Attempt to extract a totally oscillating sequence from the successive convergents of a continued fraction

$$C_i = \frac{a_1}{b_1 +} \cdots \frac{a_i}{b_i} = \sum_{j=1}^{i} (-1)^{j-1} \frac{a_1 \cdots a_j}{B_{j-1} B_j}$$

$$C_{2i+1} - C_{2i+3} = \frac{a_1 \cdots a_{2i+2} b_{2i+3}}{B_{2i+1} B_{2i+3}}$$

Special case: fix $\rho \in (1, \infty)$; set $a = \rho(\rho-1)$

$a_i = a \quad b_i = 1 \qquad C_i \to \rho - 1$

$A_0 = 0$, $A_1 = a$, $B_0 = B_1 = 1$

$$A_i = \frac{\rho(\rho-1)}{2\rho-1}\{\rho^i - (1-\rho)^i\}, \quad B_i = \frac{1}{2\rho-1}\{\rho^{i+1} - (1-\rho)^{i+1}\}$$

$A_{i+1} = \rho(\rho-1)B_i$

$B_{i+1} - (1-\rho)B_i = \rho B_i + A_i = \rho^{i+1}$

$B_{i+1} - \rho B_i = (1-\rho)B_i + A_i = (1-\rho)^{i+1}$

set $x = \frac{1-\rho}{\rho}$ $(-1 < x < 0)$: $\dfrac{1}{B_{i-1}} = (2\rho-1)\rho^{-i} \sum_{j=0}^{\infty} (x^i)^j$

$\frac{1}{B_{2i+1}}$ $(i = 0, 1, \ldots)$ is totally monotone

$C_{2i+1} - C_{2i+3} =$

$$\frac{\rho^{2i+2}(1-\rho)^{2i+2}(2\rho-1)^2}{\rho^{2i+2+2i+4}}\left\{\sum_{j=0}^{\infty} x^{2j}(x^{2j})^i\right\}\left\{\sum_{j=0}^{\infty} x^{4j}(x^{2j})^i\right\}$$

$$\{..\}\{..\} = \sum_{\nu=0}^{\infty}\sum_{z=0}^{\infty} x^{2\nu}x^{4z}(x^{2\nu+2z})^i$$

$$= \sum_{k=0}^{\infty}(x^{2k})^i \sum_{j=0}^{k} x^{2j}x^{2k} = \sum_{k=0}^{\infty}(x^{2k})^{i+1}\frac{1-x^{2k+2}}{1-x^2}$$

$$\frac{2\rho-1}{\rho} = 1-x \;; \quad \frac{(1-\rho)^{2i+2}}{\rho^{2i+4}}(2\rho-1)^2 = x^{2i+2}(1-x)^2$$

$$C_{2i+1} - C_{2i+3} = \frac{1-x}{1+x}x^{2i+2}\sum_{k=0}^{\infty}(1-x^{2k+2})(x^{2k})^{i+1}$$

$$= \frac{1-x}{1+x}\sum_{k=0}^{\infty}\{(-x)^{k+1}\}^{2i+2}\{1-(-x)^{k+1}\}\{1+(-x)^{k+1}\}$$

$$= \int_0^1 u^{2i+1}(1-u)\,d\sigma(u) \quad (i=0,1,\ldots)$$

σ has a jump of magnitude $\frac{1-x}{1+x}\{1+(-x)^{j+1}\}(-x)^{j+1}$ at $u = (-x)^{j+1}$ $(j=0,1,\ldots)$

$C_{2i+1} - C_{2i+3}$ $(i=0,1,\ldots)$ is totally monotone

If $T_i = \int_0^1 u^{2i+1}(1-u)\,d\sigma(u)$ $(i=0,1,\ldots)$ then

$$\sum_{i=0}^{\infty} T_i = \int_0^1 \frac{u^r}{1+u}\,d\sigma(u) = (-1)^r\left[\int_0^1 \frac{d\sigma(u)}{1+u} - \sum_{k=0}^{r-1}(-1)^k \int_0^1 u^k\,d\sigma(u)\right]$$

