(\xi) \cap I(\rho)$ i.e. $a \in I_2$

$a \notin I(\xi) \cap I(\mu) \to a \in I(\xi) \cap I(\rho)$ $a \notin I(\mu)$ $M(\tau,a) = \overset{=M_2(a)}{M(\xi,a) \cap M(\rho,a)}$

$a \notin I(\beta)$: either $a \notin I(\xi) \cap I(\rho)$ or $a \in I(\xi) \cap I(\rho)$ but $M(\xi,a) \cap M(\rho,a)$

$= M_1(a)$ is void $a \notin I(\xi) \cap I(\rho) \to a \in I(\xi) \cap I(\mu)$ $a \notin I(\rho)$ i.e. $a \in I_1$

$M(\tau,a) = M(\xi,a) \cap M(\mu,a) = M_1(a)$

$a \in I(\kappa) \cap I(\beta) \to a \in I(\xi) \cap I(\mu) \cap I(\rho)$ $M(\tau,a) = M_3(a)$

$I(\tau)$ is a subset of $I'(\kappa) \cup I'(\beta) = I$. When $a \notin I(\beta)$ ~~in this rang~~ either $a \notin I(\xi) \cap I(\rho)$ or $a \in I(\xi) \cap I(\rho)$ but $M'(\tau,a) = M(\xi,a) \cap M(\rho,a)$ is void. For $a \notin I(\xi) \cap I(\rho)$ in the range $a \in I$, $a \in I(\xi) \cap I(\mu)$ but $a \in I(\rho)$, i.e. $a \in I_1$ and $M(\tau,a) = M(\xi,a) \cap M(\mu,a) = M_1(a)$. When $a \notin I(\kappa)$ in the range $a \in I$, either $M'(\tau,a)$ is void or $a \in I_2$ and $M(\tau,a) = M_2(a)$. Lastly, when $a \in I(\kappa) \cap I(\beta)$, $a \in I(\xi) \cap I(\mu) \cap I(\rho) = I_3$ and $M(\tau,a) = M_3(a)$.

The members of I_1 belonging to $I(\omega)$ and $I(\tau)$ are the same, and for these a, $M(\omega,a) = M(\tau,a)$. For the two further ranges I_2 and I_3 the same result holds: $\omega = \tau$.

(viii) Let $\mu, \rho, \xi \in A[R, S]$; then $\xi \vee (\mu \wedge \rho) = (\xi \vee \mu) \wedge (\xi \vee \rho)$.

$I =$ ~~$\{I(\xi) \cup I(\mu)\}$~~ $I(\xi) \cup \{I(\mu) \cap I(\rho)\} = \{I(\xi) \cup I(\mu)\} \cap \{I(\xi) \cap I(\rho)\}$ may be decomposed into three mutually disjoint subsets: I_1, all $a \in I(\xi) \setminus \{I(\mu) \cap I(\rho)\}$; I_2, all $a \in \{I(\mu) \cap I(\rho)\} \setminus I(\xi)$ and I_3; all $a \in I(\xi) \cap I(\mu) \cap I(\rho)$. Set $M_1(a) = M(\xi, a)$ for $a \in I_1$, $M_2(a) = M(\mu, a) \cap M(\rho, a)$ for $a \in I_2$ and $M_3(a) = M(\xi, a) \cup \{M(\mu, a) \cap M(\rho, a)\} = \{M(\xi, a) \cup M(\mu, a)\} \cap \{M(\xi, a) \cup M(\rho, a)\}$ for $a \in I_3$.

Set $\kappa = \mu \wedge \rho$. $I(\kappa)$ is the subset of $a \in I'(\kappa) = I(\mu) \cap I(\rho)$ for which $M'(\kappa, a) = M(\mu, a) \cap M(\rho, a)$ is nonvoid, and $M(\kappa, a) = M'(\kappa, a)$ for $a \in I(\kappa)$. Set $\omega = \xi \vee \kappa$. $I(\omega) = I(\xi) \cup I(\kappa) \subseteq I(\xi) \cup \{I(\mu) \cap I(\rho)\}$ and, for all $a \in I(\omega)$, $M(\omega, a) = M(\xi, a)$ for all $a \notin I(\kappa)$, $M(\omega, a) = M(\kappa, a)$ for all $a \notin I(\xi)$ and $M(\omega, a) = M(\xi, a) \cup M(\kappa, a)$ for all $a \in I(\kappa) \cap I(\xi)$.

$I(\omega)$ is a subset of I. For all $a \in I(\omega)$ belonging to I_1, $M(\omega, a) = M_1(a)$. For all $a \in I(\omega)$ belonging to I_2, $M(\omega, a) = M(\mu, a) \cap M(\rho, a) = M_2(a)$. For all $a \in I(\omega)$ belonging to I_3, $M(\omega, a) = M(\xi, a) \cup \{M(\mu, a) \cap M(\rho, a)\} = M_3(a)$.

Set $\zeta = \xi \vee \mu$. $I(\zeta) = I(\xi) \cup I(\mu)$ and, for all $a \in I(\zeta)$, $M(\zeta, a) = M(\xi, a)$ for all $a \notin I(\mu)$, $M(\zeta, a) = M(\mu, a)$ for all $a \notin I(\xi)$, and $M(\zeta,a)=M(\xi,a)\cup M(\mu,a)$ for all $a\in I(\xi)\cap I(\mu)$. $\lambda=\xi\vee\rho$ is similarly defined. Set $\tau=\zeta\wedge\lambda$. $I(\tau)$ is the subset of $a\in$

$I'(\tau)=I(\zeta)\cap I(\lambda)=\{I(\xi)\cup I(\mu)\}\cap\{I(\xi)\cup I(\rho)\}=I(\xi)\cup\{I(\mu)\cap I(\rho)\}$

for which $\overset{M'(\tau,a)=}{M(\zeta,a)\cap M(\lambda,a)}$ is nonvoid, and $M(\tau,a)=M'(\tau,a)$

for $\tau\in I(\tau)$.

? $I'(\tau)$ is I.

For all $a\notin I(\xi)$ ~~in the range~~ belonging to the subset $I(\xi)\cup\{I(\mu)\cap I(\rho)\}$, $a\in$ $\cancel{I(\xi)}$ $\{I(\mu)\cap I(\rho)\}\setminus I(\xi)\equiv I_2$ and for such a, $M'(\tau,a)=M(\zeta,a)\cap M(\lambda,a)=M(\mu,a)\cap M(\rho,a)\equiv M_2(a)$: for all $a\in I(\tau)$ belonging to I_2, $M(\tau,a)=M_2(a)$.

For all a for which either $a\notin I(\mu)$ or $a\notin I(\rho)$ ~~in the range~~ belonging to the subset $I(\xi)\cup\{I(\mu)\cap I(\rho)\}$, $a\in I(\xi)\setminus\{I(\mu)\cap I(\rho)\}\equiv I_1$ and for such a, $M'(\tau,a)=M(\xi,a)\equiv M_1(a)$: for all $a\in I(\tau)$ belonging to I_1, $M(\tau,a)=M_1(a)$. For all $a\in I(\xi)\cap I(\mu)\cap I(\rho)\equiv I_3$ ~~in the range~~ belong to I, $M'(\tau,a)=\{M(\xi,a)\cup M(\mu,a)\}\cap\{M(\xi,a)\cup M(\mu,a)\}$ $=M_3(a)$.

$I(\omega)$ and $I(\tau)$ are subsets of I. The subsets of $I(\omega)$ and $I(\tau)$ belonging to I_1 are those values of a for which $\cancel{M_1(a)}$ $M_1(a)\equiv M'(\omega,a)\equiv M'(\tau,a)$ are nonvoid, and for these a, $M(\omega,a)=M(\tau,a)$. Similar ~~remarks~~ results hold with regard to the subsets I_2 and I_3: $\omega=\tau$.

(viii) Let $\mu, \rho, \xi \in A[R,S]$. If $\rho \le \xi$ then $\mu \wedge \rho \le \mu \wedge \xi$.

Let $\omega = \mu \wedge \rho$. $I(\omega)$ is the set of $a \in I'(\omega) = I(\mu) \cap I(\rho)$ for which $M'(\omega, a) = M(\mu, a) \cap M(\rho, a)$ is nonvoid, and $M(\omega, a) = M'(\omega, a)$ for all $a \in I(\omega)$. $z = \mu \wedge \xi$ is similarly defined.

~~$M(\mu,a) \cap M(\rho,a) \le M(\mu,a) \cap M$ If ρ~~ Since $\rho \le \xi$, $I(\rho) \subseteq I(\xi)$ and $M(\rho, a) \le M(\xi, a)$ for all $a \in I(\rho)$

$M(\mu, a) \cap M(\rho, a) \le M(\mu, a) \cap M(\xi, a)$ for all $a \in I(\rho) \cap I(\mu)$. ~~If for any $a \in I(\mu) \cap I(\rho)$, $M(\mu,a) \cap M(\rho,a)$~~

For all $a \in I(\mu) \cap I(\rho)$ for which $M(\mu, a) \cap M(\rho, a)$ is nonvoid, i.e. for all $a \in I(\omega)$, $M(\mu, a) \cap M(\xi, a)$ is also nonvoid, a belongs to $I(z)$, and $M(\omega, a) \le M(z, a)$: $\omega \le z$

(ix) Let $\mu, \rho, \xi \in A[R,S]$. If $\rho \le \xi$ then $\mu \vee \rho \le \mu \vee \xi$.

Let $\lambda = \mu \vee \rho$. $I(\lambda) = I(\mu) \cup I(\rho)$ and, for all $a \in I(\lambda)$, $M(\lambda, a) = M(\mu, a)$ for all $a \notin I(\rho)$, $M(\lambda, a) = M(\rho, a)$ for all $a \notin I(\mu)$, and $M(\lambda, a) = M(\mu, a) \cup M(\rho, a)$ for all $a \in I(\mu) \cap I(\rho)$. $\zeta = \mu \vee \xi$ is similarly defined.

Since $\rho \le \xi$, $I(\rho) \subseteq I(\xi)$ and $M(\rho, a) \le M(\xi, a)$ for all $a \in I(\rho)$.

$I(\lambda) \le I(\mu) \cup I(\xi) = I(\zeta)$. $I(\lambda) = I(\mu) \cup I(\rho)$ may be decomposed into four mutually exclusive subsets: I_1, all a for which $a \notin I(\xi)$ (when $a \notin I(\rho)$ also); I_2, all a for which $a \in I(\xi) \cap I(\mu)$ but $a \notin I(\rho)$; I_3, all $a \in I(\xi) \cap I(\mu) \cap I(\rho)$ (when $a \in I(\rho) \cap I(\mu)$ and $a \in I(\xi) \cap I(\mu)$); I_4; all $a \notin I(\mu)$ for which $a \in I(\rho)$ and $a \in I(\xi)$. For $a \in I_1$, $M(\lambda, a) = M(\kappa, a) = M(\mu, a)$. For $a \in I_2$, $M(\lambda, a) = M(\mu, a)$ and $M(\kappa, a) = M(\mu, a) \cup M(\xi, a)$. For $a \in I_3$, $M(\lambda, a) = M(\mu, a) \cup M(\rho, a)$ and $M(\kappa, a) = M(\mu, a) \cup M(\xi, a)$. For $a \in I_4$, $M(\lambda, a) = M(\rho, a)$ and $M(\kappa, a) = M(\xi, a)$. For all a in each subset, $M(\lambda, a) \subseteq M(\kappa, a)$: $\lambda \le \kappa$.

(xi) Let $\mu, \rho \in A[R, S]$; then $\mu \wedge \rho \le \mu$ ~~and~~ $\le \mu \vee \rho$

~~Set~~ Let $\omega = \mu \wedge \rho$. $I(\omega)$ is the subset of $I'(\omega) = I(\mu) \cap I(\rho)$ for which $M'(\omega, a) = M(\mu, a) \cap M(\rho, a)$ is nonvoid, and $M(\omega, a) = M'(\omega, a)$ for $a \in I(\omega)$. $I(\omega) \subseteq I(\mu)$ and $M(\omega, a) \subseteq M(\mu, a)$ for all $a \in I(\omega)$: $\omega \le \mu$.

Let $\lambda = \mu \vee \rho$. $I(\lambda) = I(\mu) \cup I(\rho)$ and, for all $a \in I(\lambda)$, $M(\lambda, a) = M(\mu, a)$ for all $a \notin I(\rho)$. $M(\lambda, a) = M(\rho, a)$ for all $a \notin I(\mu)$, and $M(\lambda, a) = M(\mu, a) \cup M(\rho, a)$ for all $a \in I(\mu) \cap I(\rho)$. $I(\mu) \subseteq I(\lambda)$, and $M(\mu, a) \subseteq M(\lambda, a)$ for all $a \in I(\mu)$: $\mu \le \lambda$.

see 2p. on

(xii). Let $\tau, \mu, \rho, \xi \in A^g[R, S]$. If $\tau \le \mu$ and $\rho \le \xi$, then $\tau \wedge \rho \le \mu \wedge \xi$ and $\tau \vee \rho \le \mu \vee \xi$.

~~If $\rho \le \xi$ then, from ( ), $\tau \wedge \rho \le \tau \wedge \xi$. From ( ), $\tau \wedge \xi = \xi \wedge \tau$. If $\tau \le \mu$ then, from ( ), $\xi \wedge \tau$~~

The conditions $\tau \le \mu$ and $\rho \le \xi$ imply, from (,) that $\tau \wedge \rho \le \tau \wedge \xi = \xi \wedge \tau \le \xi \wedge \mu = \mu \wedge \xi$. From ( ) $\tau \wedge \rho \le \mu \wedge \xi$. The second result is dealt with in the same way.

(xiii) Let $\mu, \rho, \xi \in A(R,S)$; ~~then~~ (1) If $\xi \le \mu$ and $\xi \le \rho$ then $\xi \le \mu \wedge \rho$. (2) If $\mu \le \xi$ and $\rho \le \xi$ then $\mu \vee \rho \le \xi$.

From (,), $\xi = \xi \wedge \xi \le \mu \wedge \rho$. From (,), $\mu \vee \rho \le \xi \vee \xi = \xi$. → full p. xiv 24. m

Sets of aggregates

( ) Let $U, V \subseteq A[R,S]$ If $U \subseteq V$ then $|V| \le |U|$.

~~Let $U = \{\sigma, \mu, \dots\}$~~

Let $\omega = |U|$. $I(\omega)$ is the subset of $a \in I'(\omega) = \bigcap_{\sigma \in U} I(\sigma)$ for which $M'(\omega, a) = \bigcap_{\sigma \in U} M(\sigma, a)$ is nonvoid, and $M(\omega, a) = M'(\omega, a)$ for each $a \in I(\omega)$. $\xi = |V|$ is similarly defined. $I'(\xi) \subseteq I'(\omega)$. Since $M'(\xi, a) \subseteq M'(\omega, a)$, $M'(\omega, a)$ is nonvoid whenever $M'(\xi, a)$ is nonvoid: all $a \in I(\xi)$ belong to $I(\omega)$, and $M(\xi, a) \subseteq M(\omega, a)$ for all $a \in I(\xi)$: $\xi \le \omega$.

( ) Let $U, V \subseteq A[R,S]$; then $|U \cup V| = |U| \wedge |V|$.

Let $\tau = |U|$, $\kappa = |V|$, $\omega = \tau \wedge \kappa$, ~~and~~ $T = |U \cup V|$ and $\xi = |T|$. $I(\xi)$ is the subset of $a \in I'(\xi) = \bigcap_{\sigma \in T} I(\sigma)$ for which $M'(\xi, a) = \bigcap_{\sigma \in T} M(\sigma, a)$ is nonvoid, and $M(\xi, a) = M'(\xi, a)$ for all $a \in I(\xi)$. τ, ~~κ and~~ and κ are similarly defined.

$I'(\xi) = \{\bigcap_{\sigma \in U} I(\sigma)\} \cap \{\bigcap_{\sigma \in V} I(\sigma)\}$ ~~$= I'(\tau) \cap I'(\kappa)$~~ and

$M'(\xi, a) = \{\bigcap_{\sigma \in U} M(\sigma, a)\} \cap \{\bigcap_{\sigma \in V} M(\sigma, a)\}$ ~~$= M'(\tau, a) \cap M'(\kappa, a)$~~

$I(\xi)$ and $I(\omega)$ are both the subsets of $a \in I'(\xi) = I'(\tau) \cap I'$ for which $M'(\xi,a) = M'(\tau,a) \cap M'(\chi,a)$ is nonvoid and then $M(\xi,a) = M(\omega,a) = M'(\xi,a)$ for all $a \in I(\xi) = I(\omega)$: $\xi = \omega$.

( ) Let $U, V \subseteq A[R,S]$; then $|U| \vee |V| \leq |U \cap V|$.

Since $U \cap V \subseteq U$, $|U| \leq |U \cap V|$, from ( ). Similarly $|V| \leq |U \cap V|$, and, from ( ), $|U| \vee |V| \leq |U \cap V|$. (That the relationship $|U| \vee |V| < |U \cap V|$ is possible is shown by taking $U = \{\mu, \rho\}$, $V = \{\xi, \rho\}$ where $\mu < \xi < \rho$. $|U| = \mu$ and $|V| = \xi$ so that, since $\mu < \xi$, $|U| \vee |V| = \xi$, from ( ). But $U \cap V = \rho = \underbrace{|U \cap V|}_{}$ and $\xi < \rho$.)

~~$\mu \leq \rho \Rightarrow \mu \vee \rho = \rho$, $\mu \wedge \rho = \mu$? $0_{R,S}$ $1_{R,S}$~~

(xi) Let $\mu, \rho \in A[R,S]$. If $\mu \leq \rho$ then $\mu \wedge \rho = \mu$ and $\mu \vee \rho = \rho$

From ( ), $\mu \wedge \rho \leq \mu$. If $\mu \leq \rho$, ~~$\mu = \mu \wedge \mu \leq \mu \wedge \rho$, from (~~ then $\mu \leq \mu \wedge \rho$, taking $\xi = \mu$ in ( ). Hence $\mu \wedge \rho = \mu$.

From ( ), $\rho \leq \mu \vee \rho$. If $\mu \leq \rho$, $\mu \vee \rho \leq \rho$, again taking $\xi = \mu$ in ( ): $\mu \vee \rho = \rho$.

(ii) $0_{R,S} \leq \mu \leq 1_{R,S}$ for all $\mu \in A[R,S]$

Since the void set is contained in $I(\mu)$ for all $\mu \in A(R,S)$, $0_{R,S} \leq \mu$ for all $\mu \in A(R,S)$. For all $\mu \in A(R,S)$, $I(\mu) \subseteq R$ and $M(\mu,a) \subseteq S$ for all $a \in I(\mu)$.

