Functional Interpolation

P. Wynn

1. Interpolation and extrapolation 1.1. Procrustean techniques

Most general theories arise from investigations of particular problems, and in this respect the theory to be described is not exceptional. By way of motivation, the problem of deriving an extrapolation method from an interpolatory formula and it's converse are considered.

It is first supposed that an interpolatory function of complex variables

(1) $\quad F_r^{(m)}(d|y;\lambda) = F_r^{(m)}(d_m,\dots,d_{m+r}|y_m,\dots,y_{m+r};\lambda)$

is available for which (a) $F_r^{(m)}(d|y;y_\nu) = d_\nu$ ($\nu \in I_{m,m+r}$; throughout the paper $I_{i,j}$ is the sequence $i, i+1, \dots, j$; I_i is the sequence $i, i+1, \dots$; I is I_0) and (b) for certain distributions of the d_ν and y_ν

$$G_r^{(m)}(d|y) = \lim F_r^{(m)}(d|y;\lambda) \qquad (\lambda \to \infty)$$

exists. The function $F_r^{(m)}$ serves as the basis of an extrapolation method: given the sequence S_ν $(\nu \in I)$ the number $G_r^{(m)}(S|y)$ obtained by setting $d_\nu = S_\nu$ $(\nu \in I_{m,m+r})$ in $G_r^{(m)}(d|y)$ is an estimate of $\lim S_\nu$ $(\nu \to \infty)$. (The numbers y_ν used are suggested by the process producing the sequence $\{S_\nu\}$: the choice $y_\nu = \nu$ $(\nu \in I)$ is natural; the choice $y_\nu = 2^\nu$ arises, for example, in Romberg's method of integration [13,14] in which the number of integration subranges is doubled at each stage, $F_r^{(m)}$ being a polynomial in λ^{-1}.) The above extrapolation method may be illustrated by taking $F_{2i}^{(m)}$ to be the quotient of two ith degree polynomials in λ; subject to certain existence conditions, the coefficient of λ^i in the denominator is unity, that in the numerator, $\rho_{2i}^{(m)}(d|y)$, being expressible as the quotient of two determinants involving d_τ, y_τ ($\tau \in I_{m,m+2i}$). In this case $G_{2i}^{(m)}(d|y) = \rho_{2i}^{(m)}(d|y)$. Replacing d_τ by S_τ, the determinantal formula for $\rho_{2i}^{(m)}(d|y)$ yields the extrapolation limit

$$(2)\quad \rho_{2i}^{(m)}(S|y) = \frac{|1, S_\tau, y_\tau, S_\tau y_\tau, \ldots, y_\tau^{i-1}, S_\tau y_\tau^{i-1}; S_\tau y_\tau^i|_{\tau \in I_{m,m+2i}}}{|1, S_\tau, y_\tau, S_\tau y_\tau, \ldots, y_\tau^{i-1}, S_\tau y_\tau^{i-1}; y_\tau^i|_{\tau \in I_{m,m+2i}}}$$

the elements in successive rows being as displayed, the index ranges being as indicated.

Subject to the conditions

$$|1, S_\tau, \ldots, y_\tau^{i-1}, S_\tau y_\tau^{i-1}; y_\tau^i|_{\tau \in I_{m,m+2i}}, \; |1, S_\tau, \ldots, y_\tau^i, S_\tau y_\tau^i|_{\tau \in I_{m,m+2i+1}} \neq 0$$

the values of $\rho_{2i}^{(m)}(S|y)$ may be computed by use of a simple recursive process deriving directly from Thiele's reciprocal difference algorithm [15, 12 Ch. 5] for computing the numbers $\rho^{(m)}_{2i}(\alpha|y)$. It involves numbers $\varepsilon_{i,j}$ which may be set at the intersections of the full rows and columns and of the half rows and columns of a chipped triangular array in which the row index i ranges over $\bar{I}_{-\frac{1}{2}}$ ($\bar{I}_k$ is the sequence $k, k+\frac{1}{2}, k+1, \ldots$; $\bar{I}$ is $\bar{I}_0$) and the column index j over the range I_i, the number $\varepsilon_{-\frac{1}{2},-\frac{1}{2}}$ being missing. The numbers $\varepsilon_{i,j}$ are constructed from the initial values $\varepsilon_{0,j} = S_j$ $(j \in I)$, $\varepsilon'_{0,j} = 0$ $(j \in I)$ (the dash is used to indicate a displacement operation acting upon numbers with two suffixes, whose effect is illustrated by the relationships $\varepsilon'_{0,j} = \varepsilon_{-\frac{1}{2}, j+\frac{1}{2}}$ and $\varepsilon'_{i,j} = \varepsilon_{i-\frac{1}{2}, j+\frac{1}{2}}$) by use of the relationship

$$(3) \qquad (\Delta_j \varepsilon_{i,j})(\Delta_i \varepsilon'_{i,j}) = w_{i,j}$$

for $i \in \bar{I}, j \in I_i$, where Δ_j is the difference operator $\Delta_j \varepsilon_{i,j} = \varepsilon_{i,j+1} - \varepsilon_{i,j}$ and Δ_i is similarly defined. With

$$w_{i,j} = (y_{i+j+1} - y_{j-i})^{-1} \qquad (i \in \bar{I}, j \in I_i)$$

$\rho_{2i}^{(m)}(S|y) = \varepsilon_{i,i+m}$ $(i, m \in I)$.

We have called an extrapolation method of the above type a Procrustean technique [18], although in fitting function values to a sequence, i.e. the bed to the victim, we are a little kinder than Procrustes is reputed to have been.

If in formula (1) λ is fixed and one of the y_ν is very large, e_ν is approximately equal to $\lim F_r^{(m)}(\lambda)$ $(\lambda \to \infty)$: if λ is fixed and the $y_\tau \neq y_\nu$ are fixed and finite, $\lim F_r^{(m)}(\lambda) = e_\nu$ $(y_\nu \to \infty)$. This observation may be made in terms of the behaviour of $G_r^{(m)}(\alpha|y)$ as y_ν tends to infinity, the other y_τ remaining finite; expressed in terms of the S_τ it is that, under suitable conditions, $\lim G_r^{(m)}(S|y) = S_\nu$ $(y_\nu \to \infty)$. This property is, for example, possessed by the determinantal quotient (2). If the cofactor of $S_\nu y_\nu^i$ in the last column is nonzero, the numerator determinant is dominated by the product of this cofactor and $S_\nu y_\nu^i$, the denominator determinant being dominated by the product of the same cofactor and y_ν^i: the quotient (2) tends to S_ν as $y_\nu \to \infty$.

1.2 **1.2 Interpolatory functions**

The steps taken in the above derivation of an extrapolation method from an interpolatory formula may be reversed. For the present and later purposes, let Z be an open set of points in $\mathbb{C}$, the finite part of the complex plane and Y, not necessarily contained in Z, be a set of limit points of Z; set $\bar{Z} = Z \cup Y$. Let $d: Y \to \mathbb{C}$ and $e, f: Y \times Z \to \mathbb{C}$ be such that for each $x \in Y$, $\lim e(y,z)$ and $\lim f(y,z)$ as z tends to x over Z are finite for each $y \in Y$, that in the former case $\lim e(y,z)\ (z \to y) = d(y)$ and that in the latter $\lim f(y,z)\ (z \to x)$ is nonzero when $x \neq y$ and is zero when $x = y$. Let $e, f: Y \times \bar{Z} \to \mathbb{C}$ be defined by use of the additional assumptions concerning the limiting behaviour of e and f. Let $y_\nu \in Y$ $(\nu \in I)$ be distinct and set $d_\nu = d(y_\nu)$, $e_\nu(z) = e(y_\nu, z)$ and $f_\nu(z) = f(y_\nu, z)$ $(\nu \in I)$.

