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~~Factor relationships of the form $c=fd$~~

In certain of the results of the preceding sections it is assumed that, c being a prescribed mapping system, further mapping systems E and d satisfy the relationship $c=Ed$. In others it is assumed that E is nonsingular over its domain of definition and d is constructed by use of the relationship $d=E^{-1}c$, so that $c=Ed$ again. Subject to appropriate conditions, relationships between homogeneous constraint systems for c and d involving E imply that $c=Ed$. Again, suitable relationships between two function systems ϕ and ψ involving E also induce the relationship $c=Ed$. Conjoint assumptions concerning homogeneous constraint systems and function systems also induce the result in question.

Before giving the results referred to it is convenient to provide some special notations and establish certain lemmata.

Preliminary definitions

Definition. In the following, where relevant, $m, n, r \geq 0$ are integers, $M \in \text{MS}\{\mathbb{B} \rightarrow K; m, n\}$, $M' \in \text{MS}\{\mathbb{B} \rightarrow K; r, m\}$, $c \in \text{col}\{\mathbb{B} \rightarrow K; m\}$, $d \in \text{col}\{\mathbb{B} \rightarrow K; n\}$, $\phi \in \text{row}\{\mathbb{B} \rightarrow K; m\}$, $b : \mathbb{B} \xrightarrow{\text{seq}} \mathbb{E}_{\mathbb{Z}}^{[m]}$, and $\beta \in \mathbb{B} \rightarrow \text{seq}[n]$ and $\beta \in \mathbb{B}$. $\mathbb{Z} \in \mathbb{B}$, $G \subseteq \mathbb{B}$

matrix and vector mapping systems

1] Sections of matrices and vectors

i) $M_{\beta}^b(z) = M_{\beta}^{\begin{bmatrix} b(z) \\ \beta(z) \end{bmatrix}}(z)$ is the b, β section of M at z . In the

special cases in which $\beta(z) = [[m]]$ and $b_{\beta}(z) = [im]$, $M_{\beta}^b(z)$ is written as $M_{\beta}^b(z)$ and $M_{\beta}^{\beta}(z)$ respectively.

ii) $c^b(z) = c^{\begin{bmatrix} b(z) \end{bmatrix}}(z)$ and $\phi_b(z) = \phi_{[b(z)]}(z)$ are the b -sections of c and ϕ respectively at z .

Mapping systems

2] Matrices of maximal section rank

i) The notation $M \in \text{MS}_{\text{MR}(\beta; G)}\{\mathbb{B} \rightarrow K; m, n\}$ indicates that for each $z \in G$, $M_{\beta}^b(z) \in \text{MS}\{\mathbb{B} \rightarrow K; m, n\}$ and that, for each $z \in G$,

$M_{\beta}^b(z)$ is of rank $\min(|b(z)|, |\beta(z)|) + 1$. In the special case in which $\beta(z) = [[n]]$ over G , the notation $M \in \text{MS}_{\text{MR}(\beta; G)}\{\mathbb{B} \rightarrow K; m, n\}$

is used. The notation $M \in \text{MS}_{\text{MR}(\beta; G)}\{\mathbb{B} \rightarrow K; m, n\}$ concerning $M_{\beta}^b(z)$ for $z \in G$ has a similar meaning.

ii) The notation $\text{col}^{\text{MR}(b;G)}\{B \rightarrow K; m\}$ indicates that $\text{col}\{B \rightarrow K; m\}$ and that, for each $z \in G$, $c^b(z)$ has at least one nonzero component. The notation $\text{row}_{\text{MR}(b;G)}\{B \rightarrow K; m\}$ concerning $\phi_b(z)$ for $z \in G$ has a similar meaning.

3] Domains of constancy

i) $\text{DC}\{G | M; c, d : b, \frac{b}{3}\}$ is the domain of joint constancy of $M_{\frac{b}{3}}, c^b$ and $d^{\frac{b}{3}}$ about G . It is the set of all $z \in B$ for which, for at least one $z \in G$, in conjunction $\{b(z')\} = \{b(z)\}$, $\{\frac{b}{3}(z')\} = \{\frac{b}{3}(z)\}$, $M_{[\frac{b}{3}(z)]}(z') = M_{[\frac{b}{3}(z)]}(z)$, $c^{[b(z)]}(z') = c^b(z)$ and $d^{[\frac{b}{3}(z)]}(z') = d^{\frac{b}{3}}(z)$.

ii) $\text{DC}'\{G | M; c, d : b, \frac{b}{3}\}$ is the domain of strict constancy of $M_{\frac{b}{3}}$, c^b and $d^{\frac{b}{3}}$ about G . It is the set of all $z \in B$ for which, for at least one $z \in B$, in conjunction $b(z') = b(z)$, $\frac{b}{3}(z') = \frac{b}{3}(z)$, $M_{\frac{b}{3}}(z') = M_{\frac{b}{3}}(z)$, $c^b(z') = c^b(z)$ and $d^{\frac{b}{3}}(z') = d^{\frac{b}{3}}(z)$.

4] Intersection

A mapping system $\hat{M} \in \text{MS}\{B \rightarrow K; m, n\}$ for which $\hat{M}_{\frac{b}{3}}^b(z) = M_{\frac{b}{3}}^b(z)$ for $z \in G$ is said to $b, \frac{b}{3}$ intersect M over G ; $\text{IN}(M; b, \frac{b}{3}; G)$ is the space of all mapping systems \hat{M} having this property.

5] Pre-quotient spaces

i) A mapping system $\hat{M} \in \text{MS}\{B \rightarrow K; m, n\}$ for which

$$\hat{M}_{\frac{b}{3}}^b(z) d^{\frac{b}{3}}(z) = c^b(z)$$

for $z \in G$ is called a $b, \frac{b}{3}$ prequotient of c by d over G ; $\text{PQ}(c/d | b, \frac{b}{3}; G)$ is the space of all mapping systems having this property.

ii) A mapping system $\hat{M} \in \text{MS}\{B \rightarrow K; r, n\}$ for which

$$\hat{M}_g(z)d^{\frac{1}{3}}(z) = M'_g(z)c^{\frac{b}{3}}(z)$$

for $z \in G$ is called a $b, \frac{1}{3}$ -^{sectional}~~quotient~~ of M', c by d over G ;

$PQ(M', c/d | b, \frac{1}{3}; G)$ is the space of all mapping systems \hat{M} having this property.

iii) A mapping system $\psi \in \text{row}\{B \rightarrow K; n\}$ for which

$$\psi_g(z)d^{\frac{1}{3}}(z) = \phi_g(z)c^{\frac{b}{3}}(z)$$

for $z \in G$ is called a $b, \frac{1}{3}$ -^{sectional}~~quotient~~ of ϕ, c by d over G ;

$PQ(\phi, c/d | b, \frac{1}{3}; G)$ is the space of mapping systems ψ having this property.

6] Relative products

i) A mapping system $\hat{M} \in \text{MS}\{B \rightarrow K; r, n\}$ for which

$$\{\hat{M}_g(z) - M'_g(z)M_{\frac{b}{3}}^{\frac{b}{3}}(z)\}d^{\frac{1}{3}}(z) = O^{[r]}$$

for $z \in G$ is called a $b, \frac{1}{3}$ -product of M' and M relative to d over G ; $RP\{M', M; d | b, \frac{1}{3}; G\}$ is the space of all mapping systems \hat{M} having this property.

ii) A mapping system $\psi \in \text{row}\{B \rightarrow K; n\}$ for which

$$\{\psi_g(z) - \phi_g(z)M_{\frac{b}{3}}^{\frac{b}{3}}(z)\}d^{\frac{1}{3}}(z) = O$$

for $z \in G$ is called a $b, \frac{1}{3}$ -product of ϕ and M relative to d over G ; $RP\{\phi, M; d | b, \frac{1}{3}; G\}$ is the space of all mapping systems ψ having this property.

(iii) the mapping $b(z)$ in step 4 in (ii) above is now given by $b(z) = \sum_{t=1}^T b_t(z) \cdot f_t$ and f_t respectively; in both cases M is the corresponding post factor.

7] Complete sequences sections

In the special cases in which the sequences $b(z)$ and $\tilde{b}(z)$ simultaneously take the complete forms $b(z) = [m]$ and $\tilde{b}(z) = [n]$ for $z \in G$, the terminologies and notations described in [3-6] above are simplified. In [4] \hat{M} is simply said to intersect M over G . In [5i] \hat{M} is called a prequotient of c by d over G . In [5ii] \hat{M} is called a prequotient of M' by d over G . In [5iii] ϕ is called a prequotient of ϕc over G . In [6i] \hat{M} is called a product of M' and M relative to d over G . In [6ii] ϕ is called a product of ϕ and M relative to d over G . The notation $DC\{G|M; c, d : b, \tilde{b}\}$ of [3] is written as ~~DC $\{G|M; c, d\}$~~ , $DC\{G|M; c, d\}$. The further abbreviated notations $IN(M|G)$, $PQ(c/d|G)$, $PQ(M'/d|G)$, $PQ(\phi c/d|G)$, $RP(M', M; d|G)$ and $RP(\phi, M; d|G)$ have similar derivations.

8] Integer frameworks

i) The notation $\{s; p, \mu\} \in SF\{B; m, n\}$ indicates that the set mapping $s: B \rightarrow \mathbb{Z}[\min(m, n)]$ and the integer mappings $p: B \times s(B) \rightarrow F[m]$, $\mu: B \times s(B) \rightarrow F[n]$ constitute an integer framework over B in the sense that for each $z \in B$ the sets $\{p(z, w)\}$ $\{w = s(z)\}$ are disjoint while $\bigcup\{p(z, w) | w = s(z)\} = [m]$ and the sets $\{\mu(z, w)\}$ $\{w = s(z)\}$ satisfy a similar condition.

ii) Let $\{s; \rho, \mu\} \in SF \{B; m, n\}$ be prescribed.

a) With $z, w \in s(z)$, $M_{[\rho(z, z)]}^{[\rho(z, z)]}(z)$, $M_{[\mu(z, w)]}^{[\rho(z, w)]}(z)$, $M_{[\mu(z, w)]}^{[\rho(z, w)]}(z)$, $M_{[\mu(z, w)]}^{[\rho(z, w)]}(z)$, $M_{[\mu(z, w)]}^{[\rho(z, w)]}(z)$, $c^{[\rho(z, w)]}(z)$ and $\phi_{[\rho(z, w)]}(z)$ are denoted by

$M_\rho^P(z; z, \omega)$, $M_\mu^P(z, \omega)$, $M^P(z, \omega)$, $M_\mu(z, \omega)$, $c^P(z, \omega)$ and $\phi_P(z, \omega)$ respectively.

b) With $z, w \in s(z)$ for $z \in G$, the notation $M \in MS_{MR(\rho; G, z)}^{MR(\rho; G, z)} \{B \rightarrow K; m, n\}$ indicates that $M \in MS \{B \rightarrow K; m, n\}$ and that, for each $z \in G$, $M_{\rho}^P(z; z, \omega)$ is of rank $\min(|\rho(z, z)|, |\mu(z, \omega)|) + 1$. The notations $M \in MS_{MR(\rho; G, z)}^{MR(\rho; G, z)} \{B \rightarrow m, n\}$ and $M \in MS_{MR(\mu; G, \omega)}^{MR(\mu; G, \omega)} \{B \rightarrow K; m, n\}$ have similar meanings with respect to the submatrices $M^P(z, \omega)$ and $M_\mu(z, \omega)$ respectively.

c) Let $w \in s(z)$ for all $z \in G$

(α) $DC \{G | M; c, d : s; \rho, \mu : w\}$ and (β) $DC' \{G | M; c, d : s; \rho, \mu : w\}$ are the complete sets of $z \in B$ for which, for at least one $z \in G$,

in conjunction $w \in s(z')$ and

(α) $\{\rho(z', \omega)\} = \{\rho(z, \omega)\}$, $\{\mu(z', \omega)\} = \{\mu(z, \omega)\}$, $M_{[\rho(z, \omega)]}^{[\rho(z, \omega)]}(z') = M_\rho^P(z, \omega)$,

(β) $\{\rho(z', \omega)\} = \{\rho(z, \omega)\}$ and $d_{[\mu(z, \omega)]}^{[\mu(z, \omega)]}(z') = d_\mu^P(z, \omega)$

(α) $\rho(z', \omega) = \rho(z, \omega)$, $\mu(z', \omega) = \mu(z, \omega)$, $M_{\rho}^P(z', \omega) = M_\rho^P(z, \omega)$, $c^P(z, \omega) =$
 $c^P(z')$ and $d_\mu^P(z', \omega) = d_\mu^P(z, \omega)$;

respectively

$IN(M; \rho, \mu; G, \omega)$ is the space of all mapping systems

$\hat{M} \in MS \{B \rightarrow K; m, n\}$ for which $\hat{M}_{\rho}^P(z, \omega) = M_\rho^P(z, \omega)$ for $z \in G$.

$PQ(c/d | \rho, \mu; G, \omega)$ is the space of all mapping systems

$\hat{M} \in M\{\mathbb{B} \rightarrow K; m, n\}$ for which $\hat{M}_\mu^P(z, \omega) d^\mu(z, \omega) = C^P(z, \omega)$ for all $z \in G$. The notations $PQ(M', c/d | \rho, \mu; G, \omega)$, $PQ(\phi, c/d | \rho, \mu; G, \omega)$, $RP(M', M; d | \rho, \mu; G, \omega)$ and $RP(\phi, M; d | \rho, \mu; G, \omega)$ have similar meanings.

iii) The notation $\{s; \rho, \mu\} \in SF_C\{\mathbb{B}; m, n\}$ indicates that the section framework $\{s; \rho, \mu\} \in SF\{\mathbb{B}; m, n\}$ is complete over \mathbb{B} in the sense that $[\cup \{p(z, \omega)\}_{\omega=s(z)}] = [m]$ for each $z \in \mathbb{B}$ and that the sets $\{\mu(z, \omega)\}$ satisfy a similar condition.

iv) a) Let $\{s; \rho, \mu\} \in SF\{\mathbb{B}; m, n\}$. For each $z \in \mathbb{B}$, let $S(z) = \omega'(z), \dots, \omega''(z)$ be a sequence formed from the members of $\{s(z)\}$. Define the mapping $b: \mathbb{B} \rightarrow F[m]$ by ~~defining~~ taking $b(z)$ to be the compound sequence $p(z, \omega'(z)), \dots, p(z, \omega''(z))$ for each $z \in \mathbb{B}$; define $\tilde{s}: \mathbb{B} \rightarrow F[n]$ in terms of μ similarly. A pair of mappings $\{b, \tilde{s}\}$ defined in this way is called a section structure derived from $\{s; \rho, \mu\}$. $SS[\mathbb{B}, \{s; \rho, \mu\}]$ is the complete system of such section structures.

b) With $\{b, \tilde{s}\} \in SS[\mathbb{B}, \{s; \rho, \mu\}]$, $S\{b, \tilde{s}\} | \{s; \rho, \mu\}$ is the sequence mapping $S: \mathbb{B} \rightarrow F[\min(m, n)]$ occurring in the above definition.

c) With $\{s; \rho, \mu\} \in SF\{B; m, n\}$, $SS_C\{s; \rho, \mu\}$ is the complete system of sequence pairs $\{b, s\} \in SS\{s; \rho, \mu\}$ for which $\{S\{b, s\} | s; \rho, \mu\}\} = s$ over B .

9] Block structure

With the integer framework $\{s; \rho, \mu\} \in SF\{B; m, n\}$ over B

specified, $BS\{B \rightarrow K; m, n | s; \rho, \mu\}$ is the space of mapping systems $\hat{M} \in MS\{B \rightarrow K; m, n\}$ for which, $\forall z \in B$ with $\{b, s\} \in SS_C\{s; \rho, \mu\}$ for each $z \in B$, the nonzero elements of $\hat{M}(z)$ are confined to the $|s(z)|+1$ blocks $\hat{M}_{\mu}(z, \omega) \{ \omega = s(z) \}$, so that

$$\hat{M}_{\mu}(z; \tau, v) = O^{\begin{bmatrix} |\rho(z, z)| \\ |\mu(z, v)| \end{bmatrix}}$$

for $z = s(z)$ and $\tau = s(z) \setminus z$.

A mapping system $\hat{M} \in BS\{B \rightarrow K; m, n | s; \rho, \mu\}$ for which $\{s; \rho, \mu\} \in SF_C\{B; m, n\}$ is said to possess complete block structure.

10] Constant mappings

i) With s a prescribed mapping from B and $G \subseteq B$, the notation $s_{\text{const}}(G)$ indicates that the value of $s(z)$ is constant for all $z \in G$. The notation $[s_{\text{const}}(G)] = \tilde{s}$ indicates that $s(z) = \tilde{s}$ for $z \in G$.

ii) With s a sequence in the preceding, the notation $\{s\}_{\text{const}}(G)$ indicates that the set from which $s(z)$ is formed is constant for all $z \in G$; the notation $[\{s\}_{\text{const}}(G)] = \hat{s}$ has a similarly extended meaning. These notations are used in connection with further structures derived from mappings and with mapping systems.

iii) With $\{s; p, \mu\} \in SF\{B; m, n\}$, the notations $\{s; p, \mu\} \in \text{const}(G)$ and $[s; \{p\}, \{\mu\}] \in \text{const}(G)$ indicates that $[s \in \text{const}(G)] = \tilde{s}$ and, setting $p(z, \omega) = p^{(\omega)}(z)$, $\mu(z, \omega) = \mu^{(\omega)}(z)$, that $p^{(\omega)}, \mu^{(\omega)} \in \text{const}(G) \{ \omega = \tilde{s} \}$.
and $[s; \{p\}, \{\mu\}]$

iii) With $\{s; p, \mu\} \in SF\{B; m, n\}$, the notations

(a) $\{s; p, \mu\} \in \text{const}(G)$ and (b) $[s; \{p\}, \{\mu\}] \in \text{const}(G)$

indicate that $[s \in \text{const}(G)] = \tilde{s}$ and, setting $p(z, \omega) = p^{(\omega)}(z)$, $\mu(z, \omega) = \mu^{(\omega)}(z)$ that

a) $p^{(\omega)}, \mu^{(\omega)} \in \text{const}(G) \{ \omega = \tilde{s} \}$ and

b) $\{p^{(\omega)}\}, \{\mu^{(\omega)}\} \in \text{const}(G) \{ \omega = \tilde{s} \}$

respectively.

In [4] above, the dimensions of the mapping system \hat{M} are those of M ; the constituents of the space $IN(M; b, \tilde{s}; G)$ are specified by the parameter list entry M in the parameter list of the notation describing this space. In [5] above, the dimensions of \hat{M} are in order those of c and d ; the dimensions of the constituents of the space $PQ(c/d | B, \tilde{s}; G)$ are specified by c and d . The dimensions of the constituents of all spaces defined above are specified in a similar way.

Concerning [6] above, it is remarked that when $b = [[m]]$, $\tilde{s} = [[n]]$, $DC(G|M; c, d; b, \tilde{s})$ = the two sets $DC(G|M; c, d; b, \tilde{s})$ and $DC(G|M; c, d; b, \tilde{s})$ = domains of constancy are identical: a special notation of the form $DC'(G|M; c, d)$ is redundant.

The ^{sextin} integer framework $\{s; p, \mu\} \in \mathbb{S}\mathcal{F}\{B; m, n\}$ specifying
 the block structure of a mapping system $M \in \mathbb{B}\mathcal{S}\{B \rightarrow k;$
 $m, n | s; p, \mu\}$ is not uniquely determined by M . For
 each $z \in B$, $s(z)$ may be replaced by a reordered
 version of this sequence. For each $z \in B$ and $w \in s(z)$,
 $p(z, w)$ may be replaced by a reordered version; and
 the same holds with regard to ~~p(z,w)~~. the subsequences
 $\mu(z, w)$. If, for some $z \in B$, $|s(z)| > 0$, $s(z)$ may be
 replaced by one of its ^{subsets} ~~subsequences~~, and certain of the
 sequences $p(z, w)$ may be ^{and} compatibly combined, and, the sequences
 $\mu(z, w)$ being reinterated in corresponding fashion, to form another
 integer framework which still describes the structure of
 M . (In this case, $\hat{p}(z, w)$ and $\hat{\mu}(z, w)$ being the combined
 sequences, $M_{\hat{\mu}}(z, w)$ contains certain of the submatrices
 $M_{\hat{\mu}}^{\hat{p}}(z, w)$ compounded from, among other constituents,
 zero submatrices.) Again Conversely it may occur that the
 structure of M is such that for some $z \in B$, $s(z)$ may be
 extended, and certain of the sequences $p(z, w), \mu(z, w)$ being
 decomposed.

The notation $MS_{MR(\mu; G, \omega)}^{MR(\rho; G, z)} \{B \rightarrow K; m, n\}$ of (8ib) above is, in particular, also used when G consists of a single member, z , and is then written as $MS_{MR(\mu; G, \omega)}^{MR(\rho; z, z)} \{B \rightarrow K; m, n\}$. The further notations of (8ib, c) are ~~then~~ used in the same way.

The condition of [9] above specifying the mapping systems $\hat{M} \in BS \{B \rightarrow K; m, n\}$ may, with $\hat{M}(z) = \hat{M}_z^B(z)$ for $z \in B$ be formulated as

$$\hat{M} \begin{bmatrix} \rho(z, \omega) \\ \mu(z, \omega) \end{bmatrix} \begin{pmatrix} z \\ \omega \end{pmatrix} = \begin{pmatrix} [|\beta(z)| - |\rho(z, \omega)| - 1] \\ [|\mu(z, \omega)|] \end{pmatrix}$$

or as

$$\hat{M} \begin{bmatrix} \rho(z, \omega) \\ \mu(z, \omega) \end{bmatrix} \begin{pmatrix} z \\ \omega \end{pmatrix} = \begin{pmatrix} [|\rho(z, \omega)|] \\ [|\beta(z)| - |\mu(z, \omega)| - 1] \end{pmatrix}$$

for each $z \in B$ and $\omega = s(z)$ in both cases.

Existence theory

The existence of domains of constancy and intersection
mapping systems

$$E \in MS\{B \rightarrow K; m, n\}$$

() Let $c \in \mathbb{S}\{B \rightarrow K; m\}$, $d \in \mathbb{S}\{B \rightarrow K; n\}$, $b: B \rightarrow F[m]$, $\beta: B \rightarrow F[n]$

and $\text{ker } b \cap G \subseteq B$

if) The space $DC\{z | c, d : b, \beta\}$ is nonvoid

if) Let $E \in MS\{B \rightarrow K; m, n\}$. The spaces $DC\{z | E; c, d : b, \beta\}$ and

ii) $IN(E; b, \beta; G)$ are nonvoid.

The spaces $DC\{z | c, d : b, \beta\}$ and $DC\{z | E; c, d : b, \beta\}$ contain G ; and $IN(E; b, \beta; G)$ contains E .

The existence of prequotient spaces

() Let $c \in \mathbb{S}\{B \rightarrow K; m\}$, $d \in \mathbb{S}\{B \rightarrow K; n\}$ and $\text{ker } b \cap G \subseteq B$

i) Let $b: B \rightarrow F[m]$ and $\beta: B \rightarrow F[n]$

ii) $PQ(c/d | b, \beta; G)$ is nonvoid if and only if, either for each $z \in G$,

a) $d^\beta(z) + O^{[1/\beta(z)]}$ or

b) $c^b(z) = O^{[1/b(z)]}$ and $d^\beta(z) = O^{[1/\beta(z)]}$

iii) If

c) $c^b(z) = O^{[1/b(z)]}$ for $z \in G$

$PQ(c/d | b, \beta; G)$ is nonvoid

iv) If condition (b) holds, $PQ(c/d | b, \beta; G) = MS\{B \rightarrow K; m, n\}$

- for $z \in G$,
- (iv) Let $|P(z)| = |\S(z)|$ and condition (a) hold. Either
for at least one $z \in G$ and then
- condition (c) holds and $E_{\frac{b}{\S}}(z)$ is singular for all $E \in PQ(c/d; b, \frac{b}{\S}; z)$ or
 - is void, or for any $z \in G$ and then $MS\{B \rightarrow K; m, n\} \cap PQ(c/d; b, \frac{b}{\S}; z)$
 - condition (c) does not hold and $PQ(c/d; b, \frac{b}{\S}; z)$ then contains E
(1) $MS_{MR\{B \rightarrow K; m, n\}} \cap MS_{MR\{b, \frac{b}{\S}\}} \cap PQ(c/d; b, \frac{b}{\S}; z)$
for which $E_{\frac{b}{\S}}(z)$ is nonsingular and (z) is of rank $\min(m, n) + 1$
is nonvoid

i) Let conditions (iv) hold. An example of a mapping system E belonging to the space described specified (1)
~~E~~ described in that ~~whereas~~ may be defined by setting $E = E(z)$
over $B \setminus G$ where $z \in B \setminus G$ is an arbitrarily chosen member of G and, for each $z \in G$,
over B where $E(z)$ is specified in the following way. Select

$t \in b(z)$ and $r \in \S(z)$ for which $c_t(z), d_r(z) \neq 0$. Let $b(z; k), \S(z; k)$
($k = [h]$), where $h = |b(z)|$, be the successive members of $b(z), \S(z)$
respectively, and determine $t', r' \in [h]$ from the relationships
 $t = b(z, t')$, $r = \S(z, r')$. Define the integer bijection $\chi: [h] \rightarrow [h]$
by setting $\chi(k) = k + r' - t' \bmod (h+1)$ ($k = [h]$) and the integer
bijection $\nu: \{b(z)\} \rightarrow \{\S(z)\}$ by setting $\nu(z) = \S(z, \chi(k))$ where
 $z = b(z, \nu(z))$ for $k = [h]$. Define $E_{\frac{b}{\S}}^b(z)$ by setting $E_{z, \nu(z)}(z) = 1$

and

$$E_{z, r}(z) = \{c_z(z) - d_{\nu(z)}(z)\} / d_r(z)$$

both for $z = b(z) \setminus t$, $E_{t, r}(z) = c_t(z) / d_r(z)$ and the remaining elements
of $E_{\frac{b}{\S}}^b(z)$ equal to zero. If $m \leq n$, select $M \in F[n]$ for which $|M| = m$
and $\S(z) \subseteq M$ and denote $E_{[m]}(z)$ by $E(M)$. If $h < m$, set
 $E(M)_{[\S(z)]}^{[b(z)]} = \mathbb{I}_{(m-h-1)}$, $E(M)_{[\S(z)]}^{[b(z)]} = \mathbb{O}_{[h]}$ and $E(M)_{[\S(z)]}^{[b(z)]} = \mathbb{O}_{[m-h-1]}$.

If $m < n$, set $E_{[m]}(z) = O_{[n-m-1]}^{[m]}$. If $n < m$, select $N \in F[m]$ for which $|N| = n$ and $b(z) \subseteq N$; denote $E^{[N]}(z)$ by $E(N)$ and use $E(N)$ in versions of the preceding allocations obtained by interchanging m and n .

2] Let $\{s; p, \mu\} \in SF\{B; m, n\}$. For each $z \in G$ and $w \in s(z)$

a) $|p(z, w)| = |\mu(z, w)| = h(w)$ and either

$$b) c^p(z, w) = d^\mu(z, w) = O^{[h(w)]} \text{ or}$$

$$c) c^p(z, w) \neq O^{[h(w)]} \text{ and } d^\mu(z, w) \neq O^{[h(w)]}.$$

(i) The intersection of the spaces

$$(2) \left\{ \begin{array}{l} BS\{B \rightarrow K; m, n | s; p, \mu\} \\ \cap [LPQ(c/d | b, \xi; G) \cap MS_{MR(b; G)}^{MR(b; G)} \{B \rightarrow K; m, n\}] \{ \{b, \xi\} = SS[G, \\ \{s; p, \mu\}] \}] \\ MS_{MR(G)} \{B \rightarrow K; m, n\} \end{array} \right.$$

is nonvoid.

ii) An example of a mapping system E belonging to the intersection of the spaces (2) may be constructed by setting $E = E$ over $B \setminus G$ where E is an arbitrary member of $BS\{B \rightarrow K; m, n | s; p, \mu\}$
 ~~$E = E(z)$ over $B \setminus G$ where z is an arbitrarily chosen member of G~~
 set $E_\mu^p(z, w) = I(h(w))$. For each $w \in s(z)$ for which condition (a) holds, construct $E_\mu^p(z, w)$ by replacing $b(z), \xi(z)$ by $p(z, w), \mu(z, w)$

in the specification of $E_{\frac{B}{S}}^P(z)$ given in the preceding part.

