

Generalisations of the B- and δ -algorithm integration processes

To approximate $I(\psi, \mu) = \int_{\mu} \psi(\mu') d\mu'$, set

(1) $I(\psi, \mu) = f(\mu) J(\mu)$ where $J(\mu)$ approximately constant

(2) $-\psi(\mu) = \partial f(\mu) \cdot J(\mu) + f(\mu) \partial J(\mu)$

Assume $\{e_\nu\}$ is

(3) $\frac{\psi(\mu)}{\partial f(\mu)} = \sum e_\nu \{f(\mu)\}^\nu$ known, and set

(4) $J(\mu) = \sum d_\nu \{f(\mu)\}^\nu$, so that

(5) $\partial J(\mu) = [\sum_1 d_\nu \nu \{f(\mu)\}^{\nu-1}] \partial f(\mu)$. From (2)

(6) $-\sum e_\nu \{f(\mu)\}^\nu = \sum d_\nu \{f(\mu)\}^\nu + \sum d_\nu \nu \{f(\mu)\}^{\nu-1}$, i.e.

(7) $d_\nu(\nu+1) = -e_\nu$ ($\nu \geq 1$) and

(8) $I(\psi, \mu) = -\sum_{\nu=1} e_\nu \{f(\mu)\}^{\nu+1}$. Setting

(9) $\Xi_\tau(\mu) = \sum_{\nu=\tau} e_\nu \nu(\nu-1)\dots(\nu-\tau+1) \{f(\mu)\}^{\nu-\tau}$, (8) becomes

(10) $I(\psi, \mu) = \sum_{\nu=1} \frac{(-1)^\nu}{\nu!} \{f(\mu)\}^\nu \Xi_{\nu-1}(\mu)$. The $\Xi_\tau(\mu)$ satisfy

(11) $\Xi_0(\mu) = \frac{\psi(\mu)}{\partial f(\mu)}$ $\Xi_{\tau+1}(\mu) = \frac{1}{\partial f(\mu)} \partial \Xi_\tau(\mu)$. Set

(12) $\Xi_\tau(\mu) = \sum_{s=0}^{\tau} A_s^{(\tau)}(\mu) \partial^s \psi(\mu)$. Then from (11)

(13) $A_0^{(\tau+1)}(\mu) = \frac{1}{\partial f(\mu)} \partial A_0^{(\tau)}(\mu)$

$$(14) A_s^{(\tau+1)}(\mu) = \frac{1}{\partial f(\mu)} \{ \partial A_s^{(\tau)}(\mu) + A_s^{(\tau)}(\mu) \} \quad s=1, \dots, \tau$$

$$(15) A_{\tau+1}^{(\tau+1)}(\mu) = \frac{1}{\partial f(\mu)} A_{\tau}^{(\tau)}(\mu) \quad \text{so that } A_{\tau}^{(\tau)}(\mu) = \frac{1}{\{\partial f(\mu)\}^{\tau+1}}$$

From (10)

$$(16) I(\psi, \mu) = \sum_1 \frac{(-1)^p}{p!} \{f(\mu)\}^p \sum_{s=0}^{p-1} A_s^{(\omega-1)}(\mu) \partial^s \psi(\mu) \quad \text{or}$$

$$(17) I(\psi, \mu) = \sum_1 \left[\sum_{j=p}^{\infty} \frac{(-1)^j}{j!} \{f(\mu)\}^j A_{j-p}^{(j-1)}(\mu) \right] \partial^{p-1} \psi(\mu), \text{ i.e.}$$

$$(18) I(\psi, \mu) = \sum_1 G_p(\mu) \partial^{p-1} \psi(\mu) \quad \text{where}$$

$$(19) G_p(\mu) = \sum_{j=p}^{\infty} \frac{(-1)^j}{j!} \{f(\mu)\}^j A_{j-p}^{(j-1)}(\mu). \quad \text{Then}$$

$$(20) \partial G_p(\mu) = - \left[\frac{(-1)^{p-1}}{(p-1)!} \partial f(\mu) \{f(\mu)\}^{p-1} A_{p-1}^{(\omega-1)}(\mu) \right]$$

$$- \sum_{j=p}^{\infty} \frac{(-1)^j}{j!} \partial f(\mu) \{f(\mu)\}^j A_{j-p}^{(j)}(\mu)$$

$$+ \sum_{j=p}^{\infty} \frac{(-1)^j}{j!} \{f(\mu)\}^j \partial A_{j-p}^{(j-1)}(\mu). \quad \text{But}$$

$$(21) \partial f(\mu) A_{j-p}^{(j)}(\mu) = \partial A_{j-p}^{(j-1)}(\mu) + A_{j-p}^{(j-1)}(\mu) \quad j=p, p+1, \dots$$

$$(22) A_{p-1}^{(\omega-1)}(\mu) = 1/\{\partial f(\mu)\}^p \quad \text{so that}$$

$$(23) \partial G_p(\mu) = \frac{(-1)^p}{(p-1)!} \left\{ \frac{f(\mu)}{\partial f(\mu)} \right\}^{p-1} - \sum_{j=p}^{\infty} \frac{(-1)^j}{j!} \{f(\mu)\}^j \partial A_{j-p}^{(j-1)}(\mu)$$

$$- \sum_{j=p}^{\infty} \frac{(-1)^j}{j!} \{f(\mu)\}^j A_{j-p}^{(j-1)}(\mu) + \sum_{j=p}^{\infty} \frac{(-1)^j}{j!} \{f(\mu)\}^j \partial A_{j-p}^{(j-1)}(\mu)$$

$$(24) \mathcal{D}G_\nu(\mu) = \frac{(-1)^\nu}{(\nu-1)!} \left\{ \frac{f(\mu)}{\mathcal{D}f(\mu)} \right\}^\nu - G_\nu(\mu) \text{ and}$$

$$(25) G_\nu(\mu) = \frac{(-1)^\nu}{(\nu-1)!} \sum_{r=0}^{\infty} (-2)^r \left\{ \frac{f(\mu)}{\mathcal{D}f(\mu)} \right\}^\nu$$

If (18) true then

$$-\psi(\mu) = \sum_1 \left[\mathcal{D}G_\nu(\mu) \mathcal{D}^{\nu-1} \psi(\mu) + G_\nu(\mu) \mathcal{D}^\nu \psi(\mu) \right]$$

i.e. if (24) true

$$(*) \quad -\psi(\mu) = \sum_1 \left[\frac{(-1)^\nu}{(\nu-1)!} \left\{ \frac{f(\mu)}{\mathcal{D}f(\mu)} \right\}^\nu \mathcal{D}^{\nu-1} \psi(\mu) \right]$$

$$- \sum_1 G_\nu(\mu) \mathcal{D}^{\nu-1} \psi(\mu) + \sum_1 G_\nu(\mu) \mathcal{D}^\nu \psi(\mu)$$

$$G_1(\mu) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \left\{ \frac{f(\mu)}{\mathcal{D}f(\mu)} \right\}^j A_0^{(j-1)}(\mu)$$

Coefft of $\psi(\mu)$ on RHS of (*) is $-\frac{f(\mu)}{\mathcal{D}f(\mu)} - G_1(\mu)$

$$\mu' = f(\mu) \quad \frac{dA(\mu')}{d\mu'} = \frac{dA}{d\mu} \cdot \frac{d\mu}{d\mu'} = \frac{1}{\mathcal{D}f(\mu)} \frac{dA}{d\mu}$$

$$A_0^0(\mu) = \frac{1}{\mathcal{D}f(\mu)} \quad A_0^{(s)}(\mu) = \left(\frac{\partial}{\partial f} \right)^s \frac{1}{\mathcal{D}f(\mu)}$$

$$= \left(\frac{d}{df} \right)^s \frac{d\mu}{df} = \frac{d^{s+1} \mu}{df^{s+1}}$$

$$G_1(\mu) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} f^j \frac{d^j \mu}{df^j}$$

$$\mu(f-f) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} f^j \frac{d^j(\mu)}{df^j} = \mu(0)$$

$$\therefore \cancel{1 + G_1(\mu) = \mu(0)} \quad \mu(f) + G_1(\mu) = \mu(0)$$

$$\text{coefft of } \psi(\mu) \text{ is } -\frac{f(\mu)}{\partial f(\mu)} \cancel{1} - \mu(0) + \mu(f)$$