$T_i = C_{2i+1} - C_{2i+3}$ $\quad C = \lim_{i\to\infty} C_{2i+1}$ $\quad r=1$

$$\sum_{i=0}^{\infty} T_i = C_1 - C = \int_0^1 \frac{u^1}{1+u}\,d\sigma(u)$$

$$= \sum_{j=0}^{\infty} \frac{1-x}{1+x}\,\frac{\{1+(-x)^{j+1}\}}{\{1+(-x)^{j+1}\}}(-x)^{2(j+1)} = \frac{1-x}{1+x}\,\frac{x^2}{1-x^2}$$

$$= \left(\frac{x}{1+x}\right)^2 = (1-\rho)^2$$

$$C = \rho(\rho-1) - (\rho-1)^2 = \rho-1$$

$$f(k) - \int_0^1 u^k\,d\sigma(u) = \sum_{j=0}^{\infty} \frac{1-x}{1+x}\left\{1+(-x)^{j+1}\right\}(-x)^{(j+1)(k+1)}$$

$$= \sum_{j=0}^{\infty} \frac{1-x}{1+x}\left[(-x)^{k+1}((-x)^{k+1})^j + (-x)^{k+2}((-x)^{k+2})^j\right]$$

$$= g(k) + g(k+1)$$

$$g(2j) = -(2\rho-1)\frac{(1-\rho)^{2j+1}}{\rho^{2j+1}+(1-\rho)^{2j+1}}, \quad g(2j+1) = (2\rho-1)\frac{(1-\rho)^{2j+2}}{\rho^{2j+2}-(1-\rho)^{2j+2}}$$

$$g(2j) = -(2\rho-1)\frac{B_{2j+1}-\rho B_{2j}}{B_{2j+1}-(1-\rho)B_{2j}+B_{2j+1}-\rho B_{2j}}$$

$$= -(2\rho-1)\rho\,\frac{\rho-1-C_{2j+1}}{2\rho(\rho-1)-C_{2j+1}}$$

$$g(2j) = -(2\rho-1)\frac{(1-\rho)B_{2j}+A_{2j}}{\rho B_{2j}+A_{2j}+(1-\rho)B_{2j}+A_{2j}}$$

$$= -(2\rho-1)\frac{1-\rho+C_{2j}}{1+2C_{2j}}$$

$$g(2j+1) = (2\rho-1)\frac{(1-\rho)B_{2j+1}+A_{2j+1}}{\rho B_{2j+1}+A_{2j+1}-(1-\rho)B_{2j+1}-A_{2j+1}} = 1-\rho+C_{2j+1}$$

$$g(2j+1) = (2\rho-1)\frac{B_{2j+2}-\rho B_{2j+1}}{B_{2j+2}-(1-\rho)B_{2j+1}-B_{2j+2}+\rho B_{2j+1}}$$

$$= \rho\,\frac{(\rho-1)B_{2j+2}-A_{2j+2}}{A_{2j+2}} = \rho\left\{\frac{\rho-1-C_{2j+2}}{C_{2j+2}}\right\}$$

$$g(2j)-g(2j+1) = \frac{a-C_{2j+1}-C_{2j+1}^2}{C_{2j+1}-2a} = \frac{a-C_{2j}-C_{2j}^2}{(2C_{2j}+1)(C_{2j}+1)} \qquad (a)$$

$$g(2j+1) - g(2j+2) = \frac{a - C_{2j+2} - C_{2j+2}^2}{C_{2j+2}(2C_{2j+2}+1)} \quad (\beta)$$

$$g(2j+1) - g(2j+3) = C_{2j+1} - C_{2j+3}$$

$$g(2j) - g(2j+2) = (2a-c)\left\{\frac{1}{1+2C_{2j}} - \frac{1}{1+2C_{2j+2}}\right\}$$

$$g(0) = \frac{1+4a-\sqrt{1+4a}}{2} \quad (\gamma)$$

$$g(2j+1) - g(2j+2) = \frac{C_{2j+2}+1}{(2C_{2j+2}+1)(C_{2j+1}-C_{2j+3})}$$

(α) (β) and (γ) can be used to compute $g(k)$ $(k=0,1,\dots)$

then $f(k) = g(k) + g(k+1)$ $(k=0,1,\dots)$ computed

then $C = C_1 + \left[\,2\{f(0) - f(2) + f(4) \dots\} - f(0)\,\right]$

$$g(2j) - g(2j+1) = \frac{C_{2j}(C_{2j-1} - C_{2j+1})}{2C_{2j}+1}$$

$$g(0) - g(1) = a$$

$$C_{2i+1} = (x-1)\left[1 + \sum_{k=0}^{\infty} x^{2k+1}(x-1)\{x^{2(k+1)}\}^{i}\right] - \int_0^1 u^i \, d\sigma(u)$$

