Connections between various classes of functions of a complex variable

Th1. $f(z)$ maps $\mathrm{Re}(z)<0$ ~~ont~~ into $|f(z)|<1$ iff

$$f(z)= e^{i\varphi}\left\{1 + \frac{2z}{A+(ic'-1)z + iz\int_{-\infty}^{\infty} \frac{\cancel{z}-it\cancel{\ }}{zt+i\cancel{\ }}\,d\sigma'(t)}\right\}$$

where $-\infty<\varphi<0$, $0\le A<\infty$ $-\infty<c'<\infty$

σ' bounded nondecreasing over $(-\infty,\infty)$

Th2. $f(z)$ maps $\mathrm{Re}(z)<0$ into $|f(z)|<1$ and

$f(z)\sim\sum t_\nu z^\nu$ as $z\to 0$ in $\frac{\pi}{2}+\delta<\arg(z)<\frac{3\pi}{2}-\delta$

(for $\delta\in(0,\frac{\pi}{2})$ fixed) iff

$$f(z)= e^{i\varphi}\left\{1 + \frac{2z}{A+(ic-1)z + z^2\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1-itz}}\right\}$$

where $-\infty<\varphi<\infty$, $0<A\le\infty$, $-\infty<c<\infty$, σ bdd nondecr over $(-\infty,\infty)$

Furthermore, if $f(z)$ such that all moments has ~~a~~ representation of the form

~~( ), then $\sum t_\nu z^\nu$ the continued fraction~~

as described ( ), then the series $\sum t_\nu z^\nu$ generates an associated continued fraction whose convergents have the form $C_0(z)=t_0$, $C_1(z)=t_0+\frac{v_1 z}{1+w_1 z}$,

$$C_r(z) = t_0 + \frac{v_1 z}{1+w_1 z+}\;\frac{v_2 z^2}{1+w_2 z+}\;\frac{v_3 z^2}{1+w_3 z+}\cdots\frac{v_r z^2}{1+w_r z}$$

and ~~where $C_r(z)$ may be expressed in the form~~

and $C_r(z)$ maps the ~~open~~ closed left half plane $Re(z) \leq 0$ into the closed disc $|C_r(z)| \leq 1$.

Th. 3. (i) $f(z)$ ~~is as descr~~ maps $Re(z) < 0$ into $|f| < 1$ ~~and~~ is real for real z with ~~$f(0) > 0$~~

$\lim_{\cancel{z \to 0}} f(z) > 0$ as $z \to 0$ in above half plane, iff

$$f(z) = 1 + \frac{2z}{A - z + z^2 \int_0^\infty \frac{(1+t^2)\,d\sigma'(t)}{1+z^2t^2}}$$

where $0 \leq A < \infty$ and σ' bounded nondecr. over $(-0, \infty)$.

(ii) $f(z)$ satisfies conditions of part (i) and $f(z) \sim \sum b_\nu z^\nu$ as $z \to 0$ in $\frac{\pi}{2} + \delta < \arg(z) < \frac{3\pi}{2} - \delta$ (for $\delta \in (0, \frac{\pi}{2}]$ fixed) iff ~~$f(z)$ has representation of form ( ) in which~~ $0 < A < \infty$

$$f(z) = 1 + \frac{2z}{A - z + z^2 \int_0^\infty \frac{d\sigma(t)}{1+z^2t^2}}$$

where $0 < A < \infty$, σ bded nondecr over $(0, \infty)$ such that all moments $\int_0^\infty t^\nu d\sigma(t)$ $(\nu = 0, 1, \dots)$ exist

Furthermore, if $f(z)$ has a representation of the form ( ) as described, then the series $\sum b_\nu z^\nu$ generates an associated continued fraction as described in Th 2, with convergents $\{C_r(z)\}$ as given in that theorem with $t_0 = 1$ $v_1 = 2/A$, $w_1 = -1/A$, $w_r = 0$ $(r = 2, 3, \ldots)$, the $\{C_r(z)\}$ also having the mapping properties described in Th. 2.