Suppose that a function of complex variables

$$G_r^{(m)}(S|y) = G_r^{(m)}(S_m, \ldots, S_{m+r} \,|\, y_m, \ldots, y_{m+r})$$

for which $\lim G_r^{(m)}(S|y) = S_\nu$ as $y_\nu \to \infty$ ($\nu \in I_{m,m+r}$), obtained either from an interpolatory formula or from an extrapolation method or in some other way is available. Set

$$H_r^{(m)}(e,f|z) = G_r^{(m)}(e_m(z),\ldots,e_{m+r}(z)|f_m(z)^{-1},\ldots,f_{m+r}(z)^{-1})$$

As $z \to y_\nu$ over Z ($\nu \in I_{m,m+r}$), $f_\tau(z)^{-1}$ ($\tau \in I_{m,m+r}; \tau \neq \nu$) remains finite, $f_\nu(z)^{-1}$ tends to infinity and $e_\nu(z) \to d_\nu$. Thus in view of the property attributed to $G_r^{(m)}(S|y)$ just described, $\lim H_r^{(m)}(e,f|z) = d_\nu$ as $z \to y_\nu$ over Z ($\nu \in I_{m,m+r}$): $H_r^{(m)}(e,f|z)$ is an interpolatory function. If the $G_r^{(m)}(S|y)$ ($r,m \in I$) may be computed by means of a recursive process, appropriate modification yields a process for computing numerical values of the $H_r^{(m)}(e,f|z)$ ($r,m \in I$).

The extrapolation estimate $\rho_{2i}^{(m)}(S|y)$ exemplifies the above. With $k \le h$, let $E^{(m)}_{h,k}(z)$ be the array whose $(\nu+1)^{th}$ row, for $\nu \in I$, contains the $h-k+1$ elements $e_{m+\nu}(z) f_{m+\nu}(z)^{h-\tau}$ $(\tau \in I_{0,h-k})$ and with $k=h+1$ let $E^{(m)}_{h,k}(z)$ be the null array; with $n \le s$, let $F^{(m)}_{n,s}(z)$ be the corresponding array whose typical row contains the elements $f_{m+\nu}(z)^{n+\tau}$ $(\tau \in I_{0,s-n})$ and with $n=s+1$ let $F^{(m)}_{n,s}(z)$ be the null array (the powers of $f_{m+\nu}(z)$ are respectively in descending and ascending order in the rows of these arrays). Let $[E^{(m)}_{h,k}(z) | F^{(m)}_{n,s}(z)]$ be the determinant formed from the first $h-k+s-n+2$ rows of the array formed by juxtaposing $E^{(m)}_{h,k}(z)$ and $F^{(m)}_{n,s}(z)$ in the indicated order.

Set

$$r^{(m)}_{i,j}(z) = N^{(m)}_{i,j}(z) / D^{(m)}_{i,j}(z) \qquad (4)$$

where

(5) $N_{i,j}^{(m)}(z) = [E_{i,0}^{(m)}(z) | F_{1,j}^{(m)}(z)]$, $D_{i,j}^{(m)}(z) = [E_{i,1}^{(m)}(z) | F_{0,j}^{(m)}(z)]$

The interpolatory function derived from the extrapolation estimate $\rho_{2i}^{(m)}(S|y)$ of formula (2) is $r_{i,i}^{(m)}(z)$.

If $D_{i,j}^{(m)}(z) \neq 0$ for all $z \in \overline{Z}$ $(m, i, j \in I)$ all functions $r_{i,j}^{(m)}(z)$ $(m, i, j \in I)$ are well defined over $\overline{Z}$ by formula (4). Select one of these functions and ν from $I_{m,m+i+j}$. Suppose that for the denominator determinant, $\lim [E_{i,1}^{(m)}(z) | F_{0,j}^{(m)}(z)] = \delta$ as $z \to y_\nu$ over Z, so that $\delta \neq 0$. As $z \to y_\nu$, the values of all elements in the row containing powers of $f_\nu(z)$ tend to zero, except in the case of the element belonging to the first column of $F_{0,j}^{(m)}(z)$, which is unity. The value of the cofactor $C(z)$ of this exceptional element tends to δ as ~~z~~ $z \to y_\nu$. The same behaviour is exhibited by the numerator determinant $[E_{i,0}^{(m)}(z) | F_{1,j}^{(m)}(z)]$. Now the exceptional element $e_\nu(z)$ lies in the last column of $E^{(m)}_{i,0}(z)$; its value tends to d_ν as $z \to y_\nu$. The cofactor of $e_\nu(z)$ in $C(z)$: $\lim N^{(m)}_{i,j}(z) = d_\nu \delta$ as $z \to y_\nu$. Hence $\lim r^{(m)}_{i,j}(z) = d_\nu$ as $z \to y_\nu$ $(\nu \in I_{m,m+i+j})$. Imposing the further condition

$$(6) \qquad e_\nu(z) - d_\nu = O\{f_\nu(z)\} \qquad (z \to y_\nu;\ \nu \in I)$$

it follows that

$$(7) \qquad r^{(m)}_{i,j}(z) - d_\nu = O\{f_\nu(z)\} \ (z \to y_\nu;\ \nu \in I_{m,m+i+j};\ m,i,j \in I)$$

This result holds in particular for the function $r^{(m)}_{i,i}(z)$ derived from $\rho^{(m)}_{2i}(S|y)$ $(m, i \in I)$.

If $[E^{(m)}_{i,1}(z) | F^{(m)}_{1,i}(z)] \neq 0$, $[E^{(m)}_{i,0}(z) | F^{(m)}_{0,i}(z)] \neq 0$ for all $z \in Z$ $(i, m \in I)$, the numerical values of the $r^{(m)}_{i,i}(z)$ for a fixed $z \in Z$ may be determined by use of formula (3).

Setting now $\varepsilon'_{0,j} = 0$, $\varepsilon_{0,j} = e_j(z)$ $(j \in I)$ and

$$w_{i,j} = \{ f_{j-i}(z)^{-1} - f_{i+j+1}(z)^{-1} \} \qquad (i \in I, j \in I_1)$$

$r_{i,i}^{(m)}(z) = \varepsilon_{i,i+m}(z)$ $(i, m \in I)$.

In this result, the conditions of the simple case in which

$$(8) \qquad e(y,z) = d(y), \quad f(y,z) = z - y$$

may be imposed upon e and f. Now $e_\nu(z) = d_\nu$ is a constant, independent of z $(\nu \in I)$ and $f_\nu(z)$ is the difference $z - y_\nu$ $(\nu \in I)$. In this case, $r_{i,i}^{(m)}(z)$ is the quotient of two ith degree polynomials in z, the rational function from which the extrapolation limit $\rho_{2i}^{(m)}(S|y)$ was derived. The above process now reduces to an algorithm for rational function interpolation due to Brezinski [4] of which a generalisation has been proposed by Cordellier [6].

The discussion of interpolatory formulae and extrapolation methods is terminated by the remark that under appropriate conditions cyclic derivation of extrapolation methods ~~and~~ from interpolatory formulae and conversely may be repeated indefinitely.