Set $E_{\mu}^P(z; z, \omega) = O \begin{cases} |\rho(z, z)| \\ |\mu(z, \omega)| \end{cases} \{z, \omega = S(z) \text{ } (z \neq \omega)\}$. With $\{\hat{b}, \hat{s}\} = S_C \{s; \rho, \mu\}$ construct that part of $E(z)$ from which $E_{\frac{B}{S}}^P(z)$ has been removed as described in clause (iv) with b, s replaced by \hat{b}, \hat{s} .

3] Let $b: B \rightarrow F[m]$ and $\hat{s}: B \rightarrow F[n]$ with $|B(z)| = |\hat{s}(z)|$ for $z \in G$. Denote by $S(G)$ the complete system of pairs $\{\hat{b}, \hat{s}\}$

of sequence mappings $\hat{b}: G \rightarrow F[m]$, $\hat{s}: G \rightarrow F[n]$ for which $\{\hat{b}(z)\} \subseteq \{b(z)\}$, $\{\hat{s}(z)\} \subseteq \{\hat{s}(z)\}$ for $z \in G$ and $|\hat{b}(z)| = |\hat{s}(z)|$ all for $z \in G$. Define the integer mapping

for $z \in G$. Furthermore, denoting the successive members of $b(z), \hat{s}(z)$ by $b(z; \kappa), \hat{s}(z; \kappa)$ assume that for each $z \in G$ and $\kappa \in [1 b(z)]$, $c_{b(z; \kappa)}(z)$ and $d_{\hat{s}(z; \kappa)}(z)$ are either both zero or both nonzero.

$\chi: G \times [m] \rightarrow [n]$ by setting $\hat{s}(z; \kappa) = \chi \{z, b(z; \kappa)\}$ for $\kappa \in [1 b(z)]$.

i) The intersection of the spaces

$$\left[\cap [PQ(c/d | \hat{b}, \hat{s}; G)] \cap MS_{\frac{MR(\hat{b}; G)}{MR(\hat{s}; G)}} \{B \rightarrow K; m, n\} \right] \{ \hat{b}, \hat{s} \} = S(G).$$

$MS_{\frac{MR}{m}} \{B \rightarrow K; m, n\}$ is nonvoid and contains E for which $E_{[x(z, z)]}^{[z]}(z) = 0_{[m-1]}, E_{[x(z, z)]}^{[z]}(z) = 0$ for $z = b(e)$.

ii) An example of such an E may be constructed by setting $E = E(z)$ over $B \setminus G$ where z is an arbitrarily chosen member of G and, for each $z \in$

each $z \in b(z)$ for which $E(z)$ is determined by setting $E_{\tau, \gamma}(z) = 1$ for which $c_\tau(z) = d_{\gamma}(z) = 0$ and $E_{\tau, x}(z) = c_\tau(z)/d_{\gamma}(z)$ for each $z \in b(z)$ for which $c_\tau(z) \neq 0, d_{\gamma}(z) \neq 0$, the remaining elements of $E^b(z)$ being set equal to zero; the remaining part of $E(z)$ is constructed as described in clause (iv), by use of the matrices $E(M)$ or $E(N)$, as described in clause (iv)

The relationship $\#$

$$(g) \quad E^b_{\frac{1}{3}}(z)d^{\frac{1}{3}}(z) = c^b(z)$$

holding for $z \in G$ characterises the mapping systems E of $PQ(c/d|b, \frac{1}{3}; z)$. If, for some $z \in G$, $E \in PQ(c/d|b, \frac{1}{3}; z)$ if and only if $E \in PQ(c/d|b, \frac{1}{3}; z)$ for all $z \in G$. Then condition (1ia) holds, $d_r(z) \neq 0$ for some $r \in \frac{1}{3}(z)$. Set

$$E_{\tau, r}(z) = c_\tau(z)/d_r(z) \text{ for } z \in b(z) \text{ and the remaining}$$

elements of $E(z)$ equal to zero. Any $E \in MS\{B \rightarrow K; m, n\}$ for which $E(z)$ is as just constructed) [n] Then E is in $PQ(c/d|b, \frac{1}{3}; z)$ which is accordingly nonvoid.

If condition (1ib) holds, any $E \in MS\{B \rightarrow K; m, n\}$ satisfies

condition (3) and is in $PQ(c/d|b, \frac{1}{3}; z)$. (This remark disposes of clause (1ii)).

Either of conditions (1ia, b) obtains if and only if

$d^{\frac{1}{3}}(z) = 0$ [1 $\frac{1}{3}(z)$] and $c^b(z) \neq 0$. In this case, no $E \in MS\{B \rightarrow K; m, n\}$

satisfies condition (1). Clause (1i) has been dealt with. If

condition (1ic) is satisfied either $d^{\frac{1}{3}}(z) = 0$ [1 $\frac{1}{3}(z)$] in which case

then condition (1ib) is satisfied, or $d^{\frac{1}{3}}(z) \neq 0$ [1 $\frac{1}{3}(z)$] in which case

condition (1ia) is satisfied. In either case $PQ(c/d|b, \frac{1}{3}; z)$ is nonvoid: the result of clause (1a) follows.

If, under the conditions of clause (iv), $c^b(z) = 0$ [1 $\frac{1}{3}(z)$], the

columns of the square matrix $E_{\xi}^b(z)$ satisfy a homogeneous linear relationships with coefficients that are not all zero: $E_{\xi}^b(z)$ is singular, the result of clause (1iv). The result of clause (1ve) follows automatically once it has been shown that the mapping system E specified whose construction is described in clause (1v) is in the space specified in clause (1ve).

The mapping χ defined in clause (1v) is evidently a bijection, as described. The argument range of χ is $\{b(z)\}$. Since χ is a bijection all members of $\xi(z)$ feature as images of ω and each value of ω corresponds to a single value argument value: ω is a bijection. $\omega(z)=r$ if and only if $b(z; \kappa) = z$ with $z = b(z; \kappa)$ where $\xi(z, \chi(\kappa)) = r$. The latter relationship holds if and only if $\chi(\kappa) = r'$ and in turn this condition holds true if and only if $\kappa = t'$, when $b(z)$ and then $b(z, \kappa) = t$: $\omega(z) = r$ if and only if $z = t$. The k elements $E_{z, \omega(z)}(z)$ — row of $E_{\xi}^b(z)$ containing part of $E^{[z]}(z)$ contains two elements not defined identically to be zero: $E_{z, \omega(z)}(z)$ and $E_{z, r}(z)$. The scalar product of such a row and $d\xi(z)$ has the value

$$1 \cdot d_{\omega(z)}(z) + [\{c_z(z) - d_{\omega(z)}(z)\} / d_r(z)] d_r(z) = c_z(z)$$

The remaining row of $E_{\xi}^b(z)$ contains only one element not defined identically to be zero: $E_{t, r}(z)$. The corresponding scalar product is

$$\{c_t(z)/d_r(z)\} \cdot d_r(z) = c_t(z)$$

Relationship (3) is satisfied and $E \in PQ(c/d|b, \frac{b}{z}; z)$. $E_{\frac{b}{z}}(z)$ may be obtained from $I(h)$ by replacing the matrix defined by

~~The matrix obtained from $E_{\frac{b}{z}}^b(z)$ by replacing the column containing part of $E_{[r]}(z)$ by the column with whose some elements are~~

$$E_{z,r}(z) = 0 \text{ for } z = b(z) \setminus t.$$

~~of $E(z) \in MS\{K; m, n\}$~~
The submatrix $\hat{E}_{\frac{b}{z}}(z)$ defined by setting $\hat{E}_{z,t}(z) = 1$ and $\hat{E}_{z,r}(z) = 0$ for $z = b(z) \setminus t$ and $\hat{E}_{t,r}(z) = 1$ is a column row permuted

version of $I(h)$. $E_{\frac{b}{z}}^b(z)$ is obtained if differs from $\hat{E}_{\frac{b}{z}}(z)$ only in the column containing part of $E_{[r]}(z)$, and in $E_{\frac{b}{z}}^b(z)$,

$$E_{t,r}(z) = c_t(z)/d_r(z). |E_{\frac{b}{z}}^b(z)|$$
 has the form $\pm c_t(z)/d_r(z)$,

the sign depending upon the permutation involved. $E_{\frac{b}{z}}^b(z)$ is nonsingular : $E \in MS_{MR(\frac{b}{z}; z)}^{UR(b; z)} \{B \rightarrow K; m, n\}$

Suppose that $m \leq n$, so that $E(M)$ is used in the construction of E . Since the submatrices $E(M)_{[\frac{b}{z}(z)]}^{[B(z)]}$ and $E(M)_{[\frac{b}{z}(z)]}^{[B(z)]}$ are zero, $|E(M)|$ has the form $\pm |E(M)_{[\frac{b}{z}(z)]}^{[B(z)]}| |E_{\frac{b}{z}}^b(z)|$, that is the form $\pm c_t(z)/d_r(z)$: $E(M)$ is of rank $m+1$ which is a submatrix

of $E(z)$ is of rank $\in M\{K; m, n\}$, is of rank $m+1$. Since

The above construction is implemented for $z = \text{diag}$. E is defined as $B \setminus G$
 $E = E(z) \setminus B$, $E \in MS_{MR\{B \rightarrow K; m, n\}}$.

as described in clause (iv). The result of that clause follows.

Clause (2i) is disposed of by dealing with clause (2ii)

The relationship Select $\{b, \frac{z}{3}\} \in SS\{B1\}^z; p, \mu\}$. The relationship and let $S = S\{b, \frac{z}{3}\}^z; p, \mu$. The relationship Select $z \in G$. The relationship (4) $E_\mu^P(z, w) d^\mu(z, w) = c^P(z, w)$ | Evidently the mapping system E prescribed in clause (2ii) is in $BS\{B1\}^z; p, \mu$.

characterises the members of $PQ(c/d | p, \mu; z, w)$. If condition (2b) holds, the matrix $E_\mu^P(z, w) = I(h(w))$ This relationship satisfies the relationship (2b) slight change as does, when condition (2c) holds, the matrix $E_\mu^P(z, w)$ constructed by replacing $b(z), \frac{z}{3}(z)$ by $p(z, w), \mu(z, w)$ in the construct described in clause (1v). Thus $E \in PQ(c/d | p, \mu; z, w)$ for all $(w \in S(z))$ $w = s(z)$ and, in particular, for $w = S(z)$ where $S \subseteq S\{b, \frac{z}{3}\}^z; p, \mu$.

The relationship characterising the members of $PQ(c/d | b, \frac{z}{3}; z)$ is

$\sum_{w \in S(z)} E_\mu^P(z; z, w) d^\mu(z, w) = c^P(z, z)$

Relationship (3) characterises the members of $PQ(c/d | b, \frac{z}{3}; z)$.

With $b, \frac{z}{3}$ synthesized from components $p(z, w), \mu(z, w)$ as described in part (2i), relationship (3) may be expressed in terms of the submatrix relationship

$$\sum_{w \in S(z)} E_\mu^P(z; z, w) d^\mu(z, w) = c^P(z, z)$$

holding for $z = s(z)$. In the construction given in

In the construction of $E(z)$ described in clause (2ii), $E_\mu^P(z; z, w) = 0$ $\begin{bmatrix} [p(z, z)] \\ [\mu(z, w)] \end{bmatrix}$ for $z = s(z), w = s(z) \setminus z$. This condition holds, in particular, for all $z, w \in S(z)$ for which $z \neq w$. Thus, using relationship (4)

$$\sum_{\omega \in S(z)} E_{\mu}^P(z; z, \omega) d^{\mu}(z, \omega) = c^P(z, z)$$

for all $z \in S(z)$. In terms of $b(z), \dot{b}(z)$ synthesised from components $p(z, \omega), \mu(z, \omega)$ as described in clause (2), these relationships may be combined in the form (3) and reveal that $E \in PQ(c, d | b, \dot{b}; z)$. This relationship holds for $z \in G$ and $\{b, \dot{b}\} = SS[G, \{s; p, \mu\}]$.

That $E_{\mu}^P(z, \omega)$ is nonsingular ($\omega \in S[z]$) follows from the proof of clause (1iv): replace $b(z), \dot{b}(z)$ by replacing $b(z), \dot{b}(z)$ by $p(z, \omega), \mu(z, \omega)$ in the proof of the corresponding result in clause (iv).

$$E \in MS_{\substack{MR(p; z, \omega) \\ MR(\mu; z, \omega)}}^{\{B \rightarrow K; m, n\}}$$

The matrix $E_{\dot{b}}^B(z)$ is compounded of a system of submatrices $E_{\mu}^P(z; z, \omega)$ for which $E_{\mu}^P(z; z, \omega)$ of which those with $z \neq 0$ are zero. Also $|p(z, \omega)| = |\mu(z, \omega)|$ for $\omega \in S$. Thus $|E_{\dot{b}}^B(z)|$ has the form $\pm \prod_{\omega \in S} |E_{\mu}^P(z; z, \omega)|$ and is accordingly nonzero: $E \in MS_{\substack{MR(b; z) \\ MR(\mu; z)}}^{\{B \rightarrow K; m, n\}}$ again for $z \in G$ and $\{b, \dot{b}\} = SS[G, \{s; p, \mu\}]$.

Treating the case in which $m=n$ in particular, the method of proof of clause (iv) also suffices to show that in the case under consideration $|E(n)| = \pm |E_{\dot{b}}^B(z)|$ and again that $E \in MS_{\substack{B \rightarrow K; m, n}}^{\{B \rightarrow K; m, n\}}$.

Part [3] is a corollary to its predecessor. The sequences $p(z, \omega), \mu(z, \omega)$ of part [2] are taken to consist of one member each; the members in question being those of the sequences $b(z), \dot{b}(z)$ of

$$\rho(z, w) = b(z; w), \mu(z, w) = \frac{1}{z}(z; w)$$

part [3]: $\rho(z, w) = b(z; w), \mu(z, w) = \frac{1}{z}(z; w)$ for $w = s(z)$

part [3]. Denoting the successive members of

part [3]. If $s(z) = [1, b(z)] = [\frac{1}{z}(z)]$ and, denoting the successive members of $b(z)$ by $b(z; k), \frac{1}{z}(z; k)$ and those of $s(z)$ by

$$s(z; k), \rho\{z, s(z; k)\} = b(z; k), \mu\{z, s(z; k)\} = \frac{1}{z}(z; k) \text{ for } k = 1, s(z)$$

Condition (2a) is satisfied, ~~and~~ and $h(w) = 0$ for $w = s(z)$. When

$c_p(z; k)(z)$ and $d_p(z; k)$ are both zero, condition (2b) is satisfied;

when they are both nonzero, condition (2c) is satisfied. In the special case being treated, the ~~set S of~~ system $S(G)$ is taken to

be $\{s(z)\}$ and the sequences $b, \frac{1}{z}$ of that clause and part [3]

become the same. The first result of clause (3i) follows

immediately from that of clause (2i). In the construction described

clause (2ii), $E^{\rho(z, w)}_{[\mu(z, w)]}(z) = O^{\frac{1}{z}(z; m+1)}_{[m-k-1]}(h(w))$ and

$$E^{\rho(z, w)}_{[\mu(z, w)]}(z) = O^{\frac{1}{z}(z; m+1)}_{[h(w)]}. \text{ The second result of clause (3i) follows.}$$

In that construction $E^{\rho}_{\mu}(z, w) \leq K$ reduces to $w = 1$ when

condition (2b) is satisfied in the present case, and to

$c_p(z, w) / d_p(z, w)$ when condition (2c) is satisfied. The construction

outlined described in clause (3ii) is a special case of that described

in clause (2ii).

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $b: B \rightarrow F[m]$, $\beta \in B \rightarrow F[n]$, $c \in \text{MS}\{B \rightarrow K; r, m\}$ and $G \subseteq B$. Set $c(C|b, \beta; z) = C_b(z)c^\beta(z)$.

1] i) $PQ(C, c/d|b, \beta; G)$ is nonvoid if and only if, for each $z \in G$, either

a) $d^\beta(z) + O^{[l\beta(z)]}$ or

b) $c(C|b, z) = O^{[r]}$ and $d^{\beta(z)}(z) = O^{[l\beta(z)]}$

ii) If $c(C|b, z) = O^{[r]}$ for $z \in G$, $PQ(C, c/d|b, \beta; G)$ is nonvoid.

iii) If condition b) holds for $z \in G$, $PQ(C, c/d|b, \beta; G) \cap \text{MS}_{MR}\{B \rightarrow K; r, n\}$ is nonvoid.

With Let $r < n$. $PQ(C, c/d|b, \beta; G) \cap \text{MS}_{MR}\{B \rightarrow K; r, n\}$ is nonvoid if and only if for each $z \in G$ either of conditions (a, b) holds.

2] Let $d \in \text{col}^{NR(\beta; G)}\{B \rightarrow K; m\}$

i) Let $r < n$.

a) If $|l\beta(z)| \leq r$ for $z \in G$, $PQ(C, c/d|b, \beta; G) \cap \text{MS}_{MR}\{B \rightarrow K; r, n\}$ contains D such that, for each $z \in G$, $D_\beta(z)$ is of rank at least $|l\beta(z)|$.

b) If $|l\beta(z)| > r$ for $z \in G$,

(i) $PQ(C, c/d|b, \beta; G) \cap \text{MS}_{MR}\{B \rightarrow K; r, n\} \cap \text{MS}_{MR(\beta; G)}\{B \rightarrow K; r, n\}$

is nonvoid

ii) Let $r \geq n$. $PQ(C, c/d|b, \beta; G)$ contains D such that, for each $z \in G$, $D(z)$ is of rank at least n and $D_\beta(z)$ is of rank at least $|l\beta(z)|$.

iii) If $c(C|b, z) + O^{[r]}$ for $z \in G$, the space (i) is nonvoid

iv) If $c(C|b, z) = O^{[r]}$ for $z \in G$, $D_\beta(z)$ is of rank $\leq |l\beta(z)|$ for each $z \in G$ and all $D \in PQ(C, c/d|b, \beta; G)$.

3] fig. 2 goes on
 Examples which suffice to confirm the validity of clauses (2i-a) 2c §(b)
 may be constructed as follows (it is supposed throughout that $d_2(z) \neq 0$ for some).

i) Under the conditions of clause (2ia) let $\xi' \in F[n]$ be such that

$\{\xi'\} \subseteq [n] \setminus \{\xi(z)\}$ and, with $\xi'' = [\xi(z) \setminus \lambda, \xi']$ that $|\xi''| = r$.

Select an integer bijection $\Upsilon: \{\xi''\} \rightarrow [r]$. Set

$$(2) \quad D_{\Upsilon(\nu), \lambda}(z) = \{c(C \mid b; z)_{\Upsilon(\nu)} - d_2(z)\} / d_2(z)$$

for $\lambda = \xi(z) \setminus \lambda$

$$D_{\Upsilon(\nu), \lambda}(z) = c(C \mid b; z)_{\Upsilon(\nu)} / d_2(z)$$

for $\lambda = \xi' \setminus \lambda$

$$(3) \quad D_{\Upsilon(\nu), \nu}(z) = 1$$

for $\lambda = \xi''$, and the remaining elements of $D(z)$ equal to zero.

ii) Under the conditions of clause (2ib), let $\xi' \in F[n]$ be

such that $\xi' \subseteq \xi(z) \setminus \lambda$ and, with $\xi'' = \xi(z) \setminus \xi'$ that $|\xi''| = r$.

Select Υ as above ^{and}, implement the allocations v for $\lambda = \xi''$.

iii) Under the conditions of clause (2ii), effect the ~~also set $r=n-1$~~

in the constructions of the two preceding subclauses (
 effect carry out the construction of (ia) above ^{with} if $|\xi(z)| \leq n-1$ with
~~if~~ $|\xi(z)| \leq n-1$ and that of (ib) above ^{with} ~~if~~ $|\xi(z)| = 1$ if $|\xi(z)| = 1$
 Also, set $r=n$ with r being replaced by $n-1$ in both cases. Also, set

$$\text{Ex. } (4) \quad D_{z,\lambda}(z) = c(C|b,z)_z / d_\lambda(z)$$

for $z = [n, r]$.

ii) Under the additional condition of clause (2iii), $c(C|b,z)_k + 0$ for some $k \in [r]$. If $r < n$ and $\frac{1}{2} |\S(z)| \leq r < n$, the bijection Γ of subclause (3ia) above is subjected to the constraint $k \in \Gamma(\frac{1}{2})$, and $D(z)$ is constructed as in that subclause. If $|\S(z)| \geq r$ and $r < n$, the construction of subclause (3ib) above suffices.

If $r \geq n$, select $\Xi \in F[r]$ such that for which $|\Xi| = n$ and $k \in \Xi$. Select $\frac{1}{2}' \in F[n]$ for which $\lambda \notin \frac{1}{2}'$ and, with

$$\frac{1}{2}'' = [b(z) \setminus \lambda, \frac{1}{2}'], \quad |\frac{1}{2}''| = n-1$$

If $|\frac{1}{2}(z)| = n$, let $\frac{1}{2}'' = \frac{1}{2}(z) \setminus \lambda$; otherwise select $\frac{1}{2}' \in F[n]$ for which $\lambda \in \frac{1}{2}'$ and, with $\frac{1}{2}'' = [b(z) \setminus \lambda, \frac{1}{2}']$, $|\frac{1}{2}''| = n-1$. Let $\underline{\underline{\Gamma}}$: Select an integer bijection $\Gamma : \{\frac{1}{2}''\} \rightarrow \{\Xi \setminus k\}$. Implement the allocations (**) for $\lambda = \frac{1}{2}(z) \setminus \lambda$, (***) for $\lambda = \frac{1}{2}''$ and (****) for $z = [r] \setminus \Gamma \{b(z) \setminus \lambda\}$.

4] Under the conditions of clause (1b), $PQ(C, c/d/b, \frac{1}{2}, \underline{\underline{\Gamma}}) = \text{MS}\{B \rightarrow K, r, n\}$.

3] Explicit constructions of mapping systems $\text{DEMS}\{B \rightarrow K, r, n\}$ which suffice in each case as an example

3] The results of subclauses (2i-2iii) may be demonstrated in each case by exhibiting an example of a mapping

system $D \in NS\{B \rightarrow K; r, n\}$ which either exhibits the properties described or is a member present in a space stated to be nonvoid. Such examples are treated in the corresponding clauses (i-iii) below. In each case certain elements of $D(z)$ are specified and it is tacitly supposed ~~both~~^(fixed now) that the remaining ~~of G~~ elements of $D(z)$ are zero, and that $D(z') = D(z)$ for $z' \in B$. Since, in section 2, $d_2^S(z) + 0$ [$|d_2^S(z)|$ in section 2], it is licit to suppose in the following that $d_2^S(z) + 0$ for some $\lambda \in S(z)$, $\lambda = \lambda(z) \in S(z)$.

Concerning the integer dimensions r and n and the sequence $S(z)$ it may occur, when $r < n$, that $|S(z)| \leq r$ or that $|S(z)| > r$ and, when $r \geq n$, that

$S(z)$

The cases in which $r < n$, $|S(z)| \leq r$ and $r \geq n$, $|S(z)| > r$ and $r \geq n$, $|S(z)|$ unrestricted exhaust the possible dispositions of the integer dimensions r and n and the sequence $S(z)$. The results of part 2 clauses (2i-ii) in part imply that $PQ(C, c/d|h, S; z)$ is nonvoid if condition (1ia) holds. These results are in turn established by verifying that the examples whose construction is specified in the corresponding clauses (3i-iii) either possess the properties described in clauses (2i-ii) or are present in the spaces mentioned. The proof begins with that of clauses (3i-ii).

The condition

$$(5) \quad D_{\frac{1}{2}}(z)d^{\frac{1}{2}}(z) = c(C|B; z)$$

holding for $z \in G$ characterises the mapping systems $\text{DEMS}\{B \rightarrow K; r, n\}$ in $PQ(C, c/d|B, \frac{1}{2}; G)$. To show that this space is nonvoid if,

for each $z \in G$, either of conditions (1ia, b) hold, construct D by taking $D_{[1-\frac{1}{2}(z)]}(z)$ to be arbitrary for each $z \in B \setminus G$, setting $D(z') = D(z)$ for $z' \in B \setminus G$ where $z \in G$ is fine for some sliced

$z \in G$ and, for each $z \in G$, $D(z)$ is specified as follows. If

$d^{\frac{1}{2}}(z) \neq 0^{[1-\frac{1}{2}(z)]}$, $d_2(z) \neq 0$ for some $\lambda \in \frac{1}{2}(z)$; set $D_{[2]}(z) = c(C|B, z)/d_2(z)$ and $D_{[2]}(z) = 0^{[r]}_{[n-1]}$. If condition (1ib) holds

$D(z) \in M\{K; r, n\}$ may be prescribed arbitrarily. The assumption

that for each $z \in G$ either of conditions (1ia, b) obtains ceases to hold if and only if $d^{\frac{1}{2}}(z) = 0^{[1-\frac{1}{2}(z)]}$ and $c(C|B; z) \neq 0^{[r]}$.

In this case, no $\text{DEMS}\{B \rightarrow K; r, n\}$ satisfies condition (5) for the exceptional values of z . Clause (1i) has been dealt with.

To dispose of clause (1ii) it is remarked that when

$c(C|B, z) = 0^{[r]}$ for $z \in G$ then for each $z \in G$ either $d^{\frac{1}{2}}(z) \neq 0^{[1-\frac{1}{2}(z)]}$ and condition (1ia) is satisfied or $d^{\frac{1}{2}}(z) = 0^{[1-\frac{1}{2}(z)]}$

and condition (1ib) holds: $PQ(C, c/d|B, \frac{1}{2}; G)$ is nonvoid.

Clause (1iii) is dismissed by the remark that if condition (1ib) holds for $z \in G$, any $\text{DEMS}\{B \rightarrow K; r, n\}$ satisfies relationship (5) for $z \in G$.

In the proof of clause (1iv) it is required in part to show that if, for each $z \in G$, either of conditions (1ia, b) holds

then the space $PQ(C, c/d/b, \frac{1}{3}; G) \cap MS_{MR}$

(b) $PQ(C, c/d/b, \frac{1}{3}; G) \cap MS_{MR} \{B \rightarrow K; r, n\}$

is now void. Decompose G into three mutually exclusive sets,

G' , G'' and G''' ; over G' , condition $\#_2(1ia)$ holds and $|z_3(z)| \leq r$ for $z \in G'$; over G'' , condition $(1ia)$ holds and $|z_3(z)| > r$ for $z \in G''$; over G''' condition $(1ib)$ holds. Assuming the result of subclause (2ia), the space (b) with G replaced by G' is now void. Assuming that of subclause (2ib), the space (b) with G replaced by G'' is now void. From clause (1ii), $PQ(C, c/d/b, \frac{1}{3}, G''') \subseteq MS \{B \rightarrow K; r, n\}$: the space (b) with G replaced by G''' is now void. Thus, assuming the result of clause (2i) the space (b) is now void. If, for some $z \in G$, the ^{assumption} _{breaks down} condition that either of conditions (1ia, b) obtains ceases to hold, the space $PQ(C, c/d/b, \frac{1}{3}; G)$ is, from clause void; and the space (b) is then also void. The result of clause (1ii) is accordingly a consequence of clause (2i), the proof of which is now considered.

The examples whose construction is specified in clauses (3i-ii) either possess the properties described in the corresponding clauses (2i-ii) or are present in the spaces mentioned. By proving part 3, clauses (2i-ii) are automatically disposed of.