~~Th. 4. Let $f_0(z)$, $f_1(z)$ be two functions having a representation of the form ( ). Then $f(z) = f_0(z)$~~

Th 4. Functions having ~~a~~ representations of the form ( ) are closed with respect to multiplication; the same result holds for functions having ~~a~~ representations of the form ( ).

~~Let~~

$$\cancel{z'(z) = i\int_{-\infty}^{\infty} \frac{z - it}{zt + i}\, d\sigma''(t) + ic + A''z^{-1}}$$

~~where~~ The function $f(z'(z))$ also has a representation of the form ( ) if and only if $f(z)$ have a representation of the form ( ), and let

Let

$$z'(z) = A''z + ic'' + \int_{-\infty}^{\infty} \frac{tz + i}{t + iz}\, d\sigma''(t)$$

where $0 \leq A'' < \infty$, $-\infty < c'' < \infty$, and σ'' is bounded and non decreasing over $(-\infty, \infty)$. ~~Then~~

~~$f(z'(z))$ also has a representation of the form~~

Let $f(z)$ have a representation of the form ( ).
The function $f\{z'(z)\}$ also has a representation of the form ( ) if and only if

$$z'(z) = A'' z^{-1} + \int_{-\infty}^{\infty} \frac{z - it}{1 - izt} \, d\sigma''(t) + ic''$$

where $0 \leq A'' < \infty$, $-\infty < c'' < \infty$ and σ'' is bounded and nondecreasing over $(-\infty, \infty)$.

Let $f(z)$ have a representation of the form ( )
The function $f\{Z(z)\}$ also has a representation of the form ( ) if and only if

$$Z(z) = ic + z \int_{-\infty}^{\infty} \frac{d\hat{\sigma}(t)}{1 - izt}$$

$\sigma := \xi$?

where $-\infty < c < \infty$, σ bded nondecr over $(-\infty, \infty)$ such that all moments $\int_{-\infty}^{\infty} t^\nu d\xi(t)$ exist

(i) Let $f(z)$ have a representation of the form ( ).
The function $f\{Z(z)\}$ also has a representation of the form ( ) if and only if

$$Z(z) = A'' z^{-1} + z \int_{\cancel{\quad}\atop 0}^{\infty} \frac{(1+t^2) \, d\sigma''(t)}{1 + z^2 t^2}$$

where $0 \leq A' < \infty$ and $\sigma' \in B(-\overset{0}{\infty}, \infty)$

(ii) Let $f(z)$ have a representation of the form ( ). The function $f\{Z(z)\}$ if

$$Z(z) = z\int_0^\infty \frac{d\xi(t)}{1+z^2t^2}$$

where $\xi \in BM(0,\infty)$

$f \in NS$ means that the function f of a complex variable is analytic at the origin and that the sequence of functions $f_n(z)$ $(n=0,1,\dots)$ obtained by setting

$$f_0(z) = f(z), \quad f_n(z) = \frac{f_{n-1}(z) - f_{n-1}(0)}{z\{1-\overline{f_{n-1}(0)}\, f_{n-1}(z)\}} \quad (n=1,2,\dots)$$

~~are such that $|f_n(0)| \leq 1$.~~
~~(the sequence terminating with $f_n(z)$ if $|f_n(0)|=1$ when $f_n(z)$ is a constant)~~

are such that either $|f_n(0)|=1$ for some finite n when the sequence $\{f_n\}(z)$ terminates with $f_n(z) = f_n(0)$ ~~for some finite~~ or $|f_n(0)| < 1$ for $n = 0,1,\dots$

In the following set $\hat{f}(z) = F(f(z))$.

Let $f \in W_1$. Then $\hat{f} \in W_1$ iff $F \in NS$.

