Interpolation by the use of rational functions

by

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1. The Thiele-Nörlund interpolation theory

A rational function $C_{2s}^{(m)}(\lambda)$ defined as the quotient of two polynomials of degree s contains $2s+1$ disposable coefficients; similarly, the rational function $C_{2s+1}^{(m)}(\lambda)$, defined as having numerator of degree $s+1$ and denominator of degree s, contains $2s+2$ disposable coefficients. It is permissible to hope that these coefficients may be determined by imposing conditions of the form

$$C_r^{(m)}(x_i) = f_i \tag{1}$$

for $i = m, m+1, \ldots, m+r$, where the x_i, f_i are prescribed

all numbers

argument and function values (these are assumed

considered in this paper are tacitly assumed to be complex;

to be complex numbers; this assumption is made functions are of a complex variable)

tacitly throughout the present paper).

The following scheme for constructing the complete set of interpolating rational functions $C_r^{(m)}(\lambda)$ ($r, m = 0, 1, \ldots$) from unboundedly many argument and function value pairs x_i, f_i ($i = 0, 1, \ldots$) has been proposed by Thiele ([11,13], [14] Ch. 15). From the initial values $\rho_{-1}^{(m)} = 0$ ($m = 1, 2, \ldots$), $\rho_0^{(m)} = f_m$ ($m = 0, 1, \ldots$) construct further numbers $\rho_r^{(m)}$ ($r = 1, 2, \ldots; m = 0, 1, \ldots$) by use of the relationships

$$(2) \qquad \rho_{r+1}^{(m)} = \rho_{r-1}^{(m+1)} + (x_{m+r+1} - x_m)\left(\rho_r^{(m+1)} - \rho_r^{(m)}\right)^{-1}$$

with $r, m = 0, 1, \ldots$. Thereafter construct the continued fraction convergents

$$(3) \quad C_r^{(m)}(\lambda) = \frac{\lambda - x_m}{\cancel{\rho^{(m)}}\,\rho_1^{(m)} +} \; \frac{\lambda - x_{m+1}}{\rho_2^{(m)} - \rho_0^{(m)} +} \; \cdots \; \frac{\lambda - x_{m+r-1}}{\rho_r^{(m)} - \rho_{r-2}^{(m)}}$$

by use of the three term recurrence relationships

$$(4)\quad N_{-1}^{(m)}(\lambda)=1,\ N_0^{(m)}(\lambda)=f_m,\ D_{-1}^{(m)}(\lambda)=0,\ D_0^{(m)}(\lambda)=1$$

$$(5)\ N_r^{(m)}(\lambda)=(\rho_r^{(m)}-\rho_{r-2}^{(m)})N_{r-1}^{(m)}(\lambda)+(\lambda-x_{m+r-1})N_{r-2}^{(m)}(\lambda)$$

$$(6)\ D_r^{(m)}(\lambda)=(\rho_r^{(m)}-\rho_{r-2}^{(m)})D_{r-1}^{(m)}(\lambda)+(\lambda-x_{m+r-1})D_{r-2}^{(m)}(\lambda)$$

for $r=1,2,\ldots;m=0,1,\ldots$, when

$$(7)\qquad C_r^{(m)}(\lambda)=\frac{N_r^{(m)}(\lambda)}{D_r^{(m)}(\lambda)}$$

for $r,m=0,1,\ldots$.

Conditions that are sufficient to ensure both that all rational functions $C_r^{(m)}(\lambda)$ can be constructed by use of the above scheme and that they have the required interpolating property (1) may be formulated in terms of determinants; determinantal formulae can also be given for the numbers and functions constructed. Such formulae are made more concise by the use of a special notation. The array, composed of an unbounded number of rows and $k+1$ columns, whose $(i+1)^{th}$ row contains the elements $1, x_i, \dots, x_i^k$ $(i=0,1,\dots)$ is denoted by X_k; $f^{(j)}(\lambda)X_k$, where $j=-1, 0$ or 1 is a similar array whose $(i+1)^{th}$ row contains the elements $f_i(\lambda-x_i)^j$, $f_i(\lambda-x_i)^j x_i, \dots, f_i(\lambda-x_i)^j x_i^k$ $(i=0,1,\dots)$ (if X_k is regarded as a matrix with an unbounded number of rows, $f^{(j)}(\lambda)$ may be regarded as a matrix with diagonal elements $f_i(\lambda-x_i)^j$ $(i=0,1,\dots)$ and zero elements otherwise, when $f^{(j)}(\lambda)X_k$ may be regarded as the indicated matrix product); fX_k