$0_{R,S} \leq 0_{R,S}$

(xiv) $0_{R,S} \wedge \mu = 0_{R,S}$, $0_{R,S} \vee \mu = \mu$, $1_{R,S} \wedge \mu = \mu$ and $1_{R,S} \vee \mu = 1_{R,S}$
for all $\mu \in A[R,S]$.

With $\omega = 0_{R,S} \wedge \mu$, $I(\omega)$ is a subset of the intersection of the void set and $I(\mu)$: $I(\omega)$ is void and $\omega = 0_{R,S}$. With $\xi = 0_{R,S} \vee \mu$, $I(\xi)$ is the union of the void set and $I(\mu)$, namely $I(\mu)$. The conditions defining $M(\xi, a)$ for $a \in I(\mu)$ are meaningful only when a lies in $I(\mu)$ alone, and then $M(\xi, a) = M(\mu, a)$: $0_{R,S} \vee \mu = \mu$. With $\omega = 1_{R,S} \wedge \mu$, $I(\omega) = R \cap I(\mu) \equiv I(\mu)$ and $M(\omega, a) = S \cap M(\mu, a) \equiv M(\mu, a)$ for all $a \in I(\mu)$: $1_{R,S} \wedge \mu \equiv \mu$. With $\xi = 1_{R,S} \vee \mu$, $I(\xi) = R \cup I(\mu) \equiv R$; $M(\xi, a) \equiv S$ when $a \in R \setminus I(\mu)$ and $M(\xi, a) \equiv S \cup M(\mu, a) \equiv S$ when $a \in R \cap I(\mu) = I(\mu)$: $\xi \equiv 1_{R,S}$

~~Closed~~ Complete classes of aggregates

Definition Let $\mathbb{P}$ be a class of aggregates. $\mathbb{P}[R,S]$ is the set of aggregates in $A(R,S)$ belonging to $\mathbb{P}$.

? $\mathbb{P}$ being a class of aggregates, ~~the notal~~ and U being a set of aggregates in $A[R,S]$, ~~$U \subseteq \mathbb{P}$~~ the notation $U \subseteq \mathbb{P}$ indicates that all members of U are in $\mathbb{P}$.

??

Definition. (1) Let $\mathbb{P}$ be a class of aggregates.

(α) $\mathbb{P}(R)$ is the subclass of aggregates $\sigma \in \mathbb{P}$ for which $I(\sigma) \subseteq \mathbb{P}$?

(β) $\mathbb{P}[R,S]$ is the set of aggregates in (R,S) belonging to $\mathbb{P}$.

~~(2α) A class of aggregates σ in (R,S) for each of which for each of which $I(\sigma) \subseteq R$ and with $\Delta : R \to \| S$ prescribed, $\Delta(a) \leq M(\sigma, a)$ for all $a \in I(\sigma)$ is said to be monovalent in (R,S)~~

(2α) A class $\mathbb{P}$ of aggregates ~~$\mathbb{P}$ in (R,S)~~ for which a mapping $\Delta : R \to \| S$ exists such that, ~~$\Delta(a) \leq M(\sigma,a)$ for~~ for each $\sigma \in \mathbb{P}[R,S]$, $\Delta(a) \leq M(\sigma, a)$ for all $a \in I(\sigma)$ is said to be monovalent in (R,S).

(β) $\mathbb{P}$ being a class of aggregates and $\Delta : R \to \| S$ being prescribed, $\mathbb{P}_\Delta$ is the subclass of aggregates $\sigma \in \mathbb{P}[R,S]$ for which $\Delta(a) \leq M(\sigma, a)$ for all $a \in I(\sigma)$. Such a subclass is said to be a monovalent subclass of $\mathbb{P}$.

(3α) An aggregate class $\mathbb{P}$ for which for each $\sigma \in \mathbb{P}[R,S]$, ~~A class of aggregates in (R,S) for each of which~~ $M(\sigma, a) = \Gamma(a)$ for all all $a \in I(\sigma)$, where $\Gamma : R \to \| S$ is prescribed, is said to be univalent in (R,S).

(β) In particular, $A^{(\Gamma)}$ is the univalent class of aggregates σ in $A[R,S]$ for which $M(\sigma, a) = \Gamma(a)$ for all $a \in I(\sigma)$.

(γ) An aggregate σ for which $M(\sigma, a) = a$ for all $a \in I(\sigma)$ is said to be degenerate. A' is the class of degenerate aggregates. $A'[R,R]$ is written simply as $A'[R]$.

(δ) $1_R^{(\Gamma)}$ is that member σ of $A^{(\Gamma)}[R,S]$ for which $I(\sigma) = R$. $1'_R$ is the corresponding member of $A'[R]$.

~~(4) An aggregate class P~~ ~~of aggregates such tha~~

(4) An aggregate class $\mathbb{P}$ for which each $\sigma \in \mathbb{P}[R,S]$ is accompanied by the set $U(\sigma) \in \mathbb{P}[R,S]$ for which

(α) $I(\mu) = I(\sigma)$ for all $\mu \in U(\sigma)$ and

(β) for any set $\{a, b, \ldots\}$ of distinct members of $I(\sigma)$ and any set of nonvoid sets $M(a), M(b), \ldots \subseteq S$, $U(\sigma)$ also contains μ with $M(\mu, a) = M(a)$, $M(\mu, b) = M(b)$, ...

is an M-unconstrained aggregate class in (R,S). The notation $\mathbb{P} \in \mathbb{U}A(R,S)$ indicates that $\mathbb{P}$ is such a class.

For all aggregates σ of a fixed univalent class, $M(\sigma, a)$ is independent of σ for all $a \in I(\sigma)$. Within the class, the aggregates σ are uniquely determined by their associated sets $I(\sigma)$ alone; In particular, this is true of the degenerate aggregates of A'; the theory of such aggregates is most conveniently presented in terms of the sets $I(\sigma)$.

(i) Let $\Gamma: R \to \| S$. $O_{R,S} \leq \mu \leq 1_R^{(\Gamma)}$, $1_R^{(\Gamma)} \wedge \mu = \mu$ and $1_R^{(\Gamma)} \vee \mu = 1_R^{(\Gamma)}$ for all $\mu \in A^{(\Gamma)}[R, S]$. Corresponding results also hold with respect to the unit aggregate $1'_R$ in $A'[R]$.

The proofs of these results are similar to those of the corresponding results of ( ).

(ii) A univalent class of aggregates in (R, S) is also monovalent in (R, S).

The mapping Γ occurring in Definition ( ) plays the role of Δ in Definition ( )

(iii) Let $\mathbb{P} \in \amalg A(R', S')$ and $R \subseteq R'$, $S \subseteq S'$

(1) $\mathbb{P} \in \amalg A(R, S)$

(2) If $\mathbb{P}[R', S']$ contains an aggregate σ for which $I(\sigma) \equiv R$, then $1_{R,S} \in \mathbb{P}[R, S]$ and, for any mapping $\Delta: R \to \| S$, $1_{R,S} \in \mathbb{P}_\Delta[R, S]$ also.

The set $\mathbb{P}[R', S']$ contains $\mathbb{P}[R,S]$ and for each $\sigma \in \mathbb{P}[R,S]$ the condition that for any set of nonvoid sets $M(a), M(b), \ldots \subseteq S'$, clause (4β) of Definition is satisfied entails that this is also true when $M(a)$, ~~$M(b)$, ... $\subseteq S$.~~
$M(b), \ldots \subseteq S$: $\mathbb{P} \in WA(R,S)$. If $\mathbb{P}[R', S']$ contains an aggregate σ for which $I(\sigma) \equiv R$, it contains, in particular, an aggregate μ for which $I(\mu) \equiv R$ and $M(\mu, a) \equiv S$ for all $a \in I(\mu)$: $1_{R,S} \in \mathbb{P}[R,S]$.
Since, when $\Delta : R \to \| S$, $\Delta(a) \subseteq S$ for all $a \in R$; ~~thus $\mathbb{P}_\Delta[R,S]$ also~~ $1_{R,S}$ is one of the aggregates ~~contained in~~ present in $\mathbb{P}_\Delta[R,S]$.

Sep '43 n.1.1.0 Special classes of aggregates

Definition . (1.6a) . . .

(1) Let $\Theta, \Xi : R \to J^{\prime\prime(\alpha)}$. An aggregate σ for which $I(\sigma) \subseteq R$ and the largest system $X \subseteq R$ satisfying the condition $\Theta(X) \subseteq \Xi\{I(\sigma)\}$ also satisfies the condition $X \subseteq I(\sigma)$ is said to be Θ-closed with respect to Ξ in R

(β) $\mathfrak{G}(\Theta, \Xi \mid R)$ is the class of such aggregates and $\mathfrak{G}(\Theta, \Xi \mid R, S)$ is the class of such aggregates in $\sqcup A[R, S]$

(γ) When $\Theta(x) = Rx$, $\Xi(a) = a$ and $\sigma \in A'[R]$, so that the largest system $X \subseteq R$ satisfying the condition $RX \subseteq I(\sigma)$ is contained in $I(\sigma)$, $I(\sigma)$ is said to be closed in R.

(2) Let $\Theta, \Xi : R \to J^{\prime\prime(\alpha)}$. An aggregate σ for which $I(\sigma) \subseteq R$ and, for all $a \in R$, $\Theta(a) \cap \Xi\{I(\sigma)\}$ is nonvoid only when $a \in I(\sigma)$ is said to be Θ-free with respect to Ξ in R.

(β) $\mathfrak{R}(\Theta, \Xi \mid R)$ is the class of such aggregates and $\mathfrak{R}(\Theta, \Xi \mid R, S)$ is the class of such aggregates in $\sqcup A[R, S]$

(γ) When $\Theta(a) = a^2$, $\Xi(a) = a$ and $\sigma \in A'[R]$, so that for all $a \in R$, $a^2 \in I(\sigma)$ only when $a \in I(\sigma)$, $I(\sigma)$ is said to be square free in R.

(3) Let $\Theta : R \to J$, $\Omega : R \times J \to K$ and $\Xi : R \to K$.

(α) An aggregate $\sigma \in A(R)$ for which, for all $a \in R$ and $b \in J$, $\Omega(a, b) \cap \Xi\{I(\sigma)\}$ is nonvoid only when $a \in I(\sigma)$ and $b \in \Theta\{I(\sigma)\}$ is said to be Ω-factored with respect to Θ, Ξ in (R, J).

(β) $\mathbb{F}(\Omega; \Theta, \Xi \mid \frac{R; J}{J; R})$ is the class of such aggregates and $\mathbb{F}(\Omega; \Theta, \Xi \mid R, S; J)$ is the class of such aggregates in $\cup A[R, S]$.

(γ) When $R \equiv J$, $\Omega(a, b) = ab$, $\Theta(a) = \Xi(a) = a$ and $\sigma \in A'[R]$, so that for all $a, b \in R$, $ab \in I(\sigma)$ only when $a, b \in I(\sigma)$, $I(\sigma)$ is said to be factored in R.

(4) Let $\Theta : R \times \ldots \times R \times R \times R \times \ldots \to J$ and $\Xi : R \to J$.

(α) An aggregate $\sigma \in A(R)$ for which for any fixed set $a, \ldots, t, v, \ldots \in I(\sigma)$ either

(a) $\Theta(a, \ldots, t, u, v, \ldots)$ is independent of u in R or

(b) for any $x \in \Xi\{I(\sigma)\}$ a unique $u = u(x; a, \ldots, t, v, \ldots) \in R$ exists for which $x \in \Theta(a, \ldots, t, u, v)$ and this u is in $I(\sigma)$

properties (a, b) holding with regard to all arguments taken from the argument set $a, \ldots, t, u, v, \ldots$ of Θ, is said to be Θ-uniquely locally soluble with respect to Ξ in R

(β) $\mathbb{UL}(\Theta, \Xi \mid R)$ is the class of such aggregates and $\mathbb{UL}(\Theta, \Xi \mid R, S)$ is the class of such aggregates in $\mathbb{AU}[R, S]$

(γ) When $\Theta(a, b) = a + b$, $\Xi(c) = c$ and $\sigma \in A'[R]$, so that when $a, c \in I(\sigma)$, the equation $a + b = c$ has a unique solution $b \in R$ and this b is in $I(\sigma)$, and when $b, c \in I(\sigma)$, the same holds with regard to the solution a of the same equation, $I(\sigma)$ is said to be additively uniquely locally soluble in R.

(5α) Let $\Theta : R \to J$, $\Omega : R \times J \to K$ and $\Xi : R \to K$.

?? An aggregate $\sigma \in A(R)$ for which $\cancel{\Omega(a,b)}\ \Omega[R \setminus I(\sigma), \Theta\{I(\sigma)\}] \subset K \setminus \Xi\{I(\sigma)\}$ ~~for all~~ $\cancel{a \in R \setminus I(\sigma)}$ ~~and~~ $\cancel{b \in \Theta\{I(\sigma)\}}$ is said to be Ω-saturated with respect to Θ, Ξ in (R, K)

(β) $\mathbb{S}(\Omega; \Theta, \Xi \mid R; K)$ is the class of such aggregates and $\mathbb{S}(\Omega; \Theta, \Xi \mid R, S; K)$ is the class of such aggregates in $\mathbb{UA}[R, S]$.

(γ) When $R \equiv J$, $\Omega(a, b) = ab$, $\Theta(a) \equiv \Xi(a) = a$ and $\sigma \in A'[R]$, so that for all $a \in R \setminus I(\sigma)$ and $b \in I(\sigma)$, $ab \notin I(\sigma)$, $I(\sigma)$ is said to be saturated in R.

(6) Let $\Theta : R \to J$, $\Omega : R\times J \to K$ and $\Xi : R \to K$.

(α) An aggregate $\sigma \in A(R)$ for which $\Omega[I(\sigma), \Theta\{I(\sigma)\}] \leq \Xi\{I(\sigma)\}$ is said to be a contractive Ω-system with respect to Θ, Ξ in R

(β) $\mathbb{CO}(\Omega; \Theta, \Xi | R)$ is the class of such aggregates and $\mathbb{CO}(\Omega; \Theta, \Xi | R, S)$ is the class of such aggregates in $\mathbb{U}A[R, S]$.

(γ) When $R \equiv J$, $\Omega(a, b) = ab$, $\Theta(a) \equiv \Xi(a) = a$ and $\sigma \in A'[R]$, so that $I(\sigma)^2 \leq I(\sigma)$, $I(\sigma)$ is said to be a multiplicative system in R.

(7) Let $\Theta, R \to J$, $\Omega : R\times J \to K$ and $\Xi : R \to K$.

(α) An aggregate $\sigma \in A(R)$ for which $\cancel{\Omega[I(\sigma), \Theta\{I(\sigma)\}]}$ $\cancel{I(\sigma)}$ $\Xi\{I(\sigma)\} \leq \Omega[I(\sigma), \Theta\{I(\sigma)\}]$ is said to be an expansive Ω-system with respect to Θ, Ξ in R.

(β) $\mathbb{E}(\Omega; \Theta, \Xi | R)$ is the class of such aggregates and $\mathbb{E}(\Omega; \Theta, \Xi | R, S)$ is the class of such aggregates in $\mathbb{U}A[R, S]$.

(γ) When $R \equiv J$, $\Omega(a, b) = ab$, $\Theta(a) \equiv \Xi(a) = a$ and $\sigma \in A'[R]$, so that $I(\sigma) \leq I(\sigma)^2$, $I(\sigma)$ is said to be an expansive multiplicative system in R.

(8) Let $\Theta, \Xi : R \to J$.

?? (α) An aggregate $\sigma \in A(R)$ for which $\Theta\{I(\sigma)\} \subseteq \Xi\{I(\sigma)\}$
[illegible] is said to be a Θ semi-ideal with respect to Ξ in R.

(β) $SI(\Theta, \Xi | R)$ is the class of such aggregates and $SI(\Theta, \Xi | R, S)$ is the class of such aggregates in ~~$A[R,S]$~~ $\sqcup A[R,S]$

(γ) When $\Theta(a) = Ra$, $\Xi(a) = a$ and $\sigma \in A'[R]$, so that $RI(\sigma) \subseteq I(\sigma)$, $I(\sigma)$ is said to be a semi-ideal in R.

The following two results offer definitions alternative to (1,2) above.

~~(i) Let $\Theta, \Xi : R \to J$. σ is Θ-closed with respect to Ξ in R if and only if~~

(i) Let $\Theta, \Xi : R \to J \mathbb{V}$ and $\sigma \in A$. Define the mapping $\Theta_{\Xi,\sigma}$ by setting $\Theta_{\Xi,\sigma}(x) = \Theta(x) \cap [J \setminus \Xi\{I(\sigma)\}]$. ~~The agg~~ σ is Θ closed with respect to Ξ in R if and only if $\Theta_{\Xi,\sigma} : R \setminus I(\sigma) \to \| J$.

If the largest system X for which $\Theta(X) \subseteq \Xi\{I(\sigma)\}$

satisfies the condition $X \subseteq I(\sigma)$, then $\Theta(x) \subseteq \Xi_i\{I(\sigma)\}$ only when $x \in I(\sigma)$, for if there is an $x \notin I(\sigma)$ for which $\Theta(x) \subseteq \Xi_i\{I(\sigma)\}$, then this x belongs to X and $X \not\subseteq I(\sigma)$. If, for the largest system X for which $\Theta(X) \subseteq \Xi_i\{I(\sigma)\}$, $X \not\subseteq I(\sigma)$, then an $x \notin I(\sigma)$ for which $\Theta(x) \subseteq \Xi_i\{I(\sigma)\}$ exists: $\sigma \in \mathfrak{G}(R,S;\Theta,\Xi)$ if and only if, for all $x \in R$, $\Theta(x) \subseteq \Xi_i\{I(\sigma)\}$ only when $x \in I(\sigma)$.

Assume that $\Theta(x) \subseteq \Xi_i\{I(\sigma)\}$ only when $x \in I(\sigma)$. Select $x \in R \setminus I(\sigma)$. $b \in \Theta(x)$ exists such that $b \notin \Xi_i\{I(\sigma)\}$, i.e. $b \in J \setminus \Xi_i\{I(\sigma)\}$: if $\Theta(x) \subseteq \Xi_i\{I(\sigma)\}$ only when $x \in I(\sigma)$, $\Theta(x) \cap [J \setminus \Xi_i\{I(\sigma)\}]$ is nonvoid for all $x \in R \setminus I(\sigma)$. Assume that $\Theta(x) \cap [J \setminus \Xi_i\{I(\sigma)\}]$ is nonvoid for all $x \in R \setminus I(\sigma)$. If $x \notin I(\sigma)$, $b \in \Theta(x)$ exists such that $b \notin \Xi_i\{I(\sigma)\}$, ~~and then $\Theta(x)$~~ i.e. $\Theta(x) \not\subseteq \Xi_i\{I(\sigma)\}$: if $\Theta(x) \cap [J \setminus \Xi_i\{I(\sigma)\}]$ is nonvoid for all $x \in R \setminus I(\sigma)$, $\Theta(x) \subseteq \Xi_i\{I(\sigma)\}$ only when $x \in I(\sigma)$.