Let $f \in W_2$ Then $\hat{f} \in W_2$ iff $F \in NS$

Let $f \in W_3$. Then $\hat{f} \in W_3$ iff $F \in NS$ and $F(z)$ is real for real z with $F(0) > 0$

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Remark on Th. 1

One of fns $f(z)$ is that obtained by setting $\phi = A = c' = \sigma'(\infty) - \sigma'(-\infty) = 0$ i.e. $f(z) = -1$

$\therefore$ mapping is not of $Re(z) < 0$ onto entire $|f| < 1$ with one exceptional pt.

if such incomplete mappings are to be considered, then $f(z) = a$ $(|a| < 1)$ maps $Re(z) < 0$ into $|f| < 1$ and this is not of form stated.

Thus only if part is false, and remainder of Th almost nugatory.

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$w_f(z)$ analytic and $\operatorname{im} f \geq 0$ for $y > 0$ iff

$$w_f(z) = Az + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} d\alpha(t) + c$$

A, c real $A \geq 0$. But $w_f(z)$ may map $Im(z) > 0$ onto subregion of $Im(w) \geq 0$. This implies derived fn. $f(z)$ maps $Re(z) < 0$ onto ~~subregion~~ subset of $|f| < 1$

becomes important to consider when w maps $\mathrm{Im}(z)>0$ into entire $\mathrm{Im}(w) \geq 0$

Presumably this is so when α is a step fn. and may even be true when α has interval of constancy

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$\sqrt{1+z^2} + z = f \qquad 1+\cancel{z^2} = \cancel{z^2} - 2fz + f^2$

$z = (f^2-1)/2f$

can $z \in \mathrm{Re}(z)<0$ be found for all $|f|<1$

$f = re^{i\theta}$

$z = (r^2e^{2i\theta}-1)/2re^{i\theta} = \frac{1}{2}\{re^{i\theta} - r^{-1}e^{-i\theta}\}$

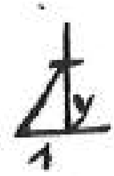
$= \frac{1}{2}\{r\cos\theta + ir\sin\theta - r^{-1}\cos\theta + i r^{-1}\sin\theta\}$

$= \frac{1}{2}\{(r - r^{-1})\cos\theta + i\sin\theta\,(r+r^{-1})$

$\mathrm{Re}(z)<0$ only for $0 \leq \cos\theta \leq 1$ ie $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$z = iy \qquad \sqrt{1-y^2} + iy = f$

image of $y \in (-1,1)$ is



$f^2 - 2fz - 1 = 0 \qquad$ $r - r^{-1}$ maps $(0,1)$ into $(-\infty,0)$

$r+r^{-1}$ maps $(0,1)$ into $(1,\infty)$

$\sqrt{(1+iz)(1-iz)} + z$

$zZ'(z) \sim \sum c_{\nu 0} z^{2\nu+1}$

$zT(iz)$ has representation of form $\int_{-\infty}^{\infty} \frac{d\Omega(t)}{z'-t}$

$\therefore$ series expansion of form $\sum_{\nu=0} C_\nu z^{\nu+1}$

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Th 1 etc. can be maintained if formulated as: pure imaginary

(i) W_1 is the class of functions which are analytic for all finite non finite values of ~~which~~ and map the open the [illegible] $\{z : \mathrm{Re}(z) < 0, \; |z| < \infty\}$ ~~right~~ left half-plane ~~$\mathrm{Re}(z) < 0$~~ ~~into $\mathrm{Re}(z) > 0$~~ into the open unit disc $\{f : |f| < 1\}$ and the finite open right half plane ~~$\mathrm{Re}(z) > 0$~~ $\{z : \mathrm{Re}(z) > 0, \; |z| < \infty\}$ into the complement of the closed unit disc $\{f : 1 < |f| < \infty\}$ ~~$\{f : |f| \leq 1\}$~~ etc. with respect to the finite part of the complex plane.