where $\mu(0) =$ that μ for which $f(\mu) = 0$

$$I(\psi, \mu) = e^{-\sigma\mu} J(\mu)$$

$$-\psi(\mu) = e^{-\sigma\mu} \partial J(\mu) - \sigma e^{-\sigma\mu} J(\mu)$$

$$\sigma J(\mu) - \partial J(\mu) = e^{\sigma\mu} \psi(\mu)$$

$$\textcircled{\alpha} \quad J(\mu) = \sum \epsilon^{-\lambda-1} \partial^\lambda \{ e^{\sigma\mu} \psi(\mu) \}$$

$$I(\psi, \mu) = f(\mu) J(\mu)$$

$$\textcircled{\ast} \quad -\psi(\mu) = \partial f(\mu) J(\mu) + f(\mu) \partial J(\mu)$$

$$\text{if } J(\mu) = \sum \left\{ -\frac{f(\mu)}{\partial f(\mu)} \right\}^{\lambda+1} \partial^\lambda \left\{ \frac{\psi(\mu)}{f(\mu)} \right\}$$

is \neq $\textcircled{\ast}$ satisfied

$$\partial_r(\psi, \mu) = e^{-\sigma\mu} \sum_{\lambda=0}^{r-1} \epsilon^{-\lambda-1} \partial^\lambda \{ e^{\sigma\mu} \psi(\mu) \}$$

mit $f(u) = e^{-\beta u}$ \otimes may also be written as

$$J(u) = \sum \epsilon^{-\beta} \mathcal{D}^{\beta} \left\{ \frac{\psi(u)}{\epsilon e^{-\beta u}} \right\} = - \dots$$

$$- \sum \left\{ - \frac{f(u)}{\mathcal{D}f(u)} \right\}^{\beta} \mathcal{D}^{\beta} \left\{ \frac{\psi(u)}{\mathcal{D}f(u)} \right\}$$

or indeed as

$$\sum \mathcal{D}^{\beta} \left\{ \frac{\psi(u)}{\epsilon^{2+\beta} e^{-\beta u}} \right\} = \sum \mathcal{D}^{\beta} \left\{ \frac{\psi(u)}{\mathcal{D}^{2+\beta} f(u)} \right\} (-1)^{\beta+1}$$

$$\# \frac{\partial I}{\partial f} = J(u) + f \frac{dJ}{df}$$

$$- \psi \cdot \frac{du}{df} = J(u) + f \frac{dJ}{df}$$

$$J(u) + \frac{f(u)}{\mathcal{D}f(u)} \mathcal{D}J(u) = J(u) + \frac{dJ(u)}{dG(u)}$$

$$\frac{f(u)}{df(u)} \cdot \frac{d}{df} = \frac{1}{dG}$$

$$dG(u) = \frac{df(u)}{f(u)}$$

$$G(u) = \int^u \frac{df(u)}{f(u)} du$$

$$= \ln \{ f(u) \}$$

$$\frac{dI}{dG} = \frac{df}{dG} J + f \frac{dJ}{dG} \quad G = \ln(f)$$

$$\frac{dG}{df} = \frac{1}{f}$$

$$\frac{dI}{dG} = \frac{dI}{du} \cdot \frac{du}{df} \cdot \frac{df}{dG}$$

$$f = e^{-\sigma\mu}$$

$$G = -\sigma\mu$$

$$-\psi(\mu) \cdot \frac{f}{\partial f} = f J + f \frac{dJ}{dG} \quad \left| \quad \frac{d}{dG} = -\frac{1}{\sigma} \frac{d}{d\mu} \right.$$

$$J + \frac{dJ}{dG} = -\frac{\psi}{\partial f}$$

$$J = \sum (-1)^{j+1} \left(\frac{d}{dG} \right)^j \left\{ \frac{\psi}{\partial f} \right\}$$

$$f = (\mu - \sigma)^{-1/\sigma} \quad G = -\ln(\mu - \sigma) \quad \frac{dG}{d\mu} = -\frac{1}{\mu - \sigma}$$

$$\frac{d}{dG} = \left\{ \frac{d}{d\mu} \right\} \frac{d\mu}{dG} = -(\mu - \sigma) \partial$$

$$\frac{\psi}{\partial f} = \frac{\psi}{2} - (\mu - \sigma)^2 \psi$$

$$\phi_1(\psi, \mu) = (\mu - \sigma)^{-1} (\mu - \sigma)^2 \psi \quad \checkmark$$

$$\phi_2(\psi, \mu) = (\mu - \sigma)^{-1} \left\{ (\mu - \sigma)^2 \psi + (\mu - \sigma) \partial \left\{ (\mu - \sigma)^2 \psi \right\} \right\}$$

$$= (\mu - \sigma)^{-1} \left\{ (\mu - \sigma)^2 \psi + 2(\mu - \sigma)^2 \psi + (\mu - \sigma)^3 \partial \psi \right\}$$

$$b_r^{(m)} = -(\mu - \epsilon)^m \left[\mathcal{D}^{m+r} \{ (\mu - \epsilon)^r I(\mu) \} \right] / (m+r)! \quad (r \geq \bar{m}, m \geq \bar{m})$$

$$b_r^{(0)} = \sum_0^{r-1} b_r^{(1)} \quad r = r+1$$

$$\hat{b}_r^{(0)m} = \sum_0^r \binom{r}{\nu} \frac{(\mu - \epsilon)^{\nu}}{\nu!} \mathcal{D}^{\nu} S(\mu)$$

$$\mathcal{D}^r \{ (\mu - \epsilon)^r S(\mu) \} / r! = \sum \binom{r}{\nu} \frac{(\mu - \epsilon)^{\nu}}{\nu!} \mathcal{D}^{\nu} S(\mu)$$

$$\mathcal{D}^{r-1} (\mu - \epsilon)^r = r(r-1) \dots (\mu)(\mu - \epsilon)^2$$

presumably

$$\hat{b}_r^{(m)} = (\mu - \epsilon)^m \left[\mathcal{D}^{m+r} \{ (\mu - \epsilon)^r S(\mu) \} \right] / (m+r)!$$

$$\hat{b}_r^{(1)} = (\mu - \epsilon) \left[\mathcal{D}^{r+1} \{ (\mu - \epsilon)^r S(\mu) \} \right] / (r+1)!$$

$$b_r^{(0)} = \cancel{S(\mu)} + \sum_{\nu=0}^{r-1} b_r^{(1)}$$

$$b_r^{(m)'} = -\epsilon^{-m-r} e^{-\epsilon\mu} \mathcal{D}^r \{ e^{\epsilon\mu} \mathcal{D}^m I(\mu) \}$$

again

$$\hat{b}_r^{(m)'} = \epsilon^{-m-r} e^{-\epsilon\mu} \mathcal{D}^r \{ e^{\epsilon\mu} \mathcal{D}^m S(\mu) \}$$

$$b_r^{(0)'} = \epsilon^{-r-1} e^{-\epsilon\mu} \mathcal{D}^r \{ e^{\epsilon\mu} \mathcal{D} S(\mu) \}$$

$$\begin{aligned}
 f_r^{(1)} &= -(\mu - \epsilon) \left[\mathcal{D}^{r+1} \{ (\mu - \epsilon)^r I(\mu) \} / (r+1)! \right] \\
 &= -(\mu - \epsilon) \sum_{\nu=0}^{r+1} \binom{r+1}{\nu} \mathcal{D}^\nu \psi(\mu) \frac{r!}{(\nu-1)!} (\mu - \epsilon)^{\nu-1} / (r+1)! \\
 \mathcal{D}^{r+1-\nu} (\mu - \epsilon)^r &= r(r-1)\dots(r-\nu+1) (\mu - \epsilon)^{\nu-1} \\
 &= \frac{r!}{(\nu-1)!} (\mu - \epsilon)^{\nu-1}
 \end{aligned}$$

$$\begin{aligned}
 f_r^{(1)} &= (\mu - \epsilon) \sum_{\nu=0}^r \frac{(r+1)!}{(\nu+1)!} \frac{r!}{(r-\nu)!} \frac{\mathcal{D}^\nu \psi(\mu) (\mu - \epsilon)^\nu}{(r+1)!} \\
 &= (\mu - \epsilon) \sum_{\nu=0}^r \binom{r}{\nu} \frac{(\mu - \epsilon)^\nu}{(\nu+1)!} \mathcal{D}^\nu \psi(\mu)
 \end{aligned}$$