~~( ) Let $\Theta: R \to \{J\}$ and $\Delta: R \to S$ and $\Xi_i: R \to \ldots$; then $\mathfrak{G}_\Delta(R,S;\Theta,\Xi_i) \in \mathfrak{C}^m(R,S)$.~~

~~Set $\mathfrak{G} = \mathfrak{G}_\Delta(R;\Theta,\Xi_i)$. Let $U \subseteq \mathfrak{G}[R,S]$ and $\omega = \cap U$. From ( ), $\sigma \in \mathfrak{G}$ if and only if $\Theta(x) \cap [J \setminus \Xi_i\{I(\sigma)\}]$ is nonvoid for all $x \in R \setminus I(\sigma)$. Select $x \in R \setminus I(\omega)$. Since $x \notin I(\omega)$ and, from ( ), $I(\omega) = \bigcap_{\sigma \in U} I(\sigma)$, there is a~~

(It is remarked that, since $\Theta(R) \subseteq J$, the mapping $\Theta_{\Xi,\sigma}$ is independent of J for all J ~~for which~~ occurring in the notation $\Theta: R \to J$.)

(ii) Let $\Theta, \Xi : R \to J$. ~~σ is~~ The aggregate σ is Θ-free with respect to Ξ in R if and only if $\Theta: R \setminus I(\sigma) \to J \setminus \Xi\{I(\sigma)\}$

Assume that for all $x \in R$, $\Theta(x) \cap \Xi\{I(\sigma)\}$ is nonvoid only when $x \in I(\sigma)$. For all $x \in R \setminus I(\sigma)$, $\Theta(x) \cap \Xi\{I(\sigma)\}$ is void, i.e. $\Theta(x) \subseteq J \setminus \Xi\{I(\sigma)\}$.
Assuming that $\Theta(x) \subseteq J \setminus \Xi\{I(\sigma)\}$ for all $x \in R \setminus I(\sigma)$, $\Theta(x) \cap \Xi\{I(\sigma)\}$ is void for all such x; $\Theta(x) \cap \Xi\{I(\sigma)\}$ is nonvoid only when $x \in I(\sigma)$.

(i) Let $\Theta, \Xi : R \to J$ and $\Theta', \Xi' : R' \to J$ with $R \subseteq R'$

~~and $\{\Theta', \Xi\} \subseteq_{(R)} \{$~~

~~and $\Theta'(x) \subseteq \Theta(x)$, $\Xi(x) \subseteq \Xi'(x)$ for all $x \in R$. If σ is Θ'-closed with respect to Ξ' in R' and $I(\sigma) \subseteq R$, σ is also Θ-closed with respect to Ξ in R~~

and $\{\Theta', \Xi\} \subseteq_{(R)} \{\Theta, \Xi'\}$. $A(R) \cap \mathbb{G}(\Theta', \Xi' \mid R) \subseteq \mathbb{G}(\Theta, \Xi \mid R)$.

If σ is Θ'-closed with respect to Ξ' in R', $I(\sigma) \subseteq R'$ and $\Theta'(x) \cap [J \setminus \Xi'\{I(\sigma)\}]$ is nonvoid for all $x \in R' \setminus I(\sigma)$, from ( ). If $R \subseteq R'$, $\Theta'(x) \subseteq \Theta(x)$ and $\Xi(x) \subseteq \Xi'(x)$ for all $x \in R$, $\Theta(x) \cap [J \setminus \Xi\{I(\sigma)\}]$ is nonvoid for all $x \in R \setminus I(\sigma)$. Hence if $I(\sigma) \subseteq R$, σ is Θ-closed with respect to Ξ in R.

(ii) Let $\Theta, \Xi : R \to J$ and $\Theta', \Xi' : R' \to J$ with $R \subseteq R'$

~~and $\Theta(x) \subseteq \Theta'(x)$, $\Xi(x) \subseteq \Xi'(x)$ for all $x \in R$. If σ is Θ'-free with respect to Ξ' in R' and $I(\sigma) \subseteq R$, σ is also Θ-free with respect to Ξ in R~~

and $\{\Theta, \Xi\} \subseteq_{(R)} \{\Theta', \Xi'\}$. $A(R) \cap \mathbb{R}(\Theta', \Xi' \mid R') \subseteq \mathbb{R}(\Theta, \Xi \mid R)$.

If σ is Θ'-free with respect to Ξ' in R', $I(\sigma) \subseteq R'$ and $\Theta'(x) \subseteq J \setminus \Xi'\{I(\sigma)\}$ for all $x \in R' \setminus I(\sigma)$, from ( ). If $R \subseteq R'$, $\Theta(x) \subseteq \Theta'(x)$ and $\Xi(x) \subseteq \Xi'(x)$ for all $x \in R$, $\Theta(x) \subseteq J \setminus \Xi\{I(\sigma)\}$ for all $x \in R \setminus I(\sigma)$. Hence if $I(\sigma) \subseteq R$, σ is Θ-free with respect to Ξ in R, from ( ) again.

~~(iii) Let $\Theta, \Xi : R \to J$ and $\Theta', \Xi' : R' \to J$ with $R \subseteq R'$~~

(iii) Let $\Theta, \Xi : R \to J$, $\Omega : R \times J \to K$, $\Xi : R \to K$, $\Theta' : R' \to J'$, $\Omega' : R' \times J' \to K$ and $\Xi' : R' \to K$, with $R \subseteq R'$, $J \subseteq J'$, ~~$\Theta'(a) \subseteq \Theta(a)$ and $\Xi(a) \subseteq \Xi'(a)$ for all $a \in R$ and $\Omega(a,b) \subseteq \Omega'(a,b)$ for all $a \in R, b \in J$.~~ If $I(\sigma) \subseteq R$ ~~and σ is Ω'-factored with respect to Θ', Ξ' in R', then σ is also Ω-factored with respect to Θ, Ξ in R~~ $\{\Theta', \Xi\} \subseteq_{(R)} \{\Theta, \Xi'\}$ and $\Omega \subseteq_{(K,J)} \Omega'$. $\mathbb{A}(R) \cap \mathbb{F}(\Omega'; \Theta', \Xi' \mid R', J') \subseteq \mathbb{F}(\Omega; \Theta, \Xi \mid R, J)$.

σ is Ω'-factored with respect to Θ', Ξ' in R' if and only if for all pairs a, b for which either $a \in R' \setminus I(\sigma)$ or $b \in J' \setminus \Theta'\{I(\sigma)\}$ (and possibly for some a, b for which both $a \in I(\sigma)$ and $b \in \Theta'\{I(\sigma)\}$) $\Omega'(a,b) \cap \Xi'\{I(\sigma)\}$ is void. If this condition holds and the stated relationships between $R, \ldots, \Omega'(a,b)$ hold then for all pairs a, b for which either $a \in R \setminus I(\sigma)$ or $b \in J \setminus \Theta\{I(\sigma)\}$, $\Omega(a,b) \cap \Xi\{I(\sigma)\}$ is void: σ is Ω-factored with respect to Θ, Ξ in R.

and, from ~~the preceding proof, $\Xi\{A\times B\times\ldots\times C,\sim\times D\times\ldots$~~
~~$\to Z$. From ( ) again, $\Xi: A\times B\times\ldots\times C.\sim\times D\ldots\to Z$.~~

—·— (Remark 2 pp. ...) ⇒

(iv) Let $\Theta': R'\times\ldots\times R'\sim\times R'\sim\times R'\sim\times\ldots\to J$,
$\Xi\subseteq(R)\Xi'$.
$\Xi: R\to J$, $\Xi': R'\to J$ with $R\subseteq R'$ and ~~$\Xi(a)\subseteq\Xi'(a)$~~
~~for all $a\in R$.~~ ~~Let σ be Θ' locally uniquely soluble with~~
~~respect to Ξ' in R' and $I(\sigma)\subseteq R$~~. If either

(1)(α) $\Theta: R\times\ldots\times R\times R\times R\times\ldots\to J$ and

(β) $\Theta'(a,\ldots,t,u,v,\ldots)\subseteq\Theta(a,\ldots,t,u,v,\ldots)$ for all
$a,\ldots,t,u,v,\ldots\in R$ or

(2)(α) $\Theta: R\times\ldots\times R\times R\times R\times\ldots\to J$ is such that for each set $S=\{a,b,\ldots,t,v,\ldots\}$ with $a,b,\ldots,t,v,\ldots\in R$ and each $x\in J$ a $u\in R$ exists for which $x\in\Theta(a,\ldots,t,u,v,\ldots)$, this solubility property holding with regard to all arguments of Θ, and

(β) $\Theta(a,\ldots,t,u,v,\ldots)\subseteq\Theta'(a,\ldots,t,u,v,\ldots)$ for all
$a,b,\ldots,t,u,v,\ldots\in R$

~~then Θ is locally uniquely soluble with respect to Ξ in R.~~ Then $A(R)\cap\mathbb{UL}(\Theta',\Xi'|R')\subseteq\mathbb{UL}(\Theta,\Xi|R)$.

σ' is Θ' uniquely locally soluble with respect to Ξ' in R' if and only if $I(\sigma)\subseteq R'$ and for ~~all~~ each set $a,b,\ldots,t,v,\ldots$ with $a,b,\ldots,t,v,\ldots\in I(\sigma)$ either (α) $\Theta'(a,\ldots,t,u,v,\ldots)$ is independent of u in R' or (β) corresponding to each $x\in\Xi'\{I(\sigma)\}$, R' contains a unique $u=u(x;a,\ldots,t,v,\ldots)$ for which $x\in\Theta'(a,\ldots,t,u,v,\ldots)$

and this u is in $I(\sigma)$, ~~property (α) or (β)~~ the property described holding with regard to all arguments and complementary argument sets of Θ'. Under the condition $\Theta': R' \times \ldots \times R'_\omega \times R \times R'_\omega \times \ldots \to J$, condition ($\alpha$) never obtains: condition (β) alone is operative.

~~Under the conditions of (1)~~, Select $a, \ldots, t, v, \ldots \in I(\sigma)$ and $x \in \Xi\{I(\sigma)\}$, so that $x \in \Xi'\{I(\sigma)\}$. ~~$a, \ldots, t, v, \ldots \in R$ and~~ For the proof of part (1), it is remarked that since ~~Since~~ Θ' is uniquely locally soluble with respect to Ξ' in R', R' contains u for which $x \in \Theta'(a, \ldots, t, u, v, \ldots)$ and ~~a unique u~~ this u is in $I(\sigma)$. Since $\Theta'(a, \ldots, t, u, v, \ldots) \subseteq \Theta(a, \ldots, t, u, v, \ldots)$, $x \in \Theta(a, \ldots, t, u, v, \ldots)$ and, since $I(\sigma) \subseteq R \subseteq R'$, $u \in R$: R contains u such that $x \in \Theta(a, \ldots, t, u, v, \ldots)$. Since $\Theta: R \times \ldots \times R \times R \times R \times \ldots \to J$, this u is unique in R; ~~and~~ or is also in $I(\sigma)$. The above holds for all selections $a, \ldots, t, v, \ldots \in I(\sigma)$ and with regard to all arguments u of Θ: σ is Θ uniquely u locally soluble with respect to Ξ in R.

For the proof of part (2) it is remarked that R contains u for which $x \in \Theta(a, \ldots, t, u, v, \ldots)$ ~~Since $x \in \Xi\{I(\sigma)\}$~~ and, since $\Theta(a, \ldots, t, u, v, \ldots) \subseteq \Theta'(a, \ldots, t, u, v, \ldots)$, $x \in \Theta'(a, \ldots, t, u, v, \ldots)$. Since $x \in \Xi'\{I(\sigma)\}$, σ is Θ' uniquely locally soluble with respect to Ξ' in R', and $\Theta': R'_\omega \times \ldots \times R'_\omega \times R'_\omega \times R'_\omega \ldots$, this u is unique in R' and is in $I(\sigma)$. Since $I(\sigma) \subseteq R \subseteq R'$, this u is unique in R and is in $I(\sigma)$: σ is Θ uniquely locally soluble with respect to Ξ in R

(v) Let $\Theta : R \to J$, $\Omega : R \times J \to K$, $\Xi : R \to K$ and $\Theta' : R' \to J$, $\Omega' : R' \times J \to K$, $\Xi' : R' \to K$ with $R \subseteq R'$, ~~$\Omega(a,b) \subseteq \Omega'(a,b)$ for all $a \in R$ and $b \in J$, $\Theta(a) \subseteq \Theta'(a)$ and $\Xi(a) \subseteq \Xi'(a)$ for all $a \in R$. If σ is Ω'-saturated with respect to Θ', Ξ' in R' and $I(\sigma) \subseteq R$, it is also Ω-saturated with respect to Θ, Ξ in R~~ $\{\Theta, \Xi\} \subseteq_{(R)} \{\Theta', \Xi'\}$ and $\Omega \subseteq_{(R,J)} \Omega'$. $A(R) \cap \mathfrak{S}(\Omega'; \Theta', \Xi' | R', K') \subseteq \mathfrak{S}(\Omega; \Theta, \Xi | R, K)$.
σ is Ω'-saturated with respect to Θ', Ξ' in R' if and only if $I(\sigma) \subseteq R'$ and $\Omega[R' \setminus I(\sigma), \Theta'\{I(\sigma)\}] \subseteq K \setminus \Xi'\{I(\sigma)\}$. Assuming this condition to hold, that $R \subseteq R'$, that $\Omega(a,b) \subseteq \Omega'(a,b)$ for all $a \in R$ and $b \in J$, and that $\Theta(a) \subseteq \Theta'(a)$, $\Xi(a) \subseteq \Xi'(a)$ for all $a \in R$, then $\Omega[R \setminus I(\sigma), \Theta\{I(\sigma)\}] \subseteq K \setminus \Xi\{I(\sigma)\}$. If $I(\sigma) \subseteq R$, σ is Ω-saturated with respect to Θ, Ξ in R.

(vi) Let $\Theta : R \to J$, $\Omega : R \times J \to K$, $\Xi : R \to K$ and $\Theta' : R' \to J$, $\Omega' : R' \times J \to K$, $\Xi' : R' \to K$ with $R' \subseteq R$, ~~$\Omega(a,b) \subseteq \Omega'(a,b)$ for all $a \in R$ and $b \in J$, $\Theta(a) \subseteq \Theta'(a)$ and $\Xi(a) \subseteq \Xi'(a)$ for all $a \in R$. If σ is a contractive Ω'-system with respect to Θ', Ξ' in R' it is also a contractive Ω-system with respect to Θ, Ξ in R.~~ $\{\Theta, \Xi\} \subseteq_{(R')} \{\Theta', \Xi'\}$ and $\Omega \subseteq_{(R',J)} \Omega'$. $\mathfrak{CO}(\Omega'; \Theta', \Xi' | R') \subseteq \mathfrak{CO}(\Omega; \Theta, \Xi | R)$.

σ is a contractive Ω'-system with respect to Θ', Ξ' in R' if and only if $I(\sigma) \subseteq R'$ and $\Omega'[I(\sigma), \Theta'\{I(\sigma)\}] \subseteq \Xi'\{I(\sigma)\}$. Assuming this condition to hold, that $R' \subseteq R$, that $\Omega(a,b) \subseteq \Omega'(a,b)$ for all $a \in R'$ and $b \in J$ and that $\Theta(a) \subseteq \Theta'(a)$, $\Xi'(a) \subseteq \Xi(a)$ for all $a \in R'$, then $I(\sigma) \subseteq R$ and $\Omega[I(\sigma), \Theta\{I(\sigma)\}] \subseteq \Xi\{I(\sigma)\}$: σ is a contractive Ω-system with respect to Θ, Ξ in R.

(vii) Let $\Theta : R \to J$, $\Omega : R \times J \to K$, $\Xi : R \to K$ and $\Theta' : R' \to J$, $\Omega' : R' \times J \to K$, $\Xi' : R' \to K$ with $R' \subseteq R$, ~~$\{\Theta', \Xi\} \leq_{(R')} \{\Theta, \Xi'\}$ and $\Omega' \leq \Omega$ i.e. $\Omega'(a,b) \subseteq \Omega(a,b)$ for all $a \in R'$ and $b \in J$, $\Theta'(a) \subseteq \Theta(a)$ and $\Xi(a) \subseteq \Xi'(a)$ for all $a \in R'$. If σ is an expansive Ω'-system with respect to Θ', Ξ' in R', it is also an expansive Ω-system with respect to Θ, Ξ in R.~~ $\{\Theta', \Xi\} \leq_{(R')} \{\Theta, \Xi'\}$ and $\Omega' \leq_{(R', J)} \Omega$. $\mathbb{E}(\Omega'; \Theta', \Xi' | R') \subseteq \mathbb{E}(\Omega; \Theta, \Xi | R)$.

σ is an expansive Ω'-system with respect to Θ', Ξ' in R' if and only if $I(\sigma) \subseteq R'$ and $\Xi'\{I(\sigma)\} \subseteq \Omega'[I(\sigma), \Theta'\{I(\sigma)\}]$. Taking into account the change in direction of the last inequality, the proof is as for that of ( ).

(viii) Let $\Theta, \Xi : R \to J$ and $\Theta', \Xi' : R' \to J$ with $R' \subseteq R$ ~~and $\Theta(x) \subseteq \Theta'(x)$, $\Xi'(x) \subseteq \Xi(x)$ for all $x \in R'$. If σ is a Θ' semi-ideal with respect to Ξ' in R', it is also a Θ-semi-ideal with respect to Ξ in R.~~ and $\{\Theta, \Xi'\} \leq_{(R')} \{\Theta', \Xi\}$. $S\mathbb{I}(\Theta', \Xi' | R') \subseteq S\mathbb{I}(\Theta, \Xi | R)$.

σ is a Θ' semi-ideal with respect to Ξ' in R' if and only if $I(\sigma) \subseteq R'$ and $\Theta'\{I(\sigma)\} \subseteq \Xi'\{I(\sigma)\}$. Assuming this condition to hold, that $R' \subseteq R$ and $\Theta(x) \subseteq \Theta'(x)$, $\Xi'(x) \subseteq \Xi(x)$ for all $x \in R'$, then $I(\sigma) \subseteq R$ and $\Theta\{I(\sigma)\} \subseteq \Xi\{I(\sigma)\}$: σ is a ~~semi~~ Θ semi ideal with respect to Ξ in R.