The rows of the matrix $D(z)$ described in subclause (3ia) belong to two sets: those with indexes featuring in the sequences

$$\underline{N} \left\{ \frac{1}{3}(z) \mid X \right\}$$

and $\mathbb{I}(\frac{1}{3})$ respectively. The rows of $D_{\frac{1}{3}}(z)$ forming parts of the rows of the first set alluded to contain two ~~more~~ elements not set auto directly set equal to zero: $D_{\mathbb{I}(w), \lambda}(z)$ and $D_{\mathbb{I}(w), \lambda}(z)$ for $\lambda = \frac{1}{3}(z) \setminus \lambda$. In the formation of the column vector $D_{\frac{1}{3}(z)}$ $D_{\frac{1}{3}}(z) d^{\frac{1}{3}}(z)$, the products involving the rows in question have the form

$$(7) \quad \left\{ c(C|B; z) \right\}_{\mathbb{I}(w)} / d_\lambda(z) \} d_\lambda(z) + \\ d_\lambda(z) + \left[\left\{ c(C|B; z) \right\}_{\mathbb{I}(w)} - d_\lambda(z) \right] d_\lambda(z) = c(C|B; z) \mathbb{I}(w)$$

for $\lambda = \frac{1}{3}(z) \setminus \lambda$. The rows of $D_{\frac{1}{3}}(z)$ featuring as parts of the second set contain one element not directly set equal to zero: $D_{\mathbb{I}(w), \lambda}(z)$ for $\lambda = \frac{1}{3}$. In the formation of the $D_{\frac{1}{3}}(z) d^{\frac{1}{3}}(z)$, the corresponding products have the form

$$\left\{ c(C|B; z) \right\}_{\mathbb{I}(w)} / d_\lambda(z) \} d_\lambda(z) = c(C|B; z) \mathbb{I}(w)$$

for $\lambda = \frac{1}{3}$. Relationship (5) is satisfied: $D \in PQ(C, c/d|B, \frac{1}{3}; z)$. As

a consequence of the substitutions (3), $D_{[\frac{1}{3}]^r}(z)$ is a row or column permuted version of $I(r)$: $D(z)$ is of rank $r+1$. Also $D_{[\frac{1}{3}(z) \setminus \lambda]}(z)$ is a row or column permuted version of $I\{1|B(z) \setminus \lambda\}$.

$I\{1|B(z) \setminus \lambda\}$: $D_{\frac{1}{3}}(z)$ is of rank at least $1|\frac{1}{3}(z)|$.

In the construction described in subclause (3ib), allocation of

the elements of $D_{[2]}(z)$ takes place according to relationship (2) alone. The proof that relationship (3) is satisfied involves formulae of the form (4) alone. That $D \in \text{MS}_{MR}^{SP} D(z)$ is of rank $r+1$ is shown demonstrated as above. Now, however,

$D_{[\frac{r}{2}]}(z)$ is a row or column permuted version of $I(r)$ and is also a submatrix of $D_{\frac{r}{2}}(z)$: $D \in \text{MS}_{MR(\frac{r}{2}; z)}^{\{B \rightarrow K; r, n\}}$.

The submatrix $D^{[n-1]}(z)$ of $D(z)$ specified in clause (3ii) is identical with that described in clause (3ia) or (3ib), as is appropriate, with $r=n-1$. In particular, the components of the column vector $D_{\frac{r}{2}}(z)d_{\frac{r}{2}}^{\frac{r}{2}}(z)$ with index $\tau (\tau=[n-1])$ agree with those of $c(C|b, z)$ for the same index values. That these rows of $D(z)$ with corresponding to the index values $\tau=[n-1]$ contain only one component not directly set equal to zero, namely $D_{\tau, n}(z)$; this component belongs to $D_{\frac{r}{2}}(z)$. Since

$$\{c(C|b, z)_\tau / d_\tau(z)\}d_\tau(z) = c(C|b, z)_\tau$$

the components of all all components of the column vector $D_{\frac{r}{2}}(z)d_{\frac{r}{2}}^{\frac{r}{2}}(z)$ agree with those of $c(C|b, z)$: $D \in \text{PQ}(C, c/d | b, \frac{r}{2}; z)$.

$D_{[[n] \setminus \lambda]}^{[n-1]}(z)$ is a row permuted version of $I(n-1)$ and is of rank at least n . As was shown in the proofs of subclauses 3.2(3ia,b), appropriate row selection from the mat

submatrix $D_{[\frac{1}{2}(z) \setminus \lambda]}^{[x-1]}$ produces a row permuted version of $I(1_{\frac{1}{2}(z)}|-1)$. The submatrix in question ~~for~~ is a submatrix of $D_{\frac{1}{2}}(z)$ which is accordingly of rank at least $|\frac{1}{2}(z)|$.

Under the conditions of clause (3ia), $\frac{1}{2}'$ is a nonvoid sequence, and the stipulation that $k \in \mathcal{N}(\frac{1}{2}')$ for some $k \in [r]$ is viable. Subject to the additional condition of clause (3ii) that $c(C|B, z)_k \neq 0$, the construction described in subclause (3ia) then entails that

$D_{k,2}(z) = c(C|B, z)_k / d_2(z) + 0$. ~~Since $D_{[\frac{1}{2}(z) \setminus \lambda]}^{[x-1]}$ has the following structure. The column consisting of part of $D_{[\lambda]}(z)$ contains the nonzero element $D_{k,2}(z)$, and the further elements in the row containing this element are zero. The submatrix obtained from the matrix whose structure is under investigation obtained by deleting the row and column just considered is a row permuted version of $I(1_{\frac{1}{2}(z)}|-1)$. The determinant of the matrix has the form $\pm D_{k,2}(z) + 0$: the matrix is nonsingular, it is of rank $|\frac{1}{2}(z)| + 1$. The matrix is a submatrix of $D_{\frac{1}{2}}(z)$: $D \in \text{MS}_{NR(\frac{1}{2}; z)}^{\{B \rightarrow K; r, n\}}$. That D is also in the first two of the spaces occurring in expression (1) was shown above.~~

That the mapping system ① whose construction is specified in subclause (3ib) below is in the space $\overset{(1)}{G}$ under the conditions of clause subclause (2ib) ^{already} has been demonstrated.

The last construction "described clause (3ii)" is an extended version of that described in clause (3ii) in which the rows $D^{[z]}(z)$ ($z = [n]$) are replaced by the rows $D^{[z]}(z)$ ($z = \overline{\Sigma}$) ~~if~~, ^{the} rest of the remaining rows in each case being similar.

$(z = [r])$ are replaced in order by $D^{[z]}(z)$ for $z = \overline{\Sigma}$ and $z = [r] \setminus \overline{\Sigma}$.

That D is still in $PQ(C, c/d \pm b, \frac{1}{3}; z)$ in the extended case is shown as above. $D^{[\overline{\Sigma}]}(z)$ has the following structure. The column consisting ~~of~~ composed of elements selected from $D_{[2]}(z)$ contains the nonzero element $D_{K, \lambda}(z)$, the further elements in the row containing this element being zero. The submatrix obtained by deleting the row and column in question from $D^{[\overline{\Sigma}]}(z)$ is a row permuted version of $I(n-1)$. The determinant of $|D^{[\overline{\Sigma}]}(z)|$ has the form $\pm D_{K, \lambda}(z)$. $D^{[\overline{\Sigma}]}(z)$ is a nonsingular matrix of rank $n+1$ and $D(z)$ is of rank $n+1$: $D \in MS_{MR}^{\{B \rightarrow K; r, n\}}$. Similarly $|D_{[\frac{1}{3}(z)]}^{[\Gamma \{ \frac{1}{3}(z) \setminus \lambda \}, K]}|$ has the form $\pm D_{K, \lambda}(z)$ and

$D \in MS_{MR}(\frac{1}{3}; z)^{\{B \rightarrow K; r, n\}}$

Clause (2iv), which remains to be dealt with follows from relationship (5): $d^{\frac{1}{3}}(z)$ is a vector containing $|\frac{1}{3}(z)|+1$ elements not all of which are zero, while $c(Clb; z)$ is a zero vector: $D_{\frac{1}{3}}(z)$ is

of rank less than $|z| + 1$.

Part [4] was dismissed in the proof of clause (1i).

When $r < n$, either $|z| \leq r$ or $|z| > r$. According to clause (2i), $PQ(C, c/d | b, z; z) \cap MS_{MR} \{B \rightarrow K; r, n\}$ is, when condition (1ia) holds, nonvoid in both cases. As has just been shown, $PQ(C, c/d | b, z; z) \equiv MS \{B \rightarrow K; r, n\}$ when condition (1ib) holds and again $PQ(C, c/d | b, z; z) \cap MS_{MR} \{B \rightarrow K; r, n\}$ is nonvoid. The first of these spaces is nonvoid only if conditions (1ia) or (1ib) holds; the same holds true for the intersection. The result of clause (1ii) is accordingly a consequence of clause (2i), the proof of which is now considered.

- () Let $c \in \{B \rightarrow K; m\}$, $d \in \{B \rightarrow K; n\}$, $b: B \rightarrow F[m]$, $\bar{z}: B \rightarrow F[n]$, $\phi \in \{B \rightarrow K; m\}$ and $\bar{z} \in B$. Set $c(\phi | b; z) = \phi_b(z) c^b(z)$
- 1] i) $PQ(\phi, c/d | b, z; \bar{z})$ is nonvoid if and only if, either
 - a) $d^{\bar{z}}(z) \neq 0$ $\left[|z| \right]$ or
 - b) $c(\phi | b, z) = 0$ and $d^{\bar{z}}(z) = 0$ $\left[|z| \right]$
 - ii) if $c(\phi | b; z) = 0$, $PQ(\phi, c/d | b, z; \bar{z})$ is nonvoid
 - iii) if condition b) holds for $z \in G$, $PQ(\phi, c/d | b, z; G) \equiv MS \{B \rightarrow K; n\}$
 - iv) Let $n > 0$. $PQ \{ \phi, c/d | b, z; G \} \cap MS_{MR} \{ B \rightarrow K; n \}$ is nonvoid if and only if, either of conditions (a, b) holds
 - for each $z \in G$,

2] Let $\phi \in \{B \rightarrow K; n\}$, decol $^{MK(S; Q)} \{B \rightarrow K; n\}$

\Rightarrow If, $\forall z \in S$ $\exists b \in B$ $c(\phi|b, z) \neq 0$ or $|z(z)| > 0$
 for each $z \in Q$,

(1) $PQ(\phi, c/d|b, z; \emptyset) \cap {}_{NR}^{\text{non}} \{B \rightarrow K; n\} \cap {}_{MR(z)}^{\text{non}} \{B \rightarrow K; n\}$

is nonvoid

ii) If $c(\phi|b, z) \neq 0$, the space (+) is nonvoid

3] Examples which serve to confirm

3] Examples of mapping systems $\psi \in {}_{\emptyset}^{\text{non}} \{B \rightarrow K; n\}$ belonging to

the space (+) under the conditions stated in part [2] may be
 constructed as follows. Assume that $\gamma d_\lambda(z) \neq 0$ for some $z \in S$. Set

$\psi = \psi(z)$ from $B \setminus Q$ for some fixed $z \in Q$, where $d_\lambda(z)$

$\psi(z') = \psi(z)$ for $z' \in B$, where

a) if $c(\phi|b, z) \neq 0$, $\psi_\lambda(z) \stackrel{\text{is set equal to}}{=} c(\phi|b, z)/d_\lambda(z)$ and or

b) for some $|z(z)| > 0$, for some $z \in S \setminus Q$ $\psi_\lambda(z) \stackrel{\text{set}}{=} 1$ and
 $\psi_\lambda(z) \stackrel{\text{to}}{=} \{c(\phi|b, z) - d_\lambda(z)\}/d_\lambda(z)$

the remaining elements of $\psi(z)$ $\stackrel{\text{being set equal to zero in}}{-}$
 both cases.

4] Under the conditions of subclause (1ib) $PQ(\phi, c/d|b, z; \emptyset) =$
 $\emptyset \{B \rightarrow K; n\}$.

The proofs propositions of the above theorem may be obtained
 from those of Theorem ...
 by setting $r=0$, replacing D by ϕ and removing the propositions
 devoid of significance (e.g. from the special case of clause (2ia) of
 Theorem , that $\psi_\lambda(z)$ is of rank at least $|z(z)|$ when $|z(z)|=0$): the

above theorem is a corollary to Theorem . Independent proof of its results are easily given

The existence of relative product spaces

() let $d \in \{B \rightarrow K; n\}$, $b: B \rightarrow F[m]$, $\beta: B \rightarrow F[n]$,

$C \in MS\{B \rightarrow K; r, m\}$, $E \in MS\{B \rightarrow K; m, n\}$ and $\{f \in B\}$. Let

$$d(C, E | b, \beta; z) = C_f(z) E_{\beta}^b(z) d^{\beta}(z).$$

1] i) $RP(C, E; d | b, \beta; \emptyset)$ is nonvoid

ii) If $d^{\beta}(z) = 0$ for $z \in G$, $RP(C, E; d | b, \beta; \emptyset) = MS\{B \rightarrow K; r, n\}$

iii) Let ~~r < n~~ If $r < n$, $RP(C, E; d | b, \beta; \emptyset) \cap MS_{NR}\{B \rightarrow K; r, n\}$ is nonvoid

2] With references to $c(C | b; z)$, $PQ(C, c | d | b, \beta; \emptyset)$ replaced consistently by corresponding references to $d(C, E | b, \beta; z)$, $RP(C, E; d | b, \beta; \emptyset)$ the results of part [2] of Theorem hold.

3] Similarly. With $d(C, E | b, \beta; z)$ used in place of $c(C | b; z)$, the constructions $\{f\}$ described in part [3] of Theorem also serve to establish the validity of the counterparts to clauses (2i-iii) of that theorem referred to in part [2] above

4] If $d^{\beta}(z) = 0$ $\forall z \in G$, $RP(C, E; d | b, \beta; \emptyset) = MS\{B \rightarrow K; r, n\}$.

That a relative product space is nonvoid follows from the observation that this space contains the direct product. Specifically,

$$(1) \quad \{D_{\beta}(z) - C_f(z) E_{\beta}^b(z)\} d^{\beta}(z) = 0^{[r]}$$

holding for $z \in G$ characterises the mapping systems $D \in MS\{B \rightarrow K; r, m\}$ belonging to $RP(C, E; d | b, \beta; \emptyset)$. The mapping system $D \in MS\{B \rightarrow K; r, n\}$ for which

for some fixed E and, for each $\varepsilon > 0$,
 $D(z') = D(z)$ for $z' \in B(z, \varepsilon)$ then $D_g(z) = C_p(z) E_g^{\frac{1}{p}}(z)$ and $D_{J_g(z)}(z)$

(4)
is arbitrary, satisfies the above relationship: $RP(C, E | B, \frac{1}{p}; z)$ is
monoid. The above Relationship (4) may be written as the form

$$D_g(z) d^{\frac{1}{p}}(z) = d(C, E | B, \frac{1}{p}; z)$$

A structural theory of the mapping systems $DEM\{B \rightarrow K; r, n\}$ which satisfying
this relationship (4) may be derived from that of the mapping systems
 D satisfying relationship (+) in the proof of theorem by
replacing $c(C | B; z)$ by $d(C, E | B, \frac{1}{p}; z)$

a form which may be obtained from relationship (5) of the
proof of theorem by replacing $c(C | B; z)$ by $d(C, E | B, \frac{1}{p}; z)$.

The structural theory of the mapping systems D in $PQ(C, c | D | B, \frac{1}{p}; z)$
given in parts [2, 3] of Theorem accordingly has a counterpart
described in parts [2, 3] of the above theorem,
concerning the mapping systems D in $RP(E, E | D | B, \frac{1}{p}; z)$. This

The required proofs are obtained by effecting the appropriate
changes in the proof of Theorem.

The result of part (i) follows from the observation that if
 $d^{\frac{1}{p}}(z) = 0$ [by clause (1ii)], relationship (4) is satisfied by any mapping
system D in $MS\{B \rightarrow K; r, n\}$.

Clause (iii) of the is the counterpart to clause (1iv) of Theorem.
It is, for reasons similar to those given in the proof of that theorem
in convenience of part [2]

() Let $d \in \text{col}\{B \rightarrow K; n\}$, $b: B \rightarrow F[m]$, $\beta: B \rightarrow F[n]$,

$\phi \in z\{B \rightarrow K; m\}$, $E \in \text{MS}\{B \rightarrow K; m, n\}$ and $\psi \in B$. Set

$$d(\phi, E | b, \beta; z) = \phi_b(z) E_{\beta}^b(z) d^{\beta}(z).$$

- 1] i) $RP(\phi, E | b, \beta; \psi)$ is nonvoid
 ii) If $d^{\beta}(z) = 0$ $\forall z \in G$, $RP(\phi, E | b, \beta; \psi)$ is nonvoid.
 iii) If $n > 0$, $RP(\phi, E | b, \beta; \psi) \cap \mathcal{Z}_{\text{MR}}^{\text{non}}\{B \rightarrow K; n\}$ is nonvoid

2] Let $d^{\beta}(z) \neq 0$ $\forall z \in G$. If (a) $d(\phi, E | b, \beta; z) \neq 0$ or (b)
 $d \in \text{col}^{\text{non}}\{B \rightarrow K; n\}$

$$|\beta(z)| > 0$$

(a) $RP(\phi, E | b, \beta; \psi) \cap \mathcal{Z}_{\text{MR}}^{\text{non}}\{B \rightarrow K; r, n\} \cap \mathcal{Z}_{\text{MR}}^{\text{non}}\{B \rightarrow K; s, n\}$
 is nonvoid

3] Examples of mapping systems $\psi \in z\{B \rightarrow K; n\}$ belonging to the space (4) under the conditions stated in part [2] may be constructed by replacing $c(\phi | b, z)$ by $d(\phi, E | b, \beta; z)$ in part [3] of Theorem.

4] ~~If $d^{\beta}(z) = 0$, $RP(\phi, E | b, \beta; \psi)$ is nonvoid.~~

The above theorem stands in relation to Theorem as Theorem does to Theorem. The modifications and simplifications introduced into the proof of Theorem that lead to the proof of Theorem may be introduced into the proof of Theorem to yield a nonvoid of the above theorem.

Semi inclusion and equivalence relationships. Such relationships for sets over which the spaces are defined.

() 1] Let $G \subseteq H \subseteq B$.

Properties of spaces.

i) Let $f: B \rightarrow F[m]$ and $g: B \rightarrow F[n]$

$$MS_{\begin{matrix} MR(f; H) \\ MR(g; H) \end{matrix}}^{\begin{matrix} MR(f; G) \\ MR(g; G) \end{matrix}} \{B \rightarrow K; m, n\} \subseteq MS_{\begin{matrix} MR(f; G) \\ MR(g; G) \end{matrix}}^{\begin{matrix} MR(f; G) \\ MR(g; G) \end{matrix}} \{B \rightarrow K; m, n\}.$$

b) Appropriate preliminary definitions being provided, similar results hold with regard to the spaces $MS_{\begin{matrix} MR(f; G) \\ MR(g; G) \end{matrix}}^{\begin{matrix} MR(f; G) \\ MR(g; G) \end{matrix}} \{B \rightarrow K; m, n\}$,

$$MS_{\begin{matrix} MR(f; G) \end{matrix}}^{\begin{matrix} MR(f; G) \end{matrix}} \{B \rightarrow K; m, n\}, \text{ col } \{B \rightarrow K; m\}, \text{ row }_{\begin{matrix} MR(f; G) \\ \{B \rightarrow K; m\} \end{matrix}} \{B \rightarrow K;$$

$$IN\{M: f, g; G\}, PQ(c/d|f, g), PQ(M', c/d|f, g),$$

$$PQ(\phi, c/d|f, g), RP(M', c/d|f, g), RP(\phi, M; d|f, g)$$

$$IN(M; G), PQ(c/d|G), PQ(M' c/d|G), PQ(\phi c/d), RP(M' M; d|G)$$

and $RP(\phi, M; d|G)$.

c) Let $\{s; p, \mu\} \in SF\{B; m, n\}$ with $\tau, \omega \in S(z)$ for $z = H$

$$MS_{\begin{matrix} MR(p; H, \tau) \\ MR(\mu; H, \omega) \end{matrix}}^{\begin{matrix} MR(p; G, \tau) \\ MR(\mu; G, \omega) \end{matrix}} \{B \rightarrow K; m, n\} \subseteq MS_{\begin{matrix} MR(p; G, \tau) \\ MR(\mu; G, \omega) \end{matrix}}^{\begin{matrix} MR(p; G, \tau) \\ MR(\mu; G, \omega) \end{matrix}} \{B \rightarrow K; m, n\}$$

Subject to the provision of appropriate preliminary definitions and to corresponding conditions, similar results hold with regard to the spaces $MS_{\begin{matrix} MR(p; G, \tau) \\ MR(\mu; G, \omega) \end{matrix}}^{\begin{matrix} MR(p; G, \tau) \\ MR(\mu; G, \omega) \end{matrix}} \{B \rightarrow K; m, n\}$, $MS_{\begin{matrix} MR(p; G, \tau) \\ MR(\mu; G, \omega) \end{matrix}}^{\begin{matrix} MR(p; G, \tau) \\ MR(\mu; G, \omega) \end{matrix}} \{B \rightarrow K; m, n\}$

$$IN(M; p, \mu; G, \omega), PQ(c/d|p, \mu; G, \omega), PQ(M', c/d|p, \mu; G, \omega),$$

$$PQ(\phi, c/d|p, \mu; G, \omega), RP(M', M; d|p, \mu; G, \omega) \text{ and }$$

$$RP(\phi, M; d|p, \mu; G, \omega).$$

ii) Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $E \in \text{MS}\{B \rightarrow K; m, n\}$,
 $f: B \rightarrow F[m]$ and $\xi: B \rightarrow F[n]$.

$$DC(G|E; c, d; f, \xi) \subseteq DC(H|E; c, d; f, \xi)$$

Appropriate preliminary definitions being provided, similar
 $DC'(G|E; c, d; f, \xi)$, $DC(G|E; c, d)$:
results hold with regard to the spaces ~~$DC(G|E; c, d)$~~ and
regards $DC'(G|E; c, d; s; p, \mu; \omega)$ and $DC(G|E; c, d; s; p)$ and
 ~~$DC(G|E; c, d; s; p, \mu; \omega)$~~

2] Let $G', G'', \dots \subseteq B$ and set $H = G' \cup G'' \cup \dots$

i) Let $f: B \rightarrow F[m]$ and $\xi: B \rightarrow F[n]$.

$$\begin{aligned} & MS_{\substack{MR(f; G') \\ MR(\xi; G')}}^{\substack{NR(f; G') \\ NR(\xi; G')}} \{B \rightarrow K; m, n\} \cap MS_{\substack{MR(f; G'') \\ MR(\xi; G'')}}^{\substack{NR(f; G'') \\ NR(\xi; G'')}} \{B \rightarrow K; m, n\} \cap \dots \\ & = MS_{\substack{MR(f; H) \\ MR(\xi; H)}}^{\substack{NR(f; H) \\ NR(\xi; H)}} \{B \rightarrow K; m, n\} \end{aligned}$$

b) Appropriate preliminary definitions being provided, similar results hold with regard to the spaces listed in subclause (1ic).

Subject to further conditions of the type introduced in subclause (1ic), with H redefined as in the present part, such results also hold with regard to the spaces mentioned in that subclause.

ii) Given the preliminary definitions of clause (1ii),
 $DC(G'|E; c, d; f, \xi) \cup DC(G''|E; c, d; f, \xi) \cup \dots =$
 $DC(H|E; c, d; f, \xi)$

This result has counterparts concerning the further sets $DC(G|E; c, d : b, \xi)$ and $DC(G|E; c, d : s, p, \mu)$ and, assuming that $\{s, p, \mu\} \in SF\{B; m, n\}$ with $w \in S(z)$ for $z = H$, with regard to $DC(G|E; c, d : s, p, \mu : w)$ and $DC'(G|E; c, d : s, p, \mu : w)$

3] Let $G', G'', \dots \subseteq B$ and set $H = G' \cap G'' \cap \dots$

i) Let $b: B \rightarrow F[m]$ and $b: B \rightarrow F[n]$.

$$\begin{aligned} & MS_{\substack{MR(b; G') \\ MR(\xi; G')}}^{MR(b; G')} \{B \rightarrow K; m, n\} \cup MS_{\substack{MR(b; G'') \\ MR(\xi; G'')}}^{MR(b; G'')} \{B \rightarrow K; m, n\} \cup \dots \\ & \subseteq MS_{\substack{MR(b; H) \\ MR(\xi, H)}}^{MR(b; H)} \{B \rightarrow K; m, n\} \end{aligned}$$

b) Appropriate preliminary definitions being provided, similar results hold with regard to the spaces listed in subclause (1ic).

c) Let $\{s, p, \mu\} \in SF\{B; m, n\}$ with $z, w \in S(z)$ for $z = G' \cup G'' \cup \dots$

$$\begin{aligned} & MS_{\substack{MR(p; G', z) \\ MR(\mu; G', w)}}^{MR(p; G', z)} \{B \rightarrow K; m, n\} \cup MS_{\substack{MR(p; G'', z) \\ MR(\mu; G'', w)}}^{MR(p; G'', z)} \{B \rightarrow K; m, n\} \cup \dots \\ & \subseteq MS_{\substack{MR(p; H, z) \\ MR(\mu; H, w)}}^{MR(p; H, z)} \{B \rightarrow K; m, n\} \end{aligned}$$

Subject to the provision of appropriate preliminary definitions and to corresponding conditions, similar results hold with regard to the further spaces listed in subclause (1ic).

ii) Given the preliminary definitions of clause (1ii)

$$DC(H|E; c, d : b, \xi) \subseteq DC(G'|E; c, d : b, \xi) \cap DC(G''|E; c, d : b, \xi) \cap \dots$$

This result has a counterpart concerning the further sets $DC(G|E; c, d : b, \xi)$ and $DC(G|E; c, d : s, p, \mu)$ and, assuming that $\{s, p, \mu\} \in SF\{B; m, n\}$ with $w \in S(z)$ for $z = G' \cup G'' \cup \dots$, with regard to $DC(G|E; c, d : s, p, \mu : w)$ and $DC'(G|E; c, d : s, p, \mu : w)$

$MS^{MR(B; G)} \{B \rightarrow K; m, n\}$ is the subspace of mapping systems
 $MR(\frac{b}{z}; G)$

$E \in MS \{B \rightarrow K; m, n\}$ for which, for each $z \in G$, $E_{\frac{b}{z}}^B(z)$ is of rank
 $\min(|b(z)|, |\frac{b}{z}(z)|) + 1$. $MS^{MR(B; H)}_{MR(\frac{b}{z}; H)} \{B \rightarrow K; m, n\}$ is similarly
defined in terms of H . If $G \subseteq H$ all mapping systems of
the second subspace belong to the first. The remaining results
of clause (1i) are proved in the same way.

$DC(G | E; c, d; b, z)$ is the set of all $z' \in B$ satisfying
certain conditions involving z for at least one $z \in G$. If $G \subseteq H$,
all such z' satisfy the conditions in question for at least one
 $z \in H$. The first result of clause (1ii) has been dealt with.
The remaining results of this clause are dismissed in the
same way.

Under the conditions of part [2], $G', G'', \dots \subseteq H$. The result
of subclause (1ia) with G replaced in turn by G', G'', \dots holds.

In consequence

$$MS^{MR(B; H)}_{MR(\frac{b}{z}; H)} \{B \rightarrow K; m, n\} \subseteq$$

$$MS^{MR(B; G')}_{MR(\frac{b}{z}; G'')} \{B \rightarrow K; m, n\} \cap MS^{MR(B; G'')}_{MR(\frac{b}{z}; G''')} \{B \rightarrow K; m, n\} \cap \dots$$

Select E belonging to the space occurring on the right hand side
of this relationship. Since $E \in MS^{MR(B; G')}_{MR(\frac{b}{z}; G'')} \{B \rightarrow K; m, n\}$, $E_{\frac{b}{z}}^B(z)$

is of rank $\min\{|b(z)|, |\frac{b}{z}(z)|\} + 1$ for each $z \in G'$. This relationship also holds for $z = G'', \dots$ and hence for $z = G' \cup G'' = H$. $E \in MS_{\frac{MR(b; H)}{MR(\frac{b}{z}; H)}}^{\frac{MR(b; H)}{MR(\frac{b}{z}; H)}}\{B \rightarrow K; m, n\}$: the above semi-inclusion relationship may be reversed and the result of subclause (2ia) has been obtained. The remaining results of clause (2i) are disposed of in the same way.

Since, under the conditions of part [2] $G', G'', \dots \subseteq H$, the first result of clause (1ii) with G replaced in turn by G', G'', \dots holds and in consequence

$$DC(G'|E; c, d : b, \frac{b}{z}) \cup DC(G''|E; c, d : b, \frac{b}{z}) \cup \dots \subseteq DC(H|E; c, d : b, \frac{b}{z}).$$

Select $z' \in DC(H|E; c, d : b, \frac{b}{z})$. H contains at least one z for which certain conditions involving z' and z hold. Such a z belongs either to G' or to G'' or Supposing that $z \in G'$, the conditions satisfied ensure that $z' \in DC(G'|E; c, d : b, \frac{b}{z})$. Hence $z' \in DC(G'|E; c, d : b, \frac{b}{z}) \cup DC(G''|E; c, d : b, \frac{b}{z}) \cup \dots$: the above semi-inclusion relationship may be reversed; the first result of clause (2ii) follows. The second result follows immediately. Subject to the last condition given in clause (2ii), we see for $z = G', G'', \dots$: the sets $DC(G'|E; c, d : s; \rho, \mu : \omega)$, ..., $DC(H|E; c, d : s; \rho, \mu : \omega)$ are well defined and the proof

of the result of clause (2ii) is as above.