Auxiliary sequence transformation before application of the ε-alg

$$\sigma^{m} v_m = S_{m+1} - S_m$$

$$\frac{a-b}{a+1} \quad \frac{a}{a+1} \quad \frac{a+b}{a+1}$$

$$\frac{-2b}{1-b} \quad \frac{-b}{1-b} \quad 0$$

$$\frac{a-b}{a+1} = -1 \qquad a-b = -a-1 \qquad 2a = b-1 \qquad a = -b$$

$$v_m = \frac{1}{\sigma^m}\left(S_{m+1} - S_m\right)$$

$$S_m = \sum_{r=1} a_r \left(\frac{1}{r}\right)^{n+2}$$

$$aS_{m+1} + S_{m+1} = (a+A)\, S_m$$

$$S_{m+1} = \left(\frac{a+A}{a+1}\right) S_m + \frac{b}{a+1}$$

$$\frac{a+b}{a-1} \qquad \frac{a+b}{a-b}$$

$$2a + 2b = a - b \qquad a = -3b$$

$$S_{m+1} = A S_m + b$$

$$S_{m+1} - S_m = A\left(S_m - S_{m-1}\right)$$

$$\left\{\frac{\sigma}{(1-\sigma)}\right\}^{r+1} \Delta^{r+1} v_m = (\sigma A - I) \cdot \frac{\sigma}{1-\sigma}\left[\frac{\sigma}{1-\sigma}\right]^r \Delta^r v_m$$

$$\frac{v_m}{\sigma^m} = \frac{A}{\sigma^{m-1}} v_{m-1}$$

$$\Delta^r v_{m-1} \qquad \Delta^{r+1} v_m \qquad \Delta^r v_m \qquad \Delta^{r+1} v_m \qquad \Delta^r v_{m+1}$$

$$T^{r+1} = (\sigma A - I) \cdot \frac{\sigma}{1-\sigma}\, T^r$$

$$v_m = \sigma A v_{m-1}$$

$$\frac{a+b}{a+1} = \frac{1}{2}$$

$$\Delta^r v_m = \sigma A \Delta^r v_{m-1}$$

$$\Delta^r v_{m+1} = \sigma A \Delta^r v_m$$

$$\Delta^{r+1} v_m = (\sigma A - I)\, \Delta^r v_m$$

$$2a + 2b = a + 1 \qquad a = 1 - 2b$$

$$\Delta^r v_m + \Delta^{r+1} v_m = \sigma A \Delta^r v_m$$

$$\frac{a}{a+1} = \frac{1-2b}{2-2b}$$

$$\frac{a-b}{a+1} = \frac{1-3b}{2+2b}$$

$\sigma^s V_s = \sigma^s \mu^s = u_s$

$\dfrac{(\mu-1)\sigma}{1-\sigma}$

If we take $\sigma > 1$

$\sigma\mu = b$

$\dfrac{\left(\frac{b}{\sigma}-1\right)\sigma}{1-\sigma}$ is +ve

$\left(\frac{b}{\sigma}-1\right)\sigma$ -ve, $1-\sigma$ -ve

12%

$\frac{12}{100}$ 25 ~~000~~

3 1000 ~~5000~~

$\mu\sigma = \frac{1}{2}$

$\sigma = \frac{3}{4}$

$\dfrac{(\mu-1)\sigma}{1-\sigma}$ $\quad \dfrac{\left(\frac{2}{3}-1\right)\frac{3}{4}}{1-\frac{3}{4}}$

power series in $\sigma\mu \Rightarrow$ power series in $\dfrac{(\mu-1)\sigma}{1-\sigma}$

$\dfrac{(2-3)}{4-3} \cdot \dfrac{1}{~~4~~} = -1$

suppose $\sigma\mu$ lies in range $a \leqslant \sigma\mu \leqslant b < 1$

$1 > \sigma > \sigma\mu$ $\quad \sigma,$ $\quad \sigma\mu \leqslant \sigma < 1$ $\quad$ means $\mu \leqslant 1$

σ +ve $\quad 1-\sigma$ +ve $\quad \dfrac{(\mu-1)\sigma}{1-\sigma}$ always -ve

we should like $\dfrac{(\mu-1)\sigma}{1-\sigma} \geqslant -1$ probably

$\sigma\mu = a \quad \mu = \frac{a}{\sigma}$ $\quad \dfrac{\left(\frac{a}{\sigma}-1\right)\sigma}{1-\sigma} = -1$

means $\dfrac{a-\sigma}{1-\sigma} = -1$

$2\sigma =$

$a - \sigma = -1 + \sigma$

$\dfrac{-b-\sigma}{1-\sigma}$ large -ve number

$\dfrac{(b+\sigma)}{1-\sigma}$

$\dfrac{(1-\sigma) + (b+\sigma)}{(1-\sigma)^2} = 0 \quad \dfrac{1+b}{(1-\sigma)^2}$ $\therefore$ take σ large as poss.