√Connections between aggregate classes

(2) Let ~~[illegible]~~ $\Omega : R\times J \to K$, $\Xi : R \to K$ and $\Theta', \Xi' : R' \to K$ with $R' \subseteq R$ and $\Xi' \subseteq_{(R')} \Xi$; ~~$\Xi'(a) \subseteq \Xi(a)$ for all $a \in R'$; let $\Theta' \subseteq \Omega$ and J be such that $\Theta'(a) \subseteq \Omega(a, J)$ for all $a \in R'$. If for any mapping $\Theta : R \to J$, σ is Ω-factored with respect to~~

With the mapping $\Omega_J : R' \to K$ defined by setting $\Omega_J(a) = \Omega(a, J)$ for all $a \in R'$, let $\Theta' \subseteq_{(R')} \Omega_J$. For any mapping $\Theta : R \to J$ ~~Ξ in R and $I(\sigma) \subseteq R'$, then σ is Θ'-free with respect to~~ $A(R') \cap F(\Omega; \Theta, \Xi \mid R; J) \subseteq R(\Theta', \Xi' \mid R$

Define the mapping $\hat{\Theta} : R \to J$ by setting $\hat{\Theta}(a) = J$ for all $a \in R$. Then for any mapping $\Theta : R \to J$, $\Theta(x) \subseteq \hat{\Theta}(x)$ for all $x \in R$ and, from ( ), if σ is Ω-factored with respect to Θ, Ξ in R, it is so with respect to $\hat{\Theta}, \Xi$ in R and, in this case, $\Omega(a,b) \cap \Xi\{I(\sigma)\}$ is nonvoid only when $a \in I(\sigma)$, $b \in J$. Assuming that $\Theta'(a) \subseteq \Omega(a, J)$, $\Xi'(a) \subseteq \Xi(a)$ for all $a \in R'$ where $I(\sigma) \subseteq R' \subseteq R$, it follows that if $\Theta'(a) \cap \Xi'\{I(\sigma)\}$ is nonvoid for some $a \in R'$ then $\Omega(a, J) \cap \Xi\{I(\sigma)\}$ is nonvoid for this a, i.e. $b \in J$ exists such that $\Omega(a,b) \cap \Xi\{I(\sigma)\}$ is nonvoid; this occurs, in particular, only when $a \in I(\sigma)$. The condition that

$\Theta'(a) \cap \Xi'\{I(\sigma)\}$ is, for all $a \in R'$, nonvoid only

?? when ~~$I(\sigma) \subseteq R'$ defines~~ $a \in I(\sigma)$ defines σ to be

Θ'-free with respect to Ξ' in R'.

(ii) Let $\Theta', \Xi' : R' \to J$ and $\Theta'', \Xi'' : R'' \to J$

with $R'' \subseteq R'$ and $\Theta'(x) \subseteq \Theta''(x)$, $\Xi''(x) \subseteq \Xi'(x)$

~~with $R'' \subseteq R'$ and $\{\Theta', \Xi''\} \subseteq_{(R'')}$~~

~~for all $x \in R''$. If σ is Θ'-free with respect~~

~~to Ξ' in R', then σ is Θ''-closed with respect~~

~~to Ξ'' in R''~~ with $R'' \subseteq R'$ and $\{\Theta', \Xi''\} \subseteq_{(R'')} \{\Theta'', \Xi'\}$.

$A(R'') \cap \mathbb{R}(\Theta', \Xi' | R') \subseteq \mathfrak{G}(\Theta'', \Xi'' | R'')$

Let σ be Θ'-free with respect to Ξ' in R'.

From ( ), $\Theta'(x) \subseteq [J \setminus \Xi'\{I(\sigma)\}]$ for all $x \in R' \setminus I(\sigma)$. But $R'' \subseteq R'$ and $\Theta''(x) \subseteq \Theta'(x)$, $\Xi''(x) \subseteq \Xi'(x)$ for all $x \in R''$ ~~[illegible]~~. Hence $\Theta''(x) \subseteq [J \setminus \Xi''\{I(\sigma)\}]$ for all $x \in R'' \setminus I(\sigma)$ and, in particular, $\Theta''(x) \cap [J \setminus \Xi''\{I(\sigma)\}]$ is nonvoid for all $x \in R'' \setminus I(\sigma)$. From ( ), σ is Θ''-closed with respect to Ξ'' in R''.

~~(The conditions $\Theta' : A \times B \times \ldots \times C_\circ \times D_\circ \times \ldots \to Z$ and~~
~~$\Theta(a, b, \ldots, c, d, \ldots) \subseteq \Theta'(a, b, \ldots, c, d, \ldots)$ alone do not suffice~~
~~to ensure that $\Theta : A \times B \times \ldots \times C_\circ \times D_\circ \times \ldots \to Z$. Take~~
~~$R = \{m, n\}$ and $\Theta'(m) \equiv \Theta'(n) = Z$, so that $\Theta' : R_\circ \to Z$.~~
~~Take $\Theta(m), \Theta(n) \subseteq Z$ with $X = \Theta(m) \cap \Theta(n)$ nonvoid.~~
~~When $x \in X$, $x \in \Theta(u)$ with $u = m$ or $u = n$.)~~

(iii)* Let $\Xi : R \to K$, $\Omega : R \times J \to K$, $R \subseteq R'$ and
$\Theta', \Xi' : R' \to K$ where ~~$\Xi'(a) \subseteq \Xi(a)$ for all $a \in R$ and~~
~~Ω, Θ' are such that $\Omega(a, J) \subseteq \Theta'(a)$ for all $a \in R$.~~
~~If σ is a Θ' semi-ideal with respect to Ξ' in R'~~
$\Xi' \subseteq_{(R)} \Xi$. With the mapping $\Omega_J : R \to K$ defined by setting
~~and $I(\sigma) \subseteq R$ then for any mapping $\Theta : R \to J$, σ is~~
$\Omega_J(a) = \Omega(a, J)$ for all $a \in R$, let $\Omega_J \subseteq_{(R)} \Theta'$. For any mapping
~~a contractive Ω system with respect to Θ, Ξ in R.~~
$\Theta : R \to J$, $A(R) \cap SI(\Theta', \Xi' | R) \subseteq CO(\Omega; \Theta, \Xi | R)$,
If σ is a Θ' semi-ideal with respect to Ξ' in R',
$\Theta'\{I(\sigma)\} \subseteq \Xi'\{I(\sigma)\}$. The conditions that $\Omega(a, J) \subseteq \Theta'(a)$
and $\Xi'(a) \subseteq \Xi(a)$ for all $a \in R$ then imply that
$\Omega\{I(\sigma), J\} \subseteq \Xi\{I(\sigma)\}$. Defining the mapping
$\hat{\Theta} : R \to J$ by setting $\hat{\Theta}(a) = J$ for all $a \in R$, it
then follows that $\Omega\{I(\sigma), \hat{\Theta}\{I(\sigma)\}\} \subseteq \Xi\{I(\sigma)\}$:
σ is a contractive Ω system with respect to $\hat{\Theta}, \Xi$
in R. For any mapping $\Theta : R \to J$, $\Theta(a) \subseteq \hat{\Theta}(a)$ for
all $a \in R$. From ( ), σ is a contractive Ω system
with respect to Θ, Ξ in R.

Definition (1) $\mathbb{P}$ being a class of aggregates and U a being set of aggregates, the notation $U \subseteq \mathbb{P}$ indicates that all members of U are in $\mathbb{P}$

(2) U being a set of aggregates $\mu, \rho, \ldots$, $|U|$ is the aggregate ω for which

(α) $I(\omega)$ is void when $I'(\omega) = \bigcap_{\sigma \in U} I(\sigma)$ is void and is otherwise the subset of $a \in I'(\omega)$ for which $M'(\omega, a) = \bigcap_{\sigma \in U} M(\sigma, a)$ is nonvoid and

(β) when $I(\omega)$ is nonvoid, $M(\omega, a) = M'(\omega, a)$ for each $a \in I(\omega)$.

($\subseteq A$??)

(i) Let $U, V \subseteq A[R,S]$. If $U \subseteq V$ then $|V| \leq |U|$

Let $\omega = |U|$. $I(\omega)$ is the subset of $a \in I'(\omega) = \bigcap_{\sigma \in U} I(\sigma)$ for which $M'(\omega, a) = \bigcap_{\sigma \in U} M(\sigma, a)$ is nonvoid, and $M(\omega, a) = M'(\omega, a)$ for each $a \in I(\omega)$. $\xi = |V|$ is similarly defined.

$I'(\xi) \subseteq I'(\omega)$. Since $M'(\xi, a) \subseteq M'(\omega, a)$, $M'(\omega, a)$ is nonvoid whenever $M'(\xi, a)$ is nonvoid: all $a \in I(\xi)$ belong to $I(\omega)$, and $M(\xi, a) \subseteq M(\omega, a)$ for all $a \in I(\xi)$: $\xi \leq \omega$.

(ii) Let $U, V \subseteq A[R,S]$. $|U \cup V| = |U| \wedge |V|$

Let $\tau = |U|$, $\kappa = |V|$, $\omega = \tau \wedge \kappa$, $T = U \cup V$ and $\xi = |T|$. $I(\xi)$ is the subset of $a \in I'(\xi) = \bigcap_{\sigma \in T} I(\sigma)$ for which $M'(\xi, a)$

$\bigcap_{\sigma\in T} M(\sigma,a)$ is nonvoid, and $M(\xi,a)=M'(\xi,a)$ for all $a\in I(\xi)$.

$I(\tau)$, $M(\tau,a)$ and $I(\kappa)$, $M(\kappa,a)$ are similarly defined.

$I'(\xi)=\{\bigcap_{\sigma\in U} I(\sigma)\}\cap\{\bigcap_{\sigma\in V} I(\sigma)\}$ and $M'(\xi,a)=\{\bigcap_{\sigma\in U} M(\sigma,a)\}\cap\{\bigcap_{\sigma\in V} M(\sigma,a)\}$. $I(\xi)$ and $I(\omega)$ are both the subsets of $a\in I'(\xi)=I'(\tau)\cap I'(\kappa)$ for which $M'(\xi,a)=M'(\tau,a)\cap M'(\kappa,a)$ is nonvoid and then $M(\xi,a)=M(\omega,a)=M'(\xi,a)$ for all $a\in I(\xi)=I(\omega)$: $\xi=\omega$.

(iii) Let $U,V\subseteq A$. ~~Then~~ $|U|\vee|V|\leq|U\cap V|$

Since $U\cap V\subseteq U$, $|U|\leq|U\cap V|$, from (i). Similarly $|V|\leq|U\cap V|$ and, from ( ), $|U|\vee|V|\leq|U\cap V|$. (That the relationship $|U|\vee|V|<|U\cap V|$ is possible is shown by taking $U=\{\mu,\rho\}$, $V=\{\xi,\rho\}$ where $\mu<\xi<\rho$. $|U|=\mu$ and $|V|=\xi$, so that, since $\mu<\xi$, $|U|\vee|V|=\xi$, from ( ). But $U\cap V=\rho=|U\cap V|$ and $\xi<\rho$.)

(iv) Let $U\subseteq A[R,S]$, $\omega=|U|$ and $\Xi: R\to J$.

(1) $\Xi\{I(\omega)\}\subseteq\bigcap_{\sigma\in U}\Xi\{I(\sigma)\}$

(2) Let $\Delta: R\to\| S$ and $U\subseteq A_{\Delta}[R,S]$. Then $\Xi\{I(\omega)\}=\bigcap_{\sigma\in U}\Xi\{I(\sigma)\}$.

(iv 1) Let $U \subseteq A[R,S]$ and $\omega = |U|$. $I(\omega) \subseteq \bigcap_{\sigma \in U} I(\sigma)$.

2) Let $\Delta : R \to\| S$, $U \subseteq A_{\Delta}[R,S]$ and $\omega = |U|$. $I(\omega) = \bigcap_{\sigma \in U} I(\sigma)$.

$I(\omega)$ is, by definition, the subset of $a \in I'(\omega) = \bigcap_{\sigma \in U} I(\sigma)$, for which $M'(\omega, a) = \bigcap_{\sigma \in U} M(\sigma, a)$ is nonvoid: the first result is already available. If $\Delta(a) \subseteq M(\sigma, a)$ for each $\sigma \in U$ and all $a \in I(\sigma)$, $M'(\omega, a)$ contains $\Delta(a)$ for all $a \in I'(\omega)$ and is accordingly nonvoid for all such a: $I(\omega) = I'(\omega)$.

(v) Let $U \subseteq A[R,S]$, $\omega = |U|$ and $\Xi : R \to J$. $\Xi\{I(\omega)\} \subseteq \bigcap_{\sigma \in U} \Xi\{I(\sigma)\}$

$I(\omega) \subseteq I(\sigma)$ for all $\sigma \in U$. Hence, from (.), $\Xi\{I(\omega)\} \subseteq \Xi\{I(\sigma)\}$ for all $\sigma \in U$ also, and $\Xi\{I(\omega)\} \subseteq \bigcap_{\sigma \in U} \Xi\{I(\omega)\}$.

(vi) Let $\Delta : R \to\| S$, $U \subseteq A_{\Delta}[R,S]$, $\omega = |U|$ and $\Xi : R. \to J$. $\Xi\{I(\omega)\} = \bigcap_{\sigma \in U} \Xi\{I(\sigma)\}$.

Set $A = \Xi\{I(\omega)\}$ and $B = \bigcap_{\sigma \in U} \Xi\{I(\sigma)\}$. From (, iv), $A \subseteq B$. Select $b \in B$. With $U = \{\sigma, \mu, \dots\}$, $b \in \Xi(a)$ for some $a \in I(\sigma)$, $b \in \Xi\{a'\}$ for some $a' \in I(\mu), \dots$. But $\Xi : R. \to J$ and the conditions $b \in \Xi(a)$, $b \in \Xi(a'), \dots$ imply that $a = a' = \dots$: $a \in \bigcap_{\sigma \in U} I(\sigma)$. But $U \subseteq A_{\Delta}[R,S]$ so that, from (iv 2), $I(\omega) = \bigcap_{\sigma \in U} I(\sigma)$:

n. 3 Complete classes of aggregates.

Definition. (1) A class $\mathbb{P}$ of aggregates for which $|U| \in \mathbb{P}$ for all aggregate sequences $U \in \mathbb{P}[R,S]$ is said to be complete in (R,S). The notation $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$ indicates that the class $\mathbb{P}$ is complete in (R,S).

(2) The notation $\mathbb{P} \in \mathbb{C}(R,S)$ indicates that $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$ and $1_{R,S} \in \mathbb{P}[R,S]$.

(i) With $R \subseteq R'$ and $S \subseteq S'$, let $\mathbb{P} \in \overline{\mathbb{C}}(R',S')$. $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$

If $|U| \in \mathbb{P}$ for all $U \subseteq \mathbb{P}[R',S']$ then $|U| \in \mathbb{P}$ for, in particular, all $U \subseteq \mathbb{P}[R,S]$.

(ii) Let $\mathbb{P} \in \overline{\mathbb{C}}(R',S')$ and $\sigma_{R,S} \in$ $\mathbb{P}[R',S']$ contains σ for which $M(\sigma,a) = S$ for all $a \in I(\sigma) = R$. $\mathbb{P} \in \mathbb{C}(R,S)$.

Since $\sigma \in \mathbb{P}[R',S']$, $R \subseteq R'$ and $S \subseteq S'$ so that, from (i) $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$. Within $\mathbb{P}[R,S]$, σ is $1_{R,S}$. Since $\mathbb{P}[R,S]$ contains this σ, $\mathbb{P} \in \mathbb{C}(R,S)$.

~~(iii) Let $\mathbb{P}', \mathbb{P}'' \in \overline{\mathbb{C}}(R,S)$ and $\mathbb{P}' \cap \mathbb{P}'' \in \overline{\mathbb{C}}(R,S)$~~ $\langle \mathbb{C}(R,S) \rangle$

$U \subseteq \mathbb{P}', \mathbb{P}''$ for all sets of aggregates in $\mathbb{P} = \mathbb{P}' \cap \mathbb{P}''$. Since $\mathbb{P}', \mathbb{P}'' \in \overline{\mathbb{C}}(R,S)$, $|U| \in \mathbb{P}' \cap \mathbb{P}''$ for all such U: $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$. Concerning the bracketed result, it is remarked that

(iii) Let $\Delta : R \to \| S$, (A) If $\mathbb{P} \in \overline{\mathbb{C}}(R,S) \langle \mathbb{C}(R,S) \rangle$ then $\mathbb{P}_\Delta \in \overline{\mathbb{C}}(R,S) \langle \mathbb{C}(R,S) \rangle$

For all $U \subseteq \mathbb{P}_\Delta[R,S]$, $U \subseteq \mathbb{P}[R,S]$ and, since $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$, $\omega = |U| \in \mathbb{P}$. Also $\Delta(a) \subseteq M(\omega, a)$ for all ~~$\in$~~ $a \in I(\omega)$, from the proof of ( ): $\omega \in \mathbb{P}_\Delta$ and $\mathbb{P}_\Delta \in \overline{\mathbb{C}}(R,S)$. If $1_{R,S} \in \mathbb{P}[R,S]$ then, since $\Delta(a) \subseteq S$ for all $a \in R$, $1_{R,S} \in \mathbb{P}_\Delta[R,S]$ also: $\mathbb{P}_\Delta \in \mathbb{C}(R,S)$.

(iv) Let ~~$\mathbb{P} \in \mathbb{P}[R'', S'']$~~ $\mathbb{P} \in \mathbb{U}A(R'', S'') \cap \overline{\mathbb{C}}(R', S')$, where $R' \subseteq R''$, $S' \subseteq S''$. Let $S \subseteq S'$ and σ be any member of $\mathbb{P}[R', S']$, with $I(\sigma) = R$.

(1) $\mathbb{P} \in \mathbb{C}(R,S)$

(2) Let $\Delta : R \to \| S$. $\mathbb{P}_\Delta \in \mathbb{C}(R,S)$.

From ( ), $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$, since $R \subseteq R'$, $S \subseteq S'$. Also, from ( ), $1_{R,S} \in \mathbb{P}[R,S]$: $\mathbb{P} \in \mathbb{C}(R,S)$. The result of (2) follows from the bracketed result of (iii).