Under the conditions of part [3], $H \leq G', G'', \dots$ The result of subclause (1ia) with G, H replaced in succession by $H, G'; H, G''; \dots$ holds and leads to the semi-inclusion relationship given in clause (3ia). The remaining results of part [3] are dismissed in the same way.

Invariance of spaces with respect to sequence rearrangement

() Let $\mathbb{G} \subseteq \mathbb{B}$.

i.) Let $b, b': \mathbb{B} \rightarrow F[m]$ and $\xi, \xi': \mathbb{B} \rightarrow F[n]$ with $\{b(z)\} = \{b'(z)\}$ and $\{\xi(z)\} = \{\xi'(z)\}$ for $z \in \mathbb{G}$.

$$MS_{\begin{smallmatrix} MR(b; \mathbb{G}) \\ MR(\xi; \mathbb{G}) \end{smallmatrix}} \{B \rightarrow K; m, n\} = MS_{\begin{smallmatrix} MR(b'; \mathbb{G}) \\ MR(\xi'; \mathbb{G}) \end{smallmatrix}} \{B \rightarrow K; m, n\}$$

ii) Appropriate preliminary definitions being provided, similar results hold with regard to the spaces $MS^{NR(b; \mathbb{G})} \{B \rightarrow K; m, n\}$

$$MS_{MR(b; \mathbb{G})} \{B \rightarrow K; m, n\}, \text{ col }^{NR(b; \mathbb{G})} \{B \rightarrow K; m\}, \text{ row}_{MR(b; \mathbb{G})} \{B \rightarrow K;$$

$$IN \{M: b, \xi; \mathbb{G}\}, PQ(c/d | b, \xi; \mathbb{G}), PQ(M', c/d | b, \xi; \mathbb{G}),$$

$$PQ(\phi, c/d | b, \xi; \mathbb{G}), RP(M', c; d | b, \xi; \mathbb{G}) \text{ and } RP(\phi, M; d | b, \xi; \mathbb{G})$$

and with regard to the regions $DC(\mathbb{G} | E; c, d : b, \xi)$ and

$$DC'(\mathbb{G} | E; c, d : b, \xi).$$

2.) Let $\{s; \rho, \mu\}, \{s'; \rho', \mu'\} \in SF \{B; m, n\}$ with $s(z) = s'(z)$ for $z \in \mathbb{G}$ and, for each $z \in \mathbb{G}$, let $\{\rho(z, w)\} = \{\rho'(z, w)\}$ and $\{\mu(z, w)\} = \{\mu'(z, w)\}$ both for $w = s(z)$.

With $z, w \in s(z)$ for $z \in \mathbb{G}$,

$$MS_{\begin{smallmatrix} MR(\rho; \mathbb{G}, z) \\ MR(\mu; \mathbb{G}, w) \end{smallmatrix}} \{B \rightarrow K; m, n\} = MS_{\begin{smallmatrix} MR(\rho'; \mathbb{G}, z) \\ MR(\mu'; \mathbb{G}, w) \end{smallmatrix}} \{B \rightarrow K; m, n\}$$

ii) Subject to the provision of appropriate preliminary definitions

and to corresponding conditions, similar results hold with regard to the spaces $MS^{MR}(p; G, z) \{B \rightarrow K; m, n\}$, $MS_{NR}(\mu; G, \omega) \{B \rightarrow K; m, n\}$, $IN(M; p, \mu; G, \omega)$, $PQ(c/d | p, \mu; G, \omega)$, $PQ(M', c/d | p, \mu; G, \omega)$, $PQ(\phi, c/d | p, \mu; G, \omega)$, $RP(M, M; d | p, \mu; G, \omega)$ and $RP(\phi, M; d | p, \mu; G, \omega)$ and to the regions $DC(G | E; c, d : s; p, \mu : \omega)$ and $DC'(G | E; c, d : s; p, \mu : \omega)$.

The two conditions characterising M as a member of the space $MS_{NR}^{\frac{B}{G}}(z; G) \{B \rightarrow K; m, n\}$ are that $M \in MS \{B \rightarrow K; m, n\}$ and that for each $z \in G$, $M_{\frac{B}{G}}^z(z)$ is of rank $\min(|B(z)|, |\frac{B}{G}(z)|)$. Select $z \in G$. If $\{b(z)\} = \{b'(z)\}$ and $\{\frac{B}{G}(z)\} = \{\frac{B'}{G}(z)\}$, $b'(z)$ and $\frac{B'}{G}(z)$ are simply rearranged forms of $b(z)$ and $\frac{B}{G}(z)$ respectively. $M_{\frac{B}{G}}^z(z)$ is a row and column rearranged form of $M_{\frac{B}{G}}^b(z)$. The rank of $M_{\frac{B}{G}}^b(z)$ is that of $M_{\frac{B}{G}}^{\frac{B}{G}}(z)$. Furthermore $|B(z)| = |B'(z)|$ and $|\frac{B}{G}(z)| = |\frac{B'}{G}(z)|$: $M_{\frac{B}{G}}^{\frac{B}{G}}(z)$ is of rank $\min(|B'(z)|, |\frac{B'}{G}(z)|) + 1$ for each $z \in G$. Any M in the above space is also in $MS_{NR}^{\frac{B'}{G}}(z; G) \{B \rightarrow K; m, n\}$. The converse result is also true. Clause (ii) has been disposed of. The results of clause (iii) concerning maximal section rank are dealt with in the same way.

The condition characterising \hat{M} as a member of the space $PQ(c/d | b, \frac{B}{G}; G)$ are that $\hat{M} \in MS \{B \rightarrow K; m, n\}$ and that

$$\hat{M}_{\frac{1}{3}}^B(z) d^{\frac{1}{3}}(z) = C^B(z)$$

for $z \in G$. By suitable row and column rearrangement of $\hat{M}_{\frac{1}{3}}^B(z)$ and corresponding rearrangements of $C^B(z)$ and $d^{\frac{1}{3}}(z)$, the latter relationship may be presented in the form

$$\hat{M}_{\frac{1}{3}'}^B(z) d^{\frac{1}{3}'}(z) = C^{B'}(z).$$

Thus $PQ(c/d|B, \frac{1}{3}; G) \subseteq PQ(c/d|B', \frac{1}{3}'; G)$. This semi-inclusion relationship may be reversed. The remaining results of clause (iii) are proved in the same way.

The results of part [2] are slight extensions of those of its predecessor.

Properties of domains of constancy

Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$ and $E \in \text{MS}\{B \rightarrow K; m, n\}$.

1] Let $z, z' \in B$.

i) Let $b: B \rightarrow F[m]$ and $\xi: B \rightarrow F[n]$. $z \in DC(z|E; c, d: b, \xi)$

a) if and only if $z \in DC(z'|E; c, d: b, \xi)$ and

b) if and only if $DC(z|E; c, d: b, \xi) = DC(z'|E; c, d: b, \xi)$

ii) Let $\{s; \rho, \mu\} \in SF\{B; m, n\}$ and $w \in \{s(z)\} \cap \{s(z')\}$.

$z' \in DC(z|E; c, d: s; \rho, \mu; w)$

a) if and only if $z \in DC(z'|E; c, d: s; \rho, \mu; w)$ and

b) if and only if $DC(z|E; c, d: s; \rho, \mu; w) = DC(z'|E; c, d: s; \rho, \mu; w)$

iii) Domains of constancy defined with respect to single members

of B are either disjoint or identical : with b, ξ as defined

in clause (i), $DC(z|E; c, d: b, \xi)$ and $DC(z'|E; c, d: b, \xi)$

are either disjoint or identical ; a similar result holds

with regard to the spaces $DC(z|E; c, d: s; \rho, \mu; w)$ and

$DC(z'|E; c, d: s; \rho, \mu; w)$ of clause (ii).

2] Let $G \subseteq B$.

i) Let $b: B \rightarrow F[m]$ and $\xi: B \rightarrow F[n]$.

$$DC(G|E; c, d: b, \xi) = [\bigcup DC(z|E; c, d: b, \xi) \ (z \in G)]$$

ii) With b, ξ as in the preceding clause, let $c, d, E, \{b\}, \{\xi\} \in \text{const}(G)$.

- a) $DC(z|E; c, d : b, \xi) = DC(z'|E; c, d : b, \xi)$ for all $z, z' \in G$ and $b, \xi \in B$.
- b) $DC(G|E; c, d : b, \xi) = DC(z|E; c, d : b, \xi)$ for $z = G$.
- iii) Let $\{s; \rho, \mu\} \in SF\{B; m, n\}$ and $c, d, E, [s; \{\rho\}, \{\mu\}] \in \text{const}(G)$. For each $z \in G$ and $\omega \in s(z)$, $DC(z|E; c, d : b, \xi; s; \rho, \mu : \omega) = DC(z'|E; c, d : s; \mu, \rho : \omega)$ for $z' = G$ and $DC(G|E; c, d : s; \mu, \rho : \omega) = DC(z|E; c, d : s; \mu, \rho : \omega)$.

- 3] Let $\{s; \rho, \mu\} \in SF\{B; m, n\}$, $G \subseteq B$, ~~and~~ $c, d, E \in \text{const}(G)$ and $z \in G$.
- i) Let $E \in BS\{B \rightarrow K; m, n | s; \rho, \mu\}$ and $[s; \{\rho\}, \{\mu\}] \in \text{const}(G)$ and ~~z~~ $\in G$.

- a) Let $\{b, \xi\} \in \text{SS}_c\{s; \rho, \mu\}$ and either $\{s\} \in \text{const}(B)$ or $\{b\}, \{\xi\} \in \text{const}(B)$.

$$[\cap DC(G|E; c, d : s; \rho, \mu : \omega) \{ \omega = s(z) \}] \subseteq DC(G|E; c, d : b, \xi)$$

- b) If $\{s; \rho, \mu\} \in SF_c\{B; m, n\}$

$$[\cap DC(G|E; c, d : s; \rho, \mu : \omega) \{ \omega = s(z) \}] \subseteq DC(G|E; c, d)$$

- ii) Let $[s; \{\rho\}, \{\mu\}] \in \text{const}(B)$ and $\{b, \xi\} \in \text{SS}_c\{s; \rho, \mu\}$
- $$DC(G|E; c, d : b, \xi) \subseteq [\cap DC(G|E; c, d : s; \rho, \mu : \omega) \{ \omega = s(z) \}]$$

- iii) If, in addition, $E \in BS\{B \rightarrow K; m, n | s; \rho, \mu\}$, ~~then~~ the preceding semi-inclusion relationship becomes one of equivalence.

4] Let $\mathbb{G} \subseteq B$.

i) Let $b : B \rightarrow F[m]$, $\beta : B \rightarrow F[n]$ and $DC''(\mathbb{G} | E; c, d : b, \beta)$ be the complete subset of $z' \in DC(\mathbb{G} | E; c, d : b, \beta)$ for which, for at least one $z \in \mathbb{G}$, $b(z') = b(z)$ and $\beta(z') = \beta(z)$.

a) $DC'(\mathbb{G} | E; c, d : b, \beta) \subseteq DC''(\mathbb{G} | E; c, d : b, \beta)$

b) If $b, \beta \in \text{const}(\mathbb{G})$, $DC'(\mathbb{G} | E; c, d : b, \beta) = DC''(\mathbb{G} | E; c, d : b, \beta)$.

ii) Let $\{s; p, \mu\} \in SF\{B; m, n\}$ with $w', \dots, w'' \in S(z)$ for $z \in \mathbb{G}$. For $w = w', \dots, w''$ let $DC''(\mathbb{G} | E; c, d : s; p, \mu; w)$ be the complete subset of $z' \in DC(\mathbb{G} | E; c, d : s; p, \mu; w)$ for which, for at least one $z \in \mathbb{G}$, $p(z', w) = p(z, w)$ and $\mu(z', w) = \mu(z, w)$.

a) For $w = w', \dots, w''$, $DC'(\mathbb{G} | E; c, d : s; p, \mu; w) \subseteq DC''(\mathbb{G} | E; c, d : s; p, \mu; w)$

b) If, for $w = w', \dots, w''$, $p(z', w) = p(z, w)$ and $\mu(z', w) = \mu(z, w)$ for all $z, z' \in \mathbb{G}$, then $DC'(\mathbb{G} | E; c, d : s; p, \mu; w) = DC(\mathbb{G} | E; c, d : s; p, \mu; w)$ for these values of w .

5i) The results of part [1] and clause (2i) hold with, where appropriate, $DC(z | E; c, d : b, \beta)$ replaced by $DC'(z | E; c, d : b, \beta)$ and further similar modifications.

ii) A similarly modified version of clause (2ii) with the condition $\{b\}, \{\beta\} \in \text{const}(\mathbb{G})$ replaced by $b, \beta \in \text{const}(\mathbb{G})$ also holds.

iii) Similarly modified versions of clauses (2iii), (3ia) and (3ii) with the condition $\{s; \{p\}, \{\mu\}\} \in \text{const}(\mathbb{G})$ replaced by $\{s; p, \mu\} \in \text{const}(\mathbb{G})$ hold.

iv) A similarly modified version of subclause (2ia) incorporating both condition changes of (ii), (iii) above also holds.

Write $DC(z|E; c, d; b, \xi)$ and $DC(z|E; c, d; s; p, \mu; \omega)$ as $DC(z)$ and $DC(z|\omega)$ respectively.

$z' \in DC(z)$ if and only if in conjunction $\{b(z')\} = \{b(z)\}$, $\{\xi(z')\} = \{\xi(z)\}$, $E_{[\xi(z)]}^{[b(z)]}(z') = E_{[\xi(z)]}^b(z)$, $c_{[\xi(z)]}^{[b(z)]}(z') = c^b(z)$ and $d_{[\xi(z)]}^{[b(z)]}(z') = d^{\xi}(z)$. The first two of these conditions hold with z and z' interchanged. These two conditions imply that $b(z')$ is simply $b(z)$ in rearranged form and that $\xi(z)$ and $\xi(z')$ are similarly related. By suitable row and column rearrangement, it then follows from the second condition that $E_{[\xi(z)]}^b(z) = E_{[\xi(z')]}^{[b(z')]}(z) = E_{[\xi(z')]}^{[b(z')]}(z)$. The third condition holds with z and z' interchanged as do the remaining two. Subclause (1ia) has been disposed of.

If $z'' \in DC(z')$ then, in particular, $\{b(z'')\} = \{b(z')\}$ and $E_{[\xi(z')]}^{[b(z')]}(z'') = E_{[\xi(z')]}^b(z')$. Thus, if $z' \in DC(z)$, $\{b(z'')\} = \{b(z)\}$.

Also, by row and column rearrangement, $E_{[\xi(z)]}^{[b(z)]}(z'') = E_{[\xi(z)]}^b(z)$. In short, if $z'' \in DC(z')$ and $z' \in DC(z)$ then $z'' \in DC(z)$: $DC(z') \subseteq DC(z)$. Similarly $DC(z) \subseteq DC(z')$

If $DC(z)$ and $DC(z')$ are disjoint, the first result of clause (1iii) holds. If $z'' \in DC(z') \cap DC(z)$ then $DC(z) = DC(z'') = DC(z')$ and the result in question again holds.

$z' \in DC(z|\omega)$ if and only if in conjunction $w \in S(z')$,
 $\{\rho(z',\omega)\} = \{\rho(z,\omega)\}$, $\{\mu(z',\omega)\} = \{\mu(z,\omega)\}$, $E_{[\mu(z,\omega)]}^{[\rho(z,\omega)]}(z') = E_{\mu}^{\rho}(z,\bar{\omega})$,
 $c_{[\rho(z,\omega)]}(z') = c^{\rho}(z,\omega)$ and $d_{[\mu(z,\omega)]}(z') = d^{\mu}(z,\omega)$. The
proof of clause (1ii) and that of the relevant part of
clause (1iii) are similar to the proofs just given.
 $z' \in DC(G)$ if and only if $z' \in DC(z)$ for at least one
 $z \in G$. All such z' belong to $\bigcup DC(z) \{z \in G\}$: $DC(G) \subseteq$
 $\left[\bigcup DC(z) \{z \in G\} \right]$. Any $z' \in \left[\bigcup DC(z) \{z \in G\} \right]$ belongs to
 $DC(z)$ for at least one $z \in G$ and hence to $DC(G)$. The result of
clause (2e) follows.

The conditions of clause (2ii) imply that for each $z \in G$, in
conjunction $\{b(z')\} = \{b(z)\}$, $\{\frac{1}{3}(z')\} = \{\frac{1}{3}(z)\}$, $E_{[\frac{1}{3}(z)]}^{[b(z)]}(z') = E_{\frac{1}{3}}^b(z)$,
 $c_{[b(z)]}(z') = c^b(z)$ and $d_{[\frac{1}{3}(z)]}(z') = d^{\frac{1}{3}}(z)$ for all $z \in G$. Thus
 $z' \in DC(z)$ for all $z, z' \in G$. The result of subclause (2iia) follows
from subclause (1ib). Since $DC(z) = DC(z')$ for some fixed
 $z \in G$ and $z \in G$, the result of subclause (2iib) follows.

The conditions of clause (2iii) imply that ~~for each $z \in G$~~
the sequence $s(z)$ is constant for $z \in G$. Taking $z \in G$ and
 ~~$w \in S(z)$ fixed~~, the condition $c, d, E, \{s; \{\rho, \mu\}\}, j \in \text{not}(G)$ implies that
 ~~$w \in S(z)$ fixed~~ in conjunction $w \in S(z')$, $\{\rho(z',\omega)\} = \{\rho(z,\omega)\}$,
 $\{\mu(z',\omega)\} = \{\mu(z,\omega)\}$, $E_{[\mu(z,\omega)]}^{[\rho(z,\omega)]}(z') = E_{\mu}^{\rho}(z,\bar{\omega})$, $c_{[\rho(z,\omega)]}(z') =$
 $d_{[\mu(z,\omega)]}(z') = d^{\mu}(z,\omega)$ for all $z \in G$. Thus, with
 $c^{\rho}(z,\omega)$ and $d^{\mu}(z,\omega)$ and $d_{[\mu(z,\omega)]}(z') = d^{\mu}(z,\omega)$ for all $z \in G$.

ω fixed, $z' \in DC(z|\omega)$ for all $z, z' \in G$. It is easily shown that if $z'' \in DC(z'| \omega)$ then $z'' \in DC(z| \omega)$: $DC(z'| \omega) \subseteq DC(z| \omega)$ and, by reversing the roles of z and z' , $DC(z'| \omega) = DC(z| \omega)$. Since $z \in G$, $DC(z| \omega) \subseteq DC(G| \omega)$. Also, any $z'' \in DC(G| \omega)$ is in $DC(z'| \omega)$ for at least one $z \in G$ and hence in $DC(z| \omega)$. Clause (2iii) has been disposed of.

In clause (3i) it is assumed that $\{s\} \subseteq \text{const}(G)$: the space occurring on the left hand side of the semi-inclusion relationship of clause (3ia) is independent of $z \in G$. Select $w \in s(z)$ and $z' \in DC\{G|E; c, d: p, \mu: w\}$. $w \in s(z')$. Furthermore G contains $z(\omega)$ for which, in conjunction, $w \in s(z(\omega))$, $\{p(z', \omega)\} = \{p(z(\omega), \omega)\}$, $\{\mu(z', \omega)\} = \{\mu(z(\omega), \omega)\}$, $E^{[p(z(\omega), \omega)]}(z') = E_p^{[p(z(\omega), \omega)]}(z(\omega), \omega)$, $c^{[\mu(z(\omega), \omega)]}(z') = c_p^{[\mu(z(\omega), \omega)]}(z(\omega), \omega)$ and $d^{[\mu(z(\omega), \omega)]}(z') = d_p^{[\mu(z(\omega), \omega)]}(z(\omega), \omega)$.

In clause (3i) it is also assumed that the sets $\{p(z, \omega)\}$ and $\{\mu(z, \omega)\}$ are independent of $z \in G$. Thus with $z \in G$ fixed as in clause (3ii), $w \in s(z)$, $\{p(z', \omega)\} = \{p(z, \omega)\}$ and $\{\mu(z', \omega)\} = \{\mu(z, \omega)\}$. In part [3] it is assumed that $c, d, E \in \text{const}(G)$. Thus $E^{[p(z, \omega)]}(z') = E_p^{[p(z, \omega)]}(z, \omega)$, $c^{[\mu(z, \omega)]}(z') = c_p^{[\mu(z, \omega)]}(z, \omega)$ and $d^{[\mu(z, \omega)]}(z') = d_p^{[\mu(z, \omega)]}(z, \omega)$. With z' belonging to the space occurring on the left hand side of the semi-inclusion relationship of

clause (3i), the relationships $wes(z')$, $wes(z)$, ... and $\bigcup_{\omega \in s(z)} \{p(z, \omega)\}$ hold for $\omega = s(z)$. The sets $\{p(z, \omega)\}$ are disjoint, as are the sets $\{p(z', \omega)\}$. The set union $\bigcup \{p(z, \omega)\} \mid \{\omega = s(z)\}$ constitutes $\{b(z)\}$. If s is constant over B , the sets $\{p(z, \omega)\}$ and $\{p(z', \omega)\}$ are the same in number and, since $\{p(z', \omega)\} \neq \{p(z, \omega)\}$ for all relevant ω , $\{b(z')\} = \left[\bigcup \{p(z', \omega)\} \mid \{\omega = s(z)\} \right] = \{b(z)\}$. If, alternatively, $\{b\}, \{\zeta\} \in \text{const}(B)$, $\{b(z')\} = \{b(z)\}$ also.

Since $E \in BS\{B \rightarrow K; m, n \mid s; p, \mu\}$, $E \frac{\{p(z, z)\}}{\{\mu(z, z)\}}(z) = O \frac{|p(z, z)|}{|\mu(z, z)|}$ for all $z, z \in s(z)$ for which $z \neq z$. Also $E \frac{\{p(z', z)\}}{\{\mu(z', z)\}}(z') = O \frac{|p(z', z)|}{|\mu(z', z)|}$ for all $z, z \in s(z')$ for which $z \neq z$. Since all $z, z \in s(z)$ feature in $s(z')$, the latter relationship holds for all $z, z \in s(z')$ for which $z \neq z$. Furthermore, $p(z', z)$ is a rearranged form of $p(z, z)$ and $|p(z, z)| = |p(z', z)|$ for $z = s(z)$. Thus

$$E \frac{\{p(z, z)\}}{\{\mu(z, z)\}}(z') = O \frac{|p(z, z)|}{|\mu(z, z)|} \text{ for all } z, z \in s(z) \text{ for which } z \neq z.$$

Also $E \frac{\{p(z, \omega)\}}{\{\mu(z, \omega)\}}(z') = E \frac{\{p(z, \omega)\}}{\{\mu(z, \omega)\}}(z)$ for $\omega = s(z)$. By combining

the last two sets of relationships, it follows that $E \frac{\{p(z, \omega)\}}{\{\zeta(z, \omega)\}}(z)$

$= E_{\frac{b}{z}}^b(z)$. Again, the relationships $c^{[P(z,w)]}(z') = c^P(z,w)$ ($w = s(z)$) may be combined in the form $c^{[\frac{b}{z}(z)]}(z') = c^{\frac{b}{z}}(z)$. Similarly $d^{[\frac{s}{z}(z)]}(z') = d^{\frac{s}{z}}(z)$. In conclusion $z \in DC\{z | E; c, d, b, \frac{s}{z}\} \subseteq DC\{G | E; c, d, b, \frac{s}{z}\}$.

When $\{s; p, \mu\} \in SF_c\{B; m, n\}$, $\{b(z)\} = [m]$ and $\{\frac{s}{z}(z)\} = [n]$ for $z \in B$, where $\{b, \frac{s}{z}\} \subsetneq S_c\{s; p, \mu\}$. Subclause (3ii) is a corollary to its predecessor.

To prove subclause (3iia), select $z \in G$ and $z' \in DC\{z | E; c, d : b, \frac{s}{z}\}$. The condition $[s; p, \mu] \in \text{const}(B)$ implies that for $w = s(z)$, in conjunction with $w \in s(z')$, $\{p(z', w)\} = \{p(z, w)\}$ and $\{u(z', w)\} = \{u(z, w)\}$. Furthermore, since $E^{[\frac{b}{z}(z)]}(z') = E^{[\frac{b}{z}(z)]}_{L^{\frac{b}{z}}(z)}$,

where $\{b, \frac{s}{z}\} \subsetneq S_c\{s; p, \mu\}$, appropriate subtraction after extraction of submatrices yields the relationships $E^{[\frac{p(z,w)}{u(z,w)}]}(z') = E^P_{\mu}(z, w)$ ($w = s(z)$).

Similarly $c^{[P(z,w)]}(z') = c^P(z, w)$ and $d^{[\frac{s}{z}(z,w)]}(z') = d^{\frac{s}{z}}(z, w)$ ($w = s(z)$). The result of subclause (3iia) has been disposed of.

The conditions of subclause (3iia) imply, in particular, that $[s; p, \mu] \in \text{const}(G)$ and $\{s\} \in \text{const}(B)$. If,

furthermore, $E \in BS\{B \rightarrow K; m, n | s; p, \mu\}$, the conditions of subclause (3iia) are satisfied, the semi-inclusion relationship of clause (3iia) may be reversed to yield the result of subclause (3iib).

The proofs of the results referred to in part [5] are similar to and slightly simpler than those of the results in parts [1-3] to which they correspond.

Conversely, let z' belong to the subset described in clause (4i).

Since $z \in DC(G)$, G contains z for which $\{b(z')\} = \{b(z)\}$, $\{\dot{b}(z')\} = \{\dot{b}(z)\}$, $E_{[\dot{b}(z)]}^{[b(z)]}(z') = E_{\dot{b}(z)}^b(z)$, $c_{[\dot{b}(z)]}^{[b(z)]}(z') = c^b(z)$ and $d_{[\dot{b}(z)]}^{[\dot{b}(z)]}(z') = d^{\dot{b}}(z)$. Furthermore G contains z'' for which $b(z') = b(z'')$ and $\dot{b}(z') = \dot{b}(z'')$. The condition $b, \dot{b} \in \text{const}(G)$

implies that $b(z'') = b(z)$ and $\dot{b}(z'') = \dot{b}(z)$ and then $E_{\dot{b}(z)}^b(z') = b, \dot{b}$.

$E_{\dot{b}(z)}^b, c^b(z') = c^b(z)$ and $d^{\dot{b}}(z') = d^{\dot{b}}(z)$ ~~and~~ $\therefore z' \in DC'(G | E; c, d; b, \dot{b})$.

Conversely, if $z' \in DC'(G | E; c, d; b, \dot{b})$, G contains z for which $b(z') = b(z)$, $\dot{b}(z') = \dot{b}(z)$, so that $\{b(z')\} = \{b(z)\}$,

$\{\dot{b}(z')\} = \{\dot{b}(z)\}$, $E_{\dot{b}(z)}^b(z') = E_{\dot{b}(z)}^b(z)$, $c^b(z') = c^b(z)$ and $d^{\dot{b}}(z') =$

$d^{\dot{b}}(z)$. The first two relationships imply that z' satisfies $d^{\dot{b}}(z)$. The first two relationships imply that z' satisfies

the condition defining the subset of $DC(G)$ and also

that $\{b(z')\} = \{b(z)\}$, $\{\dot{b}(z')\} = \{\dot{b}(z)\}$. Also ~~since~~ ~~since~~

$E_{[\dot{b}(z)]}^{[b(z)]}(z') = E_{\dot{b}(z)}^b(z)$, $c_{[\dot{b}(z)]}^{[b(z)]}(z') = c^b(z)$

$b(z') = b(z)$, $\dot{b}(z') = \dot{b}(z)$ and $d_{[\dot{b}(z)]}^{[\dot{b}(z)]}(z') = d^{\dot{b}}(z)$; $z' \in DC(G)$. Subclause (4ia) has been disposed of.

This result, combined with that of subclause (4ia), disposes of subclause (4ib).

The proof of clause (4ii) is a slight extension to that of its predecessor.