2. Approximants of general order

In the simple case (8), Laplace expansion of the determinants in formulae (5) with minors and complementary minors taken from the E and F arrays respectively, together with the remark that the value of a Vandermonde determinant formed from elements of the form $(z-y_s)^\tau$ is independent of z, reveal that $D_{i,j}^{(m)}(z)$ and $N_{i,j}^{(m)}(z)$ are polynomials in z of degrees i and j respectively. When the simple case does not obtain, the general system of approximants $r^{(m)}_{i,j}(z)$ $(m,i,j \in I)$ stands in relation to the special system $r^{(m)}_{i,i}(z)$ $(m,i \in I)$ as do general rational functions to special rational functions whose numerators and denominators are of equal degree. It is proposed to study the more general system of approximants.

2.1 Nonuniform approximation

It is first remarked that in the simple case (8), relationship (7) reduces to $r^{(m)}_{i,j}(z) - a_\nu = O(z-y_\nu)$: approximation is uniform, the form of the function $z-y_\nu$ being the same for all relevant y_ν. It may fortuitously occur that for certain examples and certain argument values y_ν, the term $O(z-y_\nu)$ may be replaced by $O\{(z-y_\nu)^n\}$ with $n>1$; nevertheless

$O\{(z-y_\nu)\}$ is the only term generally applicable. By suitable choice of the functions $f_\nu(z)$ in the nonsimplified theory given above, with for example $f_\nu(z)=O\{(z-y_\nu)^{n(\nu)}\}$ $\nu\in I$, the $n(\nu)$ being positive real numbers, nonuniform approximation is possible.

2.2 Remainder term formulae

In certain circumstances an interpolation property of the form (7) holding at points induces on the function possessing it a corresponding property of approximation over a set containing the points. By imposing severe restrictions upon the functions e and f it is possible in a few lines to exhibit the $r_{i,j}^{(m)}$ as approximations to a function defined over Z, and to provide associated remainder terms. Let Z be a finite segment of the real axis, and let $Y \equiv Z$.

Let e and f be subject to the conditions of the semi-simple case in which

(9) $\qquad e(y,z) = d(y), \quad f(y,z) = \phi(z) - \phi(y)$

(This case arises from the simple case by a change of variables from y, z to $\phi(y), \phi(z)$ and a redefinition of d.) Let $d, \phi : Z \to R$ possess derivatives of all orders over Z, ϕ being such that $d\phi(y)/dy \neq 0$ for all $y \in Z$.

Let

(10) $$\Delta_r^{(m)}(z) = \prod_{\nu=0}^{r-1} f_{m+\nu}(z)$$

and θ_i be an arbitrary ith degree polynomial in ϕ with real coefficients for which $\theta_i(z) \neq 0$ for all $z \in Z$.

Set $R_{i,j}^{(m)}(y) = d(y) - r_{i,j}^{(m)}(y)$ and, for a fixed $z \in Z$ not equal to y_ν $(\nu \in I_{m,m+r})$, let

$$R^{(m)}_{i,j}(z) = A(z)\Lambda^{(m)}_{i+j+1}(z) / \{D^{(m)}_{i,j}(z)\Theta_i(z)\}$$

where $A(z)$ is a real number. The function

$$g(y) = D^{(m)}_{i,j}(y)\Theta_i(y)d(y) - \Theta_i(y)N^{(m)}_{i,j}(y) - A(z)\Lambda^{(m)}_{i+j+1}(y)$$

vanishes at the $i+j+2$ points $y = y_\nu$ $(\nu \in I_{m,m+i+j})$ and $y = z$ in Z: $g(y)$ vanishes when $\phi(y)$ assumes any one of the $i+j+2$ distinct values $\phi(y_\nu)$ $(\nu \in I_{m,m+i+j})$ and $\phi(z)$. Thus, by Rolle's theorem, $\mathcal{D}_\phi^{i+j+1} g(y)$, where $\mathcal{D}_\phi$ is the ϕ-derivative $\{d\phi(y)/dy\}^{-1} d/dy$, is zero for some value ξ of $y \in Z$. $\Theta_i(y)$ and $N^{(m)}_{i,j}(y)$ are polynomials of degrees i and j in $\phi(y)$ respectively. $\Lambda^{(m)}_{i+j+1}(y)$ is a polynomial of degree $i+j+1$ in $\phi(y)$, the coefficient of $\phi(y)^{i+j+1}$ being unity. Hence $\mathcal{D}_\phi^{i+j+1}\{D^{(m)}_{i,j}(y)\Theta_i(y)d(y)\} = (i+j+1)!\,A(z)$ when $y = \xi$ for some $\xi \in Z$, and accordingly

(11) $$R^{(m)}_{i,j}(z) = \Lambda^{(m)}_{i+j+1}(z)\,\mathcal{D}_\phi^{i+j+1}\{D^{(m)}_{i,j}(y)\Theta_i(y)d(y)\} / \{(i+j+1)!\,D^{(m)}_{i,j}(z)\Theta_i(z)\}$$

for $y=\xi$.

The preceding is a straightforward adaptation of an argument due to Nörlund ([12] Ch. 15 §3) concerning the simple case in which $\phi(z)=z$ and either $j=i$ or $j=i+1$. Nörlund's discussion of the case, subject to the same restrictions, in which d is a function of a complex variable may be extended in the same way. It is now supposed that Z is a bounded open region of the complex plane, that $Y\equiv Z$, that d and ϕ are analytic over an open domain which contains the closure of Z, and that $\phi(z)=\phi(y)$ only when $z=y$ for all $y,z\in Z$. In this case

$$(12)\quad R_{i,j}^{(m)}(z)=-\left\{\frac{1}{2\pi i}\int_C \frac{D_{i,j}^{(m)}(y)\theta_i(y)d(y)\phi'(y)}{f(y,z)\Lambda_{i+j+1}^{(m)}(y)}dy\right\}\left\{\frac{\Lambda_{i+j+1}^{(m)}(z)}{D_{i,j}^{(m)}(z)\,\theta_i(z)}\right\}$$

where C is the boundary of Z and again θ_i is a disposable ith degree polynomial in ϕ for which $\theta_i(z) \neq C$ for all $z \in Z$.

2.3 Algorithms for approximant evaluation

Algorithms for determining the approximant values $r_{i,j}^{(m)}(z)$ for a fixed $z \in Z$ are now given.

Subject to the existence condition

$$f_\nu(z) \neq f_\tau(z) \qquad (\nu \in I, \tau \in I_\nu) \tag{13}$$

the values of the Lagrange forms

$$L_j^{(m)}(z) = r_{0,j}^{(m)}(z) = \sum_{\nu=0}^{j} e_{m+\nu}(z) \prod_{\tau=0}^{j\,[\nu]} \frac{f_{m+\tau}(z)}{f_{m+\tau}(z) - f_{m+\nu}(z)} \qquad (m, j \in I)$$

where the bracketed index denotes the term deleted from the product, may be determined from the initial values $L_0^{(m)}(z) = e_m(z)$ $(m \in I)$ by use of the relationship

(14) $L_{j+1}^{(m)}(z) = \{f_{m+j+1}(z) L_j^{(m)}(z) - f_m(z) L_j^{(m+1)}(z)\} / \{f_{m+j+1}(z) - f_m(z)\}$

for $j, m \in I$. The divided differences

$$\delta_f^j\{e_m(z)\} = (-1)^j \frac{[E_{0,0}^{(m)}(z) | F_{0,j-1}^{(m)}(z)]}{[F_{0,j}^{(m)}(z)]}$$

(15) $$= \sum_{\nu=0}^{j} e_{m+\nu}(z) \prod_{\tau=0}^{j\ [\nu]} \{f_{m+\tau}(z) - f_{m+\nu}(z)\}^{-1} \quad (m, j \in I)$$

may be determined from the initial values $\delta_f^0\{e_m(z)\} = e_m(z)$ ($m \in I$) by use of the relationship