(v) Let $\mathbb{P}', \mathbb{P}'' \in \overline{\mathbb{C}}(R,S) \langle \mathbb{C}(R,S) \rangle$. $\mathbb{P}' \cap \mathbb{P}'' \in \overline{\mathbb{C}}(R,S) \langle \mathbb{C}(R,S) \rangle$.

$U \subseteq \mathbb{P}', \mathbb{P}''$ for all sets U of aggregates in $\mathbb{P} = \mathbb{P}' \cap \mathbb{P}''$. Since $\mathbb{P}', \mathbb{P}'' \in \overline{\mathbb{C}}(R,S)$, $|U| \in \mathbb{P}' \cap \mathbb{P}''$ for all such U: $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$. Concerning the bracketed result, it is remarked that $1_{R,S}$ belongs to both $\mathbb{P}'[R,S]$ and $\mathbb{P}''[R,S]$ and therefore to $\mathbb{P}[R,S]$.

(vi$\frac{1}{2}$) Let $\mathbb{P}', \mathbb{P}'' \in \overline{\mathbb{C}}(R,S)$. Assume either that for all σ such that $\sigma \leq \mu \in \mathbb{P}'$, $\sigma \in \mathbb{P}'$ also, or that the class $\mathbb{P}''$ possesses this property. $\mathbb{P}' \cup \mathbb{P}'' \in \overline{\mathbb{C}}(R,S)$.
(2) If, in addition, either $\mathbb{P}' \in \mathbb{C}(R,S)$ or $\mathbb{P}'' \in \mathbb{C}(R,S)$, then $\mathbb{P}' \cup \mathbb{P}'' \in \mathbb{C}(R,S)$

Set $\mathbb{P} \in \mathbb{P}' \cup \mathbb{P}''$. Any set $T \subseteq \mathbb{P}[R,S]$ has a decomposition of the form $T \equiv U \uplus V$ where $U \in \mathbb{P}'[R,S]$ and $V \subseteq \mathbb{P}''[R,S]$. From ( ), $|T| = |U| \wedge |V| \leq |U|, |V|$. If $\mathbb{P}'$ possesses the property described, $|T| \in \mathbb{P}' \subseteq \mathbb{P}$ since $|T| \leq |U|$; if $\mathbb{P}''$ possesses the stated property, $|T| \in \mathbb{P}$ also. Thus $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$.

If $\mathbb{P}' \in \mathbb{C}(R,S)$, then $1_{R,S} \in \mathbb{P}'[R,S] \subseteq \mathbb{P}[R,S]$: $\mathbb{P} \in \mathbb{C}(R,S)$. If $\mathbb{P}'' \in \mathbb{C}(R,S)$, $\mathbb{P} \in \mathbb{C}(R,S)$ similarly.

(vii) Let $\mathbb{P}', \mathbb{P}''$ be two ~~classes of~~ aggregate classes for which $\mathbb{P}'' \subseteq \mathbb{P}'$, $\mathbb{P}' \in \overline{\mathbb{C}}(R,S)$ and $\mathbb{P}'' \in \mathbb{C}(R,S)$. $\mathbb{P}' \in \mathbb{C}(R,S)$.

If $\mathbb{P}'' \in \mathbb{C}(R,S)$, $1_{R,S} \in \mathbb{P}''[R,S]$; ~~and~~ if $\mathbb{P}'' \subseteq \mathbb{P}'$, $1_{R,S} \in \mathbb{P}'[R,S]$ also: $\mathbb{P}' \in \mathbb{C}(R,S)$.

(i1) $A \in \mathbb{C}(R,S)$.

(2) Let $\Delta : R \to \| S$. $A_\Delta \in \mathbb{C}(R,S)$

For all $U \subseteq A[R,S]$, $\omega = | U | \in A[R,S]$: $A \in \overline{\mathbb{C}}(R,S)$. $1_{R,S}$ is present in $A[R,S]$. Part (2) is a corollary to ( ).

(ii1) Let $\Phi : R^2 \times S^2 \to T$, ~~class~~ $\Theta : R \times S \times S. \to T$ and $\mathbb{P}$ be the class of aggregates σ for which

(*) $\Phi\{a, I(\sigma), b, M(\sigma,a)\} \subseteq \Theta\{a, b, M(\sigma,a)\}$

for each $a \in I(\sigma)$ and all $b \in M(\sigma,a)$.

(α) $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$.

(β) If in addition $\Theta(a, b, S) = T$ for all $a \in R$, ~~$b \in S$~~ and $b \in S_a$, then $\mathbb{P} \in \mathbb{C}(R,S)$.

(2) Let $\Phi : R^2 \times S^3 \to T$, $\Theta : R \times S \times (S.)^2 \to T$, $\Upsilon : R. \to S$ and $\mathbb{P}$ be the class of aggregates σ for which

(**) $\Phi[a, I(\sigma), b, M(\sigma,a), M\{\sigma, \Upsilon^{-1}(b)\}] \subseteq \Theta[a, b, M(\sigma,a), M(\sigma, \Upsilon^{-1}(b))]$

for each $a \in I(\sigma)$ and all $b \in \Upsilon\{I(\sigma)\} \cap M(\sigma,a)$.

(α) $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$

(β) If, in addition $\Theta(a, b, S, S) = T$ for all $a \in R$ and $b \in S_a$, then $\mathbb{P} \in \mathbb{C}(R,S)$.

Part (2) is slightly the more complicated of the two results and is therefore proved in detail. It is first remarked that

When $\Upsilon: R \to S$ and b belongs to a subset of $\Upsilon\{I(\sigma)\}$, $a' = \Upsilon^{-1}(b)$ belongs to a subset of $I(\sigma)$, from ( ). $M(\sigma, a')$ is defined for all $a' \in I(\sigma)$. The constituents of relationship (※) are well defined for $a \in I(\sigma)$ and $b \in \Upsilon\{I(\sigma)\} \cap M(\sigma, a)$. Select $U \subseteq \mathbb{P}[R, S]$; let $\omega = |U|$ and select any $a \in I(\omega)$, $b \in \Upsilon\{I(\omega)\} \cap M(\omega, a)$. Select $f \in \Phi[a, I(\omega), b, M(\omega, a), M\{\omega, \Upsilon^{-1}(b)\}]$; $c \in I(\omega)$, $d \in M(\omega, a)$ and $e \in M\{\omega \circ \Upsilon^{-1}(b)\}$ exist such that $f \in \Phi(\cancel{a, b, c, a,}\ a, c, b, d, e)$. Since $a \in I(\omega)$, $a \in I(\sigma)$ for all $\sigma \in U$. Since $I(\omega) \subseteq I(\sigma)$ for all $\sigma \in U$, $\Upsilon\{I(\omega)\} \subseteq \Upsilon\{I(\sigma)\}$ for all $\sigma \in U$: $b \in \Upsilon\{I(\sigma)$ for all

$c \in \Upsilon\{I(\sigma)\} \cap M(\sigma, a)$ for all $\sigma \in U$. Since all $\sigma \in U$

~~are in P, corresponding to each $\sigma \in U$, $c(\sigma) \in M(\sigma, a)$ and $d(\sigma) \in M$~~

$\sigma \in U$. Since $b \in M(\omega, a)$, $b \in M(\sigma, a)$ for all $\sigma \in U$ also: $b \in \Upsilon\{I(\sigma)\} \cap M(\sigma, a)$ for all $\sigma \in U$. Similarly $c \in I(\sigma)$ and $d \in M(\sigma, a)$ for all $\sigma \in U$. Since $b \in \Upsilon\{I(\omega)\} \subseteq \Upsilon\{I(\sigma)\}$

~~$a' = \Upsilon^{-1}(b)$ is in $I(\omega)$ and therefore in $I(\sigma)$ for all $\sigma \in U$~~

$a' = \Upsilon^{-1}(b) \in I(\sigma)$ for all $\sigma \in U$. Since $e \in M(\omega, a')$, $e \in M(\sigma, a')$ for all $\sigma \in U$. Since all $\sigma \in U$ are in P, corresponding to each $\sigma \in U$, $c(\sigma) \in M(\sigma, a)$ and $d(\sigma) \in M(\sigma, \Upsilon^{-1}(b))$ exist such that $f \in \Theta\{a, b, c(\sigma), d(\sigma)\}$.

Assuming Θ not to be independent of its last two arguments; only one $c' \in S$ and one $d' \in S$ exist for which $f \in \Theta(a, b, c', d')$ when $f \in \Theta\{a, b, c(\sigma), d(\sigma)\}$ for one pair $c(\sigma), d(\sigma) \in S$: $c(\sigma) = c'$ and $d(\sigma) = d'$ for all ~~$\sigma \in U$. Since $M(\omega, a)$ is nonvoid, $M(\omega, a) = \bigcap_{\sigma \in U} M(\sigma, a)$~~ $\sigma \in U$, and $c' = c(\sigma) \in M(\sigma, a)$, $d' = d(\sigma) \in M(\sigma, \Upsilon^{-1}(b))$ for all $\sigma \in U$. Since $M(\omega, a)$ is nonvoid, $M(\omega, a) = \bigcap_{\sigma \in U} M(\sigma, a)$ and $c' \in M(\omega, a)$. Similarly $d' \in M\{\omega, \Upsilon^{-1}(b)\}$: $f \in \Theta[a, b, M(\omega, a), M\{\sigma, \Upsilon^{-1}(b)\}]$. ω also belongs to P.

The aggregates of U are now partitioned into four mutually ~~exclusive~~ disjoint subsets $U_1, \dots, U_4$, membership of a $\sigma \in U$ in one of the subsets being decided by the numbers $c(\sigma), d(\sigma)$. U_4 is the set of $\sigma \in U$ for which (α) $\Theta\{a, b, c', d(\sigma)\}$ is independent of $c' \in S$ and ~~(β)~~ also (β) $\Theta\{a, b, c(\sigma), d'\}$ is independent of $d' \in S$. U_3 is the set of $\sigma \in U$ for which (α) is true but (β) false. U_2 is determined by the converse assertion. For the aggregates σ of U_1, both (α, β) are false. ~~Assume that the set U_1 is nonvoid.~~ Since $\Theta : R \times S \times (S.)^2$ it follows that, with $\sigma, \mu \in U_1$, the two conditions $f \in \Theta\{a, b, c(\sigma), d(\sigma)\}$, $f \in \Theta\{a, b, c(\mu), d(\mu)\}$ imply that $c(\sigma) = c(\mu)$, $d(\sigma) = d(\mu)$: for all $\sigma \in U_1$, $c(\sigma) = c_1$ and $d(\sigma) = d_1$, where c_1 and d_1 are fixed members of S. For the aggregates $\sigma \in U_2$, $\Theta\{a, b, c(\sigma), d'\}$ is independent of $d' \in S$: $\Theta\{a, b, c(\sigma), d(\sigma)\} = \Theta\{a, b, c(\sigma), d_1\}$: ~~$d'$ may be replaced by d_1~~: for the aggregates σ of U_2, $f \in \Theta\{a, b, c(\sigma), d_1\}$. Since $\Theta : R \times S \times (S.)^2$ and it is known that $f \in \Theta\{a, b, c_1, d_1\}$, $c(\sigma)$ may ~~be~~ have ~~assume~~ only one value in S, namely c_1: $c(\sigma) = c_1$ for all $\sigma \in U_2$. With a, b, c prescribed, ~~whether~~ the invariance of $\Theta(a, b, c, d')$ with respect to d' is decided by a, b and c. The functions $\Theta(a, b, c(\sigma), d')$ with $\sigma \in U_2$ are invariant with respect to d'; those with $\sigma \in U_1$ are not. But the values of $c(\sigma)$ in both cases are all the same, being equal to ~~namely~~ c_1. ~~It was assumed initially that U_1 is nonvoid~~ ~~If U_2~~

U_1 and U_2 cannot both be nonvoid. Similar reasoning establishes that of the sets $U_1, \ldots, U_4$ only one is nonvoid, so that either $U = U_1$ or ... or $U = U_4$.

Assume that U_1 is nonvoid, so that $c(\sigma) = c_1$ and $d(\sigma) = d_1$ for all $\sigma \in U$ and in consequence $c_1 = c(\sigma) \in M(\sigma, a)$, $d_1 = d(\sigma) \in M(\sigma, \Upsilon^{-1}(b))$ for all $\sigma \in U$. Since $M(\omega, a)$ is nonvoid, $M(\omega, a) = \bigcap_{\sigma \in U} M(\sigma, a)$ and $c_1 \in M(\omega, a)$. Similarly $d_1 \in M\{\omega, \Upsilon^{-1}(b)\}$: $f \in \Theta\{a, b, M(\omega, a), M\{\omega, \Upsilon^{-1}(b)\}\}$; ω also belongs to $\mathbb{P}$.

Assuming that U_2 is nonvoid, it is shown again that $c_1 \in M(\omega, a)$, so that now $f \in \Theta\{a, b, c_1, d'\}$ where $\Theta\{a, b, c_1, d'\}$ is invariant with respect to d': d' may be replaced by any member d_1 of $M\{\omega, \Upsilon^{-1}(b)\}$. Once again ω also belongs to $\mathbb{P}$. The cases in which U_3 and U_4 are nonvoid are dealt with similarly.

Under the assumption concerning Θ made in part (2,3),

$$\Phi(a, R, b, S, S) \subseteq T = \Theta(a, b, S, S)$$

for all $a \in R$ and $b \in S$; in particular for each $a \in R$ and all $b \in \Upsilon(R) \cap S$. $\mathbb{P}[R, S]$ contains an aggregate σ for which $I(\sigma) = R$ and $M(\sigma, a) = S$ for each $a \in I(\sigma)$: $1_{R,S} \in \mathbb{P}[R, S]$; $\mathbb{P} \in \mathbb{C}(R, S)$.

It follows as a simple corollary to the above result that (taking $\Phi(a,c,b,d)=c$, $\Theta(a,b,f)=f$ in (1)) the class of aggregates σ for which $I(\sigma)\subset M(\sigma,a)$ for all $a\in I(\sigma)$ is complete in (R,R). Again (taking $\Phi(a,c,b,d)=ad$, $\Theta(a,b,f)=f$ ~~in (1)~~ and $RS\subseteq S$ in (1)) the class of aggregates σ for which $aM(\sigma,a)\subset M(\sigma,a)$ for all $a\in I(\sigma)$ is complete in (R,S), as is also that in (R,R) for which for each $a\in I(\sigma)$ and all $b\in I(\sigma)$ for which $b\in M(\sigma,a)$, $bM(\sigma,b)\subseteq M(\sigma,b)$.

Unless auxiliary assumptions are introduced, the argument sets $a,b,M(\sigma,a)$ and $a,b,M(\sigma,a),M\{\sigma,\Upsilon^{-1}(b)\}$ of $\Phi\ \Theta$ in (*, **) of (ii) may not be extended to $a,I(\sigma),b,M(\sigma,a)$ and $a,I(\sigma),b,M(\sigma,a),M\{\sigma,\Upsilon^{-1}(b)\}$ respectively. Indeed, the results obtained by such extension are false. (A special case of such an extended result is that the class of aggregates σ ~~in (R,R)~~ for which $M(\sigma,a)\subseteq I(\sigma)$ for all $a\in I(\sigma)$ is complete in (R,R). Defining σ by $I(\sigma)=\{a,b\}$, $M(\sigma,a)=\{b\}$, $M(\sigma,b)=\{a,b\}$ and μ by $I(\mu)=\{a,b\}$, $M(\mu,a)=\{a\}$, $M(\mu,b)=\{a,b\}$ and taking $U=\{\sigma,\mu\}$, $\omega=|U|$ is the aggregate for which $I(\omega)=b$, $M(\omega,b)=\{a,b\}$:

$M(\sigma,a) \subseteq I(\sigma)$ for all $a \in I(\sigma)$, $M(\mu,a) \subseteq I(\mu)$ for all $a \in I(\mu)$, but $M(\omega,a) \nsubseteq I(\omega)$ for all $a \in I(\omega)$.

(iii + proof, foll p. ; end of proof:)

$\to I(\omega)$: $\omega \in \mathbb{P}_\Delta[R,S]$ and $\mathbb{P}_\Delta \in \overline{\mathbb{C}}(R,S)$. That, subject to the additional condition stated in (2β), $\mathbb{P}_\Delta \in \mathbb{C}(R,S)$, is proved as in (ii).

From the result (2) above, it follows, for example, that, $\mathbb{P}$ being the class of aggregates σ for which $M(\sigma,a) \subseteq I(\sigma)$ for all $a \in I(\sigma)$, $\mathbb{P}_\Delta \in \mathbb{C}(R,S)$, and that for the ~~further~~ subclass $\mathbb{P}'$ of $\mathbb{P}$ for which for each $a \in I(\sigma)$ and each $b \in M(\sigma,a)$, $M(\sigma,b) \subseteq M(\sigma,a)$, $\mathbb{P}'_\Delta \in \mathbb{C}(R,S)$ also.

(iv) Let $\Theta, \Xi : R \to J$ and $\Delta : R \to \| S$. $\mathbb{G}_\Delta(\Theta,\Xi|R,S) \in \mathbb{C}(R,S)$.

Set $\mathbb{G} = \mathbb{G}_\Delta(\Theta,\Xi|R,S)$. Let $U \subseteq \mathbb{G}[R,S]$ and $\omega = |U|$.

From ( ), $\sigma \in \mathbb{G}$ if and only if $\Theta(x) \cap [J \setminus \Xi\{I(\sigma)\}]$ is nonvoid for all $x \in R \setminus I(\sigma)$. Select $x \in R \setminus I(\omega)$. Since $x \notin I(\omega)$ and, from ( ), $I(\omega) = \bigcap_{\sigma \in U} I(\sigma)$, there is a

(2pp. m)

(iii 1) Let $\Phi : R^2 \times S^2 \to T$, ~~and~~ $\Theta : R \times R_\circ \times S \times S_\circ \to T$, ~~be~~ map $\Delta : R \to S$,
and $\mathbb{P}$ be the class of aggregates $\in \mathbb{A}^*(R,S)$ for which

$$\Phi\{a, I(\sigma), b, M(\sigma,a)\} \subseteq \Theta\{a, I(\sigma), b, M(\sigma,a)\}$$

for each $a \in I(\sigma)$ and all $b \in M(\sigma, a)$. ~~Then~~ (α) $\mathbb{P}_\Delta \in \overline{\mathbb{C}}(R,S)$.