~~When the restriction that the set G should contain a single member $z \in B$ is imposed, the above theorem experiences considerable simplification. Part [2] loses meaning. The conditions of part [3] reduce to $\{s; p, \mu\} \in SF\{B; m, n\}$ and those of clause (3i) to $E \in BS\{B \rightarrow K; m, n \mid s; p, \mu\}$. The result of clause (3ia) becomes $[ADC(z \mid E; c, d : b, \frac{1}{3}) \quad (\omega = s(z))] \subseteq DC(z \mid E; c, d : b, \frac{1}{3})$ and that of subclause (3ib) of subclause (3ia) and that in clause (4i) the condition $b, \frac{1}{3} \in \text{const}(G)$ is automatically fulfilled and the clause reduces to the single result that $DC'(z \mid E; c, d : b, \frac{1}{3}) = DC''(z \mid E; c, d : b, \frac{1}{3})$. Clause (4ii) is similarly simplified.~~

When the restriction that the set G should contain a single member $z \in B$ is imposed, the above theorem experiences considerable simplification. Part [2] ~~loses meaning~~ becomes ~~meaningless~~. The conditions of part [3] reduce to $\{s; p, \mu\} \in SF\{B; m, n\}$ and those of clause (3i) to $E \in BS\{B \rightarrow K; m, n \mid s; p, \mu\}$. In clause (4i) the condition $b, \frac{1}{3} \in \text{const}(G)$ of subclause (4ia) is automatically fulfilled and the clause reduces to the single result that $DC'(z \mid E; c, d : b, \frac{1}{3}) = DC''(z \mid E; c, d : b, \frac{1}{3})$. Clause (4ii) is simplified similarly simplified.

Properties of the preinvariant spaces with respect to the preinvariant systems
and domain/col, of consistency and (col) dominance of constraints in two such systems

() Let $c \in \{B \rightarrow K; m\}$, $d \in \{B \rightarrow K; n\}$, $E \in S$ and $f \in B$

1] Let $b: B \rightarrow F[m]$ and $\beta: B \rightarrow F[n]$

If $E \in PQ(c/d/b, \beta/f)$ then

i) $E \in PQ(c/d/b, \beta; DC(f|E; c, d; b, \beta))$ and

ii) $IN(E | b, \beta; f) \subseteq PQ(c/d/b, \beta; f)$

2] Let $\{s; \rho, \mu\} \in SF\{B; m, n\}$ and $w \in s(z)$ then for $z \in G$

If $E \in PQ(c/d/\rho, \mu; f, w)$ then

i) $E \in PQ(c/d/b, \beta; DC(f|E; c, d; s; \rho, \mu; w))$

ii) $IN(E | \rho, \mu; f, w) \subseteq PQ(c/d/b; \rho, \mu; f, w)$

The relationship

$$(1) \quad E^{[b(z)]}_{[\beta(z)]}(z)d^{[\beta(z)]}_{[b(z)]}(z) = c^{[b(z)]}_{[\beta(z)]}(z)$$

holding for all $z \in G$, characterises the mapping systems E in $PQ(c/d/b, \beta; z)$.
 $E \in PQ(c/d/b, \beta; z)$ if and only if $E \in PQ(c/d/b, \beta)$ for $z \in G$.

$DC(f|E; c, d; b, \beta)$ is the complete ~~subset~~ of $z \in B$ for
 for at least one $z \in G$, in conjunction

which, $\{b(z')\} = \{b(z)\}$, $\{\beta(z')\} = \{\beta(z)\}$, $c^{[b(z)]}_{[\beta(z)]}(z') = c^{[b(z)]}_{[\beta(z)]}(z)$,

Select $z' \in DC(f|E; c, d; b, \beta)$ and $E^{[b(z)]}_{[\beta(z)]}(z') = E^{[b(z)]}_{[\beta(z)]}(z)$. Evidently
 $d^{[\beta(z)]}_{[b(z)]}(z') = d^{[\beta(z)]}_{[b(z)]}(z')$ and $E^{[b(z)]}_{[\beta(z)]}(z') = E^{[b(z)]}_{[\beta(z)]}(z)$. Evidently
 $E^{[b(z)]}_{[\beta(z)]}(z')d^{[\beta(z)]}_{[b(z)]}(z') = c^{[b(z)]}_{[\beta(z)]}(z')$ be a companion in the preceding sense.

$$E^{[b(z)]}_{[\beta(z)]}(z')d^{[\beta(z)]}_{[b(z)]}(z') = c^{[b(z)]}_{[\beta(z)]}(z')$$

for all such z' . The conditions $\{b(z')\} = \{b(z)\}$, $\{\beta(z')\} = \{\beta(z)\}$

imply that $b(z'), \beta(z')$ are $b(z), \beta(z)$ in rearranged order. Relationship
 (1) with z replaced throughout by z' for all such z' : if $E \in PQ(c/d/b, \beta; z)$
 holds for

then $E \in PQ(c/d | p, s; z)$ for all $z \in DC(\mathbb{G} | E; c, d : p, s)$. The result of clause (1i) follows.

$\text{IN}(E | p, s; \mathbb{G})$ is the complete system of mapping systems $\hat{E} \in MS\{B \rightarrow K; m, n\}$ for which $\hat{E}_{[s(z)]}^{[p(z)]}(z) = E_{[s(z)]}^{[p(z)]}(z)$ for all such \hat{E} , relationship (1) with E replaced by \hat{E} is satisfied. The result of clause (1ii) follows.

The relationship With $w \in s(z)$, the relationship

$$E_{[\rho(z, w)]}^{[\rho(z, w)]}(z) d_{[\mu(z, w)]}^{[\mu(z, w)]}(z) = c_{[\rho(z, w)]}^{[\rho(z, w)]}(z)$$

holding for $z \in G$, characterises the mapping systems E in $PQ(c/d | p, \mu; \mathbb{G}, \omega)$

With $\{s; \rho, \mu\} \in SF\{B; m, n\} \cap DC$ and $w \in s(z)$, $DC(\mathbb{G} | E; c, d : s; \rho, \mu : \text{true})$ is the complete system of $z' \in B$ for which $w \in s(z')$

for at least one $w \in G$ in conjunction also, and $\{\rho(z', w)\} = \{\rho(z, w)\}$, $\{\mu(z', w)\} = \{\mu(z, w)\}$, $c_{[\rho(z, w)]}^{[\rho(z, w)]}(z') = c_{[\rho(z, w)]}^{[\rho(z, w)]}(z)$, $d_{[\mu(z, w)]}^{[\mu(z, w)]}(z') = d_{[\mu(z, w)]}^{[\mu(z, w)]}(z)$ and $E_{[\mu(z, w)]}^{[\rho(z, w)]}(z') = E_{[\mu(z, w)]}^{[\rho(z, w)]}(z)$.

Since $w \in s(z')$, the space $PQ(c/d | p, \mu; z', w)$ is defined for all such z' . The detailed proof of clause (2i) is as for that of clause (1i). The proof of clause (2ii) is also that of clause (1ii).

The equivalence of prequotient and relative product spaces resulting from a factorisation property; a factorisation property depending upon the intersection of a prequotient and a relative product space; that these spaces are either disjoint or identical

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $C \in \text{MS}\{B \rightarrow K; km\}$, $E \in \text{MS}\{B \rightarrow K; m, n\}$ and $G \subseteq B$.

1] Let $f: B \rightarrow F[m]$ and $\tilde{f}: B \rightarrow F[n]$

i) If $E \in PQ(c/d | f, \tilde{f}; G)$,

$$PQ(C, c/d | f, \tilde{f}; G') = RP(C, E; d | f, \tilde{f}; G')$$

for all $G' \subseteq DC(G | E; c, d : f, \tilde{f})$

ii) Let $C \in \text{MS}_{NR(f; G)}\{B \rightarrow K; r, n\}$ and $r \geq |f(z)|$ for $z \in G$.

a) If the intersection of the two spaces $PQ(C, c/d | f, \tilde{f}; G)$ and $RP(C, E; d | f, \tilde{f}; G)$ is nonvoid, $E \in PQ(c/d | f, \tilde{f}; DC(G | E; c, d : f, \tilde{f}))$.

b) The two spaces $PQ(C, c/d | f, \tilde{f}; G)$ and $RP(C, E; d | f, \tilde{f}; G)$ are either disjoint or identical.

2] Let $\{s; \rho, \mu\} \in SF\{B; m, n\}$ and $w \in s(z)$ for $z \in G$.

i) If $E \in PQ(c/d | \rho, \mu; G, w)$,

$$PQ(C, c/d | \rho, \mu; G', w) = RP(C, E; d | \rho, \mu; G', w)$$

for all $G' \subseteq DC(G | E; c, d : s; \rho, \mu : w)$

ii) Let $C \in \text{MS}_{NR(\rho; G, w)}\{B \rightarrow K; r, n\}$ and $r \geq |\rho(z, w)|$ for $z \in G$

a) If the intersection of the two spaces $PQ(C, c/d | \rho, \mu; G, w)$ and $RP(C, E; d | \rho, \mu; G, w)$ is nonvoid, $E \in PQ(c/d | \rho, \mu; DC(G | E; c, d : s; \rho, \mu : G, w))$.

b) The two spaces $PQ(C, c/d | p, \mu; G, \omega)$ and $RP(C, E; d | p, \mu; G)$ are either disjoint or identical.

The conditions

$$(1) \quad E_{\frac{b}{3}}^b(z) d^{\frac{1}{3}}(z) = C_b^b(z)$$

$$(2) \quad D_{\frac{b}{3}}(z) d^{\frac{1}{3}}(z) = C_b(z) C_b^b(z)$$

$$(3) \quad D_{\frac{b}{3}}(z) d^{\frac{1}{3}}(z) = C_b(z) E_{\frac{b}{3}}^b(z) d^{\frac{1}{3}}(z)$$

holding for $z \in G$ in each case characterise the mapping system $E \in PQ(C, c/d | b, \frac{1}{3}; G)$, $D \in PQ(C, c/d | b, \frac{1}{3}; G)$ and $D \in RP(C, E; d | b, \frac{1}{3}; G)$ respectively. If $E \in PQ(C, c/d | b, \frac{1}{3}; G)$, relationship (1) holds with z replaced by \bar{z} for all $\bar{z} \in DC(G|E; c, d : b, \frac{1}{3})$. From clause (1i) of Theorem . If $D \in PQ(C, c/d | b, \frac{1}{3}; z')$ for all $z' \in G$'s $DC(G|E; c, d : b, \frac{1}{3})$ relationship (2) is satisfied with z replaced by \bar{z}' for all such \bar{z}' . Replacement of $C_b(z')$ by use of relationship (1) reproduces relationship (3) with z replaced by \bar{z}' . $PQ(C, c/d | b, \frac{1}{3}; G') \subseteq RP(C, E; d | b, \frac{1}{3}; G)$. That the reversed form of this semi-inclusion relationship holds is shown in a similar way: clause (1i) has been disposed of.

If relationships (2,3) are satisfied for $z \in G$ by some

$\text{DEMS}\{B \rightarrow K; r, n\}$, subtraction yields the relationship

$$C_B(z) \left\{ E_B(z) d^B(z) - C_B(z) \right\} = 0^{[r]}$$

holding for $z = B$. For any V and in turn the resulting relationship obtained by replacing $C_B(z), 0^{[r]}$ by $C_{B'}(z), 0^{[L \equiv \{V\}]}$ where $\equiv : \mathbb{F} \rightarrow F[V]$, also holding for $z = B'$.

Under the conditions of part [2], \equiv may be so chosen that $|L \equiv| = |B(z)|$ and $C_{B'}^{z \equiv B}(z)$ is nonsingular. The derived relationship may then be multiplied throughout by $\{C_{B'}^{z \equiv B}(z)\}^{-1}$ to yield the relationship characterising E as

a member of $\text{RP}(PQ(c/d | b, \frac{1}{3}; \mathbb{F}))$ and hence of $\text{RP}(c/d | b, \frac{1}{3}; \mathbb{F})$ for all $z \in PQ(c/d | b, \frac{1}{3}, DC(G | E; c, d; b, \frac{1}{3}))$.

The result of clause (iii) follows directly from part L1 of clause (1i) and subclause (1ii). If the two spaces concerned are disjoint, the two spaces are disjoint. If the intersection is nonvoid, $E \in PQ(c/d | b, \frac{1}{3}; \mathbb{F})$ and in turn $PQ(c, c/d | b, \frac{1}{3}; \mathbb{F}) = RP(c, E; d | b, \frac{1}{3}; \mathbb{F})$

() Let $c \in \{B \rightarrow K; m\}$, $d \in \{B \rightarrow K; n\}$, $b : B \rightarrow F[m]$, $\frac{1}{3} : B \rightarrow F[n]$ $\phi \in \{B \rightarrow K; m\}$, $E \in \text{MS}\{B \rightarrow K; m, n\}$ and $\mathbb{F} \subseteq B$. With $z_1 \in B$ and $z_2(z) \in B$ ($z \in [1_B(z)]$) prescribed, $\bar{\Phi}[b, \{z_2\}]$ is the matrix whose $(z+1)^{\text{th}}$ row is $\phi_B(z_2)(z \in [1_B(z)])$. S is a set of points in $DC(E; c, d | b, \frac{1}{3}; z)$. $\Phi[B(z)]^{\{z_2(E)\}}$!! ordering $b(z_2)$ changes with z_2

The results of part [2] are slight extensions of their counterparts in part [1] ^{and} they are proved as their proofs are as above similar to those given above.

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\phi \in \text{row}\{B \rightarrow K; m\}$, $E \in \text{MS}\{B \rightarrow K; m, n\}$ and $G \subseteq B$. With $z \in B$ and $z_z(z) \in B$ ($z = [1 b(z)]$) prescribed, $\underline{\Phi}[b, z, \{z_z\}]$ is the matrix whose $(z+1)^{\text{th}}$ row is $\underline{\Phi}[B(z)]$ ($z = [1 B(z)]$).

[1] Let $b: B \rightarrow F[m]$ and $\beta: B \rightarrow F[n]$.

i) If $E \in PQ(c/d | b, \beta; G)$,

$$PQ(\phi, c/d | b, \beta; G) = RP(\phi, E; d/b, \beta; G')$$

for all $G' \subseteq DC(G | E; c, d : b, \beta)$

ii) For each $z \in G$, let $DC'(z | E, c, d : b, \beta)$ contain

$z_z(z)$ ($z = [1 b(z)]$) for which $\underline{\Phi}[b, z, \{z_z\}]$ is nonsingular
the matrix whose $(z+1)^{\text{th}}$ row is $\underline{\Phi}_b(z_z(z))$ ($z = [1 b(z)]$ is nonsingular)

a) If, for each $z \in G$, the intersection of the two spaces

$PQ(\phi, c/d | b, \beta; z_z(z))$ and $RP(\phi, E; d/b, \beta; z_z(z))$ is nonvoid ($z = [1 b(z)]$), $E \in PQ(c/d | b, \beta; DC(G | E; c, d : b, \beta))$

b) The two spaces $PQ(\phi, c/d | b, \beta; G)$ and $RP(\phi, E; d/b, \beta; G)$ are either disjoint or identical

[2] Let $\{s; p, \mu\} \in SF\{B; m, n\}$ and $w \in s(z)$ for $z = G$.

i) If $E \in PQ(c/d | p, \mu; G, w)$,

$$PQ(\phi, c/d | \rho, \mu; \mathbb{G}, \omega) = RP(\phi, E; d | \rho, \mu; \mathbb{G}, \omega)$$

for all $\mathbb{G}' \subseteq DC(G | E; c, d : s; \rho, \mu; \omega)$

i) ~~Let~~ For each $z \in \mathbb{G}$, let $DC'(z | E; c, d : \rho, \mu : s; \rho, \mu : \omega)$ contain $z_{\tau}(z)$ ($z = [|\rho(z, \omega)|]$) for which ~~$\Phi_{\rho}^{[b, z, \{z_{\tau}\}]}$~~ non-singular, the matrix whose $(z+1)^{\text{th}}$ row is $\Phi_{\rho}^{(z_{\tau}(z), \omega)}$ ~~is non-singular~~ $z = [|\rho(z, \omega)|]$ is non-singular.

a) If, for each $z \in \mathbb{G}$, the intersection of the two spaces

$PQ(\phi, c/d | \rho, \mu; \mathbb{G}, \omega) \cap z_{\tau}(z), \omega)$ and $RP(\phi, E; d | \rho, \mu; z_{\tau}(z), \omega)$ is nonvoid ($z = [|\rho(z, \omega)|]$), $E \in PQ(c/d | \rho, \mu; DC(G | E; c, d : s; \rho, \mu : \omega); \omega)$.

b) The two spaces $PQ(\phi, c/d | \rho, \mu; \mathbb{G}, \omega)$ and $RP(\phi, E; d | \rho, \mu; \mathbb{G}, \omega)$ are either disjoint or identical.

The result of clause (1i) is a corollary to clause (1ii) of

Theorem : set $r=0$ and replace $C(z)$ by $\phi(z)$.

Under the conditions of subclause (1ii), for each $z \in \mathbb{G}$ now $\{B \rightarrow K; n\}$ contains $\psi^{(z)} (z = [|\rho(z)|])$ for which, setting

$$z_{\tau}(z) = z_{\tau},$$

$$\psi_{\frac{1}{3}}^{(z)}(z_{\tau}) d^{\frac{1}{3}}(z_{\tau}) = \phi_{\rho}(z_{\tau}) c^b(z_{\tau})$$

$$\psi_{\frac{2}{3}}^{(z)}(z_{\tau}) d^{\frac{2}{3}}(z_{\tau}) = \phi_{\rho}(z_{\tau}) E_{\frac{2}{3}}^b(z_{\tau}) d^{\frac{2}{3}}(z_{\tau})$$

for $z = [|\rho(z)|]$. Since $z_{\tau}(z) \in DC'(z | E; c, d : \rho, \frac{2}{3})$, $E_{\frac{2}{3}}^b(z_{\tau}) = \bar{E}_{\frac{2}{3}}^b$ $c^b(z_{\tau}) = c^b(z)$ and $d^{\frac{2}{3}}(z_{\tau}) = d^{\frac{2}{3}}(z)$ ($z = [|\rho(z)|]$). Eliminating the

terms $\psi_{\frac{1}{3}}(z_2) d^{\frac{1}{3}}(z_2)$, the relationship

$\bar{\Phi}[b, z, \{z_2\}] \{E_{\frac{1}{3}}^b(z) d^{\frac{1}{3}}(z) - C^b(z)\} = [1 b(z)]$
 where $\bar{\Phi}[b, z, \{z_2\}]$ is the matrix described in clause (1ii), is obtained.
~~is obtained.~~ Since $\bar{\Phi}[b, z, \{z_2\}]$ is assumed to be nonsingular

$$(1) \quad E_{\frac{1}{3}}^b(z) d^{\frac{1}{3}}(z) = C^b(z)$$

for $z \in G$ and $E \in PQ(c/d | b, \frac{1}{3}; G)$. From clause (1a) of Theorem
 $E \in PQ(c/d | b, \frac{1}{3}; DC(G | E; c, d : b, \frac{1}{3}))$.

If, under the conditions of clause (1ii), the two spaces
 $PQ(\phi, c/d | b, \frac{1}{3}; G)$ and $RP(\phi, E; d | b, \frac{1}{3}; G)$ are not disjoint
 row $\{B \rightarrow K; n\}$ contains ϕ for which, for each $z \in G$, setting

$$z_2 = z_2(z),$$

$$\psi_{\frac{1}{3}}(z_2) d^{\frac{1}{3}}(z_2) = \phi_B(z_2) C^b(z_2)$$

$$\psi_{\frac{1}{3}}(z_2) d^{\frac{1}{3}}(z_2) = \phi_B(z_2) E_{\frac{1}{3}}^b(z_2) d^{\frac{1}{3}}(z_2)$$

for $z = [1 b(z)]$. These relationships, as was shown above, induce
 relationship (1) holding for $z = G : E \in PQ(c/d | b, \frac{1}{3}; G)$. The result
 of clause (1i) holds and, in particular, $PQ(\phi, c/d | b, \frac{1}{3}; G) =$
 $RP(\phi, E; d | b, \frac{1}{3}; G)$.

The results of part [2] are slight extensions of their counterpart
 in part [1] and their proofs are similar to those given above.

(c) Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $b : B \rightarrow F[m]$, $\frac{1}{3} : B \rightarrow F[n]$
 $E \in MS\{B \rightarrow K; m, n\}$ and $G \subseteq B$

1] Let $r \geq 1 b(z)$ for $z \in G$.

$C \in MS_{MR}(b, G) \{B \rightarrow K; r, n\}$ and $D \in PQ(c, c/d | b, \frac{1}{3}; G) \cap RP(C, E;$
 $b, \frac{1}{3}; G)$ exist if and only if
 a) For each $z \in G$, $M_{MR}\{K; 1 b(z)\}$ and $M\{K; 1 b(z)\}, 1 \frac{1}{3}(z)\}$ contain $\hat{C}(z)$ a

$$(1) \quad \hat{C}(z) c^b(z) = \hat{D}(z) d^{\frac{b}{2}}(z) \quad \text{for which}$$

and

$$(2) \quad \{\hat{D}(z) - \hat{C}(z) E_{\frac{b}{2}}^b(z)\} d^{\frac{b}{2}}(z) = O^{[|B(z)|]}$$

for $z \in G$.

2] It is supposed in the following

a) that $G' \subseteq G$ is such that the sets $DC(z'|E; c, d; b, \frac{b}{2})$ $z' \in G'$ are disjoint and that their union forms G , and

b) that $|B(z')|+1$ members $z_z(z')$ ($z = [B(z')]$) of $DC(z'|E; c, d; b, \frac{b}{2})$

b) that for each $z' \in G'$, $|B(z')|+1$ distinct members $z_z(z') = z_z(z')$ ($z = [B(z')]$) of $DC(z'|E; c, d; b, \frac{b}{2})$ are associated with all $z \in DC(z'|E; c, d; b, \frac{b}{2})$.

Now $\{B \rightarrow K; m\}$ contains ϕ for which, for each $z \in G$, the matrix whose $(z+1)^{th}$ row is $\phi_{[B(z)]}(z_z(z))$ ($z = [B(z)]$) is nonsingular and

c) now $\{B \rightarrow K; m\}$ contains ψ for which

(3) $\psi \in PQ(\phi, c/d | b, \frac{b}{2}; z_z(z)) \cap RP(\phi, E; d | b, \frac{b}{2}; z_z(z))$
 for $z = [B(z)]$ exists
 exists if and only if condition (a) holds

ii) Replace $DC(z'|E; c, d; b, \frac{b}{2})$ by $DC'(z'|E; c, d; b, \frac{b}{2})$ in conditions (a, b) of the preceding clause.

now $\{B \rightarrow K; m\}$ contains ϕ for which, for each $z \in G$, the matrix whose $(z+1)^{th}$ row is $\phi_B(z_z(z))$ ($z = [B(z)]$) is nonsingular and the mapping system ψ of formula (3) exists as described if and only if condition (a) holds.

$$(1) \quad \hat{C}(z)c^b(z) = \hat{D}(z)d^{\frac{1}{2}}(z)$$

and

$$(2) \quad \left\{ \hat{D}(z) - \hat{C}(z)E^{\frac{1}{2}}(z) \right\} d^{\frac{1}{2}}(z) = O^{[IB(z)]}$$

for $z \in G$.

[2] It is supposed in the following that for each $z \in G$, $z_z(z) \in DC(z|E; c, d; b, \frac{1}{2})$ ($z = [IB(z)]$)

i(b) row $\{B \rightarrow K; m\}$ contains ϕ for which, for each $z \in G$, the matrix whose $(z+1)^{th}$ row is $\psi_{IB(z)}^{(z, z)}(z_z(z))$ ($z = [IB(z)]$) is nonsingular and

c) row $\{B \rightarrow K; n\}$ contains $\psi^{(z, z)}(z = G, z = [IB(z)])$ for which

$$(3) \quad \psi^{(z, z)} \in PQ(\phi, c/d | b, \frac{1}{2}; z_z(z)) \cap RP(\phi, E; d | b, \frac{1}{2}; z_z(z))$$

for $z \in G, z = [IB(z)]$

if and only if

d) for each $z \in G$ the $z_z(z)$ are distinct and the further conditions of (a) above hold

ii) Condition (b) holds and

c') row $\{B \rightarrow K; n\}$ contains a single mapping system ψ for which relationship (3) with $\psi^{(z, z)}$ replaced by ψ holds

if and only if condition (d) holds as stated.

Assuming conditions (a) of part [1] to hold, construct $C \in NS\{B \rightarrow K; r, m\}$

by selecting $\Xi: G \rightarrow F[r]$ with $| \Xi(z) | = | B(z) |$ for $z \in G$,

and setting $C_{\Xi}^{\Xi}(z) = \hat{C}(z)$ for $z \in G$, $C_{[B(z)]}^{[\Xi(z)]} = O^{[r-IB(z)-1]}_{[IB(z)]}$ for

all $z \in G$ for which $| B(z) | < r$, $C_{[B(z)]}^{[z]} = O^{[r]}_{[m-IB(z)-1]}$ for

all $z \in G$ for which $| B(z) | \leq m$, and $C_{\Xi} = O_{[m]}^{[r]}$ over $B \setminus G$.

follows:

Construct $D \in NS\{B \rightarrow K; r, m\}$ by setting $D_{\Xi}^{\Xi}(z) = \hat{D}(z)$ for $z \in G$,

and imposing further conditions similar to those just imposed upon C . The ~~char~~ Relationships of $\underline{\underline{C}}^{(1,2)}_{\underline{\underline{B}}^{(1,2)}}$ may be presented as

$$(4) \quad C_{\underline{\underline{B}}^{(1,2)}}^{(1,2)}(z) C_{\underline{\underline{B}}^{(1,2)}}^{\underline{\underline{B}}}(z) = D_{\underline{\underline{B}}^{(1,2)}}^{(1,2)} d_{\underline{\underline{B}}^{(1,2)}}^{\underline{\underline{B}}}(z)$$

$$(5) \quad \left\{ D_{\underline{\underline{B}}^{(1,2)}}^{(1,2)}(z) - C_{\underline{\underline{B}}^{(1,2)}}^{(1,2)} E_{\underline{\underline{B}}^{(1,2)}}^{\underline{\underline{B}}}(z) \right\} d_{\underline{\underline{B}}^{(1,2)}}^{\underline{\underline{B}}}(z) = 0^{[1_B(z)]}$$

holding for $z \in G$.

Since the rows of $B, C_B(z)$ and $D_{\underline{\underline{B}}}^{(1,2)}(z)$ no with index not
 for each $z \in G$, featuring in the sequence $\underline{\underline{B}}^{(1,2)}$ are identically zero, these two relationships may be extended to the form

$$(6) \quad C_B(z) C_{\underline{\underline{B}}}(z) = D_{\underline{\underline{B}}}(z) d_{\underline{\underline{B}}}(z)$$

$$(7) \quad \left\{ D_{\underline{\underline{B}}}(z) - C_B(z) E_{\underline{\underline{B}}}(z) \right\} d_{\underline{\underline{B}}}(z) = 0^{[1_B(z)]}$$

With C as constructed, the first relationships characterize D as a member of $PQ(C, c/d | B, \underline{\underline{B}}; G)$ and $RP(C, E; d | B, \underline{\underline{B}}; G)$ respectively.

For each $z \in G$, a ~~PR nonsingular~~ submatrix of order $|B(z)|+1$, namely $C_{\underline{\underline{B}}^{(1,2)}(z)}^{(1,2)} = C$, is determined by suitable row selection from $C_B(z)$: the latter matrix is of ~~order~~ rank $|B(z)|+1$: $C \in MS_{MR}(B; G) \{B \rightarrow k; r, n\}$

Conversely, suppose that C and D as described in part [1] exist. Relationships (6, 7) hold. Since $C_B(z)$ is of rank $|B(z)|+1$, a mapping $\underline{\underline{B}}^{(1,2)} \rightarrow F[r]$ for which $|B(z)| = |B(z)|$ exists and such that sequence $\underline{\underline{B}}^{(1,2)}(z)$ is nonsingular exists. Appropriate row selection from

Domains of constancy are either identical or disjoint. From the matrices occurring in relationships (6,7) lead to relationships (4,5) which in turn may be presented in the form (1,2) where, in particular, $\hat{C}(z)$ is nonsingular for $z \in \mathbb{G}$.