is $f^{(0)}(\lambda)X_k$; the symbol $[X_k, f^{(j)}(\lambda)X_\ell]_m$ $(k, \ell \geq 0)$ denotes the determinant formed from the juxtaposition of $k+\ell+2$ rows taken in succession from X_k and $f^{(j)}X_\ell$, beginning with row $(m+1)$ of each of these arrays: the $(i+1)^{th}$ row of this determinant contains the elements $1, x_{m+i}, \ldots, x_{m+i}^k$, $f_{m+i}(\lambda - x_{m+i})^j, f_{m+i}(\lambda - x_{m+i})^j x_{m+i}, \ldots, f_{m+i}(\lambda - x_{m+i})^j x_{m+i}^\ell$ $(i = 0, \ldots, k+\ell+1)$; this notation is extended to the case in which $\ell < 0$: in this case the determinant in question is formed from rows of X_k alone (undefined arrays are tacitly discarded from determinantal formulae throughout); $\Pi_k^{(m)}(\lambda)$ is the continued product $(\lambda - x_m)(\lambda - x_{m+1}) \ldots (\lambda - x_{m+k})$.

Concerning the behaviour of the determinants $[X_k, fX_\ell]_m$ and the implied structure of the arrays of interpolating functions $C_r^{(m)}(\lambda)$, three cases of principal interest may be distinguished. In the first, all determinants

$$(8) \quad [X_{r-1}, fX_{r-1}]_m, \quad [X_r, fX_{r-1}]_m$$

for $r = 1, 2, \ldots; m = 0, 1, \ldots$ are nonzero; the notation $\{x, f\} \in E$ is used to indicated that this condition holds. Secondly, all determinants (8) for $r = 1, \ldots, n; m = 0, 1, \ldots$ are nonzero, but all determinants $[X_n, fX_n]_m$ $(m = 0, 1, \ldots)$ are zero, a set of conditions indicated by the symbol $\{x, f\} \in E_{2n}$. Thirdly, all determinants (8) for $r = 1, \ldots, n; m = 0, 1, \ldots$ and also $[X_n, fX_n]_m$

$(m=0,1,\ldots)$ are nonzero, but all determinants $[X_{n+1}, fX_n]_m$ $(m=0,1,\ldots)$ are zero, a set of conditions indicated by the symbol $\{x,f\} \in E_{2n+1}$. When $\{x,f\} \in E$, no rational function f for which $f_i = f(x_i)$ $(i=0,1,\ldots)$ exists; when $\{x,f\} \in E_{2n}$, $f_i = f_{2n}(x_i)$ $(i=0,1,\ldots)$ where $f_{2n}(\lambda)$ is the irreducible quotient of two polynomials of degree n, and the coefficient of λ^n in the denominator is nonzero; when $\{x,f\} \in E_{2n+1}$, $f_i = f_{2n+1}(x_i)$ $(i=0,1,\ldots)$ where $f_{2n+1}(\lambda)$ is now an irreducible quotient of two polynomials, of degrees $n+1$ and n in the numerator and denominator respectively, and the coefficient of λ^{n+1} in the numerator is nonzero. When $\{x,f\} \in E$, all numbers $\rho_r^{(m)}$ and functions $N_r^{(m)}$, $D_r^{(m)}$ and $C_r^{(m)}$

of formulae (2 - 7) can be constructed; when $\{x, f\} \in E_k$, all numbers $\rho_r^{(m)}$ and functions $N_r^{(m)}, D_r^{(m)}$ and $C_r^{(m)}$ for $r = 1, \ldots, k; m = 0, 1, \ldots$ can be constructed, the numbers $\rho_k^{(m)}$ $(m = 0, 1, \ldots)$ are identically equal, and $C_k^{(m)}(\lambda) = f_k(\lambda)$ for $m = 0, 1, \ldots$ $(k = 2n, 2n+1)$. All functions $C_r^{(m)}$ that can be constructed under any of the three preceding conditions have the interpolation property (1).