(β) If in addition $\Theta(a, R, b, S) \equiv T$ for all $a \in R$ and $b \in S$,

~~$\mathbb{P}(R,S)$~~ $\mathbb{P}_\Delta \in \mathbb{C}^W(R,S)$

(2b) Let $\Phi : R^2 \times S^3 \to T$, $\Theta : R \times R_\circ \times S \times (S_\circ)^2 \to T$, ~~and~~ $\Delta : R \to S$ and
$\Upsilon : R_\circ \to S$, ~~and~~ $\mathbb{P}$ be the class of aggregates $\in \mathbb{A}(R,S)$

for which

$$\Phi[a, I(\sigma), b, M(\sigma,a), M\{\sigma, \Upsilon^{-1}(b)\}] \subseteq \Theta[a, I(\sigma), b, M(\sigma,a), M\{\sigma, \Upsilon^{-1}(b)\}]$$

for each $a \in I(\sigma)$ and all $b \in \Upsilon\{I(\sigma)\} \cap M(\sigma,a)$. ~~Then~~ (α) $\mathbb{P}_\Delta(R,S) \in \overline{\mathbb{C}}(R,S)$.

(β) If in addition $\Theta(a, R, b, S, S) \equiv T$ for all $a \in R$ and $b \in S$,

$\mathbb{P}_\Delta \in \mathbb{C}^W(R,S)$.

Part (2), slightly the more complicated of the two results, is dealt with.

As in the proof of (ii), it is shown that with ~~$a \in I$~~ $U \subseteq \mathbb{P}_\Delta[R,S]$ and $\omega = |U|$, and $a \in I(\omega)$, $b \in \Upsilon\{I(\omega)\} \cap M(\sigma, a)$ any $f \in \Phi[a, I(\omega), b, M(\omega,a), M\{\omega, \Upsilon^{-1}(b)\}]$ satisfies the relationship $f \in \Theta[a, e', b, c', d']$ where, as before, $c' \in M(\omega, a)$ and $d' \in M\{\omega, \Upsilon^{-1}(b)\}$ and now $e' \in \bigcap_{\sigma \in U} I(\sigma)$. $\boxed{I'(\omega) =}$ prev. p.
Since $U \subseteq \mathbb{P}_\Delta[R,S]$, ~~we~~ $I(\omega) = \bigcap_{\sigma \in U} I'(\omega)$ and accordingly $e \in$

$\mu\in U$ for which $x\notin I(\mu)$ and, since this μ is in $\mathfrak{G}$, (6)

$\Theta(x)\cap[J\setminus\Xi_1\{I(\mu)\}]$ is nonvoid for the x in question.

But, from ( ), $\Xi_1\{I(\omega)\}\subseteq\bigcap_{\sigma\in U}\Xi_1\{I(\sigma)\}$ so that

$J\setminus\bigcap_{\sigma\in U}\Xi_1\{I(\sigma)\}\equiv\bigcup_{\sigma\in U}[J\setminus\Xi_1\{I(\sigma)\}]\subseteq J\setminus\Xi_1\{I(\omega)\}$ and

$$\Theta(x)\cap[J\setminus\Xi_1\{I(\mu)\}]\subseteq\Theta(x)\cap\Big[\bigcup_{\sigma\in U}[J\setminus\Xi_1\{I(\sigma)\}]\Big]$$
$$\subseteq\Theta(x)\cap[J\setminus\Xi_1\{I(\omega)\}]$$

Accordingly $\Theta(x)\cap[J\setminus\Xi_1\{I(\omega)\}]$ is nonvoid for all $x\in R\setminus I(\omega)$: $\omega\in\mathfrak{G}$ and $\mathfrak{G}\in\overline{\mathfrak{C}}(R,S)$. ~~$M(\sigma,a)=S$ for all $a\in I(\sigma)$~~

The aggregate σ for which $I(\sigma)\equiv R$, ~~set R is associated by default with an~~ is ~~belongs~~ by default in $[R,S]$ ~~to~~ $\mathfrak{G}$: ~~since In this case~~ the largest system $X\subseteq R$ satisfying the condition $\Theta(X)\subseteq\Xi_1\{I(\sigma)\}$ is indeed in $I(\sigma)\equiv R$: ~~Also, the aggr since $\Delta(a)\subseteq S$ for all~~ $\sigma\in\mathfrak{G}(R,S)$. From ( ), $\mathfrak{G}\in\mathfrak{C}(R,S)$ (it is remarked parenthetically that if $R'\subseteq R$, $S\subseteq S'$ then $\mathfrak{G}(R,S;\Theta,\Xi)\subseteq\mathfrak{G}(R',S';\Theta,\Xi)$)
~~$a\in R$: $\sigma\in\mathfrak{G}[R,S]$: $\mathfrak{G}\in\mathfrak{C}(R,S)$.~~
The ~~free system~~ closure

Definition (1) Let $\Theta,\Xi_1:R\to J$. An aggregate of $\mathfrak{A}(R,S)$ for which $I(\sigma)\subseteq R$ and ~~$\Theta(a)\cap\Xi_1\{I(\sigma)\}$ is~~, for all $a\in R$, $\Theta(a)\cap\Xi_1\{I(\sigma)\}$ is nonvoid only when $a\in I(\sigma)$ is said to Θ-free with respect to Ξ_1 in R. $\mathfrak{R}^{(2)}(R,S;\Theta,\Xi_1)$ is the class of such aggregates in $\mathfrak{A}[R,S]$.

(3) When $\Theta(a)=a^2$, $\Xi_1(a)=a$ and $a\in A'[R]$, so that

for all $a\in R$, $a^2\in I(\sigma)$ only when $a\in I(\sigma)$, $I(\sigma)$ is said to be square free in R.

( ) Let $\Theta, \Xi : R \to J$. σ is Θ-free with respect to Ξ_1 in $\mathbb{R}$ ~~$\sigma\in\mathbb{R}(R;\Theta,\Xi_1)$~~ if and only if $\Theta(x)\subseteq J\setminus \Xi_1\{I(\sigma)\}$ for all $x\in R\setminus I(\sigma)$.

Assume that for all $x\in R$, $\Theta(x)\cap \Xi_1\{I(\sigma)\}$ is nonvoid only when $x\in I(\sigma)$. For all $x\in R\setminus I(\sigma)$, $\Theta(x)\cap \Xi_1\{I(\sigma)\}$ is void, i.e. $\Theta(x)\subseteq J\setminus \Xi_1\{I(\sigma)\}$. Assuming that $\Theta(x)\subseteq J\setminus \Xi_1\{I(\sigma)\}$ for all $x\in R\setminus I(\sigma)$, $\Theta(x)\cap\Xi_1\{I(\sigma)\}$ ~~is void for all such x: $\Theta(x)\cap\Xi_1\{I(\sigma)\}$~~ is nonvoid only when $x\in I(\sigma)$.

(v) Let $\Theta, \Xi_1 : R\to J$ and $\Delta : R \to S$. $\mathbb{R}_\Delta(\Theta,\Xi\mid R,S)\in\mathbb{C}(R,S)$.

Set $\mathbb{R}=\mathbb{R}_\Delta(R,S;\Theta,\Xi_1)$. Let $U\subseteq\mathbb{R}[R,S]$ and $\omega=|U|$. From ( ), $\sigma\in\mathbb{R}$ if and only if $\Theta(x)\subseteq J\setminus\Xi_1\{I(\sigma)\}$ for all $R\setminus I(\sigma)$. Select $x\in R\setminus I(\omega)$. Since $x\notin I(\omega)$ and, from ( ), $I(\omega)=\bigcap_{\sigma\in U} I(\sigma)$, there is a $\mu\in U$ for which $x\in R\setminus I(\mu)$ and, since this μ is in $\mathbb{R}$,

$$\Theta(x)\subseteq J\setminus\Xi\{I(\mu)\}\subseteq\bigcup_{\sigma\in U}\{J\setminus\Xi\{\sigma\}\}=J\setminus\bigcap_{\sigma\in U}\Xi_1\{I(\sigma)\}.$$

But, from ( ), $\Xi_1\{I(\omega)\}\subseteq\bigcap_{\sigma\in U}\Xi_1\{I(\sigma)\}$ so that $\Theta(x)\subseteq J\setminus\Xi\{I(\omega)\}$: $\omega\in\mathbb{G}$ and $\mathbb{G}\in\overline{\mathbb{C}}(R,S)$.

Since $\Theta(a)\cap\Xi_1\{R\}$ is, for all $a\in R$, nonvoid only when $a\in R$, $\mathbb{R}[R,S]$ contains σ for which $I(\sigma)=R$. From ( ), $\mathbb{R}\in\mathbb{C}(R,S)$

(viii)
(vi) Let $\Theta : R. \to J$, $\Omega : R \times J \to K$, $\Xi : R \to K$ and $\Delta : R \to\!\!| S$.

(1) $\mathbb{F}_{\Delta}(\Omega ; \Theta, \Xi | R, S ; J) \in \overline{\mathbb{C}}(R, S)$

(2) If, in addition, $\Theta : R \to | J$, then $\mathbb{F}_{\Delta}(\Omega ; \Theta, \Xi | R, S ; J) \in \mathbb{C}(R, S)$.

proof fill p.

(vii) Let $\Theta : R. \times \ldots \times R \times R. \times R. \times \ldots \to J$, $\Xi : R \to J$ and $\Delta : R \to\!\!| S$.

(1) $\mathbb{U}\mathbb{L}_{\Delta}(\Theta, \Xi | R, S) \in \overline{\mathbb{C}}(R, S)$.

(2) If, in addition, $\Xi : R \to | J$, then $\mathbb{U}\mathbb{L}_{\Delta}(\Theta, \Xi | R, S) \in \mathbb{C}(R, S)$.

Set $\mathbb{L} = \mathbb{U}\mathbb{L}_{\Delta}(\Theta, \Xi | R, S)$. Let $U \subseteq \mathbb{L}[R, S]$ and $\omega = |U|$. Select $a, \ldots, t, v, \ldots \in I(\omega)$. If $\Theta(a, \ldots, t, u, v, \ldots)$ is not independent of u in R, it is first remarked that, from ( ), $I(\omega) = \bigcap_{\sigma \in U} I(\sigma)$, so that $a, \ldots, t, v, \ldots \in I(\sigma)$ for all $\sigma \in U$. Select $x \in \Xi\{I(\omega)\}$. From ( ), $\Xi\{I(\omega)\} \subseteq \bigcap_{\sigma \in U} \Xi\{I(\sigma)\}$, so that $x \in \Xi\{I(\sigma)\}$ for all $\sigma \in U$. Since $a, \ldots, t, v, \ldots \in I(\sigma)$, $x \in \Xi\{I(\sigma)\}$ and $\sigma \in \mathbb{L}$ for all $\sigma \in U$, $u(\sigma ; x)$ exists such that $x \in \Theta(a, t, \ldots, u(\sigma ; x), v, \ldots)$ for all $\sigma \in U$. But, from ( ), $\Theta : R \times \ldots \times R \times R. \times R \times \ldots \to J$ where $R.$ in this notation corresponds to u in the list $a, \ldots, t, u, v, \ldots$. There is only one ~~u∈R~~ $u(x) \in R$ for which $x \in \Theta(a, \ldots, t, u(x), v, \ldots)$ when $x \in \Theta(a, \ldots, u(\sigma ; x), v, \ldots)$

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Let $\mathbb{F} = \mathbb{F}_\Delta(R, S; \Omega; \Theta, \Xi)$ $(R, S; J)$. Let $U \subseteq \mathbb{F}[R, S]$ and $\omega = |U|$.

Select $a \in R$, $b \in J$. If $\Theta(a, b) \cap \Xi\{I(\omega)\}$ is nonvoid, $c \in K$ exists such that $c \in \Theta(a, b)$ and $c \in \Xi\{I(\omega)\}$.

Since, from ( ), $\Xi\{I(\omega)\} \subseteq \bigcap_{\sigma \in U} \Xi\{I(\sigma)\}$, $c \in \Xi\{I(\sigma)\}$ for all $\sigma \in U$. But $\sigma \in \mathbb{F}$ for all $\sigma \in U$; hence $a \in I(\sigma)$, $b \in \Theta\{I(\sigma)\}$ for all $\sigma \in U$. Thus ~~$a \in I(\omega)$ and~~ since $I(\omega) = \bigcap_{\sigma \in U} I(\sigma)$, $a \in I(\omega)$ and; since from ( ),

$\bigcap_{\sigma \in U} \Theta\{I(\sigma)\} = \Theta\{I(\omega)\}$ when $\Theta : R \to J$, $b \in \Theta\{I(\omega)\}$: $\omega \in \mathbb{F}$ and $\mathbb{F} \in \overline{\mathbb{C}}(R, S)$.

If $\Theta : R \to J$, then $\Theta\{R\} = J$. For all $a \in R$ and $b \in J$, $\Theta(a, b) \cap \Xi\{R\}$ is nonvoid only when $a \in R$ and $b \in \Theta(R) = J$. $\mathbb{F}[R, S]$ contains σ for which $I(\sigma) = R$. From ( ), ~~and $M(\sigma, a) = S$ for all $a \in I(\sigma)$~~; $\mathbb{F} \in \mathbb{C}^*(R, S)$.

The saturated system closure

Definition (1) Let $\Theta : R \to J$, $\Omega : R \times J \to K$ and $\Xi : R \to K$.
An aggregate $\sigma \in A(R, S)$ for which $I(\sigma) \subseteq R$ and for all $a \in R \setminus I(\sigma)$ and $b \in \Theta\{I(\sigma)\}$, $\Omega(a, b) \subseteq K \setminus \Xi\{I(\sigma)\}$ is said to be Ω-saturated with respect to Θ, Ξ in R in $UA[R, S]$.

(2) $S(R, S; \Omega, \Theta, \Xi)$ is the class of such aggregates.

(3) When $\Omega(a, b) = ab$, $\Theta(a) = \Xi(a) = a$ and $\sigma \in A[R]$,

for some ~~x∈~~ $u(\sigma,x)\in R$. Thus $u(\sigma,x)=u(x)\in I(\sigma)$ for all $\sigma\in U$: $u(x)\in I(\omega)$. For any selection $a,..,t,v,...\in I(\omega)$ either $\Theta(a,..,t,u,v,..)$ is independent of u in R or, for any $x\in\Xi\{I(\omega)\}$, a unique $u(x)=u(x;a,..,t,v,...)\in R$ exists for which $x\in\Theta(a,..,t,u(x),v,..)$ and this $u(x)$ is in $I(\omega)$. The property with respect to u, just described holds with regard to any argument selected from the set $a,..,t,u,v,...$: $\omega\in\mathbb{L}$ and $\mathbb{L}\in\overline{\mathbb{C}}(R,S)$.

The conditions characterising the mapping $\Theta: R.\times...\times R.\times R.\times R.\times...\to J$ are the properties (α,β) of Definition modified by replacing $I(\sigma)$ by R and $\Xi\{I(\sigma)\}$ by J. Accordingly, when $\Xi(R)=J$, $\mathbb{L}$ contains an aggregate σ for which $I(\sigma)\equiv R$. From ( ), $\mathbb{L}\in\mathbb{C}(R,S)$.

(viii) Let $\Theta:R\to J$, $\Omega:R\times J\to K$, $\Xi:R\to K$ and $\Delta:R\to\|S$.

$\mathbb{S}_\Delta(\Omega;\Theta,\Xi|R,S;K)\in\mathbb{C}(R,S)$.

so ~~that, for all~~ $\cancel{a \in R \setminus I(\sigma)}$ ~~and~~ $\cancel{b \in I(\sigma)}$, $\cancel{\Theta(a,b) \in}$ [illegible]

$\boxed{65}$

~~$I(\sigma)$ is said to be saturated in R.~~

~~( ) Let $\Theta : R \to J$, $\Omega : R \times J \to J$ and $\Xi : R \to K$ and $\Delta : R \to S$.~~

~~Then $S_\Delta(R, S;\ \Omega, \Theta, \Xi) \in \mathbb{C}(R, S)$.~~ [illegible]

Set $\mathcal{S} = S_\Delta(R, S; K)$. Let $U \subseteq \mathcal{S}[R, S]$ and $\omega = |U|$. Select $a \in R \setminus I(\omega)$ and $b \in \Theta\{I(\omega)\}$. Since $I(\omega) \equiv \bigcap_{\sigma \in U} I(\sigma)$, ~~there is~~ and $a \notin I(\omega)$, there is a $\mu \in U$ for which $a \notin I(\mu)$.

From ( ), $\Theta\{I(\omega)\} \subseteq \bigcap_{\sigma \in U} \Theta\{I(\sigma)\}$ so that, in particular, $b \in \Theta\{I(\mu)\}$. But $\mu \in \mathcal{S}$, so that $\Omega(a, b) \subseteq K \setminus \Xi\{I(\mu)\}$

From ( ), $\Xi\{I(\omega)\} \subseteq \bigcap_{\sigma \in U} \Xi\{I(\sigma)\}$, so that

$$K \setminus \Xi\{I(\mu)\} \subseteq \bigcup_{\sigma \in U} [K \setminus \Xi\{I(\sigma)\}] \equiv K \setminus \bigcap_{\sigma \in U} \Xi\{I(\sigma)\} \subseteq K \setminus \Xi\{I(\omega)\}$$

and $\Theta(a, b) \subseteq K \setminus \Xi\{I(\omega)\}$: $\omega \in \mathcal{S}$ and $\mathcal{S} \in \overline{\mathbb{C}}(R, S)$.

When $I(\sigma) \equiv R$, $\Omega(a, b) \subseteq K \setminus \Xi\{I(\sigma)\}$ whenever $a \in R \setminus I(\sigma)$ and $b \in \Theta\{I(\sigma)\}$, since in this case ~~there are no~~ $a \in R \setminus I(\sigma)$ does not exist. $\mathcal{S}[R, S]$ contains an aggregate σ for which $I(\sigma) \equiv R$. ~~and $M(\sigma, a) = S$ for all $a \in T(\sigma)$: $\mathcal{S} \in \mathbb{C}'(R, S)$.~~ From ( ), $\mathcal{S} \in \mathbb{C}(R, S)$

(xx) Let $\Theta : R \to J$, $\Omega : R \times J \to K$, $\Xi : R. \to K$ and $\Delta : R \to\!\!\| S$.