From clause (iii) of theorem two domains of constancy $DC(z | E; c, d; b, \frac{1}{3})$ with $z = z', z'' \in \mathbb{B}$ are either identical or disjoint: \mathbb{G} has a decomposition of the form prescribed in condition (2c). Select $z \in \mathbb{G}$, $z \in DC(z' | E; c, d; b, \frac{1}{3})$ for one z' and only one $z' \in \mathbb{G}$, and, in particular, $\{b(z)\} = \{b(z')\}$, $\{\frac{1}{3}(z)\} = \{\frac{1}{3}(z')\}$. Also $\{z_2(z)\} \subset DC(z' | E; c, d; b, \frac{1}{3})$ and hence $\{b(z_2(z))\} \subset \{b(z')\}$ and $\{\frac{1}{3}(z_2(z))\} = \{\frac{1}{3}(z')\}$, $E^{[b(z)]} (z_2(z)) = E^{[b(z')]} (z_2(z))$, $C^{[b(z)]} (z_2(z)) = C^{[b(z')]} (z_2(z))$ and $d^{[b(z)]} (z_2(z)) = d^{[b(z')]} (z_2(z))$ all for $z = [1 b(z) 1]$. The $z_2(z)$ are distinct. Furthermore, no $z_2(z)$ associated with $z \in DC(z' | E; c, d; b, \frac{1}{3})$ is equal to a $z_2(z)$ associated with a $z \in DC(z'' | E; c, d; b, \frac{1}{3})$, where z' and z'' are distinct members of \mathbb{G}' , since the two domains of constancy in question are disjoint. It is possible to define $\phi \in \text{row}\{B \rightarrow k\}$

by setting $\phi^{(z)}_{[b(z')]} (z_2(z)) = \hat{C}^{(z)} (z')$ and, when $|b(z)| < m$, $\phi_{[b(z')]} (z_2(z)) = O_{[m - |b(z')| - 1]}$, both for $z' \in \mathbb{G}'$ and $z = [1 b(z') 1]$, and $\phi = O_{[m]}$ over $B \setminus [\cup \{z_2(z')\} \{z' \in \mathbb{G}'\}]$. Similarly $\psi \in \text{row}\{B \rightarrow k; n\}$ may be defined by setting $\psi^{(z)}_{[\frac{1}{3}(z')]} (z_2(z)) = \hat{D}^{(z)} (z')$ and,

when $|b(z)| < n$, $\psi_{[\frac{1}{3}(z')]}\left(z_2(z)\right) = \phi_{[n-1\frac{1}{3}(z')-1]}$, both
 for $z' = G'$ and $z = [1 \frac{1}{3} b(z')]$, and $\psi = \phi_{[n]}$ over
 $B \setminus [\cup \{z_2(z')\} \cup \{z' = G'\}]$.

Relationship (1) may be decomposed in the form as
 $|b(z)|+1$ relationships

$$(10) \quad \hat{C}^{[z]}(z') c^b(z') = \hat{D}^{(z)}(z') d^{\frac{1}{3}}(z')$$

for $z = [1 b(z)]$ or, since $z_2(z) \in DC(z' | E; c, d; b, \frac{1}{3})$,

$$(11) \quad \phi_{[b(z')]}(z_2(z)) \in^{[b(z')]}(z_2(z)) = \psi_{[\frac{1}{3}(z')]}^{[b(z')]}(z_2(z))$$

or, by rearrangement of the vector products,

$$(12) \quad \phi_b(z_2(z)) c^b(z_2(z)) = \psi_{\frac{1}{3}}(z_2(z)) d^{\frac{1}{3}}(z_2(z))$$

for $z = [1 b(z)]$ in each case.

Similarly, relationship (2) leads to the relationships

$$\{ \hat{D}^{(z)}(z') - \hat{C}^{(z)}(z') E_{\frac{1}{3}}^b(z') \} d^{\frac{1}{3}}(z') = 0$$

$$\{ \psi_{[\frac{1}{3}(z')]}(z_2(z)) - \phi_{[b(z')]}(z_2(z)) E_{[\frac{1}{3}(z')]}^{[b(z')]}(z_2(z)) \} d^{[\frac{1}{3}(z')]}(z_2(z)) = 0$$

and

$$(13) \quad \{ \psi_{\frac{1}{3}}(z_2(z)) - \phi_b(z_2(z)) E_{\frac{1}{3}}^b(z_2(z)) \} d^{\frac{1}{3}}(z_2(z)) = 0$$

again for $z = [1 b(z)]$ in each case. Relationships (8, 9) imply the inclusion relationship (3).

The matrix whose $(z+i)^{th}$ row is $\phi_{[B(z)]}(z_2(z))$ ($z = [1 \ b(z) 1]$) is a column permuted version of the matrix whose $(z+i)^{th}$ row is $\phi_{[B(z')]}(z_2(z))$ ($z = [1 \ b(z) 1]$). The latter is $\hat{C}(z')$ and is nonsingular; the former is also nonsingular.

Conversely, assume that $\text{row}\{B \rightarrow K; m\}$ and $\text{row}\{B \rightarrow K; n\}$ contain ϕ and ψ respectively for which the matrix whose $(z+i)^{th}$ row is $\phi_{[B(z)]}(z_2(z))$ ($z = [1 \ b(z) 1]$) is nonsingular and relationships $\begin{matrix} (12) \\ (3, 2, 13) \\ (3, 2) \end{matrix}$ hold for $z = G$ and $z = [1 \ b(z) 1]$, Select $z \in G$, $z_2 \in G$ and $z' \in G'$ for which $z \in DC(z'|E; c, d : b, \xi)$. Define $\hat{C}(z') \in M\{K; |b(z')|\}$ and $\hat{D}^{(t)} \in M\{K; |b(z')|, |\xi(z')|\}$ by setting use of relationships $(8, 9)$ for $z = [1 \ b(z) 1]$. By rearrangement of the vector products occurring in relationship $\begin{matrix} (12) \\ (13) \end{matrix}$, relationship $\begin{matrix} (11) \\ (10) \end{matrix}$ is derived and, since $z_2(z) \in DC(z'|E; c, d : b, \xi)$, relationships $\begin{matrix} (10) \\ (11) \end{matrix}$, holding for, $z = [1 \ b(z) 1]$, follows. The latter set may be combined in the form

$$\hat{C}(z') c^b(z') = \hat{D}(z') d^{\xi}(z')$$

Since $z \in DC(z'|E; c, d : b, \xi)$ then, in addition to the relationships $\{b(z)\} = \{b(z')\}$, $\{b(z')\} = \{b(z)\}$ already used, the further relationships $E^{[b(z')]}(z) = E^b_{\xi}(z')$, $c^{[b(z')]}(z) = c^b(z')$ and $d^{[\xi(z')]}(z) = d^{\xi}(z')$ hold. Thus

$$\hat{C}(z') c^{[b(z')]}(z) = \hat{D}(z') d^{[\xi(z')]}(z)$$

$b(z), \dot{b}(z)$ are rearrangements of the sequences $b(z'), \dot{b}(z')$ respectively. $\hat{C}(z) \in M\{K; |b(z)|\}$ and $\hat{D}(z) \in M\{K; |b(z)|, |\dot{b}(z)|\}$ are defined by setting $\hat{C}(z) = C^{(b,z)}(z')$, where $C^{(b,z)}(z')$ is a suitably column permuted form of $C(z')$, and defining $\hat{D}(z)$ similarly. Relationship (1) is obtained. Relationship (2) is derived in the same way from relationship (13). $\hat{C}(z)$ is a column permuted form of the matrix whose $(z+1)^{\text{th}}$ row is $\phi_{[b(z)]}(z_2(z))$ ($z = [1 b(z) 1]$): $\hat{C}(z)$ is nonsingular for $z \in G$.

The matrix with rows $\phi_{[b(z)]}(z_2(z))$ considered in clause (2i) is nonsingular. If $z, z_2(z) \in DC(z' | E; c, d; b, \dot{b})$ ($z = [1 b(z) 1]$) $\{b(z)\} = \{b(z_2(z))\} = \{b(z')\}$ ($z = [1 b(z) 1]$). The matrix with rows $\phi_b(z_2(z))$ is obtained from the former matrix by reordering of individual rows; the reorderings may not be common: the second matrix may be singular.

The proof of clause (2ii) is similar to and slightly simpler than that of its predecessor. Now, however, $z, z_2(z) \in DC'(z' | E; c, d; b, \dot{b})$ so that $b(z) = b(z_2(z)) = b(z')$ ($z = [1 b(z) 1]$). The reorderings are common: the second matrix is nonsingular. Results holding for complete section frameworks and mapping systems with complete block structure

(*) Let $G \subseteq B$. The structure of nonsingular block mapping systems
 i) Let $\{s; p, \mu\} \in SF\{B; m, n\}$, $\{b, \dot{b}\} \in SS[G, \{s; p, \mu\}], |b(z)|$

$|\dot{b}(z)|$ for $z \in G$ and $S = S\{b, \dot{b} | s; p, \mu\}$.

If $BS\{B \rightarrow K; m, n\} \cap \{s; p, \mu\} \cap MS_{NR(b; G)}^{|B \rightarrow K; m, n|}$
 is nonvoid, then $|p(z, w)| = |\mu(z, w)|$ ($z \in G, w = S(z)$)

ii) Let $\{s; p, \mu\} \in SF_C\{B; n\}$. If

$BS\{B \rightarrow K; m\} \cap \{s; p, \mu\} \cap MS_{NR(\dot{b}; G)}^{|B \rightarrow K; n|}$
 is nonvoid, then $|p(z, w)| = |\mu(z, w)|$ ($z \in G, w = S(z)$).

$$|B(z)| = |\zeta(z)| \text{ for } z \in G, \text{ and } S = S\{\beta, \zeta | s; p, \mu\}$$

(i) Let $G \subseteq B$.

(ii) Let $\{s; p, \mu\} \in SF_D\{B; m, n\}$ and $z \in B$. If $BS\{B \rightarrow K; n, m | s, p, \mu\}$ contains E for which $E(z)$ is

non-singular, then $|p(z, w)| = |\mu(z, w)|$ ($w = [S(z)]$)

Select $z \in G$. To prove clause (i) it is remarked that if $E \in BS\{B \rightarrow K; n, m | s, p, \mu\}$ the nonzero elements of

$E(z)$ are confined to the blocks $E_{\mu}^p(z, p\bar{w})$ ($w = [S(z)]$).

Suppose that the condition $|p(z, w)| = |\mu(z, w)|$ ($w = [S(z)]$) is violated.

There is an $r \in S(z)$ for which with $p(r) = \frac{1}{2} |p(z, r)|$ and

and $\mu(r) = \sum_{w=0}^n |\mu(z, w)| + r + 1$, $p(r) > \mu(r)$ or $\mu(r) > p(r)$. In the

former case, expand $|E(z)|$ according to minors taken from

the $p(r)$ rows of $E(z)$ with $z \in p(z, r)$ ($w = [r]$) and cofactors

taken from the remaining $n - p(r) - 1$ rows. Since $\mu(r) < p(r)$ each minor is formed from an array containing at least one column

consisting exclusively of zero elements: $|E(z)| = 0$ and $E(z)$

is singular. If $\mu(r) > p(r)$, an expansion by columns yields

the same result.

Clause (ii) is a corollary to its predecessor: Let $\{\beta, \zeta\} = S_C\{s; p, \mu\}$.

(iii) Let $\{s; p, \mu\} \in SF_C\{B; n, m\}$. If $BS\{B \rightarrow K; n, m | s, p, \mu\}$ contains E for which $E(z)$ is non-singular for $z \in G$ then $|p(z, w)| = |\mu(z, w)|$ for $z \in G$ and $w \in S(z)$.

The existence of nonsingular prequotient spaces with complete block structure

(*) Let $c, d \in \text{col}\{B \rightarrow K; n\}$, $\{s; \rho, \mu\} \in SF_c\{B; n\}$ and $G \subseteq B$. For each $z \in G$ and $w = s(z)$ let

a) $|\rho(z, w)| = |\mu(z, w)| = h(w)$ and either

b) $c^\rho(z, w) = d^\mu(z, w) = O^{[h(w)]}$ or

c) $c^\rho(z, w), d^\mu(z, w) \neq O^{[h(w)]}$

1] The intersection of the spaces

$BS\{B \rightarrow K; n \mid s; \rho, \mu\}$

$PQ(c/d \mid G)$

$\left[\cap [PQ(c/d \mid b, \frac{1}{3}; G) \cap MS_{MR(\frac{1}{3}; G)}^{MR(b; G)}\{B \rightarrow K; n\}] \right] \left\{ \{b, \frac{1}{3}\} = SS[G, \{s; \rho, \mu\}] \right\} \right]$

$MS_{MR(G)}\{B \rightarrow K; n\}_s$

is nonvoid.

~~BS $\{B \rightarrow K; n, n \mid S, p, q\}$, i.e. PQ(c/d/G)~~

~~PQ(c/d/G), PQ(c/d/p, z, x), (z \in S(z)), PQ(c/d/B, S, G)~~

~~MS_{MR} {B → K; n, n}, M_{MR(P; G, w)} {B → K; n, n}, M_{MR(U; G, w)} {B → K; n, n}, M_{MR(B; G)} {B → K; n, n}, M_{MR(S; G)} {B → K; n, n}~~

~~is nonvoid. (n[PQ(c/d/B, S; G) \cap M_{MR(B; G)} {B → K; n, n}] \cap S(z)) \cap S(z) \neq \emptyset~~

2] An example of a mapping system E belonging to the above

intersection may be constructed by setting $E = E(z)$ over $B \setminus \{G\}$ where $z \in G$ is fixed arbitrarily and, for each $z \in G$,

$E(z)$ is specified in the following way. For each $w \in S(z)$ for which

condition (a) holds, set $E_{\mu}^P(z, w) = I(h(w))$. For each $w \in S(z)$ for

which condition (b) holds $E_{\mu}^P(z, w)$ is constructed as follows. Select

$t(w)$ $\in \rho(z, w)$ and $r^{(w)} \in \mu(z, w)$ for which $c_{t(w)}(z) > d_{r^{(w)}}(z) \neq 0$. Let $p(z, w; \kappa)$,

$\mu(z, w; \kappa)$ ($\kappa \in [h(w)]$) be the successive members of $\rho(z, w)$, $\mu(z, w)$

respectively, and determine $t'(w), r'(w) \in [h(w)]$ from the relationships

$t = p(z, w)$, $t(w) = p(z, w; t'(w))$, $r(w) = \mu(z, w; r'(w))$. Define the integer

bijection $\chi: [h(w)] \rightarrow [h(w)]$ by setting $\chi(\kappa) = \kappa + r'(w) - t'(w) \pmod{h(w)+1}$

($\kappa \in [h(w)]$) and the integer bijection $\nu: \{\rho(z, w)\} \rightarrow \{\mu(z, w)\}$ by

setting $\nu(z) = \mu(z, w; \chi(\kappa))$ where $z = p(z, w; \kappa)$ for $\kappa \in [h(w)]$. Set

$E_{t, D(G)}(z) = 1$ and

$$E_{z, r}(z) E_{z, r(w)}(z) = \{c_z(z) - d_{\nu}(z)\} / d_{r(w)}(z)$$

both for $z \in \rho(z, w) \setminus t$, $E_{t(w), r(w)}(z) = c_{t(w)}(z) / d_{r(w)}(z)$ and

the remaining elements of $E_{\mu}^P(z, w)$ to zero. For $w \in S(z)$ and

$z \in s(z) \setminus \omega$, set $E_\mu^P(z; z, \omega) = O_{[h(\omega)]}^{[h(z)]}$

With the exception of the reference to the space $PQ(c/d|G)$ the results of the above theorem are special cases, obtained by setting $m=n$ and taking $\{s; p, \mu\} \in SF_C \{B; m\}$ of part [2] of Theorem . It has only to be remarked that $\{h, \xi\} \in S_C \{s; p, \mu\}$ features in $SS[G, \{s; p, \mu\}]$. The result that $EPQ(c/d|h, \xi; G)$ for this pair implies that $EPQ(c/d|G)$, Relationships between sectional and complete prequotient spaces

(*) Let $c \in \text{col}\{B \rightarrow k; m\}$, $d \in \text{col}\{B \rightarrow k; n\}$, $\{s; p, \mu\} \in SF_C \{B; m, n\}$ and $G \subseteq B$. Set $\tilde{S} = BS\{B \rightarrow k; m, n | s; p, \mu\}$

$$i) \quad \tilde{S} \cap PQ(c/d|G) = \tilde{S} \cap [\cap PQ(c/d|p, \mu; z, \omega) \{z=G, \omega=s(z)\}]$$

$$ii) \quad \text{Let } [\{s\} \in \text{const}(G)] = \hat{s}.$$

$$\tilde{S} \cap PQ(c/d|G) = \tilde{S} \cap [\cap PQ(c/d|p, \mu; G, \omega) \{ \omega = \hat{s} \}]$$

Any member E of $PQ(c/d|G)$ satisfies the relationship $E(z)d(z)=c(z)$ for $z \in G$ which, in terms of the submatrices $\{E_\mu^P(z; z, \omega)\}$ may be expressed as

$$(1) \quad \sum_{\mu \in s(z)} E_\mu^P(z; z, \omega) d^\mu(z, \omega) = c^P(z; z)$$

for $z \in G$ and $z = s(z)$, & for all $E \in \tilde{S}$

$$(2) \quad E_\mu^P(z; z, \omega) = O_{[\mu(p(z; z))]}^{[\mu(p(z; z))]}$$

for $z \in G$ when $z \neq \omega$ for all relevant z, ω :

$$(3) \quad E_\mu^P(z; \omega) d^\mu(z; \omega) = c^P(z; \omega)$$

for $z \in G, \omega = s(z)$. Hence $E \in PQ(c/d|p, \mu; z, \omega)$ for $z \in G, \omega = s(z)$ in

$$(4) \quad S \cap PQ(c/d | G) \subseteq S \cap \left[\bigcup_{\substack{\text{PQ}(c/d | p, \mu; G, \omega) \\ \omega \in S(t)}} \right]$$

Any E belonging to all spaces $PQ(c/d \mid p, \mu; z, w)$ ($z \in G$, $w \in s(z)$) satisfies relationship (3) for $z \in G$, $w \in s(z)$. If $E \in S$ also, relationship (2) holds for $z \in G$, $z \in s(z)$ and $v \in s(z) \setminus z$. Relationships (2, 3) may be combined in the form (1) holding for $z \in G$, $z \in s(z)$ and in the composite form $E(z)d(z) = c(z)$ for $z \in G$; the semi-inclusion relationship (4) may be reversed. Clause (i) has been disposed of.

If $\{s(z)\}$ is a constant function G , with $\{s\}$ set \hat{s} for $\hat{G} = G$,
 $[PQ(c/d | p, \mu; z, w) \{z = G, w = s(z)\}]$ may be expressed as
 $\nabla [PQ(c/d | p, \mu; z, w) \{z = G\}] \{w = \hat{s}\}$ i.e. as \hat{s}
 $\nabla [PQ(c/d | p, \mu; G, w) \{w = \hat{s}\}]$. The result of clause (ii) follows.

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\{s; p, \mu\} \in SF_c \{B; m, n\}$,
 $c \in MS\{B \rightarrow K; r, m\}$, $G \subseteq B$ and $[s \text{econst}(G)] = \hat{s}$.

Any mapping system D belonging to $|\hat{s}|+1$ of the $|\hat{s}|+2$ spaces $PQ(C, c/d \mid p, \mu; G, \omega)$ ($\omega = \hat{s}$) and $PQ(C, c/d \mid G)$ also belongs to the remaining space.

relationship (3) for $\omega = [s(z)]$. If $E \in \tilde{S}$ also, relationships (2) hold when $z \neq 2$ for odd $z = [s(t)]$, $\omega = [s(z)] \setminus z$. Relationships (2,3) may be combined in the form (1) holding for $z = [s(z)]$ and in the composite form $E(z)d(z) = c(z)$: the semi-inclusion relationship (4) may be reversed.

() Let $c \in \overset{\text{col}}{\{B \rightarrow K; m\}}$, $d \in \overset{\text{col}}{\{B \rightarrow K; n\}}$, $\{s; \rho, \mu\} \in \overset{\text{com}, C \in M}{SF}\{B; m, n\}$, $C \in MS\{B \rightarrow K; m\}$ and $z \in B$. ~~besides that~~ Let s and G be such that ~~for each $z \in G$, $\{s(z)\}$ is a constant set, $s(z) \in S$ for $z \in G$.~~ For each $z \in G$, $\{s(z)\}$ is a constant set, $s(z) \in S$ for $z \in G$. Any mapping system D belonging to $\{s(z)\}_{z \in G} + 1$ of the $\lceil s(z) \rceil + 2$ spaces $PQ(C, c/d | \rho, \mu; G, \omega)$ ($\omega = \{s(z)\}$) and $PQ(c/d | G)$ belongs to the remaining space.

The notation $\{s; \rho, \mu\} \in \overset{\text{com}, C}{SF}\{B; m, n\}$ indicates that $s: B \rightarrow \llbracket \min(m, n) \rrbracket$ and that for each $z \in B$ and $s(z) \in \llbracket \min(m, n) \rrbracket$, that $\rho(z, \omega) \in F[m]$ ($\omega = \{s(z)\}$) with $\{\cup \{\rho(z, \omega)\} \mid \omega = \{s(z)\}\} = [m]$, and that

the sequences $\mu(z, \omega)$ ($\omega = \{s(z)\}$) satisfy similar conditions. The condition $[s(z) \in \text{const}(G)] = \hat{s}$ asserts that for each $z \in G$, $\{s(z)\}$ is the constant set \hat{s} . The condition $D \in PQ(c/d | G, \omega)$ holding for $\omega = [s(z)]$ asserts

that for each $w \in \hat{s}$

$$(1) \quad C\rho(z, w)c^P(z, w) = D\mu(z, w)d^\mu(z, w)$$

for $z \in G$. The condition $D \in PQ(c/d | G)$ asserts that

$$(2) \quad C(z)c(z) = D(z)d(z)$$

for $z \in G$,
a relationship which may be expressed in summation form as

$$(3) \sum_{w \in \hat{S}} C_p(z, w) c^p(z, w) = \sum_{w=0}^{s(z)} D_p(z, w) d^p(z, w)$$

If condition (1) holds for $w = \hat{s}(z)$,^{and $z \in G$} the relationships in question may be additively combined in the form (3) and induce condition (2): $D \in PQ(c/d|G)$. Again, if condition (1) holds for all but one of the values of w in \hat{S} and condition (3) also holds, suitable subtraction yields the result that condition (1) also holds for the exceptional value of w and $D \in PQ(c/d|p, \mu; G, w)$ for this value of w .

() Let $c \in \{B \rightarrow K; m\}$, $d \in \{B \rightarrow K; n\}$, $\{s; p, \mu\} \in IF\{B; m, n\}$,
 now $G \subseteq B$ ~~iff $\{s; p, \mu\} \in const(G)$~~
~~iff $\{s(z)\}$ is a constant set.~~
 $\phi \in \{B \rightarrow K; m\}$ and, $\exists z \in B$. ~~Let s and G be such that $\{s(z)\}$ is a constant set.~~
~~iff $s(z) = s$ for $z = G$.~~

For each $z \in G$,
 every function system ψ belonging to \hat{S} of the $\hat{S}+2$ function system spaces $PQ(\phi, c/d|p, \mu; G, w)$ ($w = \hat{s}(z)$) and $PQ(\phi, c/d|G)$ also belongs to the remaining space.

The above theorem is a corollary to its predecessor: set $r=0$ and replace C, D by ϕ, ψ .

Relationships between sectional and complete relative product spaces
~~iff $\{s(z)\}$ is a constant set for $z \in G$.~~

() Let $c \in \{B \rightarrow K; m\}$, $d \in \{B \rightarrow K; n\}$, $\{s; p, \mu\} \in IF\{B; m, n\}$,
 $G \subseteq B$ ~~iff $\{s; p, \mu\} \in const(G)$~~
 $C \in MS\{B \rightarrow K; r, m\}$, $E \in BS\{B \rightarrow K; m, n | s; p, \mu\}$ and ~~iff $\{s; p, \mu\} \in const(G)$~~
~~iff $\{s(z)\}$ is a constant set for each $z \in G$.~~

every mapping system D belonging to $\hat{S}+1$ of the $\hat{S}+2$ spaces $RP(C, E; d|p, \mu; G, w)$ ($w = \hat{s}(z)$) and $RP(C, E; d|G)$ also belongs to

the remaining space.

The condition $D \in RP(C, E; d|p, \mu; \hat{s}, \omega)$ holding for $\omega \in \hat{S} \setminus \{s\}$ asserts

that for each $\omega \in \hat{S}$

$$(1) \quad \{D_\mu(z; \omega) - C_p(z; \omega) E_\mu^p(z; \omega)\} d^\mu(z; \omega) = 0 \quad [r]$$

for $\frac{z \in G}{\omega \in S \setminus \{s\}}$. The condition that $D \in RP(C, E; d|p)$ asserts that

$$\{D(z) - C(z) E(z)\} d(z) = 0 \quad [r]$$

for $\frac{z \in G}{}$)

The additional condition $E \in BS\{B \rightarrow K; m, n | s; p, \mu\}$ implies

that $E_\mu^p(z; z, \omega) = 0 \quad \begin{matrix} [l(p(z, z))] \\ [l(\mu(z, \omega))] \end{matrix}$ for $\omega \in \hat{S} \setminus \{s\}$, $\frac{\omega \in \hat{S} \setminus \{s\}}{z \in G \setminus \{s\}}$. Setting

$$\hat{E}(z) = C(z) E(z),$$

$$\begin{aligned} \hat{C}_\mu(z, \omega) &= C(z) \hat{E}_\mu(z; \omega) = C_p(z; \omega) E_\mu^p(z; \omega) \\ &= \sum_{\substack{z \in G \\ \omega \in \hat{S}}} C_p(z; z) E_\mu^p(z; z, \omega) \\ &= C_p(z; \omega) E_\mu^p(z; \omega) \end{aligned}$$

for $\omega \in \hat{S}$ and $z \in G$.

Condition (1) may be expressed as

$$\sum_{\substack{\omega \in \hat{S} \\ \omega \neq s}} \{D_\mu(z; \omega) - \hat{C}_\mu(z; \omega)\} d^\mu(z; \omega) = 0 \quad [r]$$

or

$$\sum_{\substack{\omega \in \hat{S} \\ \omega \neq s \\ \omega = 0, z \in \hat{S}}} \{D_\mu(z; \omega) - C_p(z; \omega) E_\mu^p(z; \omega)\} d^\mu(z; \omega) = 0 \quad [r]$$

holding for $\frac{z \in G}{}$. The remainder of the proof is similar to that of Theorem .

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\{s; \rho, \mu\} \in SF_C\{B; m, n\}$
 $\phi \in \text{row}\{B \rightarrow K; m\}$, $E \in BS\{B \rightarrow K; m, n | s; \rho, \mu\}$, $\hat{G} \subseteq B$ and
 $[s] \in \text{const}(\hat{G})] = \hat{s}$.

Any function system ψ belonging to $|\hat{s}|+1$ of the $|\hat{s}|+2$
function system spaces $RP(\phi, E; d | \rho, \mu; \hat{G}, \hat{\omega})$ ($\omega = \hat{s}$) and
 $RP(\phi, E; d | \hat{G})$ also belongs to the remaining space.

The above theorem is a corollary to its predecessor; set $r=0$

and replace C, D by ϕ, ψ .
Relationships between sectional prequotient and relative product spaces
based upon an assumption concerning one sector of a post factor
mapping system

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $C \in MS\{B \rightarrow K; r, m\}$,

$\{s; \rho, \mu\} \in SF_C\{B; m, n\}$, and $\hat{G} \subseteq B$ and $[s] \in \text{const}(\hat{G})] = \hat{s}$.