suppose $\sigma\mu$ lies in

$-b, b \qquad \sigma > b$

$\mu = \frac{\sigma\mu}{\sigma}$

$\sigma\mu = -b \quad \dfrac{(\mu-1)\sigma}{1-\sigma} = \dfrac{-b-\sigma}{1-\sigma}$

$0 \qquad -\dfrac{\sigma}{1-\sigma}$

$b \qquad \dfrac{b-\sigma}{1-\sigma}$

$\frac{\sigma\mu}{\sigma}$ ~~$-\sigma$~~

$\dfrac{\left(\frac{\sigma\mu}{\sigma}-1\right)\sigma}{1-\sigma}$ $\quad \frac{b}{\sigma}$ $\quad - \dfrac{\frac{3}{4}}{\frac{1}{4}} = -3$

$b = $ ~~-0.99~~ $\quad 1 - 10^{-5} = \sigma$

$\dfrac{(\mu-1)\sigma}{1-\sigma} = \dfrac{-2-2.10^{-5}}{10^{-5}} = -2.10^{-5}$

1 $S_{2n} = S + \sum_{r=0}^{\infty}{}' a_r \lambda_r^n \qquad S_{2n+1} = 0$

$S_n = S + \sum{}' a_r \lambda_r'^n + a_r'(-\lambda_r')^n \qquad A'(-1)$

$\lambda'^2 = \lambda$

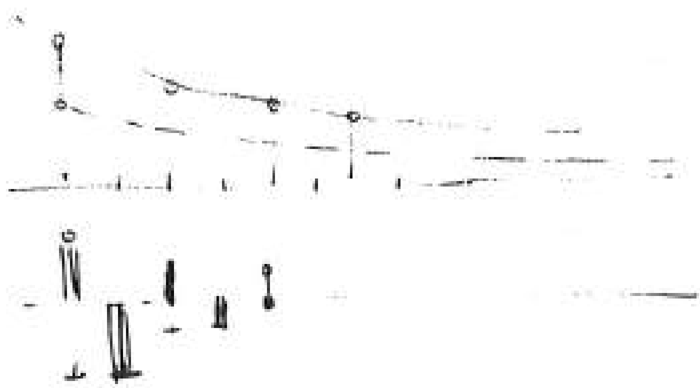
$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\ 1 & 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 & \frac{1}{4} & 0 \end{array}$$

$S_n = \frac{1}{\frac{n}{2}+2} \cdot \{1 + (-1)^n\}$

Transformation of monotonic sequences by means of the ε-alg.

$S_0 = 0 \quad S_1 = \ldots \quad S_{2n} = 0 \quad S_{2n+1} = S + \sum_{r=0}^{\infty}{}' a_r \lambda_r^n$

$$S_n = \frac{1}{2}S - \frac{1}{2}S(-1)^n + \sum_{r=0}^{}{}' \frac{1}{2}\left(a_r(\lambda_r^{\frac{1}{2}})^n - a_r(-\lambda_r^{\frac{1}{2}})^n\right)$$



$$\frac{1}{4} \cdot \left\{1 + \frac{n+1}{n+2} + (-1)^n \left\{1 - \frac{n+1}{n+2}\right\}\right\}$$

$$\frac{1}{4}\left\{\frac{2n+3}{n+2} + (-1)^n \cdot \frac{1}{n+2}\right\}$$

$S_{2n} = \frac{1}{2}\left\{S + \sum_{r=0}^{\infty}{}' a_r \lambda_r^n\right\} \qquad S_{2n+1} = \left\{\frac{1}{2}S + \sum_{r=0}{}' a_r \lambda_r\right\}$

$S_n = \frac{1}{2}S(-1)^n - \frac{1}{2}\sum_{r=0}^{\infty}{}' a_r (-\lambda_r)^n$

S o S o S

$S_n = \frac{1}{4} \cdot \{1 + (-1)^n\}$

$\frac{1}{2}S$	$\frac{1}{3}S$	$\frac{1}{4} \cdot 2S$	$\frac{1}{5} \cdot 2S$	$\frac{1}{6} \cdot 3S$	$\frac{1}{7}3S$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{7}$
$n=0$	1	2	3	4	5