The numbers and constituent polynomials that may be constructed have the following representations

$$(9) \quad \rho_{2s-1}^{(m)} = \frac{[X_s, fX_{s-2}]_m}{[X_{s-1}, fX_{s-1}]_m}, \quad \rho_{2s}^{(m)} = \frac{[X_{s-1}, fX_s]_m}{[X_s, fX_{s-1}]_m}$$

$$(10) \quad N_{2s-1}^{(m)}(\lambda) = \frac{[X_{s-1}, f^{(-1)}X_{s-1}]_m \overline{\Pi}_{2s-1}^{(m)}}{[X_{s-1}, fX_{s-1}]_m}, \quad N_{2s}^{(m)}(\lambda) = \frac{[X_{s-1}, f^{(-1)}X_s]_m \overline{\Pi}_{2s}^{(m)}}{[X_s, fX_{s-1}]_m}$$

$$(11)\quad D^{(m)}_{2s-1}(\lambda) = \frac{[X_s, f^{(1)}X_{s-2}]_m}{[X_{s-1}, fX_{s-1}]_m}, \quad D^{(m)}_{2s}(\lambda) = \frac{[X_s, f^{(1)}X_{s-1}]_m}{[X_s, fX_{s-1}]_m}$$

for $s = 1, 2, \dots; m = 0, 1, \dots$. (Short formulae are written two to a line where convenient, and distinguished in reference by the letters a, b; the ~~formulae~~ formula for $D^{(m)}_{2s-1}(\lambda)$, for example, is (11 a)).

in the above representations

Nörlund ([14] Ch. 15 p. 420) observes that the

~~in the representations of~~

denominators of $\rho^{(m)}_{2s-1}$, $N^{(m)}_{2s-1}(\lambda)$ and $D^{(m)}_{2s-1}(\lambda)$ are

(to $d^{(m)}_{2s-1}$, as indicated; this a property exploited to shorten the ~~above formulae~~)

all equal, that the numerator of $N^{(m)}_{2s-1}(\lambda)$ is obtained

in that

from the denominator of $\rho^{(m)}_{2s-1}$ by replacing f by $f^{(-1)}(\lambda)$ and multiplying by $\Pi^{(m)}_{2s-1}(\lambda)$, that the numerator of $D^{(m)}_{2s-1}(\lambda)$ is obtained from the numerator of $\rho^{(m)}_{2s-1}$ by replacing f by $f^{(1)}(\lambda)$ and that these three remarks with $2s-1$ consistently replaced by $2s$ also hold true. He then makes the following remark: Aus jedem Ergebnis über reziproke Differenzen können wir daher leicht auch eine Aussage für die Näherungsbrüche herleiten; he proceeds, in masterly fashion, to construct much theory upon the basis of this ~~above~~ observation. Nevertheless, supported by a single observation alone, the statement made is perhaps a little exuberant. It is the principal purpose of this paper to point out that, using another very simple relationship (namely, in particular, that if x_i is replaced by $(\lambda - x_i)^{-1}$, $\rho_{2r}^{(m)}$ becomes $C_{2r}^{(m)}(\lambda)$) many results suggested by the behaviour of reciprocal differences may be obtained for convergents.