For a fixed $\hat{\omega} \in \hat{s}$, let $\hat{G} \subseteq \hat{G}$ be such that $\hat{G} \subseteq DC(G | E; c, d; s; \rho, \mu)$
and $E \in BS\{B \rightarrow K; m, n | s; \rho, \mu\}$ $PQ(c/d | \rho, \mu; \hat{G}, \hat{\omega})$ be

fixed.

i) With $\tilde{s} = \cap PQ(c, c/d | \rho, \mu; \hat{G}, \hat{\omega})$ ($\omega = \hat{s} \setminus \hat{\omega}$)

$$\tilde{s} \cap PQ(c, c/d | \hat{G}) = \tilde{s} \cap RP(C, E; d | \rho, \mu; \hat{G}, \hat{\omega})$$

ii) Let $E \in BS\{B \rightarrow K; m, n | s; \rho, \mu\}$.

a) With $\tilde{s} = \cap RP(C, E; d | \rho, \mu; \hat{G}, \omega)$ ($\omega = \hat{s} \setminus \hat{\omega}$)

$$\tilde{s} \cap PQ(c, c/d | \rho, \mu; \hat{G}, \hat{\omega}) = \tilde{s} \cap RP(C, E; d | \hat{G})$$

b) Let $\tilde{\omega} \in \tilde{s}$ also be fixed. With

↓
↓ with

$$\tilde{s} = PQ(c, c/d | \hat{G}) \cap [\cap PQ(c, c/d | \rho, \mu; \hat{G}, \omega) \{ \omega = \hat{s} \setminus \hat{\omega}, \tilde{\omega} \}]$$

$$\hookrightarrow \tilde{S} \cap PQ(C, c/d|p, \mu; \hat{G}, \tilde{\omega}) = \tilde{S} \cap RP(C, E; d|p, \mu; \hat{G}, \tilde{\omega})$$

b) Now let $\tilde{\omega} \in \hat{S}$ also be fixed. With

$$\hat{S} = RP(C, E; d|\hat{G}) \cap [\cap RP(C, E; d|p, \mu; \hat{G}, \omega) \quad (\omega = \hat{S} \setminus \tilde{\omega}, \tilde{\omega})]$$

$$\tilde{S} \cap PQ(C, c/d|p, \mu; \hat{G}, \tilde{\omega}) = \tilde{S} \cap RP(C, E; d|p, \mu; \hat{G}, \tilde{\omega})$$

In the following proofs the spaces $PQ(C/d|p, \mu; \hat{G}, \omega)$, $PQ(C, c/d|p, \mu; \hat{G}, \omega)$, $PQ(C, c/d|\hat{G})$, $RP(C, E; d|p, \mu; \hat{G}, \omega)$ and $RP(C, E; d|\hat{G})$ are written in abbreviated form as $PQ'(\hat{G}, \tilde{\omega})$, $PQ(\hat{G}, \tilde{\omega})$, $PQ(\hat{G})$, $RP(\hat{G}, \tilde{\omega})$ and $RP(\hat{G})$ respectively.

The condition $E \in PQ'(\hat{G}, \tilde{\omega})$ implies, from clause (2i) of Theorem, that $E \in PQ'(\hat{G}, \tilde{\omega})$ when $\hat{G} \subseteq DC(G|E; c, d; \tilde{S}; p, \mu; \tilde{\omega})$ and in turn, from clause (1b) of Theorem, that $E \in PQ'(\hat{G}, \tilde{\omega})$ when \hat{G} is any subset of this domain of constancy.

To prove clause (ia) it is remarked that if

$D \in PQ(\hat{G}, \tilde{\omega})$ ($\omega = \hat{S} \setminus \tilde{\omega}$) and $D \in PQ(\hat{G})$ then, from

[It follows from From clause (2i) of Theorem]

Theorem, $D \in PQ(\hat{G}, \tilde{\omega})$ also. In turn Replacing ~~it follows that~~ in part [1] of Theorem, it may be deduced that if

~~it follows that~~ $E \in PQ'(\hat{G}, \tilde{\omega})$ and $D \in PQ(\hat{G}, \tilde{\omega})$, then $D \in RP(\hat{G}, \tilde{\omega})$ also.

Conversely, if $D \in RP(\hat{G}, \tilde{\omega})$ then, from ~~part [1]~~ of Theorem again, $D \in PQ(\hat{G}, \tilde{\omega})$ and for $\omega = \tilde{\omega}$ and, since this condition holds for $\omega = \hat{S} \setminus \tilde{\omega}$, $D \in PQ(\hat{G})$, from Theorem.

Subclause (ii) is a variation of its predecessor. In subclause (ia) it is assumed that $\text{D}\in \text{PQ}(\hat{G}, w)$ for all $w \in \hat{\mathcal{S}} \setminus \hat{w}$ including $w = \hat{w}$ and the result concerns the location of D in the space $\text{PQ}(\hat{G})$. In subclause (ib) it is assumed that $\text{D}\in \text{PQ}(\hat{G})$ and the result concerns location of D in the space $\text{PQ}(\hat{G}, \hat{w})$. The proof again uses Theorem .

Subclause (iiia) is a reflection of subclause (ia). The spaces $\text{PQ}(\hat{G}, \hat{w})$, $\text{PQ}(\hat{G})$ and $\text{RP}(\hat{G}, \hat{w})$ are replaced by $\text{RP}(\hat{G}, \hat{w})$, $\text{RP}(\hat{G})$ and $\text{PQ}(\hat{G}, \hat{w})$ respectively. Theorem , which requires the additional condition $E \in \text{BS}\{B \rightarrow K; m, n\}$, is used in place of Theorem . Subclause (iib) is a variation of its predecessor, similar to subclause (ib).

(i) Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\{s; p, \mu\} \in \text{SF}_c\{B; m, n\}$, $\hat{G} \subseteq \hat{G}$, $\hat{G} \subseteq B$ and $[s; \text{const}(\hat{G})] = \hat{s}$. For a fixed $\hat{w} \in \hat{\mathcal{S}}$, let $G \subseteq \hat{G}$, $C \in \text{MS}_{\text{NR}(p; G, \hat{w})}\{B \rightarrow K; r, m\}$ and $E \in \text{MS}\{B \rightarrow K; m, n\}$ be such

that $\hat{G} \subseteq \text{DC}(G | E; c, d; s; p, \mu; \hat{w})$ and

(1) $\text{PQ}(C, c/d | p, \mu; G, \hat{w}) \cap \text{RP}(C, E; d/p, \mu; G, \hat{w})$

is nonvoid.

The results of clauses (i, ii) of the preceding theorem hold.

Replacing $p, d, B(z), \beta(z)$ by $p(z, \hat{w}), \mu(z, \hat{w})$ respectively, from clause (2a) of Theorem it follows that if the space (1) is nonvoid, clause (2b) of Theorem

$E \in PQ(c/d | \rho, \mu; \hat{G}, \hat{\omega})$. All conditions of the preceding theorem are satisfied and its results follow.

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\phi \in \text{row}\{B \rightarrow K; m\}$, $\{s; \rho, \mu\} \in SF_c\{B; m, n\}$, $\hat{G} \subseteq B$ and $[\{s\} \in \text{const}(\hat{G})] = \hat{s}$. For a fixed $\hat{\omega} \in \hat{s}$, let $G \subseteq \hat{G}$ and $E \in MS\{B \rightarrow K; m, n\}$ be such that $\hat{G} \subseteq DC(G | E; c, d; s; \rho, \mu; \hat{\omega})$ and $E \in PQ(c/d | \rho, \mu; G, \hat{\omega})$.

With the spaces $PQ(C, c/d | \rho, \mu; \hat{G}, \hat{\omega})$, $PQ(C, c/d | \hat{G})$, $RP(C, E; d | \rho, \mu; \hat{G}, \hat{\omega})$ and $RP(C, E; d | \hat{G})$ replaced by $PQ(\phi, c/d | \rho, \mu; \hat{G}, \hat{\omega})$, $PQ(\phi, c/d | \hat{G})$, $RP(\phi, E; d | \rho, \mu; \hat{G}, \hat{\omega})$ and $RP(\phi, E; d | \hat{G})$ the results of Theorem clauses (i, ii) of Theorem hold.

Replace Theorems , in the proof of Theorem by Theorems , .
 Replace Theorems , in the proof of Theorem by Theorems , .
 () Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\{s; \rho, \mu\} \in SF_c\{B; m, n\}$, $\hat{G} \subseteq B$ and $[\{s\} \in \text{const}(\hat{G})] = \hat{s}$. For a fixed $\hat{\omega} \in \hat{s}$, let $G \subseteq \hat{G}$, $\phi \in \text{row}\{B \rightarrow K; m\}$ and $E \in BS\{B \rightarrow K; m, n\}$ be such that

~~such that~~

~~a) $\hat{G} \subseteq DC(G | E; c, d; s; \rho, \mu; \hat{\omega})$ and~~

~~b) for each $z \in \hat{G}$, $DC\{z | E; c, d; s; \rho, \mu; \hat{\omega}\}$ contains~~

~~$z_2(z)$ ($z = [\begin{smallmatrix} 1 & p(z, \hat{\omega}) \\ 0 & 1 \end{smallmatrix}]$) for which (a) the matrix whose $(z+1)^{th}$ row is $\phi_{p(z_2(z), \hat{\omega})}(z) = [\begin{smallmatrix} 1 & p(z_2(z), \hat{\omega}) \\ z_2(z) & 1+p(z_2(z), \hat{\omega}) \end{smallmatrix}]$ is nonsingular and (b) for $\frac{z \in [1, p(z, \hat{\omega})]}{z \in [1, p(z_2(z), \hat{\omega})]}$~~

~~(1) $PQ(\phi, c/d | \rho, \mu; z_2(z), \hat{\omega}) \cap RP(\phi, E; d | \rho, \mu; z_2(z), \hat{\omega})$~~

The results of the preceding theorem hold
 Replacing $b(\cdot, \cdot)$ by $p(\cdot, \cdot)$, $\mu(\cdot, \cdot)$ respectively, clause (iii)
 It follows from Clause (iiia) of Theorem 1 that if conditions (a),
 of Theorem 1 holds that if the spaces (V, ω) and (E, ρ) exist
 above, induce the result that $EePQ(c/d | p, \mu; \hat{G}, \omega)$
 satisfying $\hat{\omega}$ for which all conditions of the preceding theorem

are satisfied and its results follow.
 Relationships between sectional prequotient and relative product spaces
based upon an assumption concerning all sectors of a post factor mapping
system Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $C \in \text{BS}\{B \rightarrow K; r, m\}$

$\{s; p, \mu\} \in SF_C\{B; m, n\}$, $\hat{G} \subseteq B$ and $[s; e \text{ const}(\hat{G})] = \hat{s}$. Let
 $E \in BS\{B \rightarrow K; m, n\}$, $\{s; p, \mu\} \cap PQ(C/d | \hat{G})$

i) For any ϵ with $\tilde{G}(\omega) \subseteq DC(G | E; c, d; p, \mu; \hat{G}, \omega)$

$$PQ(C, c/d | p, \mu; \tilde{G}(\omega), \omega) = RP(C, E; d | p, \mu; \tilde{G}(\omega), \omega)$$

for $\omega = \hat{s}$.

$$\begin{aligned} \text{i)} & \text{ Let } \hat{G} \subseteq G. \\ & [PQ(C, c/d | \hat{G}) \cup RP(C, E; d | \hat{G})] \cup \\ & [PQ(C, c/d | p, \mu; \hat{G}, \omega) \cup RP(C, E; d | p, \mu; \hat{G}, \omega)] \quad \{\omega = \hat{s}\} \\ & = PQ(C, c/d | \hat{G}) \cap RP(C, E; d | \hat{G}) \cap \end{aligned}$$

$$[PQ(C, c/d | p, \mu; \hat{G}, \omega) \cap RP(C, E; d | p, \mu; \hat{G}, \omega)] \quad \{\omega = \hat{s}\}$$

ii) Let $\hat{\omega} \in \hat{s}$. With

$$\tilde{s} = [PQ(C, c/d | p, \mu; \hat{G}, \hat{\omega}) \cup RP(C, E; d | p, \mu; \hat{G}, \hat{\omega})] \quad \{\omega = \hat{s} \setminus \hat{\omega}\}$$

$$\begin{aligned} \tilde{s} \cap PQ(C, c/d | p, \mu; \hat{G}, \hat{\omega}) &= \tilde{s} \cap RP(C, E; d | p, \mu; \hat{G}, \hat{\omega}) \\ &\equiv \tilde{s} \cap PQ(C, c/d | \hat{G}) \\ &\equiv \tilde{s} \cap RP(C, E; d | \hat{G}) \end{aligned}$$

iii) Let $\hat{\omega}, \tilde{\omega} \in \hat{s}$. With

$$\tilde{S} = \left[\bigcap \{ PQ(C, c/d | p, \mu; \hat{G}, \omega) \cup RP(C, E; d | p, \mu; \hat{G}, \bar{\omega}) \} \{ \omega = \hat{S} \setminus \hat{W}, \tilde{\omega} \} \right] \\ \cap \{ PQ(C, c/d | \hat{G}) \cup RP(C, E; d | \hat{G}) \}$$

$$\tilde{S} \cap PQ(C, c/d | p, \mu; \hat{G}, \hat{\omega}) = \tilde{S} \cap RP(C, E; d | p, \mu; \hat{G}, \tilde{\omega})$$

In the following proofs of the spaces $PQ(c/d | p, \mu; \hat{G}, \omega)$, $PQ(c/d | \hat{G})$, $PQ(c/d | \hat{G})$, $PQ(C, c/d | p, \mu; \hat{G}, \omega)$, $PQ(C, c/d | \hat{G})$, $RP(C, E; d | p, \mu; \hat{G}, \omega)$ and $RP(C, E; d | \hat{G})$ are written in abbreviated form as $PQ'(\hat{G}, \omega)$, $PQ'(\hat{G})$, $PQ(\hat{G}, \omega)$, $PQ(\hat{G})$, $RP(\hat{G}, \omega)$ and $RP(\hat{G})$ respectively.

From Theorem, the two conditions $E \in BS \{ B \rightarrow K ; m, n | s ; p, \mu \}$ and $E \in PQ'(\hat{G}, \omega)$ imply that $E \in PQ'(\hat{G}, \omega)$ ($\omega = \hat{S}$). In turn, from clause (2i) of Theorem, $E \in PQ'(\hat{G}(\omega), \omega)$ ($\omega = \hat{S}$) where $\hat{G}(\omega) \subseteq DC(G | E; c, d ; p, \mu = s ; p, \mu : \omega) \{ \omega = \hat{S} \}$ and, from clause (1ib) of Theorem, $E \in PQ'(\hat{G}(\omega), \omega)$ when $\hat{G}(\omega)$ is any set a subset of the domain of constancy ($\omega = \hat{S}$), replacing $f_1(z), f_2(z)$ in \hat{G} by $p(z, \omega), \mu(z, \omega) \{ \omega = \hat{S} \}$ in part [1] of

Theorem, The result of part [1] of the present theorem follows.

From clause (2i) of Theorem, since $E \in PQ'(\hat{G})$, $PQ(\hat{G}, \omega) = RP(\hat{G})$, from

$$\text{part [1] of Theorem. Since } PQ(\hat{G}, \omega) = RP(\hat{G}, \omega) \{ \omega = \hat{S} \} \\ \left[\bigcap \{ PQ(\hat{G}, \omega) \cup RP(\hat{G}, \omega) \} \{ \omega = \hat{S} \} \right] = \left[\bigcap \{ PQ(\hat{G}, \omega) \cap RP(\hat{G}, \omega) \} \{ \omega = \hat{S} \} \right] \\ = \left[\bigcap PQ(\hat{G}, \omega) \{ \omega = \hat{S} \} \right].$$

From Theorem , $[\cap PQ(\hat{G}, \omega) \{ \omega \in \hat{S} \}] = PQ(\hat{G})$. The result of clause (2i) follows.

For the space \tilde{S} defined in clause (2ii)

$$\tilde{S} = [\cap PQ(\hat{G}, \omega) \{ \omega = \hat{S} \setminus \hat{\omega} \}] = [\cap RP(\hat{G}, \omega) \{ \omega = \hat{S} \setminus \hat{\omega} \}]$$

From Theorems and ,

$$\tilde{S} \cap PQ(\hat{G}, \hat{\omega}) = \overset{\text{by}}{PQ}(\hat{G}), \quad \tilde{S} \cap RP(\hat{G}, \hat{\omega}) = \overset{\text{by}}{RP}(\hat{G})$$

and as above $PQ(\hat{G}) = RP(\hat{G})$.

The result of clause (2iii) is a variation of its predecessor.

Now

$$\tilde{S} = [\cap PQ(\hat{G}, \omega) \{ \omega = \hat{S} \setminus \hat{\omega}, \tilde{\omega} \}] \cap PQ(\hat{G})$$

$$= [\cap RP(\hat{G}, \omega) \{ \omega = \hat{S} \setminus \hat{\omega}, \tilde{\omega} \}] \cap RP(\hat{G})$$

~~and~~ Any $D \in \tilde{S} \setminus PQ(\hat{G}, \hat{\omega})$ is in $PQ(\hat{G}, \tilde{\omega}) = RP(\hat{G}, \tilde{\omega})$.

~~Any $D \in \tilde{S} \setminus RP(\hat{G}, \hat{\omega})$ is in $PQ(\hat{G}, \tilde{\omega})$.~~ Conversely any D in \tilde{S} and

~~$PQ(\hat{G}, \tilde{\omega})$ is in $RP(\hat{G}, \hat{\omega}) = PQ(\hat{G}, \hat{\omega})$.~~

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\phi \in \text{row}\{B \rightarrow K; m\}$

$\{s; \rho, \mu\} \in SF_c\{B; m, n\}$, $G \subseteq B$ and $[s; \text{const}(G)] = \hat{S}$. Let

$$E \in BS\{B \rightarrow K; m, n | s; \rho, \mu\} \cap PQ(c/d|G).$$

With the spaces $PQ(C, c/d|\rho, \mu; G, \omega)$, $PQ(C, c/d|G)$, $RP(C, E; d|\rho, \mu; G, \omega)$ and $RP(C, E; d|G)$ replaced by

$$PQ(\phi, c/d|\rho, \mu; G, \omega), PQ(\phi, c/d|G), RP(\phi, E; d|\rho, \mu; G, \omega)$$

and $RP(\phi, E; d|G)$ respectively, the results of Theorem hold.

Replace Theorems , , in the proof of Theorem by Theorems , .

Complete factorisation results

Complete factorisation results based upon assumptions concerning sectional prequotient and relative product space:

In Theorem an assumption of the form $E \in PQ(c/d\mid p, \mu; \mathbb{G}, \hat{\omega})$ concerning one sector $E_\mu^P(\mathbb{G}, \hat{\omega})$ of E is made; the results of the theorem concern relationships between sectional prequotient and relative product spaces defined in terms of a general mapping system C . The same assumption is made in Theorem , which is concerned with sectional prequotient and relative product spaces defined in terms of a general function mapping system ϕ .

In Theorem it is assumed that for a special mapping system \hat{C} the intersection of a sectional prequotient space and a sectional relative product space, both defined in terms of a special mapping system \hat{C} , is nonvoid. This assumption induces the condition $E \in PQ(c/d\mid p, \mu; \mathbb{G}, \hat{\omega})$ upon which Theorem is based and the results of that theorem are those of Theorem . In Theorem a special function mapping

system $\hat{\phi}$ is used in a similar way.

In Theorem the more sweeping assumption $E \in PQ(c/dl\mathbb{G})$ concerning all sectors of a mapping system E with block structure is made; further results concerning sectional prequotient and relative product spaces defined in terms of a general mapping system C follow. The same assumption is made in Theorem which concerns sectional prequotient and relative product spaces defined in terms of a general function mapping system ϕ .

In Theorems, to follow of this section it is assumed that $E \in PQ(c/dl\mathbb{p}, \mu: \mathbb{G}, \omega)$ for a subset of values of ω and that for each ω belonging to a complementary subset $\text{ins}(\omega)$ the intersection of a sectional prequotient space and a relative product space, both defined in terms of a special mapping system $\hat{C}^{(\omega)}$, is nonvoid. These assumptions induce the complete factorisation result $E \in PQ(c/dl\mathbb{E})$ and in turn certain consequences among which feature the results of Theorem. Theorems, stand in relation to Theorem as does Theorem to Theorem.

In Theorems, to follow a special function mapping systems $\hat{\phi}^{(\omega)}$ are used in a similar way. Theorems, stand in relation to Theorem as does Theorem to Theorem.

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\{s; p, \mu\} \in \text{SF}_c\{B; m, n\}$, $E \in \text{BS}\{B \rightarrow K; m, n | s; p, \mu\}$ and $\hat{G} \subseteq B$. For each $z \in \hat{G} = m, n\}$,

a) Let $s(z) = M_1(z) \cup M_2(z)$ where $M_1(z)$ and $M_2(z)$ are disjoint sets of integers

b) Let $E \in PQ(c/d | p, \mu; z, \omega)$ for $\omega = M_1(z)$ and

c) for each $w \in M_2(z)$, let $|p(z, w)| \leq \hat{r}(w)$ and $MS_{MR(p; z, w)}^{\{B \rightarrow K\}}$

$r(w), m\}$ contain $C^{(\omega)}$ for which

$$PQ(C^{(\omega)}, c/d | p, \mu; z, \omega) \cap RP(C^{(\omega)}, E; d | p, \mu; z, \omega)$$

is ~~be~~ nonvoid

i) For each $z \in \hat{G}$ and $\omega = s(z)$

$$E \in PQ(c/d | p, \mu; DC\{z | E; c, d; s; p, \mu; \omega\}, \omega)$$

and

$$IN(E; p, \mu; z, \omega) \subseteq PQ(c/d | p, \mu; z, \omega)$$

ii) $E \in PQ(c/d | DC\{\hat{G} | E; c, d\})$ and $IN(E; \hat{G}) \subseteq PQ(c/d | \hat{G})$

iii) Let $C \in MS\{B \rightarrow K; r, m\}$. For all $G \subseteq \hat{G}$

$$PQ(C, c/d | G) = RP(C, E; d | G)$$

b) Let $\phi \in \text{row}\{B \rightarrow K; m\}$. For all $G \subseteq \hat{G}$

$$PQ(\phi, c/d | G) = RP(\phi, E; d | G)$$

iv) Let $[s \text{ const } \hat{G}] = \hat{s}$. The results of Theorem hold as do the counterparts to these results in Theorem .

It follows from subclause (2ii) of theorem that condition (c) above implies that $E \in PQ(c/d|p, \mu; z, \omega)$ for each $z \in \hat{G}$ and $\omega = \mathbb{M}_2(z)$. Thus, in view of condition (b), $E \in PQ(c/d|p, \mu; z, \omega)$ for each $z \in \hat{G}$ and $\omega = s(z)$. The results of clause (i) follow from clauses (2i, ii) of Theorem. Since $E \in BS\{B \rightarrow K; m, n | s; p, \mu\}$ and $E \in PQ(c/d|p, \mu; z, \omega)$ for each $z \in \hat{G}$ and $\omega = s(z)$, it follows from Theorem that $E \in PQ(c/d|z)$ for $z \in \hat{G}$ and hence that $E \in PQ(c/d|\hat{G})$. The results of clause (ii) follow from clauses (1i, ii) of Theorem, ~~thus~~ and those of clause (iii) from clauses (1i) of Theorems, respectively. The results of Theorems , are based upon the assumptions that $[se \text{ const}(\hat{G})] - \hat{s}$ and $E \in BS\{B \rightarrow K; m, n | s; p, \mu\} \cap PQ(c/d|\hat{G})$. Clause (iv) has been disposed of.

The relevant sections of the mapping systems $C^{(w)}$ referred to in condition (c) of the above theorem may naturally be extended in dimension and suitably combined to form a single mapping system $\hat{C} \in MS\{B \rightarrow K; \hat{r}, \hat{n}\}$ in terms of which condition (c) may be reformulated. The general mapping system C in the result of clause (iiia) of the above theorem may be taken to be any one of the $C^{(w)}$ inducing this result or the mapping system \hat{C} just referred to. Similar remarks may be made concerning the following theorem.

The sections of the function mapping systems $\phi^{(w)}$ referred to in condition (c) of Theorem may also be combined as a single system $\hat{\phi}$ and the results of the counterpart to clause (iib) of the above theorem i

~~for $z \in G$ and hence that $\mathbb{E}PQ(c/d|z)$~~ . The results of clause (ii) follow from theorem $\mathbb{E}PQ(c/d|z)$. The results of Theorems , are based upon the assumptions that $\{\mathbb{E}\text{const}(G)\} = S$ and $\forall E \in BS \{B \rightarrow K; m, n | s; p, \mu\} \cap PQ(c/d|z)$. Clause (iii) has

The been disposed of. Theorem may also be taken to refer to one of the $\phi^{(0)}$ or the combination ϕ of all of them. A similar remark may be made concerning Theorem the conditions of the above theorem apply to each point

of a set G . The sequence $s(z)$ may vary with $z \in G$ and, since a given integer ω may not belong to $s(z)$ for all $z \in G$, the spaces $PQ(c/d | p, \mu; G, \omega)$ and $RP(C, E; d | p, \mu; G, \omega)$ may not be defined. The principal results of clause (i) also refer to single values of $z \in G$. These results are assembled for all $w \in s(z)$ in forms which become independent of $s(z)$ and which may be presented with reference to the set G , as in clause (ii). Restrictive assumptions which permit formulation of both conditions and results in terms of the set G are made in the following theorem.

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $C \in MS\{B \rightarrow K; m, n\}$, $\{s; p, \mu\} \in SF_C\{B; m, n\}$ and $E \in BS\{B \rightarrow K; m, n | s; p, \mu\}$. Let $G \in B$ and $\{\mathbb{E}\text{const}(G)\} = S$.

a) Let $S = M_1 \cup M_2$ where M_1 and M_2 are disjoint sets of integer

b) let $E \in PQ(c/d | p, \mu; G, \omega)$ for $\omega \in M_1$, and

c) for each $\omega \in M_2$, let $|p(z, \omega)| \leq r$ for $z \in G$ and
 $C \in MS_{MR(p; G, \omega)} \{B \rightarrow K; r, m\}$ and

$$PQ(c/d | p, \mu; G, \omega) \cap RP(C, E; d | p, \mu; G, \omega)$$

be nonvoid.

i) For $\omega = S$, $E \in PQ(c/d | p, \mu; DC\{G | E; c, d : s; p, \mu; \omega\}, \omega)$ and

$$IN(E; p, \mu; G, \omega) \subseteq PQ(c/d | p, \mu; G, \omega)$$

ii) The results of clauses (ii, iii) of Theorem hold.

With $\{s(z)\}$ taken to be a constant set S for $z \in G$, and
 $M_1(z), M_2(z)$ similarly taken to be constant sets, condition (b)

of Theorem becomes the assertion that for each $\omega \in M_1$,
 $E \in PQ(c/d | p, \mu; z, \omega)$ for $z \in G$; for $\omega \in M_2$, $E \in PQ(c/d | p, \mu; G, \omega)$
this is condition (b) of the present theorem. The condition imposed
upon p in (c) of Theorem reappears in alternative formulation
in the present theorem. The remaining constituents of condition

(c) of Theorem may be reformulated as in (c) above.

Subject to the additional restrictions introduced in the
present theorem, the first result of clause (i) of Theorem
may be presented in the following way. For each $w \in S$,

$$E \in PQ(c/d | p, \mu; DC\{z | E; c, d : s; p, \mu; w\}, \omega)$$

for $z \in G$. In alternative formulation

$$E \in PQ(c/d | p, \mu; [UDC\{z | E; c, d; s; p, \mu; w\} \{z \in G\}], \omega)$$

The first result of clause (i) of the present theorem follows from clause (.) of Theorem.

The second result of clause (i) of Theorem may, subject to the current conditions, be formulated as follows: for each $w \in S$

$$IN(E; p, \mu; z, w) \subseteq PQ(c/d | p, \mu; z, w)$$

for $z \in G$. Thus, for each $w \in S$

$$[\cap IN(E; p, \mu; z, w) \{z \in G\}] \subseteq [\cap PQ(c/d | p, \mu; z, w) \{z \in G\}]$$

The second result of clause (i) follows from clause (.) of Theorem.

The results of clause (i) of Theorem still hold in the case of the present theorem; the results of clauses (ii, iii) still follow.

$E \in PQ(c/d | z)$. The results of clause (ii) follow from Theorem .
 The results of Theorems , are based upon the assumption that
 ~~$E \in BS\{B \rightarrow K; m, n | s; p, \mu\} \cap PQ(c/d | \rho, \omega)$~~ . Clause (iii) has
 been disposed of.

In det. of $DC\{z | z; p, \mu; \omega\}$ require $\{\rho(z, \omega)\} = \rho(z^*, \omega)$, not $\rho(z^*, \omega) = \rho(z, \omega)$
 Possibly set $R = \rho(z, \omega)$ $M = \mu(z, \omega)$ all z' such that $d^{[R]}(z') = c(z)$
 ~~$d^{[R]}(z') = d^{[R]}(z)$ and $E_{[M]}^{[R]}(z') = E_{[M]}^{[R]}(z)$~~

() Let $c \in \overset{col}{\{B \rightarrow K; m\}}$, $d \in \overset{col}{\{B \rightarrow K; n\}}$, $\phi \in \overset{row}{\{B \rightarrow K; m\}}, \{s; p, \mu\} \in$
 ~~$SIF\{B; m, n\}$, and $E \in BS\{B \rightarrow K; m, n | s; p, \mu\}$~~ Let $\{s(z)\} = M_1(z) \cup M_2(z)$
 where $M_1(z)$ and $M_2(z)$ are disjoint sets of integers

b(a) Let $E \in PQ(c/d | p, \mu; z, \omega)$ for $\omega \in M_1(z)$ and
 b(b) In ~~for each~~ for each $\omega \in M_2(z)$, let $DC\{z | E; c, d; s; p, \mu; \omega\}$ contain
 $\frac{(w)}{z_z}(z = |\rho(z, \omega)|)$ for which the matrix whose $(z+1)^{th}$ row is
 $\phi_p(\frac{(w)}{z_z}(z))(z = [|\rho(z, \omega)|])$ is nonsingular and $PQ(\phi, c/d | p, \mu; z, \omega)$
 $\cap RP(\phi, E; d | p, \mu; z, \omega)$ is nonvoid

The results of clauses (i-iii) of Theorem hold.