$$\begin{aligned}
 f_r^{(1)'} &= \epsilon^{-r-1} e^{-\epsilon\mu} \mathcal{D}^r \{ e^{\epsilon\mu} \psi(\mu) \} \\
 &= \epsilon^{-r-1} e^{-\epsilon\mu} \sum_{\nu=0}^r \epsilon^{r-\nu} \mathcal{D}^\nu \psi(\mu) \binom{r}{\nu} e^{\epsilon\mu} \\
 &= \sum_{\nu=0}^r \binom{r}{\nu} \epsilon^{-\nu-1} \mathcal{D}^\nu \psi(\mu) \tag{2}
 \end{aligned}$$

$$f_r^{(1)} = (\mu - \epsilon)^{-1} \sum_{\nu=0}^r \binom{r}{\nu} \frac{(\mu - \epsilon)^{\nu+2}}{(\nu+1)!} \mathcal{D}^\nu \psi(\mu) \tag{1}$$

$$\begin{aligned}
 f(\mu) &= (\mu - \epsilon)^{-1} \quad \mathcal{D}^\nu f(\mu) = \frac{\nu!}{(\mu - \epsilon)^{\nu+1}} (-1)^\nu \\
 \mathcal{D}^{\nu+1} f(\mu) &= \frac{(\nu+1)!}{(\mu - \epsilon)^{\nu+2}} (-1)^{\nu+1}
 \end{aligned}$$

$$(1) = f(\mu) \sum_{\nu=0}^r \binom{r}{\nu} \frac{\partial^{\nu} \psi(\mu)}{\partial^{\nu+1} f(\mu)} (-1)^{\nu+1} \quad (3)$$

with $f(\mu) = (\mu - \epsilon)^{-1}$

$$f(\mu) = e^{-\epsilon\mu} \quad \partial^{\nu+1} f(\mu) = (-1)^{\nu+1} \epsilon^{\nu+1} e^{-\epsilon\mu}$$

(2) = (3) with $f(\mu) = e^{-\epsilon\mu}$

$$\text{If } \frac{\partial^{\nu} \psi(\mu)}{\partial^{\nu+1} f(\mu)} = \sum_{\tau=0}^{r'} a_{\tau} \nu^{\tau} \quad (3) = 0 \text{ for } \nu > r'$$

$$= \sum_{\tau=0}^{r'} a_{\tau}(\mu) \nu^{\tau}$$

$$\sum_{r=0}^{\infty} f(\mu) \sum_{\nu=0}^r \binom{r}{\nu} \frac{\partial^{\nu} \psi(\mu)}{\partial^{\nu+1} f(\mu)} (-1)^{\nu+1}$$

$$= f(\mu) \sum_{\nu=0}^{\infty} \frac{\partial^{\nu} \psi(\mu)}{\partial^{\nu+1} f(\mu)} \sum_{r=0}^{\infty} \binom{r}{\nu}$$

$$\bullet \sum_{\nu=0}^{\infty} (-1)^{\nu} \epsilon^{\nu} u_0 = \frac{1}{1+\epsilon} u_0 = \frac{1}{2+\Delta} u_0 =$$

$$\frac{1}{2} \sum_{\nu=0}^{\infty} \{ \epsilon^{\nu} \Delta^{\nu} u_0 \} \frac{1}{2^{\nu}} (-1)^{\nu}$$

$$(3) = -f(\mu) \Delta^r \left\{ \frac{\partial^{\nu} \psi(\mu)}{\partial^{\nu+1} f(\mu)} \right\}$$

$$\mathcal{D}^{\nu} \psi(\mu) = \mathcal{D}^{\nu+1} f(\mu) \left\{ \sum_{\tau=0}^{\nu} a_{\tau}(\mu) \mu^{\tau} \right\}$$

$$\psi(\mu) = \sum_{i=1}^r A_i (\mu - \epsilon)^{-i-1} i!$$

$$\mathcal{D}^{\nu} \psi(\mu) = \sum_{i=1}^r A_i (i+\nu)! (\mu - \epsilon)^{-i-\nu-1}$$

$$f(\mu) = (\mu - \epsilon)^{-1} \quad \mathcal{D}^{\nu+1} f(\mu) = (\nu+1)! (\mu - \epsilon)^{-\nu-2}$$

$$\sum_{\tau=0}^{\nu} a_{\tau}(\mu) \mu^{\tau} = \sum_{i=1}^r A_i \frac{(i+\nu)!}{(\nu+1)!} (\mu - \epsilon)^{-i-\nu-1+\nu+2}$$

$$= \sum_{i=1}^r A_i \prod_{j=2}^i (\nu+j) (\mu - \epsilon)^{-i+1}$$

$$\psi(\mu) = e^{-\epsilon\mu} \sum_{i=0}^{r-1} A_i \mu^i$$

$$\mathcal{D}^{\nu} \psi(\mu) = e^{-\epsilon\mu} \sum_{i=0}^{r-1} A_i \sum_{j=0}^{\nu} \frac{i!}{(i-j)!} \mu^{i-j} (-\epsilon)^{\nu-j} \binom{\nu}{j}$$

$$f(\mu) = e^{-\epsilon\mu} \quad \mathcal{D}^{\nu+1} f(\mu) = (-\epsilon)^{\nu+1} e^{-\epsilon\mu}$$

$$\sum_{\tau=0}^{\nu} a_{\tau}(\mu) \mu^{\tau} = \sum_{i=0}^{r-1} A_i \sum_{j=0}^{\nu} \frac{i!}{(i-j)!} \mu^{i-j} (-\epsilon)^{\nu-j-1} \binom{\nu}{j}$$

$$= \sum_{i=0}^{r-1} A_i \sum_{j=0}^{\nu} \binom{i}{j} \mu^{i-j} (-\epsilon)^{\nu-j-1} \nu(\nu-1)\dots(\nu-j+1)$$

↑
degree j

j > i and i > r-1.

$$\psi(\mu) = \sum_{i=1}^r A_i i! (\mu - \epsilon)^{-i-1}$$

$$f(\epsilon, \mu) = \frac{1}{\mu - \epsilon} \quad \frac{\partial f(\epsilon, \mu)}{\partial \epsilon} = \frac{1}{(\mu - \epsilon)^2}$$

$$\frac{1}{\mu - \epsilon} = \frac{\partial f(\epsilon, \mu)}{\partial \epsilon} / f(\epsilon, \mu)$$

$$\psi(\mu) = f(\epsilon, \mu) \sum_{i=1}^r A_i i! \left\{ \frac{\partial \epsilon f}{f} \right\}^i$$

$$\psi(\mu) = e^{-\epsilon \mu} \sum_{i=0}^{r-1} A_i \mu^i$$

$$f(\epsilon, \mu) = e^{-\epsilon \mu} \quad \frac{\partial \epsilon f}{f} = -\mu$$

$$\psi(\mu) = f(\epsilon, \mu) \sum_{i=0}^{r-1} A_i (-1)^i \left\{ \frac{\partial \epsilon f}{f} \right\}^i$$

$$\psi(\mu) = f \sum_{i=0}^r c_i \left(\frac{\partial \epsilon f}{f} \right)^i$$

$$\partial \psi = \partial f \cdot \sum_{i=0}^r c_i \left(\frac{\partial \epsilon f}{f} \right)^i + f \sum_{i=0}^r c_i f^i \left(\frac{\partial \epsilon f}{f} \right)^{i-1} \partial \left(\frac{\partial \epsilon f}{f} \right)$$

$$+ f \sum_{i=0}^r c_i i \frac{f^i (\partial \epsilon f)^{i-1} \partial \partial \epsilon f - (\partial \epsilon f)^i f^{i-1} \partial f}{f^{2i}}$$