2. The ε- and μ-algorithms

A further notation is introduced: the array, composed of an unbounded number of rows and $j-k+1$ columns, whose $(i+1)^{th}$ row contains the elements $(\lambda-x_i)^k, (\lambda-x_i)^{k+1}, \ldots, (\lambda-x_i)^{j}$ $(i=0,1,\ldots)$ is denoted by $X_{j,k}(\lambda)$; the elements of the array $fX_{j,k}(\lambda)$ are formed by multiplying successive rows of $X_{j,k}(\lambda)$ throughout by f_i $(i=0,1,\ldots)$ as for the formation of fX_k above; $X_k(\lambda)$ and $fX_k(\lambda)$ denote $X_{0,k}(\lambda)$ and $fX_{0,k}(\lambda)$ respectively. The arrays X_k and $X_k(\lambda)$ are related by means of the symbolic equation

(12) $$\text{column } (j+1) \text{ of } X_k(\lambda) = (-1)^j \sum_{s=0}^{j} \binom{j}{s} (-\lambda)^{j-s} \{\text{column } (s+1) \text{ of } X_k\}$$

for $j=0,1,\ldots,k$; fX_k and $fX_k(\lambda)$ are similarly related.

The arrays $\mathbb{X}_{j,k}^{(\lambda)}, \dots, f\mathbb{X}_k^{(\lambda)}$ are used in conjunction with the displacement index m in determinantal expressions in a manner analogous to that concerning the arrays X_k and fX_k in formulae (9 - 11). By use of relationship (12) and others of its kind, multiplication by powers of -1 and where appropriate by factors of the form $(\lambda - x_i)$, a set of formulae alternative to (9 - 11) may be derived; the formulae may be obtained as follows from those displayed: replace expressions of the form $[X_j, fX_k]_m$ consistently by

$[fX_k(\lambda), X_j(\lambda)]_m$

$\cancel{[\mathbb{Y}_j, f\mathbb{X}_k]_m}$; replace the numerators of $N_{2s-1}^{(m)}(\lambda)$ and $N_{2s}^{(m)}(\lambda)$, both of which have the form $\cancel{[X_j, f^{(-1)} X_k]_m}$ $[X_j, f^{(-1)}(\lambda) X_k]_m \Pi_{j+k-1}^{(m)}(\lambda)$ by the corresponding expressions

$[fX_k(\lambda), X_{1,j}(\lambda)]_m$

$\cancel{[Y_{1,j+1}, fY_k]_m}$; replace the numerators of $D_{2s-1}^{(m)}(\lambda)$ and $D_{2s}^{(m)}(\lambda)$, both of which have the form $[X_j, f^{(1)}(\lambda)X_k]_m$ by the corresponding expressions $[fX_{1,k+1}(\lambda), fX_j(\lambda)]_m$ $\cancel{[Y_j, fY_{1,k+1}]_m}$. By division, the following further two formulae may be derived:

$$(13)\quad C_{2s-1}^{(m)}(\lambda) = \frac{[Y_{1,s}, fY_{s-1}]_m}{[Y_s, fY_{1,s-1}]_m}, \quad C_{2s}^{(m)}(\lambda) = \frac{[Y_{1,s}, fY_s]_m}{[Y_s, fY_{1,s}]_m}$$

for $s = 1, 2, \ldots$; $m = 0, 1, \ldots$.

Methods for the evaluation of the rational functions $C_r^{(m)}(\lambda)$ alternative to the use of formulae (2-7) can now be described.

Theorem 1. Let λ be a fixed finite complex number unequal to x_i $(i = 0, 1, \ldots)$. Set