Slight changes to the proof of part [2] of Theorem reveal that
 condition (b) implies that $E \in PQ(c/d | p, \mu; z, \omega)$ for $\omega = M_2(z)$. The
 remainder of the proof is as that of Theorem .

The following theorem stands in relationship to the preceding theorem as does Theorem to its predecessor

() Let $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$, $\phi \in \text{row}\{B \rightarrow K; m\}$
 $\{s; p, \mu\} \in SF_C\{B; m, n\}$ and $E \in BS\{B \rightarrow K; m, n | s; p, \mu\}$. Let

$G \subseteq B$ and $\{\{s\} \in \text{const}(G)\} = S$.

a) Let $S = M_1 \cup M_2$, where M_1 and M_2 are disjoint sets of integers,

b) Let $E \in PQ(c/d | p, \mu; G, \omega)$ for $\omega \in M_1$,

c) ~~Let $E \in PQ(c/d | p, \mu; G, \omega)$~~

c) for each $\omega \in M_2$ and each $z \in G$, let $D\{z | E; c, d; s; p, \mu; \omega\}$

contain $z^{(\omega)}_z(z)$ ($z = |\rho(z, \omega)|$) for which the matrix whose

$(z+1)^{\text{th}}$ row is $\phi_p\{z^{(\omega)}_z(z)\}$ ($z = |\rho(z, \omega)|$) is nonsingular, and

d) for each $\omega \in M_2$ let

$$PQ(\phi, c/d | p, \mu; G, \omega) \cap RP(\phi, E; d | p, \mu; G, \omega)$$

be nonvoid

The results of clause (i) of Theorem and those of clauses (ii, iii) of Theorem hold.

If $\{s(z)\}$ is taken to be a constant set S for $z \in G$ and $M_1(z), M_2(z)$ are similarly taken to be constant sets, condition (b) of Theorem may be given the special formulation of (b) above. Condition (c) of Theorem may also be expressed as conditions (c,d) above. Thus, if conditions (a-d) above hold,

it follows from Theorem that, in particular, the results of clause (i) of Theorem hold. As in the proof of Theorem , it may be shown that these results may be presented in the form given in clause (i) of Theorem . That clauses (ii,iii) of Theorem also follow is demonstrated as in the proof of that theorem.

A complete factorisation result based upon properties of a polynomial, of a homogeneous constraint system for its coefficient system and of its function system.

Theorem may be based upon an assumption concerning a mapping system $\hat{C} \in MS\{B \rightarrow K; \hat{r}, m\}$; its results concern in the first instance a factorisation property $c = Ed$ over $\hat{G} \subseteq B$ of the mapping system $E \in MS\{B \rightarrow K; m, n\}$ which has the block structure specified by the section framework $\{s; p, \mu\} \in SF_c\{B; m, n\}$. It is assumed in part that for each $z \in \hat{G}$ a condition of the form $|p(z, \omega)| \leq \hat{r}$ holds for all $\omega \in M_2(z)$ where $M_2(z)$ is a subset of $s(z)$. If $M_2(z)$ is nonvoid for some $z \in \hat{G}$, so that the last condition is operative, and $\hat{r} < m$, $s(z)$ consists of more than one member, ω , then for if for the z in question, $s(z)$ were to consist of one member ω , the condition $\{s; p, \mu\} \in SF_c\{B; m, n\}$ would imply that $|p(z, \omega)| = m$, the condition $|p(z, \omega)| \leq \hat{r}$ being violated. Thus for the z in question $E(z)$ has proper block structure; certain of its submatrices consist of zero elements. The case in which $\hat{r} < m$, the condition holding for ω in $M_2(z)$ is operative for some $z \in \hat{G}$ and E is a general member of $MS\{B \rightarrow K; m, n\}$ is not one of the permitted variants of the theorem.

In Theorem , which may be based upon the behaviour of a function mapping system ϕ over a system of sets in B ,

the condition holding for ω in $M_2(z)$ is not restricted in the same way, and the case in which E is a general member of $MS\{B \rightarrow K; m, n\}$ is permitted.

The account given in the penultimate paragraph holds true, in particular, when $r=n-1$, $m=n$ and $\widehat{C} \in HC\{B \rightarrow K; n\}^c$. In this case it is possible to make additional use of an assumption, less sweeping than that referred to in the preceding paragraphs, concerning the behaviour of a function mapping system $\widehat{\phi}$ and in this way derive a factorisation result of the form $c = Ed$ in which E is a general member of $MS\{B \rightarrow K; m, n\}$. This result is described in the following theorem.

A complete factorisation result based upon properties of a polynomial of a homogeneous constraint system for its coefficient system and of its function system [out].

(*) Let $c, d \in \hat{\{B \rightarrow K; n\}}$ (Grothendieck ring), $\phi \in \hat{\{B \rightarrow K; n\}}$ and $\psi \in \hat{\{B \rightarrow K; n\}}$

Set $P = \hat{\phi}c$ over B . Let $\hat{G} \in \hat{B}$ be such that $P(z) + O$. Let

for $z \in \hat{G}$, and let $\hat{C} \in HC_{MRC(\hat{G})}\{\hat{B} \rightarrow K; n|c\}$. Let

$HC\{\hat{B} \rightarrow K; n|d\} \cap RP\{\hat{C}, E; d|\hat{G}\}$ and $PQ(P/d|\hat{G}) \cap RP\{\hat{\phi}, E; d|\hat{G}\}$

be nonvoid.

(i) The results of clauses (ii, iii) of Theorem hold.

(ii) $E \in PQ(c \otimes d|\hat{G}; E; c, d)$ and $IP(E; \hat{G}) \subseteq PQ(c \otimes d|\hat{G})$

(iii) Let $E \in BS\{\hat{B} \rightarrow K; n|s; p, \mu\}$ for some integer s like section for which $[secant(\hat{G})] = \hat{s}$.
 framework $\{s; p, \mu\} \in SF_c\{\hat{B}; n\}$. With z taken to be the value of z for which the above conditions hold, the results of Theorem hold, as do the counterparts to these results in Theorem for each $z \in DC\{z' | E; c, d\}$ in both cases.

Select $z \in \hat{G}$.

The conditions $C \in HC\{\hat{B} \rightarrow K; n|c\}$ and $P = \hat{\phi}c$ over B imply

that $C(z)c(z) = O^{[n-1]}$ and $\phi(z)c(z) = O$. Thus, setting $\Upsilon_n(C, \phi; z) = [C(z) \parallel \phi(z)]$ and $O_{P(z)}^{[n-1]} = [O^{[n-1]} \parallel P(z)]$,

$$(1) \quad \Upsilon_n(C, \phi; z)c(z) = O_{P(z)}^{[n-1]}(z)$$

Selecting $D \in HC\{\hat{B} \rightarrow K; n|d\}$, $D(z)d(z) = O$. Selecting

$\psi \in PQ(P/d|z) \cap RP\{\hat{\phi}, E; d|z\}$, $\psi(z)d(z) = P(z)$. Hence

$$(2) \quad \Upsilon_n(D, \psi; z)d(z) = O_{P(z)}^{[n-1]}(z)$$

Also

$$\{D(z) - C(z)E(z)\}d(z) = O^{[n-1]}$$

$$\{\psi(z) - \phi(z)E(z)\}d(z) = O$$

so that

$$(3) \quad \{\Upsilon_n(D, \psi; z) - \Upsilon_n(C, \phi; z)E(z)\}d(z) = O^{[n]}$$

Eliminating $O_{P(z)}^{[n-1]}(z)$ and $\Upsilon_n(D, \psi; z)$ from relationships (1-3), it

follows that

$$\Upsilon_n(C, \phi; z) \{c(z) - E(z)d(z)\} = O^{[n]}$$

Since $C \in HC_{MR}\{B \rightarrow K; n|c\}$, $\forall r \in [n]$ exists z_r for which $c_r(z)$

Since $z \in B$ is such that $P(z) \neq 0$, $c_r(z) \neq 0$ for some $r \in [n]$ and,

since $C \in HC_{MR}\{B \rightarrow K; n|c\}$, $|C_{JrL}(z)| \neq O^{[n-1]}$ for this z_r .

Furthermore, from Theorem ,

$$P(z) = \frac{(-1)^{n+r+1} |\Upsilon_n(C, \phi; z)| c_r(z)}{|C_{JrL}(z)|}$$

Since $P(z) \neq 0$, $|\Upsilon_n(C, \phi; z)| \neq 0$, $\Upsilon_n(C, \phi; z)$ is nonsingular, and

relationship (4) holds for all $z \in B$:
relationship (4) reveals that $c(z) = E(z)d(z) \forall E \in PQ(c/d|z)$. The

results of clause (i) follows from Theorem . The results of Theorems

and are based upon the assumptions that E has the block
structure described in clause (ii) and that $E \in PQ(c/d|z)$. Clause

(ii) has been disposed of.

The general mapping systems C and ϕ referred to in
the counterparts to subclauses (iii(a) and (iii(b)) respectively
of Theorem in the above theorem may be taken to be
the \hat{C} and $\hat{\phi}$ inducing the results of the above theorem.

The case in which coefficient and other mapping systems and sequences defining sectors are taken to be constant

In the above theory it is permissible to take, where appropriate, the coefficient mapping systems $c \in \text{col}\{B \rightarrow K; m\}$, $d \in \text{col}\{B \rightarrow K; n\}$ and the further mapping system's $C \in \text{MS}\{B \rightarrow K; r, m\}$ and $E \in \text{MS}\{B \rightarrow K; m, n\}$ to be constant over B . The sequence mappings $b: B \rightarrow F[m]$ and $\beta: B \rightarrow F[n]$ defining sections such as c_b and d_β may also be taken to be constant and the same holds true with regard to the set $s(z)$ and the component sequences $p(z, \omega), \mu(z, \omega)$ of a section framework $\{s; p, \mu\}$.

Subject to constraints of this sort, sections of mapping systems such as E_B^β , c_b^β and d_β^β are also constant over B . Mapping systems of maximal section rank, such as those belonging to the space $\text{MS}_{\text{MR}(\frac{1}{2}, G)}^{MR(B, G)}$ $\{B \rightarrow K; m, n\}$, defined to be so over a region $G \subseteq B$, automatically are so over B . Domains of constancy such as $\text{DC}(G \cap E; c, d: b, \beta)$ automatically extend

to \mathbb{B} . Intersection mapping systems intersect over \mathbb{B} .
A pre-quotient space, such as $PQ(c/d | b, \frac{1}{3}; G)$
defined over $G \subseteq \mathbb{B}$ contains a space of constant matrices
defined automatically over \mathbb{B} itself. Similar observations

concerning further spaces are valid.
Block structures are sustained uniformly over \mathbb{B} .

In the theorems given above conditions upon $c, d,$
 E, C, \dots stated to hold over a region $G \subseteq \mathbb{B}$ may be
weakened so as to hold for a single $z \in \mathbb{B}$; results
stated to hold over $DC(G | E; c, d; b, \frac{1}{3})$ may be
strengthened so as to hold over \mathbb{B} .

In many cases references to sets over which the
conditions hold and the results apply may be discarded
the results concerned may be given an alternative for
simpler formulation. Thus, in the case of clause (1i)
of theorem in which the condition $E \in PQ(c/d | b, \frac{1}{3}; \hat{G})$,
induces the result $PQ(C, c/d | b, \frac{1}{3}; G) = RP(C, d | b, \frac{1}{3}; G)$
 (G) for all $G \subseteq DC(\hat{G} | E; c, d; b, \frac{1}{3})$, it may be stated
subject to appropriate preliminary declarations of
matrices c, d and matrices E, C and D , that if

$c = Ed$ then $Cc = Dd$ if and only if $(D - CE)d = 0^U$.

In further cases in which variability of certain mapping systems such as $\phi \in MS\{B \rightarrow K; m\}$ and $\psi \in MS\{B \rightarrow K; n\}$ is retained, simplification is still possible. Thus, in the case of clause (1e) of Theorem in which the condition $E \in PQ(c/d | h, \hat{s}; \hat{G})$ induces the result $PQ(\phi, c/d | h, \hat{s}; \hat{G})$ $\Rightarrow PQ(c/d | h, \hat{s}; \hat{G})$ for all $\hat{G} \subseteq DC(\hat{G} | E; c, d; \hat{s}, \hat{z})$ $\equiv RP(C, E; d | h, \hat{s}; \hat{G})$ it may be stated that if $c = Ed$ then $\phi(z)c = \psi(z)d$ ($z \in \hat{G}$) if and only if $\{\psi(z) - \phi(z)E\}d = 0$ ($z \in \hat{G}$) for all $\hat{G} \subseteq B$.

Relative and simple products

Theorems (), (), and have in common that in each case a factorisation result of the form $C = Ed$ is established and common consequences of this relationship are deduced. A selection from the results of Theorem () reveals that if special mapping systems $\hat{C}^{(w)}$ and $\hat{D}^{(w)}$ are such that over a prescribed system of sections $\hat{D}^{(w)}$ is at the same time a prequotient of $\hat{C}^{(w)}$ and c by d and a product of $\hat{C}^{(w)}$ and E relative to Bd then any function mapping system ψ which is a prequotient of ϕ and c by d is a product of ϕ and E relative to d . In Theorem () it is shown, in particular, that if special function mapping systems $\hat{\phi}$ and $\hat{\psi}$ are such that $\hat{\psi}$ is simultaneously over a system of sets α in B a prequotient of $\hat{\phi}$ and c by d and a product of $\hat{\phi}$ and E relative to d then any mapping system D which is a prequotient of C and c by d is a product of C and E relative to d . The conditions of Theorem are based upon the existence of a pair of homogeneous constraint systems, \hat{C} for c and \hat{D} for d , for which \hat{D} is a product of \hat{C} and E relative to d and upon the existence of two function systems, $\hat{\phi}$ and $\hat{\psi}$, for which $\hat{\phi}c$ and $\hat{\psi}d$ both represent the same polynomial and $\hat{\psi}$ is a product

of $\hat{\phi}$ and E relative to d . It is shown, in particular, that any mapping system D which is a prequotient of C and c by d is a product of C and E relative to d and that any function mapping system ψ which is a prequotient of ϕ and c by d is a product of ϕ and E relative to d .

The results selected from Theorems (), () and () described in the preceding paragraph naturally remain true if in the case of Theorems (), the condition that, over a prescribed system of sections, $\hat{D}^{(\omega)}$ should be a product of $\hat{C}^{(\omega)}$ and E with respect to d is replaced by the stronger condition that $\hat{D}^{(\omega)}$ should be the simple product of $\hat{C}^{(\omega)}$ and E over the same sections, in the case of Theorems (), the condition that $\overset{\wedge}{\psi}$ should be a product of $\overset{\wedge}{\phi}$ and E relative to d is replaced by the stronger condition that $\overset{\wedge}{\psi}$ should be the product $\overset{\wedge}{\phi}E$ and in the case of Theorem that both conditions of the sort just referred to should be strengthened in the same way.

The results of Theorems (), do not permit the sharper inference that if, in the account given above, any function mapping system ψ which is a prequotient of ϕc by d is also the simple product ϕE . The results of Theorems (), described may not be replaced by the sharper result that any mapping system D which is a prequotient of Cc by d is the simple product CE . The

described result of Theorem may not be replaced by sharper versions similar to those just described stated.

It is proposed to show, by means of simple examples, that in each of the three cases dealt with above, the stronger conditions do not induce the corresponding sharper results.

In order to establish contact with the theories of polynomials and rational functions, the cases in which \hat{C}, C and \hat{D}, D are homogeneous constraint systems for c and d respectively are considered. For simplicity in exposition, K being a suitable field, $c, d \in \text{col}\{B \rightarrow K; n\}$, $\hat{C}, \hat{C}, \hat{D}, D \in \text{MS}\{B \rightarrow K; n-1, n\}$ and $E \in \text{MS}\{B \rightarrow K; n\}$ are all taken to be constant over $B \subseteq K$. $\hat{\phi}, \phi, \hat{\psi}, \psi$ are taken to be general mapping systems in row $\{B \rightarrow K; n\}$. In each of the numerical examples, $n=1$.

In the treatment of Theorems (), it is assumed that $E \in \text{BS}\{B \nrightarrow K; n | s; p, \mu\}$ where $\{s; p, \mu\} \in \text{SF}_c\{B; n\}$ and that $s(z)$ is a constant set \hat{s} , that $p(z, w)$ ($w = \hat{s}$) and $\mu(z, w)$ ($w = \hat{s}$) are constant sequences for $z \in B$ and that M_1 and M_2 are two nonintersecting sets whose union forms \hat{s} .

It is assumed that for each $w \in M_1$,

$$(1) \quad E_p^P(z, w) d^M(z, w) = c^P(z, w)$$

for $z \in B$ and that, for each $w \in M_2$, $\text{MS}_{\text{NR}}(p; G, w) \{B \rightarrow K; n-1, n\}$

contains $\hat{C}^{(\omega)}$ and $MS\{B \rightarrow K; n-1, n\}$ contains $\hat{D}^{(\omega)}$ such that

$$(2) \quad \hat{C}_\rho^{(\omega)}(z, \omega)c^\rho(z, \omega) = \hat{D}_\mu^{(\omega)}(z, \omega)d^\mu(z, \omega)$$

and

$$(3) \quad \left\{ \hat{D}_\mu^{(\omega)}(z, \omega) - \hat{C}_\rho^{(\omega)}E_\mu^\rho(z, \omega) \right\} d^\mu(z, \omega) = 0 \quad [m.t.]$$

for $z \in B$. It is shown in succession that

$$(4) \quad Ed = 0$$

and that for any ϕ and ψ such that $\phi(z) \neq 0$

$$(5) \quad \phi(z)c = \psi(z)d$$

for $z \in B$ it follows that

$$(6) \quad \{\psi(z) - \phi(z)E\}d = 0$$

for $z \in B$. Condition (3) is implied by the stronger condition

$$(7) \quad \hat{D}_\mu^{(\omega)}(z, \omega) = \hat{C}_\rho^{(\omega)}(z, \omega)E_\mu^\rho(z, \omega)$$

holding for each $\omega \in M_2$ and $z \in B$. It has to be shown that this condition does not imply the sharper result

$$(8) \quad \psi(z) = \phi(z)E$$

holding for $z \in B$ in place of (6).

Take $B = K$, $S = \{0, 1\}$, $M_1 = \{1\}$, $M_2 = \{0\}$, $c(z, 0) = \mu(z, 0) = 0$, $\rho(z, 1) = \mu(z, 1) = 1$, $c = [1 \parallel 1]$, $d = [-1 \parallel 1]$, $E = [-1 \mid 0] \parallel [0 \mid 1]$, $\hat{C}^{(0)} = [1 \mid -1]$, $\hat{D}^{(0)} = [-1 \mid -1]$,

$\phi(z) = [z \mid -1]$ and $\psi(z) = [1 \mid z]$. Then $E_\mu^P(z, 1) = 1$, $d^H(z, 1) = 1$ and $c^P(z, 1) = 1$: the condition involving relationship (1) is satisfied. $\hat{C}_P^{(0)}(z, 0) = 1$, $c^P(z, 0) = 1$, $\hat{D}_\mu^{(0)}(z, 0) = -1$ and $d^H(z, 0) = -1$: $C^{(0)} \in \text{MS}_{\text{MR}(\rho; B, 0)} \{B \rightarrow K; 0, 1\}$ and condition (2) is satisfied as required. $E_\mu^P(z, 0) = -1$: $\{-1 - 1(-1)\}(-1) = 0$ and condition (3) is satisfied. The result (4) is evidently correct. $\phi(z)c = z - 1 = \psi(z)d$: condition (5) is satisfied as required. $\phi(z)E = [[-z] \mid -1]$, so that $\psi(z) - \phi(z)E = [1+z \mid 1+z]$: the result (6) is correct. Since $\hat{D}_\mu^{(0)}(z, 0) = -1$, $\hat{C}_P^{(0)}(z, 0) = 1$ and $E_\mu^P(z, 0) = -1$, the stronger condition (7) is satisfied. But $\psi(z) = [1 \mid z]$ while $\phi(z)E = [-z \mid -1]$: the sharper result (8) does not follow. It is remarked in passing that $\hat{C}^{(0)}$ and $\hat{D}^{(0)}$ above are homogeneous constraint systems for c and d respectively.

In the treatment of Theorems (), $s(z) = M_2(z)$ is taken to be a constant set consisting of one member w , so that $\rho(z, w) = \mu(z, w) = [[n]]$ for $z \in B$ and E is not required to have proper block structure. Since E, c and d are taken to be constant, the domain of constancy $D\mathcal{S}'(z \mid E; c, d; s; \rho, \mu; w)$ referred to in condition (c) of

Theorem is \mathbb{B} itself. It is assumed that \mathbb{B} contains z_z ($z \in [n]$) for which the matrix whose $(z+1)^{\text{th}}$ row is $\phi(z_z)$ ($z \in [n]$) is nonsingular and that relationships (5,6) above are satisfied for $z \in \mathbb{B}$. It is shown in succession that relationship (4) holds and that for any C for which

$$(9) \quad Cc = Dd$$

it follows that

$$(10) \quad \{D - CE\}d = 0^{[n-1]}$$

Condition (6) is implied by the stronger condition (8) holding for $z \in \mathbb{B}$. It has to be shown that the latter condition does not imply the sharper result

$$(11) \quad D = CE$$

Take $\mathbb{B} = K$, $\phi(z) = [1|z]$ and $z_0 = 0, z_1 = 1$. The matrix whose $(z+1)^{\text{th}}$ row is $\phi(z_z)$ ($z \in [1]$) is $[[1|0] \parallel [1|1]]$ and is nonsingular. Take $c = [0 \parallel 1], d = [1 \parallel 1]$ and $\psi(z) = [z-1|1]$

$\phi(z)c = \psi(z)d = z$: condition (5) is satisfied as required.

Take $E = [-1|1] \parallel [1|0]$. Since $\psi(z) = \phi(z)E$ over \mathbb{B} , condition (6) holds as stated. The result (4) is evidently correct. Take $C = [1|0]$ and $D = [1|-1]$ so that C and D are homogeneous constraint systems for c and d respectively and relationship (9) is automatically satisfied. Furthermore,

$CE = [-111]$, $D - CE = [21-2]$ and relationship (10) is satisfied. That the stronger condition (8) holds for $z \in B$ has already been remarked. But $D = [11-1]$, $CE = [-111]$ and the sharper result does not follow.

In the treatment of Theorem it is supposed that, with P defined by $P(z) = \hat{\phi}(z)c$ over B , $P(z) \neq 0$ for $z \in B$. \hat{C} and \hat{D} are homogeneous constraint systems for c and d respectively, so that $\hat{C}c = \hat{D}d = 0^{[n-1]}$ and it is further supposed that

$$(12) \quad (\hat{D} - \hat{C}E)d = 0^{[n-1]}$$

Concerning the function mapping system $\hat{\psi}$, it is supposed that $P(z) = \hat{\psi}(z)d$ over B and that

$$(13) \quad \{\hat{\psi}(z) - \hat{\phi}(z)E\}d = 0$$

over B . It is shown in succession that relationship (4) holds, that for any C relationship (9) induces relationship (10) and that for any ψ , relationship (5) induces relationship (6). Conditions (12, 13) are implied by the stronger conditions

$$(14) \quad \hat{D} = \hat{C}E$$

and

$$(15) \quad \hat{\psi}(z) = \hat{\phi}(z)E$$

over B . It has to be shown that replacement of condition (12)

by condition (14) does not induce the sharper result (8) in place of (6) for $z \in B$ and that replacement of condition (13) by condition (15) does not induce the sharper result (11) in place of (10).

Take $B = K \setminus 0$, $c = [0 \mid 1]$ and $\phi(z) = [1 \mid z] : P(z) = z$ to over B . Take $d = [1 \mid 1]$, $\hat{C} = [1 \mid 0]$ and $\hat{D} = [-1 \mid 1]$ so that \hat{C} and \hat{D} are homogeneous constraint systems for c and d respectively. Take $E = [-1 \mid 1] \parallel [1 \mid 0]$ so that relationship (14), and in turn (12), are satisfied. Take $\hat{\psi}(z) = [z - 1 \mid 1]$ so that $P(z) = z = \hat{\psi}(z)d$ over B . Relationship (15) and, in consequence, (13) are satisfied over B . The result (4) is evidently correct. $\psi(z) = [0 \mid z]$ is another function mapping system for which $\psi(z)d = P(z)$ over B so that, with $\phi = \hat{\phi}$ and ψ as just given, condition (5) is satisfied. Also $\phi(z)E = [z - 1 \mid 1]$ so that $\psi(z) - \phi(z)E = [1 - z \mid z - 1]$. Relationship (6) is satisfied, but $\psi(z) + \phi(z)E = D + CE$: the sharper result (8) is not attained. $C = [-1 \mid 0]$ is also a homogeneous constraint system for c : with $D = \hat{D}$, relationship (9) is satisfied. $CE = [1 \mid -1]$. Relationship (10) with C as just given and $D = \hat{D}$ is satisfied, but $D + CE$: the sharper result (11) is not attained.

Although the stronger conditions referred to above do not induce the sharper results as described, simple examples in which the sharper results hold in conjunction with the stronger conditions are easily contrived. In the case of Theorems (,) let the stronger condition (7) hold. With $\phi(z)$ prescribed, construct ψ by use of the relationship $\psi(z) = \phi(z)E$, so that the sharper result (8) holds. Since $Ed = c$, it follows that $\phi(z)c = \phi(z)Ed = \psi(z)d$. Condition (5) holds as required. In this case the stronger condition (7) and the sharper result (8) hold in conjunction but the former does not in general imply the latter.

Theorems (,) are treated in the same way by constructing \mathcal{D} by use of the relationship $\mathcal{D} = CE$.

Spaces of ordered pairs of mapping systems.

In the above, the spaces $PQ(C, c/d/b, \frac{1}{3}; G)$, $RP(C, E; d/b, \frac{1}{3})$, ... are defined in terms of a fixed mapping system $C \in MS\{B \rightarrow K; r, m\}$. Results concerning them are presented in terms of a prescribed C . Thus in the case of clause (1i) of Theorem , C is prescribed and the condition $E \in PQ(c/d/b, \frac{1}{3}; G)$ is stated to imply the equivalence relationship $PQ(C, c/d/b, \frac{1}{3}; G) = RP(C, E; d/b, \frac{1}{3}; G)$.

It is possible to define a space, $PQ_r(c/d/b, \frac{1}{3}; G)$ say, of ordered pairs $\{C, D\}$ of mapping systems for which $D \in PQ(C, c/d/b, \frac{1}{3}; G)$, and define further spaces $RP_r(E; d/b, \frac{1}{3}; G)$... of ordered pairs of mapping systems similarly. Clause (1i) of Theorem then becomes the assertion that subject to the stated condition, $PQ_r(c/d/b, \frac{1}{3}; G) = RP_r(E; d/b, \frac{1}{3}; G)$. In the foregoing exposition, references to the spaces $PQ(C, c/d/b, \frac{1}{3}; G)$ and $RP(C, E; d/b, \frac{1}{3}; G)$ may be consistently be replaced by corresponding modified references to the spaces $PQ_r(c/d/b, \frac{1}{3}; G)$ and $RP_r(E; d/b, \frac{1}{3}; G)$, and references to $PQ(\phi, c/d/b, \frac{1}{3}; G)$ and $RP(\phi, E; d/b, \frac{1}{3}; G)$ by references to $PQ_r(c/d/b, \frac{1}{3}; G)$ and $RP_r(E; d/b, \frac{1}{3}; G)$.

However, in many of the theorems given above, preliminarily

conditions are imposed upon C and ϕ and the exposition adopted is perhaps the most felicitous available. Nevertheless, if subsequent application of the theory is presented exclusive in terms of references to ordered pairs of mapping systems, reformulation of the above theory may be advisable.