Simpler: $\psi = f \sum c_i \frac{\partial \epsilon f}{f}$ shows $\frac{\partial' \psi}{\partial' \mu f} = \text{const}$

$$\psi = c \partial \epsilon f \quad \partial' \psi = c \partial' \partial \epsilon f$$

is const (i.e. ind. of ν) if $c' \partial' \partial \epsilon f = \partial' \psi$

$$f^r \psi = \sum a_i f^{i+r} (\partial_0 f)$$

$$f^r \psi = \sum_{i=0}^r c_i f^{r+1-i} (\partial_0 f)^i$$

$$\partial^j \{f^r \psi\} = \sum_{i=0}^r c_i \sum_{j=0}^i \binom{i}{j} \partial^j \{(\partial_0 f)^i\} \partial^{i-j} (f^{r+1-i})$$

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$$\psi = c_0 f + c_1 \partial_0 f$$

$$\partial^2 \psi = c_0 \partial^2 f + c_1 \partial^2 \partial_0 f$$

$$f \psi = c_0 f^2 + c_1 \partial f \partial_0 f + c_2 (\partial_0 f)^2$$

$$\underline{\partial^2 f \cdot \psi +}$$

$$c_1(\mu) = \frac{\partial c_0(\mu) \partial f(\mu)}{\partial^2 f(\mu)}$$

$$\underline{\psi(\mu) = \sum_{i=0}^{r-1} c_i \partial_0^i f(\mu)} \quad \parallel \quad 2 \partial c_0(\mu) \partial^2 f(\mu) + \partial^2 c_0(\mu) \partial f(\mu)$$

$$= 2 c_1(\mu) \partial^2 f(\mu)$$

$$= 2 \partial c_0(\mu) \partial f(\mu)$$

$$\psi(\mu) = \partial \partial f(\mu) \quad : (3) \text{ is zero}$$

$$\text{when } \partial^2 \psi(\mu) = \{c_0 + c_1 \nu\} \partial^{2+\nu} f(\mu)$$

$$\psi(\mu) = c_0(\mu) \partial f(\mu) \quad | \quad \partial \psi(\mu) = \partial c_0(\mu) \partial f(\mu) + c_0(\mu) \partial^2 f(\mu)$$

$$\partial \psi(\mu) = \{c_0(\mu) \partial f + c_1(\mu)\} \partial^2 f(\mu)$$

$$\partial^2 \psi(\mu) = \{c_0(\mu) + 2c_1(\mu)\} \partial^3 f(\mu) \text{ etc}$$

$$\partial^2 \psi(\mu) = c_0 \partial^3 f(\mu) + 2 \partial c_0(\mu) \partial^2 f(\mu) + \partial^2 c_0(\mu) \partial f(\mu)$$

$$2 \partial_{\mu} \psi \{ \partial^2 f(\mu) - \partial f(\mu) \} + \partial^2 \psi(\mu) \partial f(\mu) = 0$$

$$\psi(\mu) = \{a + bf\} \partial f(\mu) \frac{df(\mu)}{d\mu}$$

$$\frac{d^2 \psi}{df^2} = \{a + bf\} \frac{d^2}{df^2} \frac{df}{d\mu} + \nu b \frac{d^{\nu-1}}{df^{\nu-1}} \frac{df}{d\mu}$$

~~$$C(\mu) = A(\mu) / f(\mu)$$~~

$$\partial C(\mu) = C(\mu) \partial f(\mu) = F(\mu)$$

$$2 C(\mu) \{ \partial F(\mu) - F(\mu) \} + \partial C(\mu) F(\mu) = 0$$

$$\frac{\partial C(\mu)}{C(\mu)} = 2 \left\{ 1 - \frac{\partial F(\mu)}{\partial F(\mu)} \right\}$$

$$\ln \{ C(\mu) \} = 2(\mu - k) - 2 \ln F(\mu)$$

$$= \ln \{ (\mu - k)^2 \} - \ln \{ F(\mu) \}^2 + \text{const} \exp 2(\mu - k)$$

$$= \ln \frac{(\mu - k)^2}{\{ F(\mu) \}^2} \ln \left\{ \frac{\exp(2(\mu - k))}{F(\mu)^2} \right\}$$

$$C(\mu) = \left\{ \frac{(\mu - k)^2}{F(\mu)} \right\}^2 \quad C(\mu) = \exp \left\{ \frac{e^{\mu - k}}{F(\mu)} \right\}^2$$

$$\bullet b_r^{(0)} = \sum_{j=1}^r \binom{r}{j} \frac{(\mu - \epsilon)^j}{j!} \partial^{j-1} \psi(\mu)$$

$$b_r^{(0)'} = \sum_{j=1}^r \binom{r}{j} \epsilon^{-j} \partial^{j-1} \psi(\mu)$$

$$f(\mu) = \frac{1}{\mu - \epsilon} \quad \partial^j f(\mu) = (-1)^j \frac{j!}{(\mu - \epsilon)^{j+1}}$$

$$b_r^{(0)} = f(\mu) \sum_{j=1}^r \binom{r}{j} \frac{\partial^{j-1} \psi(\mu)}{\partial^j f(\mu)} (-1)^j \quad (*)$$

$$f(\mu) = e^{-\sigma \mu} \quad \partial^j f(\mu) = (-1)^j \sigma^j e^{-\sigma \mu} \rightarrow (**)$$

$\frac{1}{\epsilon}$

$$b_r^{(0)} = 0 \quad b_0^{(0)} = (-1)^m \partial^{m-1} \psi(\mu) / \partial^m f(\mu)$$

$$\frac{\partial^{(0)}}{\partial r} = \frac{\partial f(\mu)}{f(\mu)} \quad -\epsilon$$

$$\partial b_r^{(0)} = \frac{\partial f(\mu)}{f(\mu)} b_r^{(0)} + f(\mu) \sum_{j=1}^r \binom{r}{j} \frac{\partial^j f \partial^j \psi - \partial^{j-1} \psi \partial^{j+1} f}{(\partial^j f)^2} (-1)^j$$

$$b_{r+1}^{(0)'} = b_r^{(0)'} + \epsilon^{-1} \{ \psi(\mu) + \partial b_r^{(0)'} \}$$

$$b_{r+1}^{(0)} - b_r^{(0)} = f(\mu) \sum_{j=0}^r \binom{r}{j} \frac{\partial^j \psi(\mu)}{\partial^{j+1} f(\mu)} (-1)^{j+1}$$

$$\partial b_r^{(0)'} = \epsilon \{ b_{r+1}^{(0)'} - b_r^{(0)'} \}' - \psi(\mu)$$

$$f_{br}^{(\omega)} = b_r^{(\omega)} + \frac{\mu - \epsilon}{r+1} \{ \psi(\mu) + \mathcal{D} b_r^{(\omega)} \}$$

$$(\mu - \epsilon) \sum_{\nu=0}^r \binom{r}{\nu} \frac{(\mu - \epsilon)^\nu}{(\nu+1)!} \mathcal{D}^\nu \psi(\mu) =$$

$$\frac{\mu - \epsilon}{r+1} \left\{ \psi(\mu) + \sum_{\nu=1}^r \binom{r}{\nu} \frac{(\mu - \epsilon)^{\nu-1}}{(\nu-1)!} \mathcal{D}^{\nu-1} \psi(\mu) + \sum_{\nu=1}^r \binom{r}{\nu} \frac{(\mu - \epsilon)^\nu}{\nu!} \mathcal{D}^\nu \psi(\mu) \right\}$$

$$\frac{f(\mu)}{b_r^{(\omega)}} \quad f(\mu) = \frac{1}{\mu - \epsilon} \quad \mathcal{D}^{r+1} f(\mu) = \frac{(r+1)!}{(\mu - \epsilon)^{r+2}} (-1)^{r+1}$$

$$\frac{\mu - \epsilon}{r+1} = - \mathcal{D}^r f(\mu) / \mathcal{D}^{r+1} f(\mu)$$

$$\text{look for } b_{r+1}^{(\omega)} = b_r^{(\omega)} - \frac{\mathcal{D}^r f(\mu)}{\mathcal{D}^{r+1} f(\mu)} \{ \psi(\mu) + \mathcal{D} b_r^{(\omega)} \}$$

$$f(\mu) \sum_{\nu=0}^r \binom{r}{\nu} \frac{\mathcal{D}^\nu \psi(\mu)}{\mathcal{D}^{\nu+1} f(\mu)} (-1)^{\nu+1} =$$

$$\frac{\mathcal{D}^r f(\mu)}{\mathcal{D}^{r+1} f(\mu)} \left[\psi(\mu) + \mathcal{D} f(\mu) \sum_{\nu=1}^r \binom{r}{\nu} \frac{\mathcal{D}^{\nu-1} \psi(\mu)}{\mathcal{D}^\nu f(\mu)} (-1)^\nu \right]$$

$$+ f(\mu) \sum_{\nu=1}^r \binom{r}{\nu} \frac{\mathcal{D}^\nu \psi(\mu)}{\mathcal{D}^\nu f(\mu)} (-1)^\nu - \frac{f(\mu)}{\mathcal{D}^0 f(\mu)} \sum_{\nu=1}^r \binom{r}{\nu} \frac{\mathcal{D}^{\nu-1} \psi(\mu) \mathcal{D}^{\nu+1} f(\mu)}{\{\mathcal{D}^\nu f(\mu)\}^2} (-1)^\nu$$