(ii) W_2 is that subclass of W_1 ~~containing functions possess an asymptotic representation of~~ containing all functions $f(z)$ ~~$f(z)$ whose members~~ that are asymptotically represented ~~by of series of the form $\sum_{\nu=0}^{\infty} b_\nu z^\nu$~~ for small values of z in any sector ~~of the form~~ $\frac{1}{2}\pi + \delta \leq \arg(z) \leq \frac{3}{2}\pi - \delta$, $0 < \delta < \frac{1}{2}\pi$ by a series of the form $\sum_{\nu=0}^{\infty} b_\nu z^\nu$, the $\{b_\nu\}$ being finite complex numbers.

(iii) W_3 is [redacted] that subclass of W_1 containing all functions that are real for real z, with $f(z) > 0$ for small real values of z

(iv) W_4 is that subclass of W_2, W_3 for which in (ii) above ~~the $\{b_\nu\}$ are real with $b_\nu > 0$. Furthermore $W_4 \subset W_3$~~ –

Notation. With $[\alpha,\beta]$ a prescribed segment of the real axis, and the integral involved being a Riemann–Stieltjes integral, ~~we set~~

$$S[g(t)|\sigma;\alpha,\beta]=\int_\alpha^\beta g(t)\,d\sigma(t).$$

~~$\sigma\in$~~ $B[\alpha,\beta]$ ~~means that σ is a~~ is the class of bounded nondecreasing functions over $[\alpha,\beta]$. ~~$\sigma\in$~~ $BM[\alpha,\beta]$ is the subclass of $B[\alpha,\beta]$ containing all ~~for whose~~ members to which all moments $S[t^\nu|\sigma;\alpha,\beta]$ $(\nu=0,1,\dots)$ exist.

Definition $W_1,\dots,W_4$ are the classes of all functions ~~representable~~ having the following representations mappings ~~with~~ of

~~Definition. Four classes $W_1,\dots,W_4$ of functions defined over the non pure imaginary part of the finite complex plane $\{z: \mathrm{Re}(z)\neq 0,\ |z|<\infty\}$ are defined~~ by inserting all permissible ~~by allowing the~~ ~~in terms of~~ real constants and functions in the following formulae to which ~~these~~ typical members ~~as follows~~ ~~take all permissible~~

$$W_1:\quad f(z)=e^{i\phi}\left\{1+\frac{2z}{A+(iB-1)z+izS\left[\frac{z-it}{z+ti}\,\middle|\,\sigma;-\infty,\infty\right]}\right\}$$

where ~~$-\infty<\phi<\infty$~~ $-\pi\leq\phi<\pi$, $0\leq A<\infty$, $-\infty<B<\infty$, $\sigma\in B[-\infty,\infty]$ and $A+\sigma(\infty)-\sigma(-\infty)\neq 0$.

$$W_2:\quad f(z)=e^{i\phi}\left\{1+\frac{az}{1+(ib-\tfrac{1}{2}a)z+z^2S[(1-izt)^{-1}|\sigma;-\infty,\infty]}\right\}$$

where $-\pi \leq \phi < \pi$, $0 < a < \infty$, $-\infty < b < \infty$, and $\sigma \in \mathbb{M}(-\infty, \infty)$.

$$W_3: \quad f(z) = 1 + \frac{2z}{A - z + z^2 \mathcal{S}\left[\frac{1+t}{1+z^2 t} \middle| \sigma; 0, \infty\right]}$$

where $0 \leq A < \infty$ ~~and~~ $\sigma \in B(0,\infty)$ and $A + \sigma(\infty) - \sigma(0) \neq 0$.

$$W_4: \quad f(z) = 1 + \frac{a}{1 - \frac{1}{2}a + z^2 \mathcal{S}\left[(1+z^2 t)^{-1} \middle| \sigma; 0, \infty\right]}$$

where $0 < a < \infty$ and $\sigma \in BM(0,\infty)$.