(16) $\delta_f^{j+1}\{e_m(z)\} = [\delta_f^j\{e_{m+1}(z)\} - \delta_f^j\{e_m(z)\}] / \{f_m(z) - f_{m+j+1}(z)\}$ $(m, j \in I)$

In terms of the divided differences and the factorial functions (10), the Lagrange forms may be expressed in Newton form as

(17) $$L_j^{(m)}(z) = \sum_{\nu=0}^{j} \delta_f^\nu\{e_m(z)\} \Lambda_\nu^{(m)}(z) \qquad (m, j \in I)$$

Assuming in addition that

(18) $$[E_{i,1}^{(m)}(z) | 1_i] \neq 0 \qquad (m \in I, i \in I_1)$$

where 1_i denotes a column array containing $i+1$ unit elements, the values of the reciprocal Lagrange forms

$$M_i^{(m)}(z) = r_{i,0}^{(m)}(z) = \left\{ \sum_{\nu=0}^{i} e_{m+\nu}(z)^{-1} \prod_{\tau=0}^{i\,[\nu]} \frac{f_{m+\tau}(z)}{f_{m+\tau}(z) - f_{m+\nu}(z)} \right\}^{-1} \quad (m, i \in$$

may be determined from the initial values $M_0^{(m)}(z) = e_0^{(m)}(z)$

($m \in I$) by use of the relationship

$$(19) \quad M_{i+1}^{(m)}(z) = \frac{f_m(z) - f_{m+i+1}(z)}{\{f_m(z)/M_i^{(m+1)}(z)\} - \{f_{m+i+1}(z)/M_i^{(m)}(z)\}} \quad (m, i \in$$

For a fixed value of m, assuming the existence conditions

$$(20) \qquad [E_{i,1}^{(m)}(z) \,|\, F_{0,j}^{(m)}(z)], \; [E_{i,1}^{(m)}(z) \,|\, F_{i,j+1}^{(m)}(z)] \neq 0$$

for $i, j \in I$, the values of $r_{i,j}^{(m)}(z)$ $(i, j \in I)$ may be determined

by application of formula (3) with

$$(21) \qquad w_{i,j} = f_{m+i+j+1}(z)^{-1}$$

now to a system of numbers $\varepsilon_{i,j}$ $(i \in \bar{I}_{-\frac{1}{2}}, j \in I_{i-[i]-1}$

which may be placed at the intersections of the full

rows and columns and the half rows and columns of a square array, the boundary values being $\varepsilon_{k,-1} = \varepsilon'_{0,k} = 0$ $(k \in I)$. Formula (3) may be applied to the initial values $\varepsilon_{0,j} = L_j^{(m)}(z)$ $(j \in I)$ in ~~the~~ row by row order, with $i \in \bar{I}$, $j \in I_{i-[i]-1}$. Alternatively, it may be applied to the initial values $\varepsilon'_{i,-1} = \sum_{\nu=0}^{i-1} M_\nu^{(m)}(z)^{-1}$ $(i \in I_1)$ in column by column order with $j \in \bar{I}_{-1}$, $i \in I_{j-[j]}$. In both cases $r_{i,j}^{(m)}(z) = \varepsilon_{i,j}$ $(i,j \in I)$. In the first case the values $\varepsilon_{i,0} = M_{i,0}(z)$ $(i \in I)$ arise in the course of the computation; in the second, these values and $\varepsilon_{0,j} = L_j^{(m)}(z)$ $(j \in I)$ do.

The function values lying at the intersections of the half rows and columns are given by

$$(22) \quad \varepsilon'_{i,j} = [E_{i,2}^{(m)}(z) \mid F_{0,j+1}^{(m)}(z)] / [E_{i,1}^{(m)}(z) \mid F_{1,j+1}^{(m)}(z)] \quad (i \in I, j \in I_{-1})$$

They assist in the computations and may be eliminated to yield a recurrence relationship involving numbers lying at the intersections of the full rows and columns of the ε-array alone. It is

$$(23) \quad \Delta_j\{w_{i,j}(\Delta_j\varepsilon_{i,j})^{-1}\} = \Delta_i\{w'_{i,j}(\Delta_i\varepsilon''_{i,j})^{-1}\}$$

where $\varepsilon''_{i,j} = \varepsilon_{i-1,j+1}$. The boundary values are now $\varepsilon_{i,-1} = 0$ $(i\in I)$, $\varepsilon_{-1,j} = \infty$ $(j\in I)$. Relationship (23) may be applied to the initial values $\varepsilon_{0,j} = L_j^{(m)}(z)$ $(j\in I)$ in the row by row order $i\in I$, $j\in I_{-1}$ or to the initial values $\varepsilon_{i,0} = M_i^{(m)}(z)$ $(i\in I)$ in the column by column order $j\in I_{-1}$ $i\in I$.

That formulae (3,21) lead via formula (22) to the construction of the determinantal quotients (4,5) is demonstrated by use of elementary determinantal identities (see, for example, [2] §§45,46). An independent proof of formula (23) may be obtained in the same way.

In the simple case of formulae (8) in which $e_\nu(z)$ and $f_\nu(z)$ are taken to be a constant function $e_\nu(z)=d_\nu$ and a simple difference $f_\nu(z)=z-y_\nu$ respectively, the Lagrange form $L_j^{(m)}(z)$ reduces to the Lagrange interpolation polynomial. Relationship (14) reduces to the Aitken-Neville recursion [1,11]. The factors $f_{m+z}(z)-f_{m\nu}(z)$ in formula (15) are independent of z: the divided differences are complex numbers as they also are in the semi-simple case of formulae (9). Formula (17) now merely expresses the Lagrange interpolation polynomial as a partial sum of a Newton series. Relationship (19) reduces to one due to Stoer[15]. The version of the ε-algorithm for the recursive computation of the $r^{(m)}_{i,j}(z)$ for fixed m described above reduces to a variant of the ε-algorithm due to Claessens [5]. Formula (23) involving numbers disposed in the form of a cross in the ε-array reduces to the relationship from which Claessens' form of the ε-algorithm was derived. (Even in the simple case, the determinantal formulae for $\varepsilon_{i,j}$ and $\varepsilon'_{i,j}$ given above are new and offer a simple direct proof of Claessens' form of the ε-algorithm.)

Concerning the derivation of an ε-algorithm relationshi[p] from a system of relationships of the form

$$K^{(1)}_{i,j}(\Delta_j \varepsilon_{i,j+1})^{-1} - K^{(2)}_{i,j}(\Delta_j \varepsilon_{i,j})^{-1} = K^{(3)}_{i,j}(\Delta_i \varepsilon_{i,j+1})^{-1} - K^{(4)}_{i,j}(\Delta_i \varepsilon_{i-1,j+\ldots}$$

for $i,j \in I$ with $\varepsilon_{-1,j} = \infty$ $(j \in I)$, $\varepsilon_{i,-1} = 0$ $(i \in I)$ and $\varepsilon_{0,j}$ $(j \in I)$ prescribed, it is remarked that if the $\kappa^{(k)}_{i,j}$ are finite and nonzero and satisfy the consistency condition

$$\kappa^{(1)}_{i,j}\kappa^{(2)}_{i+1,j+1}\kappa^{(3)}_{i,j+1}\kappa^{(4)}_{i+1,j} = \kappa^{(1)}_{i+1,j}\kappa^{(2)}_{i,j+1}\kappa^{(3)}_{i,j}\kappa^{(4)}_{i+1,j+1} \quad (i,j \in I)$$

then the same numbers $\varepsilon_{i,j}$ may be produced by means of the lozenge algorithm relationship (3) from the initial values $\varepsilon'_{0,j} = 0$ $(j \in I_{-1})$ with $\varepsilon_{-1,j}$ and $\varepsilon_{i,-1}$ as before, where the $w_{i,j}$ are obtained by setting $\chi_{0,0} = 1$, $\chi_{0,j+1} = \chi_{0,j}\kappa^{(1)}_{0,j} / \kappa^{(2)}_{0,j+1}$ $(j \in I)$ and thereafter $\chi_{i+1,j} = \chi_{i,j}\kappa^{(3)}_{i,j} / \kappa^{(4)}_{i+1,j}$ $(i,j \in I)$ when $w_{i,j} = \chi_{i,j}\kappa^{(2)}_{i,j}$ $(i,j \in I)$ and $w'_{i,j} = \chi_{i,j}\kappa^{(4)}_{i,j}$ $(i \in I_1, j \in I)$.