ψ alone

$$\frac{f(\mu)}{\partial f(\mu)} = \frac{\partial^r f(\mu)}{\partial^{\mu} f(\mu)} \left\{ \frac{1}{\partial f(\mu)} - r + f(\mu) + \frac{\partial^2 f(\mu)}{(\partial f(\mu))^2} \right\}$$

$$f(\mu) = (\mu - \epsilon)^{-1}$$

$$\rightarrow -(\mu - \epsilon) = -\frac{\mu - \epsilon}{r+1} \{1 - r + 2r\}^r$$

$$f(\mu) = e^{-\epsilon/\mu} \quad \frac{1}{-\epsilon} = \frac{1}{-\epsilon} \{1 - r + r\}^r$$

$$\partial f(\mu) \cdot f(\mu) \cdot \partial^{r+1} f(\mu) = \partial^r f(\mu) \left[r f(\mu) \partial^2 f(\mu) - (r-1) \{\partial f(\mu)\}^2 \right]$$

$r=0$ f unconstrained

$r=1$ "

$$\boxed{r=2 \quad f(\mu) \partial f(\mu) \partial^3 f(\mu) = \partial^2 f(\mu) \{ 2 f(\mu) \partial^2 f(\mu) - \{\partial f(\mu)\}^2 \}}$$

$$\{\partial f(\mu)\}^2 \partial^3 f(\mu) + f(\mu) \partial^2 f(\mu) \partial^3 f(\mu) + f(\mu) \partial f(\mu) \partial^4 f(\mu)$$

$$= \partial^2 f(\mu) \{ \partial^2 f(\mu) \}^2 + 4 f(\mu) \partial^2 f(\mu) \partial^3 f(\mu)$$

$$- \partial^3 f(\mu) \{ \partial f(\mu) \}^2 - 2 \partial f(\mu) \{ \partial^2 f(\mu) \}^2$$

$$f(\mu) \partial f(\mu) \partial^4 f(\mu) = \partial^3 f(\mu) \{ 3 f(\mu) \partial^2 f(\mu) - 2 \{ \partial f(\mu) \}^2 \}$$

$$\begin{aligned}
 & \{ \partial f(u) \}^2 \partial^{r+1} f(u) + f(u) \partial^2 f(u) \partial^{r+1} f(u) + f(u) \partial f(u) \partial^{r+2} f(u) \\
 &= r \partial f(u) \partial^2 f(u) \partial^r f(u) + r f(u) \partial^3 f(u) \partial^r f(u) + r f(u) \partial^2 f(u) \partial^{r+1} f(u) \\
 &- \cancel{2} \{ 2(r-1) \partial f(u) \partial^2 f(u) \partial^r f(u) \} - (r-1) \{ \partial f(u) \}^2 \partial^{r+1} f(u)
 \end{aligned}$$

$$f(u) \partial f(u) \partial^{r+2} f(u) =$$

~~$$\partial^2 f(u) \left\{ (r-1) f(u) \partial^{r+1} f(u) + (2r-2) \partial f(u) \partial^r f(u) \right\}$$~~

$$\partial^2 f(u) \left[(r-1) f(u) \partial^{r+1} f(u) - (r-2) \partial f(u) \partial^r f(u) \right]$$

$$- r \{ \partial f(u) \}^2 \partial^{r+1} f(u) + r f(u) \partial^3 f(u) \partial^r f(u)$$

$$f(u) \left[\partial f(u) \partial^3 f(u) - \partial^2 f(u) \right] = \partial^2 f(u) \left[f(u) \partial^2 f(u) - \partial f(u)^2 \right]$$

$$f(u) \partial \left\{ \frac{\partial^2 f(u)}{\partial f(u)} \right\} \cdot \{ \partial f(u) \}^2 = \partial^2 f(u) \left\{ \partial \frac{\partial f(u)}{f(u)} \right\} f(u)^2$$

$$\left| \begin{array}{cc}
 f(u) \partial \left\{ \frac{\partial f(u)}{f(u)} \right\} & \partial f(u) \partial \left\{ \frac{\partial^2 f(u)}{\partial f(u)} \right\} \\
 \partial f(u) & \partial^2 f(u)
 \end{array} \right| = 0$$

$$f(x) \partial f(x) \partial^2 f(x) = \partial^0 f(x) f(x) \partial^2 f(x)$$

$$f(x) \{\partial^2 f(x)\}^2 + \{\partial f(x)\}^2 \partial^2 f(x) + f(x) \partial f(x) \partial^3 f(x)$$

$$= \partial \{f(x) \partial f(x) \partial^2 f(x)\}$$

$$\partial \{f(x) \partial f(x) \partial^2 f(x)\} - f(x) \{\partial^2 f(x)\}^2 - \{\partial f(x)\}^2 \partial^2 f(x)$$

$$= 2 f(x) \{\partial^2 f(x)\}^2 - \{\partial^2 f(x) \partial f(x)\}^2$$

i.e. condition imposed for $r=2$ is equivalent to

$$\partial \{f(x) \partial f(x) \partial^2 f(x)\} = 3 f(x) \{\partial^2 f(x)\}^2$$

accepting condition of p. 23 examine coeffs of $\partial^{p-1} \psi(x)$ with $p > 1$

$$f(x) \sum_{\nu=0}^{r+1} \binom{r}{\nu-1} \frac{\partial^{\nu-1} \psi(x)}{\partial^\nu f(x)} (-1)^{\nu-1} =$$

$$\frac{\partial^r f(x)}{\partial^{r+1} f(x)} \left[\psi(x) + \partial f(x) \sum_{\nu=1}^r \binom{r}{\nu} \frac{\partial^{\nu-1} \psi(x)}{\partial^\nu f(x)} (-1)^\nu \right]$$

$$+ f(x) \sum_{\nu=2}^{r+1} \binom{r}{\nu-1} \frac{\partial^{\nu-2} \psi(x)}{\partial^{\nu-1} f(x)} (-1)^{\nu-1} - f(x) \sum_{\nu=1}^r \binom{r}{\nu} \frac{\partial^{\nu-1} \psi(x)}{\{\partial^\nu f(x)\}^2} \partial^{2r} f(x) (-1)^\nu$$

$$- f(x) \binom{r}{\nu-1} \frac{1}{\partial^\nu f(x)} = \frac{\partial^r f(x)}{\partial^{r+1} f(x)} \left[\partial f(x) \binom{r}{\nu} \frac{1}{\partial^\nu f(x)} \right]$$

$$- \left[f(x) \binom{r}{\nu-1} \frac{1}{\partial^{\nu-1} f(x)} - f(x) \binom{r}{\nu} \frac{\partial^{\nu+1} f(x)}{\{\partial^\nu f(x)\}^2} \right]$$

$$\binom{r}{\nu} = \frac{r!}{(r-\nu)! \nu!} \binom{r}{\nu-1} = \frac{r!}{(r-\nu+1)! (\nu-1)!} \quad \binom{r}{\nu} = \frac{(r-\nu+1)}{\nu} \binom{r}{\nu-1}$$

$$\frac{f(x)}{\partial^\nu f(x)} = \frac{\partial^r f(x)}{\partial^{r+\nu} f(x)} \left[\frac{r-\nu+1}{\nu} \frac{\partial f(x)}{\partial^2 f(x)} - \frac{f(x)}{\partial^{\nu-1} f(x)} - \frac{r-\nu+1}{\nu} \frac{\partial^{\nu+1} f(x)}{\{\partial^2 f(x)\}^2} \right]$$