We mention the existence of a further class of functions W_0 defined as for W_1 above but with the condition $A + \sigma(\infty) - \sigma(-\infty)$ discarded. W_0 consists of ~~W_1~~ all members of W_1 together with all constant functions having a representation of the form $f(z) = e^{i\phi}(iB+1)/(iB-1)$; the additional functions merely map the entire complex plane into a fixed point on the unit circle.

~~Th. In the following let $\hat{f}(z) = f\{Z(z)\}$.~~

~~(i) Let $f \in W_1$. Then $\hat{f} \in W_1$ if and only if~~

$$Z_1: \quad Z(z) = Cz^{-1} + iD + S\left[\frac{z-it}{1-izt} \,\middle|\, \xi; -\infty, \infty\right]$$

where $0 \leq C < \infty$, $-\infty < D < \infty$ and $\xi \in B(-\infty, \infty)$.

~~(ii) Let $f \in W_2$. Then $\hat{f} \in W_2$ if and only if~~

$$Z_2: \quad Z(z) = iD + zS\left[(1-izt)^{-1} \,\middle|\, \xi; -\infty, \infty\right]$$

where $-\infty < D < \infty$ and $\xi \in BM(-\infty, \infty)$

~~(iii) Let $f \in W_3$. Then $\hat{f} \in W_3$ if and only if~~

$$Z_3: \quad Z(z) = Cz^{-1} + zS\left[\frac{1+t}{1+z^2 t} \,\middle|\, \xi; 0, \infty\right]$$

where $0 \leq C < \infty$ and $\xi \in B(0, \infty)$

~~(iv) Let $f \in W_4$. Then $\hat{f} \in W_4$ if and only if~~

$$Z_4, \quad Z(z) = zS\left[(1+z^2 t)^{-1} \,\middle|\, \xi, 0, \infty\right]$$

where $\xi \in BM(0, \infty)$.

~~Th~~ For $1 \leq i \leq 4$

Th. Let $f_\xi \in W_i$. Then ~~$f\{Z(z)\}$~~, setting $\hat{f}(z) = f\{Z(z)\}$ $\hat{f} \in W_i$ if and only if $Z \in Z_i$ where Z_i is the class of functions having the following representations

$$(1+z)^{\frac{1}{2}}(1-z)^{-1/2} = \left\{1+\tfrac{1}{2}z+\quad\right\}\left\{1+\tfrac{1}{2}z+\cdots\right\} \sim 1+z$$

$$= 1 + \frac{z}{1-\frac{1}{2}z+z^2S(z)} = \left\{\frac{1+z}{1-z}\right\}^{1/2}$$

answer to all - this is that $\left(\frac{1+z}{1-z}\right)^{1/2}$ is not analytic in left and right half plane

$$\frac{(1+z)^{1/2}-(1-z)^{1/2}}{(1-z)^{1/2}} = \frac{z}{\cdots}$$

$$1-\tfrac{1}{2}z+z^2S(z) = \frac{\cancel{\varepsilon}\, z(1-z)^{1/2}}{(1+z)^{1/2}-(1-z)^{1/2}}$$

$$z^2S(z) = \frac{z(1-z)^{1/2}-(1-\frac{1}{2}z)\left\{(1+z)^{1/2}-(1-z)^{1/2}\right\}}{(1+z)^{1/2}-(1-z)^{1/2}}$$

$$= \frac{z(1-z)^{1/2}-(1+z)^{1/2}+\underline{(1-z)^{1/2}}+\tfrac{1}{2}z(1+z)^{1/2}-\tfrac{1}{2}z(1-z)^{1/2}}{}$$

$$\frac{(1+\frac{1}{2}z)(1-z)^{1/2}-(1-\frac{1}{2}z)(1+z)^{1/2}}{(1+z)^{1/2}-(1-z)^{1/2}}$$

$$Z = (1+z)^{1/2}-(1-z)^{1/2} \qquad Z^2 = 2-2(1+z)^{1/2}(1-z)^{1/2}$$

$$\left(\frac{Z^2-2}{4}\right)^2 = 1-z^2 \qquad z^2 = \frac{(2-Z^2+2)(2+Z^2-2)}{4}$$