Subject to the existence condition (13), divided differences of any sequence of finite function values are

are well defined. $g_\nu(z), h_\nu(z)$ $(\nu \in I)$ being two such sequences

$$(24)\quad \delta_f^j\{g_m(z)h_m(z)\} = \sum_{\nu=0}^{j} \delta_f^\nu\{g_m(z)\}\delta_f^{j-\nu}\{h_{m+\nu}(z)\} \quad (j \in I)$$

This result may be used to show that if

$$g_\nu(z) = \sum_{\tau=0}^{n} a_\tau(z) f_\nu(z)^\tau \qquad (\nu \in I)$$

with $a_\tau : Z \to \mathbb{C}$ $(\tau \in I_{0,n})$ then $\delta_f^j\{g_m(z)\} = 0$ $(j \in I_{n+1}, m \in I)$.

If $f_\nu(z) \neq 0$ $(\nu \in I)$, $\delta_f^j\{f_m(z)^{-1}\} = \{\Lambda_{j+1}^{(m)}(z)^{-1}\}$ $(m, j \in I)$.

This result and formulae (17, 24) lead to the expression for the Lagrange forms

$$(25)\quad L_j^{(m)}(z) = \Lambda_{j+1}^{(m)}(z)\, \delta_f^j\{e_m(z)/f_m(z)\} \qquad (m, j \in I)$$

and if in addition condition (18) obtains, to that for the reciprocal Lagrange forms

$$M_j^{(m)}(z) = \{\Lambda_{j+1}^{(m)}(z)\, \delta_f^j\{\{e_m(z)/f_m(z)\}^{-1}\}\}^{-1} \quad (m, j \in I)$$

By using these results, the determinantal formulae (4, 5) for $r^{(m)}_{i,j}(z)$ may be transformed to yield numerous alternative expressions. In the simple case of formulae (8), formula (24) reduces to one made use of by Jacobi [9] and Nörlund [12]. Relationship (25) reduces to one used by Jacobi [9] to derive a variety of expressions for rational interpolatory functions. It is remarked in passing that it is possible to extend not only Jacobi's work but much of the classical theory of finite differences (see for example [8, 12]) by the use of functional divided differences.

2.4 Termination

In the special cases in which $e(y,z)$ is a polynomial or rational function of $f(y,z)$, termination of the above algorithms may be demonstrated. When

$$(26) \qquad e(y,z) = \sum_{\nu=0}^{k} a_\nu(z) f(y,z)^\nu$$

with $a_\nu : Z \to \mathbb{C}$ $(\nu \in I_{0,k})$ and condition (13) holds, $r_{0,k}^{(m)}(z) = a_0(z)$ $(m \in I)$ and recursion (14) may be applied for $j \in I_{0,k-1}$, $m \in I$ to produce these functions. When

$$(27) \qquad e(y,z) = \left\{ \sum_{\nu=0}^{h} b_\nu(z) f(y,z)^\nu \right\}^{-1}$$

with $b_\nu : Z \to \mathbb{C}$ $(\nu \in I_{0,h})$, $b_0(z) \neq 0$ and condition (18) holds for $i \in I_{0,h}$, $m \in I$, $r_{h,0}^{(m)}(z) = b_0(z)^{-1}$ $(m \in I)$ and recursion (19) may be applied for $i \in I_{0,h-1}$, $m \in I$ to produce these functions. When $e(y,z)$ is a rational function of $f(y,z)$, being the product of the two expressions given in formulae (26,27) and conditions (20) hold for a fixed m and all i, j such that either $i \leq h$ or $j \leq k$, relationship (3) may be used as described to produce all function values $r_{i,j}^{(m)}(z)$ for these values of i and j, and when $i=h$, $j \geq k$ or $i \geq h$, $j=k$, $r_{i,j}^{(m)}(z) = a_0(z)/b_0(z)$. These results hold in particular in the simple case of formulae (8). If, for example, $e(y,z) = d(y)$ is a k^{th} degree polynomial in y, $e(y,z)$ has a representation of the form (26) in which $f(y,z) = z-y$, $a_0(z)$ being $d(z)$. Similar observations hold with regard to the other two cases.

When $Y \equiv Z$, e and f are as in formulae (9), and in the special cases to which they relate, the remainder term formulae (11,12) confirm the above analysis. In formula (11), $D_{i,j}^{(m)}(y)$ is an i^{th} degree polynomial in $\phi(y)$ and $\theta_i(y)$ is a disposable i^{th} degree polynomial of the same type. If $d(y)$ is a rational function of $\phi(y)$ with numerator and denominator of degrees k

and h respectively, and $i=h$, $\theta_i(y)$ may be taken to be the denominator polynomial of $d(y).D_{i,j}^{(m)}(y)\theta_i(y)d(y)$ is then a polynomial of degree $i+k$ in $\phi(y)$. When $j\geq k$ the ϕ-derivative of order $i+j+1$ of this polynomial is zero, and $R_{i,j}^{(m)}(z)=0$. The same analysis holds for the case in which $j=k$ and $i\geq h$.

In the case of the remainder term formula (12), d and ϕ above are analytic within and upon a closed contour C, and $\phi(z)=\phi(y)$ only when $z=y$ for all y,z within and upon C. It is remarked that, x_ν $(\nu\in I)$ being a prescribed sequence of points within C, $\psi_\nu^{(m)}(y)$ being the $(\nu+1)^{th}$ degree polynomial in $\phi(y)$

(28) $$\psi_\nu^{(m)}(y)=\prod_{\tau=0}^{\nu}\{\phi(y)-\phi(x_{m+\tau})\} \quad (m,\nu\in I)$$

f being analytic within and upon C, and $f_\nu(z)=f(x_\nu)$

being independent of z $(\nu\in I)$, the divided differences defined by formula (15), now formed with respect to the sequence x_ν $(\nu\in I)$, may be expressed as

$$\delta_f^j\{f_m(z)\}=\frac{1}{2\pi i}\int_C \frac{f(y)\phi'(y)}{\psi_j^{(m)}(y)}dy \qquad (m,j\in I) \tag{29}$$

They are independent of z. If $f(y)$ is a polynomial of degree n in $\phi(y)$, $\delta_f^j\{f_m(z)\}=0$ $(j\in I_{n+1}, m\in I)$. The remarks made in the preceding paragraph concerning $D_{i,j}^{(m)}$ and θ_i in formula (11) apply with equal force to formula (12). Again when $d(y)$ is a rational function of $\phi(y)$, with numerator and denominator of degrees k and h respectively, suitable choice of θ_i again renders $D_{i,j}^{(m)}(y)\theta_i(y)d(y)$ a polynomial of degree $i+k$ in $\phi(y)$. The denominator term in the integrand in formula (12) has, apart from a change of sign, the form (28), with $x_\nu = y_\nu$ $(\nu \in I_{m,m+i+j})$, $x_{m+i+j+1} = z$. Subject to suitable choice of Θ_i, the integral in formula (12) represents a divided difference of order $i+j+1$ of a polynomial of degree $i+k$ in $\phi(y)$: when $j \geq k$ its value is zero, and $R^{(m)}_{i,j}(z) = 0$. Again the same analysis holds for the case in which $j=k$ and $i \geq h$.