$r=2 \quad \nu=1$

$$-\frac{f(x)}{\partial f(x)} = \frac{\partial^2 f(x)}{\partial^3 f(x)} \left[2-1 - 2 \frac{\partial^2 f(x) f(x)}{\{\partial f(x)\}^2} \right]$$

$$\frac{f(x)}{\partial f(x)} = \frac{\partial^2 f(x)}{\partial^3 f(x)} \left[-1 + 2 \frac{\partial^2 f(x) f(x)}{\{\partial f(x)\}^2} \right]$$

$$f(x) \partial f(x) \partial^3 f(x) = \partial^2 f(x) \left[2 \frac{\partial^2 f(x) f(x)}{\{\partial f(x)\}^2} - \{\partial f(x)\}^2 \right]$$

$\nu=1 \quad \{0, 0, 1, r-1\}$

$$\cancel{\nu f(x) \partial^\nu f(x) \partial^{r+\nu} f(x)} = \{r\} \left[r \{1, 0, 1\} - \{0, 1, 1\} - r \{0, 0, 2\} \right]$$

~~$\partial^r f(x)$~~

$$\boxed{\nu f(x) \partial^{\nu-1} f(x) \partial^\nu f(x) \partial^{r+\nu} f(x) = \partial^r f(x) \left[(r-\nu+1) \partial f(x) \partial^{\nu-1} f(x) \partial^\nu f(x) - \nu f(x) \{\partial^2 f(x)\}^2 - (r-\nu+1) f(x) \partial^{\nu-1} f(x) \partial^{\nu+1} f(x) \right]}$$

Review of // Wynn P.; The evaluation of singular and highly oscillatory integrals by use of the anti-derivative, *Calcolo*, 15 Fasc. 4 bis, 1-103 (1978) (1)

A: An anti-derivative of the function ψ of a complex variable is a function $T(\psi, \mu)$ for which $D T(\psi, \mu) = \psi(\mu)$, where $D \equiv d/d\mu$, for values of μ in a domain over which ψ is analytic. Such functions T occur naturally in many problems of applied mathematics. For example, when ψ represents the complex velocity components of irrotational ~~flow~~ fluid flow vanishing at infinity, the real and imaginary parts of T normalised by the condition that $T(\psi, \mu)$ tends to zero for large μ are the velocity potential and stream function of the flow. Again, solutions of the differential equation $p_n(D)f(\mu) = \psi(\mu)$, where p_n is a polynomial with complex number coefficients, may be exhibited, by use of the Heaviside calculus, in the form $\sum_{i=1}^m e^{\alpha_i \mu} T^{(i)}(\psi_i, \psi)$, where $T^{(i)}$ is a first or higher order anti-derivative: $T^{(i)}(\psi_i, \mu) = \Delta^{-k(i)} \psi_i(\mu)$, $\psi_i(\mu) = e^{-\alpha_i \mu} \psi(\mu)$, the $k(i)$ and α_i depending upon p_n . The same method of treatment may be extended to certain

differential equations of more complex structure, and in this way such equations, and systems of such equations, may be treated by purely algebraic methods (see, for example, the excellent work: Maslov P.; Operational methods, Moscow (1973)).

B: Subject to appropriate restrictions upon ψ , the integral (a) $I(\psi, \mu) = \int_{\mu}^{\infty} \psi(t) dt$ satisfies the equation (b) $\Delta I(\psi, \mu) = -\psi(\mu)$. $-I(\psi, \mu)$ is then an anti-derivative. Under suitable conditions,

the following function pairs ψ, I satisfy relationship (b): (B1)

$$\psi(\mu) = \sum_{z=0}^n A_z (\mu - \alpha)^{-z-2}, \quad I(\psi, \mu) = \sum_{z=0}^n (z+1)^{-1} A_z (\mu - \alpha)^{-z-1}; \quad (B2)$$

$$\psi(\mu) = \sum_{j=0}^N \sum_{z=0}^{N(j)} A_{j,z} (\mu - \alpha_j)^{-z-2}, \quad I(\psi, \mu) = \sum_{j=0}^N \sum_{z=0}^{N(j)} A_{j,z} (z+1)^{-1} (\mu - \alpha_j)^{-z-1};$$

$$B(3) \quad \psi(\mu) = e^{-\alpha\mu} \sum_{z=0}^n A_z \mu^z, \quad I(\psi, \mu) = \alpha^{-1} e^{-\alpha\mu} \sum_{z=0}^n A_z \sum_{j=0}^z \binom{z}{j} j! \alpha^{-j} \mu^{z-j};$$

$$B(4) \quad \psi(\mu) = \sum_{j=1}^N e^{-\alpha_j \mu} \sum_{z=0}^{N(j)} A_{j,z} \mu^z, \quad I(\psi, \mu) =$$

$$\sum_{j=1}^N e^{-\alpha_j \mu} \alpha_j^{-1} \sum_{z=0}^{N(j)} A_{j,z} \sum_{k=0}^z \binom{z}{k} k! \alpha_j^{-k} \mu^{z-k}. \quad \text{When the}$$

~~integral~~ integral in relationship (a) does not exist in, for example, a Riemann sense (e.g. when $-\infty < \mu < \alpha < \infty$ for the function ψ of example (B1), and for further obvious cases of the other examples) an anti-derivative may

serve to define a singular or divergent integral. When the integral exists but the integrand is highly oscillatory upon the path of integration (eg. when $\text{Re}(\alpha) \geq 0$ but $|\text{Im}(\alpha)|$ is large for the function ψ of example (B3)) so that numerical treatment by quadrature methods is ineffective, the anti-derivative may be used to assist computation. (3)

C: Four methods for constructing an anti-derivative using the value of ψ and those of its derivatives at the point μ are considered. They involve the sequences of functions

$$(C1) \phi_r(\epsilon|\psi, \mu) = \sum_{\nu=0}^r \binom{r}{\nu} (\mu - \epsilon)^{\nu} \Delta^{\nu-1} \psi(\mu) / \nu!; (C2) \rho_{2r}(\nu, \mu) =$$

$$H[\Delta^{\nu-1} \psi(\mu) / \nu!]_r / H[\Delta^{\nu+1} \psi(\mu) / (\nu+2)!]_{r-1}; \text{ where } H[\Delta^{\nu+m} \psi(\mu) / (\nu+m+1)!]_r \text{ is the } r^{\text{th}} \text{ order determinant}$$

with element $\Delta^{m+i+j-2} \psi(\mu) / (m+i+j-1)!$ in the i^{th} row and j^{th} column ($1 \leq i, j \leq r$) with $\Delta^{\nu+m} \psi(\mu) / (\nu+m+1)!$ set equal to zero when $\nu+m < 0$, and $H[\dots]_0 = 1$; (C3) $\gamma_r(\epsilon|\psi, \mu) =$

$$\sum_{\nu=1}^r \binom{r}{\nu} \epsilon^{-\nu} \Delta^{\nu-1} \psi(\mu); \text{ and } (C4) \epsilon_{2r}(\nu, \mu) = H[\Delta^{\nu-1} \psi(\mu)]_r / H[\Delta^{\nu+1} \psi(\mu)]_{r-1}$$

the determinants being defined in analogy with (C2); in each case $r=1, 2, \dots$ and in cases (C1, 3) ϵ is a disposable parameter.