$$z = \frac{Z}{2}(2-Z)^{1/2}(2+Z)^{1/2}$$

denom $\sim z$; num $(1+\frac{1}{2}z)\{1-\frac{1}{2}z-\frac{1}{8}z^2$

$$1-(\tfrac{1}{4}+\tfrac{1}{8})z^2$$

num $\sim -(\frac{1}{2}+\frac{1}{4})z^2$

$$F_n(\zeta) = \frac{F_{n-1}(\zeta) - F_{n-1}(0)}{\zeta\{1 - \overline{F_{n-1}(0)}\, F_{n-1}(\zeta)\}}$$

$$f^{(n)}(z) = \frac{z\{f^{(n-1)}(z) - f^{(n-1)}(\infty)\}}{1 - \overline{f^{(n-1)}(\infty)}\, f^{(n-1)}(z)} \qquad f^{(0)}(z) = 1/f(z)$$

$$f(z) = z^2 \qquad f^{(0)}(\infty) = 0$$

$$f^{(1)}(z) = \frac{z\{\frac{1}{z^2} - 0\}}{1 - 0} = \frac{1}{z}$$

$$f^{(2)}(z) = \frac{z\{\frac{1}{z} - 0\}}{1 -} = 1$$

$$f(z) = \tfrac{1}{2}(1+z) \qquad f^{(0)}(z) = \frac{2}{1+z} \qquad f^{(0)}(\infty) = \cancel{2}\ 0$$

$$f^{(1)}(z) = z\left\{\frac{2}{1+z}\right. = \frac{2z}{1+z} \quad f^{(1)}(\infty) = 2$$

NS is the class of functions f of a complex variable ~~such that~~ analytic at the origin, with $1/f(z)$ analytic at infinity. ~~Such that~~ for which the functions f_i $(i = 0, 1, \ldots)$ $f^{(j)}$ $(j = 0, 1, \ldots)$ obtained by setting

$$f_0(z) = f(z), \quad f_{i+1}(z) = \frac{f_i(z) - f_i(0)}{z\{1 - \overline{f_i(0)}\, f_i(z)\}} \qquad (i = 0, 1, \ldots)$$

$$f^{(0)}(z) = 1/f(z), \quad f^{(j+1)}(z) = \frac{z\{f^{(j)}(z) - f^{(j)}(\infty)\}}{1 - \overline{f^{(j)}(\infty)}\, f^{(j)}(z)}$$

are such that either $|f_I(0)| = 1$ for some finite I, when (the sequence f_i then terminates with $f_I(z) = f_I(0)$) or $|f_i(0)| < 1$ for $i = 0, 1, \ldots$, ~~&~~ similar conditions obtaining for the $\{f^{(j)}\}$.

Th. In the following, set $\tilde{f}(z)=F\{f(z)\}$. Let $f\in W_i$ and

(i) ~~Let $f\in W_i$, then~~ $\tilde{f}\in W_i$ if and only if $F\in NS$ $(i=1,2)$.

(ii) ~~Let $f\in W_i$,~~ $\tilde{f}\in W_i$ if and only if $F\in NS$ NS' is real and f also for small real values of its argument $(i=3,4)$.

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Th. Two ~~function~~ W_i functions f_1 and f_2 map the finite open left half plane conformally onto ~~or~~ the same open domain if and only if $f_1(z)=f_2\{Z(z)\}$ where $Z\in Z_i$ $(i=1,\dots,4)$.