(1) $\mathbb{CO}_\Delta(\Omega; \Theta, \Xi \mid R, S) \in \overline{\mathbb{C}}(R, S)$

(2) If, in addition, $\Xi : R \to\!| K$, then $\mathbb{CO}_\Delta(\Omega; \Theta, \Xi \mid R, S) \in \mathbb{C}(R, S)$

Set $\mathbb{CO} = \mathbb{CO}_\Delta(\Omega; \Theta, \Xi \mid R, S)$. Let $U \subseteq \mathbb{CO}[R, S]$ and $\omega = |U|$. Select $a \in I(\omega)$, so that, since $I(\omega) \subseteq \bigcap_{\sigma \in U} I(\sigma)$, $a \in I(\sigma)$ for all $\sigma \in U$, and $b \in \Theta\{I(\omega)\}$. From ( ), $\Theta\{I(\omega)\} \subseteq \bigcap_{\sigma \in U} \Theta\{I(\sigma)\}$, so that $b \in \Theta\{I(\sigma)\}$ for all $\sigma \in U$. Since $\sigma \in \mathbb{CO}$ for all $\sigma \in U$, $\Omega(a, b) \subseteq \Xi\{I(\sigma)\}$ for all $\sigma \in U$. But $\Xi : R. \to K$ so that, from ( ), $\bigcap_{\sigma \in U} \Xi\{I(\sigma)\} \equiv \Xi\{I(\omega)\}$. Accordingly $\Omega(a, b) \subseteq \Xi\{I(\omega)\}$ for all $a \in I(\omega)$ and $b \in \Theta\{I(\omega)\}$: $\omega \in \mathbb{CO}$ and $\mathbb{CO} \in \overline{\mathbb{C}}(R, S)$.

When $\Xi : R \to\!| K$, $\Xi(R) \equiv K$ and $\Omega(R, \Theta(R)) \subseteq \Xi(R)$. $\mathbb{CO}[R, S]$ contains σ for which $I(\sigma) \equiv R$. From ( ), $\mathbb{CO} \in \mathbb{C}(R, S)$.

(xx) Let $\Theta : R. \to J$, $\Omega : R{:} \times J{:} \to K$, $\Xi : R \to K$ and $\Delta : R \to\!\!\| S$

(1) $\mathbb{E}_\Delta(\Omega; \Theta, \Xi \mid R, S) \in \overline{\mathbb{C}}(R, S)$

(2) If, in addition, $\Theta : R \to\!\| J$ and $\Omega : R_{\S} \times J_{\S} \to\!| K$ then $\mathbb{E}_\Delta(\Omega; \Theta, \Xi \mid R, S) \in \mathbb{C}(R, S)$

The expansive system closure

Definition. (1) Let $\Theta : R \to_{\S} J$, $\Omega : R \times J \to_{\S} K$ and $\Xi : R \to_{\S} K$. An aggregate $\sigma \in A(R, S)$ for which $I(\sigma) \subseteq R$ and ~~that~~ $\Xi\{I(\sigma)\} \subseteq \Omega[I(\sigma), \Theta\{I(\sigma)\}]$ is said to be an expansive Ω-system with respect to Θ, Ξ in R. (2) $E(R, S; \Omega, \Theta, \Xi)$ is the class of such aggregates in $\amalg A[R, S]$.

(3) When $\Omega(a, b) = ab$, $\Theta(a) = \Xi(a) = a$ and $\sigma \in A'[R]$ so that $I(\sigma) \subseteq I(\sigma)^2$, $I(\sigma)$ is said to be an expansive multiplicative system in R

( )(4) Let $\Theta : R. \to_{\S} J$, $\Omega : R. \times J. \to_{\S} K$, and $\Delta : R; \to S$, and $\Xi : R \to_{\S} K^{\vee}$. Then $E_{\Delta}(R, S; \Omega, \Theta, \Xi) \in \overline{C}^{(2)}(R, S)$. If in addition $\Theta : R; \to ; J$ and $\Omega : \overset{?}{R}{}^{\vee}; \times \overset{?}{J}; \to_{\S} K$ then $E_{\Delta}(R, S; \Omega, \Theta, \Xi) \in C^{\vee}(R, S)$.

Set $E = E_{\Delta}(R, S; \Omega; \Theta, \Xi)$. Let $U \subseteq E[R, S]$ and $\omega = |U|$. Select $b \in \Xi\{I(\omega)\}$. Since, from ( ), $\Xi\{I(\omega)\} \subseteq \bigcap_{\sigma \in U} \Xi\{I(\sigma)\}$, $b \in \Xi\{I(\sigma)\}$ for all $\sigma \in U$. All $\sigma \in U$ are in $E[R, S]$:

$b \in \Omega\{a(\sigma), c(\sigma)\}$ where $a(\sigma) \in I(\sigma)$ and $c(\sigma) \in \Theta\{I(\sigma)\}$ for each $\sigma \in U$. Since $\Omega : R: \times J: \to_{\S} K$, ~~$a \in R$ and~~ only one pair $a \in R$, $c \in J$ exist for which $b \in \Omega\{a, c\}$ when $b \in \Omega\{a(\sigma), b(\sigma)\}$ for ~~one~~ a pair ~~$a(\sigma), b(\sigma)$~~ $a(\sigma) \in R$, $c(\sigma) \in J$: $a(\sigma) = a$ and $c(\sigma) = c$ for all $\sigma \in U$. ~~$c \in J$ exist such that $a = a(\sigma)$, $c = c(\sigma)$ for all $\sigma \in U$~~

Since $I(\omega) = \bigcap_{\sigma \in U} I(\sigma)$, $a \in I(\omega)$. Since $c \in \Theta\{I(\sigma)\}$ for all $\sigma \in U$, $d(\sigma) \in I(\sigma)$ exists for which $c \in \Theta\{d(\sigma)\}$ for each $\sigma \in U$. But $\Theta : R. \to_{\S} J$ so that again $d = d(\sigma)$ for all $\sigma \in U$ and $d \in I(\omega)$: $c, d \in I(\omega)$ exist ~~such that~~

for which $b \in \Omega\{c, \Theta(d)\}$: $\Xi_1\{I(\omega)\} \subseteq \Omega[I(\omega), \Theta\{I(\omega)\}]$ $\boxed{68}$

and $\omega \in E$; $E \in \overline{\mathbb{C}}(R,S)$.

~~If $\Omega : R_i \times J_i \to K$ so that, in particular,~~

~~$\Omega(R,$~~

If $\Theta(R) = J$ then corresponding to any $c \in J$, $d \in R$ exists such that $c \in \Theta(d)$. If, furthermore $\Omega : R^{\xi_1'}_{\xi} \times J^{\xi_1'}_{\xi} \to| K$ then corresponding to any $b \in K$, $a \in R$ and $c \in J$ exist such that $b \in \Omega(a, c)$. If both conditions obtain, corresponding to any $b \in K$, ~~a, b~~ R contains a and ~~b~~ d for which $b \in \Omega(a, \Theta(d))$. ~~Since~~ In particular, this ~~holds at~~ condition holds ~~when b ∈ Ξ~~ with regard to any $b \in \Xi_1(R)$. Thus if $\Theta : R_\xi \to| J$ and $\Omega : R_\xi \times J_\xi \to| K$ $E[R,S]$ contains an aggregate σ for which $I(\sigma) = R$. From ( ), ~~and $M(\sigma, a) = S$ for all $a \in I(\sigma)$~~, $E \in \mathbb{C}^u(R,S)$.

Relationships between classes replace $\mathbb{R}$, $\mathbb{G}$ by description

If σ is Θ'-free with respect to Ξ' in R, it is also Θ-closed with respect to Ξ in R

(ii) Let $\Theta, \Xi_1, \Theta', \Xi' : R \to_{\xi_1} J$, with $\Theta(a) \subseteq \Theta'(a)$ and $\Xi_1(a) \subseteq \Xi'(a)$ for all $a \in R$. ~~$\mathbb{R}(R;\Theta', \Xi') \subseteq \mathbb{G}(R, \Theta, \Xi)$~~.

From ( ), ~~(R,S;~~ σ is Θ'-free with respect to Ξ' in R ~~$\sigma \in \mathbb{R}(R, \Theta', \Xi')$~~ if and only if $\Theta(x) \subseteq J \setminus \Xi_1\{I(\sigma)\}$ for all $x \in R \setminus I(\sigma)$. From ( ), ~~(R,S;~~ σ is Θ closed with respect to Ξ in R ~~$\sigma \in \mathbb{G}(R, \Theta, \Xi)$~~ if and only if $\Theta(x) \cap [J \setminus \Xi\{I(\sigma)\}]$ is nonvoid for all $x \in R \setminus I(\sigma)$.

Let $\sigma \in \mathbb{R}$ ~~(R, Θ', Ξ')~~ (R,S; be Θ'-free with respect to Ξ' in R. Then $\Theta(x) \subseteq \Theta(x') \subseteq J \setminus \Xi\{I(\sigma)\} \subseteq J \setminus \Xi_1\{I(\sigma)\}$ for all $x \in R \setminus I(\sigma)$. In particular

(xi) Let $\Theta: R \to J$, $\Xi: R \to J$ and $\Delta: R \to \| S$.

(1) $SI_\Delta(\Theta, \Xi | R, S) \in \overline{\mathbb{C}}(R,S)$

(2) If, in addition, ~~$\Xi: R \to$~~ $\Xi: R \to |J$, then $SI_\Delta(\Theta, \Xi | R; S) \in \mathbb{C}(R,S)$

Set $SI = SI_\Delta(\Theta, \Xi | R, S)$. Let $U \in SI[R,S]$ and $\omega = |U|$.

since $\Xi: R \to J$.

$I(\omega) = \bigcap_{\sigma \in U} I(\sigma)$ and, from ( ), $\Xi\{I(\omega)\} = \bigcap_{\sigma \in U} \Xi\{I(\sigma)\}$

Select $a \in I(\omega)$ so that $a \in I(\sigma)$ for all $\sigma \in U$ and, since $U \in SI[R,S]$, $\Theta(a) \subseteq \Xi\{I(\sigma)\}$ for all $\sigma \in U$. Hence $\Theta(a) \subseteq \bigcap_{\sigma \in U} \Xi\{I(\sigma)\} = \Xi\{I(\omega)\}$ for all $a \in I(\omega)$; $\omega \in SI[R,S]$ and $SI \in \overline{\mathbb{C}}(R,S)$

If it is further assumed that $\Xi: R \to |J$, i.e. that $\Xi(R) = J$ then, since $\Theta: R \to J$, $\Theta(R) \subseteq \Xi(R)$. $SI[R,S]$ contains σ for which $I(\sigma) = R$. From ( ), $SI \in \mathbb{C}(R,S)$.

n.4 Class closures of aggregates.

Definition. With $\xi \in A[R,S]$ prescribed, and $\mathbb{P}$ a given class of aggregates, the aggregate $\omega = \overline{\mathbb{P}}(\xi) \in \mathbb{P}[R,S]$ which

(α) includes ξ and

(β) is minimal in the sense that $\omega < \sigma$ for all $\sigma \in \mathbb{P}[R,S]$ for which $\omega \neq \sigma$ and $\xi \leq \sigma$

is the $\mathbb{P}$ closure of ξ in (R,S)

(i)-(vi) foll pp.

(iii) $\mathbb{P}\in\mathbb{C}^{**}(R,S)$ if and only if $\overline{\mathbb{P}}(\xi)$ is defined for any $\xi\in A[R,S]$. // (B) If $\mathbb{P}\in\mathbb{C}^{**}(R,S)$ then, for any $\xi\in A[R,S]$, $\overline{\mathbb{P}}(\xi)=|\mathbb{P}_{\xi}[R,S]|$ where $\mathbb{P}_{\xi}[R,S]$ is the complete set of aggregates $\sigma\in\mathbb{P}[R,S]$ for which $\xi\leq\sigma$.

It is first shown that $\mathbb{P}\in\mathbb{C}^{**}(R,S)$ if $\overline{\mathbb{P}}(\xi)$ is defined for all $\xi\in A[R,S]$. ~~Assuming the latter condition to hold, $\overline{\mathbb{P}}(\xi)\in\mathbb{P}$ for all $\xi\in A(R,S)$, by definition. If $\overline{\mathbb{P}}(|U|)=|U|$ for all $U\subseteq\mathbb{P}(R,S)$ then $|U|\in\mathbb{P}$~~ for all such U, ~~and $\mathbb{P}\in\overline{\mathbb{C}}(R,S)$.~~ Suppose that for some $U\subseteq\mathbb{P}[R,S]$, $\xi\neq\omega$, where ~~$|U|$~~ $\xi=|U|$ and $\omega=\overline{\mathbb{P}}(\xi)$. Since, from clause ( ) of Definition (, $\xi\leq\omega$, the supposition ~~implies~~ entails that $\xi<\omega$. That $\omega\leq\sigma$ for all $\sigma\in U$ is impossible, for if this were so, it would then follow that $\omega\leq\xi=|U|$, and then $\xi\nless\omega$: U contains at least one σ for which $\omega\nleq\sigma$, i.e. $\omega\neq\sigma$ and $\omega\nless\sigma$. Since $U\subseteq\mathbb{P}[R,S]$, $\sigma\in\mathbb{P}[R,S]$ in particular. Also $\xi\leq\sigma$, since $\sigma\in U$. From clause ( ) of Definition , $\omega<\sigma$ for all $\sigma\in\mathbb{P}[R,S]$ for which $\omega\neq\sigma$ and $\xi\leq\sigma$. However $\omega\nless\sigma$, so that $\omega\neq\overline{\mathbb{P}}(\xi)$ conflicting with the definition of ω: there is no $U\subseteq\mathbb{P}[R,S]$ for which ~~$|U|\neq$~~ $\overline{\mathbb{P}}(|U|)\neq|U|$. Since $\overline{\mathbb{P}}(\xi)\in\mathbb{P}$ for all $\xi\in A(R,S)$ by definition, and $\overline{\mathbb{P}}(|U|)=|U|$ for all $U\subseteq\mathbb{P}(R,S)$, $|U|\in\mathbb{P}$ for all such U and $\mathbb{P}\in\overline{\mathbb{C}}(R,S)$.

~~$\overline{\mathbb{P}}(\xi)$ is defined and belongs to $\mathbb{P}(R,S)$ for all $\xi \in A(R,S)$, in particular for~~ Set $\xi = 1_{R,S}$. $A[R,S]$ contains one aggregate ω for which $\xi \le \omega$, namely $\omega = 1_{R,S}$. It contains no σ for which $1_{R,S} < \sigma$: there is no $\sigma \in A[R,S]$ for which $\omega \ne \sigma$ and $\xi \le \sigma$. Since $\mathbb{P}[R,S] \subseteq A[R,S]$, $\omega < \sigma$ for all $\sigma \in \mathbb{P}[R,S]$ for which $\omega \ne \sigma$ and $\xi \le \sigma$. $\omega = 1_{R,S}$ is the only aggregate in $A[R,S]$ possessing properties (a, b) of Definition with respect to $\xi = 1_{R,S}$. $\overline{\mathbb{P}}(\xi)$ is defined in particular for $\xi = 1_{R,S}$ and belongs to $\mathbb{P}$: $1_{R,S} \in \mathbb{P}[R,S]$ and accordingly $\mathbb{P} \in \mathbb{C}^{M}(R,S)$

That, when $\mathbb{P} \in \mathbb{C}^{M}(R,S)$, $\overline{\mathbb{P}}(\xi)$ is defined for all $\xi \in A[R,S]$ is a consequence of the result (2) which gives an explicit definition of $\overline{\mathbb{P}}(\xi)$ valid under this assumption. With regard to part (2) it is first remarked

~~The existence result of (1) is a consequence of (2). With regard to the latter it is first remarked~~ that, since $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$, $1_{R,S} \in \mathbb{P}$: ~~ξ~~ $\xi \le 1_{R,S}$ for all $\xi \in A[R,S]$: the set $\mathbb{P}_\xi[R,S]$ is nonvoid. Whatever properties may be ascribed to it, $\omega = |\mathbb{P}_\xi[R,S]|$ is at least well defined by the set $\mathbb{P}_\xi[R,S]$. ω is the aggregate ~~over~~ in (R,S) for which $I(\omega)$ is the complete subset of $a \in I'(\omega) = \bigcap_{\sigma \in \mathbb{P}_\xi(R,S)} I(\sigma)$ for which $M'(\omega,a) = \bigcap_{\sigma \in \mathbb{P}_\xi(R,S)} M(\sigma,a)$ is nonvoid, and $M(\omega,a) = M'(\omega,a)$ for all $a \in I(\omega)$. ~~Since $\xi \le \sigma$ for all $\sigma \in \mathbb{P}_\xi(R,S)$,~~ Select $a \in I(\xi)$ and $b \in M(\xi,a)$. Since $\xi \le \sigma$ for all $\sigma \in \mathbb{P}_\xi[R,S]$, $a \in I(\sigma)$ and $b \in M(\sigma,a)$ for all $\sigma \in \mathbb{P}_\xi[R,S]$. Hence $a \in I'(\omega)$, and, since $b \in M'(\omega,a)$, $M'(\omega,a)$ is nonvoid for this a: $a \in I(\omega)$ and $b \in M(\omega,a)$. The choices of a and b are arbitrary: $\xi \le \omega$. Furthermore, since $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$, $\omega \in \mathbb{P}[R,S]$. Suppose that ω does not possess the minimality property (b) of Definition : $\sigma \in \mathbb{P}[R,S]$ for which $\xi \le \sigma$ but $\omega \ne \sigma$ and $\omega \nleq \sigma$ exists. For any such σ, $M(\omega,a)$ contains b such that $b \notin M(\sigma,a)$ for some $a \in I(\omega)$. But $\sigma \in \mathbb{P}$ includes ξ: $\sigma \in \mathbb{P}_\xi[R,S]$. $b \notin \bigcap_{\sigma \in \mathbb{P}_\xi[R,S]} M(\sigma,a)$ and accordingly $b \notin M(\omega,a)$. Hence no $\sigma \in \mathbb{P}[R,S]$ violating the conditions $\omega \ne \sigma$, $\xi \le \sigma$, $\omega < \sigma$ exists.

(ii) Let $\mathbb{P}', \mathbb{P}''$ be two classes of aggregates such that:

" $\mathbb{P}'' \subseteq \mathbb{P}'$, $\mathbb{P}' \in \overline{\mathbb{C}}(R,S)$ and $\mathbb{P}'' \in \mathbb{C}^{**}(R,S)$. Then $\cancel{\mathbb{P}' \in \mathbb{C}^*(R,S)}$,

(1) $\overline{\mathbb{P}}''(\xi) \in \mathbb{P}'$ for all $\xi \in A[R,S]$ and (2) $\overline{\mathbb{P}}'(\xi) \leq \overline{\mathbb{P}}''(\xi)$ for all $\xi \in A[R,S]$.