D: These methods possess the following algebraic properties

(D1) below concerns the functions ψ, I of (B1) and ϕ_r of (4)

(C1) above, and so on): (D1) when $\mu \neq \alpha$, $\phi_r(\alpha|\nu, \mu) = I(\nu, \mu)$

($r > n$); (D2) when $A_{j, N(j)} \neq 0, \mu \neq \alpha_j (1 \leq j \leq N)$ and with $n =$

$\sum_{j=1}^N \{N(j)+1\}$, $\rho_{2n}(\nu, \mu) = I(\nu, \mu)$; (D3) $\alpha_r(\alpha|\nu, \mu) = I(\nu, \mu) (r > n)$;

(D4) with $A_{j, N(j)}$ and n as in (D2), $\varepsilon_{2n}(\nu, \mu) = I(\nu, \mu)$.

in addition to those provided by the algebraic results of (D),

E: Formulae (C1-4) may be given the following motivations. (E1)

The first r terms of Bürmann's series expressing $K(z) - K(\mu)$

in terms of powers of a suitable function $\mathfrak{F}(z, \mu)$ yield the

approximate relationship $K(z) - K(\mu) = \sum_{\nu=0}^{r-1} t_{\nu}(\mu, z)$ with $t_{\nu}(z, \mu) =$

$$\{(2+1)!\}^{-1} \{ \mathfrak{F}(z, \mu)^{2+1} - \mathfrak{F}(\mu, \mu)^{2+1} \} \mathcal{D}_t^{\nu} \{ \Theta(t, \mu)^{2+1} \mathcal{D}_t K(t) \}_{t \rightarrow \mu}$$

where $\mathcal{D}_t \equiv d/dt$ and $\Theta(t, \mu) = (t - \mu) / \mathfrak{F}(t, \mu)$. Taking $\mathfrak{F}(t, \mu) = (t - \mu) / (t - \mu)$

(so that $\mathfrak{F}(\mu, \mu) = 0$ and $\Theta(t, \mu) = t - \mu$) and $K(t) = \int_{\mu}^t \psi(x) dx$

(so that $K(\mu) = 0$) and letting z tend to infinity (so that

$K(z) \rightarrow I(\psi, \mu)$, $\mathfrak{F}(z, \mu) \rightarrow 1$) the approximation $I(\nu, \mu) = \phi_r(\varepsilon|\psi, \mu) =$

$\sum_{\nu=0}^{r-1} t_{\nu}(\varepsilon|\psi, \mu)$, where $t_{\nu}(\varepsilon|\psi, \mu) = \mathcal{D}^{\nu} \{ (\mu - \varepsilon)^{2+1} \psi(\mu) \} / (2+1)!$ is

obtained. (E2) Let $C_r \{c_{\nu} | \lambda\}$ be the r th convergent $\frac{a_1}{\lambda + b_1} \dots \frac{a_r}{\lambda + b_r}$

of the continued fraction associated with the series

assuming the C_r to be well defined,

$\sum_{\nu=0}^{\infty} c_{\nu} \lambda^{-\nu-1}$; then $\rho_{2r}(\nu, \mu) = C_r \{ t_{\nu}(\varepsilon|\psi, \mu) / 1 \}$, independently of ε . (E3a)

The first r terms of Taylor's series for $e^{\epsilon z} \psi(z)$ yield the approximation $\psi(z) = \sum_{\nu=0}^{r-1} (z-\mu)^\nu e^{-\epsilon z} \Delta^\nu \{e^{\epsilon \mu} \psi(\mu)\} / \nu!$; integrating

with respect to z over the range (μ, ∞) , the approximation

$$\bar{I}(\nu, \mu) = \delta_r(\epsilon | \nu, \mu) = \sum_{\nu=0}^{r-1} \nu_\nu(\epsilon | \nu, \mu), \text{ where } \nu_\nu(\epsilon | \nu, \mu) = \epsilon^{-\nu-1} e^{-\epsilon \mu} \Delta^\nu \{e^{\epsilon \mu} \psi(\mu)\}$$

is obtained. (E3b) Suppose that $\bar{I}(\nu, \mu)$ is dominated by the exponential term $e^{-\epsilon \mu}$; divide out this term

by setting $\bar{I}(\nu, \mu) = e^{-\epsilon \mu} G(\mu)$. G satisfies the equation $\epsilon G(\mu) - \Delta G(\mu) = e^{\epsilon \mu} \psi(\mu)$ and has in consequence the formal expansion

$$\sum_{\nu=0}^{\infty} \epsilon^{-\nu-1} \Delta^\nu \{e^{\epsilon \mu} \psi(\mu)\}; \delta_r(\epsilon | \nu, \mu) \text{ is the approximation to } \bar{I}(\nu, \mu)$$

derived from the first r terms of this expansion. (E4) Using the notations of (E2), $\epsilon_{2r}(\nu, \mu) = C_r \{ \nu_\nu(\epsilon | \nu, \mu) | 1 \}$, again independently

of ϵ . (The functions ϕ_r and δ_r are connected by relationships of the form $\phi_r(\epsilon | \nu, \mu) = (2\pi i)^{-1} \int_C z^{-1} e^{(\mu-\epsilon)z} \delta_r(z | \nu, \mu) dz$, C being a circle enclosing the origin, and $\delta_r(\epsilon | \nu, \mu) = \epsilon \int_{-\infty}^{\mu} e^{(\epsilon-\mu)z} \phi_r(z | \nu, \mu) dz$ with $\text{Re}(\epsilon) > \text{Re}(\mu)$. The above motivations indicate that formulae (C1-4) are each simple representatives of general classes of anti-derivative formulae.)

F: The main thrust of the paper is towards the derivation of convergence theory applicable to functions if not having the

special forms of (A1-4). (E1) With the possible singularities and ∞ branch cuts of ψ confined to a bounded convex open domain D in the complex plane, $\psi(z) = O(z^{-2})$ for large z , and

$$\bar{I}(\psi, \mu) = (2\pi i)^{-1} \int_L \ln(\mu - z) \psi(z) dz,$$

L being a semi-infinite loop lying in a half-plane in which ψ is analytic and encircling the point $z = \mu$, regions in the μ -plane (σ -plane) for which $\phi_r(\sigma | \psi, \mu) \rightarrow \bar{I}(\psi, \mu)$ for a fixed σ (μ) are derived. With \bar{D} being the smallest convex closed domain containing the singularities of ψ , the optimal value of σ inducing the most rapid convergence of the sequence $\{\phi_r(\sigma | \psi, \mu)\}$ for a fixed μ is derived. The special case in which $\psi(\mu) = \sum_{z=0}^{\infty} \int_{\alpha(z)}^{\beta(z)} (\mu - t)^{-z-2} d\zeta_z(t)$, the real and imaginary parts of the ζ_z being of bounded variation over $[\alpha(z), \beta(z)] \subseteq [\alpha, \beta] \subset (-\infty, \infty)$, is considered in detail. In particular, it is shown that if f is an entire function, $\mu \neq \alpha$ and $\psi(\mu) = (\mu - \alpha)^{-2} f\{(\mu - \alpha)^{-1}\}$, $|\phi_r(\sigma | \psi, \mu) - \bar{I}(\psi, \mu)|$ tends to zero faster than the terms of any geometric progression. (F2) It is shown that with $\psi(\mu) = \int_{\alpha}^{\beta} (\mu - t)^{-2} d\zeta(t)$, $\bar{I}(\psi, \mu) = \int_{\alpha}^{\beta} (\mu - t)^{-1} d\zeta(t)$, ζ being a nondecreasing real valued function of bounded variation over $[\alpha, \beta] \subset (-\infty, \infty)$, $\phi_{2r}(\psi, \mu) \rightarrow \bar{I}(\psi, \mu)$ for $\mu \in (-\infty, \infty) \setminus [\alpha, \beta]$.

(F3) With \mathbb{C} a certain point set not containing the origin, and \textcircled{P}

$\psi(\mu) = \int_{\mathbb{C}} e^{-\mu z} d\zeta(z)$, $I(\psi, \mu) = \int_{\mathbb{C}} z^{-1} e^{-\mu z} d\zeta(z)$, the real and imaginary parts of ζ being of bounded variation over \mathbb{C} , regions in the μ -plane (σ -plane) for which $\varepsilon_r(\sigma/\psi, \mu) \rightarrow$

$I(\psi, \mu)$ for a fixed σ (μ fixed) are derived. (F4) With

$\psi(\mu) = e^{-\lambda\mu} \int_{\alpha}^{\beta} e^{\mu t} d\zeta(t)$, $I(\psi, \mu) = e^{-\lambda\mu} \int_{\alpha}^{\beta} (\lambda-t)^{-1} e^{\mu t} d\zeta(t)$

where ζ is as in (F2) over $[\alpha, \beta] \subseteq [-\infty, \infty]$ and $\lambda \notin [\alpha, \beta]$, $\varepsilon_{2r}(\psi, \mu) \rightarrow I(\psi, \mu)$ for all $\mu \in (-\infty, \infty)$ for which the derivatives of ψ are defined. (In all of the above cases a priori error bounds are obtained.)