onto the same open domain ~~lying~~ of a Riemann surface

Th. ~~Every rational W_2 function is a rational NS function~~

~~of $(1+\bar{a}z)/(1-\bar{a}z)$ for some $(e^{i\theta}+z)/(e^{i\theta}-z)$~~

~~of $\{A+(B+1)z\}/\{A+(B-1)z\}$ for some $A\in[0,\infty)$~~

~~$1+2z/\{A+(B-1)z\}$~~

~~$B\in(-\infty,\infty)$.~~

Every rational W_i function is a rational NS function

~~of . for some $A\in(0,\infty)$ $B\in(-\infty,\infty)$~~

~~W_2~~

of $1+az/\{1+(b-\frac{1}{2}a)z\}$ for some $a\in(0,\infty)$, $b\in(-\infty,\infty)$

$(i=1,2)$

Every rational W_i function is a rational NS' function

of ~~$1+2z/(A-$~~ $(A+z)/(A-z)$ for some $A\in(0,\infty)$ $(i=3,4)$

~~B~~

W_2 is the class of functions which
(i) are asymptotically represented ~~by a series~~ for small values of z in a sector ~~of the form~~ $\frac{1}{2}\pi+\delta \le \arg(z) \le \frac{3}{2}\pi-\delta$, $\{0<\delta<\frac{1}{2}\pi$,
by a series $\sum_{\nu=0}^{\infty} f_\nu z^\nu$ of the form $\sum_{\nu=0}^{\infty} f_\nu z^\nu$ which
(ii) generates an associated continued fraction whose

successive convergents

$$\cancel{C_1(z)} \quad \cancel{C_1(z)} \quad C_1(z) = f_0 + \frac{v_1 z}{1+w_1 z}$$

$$C_\nu(z) = f_0 + \frac{v_1 z}{1+w_1 z+} \; \frac{v_2 z^2}{1+w_2 z+} \; \frac{v_\nu z^2}{1+w_\nu z} \qquad (\nu=2,3,\dots)$$

(iii) maps the ~~closed unit disc into~~ complete closed left half plane into the closed unit disc

W_4 is the subclass of W_3 functions for which, in addition ~~to the above~~, described above, the $\{f_\nu\}$ are real with $f_0>0$

$$f_{i+1}(z) = \frac{f_i(z) - f_i(0)}{z\{1-\overline{f_i(0)}\, f_i(z)\}}$$

$$f_i(z) = \sum_{\tau=0}^{n} N_\tau z^\tau \Big/ \sum_{\tau=0}^{d} D_\tau z^\tau$$

$$f_{i+1}(z) = \frac{\sum_{\tau=0}^{n} N_\tau z^\tau - \frac{N_0}{D_0} \sum_{\tau=0}^{d} D_\tau z^\tau}{z\left\{\sum_{\tau=0}^{d} D_\tau z^\tau - \frac{\overline{N_0}}{\overline{D_0}} \sum_{\tau=0}^{n} N_\tau z^\tau\right.}$$

$n = d$

$f_{i+1} \quad \frac{n-1}{n}$

$$f_i(z) = \frac{N^{(i)}(z)}{D^{(i)}(z)} \qquad N^{(i+1)}(z) = \{\hat{D}_0 N^{(i)}(z) - \hat{N}_0^{(i)} D^{(i)}(z)\} \overline{D}_0^{(i)}$$

$$D^{(i+1)}(z) = \{\overline{D}_0^{(i)} D^{(i)}(z) - \overline{N}_0^{(i)} N^{(i)}(z)\}$$

$$f_{i+1}(z) = \qquad N_0^{(i+1)} = \{D_0^{(i)} N_1^{(i)} - N_0^{(i)} D_1^{(i)}\} \overline{D}_0^{(i)}$$

$$D_0^{(i+1)} = \{\overline{D}_0^{(i)} D_0^{(i)} - \overline{N}_0^{(i)} N_0^{(i)}\} D_0^{(i)}$$