2.5 Confluence

Throughout the above analysis it has been assumed that the argument values y_ν are distinct. When y_μ is associated with $n(\mu)$ confluent values $(\nu \in I_{\mu,\mu+n(\mu)-1})$ including itself, $f_\mu(z)$ is associated with $n(\mu)$ corresponding functions $f_\nu(z)$ $(\nu \in I_{\mu,\mu+n(\mu)-1})$ and, for $j \in I_{0,n(\mu)-1}$, $\tau \in I_{0,n(\mu)-j-1}$ the Newton series representations of the corresponding Lagrange forms appear as

(30) $$L_j^{(\mu+\tau)}(z) = e(y_\mu, z) - \sum_{\nu=1}^{j} d_\nu(y_\mu, z) f(y_\mu, z)^\nu$$

the coefficients $d_\nu(y_\mu, z)$ being confluent forms of functional divided differences. For the relevant index values, relationships (14,16) reduce to difference-differential recursions of limited utility. The problem of confluence may be dealt with by assuming that e and f are such that $x : Y \times Z \to C$ exists for which $\lim x(y,z) = 0$ as $z \to y$ in Z for each $y \in Y$ and, for such y and z

(31) $$f(y,z) - \sum_{\nu=1}^{n-1} c_\nu(y,z)\, x(y,z)^\nu = O\{x(y,z)^n\}$$

(32) $$e(y,z) - d(y) - \sum_{\nu=1}^{n-1} b_\nu(y,z)\, x(y,z)^\nu = O\{x(y,z)^n\}$$

for $n \in I_2$, where $c_\nu, d_\nu : Y \times Z \to \mathbb{C}$ and, in particular, $c_1(y,z) \neq 0$ for all $y \in Y, z \in Z$. The coefficients $d_\nu(y,z)$

occurring in formula (30) are obtained by evaluating $c_{h,k}$ $(h \in I_{1,n-1}, k \in I_{h,n-1})$ by setting $c_{h,1} = c_h(y,z)$ $(h \in I_{1,n-1})$ and thereafter

$$(33) \quad c_{h,k} = \sum_{\tau=1}^{h-k+1} c_{\tau,1} c_{h-\tau,k-1} \quad (k \in I_{2,n-1}, h \in I_{k,n-1})$$

when $d_1(y,z) = b_1(y,z)/c_1(y,z)$ and

$$(34) \quad d_\nu(y,z) = \{b_\nu(y,z) - \sum_{\tau=1}^{\nu-1} d_\tau(y,z) c_{\nu,\tau}\} c_1(y,z)^{-\nu} \quad (\nu \in I_{2,n-1})$$

y and n being y_μ and $n(\mu)$.

Formulae (33,34) merely implement a truncated composition of polynomials: given two polynomials $e[x]$ and $f[x]$, $f[x]$ having zero constant term, a truncated version of $e[x]$ is exhibited as a truncated formal power series in powers of $f[x]$. e and f satisfying relationships (31,32) and the $d_\nu(y,z)$ having been obtained.

$$(35) \quad e(y,z) - d(y) - \sum_{\nu=1}^{n-1} d_\nu(y,z) f(y,z) = O\{f(y,z)^n\}$$

Expressing $L_j^{(m+\tau)}(z) - d(y_\mu)$ as $L_j^{(m+\tau)}(z) - e(y_\mu,z) + e(y_\mu,z) - d(y_\mu)$, it then follows from formulae (30,35) that

$$L_j^{(\mu+\tau)}(z) - d(y_\mu) = O\{f(y_\mu,z)^{j+1}\} \quad (j \in I_{0,n(\mu)-1}, \tau \in I_{0,n(\mu)-j-1})$$

Assuming that the y_ν are so ordered that values of y_ν belonging to confluent subsets are grouped together, the values of the divided differences determined by use of formulae (30,33) are given by $\delta_f^0\{e_{\mu+\tau}(z)\} = e_\mu(z)$ $(\tau \in I_{0,n(\mu)-1})$ and $\delta_f^\nu\{e_{\mu+\tau}(z)\} = -d_\nu(y_\mu,z)$ $(\nu \in I_{1,n(\mu)-1}, \tau \in I_{0,n(\mu)-j-1})$. Subject to the existence condition (13), where now ν and τ refer to values of y_ν and y_τ belonging to different confluent subsets, the remaining divided differences $\delta_f^{j+1}\{e_m(z)\}$ are computed by use of recursion (16); for those values of m and j

for which formula (16) is used, $f_m(z)$ and $f_{m+j+1}(z)$ have distinct values.

The Lagrange form values $L_j^{(m)}(z)$ are determined by use of formulae (33, 34, 30) and subsequent use of recursion (14) subject to the existence condition (13) as described above, in the same way.

In terms of divided differences and Lagrange forms as defined in the preceding two paragraphs, and of the factorial functions (10) which remain unchanged, the general approximants $r_{i,j}^{(m)}$ are defined as follows. Let $\overset{\cdot}{\underline{\text{H}}}{}^{(m)}_{n,s|h,k}(z)$ be the $(h+1)\times(k+1)$ array whose $(\nu+1)^{th}$ row contains the elements $\delta_f^{s-n+\tau+\nu}\{e_{m+n-\tau}(z)\}$ $(\tau\in I_{0,k})$ for $\nu\in I_{0,h}$. Let $L_{j|i}^{(m)}(z)$ be the column array containing in succession the $i+1$ elements

$\Lambda_{i-\nu}^{(m+j+\nu+1)}(z) L_{j+\nu}^{(m)}(z)$ $(\nu \in I_{0,i})$. Let $\Delta_{|i}^{(m)}(z)$ be the row array containing the $i+1$ elements $\Delta_{i-\tau}^{(z)}$ $(\tau \in I_{0,i})$.

$r_{i,j}^{(m)}$ is given by formula (4) where now

(36) $N_{i,j}^{(m)}(z) = [\Xi_{i-1,j|i,i-1}^{(m)}(z) \,|\, L_{j|i}^{(m)}(z)]$

(37) $D_{i,j}^{(m)}(z) = [\Xi_{i,j+1|i-1,i}^{(m)}(z) \,||\, \Delta_{|i}^{(m)}(z)]$

The $(i+1)^{th}$ order determinant $N_{i,j}^{(m)}(z)$ is formed from the array obtained by juxtaposing the array $\Xi_{i-1,j|i,i-1}^{(m)}(z)$ and the column $L_{j|i}^{(m)}(z)$ in the indicated order, the $(i+1)^{th}$ order determinant $D_{i,j}^{(m)}(z)$ being defined by row-wise juxtaposition in the same way (when $i=0$, the Ξ-arrays are taken to be void).