$\frac{n+1}{4(n+2)}\left\{1 - (-1)^n\right\}$

1

0 -1 $+\frac{1}{3}$

$+2$ $2-12=-10$

$\frac{1}{2}$ $+\frac{1}{4}$ $\frac{1}{4}-\frac{1}{12}=\frac{1}{6}$

-2 $-2-20=-22$ $-22-6=-28$

0 $+\frac{1}{5}$ $\frac{1}{5}-\frac{1}{5}=0$ $+\frac{1}{10}$

$+3$ $+3-30=-27$ $-27+9=-18$ $\frac{1}{9}-\frac{1}{36}=+\frac{1}{12}$

$\frac{1}{3}$ $+\frac{1}{6}$ $\frac{1}{6}-\frac{1}{18}=\frac{1}{9}$ $-45-9=-54$

-3 $-3-42=-45$ $+\frac{1}{14}$

0 $+\frac{1}{7}$ $\frac{1}{7}-\frac{1}{7}=0$ $-52+12=-40$

$+4$ $+4-56=-52$

$\frac{1}{4}$ $+\frac{1}{8}$ $\frac{1}{8}-\frac{1}{24}=\frac{1}{12}$

-4 $-4-72=-76$

0 $+\frac{1}{9}$

$+5$ $\frac{1}{10}$

$\frac{1}{5}$

-18 $-60-18=-78$

$\frac{1}{12}$ $\frac{1}{12}-\frac{1}{60}=\frac{1}{15}$

-54 $-84-54=-138$

$\frac{1}{14}$

-40

	1								
1	0	$\frac{1}{3}$							
	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$						
0	0	$\frac{1}{5}$	0	$\frac{1}{10}$					
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{15}$				
	0	$\frac{1}{7}$	0	$\frac{1}{14}$	0	$\frac{1}{21}$			
0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{1}{20}$	$\frac{1}{24}$	$\frac{1}{28}$		
$\frac{1}{3}$	0	$\frac{1}{9}$	0	$\frac{1}{18}$	0	$\frac{1}{27}$	0	$\frac{1}{36}$	
	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{20}$	$\frac{1}{25}$	$\frac{1}{30}$	$\frac{1}{35}$	$\frac{1}{40}$	$\frac{1}{45}$

	$\frac{1}{2} \to \frac{1}{3}$		$\frac{1}{1.3}$
1	$\frac{1}{3} \to \frac{1}{6}$		$\frac{1}{2.3}$
	$\frac{1}{4} \to \frac{1}{10}$		$\frac{1}{2.5}$
2	$\frac{1}{5} \to \frac{1}{15}$		$\frac{1}{3.5}$
	$\frac{1}{6} \to \frac{1}{21}$		$\frac{1}{3.7}$
3	$\frac{1}{7} \to \frac{1}{28}$		$\frac{1}{4.7}$

$$\frac{1}{2n'+1} \to \frac{1}{(n'+1)(2n'+1)}$$

$\frac{1}{5}$ $\frac{1}{10}$ $\frac{1}{20}$

0 0 $\frac{1}{21}$ 0

$\frac{1}{6}$ $\frac{1}{11}$ $\frac{1}{22}$ $\frac{1}{33}$ $\frac{1}{44}$

0 0 $\frac{1}{23}$ 0 $\frac{1}{46}$

$\frac{1}{12}$ $\frac{1}{24}$ $\frac{1}{36}$ $\frac{1}{48}$ $\frac{1}{60}$ $\frac{1}{72}$

0 $\frac{1}{25}$ 0 $\frac{1}{50}$ 0 $\frac{1}{75}$

$\frac{1}{13}$ $\frac{1}{26}$ $\frac{1}{39}$ $\frac{1}{52}$

n' $\frac{1}{10} \to \frac{1}{10}$

$$\frac{1}{10+2n'+1} = \frac{1}{n'+1} \cdot \frac{1}{10+2n'+1 -}$$

0 $\frac{1}{11} \to \frac{1}{21}$ $\frac{1}{3.7}$

$$\frac{1}{n'+1} \cdot \frac{1}{20+2n'+1}$$

$\frac{1}{12} \to \frac{1}{33}$

$$\frac{1}{m+2n'+1} \to \frac{1}{n'+1} \cdot \frac{1}{2} \cdot \left(\frac{1}{m+2n'+1}\right)$$

1 $\frac{1}{13} \to \frac{1}{46}$ $\frac{1}{2.23}$

2 $\frac{1}{15}$ $\frac{1}{75}$ $\frac{1}{25}$