From ( ), $\mathbb{P}' \in \mathbb{C}(R,S)$. If $\mathbb{P}'' \subseteq \mathbb{P}'$, ~~If $\mathbb{P}' \in \mathbb{C}^*(R,S)$, $1_{R,S} \in \mathbb{P}''[R,S]$. If $\mathbb{P}'' \subseteq \mathbb{P}'$, $1_{R,S} \in \mathbb{P}'[R,S]$ also: $\mathbb{P}' \in \mathbb{C}^*(R,S)$.~~ $\overline{\mathbb{P}}''(\xi) \in \mathbb{P}''[R,S] \subseteq \mathbb{P}'[R,S]$ for all $\xi \in A[R,S]$ ~~If $\mathbb{P}'' \subseteq \mathbb{P}'$~~: $\overline{\mathbb{P}}''(\xi)$ is in $\mathbb{P}'$. If $\mathbb{P}'' \subseteq \mathbb{P}'$, all members of the set $\mathbb{P}''_{\xi}[R,S]$ of $\sigma \in \mathbb{P}''[R,S]$ for which $\xi \leq \sigma$ also belong to $\mathbb{P}'_{\xi}[R,S]$, the similarly defined set of aggregates in $\mathbb{P}'[R,S]$. $\mathbb{P}''_{\xi}[R,S] \subseteq \mathbb{P}'_{\xi}[R,S]$ and hence, from ( ), $|\mathbb{P}'_{\xi}[R,S]| \leq |\mathbb{P}''_{\xi}[R,S]|$.

(iii) Let $\mathbb{P}', \mathbb{P}'' \in \mathbb{C}^{**}(R,S)$ and $\mathbb{P} = \mathbb{P}' \cap \mathbb{P}''$. Then $\mathbb{P} \in \mathbb{C}^{**}(R,S)$ ~~and~~ $\overline{\mathbb{P}}'(\xi), \overline{\mathbb{P}}''(\xi) \leq \overline{\mathbb{P}}(\xi)$ for all $\xi \in A[R,S]$.

From ( ), $\mathbb{P} \in \mathbb{C}^{**}(R,S)$ ~~$U \subseteq \mathbb{P}', \mathbb{P}''$ for all sets U of aggregates in $\mathbb{P}$.~~ ~~Since $\mathbb{P}', \mathbb{P}'' \in \overline{\mathbb{C}}(R,S)$, $|U| \in \mathbb{P}' \cap \mathbb{P}'' = \mathbb{P}$ for all such U: $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$. $1_{R,S}$ belongs to ~~is in $\mathbb{P}'$ and $\mathbb{P}''$, and there is in $\mathbb{P}'[R,S]$ and $\mathbb{P}''[R,S]$ and therefore to $\mathbb{P}[R,S]$. $\mathbb{P} \in \mathbb{C}^*(R,S)$.~~ Since $\mathbb{P} \subseteq \mathbb{P}', \mathbb{P}''$, it follows that $\overline{\mathbb{P}}'(\xi), \overline{\mathbb{P}}''(\xi) \leq \overline{\mathbb{P}}(\xi)$ for all $\xi \in A[R,S]$, from ( ).

(iv) Let $\mathbb{P}' \in \mathbb{C}^{w}(R,S)$ and $\mathbb{P}'' \in \overline{\mathbb{C}}^{*}(R,S)$. Assume either that for all σ such that $\sigma \le \mu \in \mathbb{P}'$, $\sigma \in \mathbb{P}'$ also, or that the class $\mathbb{P}''$ possesses this property. Let $\mathbb{P} = \mathbb{P}' \cup \mathbb{P}''$. Then $\overline{\mathbb{P}}(\xi) \le \overline{\mathbb{P}}'(\xi)$ for all $\xi \in A[R,S]$.

From ( ), $\mathbb{P} \in \overline{\mathbb{C}}(R,S)$.

~~Any set $T \subseteq \mathbb{P}(R,S)$ has a decomposition of the form $T = U \cup V$ where $U \subseteq \mathbb{P}'(R,S)$ and $V \subseteq \mathbb{P}''(R,S)$. From (,), $|T| = |U| \wedge |V| \le |U|, |V|$. If $\mathbb{P}'$ possesses the property described, $|T| \in \mathbb{P}' \subseteq \mathbb{P}$ since $|T| \le |U|$; if $\mathbb{P}''$ possesses the stated property, $|T| \in \mathbb{P}$ also. Thus $\mathbb{P} \in \overline{\mathbb{C}}(R)$. Since $1_{R,S} \in \mathbb{P}'[R,S]$, $1_{R,S} \in \mathbb{P}[R,S]$ also.~~

Since $\mathbb{P}' \subseteq \mathbb{P}$, $\overline{\mathbb{P}}^{*}(\xi) \le \overline{\mathbb{P}}'(\xi)$ for all $\xi \in A[R,S]$, from ( ).

(v). Let $\mathbb{P} \in \mathbb{C}^{w}(R,S)$. "(1) If $\sigma, \mu \in A[R,S]$ with $\sigma \le \mu$, then $\overline{\mathbb{P}}(\sigma) \le \overline{\mathbb{P}}(\mu)$." (2) Let $\xi \in A[R,S]$. $\xi \in \mathbb{P}$ if and only if $\overline{\mathbb{P}}(\xi) = \xi$." (3) $\overline{\mathbb{P}}\{\overline{\mathbb{P}}(\xi)\} = \overline{\mathbb{P}}(\xi)$ for all $\xi \in A[R,S]$. "(4) Let $\sigma, \mu \in A[R,S]$. If $\sigma \le \mu \le \overline{\mathbb{P}}(\sigma)$, then $\overline{\mathbb{P}}(\mu) = \overline{\mathbb{P}}(\sigma)$.

Let $U = \mathbb{P}_{\sigma}[R,S]$ and $V = \mathbb{P}_{\mu}[R,S]$ be the sets of all $\tau \in \mathbb{P}[R,S]$ for which $\sigma \le \tau$ and $\mu \le \tau$ respectively. If $\sigma \le \mu$, $\mathbb{P}_{\mu}[R,S] \subseteq \mathbb{P}_{\sigma}[R,S]$ and, from ( ), $\overline{\mathbb{P}}(\sigma) = |U| \le |V| = \overline{\mathbb{P}}(\mu)$.

$\overline{\mathbb{P}}(\xi) \in \mathbb{P}$. Hence if $\overline{\mathbb{P}}(\xi) = \xi$, $\xi \in \mathbb{P}$ also. $\xi \le \overline{\mathbb{P}}(\xi)$ by definition. Also $\overline{\mathbb{P}}(\xi) \le \sigma$ for any $\sigma \in \mathbb{P}[R,S]$ for which

$\xi \leq \sigma$. Thus if $\xi \in \mathbb{P}[R,S]$, $\overline{\mathbb{P}}(\xi) \leq \xi$, since $\xi \leq \xi$. Accordingly $\overline{\mathbb{P}}(\xi) = \xi$ if $\xi \in \mathbb{P}$: $\xi \in \mathbb{P}$ if and only if $\overline{\mathbb{P}}(\xi) = \xi$. Since $\overline{\mathbb{P}}(\xi) \in \mathbb{P}$, $\overline{\mathbb{P}}\{\overline{\mathbb{P}}(\xi)\} = \overline{\mathbb{P}}(\xi)$, from the preceding result. When $\sigma \leq \mu$, $\overline{\mathbb{P}}(\sigma) \leq \overline{\mathbb{P}}(\mu)$. When $\mu \leq \overline{\mathbb{P}}(\sigma)$, $\overline{\mathbb{P}}(\mu) \leq \overline{\mathbb{P}}\{\overline{\mathbb{P}}(\sigma)\} = \overline{\mathbb{P}}(\sigma)$. The last result follows.

~~(v) $A \in \mathbb{C}^{m}(R,S)$. ($A \in \overline{\mathbb{C}}(R,S)$ and $1_{R,S} \in A[R,S]$.)~~;

(vi) $\overline{A}(\xi) = \xi$ for all $\xi \in A[R,S]$.

ξ itself includes ξ and is minimal in the sense that for any $\sigma \in A(R,S)$ for which $\sigma \neq \xi$ and $\xi \leq \sigma$, $I(\xi) \subseteq I(\sigma)$, $M(\xi,a) \subseteq M(\sigma,a)$ for all $a \in I(\xi)$ and $M(\sigma,a) \setminus M(\xi,a)$ is nonvoid for at least one $a \in I(\omega)$: $\overline{A}(\xi) = \xi$ for all $\xi \in A(R,S)$.

( )1(a) Let $\Phi: R^2 \times S^2 .\to; T$ and $\Theta: R \times S \times S .\to; T$ with $\Theta(a,b,S) \equiv T$ for all $a \in R$, $b \in S$. Let $\mathbb{P}$ be the class of aggregates σ $\in A(R,S)$ for which condition (*) of (1) holds

~~$\Phi\{a, I(\sigma), b, M(\sigma,a)\} \subseteq \Theta\{a, b, M(\sigma,a)\}$~~

for each $a \in I(\sigma)$ and all $b \in M(\sigma,a)$. Then $\mathbb{P}(R,S) \in \mathbb{C}^{m}(R,S)$.

(b) Let $\Phi: R^2 \times S^3 .\to; T$, $\Upsilon: R .\to; S$, and $\Theta: R \times S \times S^2 .\to T$ with $\Theta(a,b,S,S) \equiv T$ for all $a \in R$, $b \in S$. Let $\mathbb{P}$ be the class of aggregates σ for which condition (**) of (2) holds

~~$\Phi[a, I(\sigma), b, M(\sigma,a), M\{\sigma, \Upsilon^{-1}(b)\}] \subseteq \Theta[a, b, M(\sigma,a), M\{\sigma, \Upsilon^{-1}(b)\}]$~~

for each $a \in I(\sigma)$ and all $b \in \Upsilon\{I(\sigma)\} \cap M(\sigma,a)$. Then $\mathbb{P}(R,S) \subseteq \mathbb{C}^{m}(R,S)$.

Definition (1) Let $\Theta : R \to ; J$, $\Omega : R \times J \to ; K$ and $\Xi : R \to ; K$.

An aggregate $\sigma \in A(R)$ ~~$\sigma \in A(R;S)$~~ for which ~~$I(\sigma) \subseteq R$ and~~ for all $a \in R \setminus I(\sigma)$ and $b \in J \setminus \Theta\{I(\sigma)\}$, $\Omega(a,b) \cap \Xi\{I(\sigma)\}$ is void ~~$\Omega(a,b) \subseteq K \setminus \Xi\{I(\sigma)\}$~~ is said to be Ω-independent with respect to Θ, Ξ in (R, J)

(2) $\amalg\left(\begin{smallmatrix} R, S; J \\ \Omega; \Theta, \Xi \end{smallmatrix}\right)$ is the class of such aggregates in $UA[R,S]$

(3) When $\Omega(a,b) = ab$, $\Theta(a) \equiv \Xi(a) = a$ and $\sigma \in A'[R]$ so that for all ~~$a \in R \setminus I(\sigma)$ and $b \in R \setminus I(\sigma)$~~ $a, b \in R \setminus I(\sigma)$ $ab \notin I(\sigma)$, $I(\sigma)$ is said to be independent in R

An attempt to prove that the class $\amalg\left(\begin{smallmatrix} R,S;J \\ \Omega; \Theta, \Xi \end{smallmatrix}\right)$ is complete in (R, S) by use of the methods used above fails. Selecting $a \in R \setminus I(\omega)$, it follows, since $I(\omega) \equiv \bigcap_{\sigma \in U} I(\sigma)$ that $\mu \in U$ exists for which $a \in R \setminus I(\mu)$.

Assuming that $\Theta : R \to ; J$, $\Theta\{I(\omega)\} \equiv \bigcap_{\sigma \in U} \Theta\{I(\sigma)\}$. Selecting $b \in J \setminus \Theta\{I(\omega)\}$ it follows that $\mu' \in U$ exists for which $b \in J \setminus \Theta\{I(\mu')\}$. But it is possible that $\mu \neq \mu'$, and a conclusion of the form $\Omega(a,b) \subseteq K \setminus \Xi\{I(\mu)\}$ may not be drawn.

Definition (1) Let $\Theta : R \to ; J$, $\Omega : R \times J \to ; K$ and $\Xi : R \to ; K$. An aggregate $\sigma \in A(R)$ ~~$\sigma \in A(R;S)$~~ for which ~~$I(\sigma) \subseteq R$~~ and for all $a \in R$ and $b \in J$, $\Omega(a,b) \cap \Xi\{I(\sigma)\}$ is nonvoid only when either $a \in I(\sigma)$ or $b \in \Theta\{I(\sigma)\}$ is said to be Ω

semi factored with respect to Θ, Ξ in R. ~~SF~~ 79

(2) $SF(\Omega; \Theta, \Xi | R, S)$ is the class of such aggregates in $UA[R,S]$

"(3) When $\Omega(a,b) = ab$, $\Theta(a) = \Xi(a) = a$ and $\sigma \in A'[R]$ so that for all $a,b \in R$, $ab \in I(\sigma)$ only when either a or b is in $I(\sigma)$, $I(\sigma)$ is said to be semi-factored in R.

The proof of completeness of the class $SF_{\Delta}(\Omega; \Theta, \Xi | R, S)$ breaks down as above

~~Definition~~

(7) ~~Denote by~~ $\mathbb{I}_F(R)$, $\mathbb{I}_{SF}(R)$, $\mathbb{I}_{QF}(R)$, & $\mathbb{I}_C(R)$ are respectively the classes of $\sigma \in A'[R]$ for which $I(\sigma)$ is factored, semi factored, square free and closed in R.

(i) ~~respectively.~~ ~~Then~~ $\mathbb{I}_F(R) \subseteq \mathbb{I}_{SF}(R) \subseteq \mathbb{I}_{QF}(R) \subseteq \mathbb{I}_C(R)$.

91 If both a and b belong to $I(\sigma)$ whenever $ab \in I(\sigma)$ then ~~that~~ either a or b belongs to $I(\sigma)$ ~~when both do~~ is ~~obvious~~ $ab \in I(\sigma)$: $\mathbb{I}_F(R) \subseteq \mathbb{I}_{SF}(R)$. If $\sigma \in \mathbb{I}_{SF}(R)$, then $a^2 \in I(\sigma)$ only when $a \in I(\sigma)$: $\mathbb{I}_{SF}(R) \subseteq \mathbb{I}_{QF}(R)$.

That $\mathbb{I}_{QF}(R) \subseteq \mathbb{I}_C(R)$ has been shown above.

Definition (1) ~~An half factored set $I(\sigma)$ is defined by the condition that if $ab = cd$ when $c \in I(\sigma)$, $d \in R$~~ aggregate σ for which $I(\sigma)$ ~~in~~ R and ~~for which~~, when $c \in I(\sigma)$, $d \in R$, $ab = cd$ only when either a or b is in $I(\sigma)$ is said to be half factored in R. ~~A completeness proof breaks down as above~~.

"(2) $\mathbb{I}_{HF}(R)$ is the class of ~~set~~ ~~$\sigma \in A'$~~ such aggregates in $A'[R]$.

"( ) $\mathbb{I}_{SI}(R) \cap \mathbb{I}_{SF}(R) \subseteq \mathbb{I}_{HF}(R)$. Suppose that $\sigma \in \mathbb{I}_{SI}(R) \cap \mathbb{I}_{SF}(R)$ and that $ab = cd$ where $c \in I(\sigma)$ (above). Since $\sigma \in \mathbb{I}_{SI}(R)$, $cd = c' \in I(\sigma)$. Since $\sigma \in \mathbb{I}_{SF}(R)$ and $ab = c'$

(A proof of completeness of the class of ~~aggregates~~ that are half factored in R breaks down as above)

Definition. $\mathbb{I}_F(R)$, $\mathbb{I}_{SF}(R)$, $\mathbb{I}_{QF}(R)$ and $\mathbb{I}_C(R)$ are respectively the classes of $\sigma \in A'[R]$ for which $I(\sigma)$ is factored, semi-factored, square free and closed in R

(i) $\mathbb{I}_F(R) \subseteq \mathbb{I}_{SF}(R) \subseteq \mathbb{I}_{QF}(R) \subseteq \mathbb{I}_C(R)$

If both a and b belong to $I(\sigma)$ whenever $ab \in I(\sigma)$ then either a or b belongs to $I(\sigma)$ when $ab \in I(\sigma)$: $\mathbb{I}_F(R) \subseteq \mathbb{I}_{SF}(R)$. If $\sigma \in \mathbb{I}_{SF}(R)$, then $a^2 \in I(\sigma)$ only when $a \in I(\sigma)$: $\mathbb{I}_{SF}(R) \subseteq \mathbb{I}_{QF}(R)$. That $\mathbb{I}_{QF}(R) \subseteq \mathbb{I}_C(R)$ has been shown above.

Definition (1) An aggregate $\sigma \in A(R)$ ~~σ for which $I(\sigma)$~~ for which, when $c \in I(\sigma)$ and $d \in R$, $ab = cd$ only when either a or b is in $I(\sigma)$ is said to be half-factored in R

(2) $\mathbb{I}_{HF}(R)$ is the class of such aggregates in $A'[R]$.

~~(ii) $\mathbb{I}_{S1}$~~ A proof of completeness of the class of aggregates that are half factored in R breaks down as above

(ii) $\mathbb{I}_{S\neq I}(R) \cap \mathbb{I}_{SF}(R) \subseteq \mathbb{I}_{HF}(R)$

Suppose that $\sigma \in \mathbb{I}_{S1}(R) \cap \mathbb{I}_{SF}(R)$ and that $ab = cd$ where $c \in I(\sigma)$. Since $\sigma \in \mathbb{I}_{S1}(R)$, $cd = c' \in I(\sigma)$. Since $\sigma \in \mathbb{I}_{SF}(R)$ and $ab = c'$

where $c' \in J(\sigma)$, either a or b is in $J(\sigma)$: $\sigma \in \mathbb{I}_{HF}(R)$.

Definition

$\mathbb{I}_{min}(R)$ is the class of aggregates $\xi \in A[R]$ for which $J(\sigma) \subseteq RJ(\sigma)$. (iii) $\mathbb{I}_{min}(R) \cap \mathbb{I}_{HF}(R) \subseteq \mathbb{I}_{SF}(R)$. Suppose that $\sigma \in \mathbb{I}_{min}(R) \cap \mathbb{I}_{HF}(R)$ and that $ab = c'$ where $c' \in J(\sigma)$. Since $\sigma \in \mathbb{I}_{min}(R)$, $c \in J(\sigma)$ and $d \in R$ exist such that $c' = cd$. Since $\sigma \in \mathbb{I}_{HF}(R)$, either a or b is in $J(\sigma)$: $\sigma \in \mathbb{I}_{SF}(R)$.

~~($\mathbb{P} \in \mathbb{C}'(R,S)$ if and only if $\mathbb{P}(\xi)$ exists for all $\xi \in A(R,S)$~~

also some properties of $P \cap P'$ etc ? ~~→ preceding section~~

(on completeness)