$\frac{1}{3} \quad \frac{2}{3} \quad \frac{2}{3}$

$$D^{(i+2)}(z) = \overline{D}_0^{(i+1)} D^{(i+1)}(z) - \overline{N}_0^{(i+1)} N^{(i+1)}(z)$$

$$D_0^{(i+2)} = \{\overline{D}_0^{(i)} D_0^{(i)} - \overline{N}_0^{(i)} N_0^{(i)}\}^2 \overline{D}_0^{(i)} D_0^{(i)}$$
$$- \{\overline{D}_0^{(i)} \overline{N}_1^{(i)} - N_0^{(i)} D_1^{(i)}\} \overline{D}_0^{(i)} \overline{D}_0^{(i)} \{D_0^{(i)} N_1^{(i)} - N_0^{(i)} D_1^{(i)}\}$$

$$|D_0^{(i)}|^2 + |N_0^{(i)}|^2 - 2 |D_0^{(i)}||N_0^{(i)}| \qquad 4 - 5\cos\theta - 3\sin\theta\, i$$

$$-|D_0^{(i)}|^2 |N_1^{(i)}|^2 - |N_0^{(i)}|^2 |D_1^{(i)}|^2 \qquad 16 - 40\cos\theta + 25\cos^2\theta + 9\sin^2\theta$$

$$25 - 40\cos\theta + 16\cos^2$$

$$\frac{\cancel{z^2 - 2i}}{} \quad \frac{i z^2 - \frac{1}{2}i}{-\frac{1}{2} i z^2 - 1} \quad \frac{z^2 - \frac{1}{2}}{\frac{1}{2} z^2 - 1} \quad \frac{2z^2 - 1}{z^2 - 2}$$

$$\frac{2(a+ib) - 1}{a+ib - 2} \qquad \frac{2e^{i\theta} - 1}{e^{i\theta} - 2}$$

$$(e^{i\theta} - 2)(e^{-i\theta} - 2) \quad 1 + 4 - 2(e^{i\theta} + e^{-i\theta}) = 5 - 4\cos\theta$$

$$(2e^{i\theta} - 1)(e^{-i\theta} - 2) = 2 + 2 - e^{-i\theta} - 4e^{i\theta}$$

$$4 - \cos\theta - 4\cos\theta + (\sin\theta - 4\sin\theta) i$$

$$f_0(z) = \frac{2z^2-1}{z^2-2} \qquad f_0(0) = \frac{1}{2}$$

$$f_1(z) = \frac{\frac{2z^2-1}{z^2-2} - \frac{1}{2}}{z\left\{1 - \frac{1}{2}\cdot\frac{2z^2-1}{z^2-2}\right\}} = \frac{4z^2-2-z^2+2}{z\left\{2z^2-4-2z^2+1\right\}}$$

$$= \frac{3z}{-3} = -z \qquad f = \frac{z+1}{z-1} \quad \frac{1+z}{1-z}$$

$$f_2(z) = \frac{-z}{z.1} = -1 \qquad \frac{f-1}{f+1} = z \quad |f|<1 \rightarrow Re(z)<0$$

$$\boxed{w(z) = \exp\left\{\frac{z-1}{z+1}\right\} \text{ maps } |z|<1 \text{ into } |w|<1}$$

$$\frac{z-\rho_n^*}{\rho_n z - 1}$$

example of nonrational NS fn.

assume $N^{(0)}(z) = \prod_{\nu=0}^{r}(z+a_\nu)$

$D^{(0)}(z) = \prod_{\nu=0}^{r}(\bar{a}_\nu z+1)$

$$D^{(1)}(z) = \left\{\bar{D}_0^{(0)} D^{(0)}(z) - \bar{N}_0^{(0)} N^{(0)}(z)\right\} D_0^{(0)}$$

coeff^t of z^r $\quad 1 \prod_{\nu=0}^{r}\bar{a}_\nu - \prod_{\nu=0}^{r}\bar{a}_\nu . 1 = 0$

Th. f rational NS fn. algorithm terminates

(all rational NS fns. may be decomposed into products of primary factors of form $\left\{\frac{z+a_\nu}{\bar{a}_\nu z+1}\right\}e^{i\phi_\nu}$ $|a_\nu|<1$)

algorithm knocks down degrees of num & denom by at least one at each stage)