\textcircled{Q} : The functions of (C1-4) may be constructed by the use of

simple recursive algorithms. (G1) Let $b_0^{(0)} = 0$, $b_0^{(m)}$

$(\mu - \sigma)^m \Delta^{m-1} \psi(\mu) / m!$ ($m \geq 1$) and $b_{r+1}^{(m)} = b_r^{(m)} + b_r^{(m+1)}$ ($m, r \geq 0$);

then $b_r^{(0)} = \phi_r(\sigma/\psi, \mu)$ ($r \geq 0$). (If $\Delta^{\nu} \psi(\mu) = 0$ ($0 \leq \nu < \tau$); $0 \leq \tau < \infty$)

slight modification of the algorithm avoids arithmetic operations upon numbers known to be zero.) (G2) When

$\Delta^{\nu} \psi(\mu) = 0$ ($\nu = 0, \dots, \tau-1$), $\Delta^{\nu} \psi(\mu) \neq 0$ ($0 \leq \nu < \infty$), set $w_{-1}^{(m+1)} = 0$, $w_0^{(m)}$

$\Delta^{m-1} \psi(\mu) / m!$ ($m \geq \tau+1$), $w_{2r+1}^{(z-r)} = 0$ ($0 \leq r \leq \tau$); if numbers $w_r^{(m)}$

can be constructed by use in alternation of the formulae

$$(a) \omega_{2r+1}^{(m)} = \omega_{2r-1}^{(m+1)} + \omega_{2r}^{(m)} / \omega_{2r}^{(m+1)} \text{ for } 0 \leq r \leq \tau, m \geq \tau - r + 1, (b) \omega_{2r+2}^{(m)} =$$

$$\omega_{2r}^{(m+1)} (\omega_{2r+1}^{(m)} - \omega_{2r+1}^{(m+1)}) \text{ for } 0 \leq r \leq \tau - 1, m \geq \tau - r \text{ and thereafter}$$

by use of (b) for $r \geq \tau, m \geq 0$ and of (a) for $r \geq \tau + 1, m \geq 0$, then

$\rho_{2r}(\psi, \mu) = \omega_{2r}^{(0)}$ ($r > \tau$). (G3) With the initial values of (G1)

changed by taking $\rho_0^{(m)} = \epsilon^{-m} D^{m-1} \psi(\mu)$ ($m \geq 1$), $\rho_r^{(0)} = \phi_r(\epsilon/\psi, \mu)$

($r > 1$). (G4) With the initial values of (G2) changed by

taking: $\omega_0^{(m)} = D^{m-1} \psi(\mu)$ ($m \geq \tau + 1$), $\epsilon_{2r}(\psi, \mu) = \omega_{2r}^{(0)}$ ($r > \tau$). (Many

numerical illustrations of the theory of (E1-4) are given,

the a priori error bounds being confirmed. Acceleration of

the sequences $\{\phi_r(\epsilon/\psi, \mu)\}$, etc... by use of the ϵ -algorithm

is also illustrated.)

H: The above work is extended to the estimation of integrals

over a finite range of the form $\int_{\mu}^{\mu+h} \psi(x) dx$ in terms of the

derivatives of ψ at the ~~two~~ endpoints μ and $\mu+h$ of the

range of integration. Favorable comparisons with the use

of the ~~same~~ Euler-Maclaurin series, which makes use of

the same derivatives, are given

I: The Euler-Maclaurin formula, expressed as

$$S = \sum_{j=0}^{\infty} \psi(\mu + jh) = \sum_{j=0}^{m-1} \psi(\mu + jh) + h^{-1} \int_{\mu}^{\mu+h} \psi(t) dt + \frac{1}{2} \psi(\mu + h) + \sum_{j=0}^{\infty} b_j h^{2j+1} \psi^{(2j+1)}(\mu)$$

where $S = \sum_{\nu=0}^{\infty} \psi(\nu, \mu)$ in conjunction with the method of (C2) for the evaluation of the integral involved, is used as a method for transforming the series S . (An example is given in which $m=0$ and the sequence $\{\rho_{2r}(\nu, \mu)\}$ of (C2) is accelerated, the sum of the series S then being estimated in terms of the derivatives of its first term alone.)

J: The above work may be extended to the estimation of the value of $\lim F(z)$ ($z \rightarrow \infty$) in terms of the values of F and its derivatives for a finite argument value μ . Letting $\Delta F(\mu) = \psi(\mu)$, it follows that $I(\nu, \mu) = F(\infty) - F(\mu)$: an approximation to $F(\mu) + I(\nu, \mu)$ is an approximation to $F(\infty)$.

The version of (C1) in this case is $\phi_r^{(0)}(\epsilon | \psi, \mu) =$

$$\sum_{\nu=0}^r \binom{r}{\nu} (\mu - \epsilon)^\nu D^\nu F(\mu) / \nu! \text{ and similarly for formulae (C2-4).}$$

Convergence theories for the modified processes are obtained by considering functions F whose derivatives ψ are dealt with in (E) above. The algorithm of (G1) is modified by setting $b_0^{(m)} = (\mu - \epsilon)^m D^m F(\mu) / m!$ ($m \geq 0$). That of (G2) is, when $D^\nu F(\mu) = 0$ ($\nu = 0, \dots, z-1$), $D^\nu F(\mu) \neq 0$ ($0 \leq \nu < \infty$),

modified by setting $\omega_{-1}^{(m+1)} = 0$, $\omega_0^{(m)} = D^m F(\mu)/m!$ ($m \geq \tau$), $\omega_{2r+1}^{(\tau-r-1)} = 0$ ($0 \leq r \leq \tau-1$) and replacing τ by $\tau-1$ in the subsequent description given in (G2); the algorithms of (G3,4) are dealt with similarly.

K: Further extension in the opposite direction to the estimation of higher order anti-derivatives is also possible. For example, the sum $\phi_r^{(j)}(\epsilon|\psi^{(j)}, \mu) = \sum_{\nu=j}^r \binom{r}{\nu} (\mu-\epsilon)^\nu \Delta^{\nu-j} \psi^{(j)}(\mu)/\nu!$ offers an approximation to a function $I^{(j)}(\psi^{(j)}, \mu)$ for which $(-D)^j I^{(j)}(\psi^{(j)}, \mu) = \psi^{(j)}(\mu)$. Corresponding convergence theories are obtained by considering functions $\psi^{(j)}$ for which

$D^{j-1} \psi(\mu) = \psi^{(j)}(\mu)$, the ψ being those functions featuring in (E).

The algorithm of (G1) is modified by setting $\rho_0^{(m)} = 0$ ($0 \leq m < j$)

$\rho_0^{(m)} = (\mu-\epsilon)^m D^{m-j} \psi(\mu)/m!$ ($m \geq j$). That of (G2) is, when $D^j \psi(\mu) = 0$

($\nu=0, \dots, \tau-1$), $D^\nu \psi(\mu) \neq 0$ ($0 \leq \nu < \infty$) modified by setting $\omega_{2r+1}^{(m+1)} = 0$,

$\omega_0^{(m)} = D^{m-j} \psi(\mu)/m!$ ($m \geq \tau+j$), $\omega_{2r+1}^{(\tau-r+j-1)} = 0$ ($0 \leq r \leq \tau+j-1$) and

replacing τ by $\tau+j-1$ in the subsequent description given in (G2). The algorithms (G3,4) are similarly modified.

L: The paper under review extends earlier work by the author (in particular, that described in Arch. Rat. Mech. Anal., 28,

83-148 (1968); Zbl. Calcolo, 9, 197-278 (1972); Zbl. 248.65007 |

Jour. Reine Angew. Math., 285, 181-208 (1976); Zbl. 326.45005). (11)

The process of formula (63) is considered for the first time. The remarks made in (E1-3a,4) and (K) are original to this review.

Primary reference: 40A10

Secondary references: 41A55 65D30 26A36 65B15

Note to the printer:

Script C D L M O S

Greek $\alpha \beta \gamma \varepsilon \zeta \theta \lambda \mu \nu \xi \pi \rho \sigma \tau \phi \chi \psi \omega$

Gothic \mathbb{D}

Superscripts and suffixes underlined in brown: $p_n \underline{T}^{(i)}$

$f^{(m)}$
 b_r