If $D_{i,j}^{(m)}(z) \neq 0$ for all $z \in \bar{Z}$ $(m,i,j \in I)$, $r_{i,j}^{(m)}(z)$ is defined by formulae (4, 36, 37) for all $z \in \bar{Z}$ $(m,i,j \in I)$. If for all ν such that y_ν does not belong to a confluent subset containing more than one member, the simple condition (6) obtains, relationship (7) still holds. When y_ν is one of n confluent values y_τ $(\tau \in I_{\mu,\mu+n-1})$, conditions (31, 32) obtain, and either $\mu+n>m$ or $\mu \le m+i+j$, relationship (7) is to be replaced by

$$r_{i,j}^{(m)}(z) - d(y_\nu) = O\{f_\nu(z)^{\xi}\}$$

where $\xi = \min(\nu+n-m, i+j-1)$ in the first case and $\xi = \min(n, m+i+j-\mu+1)$ in the second.

Subject to the existence condition $D_{i,0}^{(m)}(z) \neq 0$ for a fixed $z \in Z$ $(m, i \in I)$, $D_{i,0}^{(m)}$ being given by formula (37), the reciprocal Lagrange forms values $M_i^{(m)}(z) = r_{i,0}^{(m)}(z)$ $(m, i \in I)$ may be computed by setting $M_0^{(m)}(z) = e_m(z)$ when y_m is not associated with a confluent subset containing more than one member. When y_m is one of $n(\mu)$ confluent values y_τ $(\tau \in I_{\mu,\mu+n(\mu)-1})$ the values $M_i^{(m)}(z)$ $(m \in I_{\mu,\mu+n(\mu)-1},\ i \in I_{0,\mu+n(\mu)-m-1})$ are obtained as follows. The coefficients $d_\nu(y_\mu,z)$ $(\nu \in I_{0,n(\mu)-1})$ are determined by use of formulae (33,34); thereafter coefficients $g_\nu(y_\mu,z)$ $(\nu \in I_{0,n(\mu)-1})$ are determined by use of the formulae $g_0(y_\mu,z) = e_\mu(z)^{-1}$,

$$g_\nu(y_\mu,z) = \Big\{\sum_{\tau=0}^{\nu-1}{}' g_\tau(y_\mu,z)\, d_{\nu-\tau}(y_\mu,z)\Big\} / e_\mu(z) \qquad (\nu \in I_{1,n(\mu)-1})$$

and then $M_i^{(m)}(z) = \Big\{\sum_{\nu=0}^{i}{}' g_\nu(y_\mu,z) f_\mu(z)^\nu\Big\}^{-1}$ for $i \in I_{0,\mu+n(\mu)-m}$

Thereafter, recursion (19) is used to obtain the values of the remaining $M_i^{(m)}(z)$.

If, for fixed m and $z \in Z$, $D_{i,j}^{(m)}(z) \neq 0$ $(i,j \in I)$ ($D_{i,j}^{(m)}$ being given by formula (37)), $f_\nu(z) \neq 0$ $(\nu \in I_m)$ and $[H_{i,j+1|i,i}^{(m)}(z)] \neq 0$ $(i,j \in I)$ the values of $r_{i,j}^{(m)}(z)$

$(i, j \in I)$ may be determined by use of formula (3,21) as described above for the nonconfluent case, the initial values $\varepsilon_{0,j}$ or $\varepsilon'_{i,0}$ now involving $L_j^{(m)}(z)$ or $M_i^{(m)}(z)$ determined as described in the preceding paragraphs. In the confluent case, certain of the $w_{i,j}$ given by formula (21) have equal values, but this causes no trouble. In addition to the conventions adopted in connection with formulae (36, 37), set $\omega_0^{(m)}(z)=0$, $\omega_1^{(m)}(z)=1$ and $\omega_k^{(m)}(z)=\sum_{\nu=0}^{k-1}\prod_{\tau=0}^{k-1}{}^{[\nu]} f_{m+\tau}(z)$ $(k \in I_2)$ and let $\Omega_{|i}^{(m)}(z)$ be the row array containing the $i+1$ elements $\omega_{i-\tau}^{(m)}(z)$ $(\tau \in I_{0,i})$. The function values lying at the intersections of the half rows and columns of the ε-array are given by

$$\varepsilon'_{i,j} = \frac{\left[\Omega_{|i}^{(m)}(z) \,\|\, \overleftarrow{\Xi}_{i,j+2|i-2,i}^{(m)}(z) \,\|\, \Lambda_{|i}^{(m)}(z)\right]}{\Lambda_{i|i+1}^{(m)}(z)\left[\Xi_{i-1,i+1|i-1,i-1}(z)\right]} \quad (i \in I_1, j \in$$

2.6 Zero finding algorithms

Subject to suitable conditions, the above algorithm for the truncated composition of polynomials functions when $e[x]$ is simply x, and then serves for the inversion of formal power series. In so doing it serves as the basis of a number of algorithms for determining the zeros of a function and motivates the use of the approximants $r_{i,j}^{(m)}(z)$ for the same purpose. Setting $e(y,z)=z-y$ and $f(y,z)=\phi(z)-\phi(y)$, ϕ being the function under treatment, relationships of the form (31,32) hold with $x(y,z)=z-y$, $c_\nu(y,z)=\phi^{(\nu)}(y)/\nu!$ $(\nu \in I_1)$ are Taylor series coefficients, $d(y)=0$, $b_1(y,z)=1$ and $b_\nu(y,z)=0$ $(\nu \in I_2)$. Subject to suitable conditions, $z-y=\sum_{\nu=1}^{\infty} q_\nu(y,z)\{\phi(z)-\phi(y)\}^\nu$, where the coefficients $d_\nu(y,z)$ obtained by use of formulae (33,34) are in order $d_1(y,z)=1/\phi'(y)$, $d_2(y,z)=-\frac{1}{2}\phi''(y)/\phi'(y)^3, \dots$.

Taking the points y_0, y_1 and y_2 to be confluent, the Lagrange forms derived from formulae (33,34,30) are $L_0^{(0)}(z)=z-y_0$,

$$L_1^{(0)}(z)=z-y_0-(\phi_0')^{-1}\{\phi(z)-\phi_0\},\ L_2^{(0)}(z)=L_1^{(0)}(z)-\frac{1}{2}(\phi_0')^{-3}\phi_0''\{\phi(z)-\phi_0\}^2$$

where $\phi_0=\phi(y_0),\dots,\phi_0''=\phi''(y_0)$. Newton's process, $z=y_0-\phi_0/\phi_0'$ is obtained from $L_1^{(0)}(z)$ by equating the latter to zero after setting $\phi(z)=0$. The third order process $z=y_0-\{\phi_0/\phi_0'\}-\{\frac{1}{2}\phi_0^2\phi_0''/\phi_0'^3\}$ is obtained from $L_2^{(0)}(z)$ in the same way. Applying relationship (3) to the initial values $\varepsilon_{0,j}=L_j^{(0)}(z)$ $(j\in I_{0,2})$ with $W_{0,0}=W_{1,1}=W_{\frac{1}{2},\frac{1}{2}}=\{\phi(z)-\phi_0\}^{-1}$ (since $y_1=y_2=y_0$) and using $\varepsilon_{1,1}=r^{(0)}_{1,1}(z)$ as just described, the further third order process $z=y_0-\phi_0\phi_0'/\{\phi_0'^2-\frac{1}{2}\phi_0\phi_0''\}$ is obtained.

The artifice described above is capable of further application. Taking y_1 and y_2 to be distinct, so that

$$L_0^{(j)}(z)=z-y_j \quad (j\in I_{1,2})$$

$$L_1^{(1)}(z)=z-y_1+\{(\phi(z)-\phi_1)(y_2-y_1)/(\phi_1-\phi_2)\}$$

from formula (14). Using $L_1^{(1)}(z)$ as described above, the regulus falsi $z=(y_2\phi_1-y_1\phi_2)/(\phi_1-\phi_2)$ is obtained. Taking y_0, y_1 to be confluent, so that $L_1^{(0)}(z)$ is as given above, and determining $L_2^{(0)}(z)$ by use of formula (14), the above modifications yield the Newton's method-regulus falsi combination

$$z=y_0-(\phi_0/\phi_0')-\{\phi_0^2(\phi_2-\phi_0-(y_2-y_0)\phi_0'\}/\{(\phi_2-\phi_0)^2\phi_0'\}$$

Applying relationship (3) to the initial values $\varepsilon_{0,j}=$

$L_j^{(0)}(z)$ $(j \in I_{0,2})$ with $w_{0,0} = \{\phi(z) - \phi_0\}^{-1}$ (since $y_0 = y_1$) and $w_{0,1} = w_{\frac{1}{2},\frac{1}{2}} = \{\phi(z) - \phi_2\}^{-1}$, and modifying $\epsilon_{1,1}$, the Newton's method – regulus falsi combination

$$z = y_0 + \frac{\phi_0}{\{\phi_0/(y_2 - y_0)\} - \{\phi_0'\phi_2/(\phi_2 - \phi_0)\}}$$

is obtained.

The preceding paragraphs have been given to provide worked examples of the general theory. In practice one simplifies the working and mechanises the processes involved, formulae (33, 34, 30, 3) then yielding an n^{th} order single point iteration process (since, in this case, $w_{i,j}$ occurring in formula (3) is constant, it may be taken to be unity). The more general theory yields multepoint processes (for a further application of the ε-algorithm to the problem of finding the zeros of a function, see [3,7]).

2.7 Extensions of the Lagrange-Bürmann expansion

The above treatment of the confluent case offers an interpretation of the theory of this paper. With C a closed contour in the complex plane, and d and ϕ analytic within and upon C, ϕ being such that $\phi(y)=\phi(z)$ only when $y=z$ for all y,z within and upon C, and y_μ being a point within C, the Lagrange–Bürmann expansion of $d(z)$ in powers of $\phi(z)-\phi(y_\mu)$, where z lies within C, is

$$d(z)=\sum_{\nu=0}^{j}\left\{\frac{1}{2\pi i}\int_C \frac{d(u)\phi'(u)}{\{\phi(u)-\phi(y_\mu)\}^{\nu+1}}du\right\}\{\phi(z)-\phi(y_\mu)\}^{\nu}+R_j(z)$$

where

$$R_j(z) = \frac{1}{2\pi i}\int_C \frac{d(u)\phi'(u)}{\{\phi(u)-\phi(z)\}\{\phi(u)-\phi(y_\mu)\}^{j+1}}du\}\{\phi(z)-\phi(y_\mu)\}^{j+1}$$

(see, for example, [10] $\underset{\sim}{1}$ Ch. 4 §5, [17] Ch. 7 §7.3). Taking $e(y,z)=d(y)$ to be independent of z, and $f(y,z)=\phi(z)-\phi(y)$ where d and ϕ are as just described, and letting y_μ be associated with $j+1$ confluent values y_ν ($\nu \in I_{\mu,\mu+j}$) the contour integral representation (29) of a functional divided difference reduces to

$$\delta_f^\nu\{g_\mu(z)\} = \frac{1}{2\pi i}\int_C \frac{d(y)\phi'(y)}{\{\phi(y)-\phi(y_\mu\}^{\nu+1}}dy$$

and is valid for $\nu \in I_{0,j}$. When $i=0$ in the remainder term formula (12), $D_{i,j}^{(m)}(z)=1$ and the disposable i^{th} degree polynomial $\Theta_i(y)$ in powers of $\phi(y)$ may also be taken to be 1. Again, when y_μ is associated with $j+1$ confluent values, formula (12) becomes $R_{0,j}^{(\mu)}(z)=R_j(z)$.

In short, using the Newton series representation (17) of the Lagrange form, the Lagrange-Bürmann expansion with remainder is

$$d(z) = L_j^{(\mu)}(z) + R_{0,j}^{(\mu)}(z)$$

Formulae (33,34) provide an algorithm for determining the coefficients in the Lagrange-Bürmann expansion from the Taylor series coefficients of $d(z)-d(y)$ and $\phi(z)-\phi(y)$ at the point $y=y_\mu$. They serve equally to determine the coefficients in an asymptotic version of the Lagrange-Bürmann expansion. The functions $e(y,z)$, $f(y,z)$ and $x(y,z)$ featuring in relationships (31,32) may be given more general form than $d(y)$, $\phi(z)-\phi(y)$ and $z-y$: an extension of the asymptotic version is thereby obtained. The associated

Lagrange forms are approximants derived from this extension. By taking a number of points of confluence into account, a multipoint form of this extension is obtained, together with associated Lagrange forms. The various functions $r_{i,j}^{(m)}$ obtained by application of relationships (3.21) to the associated Lagrange forms as described above are rational approximants associated with the multipoint form of the extended version of the asymptotic version of the Lagrange-Bürmann expansion.

References

1. Aitken A.C., On interpolation by proportional parts, without use of differences, Proc. Edin. Math. Soc., $\underset{\sim}{3}$ (1932) 56-76

2. Aitken A.C., Determinants and matrices, Oliver and Boyd, Edinburgh (1949)

3. Brezinski C., Sur un algorithme de résolution des systèmes non linéaires, Comptes Rendus de l'Acad. Sci. (Paris) Ser. A **272** (1971) 145–148

4. Brezinski C., Généralisation des extrapolations polynomiales et rationnelles, RAIRO, R1 (1972) 61-66

5. Claessens G., A useful identity for the rational Hermite interpolation table, Num. Math., **29** (1978) 227–231

6. Cordellier F., Utilisation de l'invariance homographique dans les algorithmes de losange, Proc. Conf. Bad Honnef 1983, Springer Lecture Notes, **1071** (1984) 62-94

7. Gekeler E., Über den ε-Algorithmus von Wynn, ZAMM, 51 (1971) 53-54

8. Gelfond A.O., The calculus of finite differences, Moscow (1959)

9. Jacobi C.G.J., Über die Darstellung einer Reihe gegebner Werthe durch eine gebrochne rationale Funktion, Jour. f.d. reine u. ang. Math., 30 (1846) 127-156

10 Markushevich A., The theory of analytic functions, Moscow (1970)

11. Neville E.H., Iterative interpolation, Jour. Ind. Math. Soc., 20 (1934) 87-120

12. Nörlund N.E., Vorlesung über Differenzenrechnung, Berlin, Springer (1937)

13. Romberg W., Vereinfachte numerische Integration,

Kon. Norsk. Videnskab. Selsk. Forhandl., **28** (1955) 30-36

14. Rutishauser H., Ausdehnung des Rombergschen Prinzips, Num. Math., **5** (1963) 48-54

15. Stoer J., Über zwei Algorithmen zur Interpolation mit rationalen Funktionen, Num. Math., **3** (1961) 285-304

16. Thiele T.N., Interpolationsrechnung, Leipzig, Teubner (1909)

17. Whittaker E.T. and G.N. Watson., A course of modern analysis, Cambridge University Press, (1965)

18. Wynn P., On a Procrustean technique for the numerical transformation of slowly convergent sequences and series, Proc. Camb. Phil. Soc., **52** (1956) 663-671