(14) $$z_i = (\lambda - x_i)^{-1} \qquad \eta(\lambda, x_i) = (\lambda - x_i)^{-1}$$

for $i = 0, 1, \dots$

i) Let $\{x, f\} \in E$ and

(15) $$[X_s, f^{(1)}(\lambda) X_{s-1}]_m \neq 0$$

for $s = 1, 2, \dots; m = 0, 1, \dots$. Numbers $\sigma_r^{(m)}(\lambda)$ $(r, m = 0, 1, \dots)$ can be constructed from the initial values $\sigma_{-1}^{(m)}(\lambda) = 0$ $(m = 1, 2, \dots)$, $\sigma_0^{(m)}(\lambda) = f_m$ $(m = 0, 1, \dots)$ by use of the relationship

$$\{\eta(\lambda, x_{m+r+1}) - \eta(\lambda, x_m)\}$$

(16) $$\sigma_{r+1}^{(m)}(\lambda) = \sigma_{r-1}^{(m+1)}(\lambda) + \{z_{m+r+1}(\lambda) - z_m(\lambda)\}\{\sigma_r^{(m+1)}(\lambda) - \sigma_r^{(m)}(\lambda)\}^{-1}$$

with $r, m = 0, 1, \dots$ and, in particular,

(17) $$\sigma_{2s}^{(m)}(\lambda) = C_{2s}^{(m)}(\lambda)$$

for $s, m = 0, 1, \dots$.

ii) Let $\{x, f\} \in E$ and

(18) $$[X_{s-1}, f^{(-1)}(\lambda) X_{s-1}]_m \neq 0$$

for $s = 1, 2, \ldots; m = 0, 1, \ldots$. Numbers $\mu_r^{(m)}(\lambda)$ $(r, m = 0, 1, \ldots)$ can be constructed from the initial values $\mu_{-1}^{(m)}(\lambda) = 0$ $(m = 1, 2, \ldots)$, $\mu_0^{(m)}(\lambda) = \overset{\nu(\lambda, z_m)}{\cancel{z_m(\lambda)}} f_m$ $(m = 0, 1, \ldots)$ by use of a relationship obtained from (16) by replacing σ by μ, for $r, m = 0, 1, \ldots$, and in particular

(19) $$\mu_{2s+1}^{(m)}(\lambda) = C_{2s+1}^{(m)}(\lambda)^{-1}$$

for $s, m = 0, 1, \ldots$.

iii) Let $\{x, f\} \in E_{2n+k}$ $(k = 0, 1)$

a) Let condition (15) hold for $s = 1, \ldots, n+k; m = 0, 1, \ldots$. Numbers $\sigma_r^{(m)}(\lambda)$ $(r = 1, \ldots, 2(n+k); m = 0, 1, \ldots)$ can be constructed as described in clause (i): relationship (17) holds for $s = 0, \ldots, n+k; m = 0, 1, \ldots$ and, in particular,

$$\sigma_{2(n+k)}^{(m)}(\lambda) = f_{2n+k}(\lambda) \quad (m = 0, 1, \ldots)$$

b) Let condition (18) hold for $s=1,\ldots,n+1; m=0,1,\ldots$.
Numbers $\mu_r^{(m)}(\lambda)$ $(r=1,\ldots,2n+1; m=0,1,\ldots)$ can be constructed as described in clause (ii); relationship (19) holds for $s=0,\ldots,n; m=0,1,\ldots$ and, in particular,
$\mu_{2n+1}^{(m)}(\lambda) = f_{2n+k}(\lambda)^{-1}$ $(m=0,1,\ldots)$.

Proof. First assume that the conditions of clause (i) hold. The algorithm of relationship (16) is obtained from that of relationship (2) simply by replacing x_i by $(\lambda-x_i)^{-1}$ $(i=0,1,\ldots)$. If the determinants obtained from the pairs (8) by substituting $(\lambda-x_i)^{-1}$ for x_i consistently are all nonzero, all numbers $\sigma_r^{(m)}(\lambda)$ can be constructed. The first determinant reduces to

$$\{\Pi_{2s-1}^{(m)}(\lambda)\}^{-s-1}[X_{s-1}(\lambda), fX_{s-1}(\lambda)]_m = \{\Pi_{2s-1}^{(m)}(\lambda)\}^{-s-1}[X_{s-1}, fX_{s-1}],$$

The requirement that its value should be nonzero is already satisfied in virtue of the imposed condition that the first of the determinants (8) is nonzero. The second determinant reduces to $\{-\Pi_{2s}^{(m)}(\lambda)\}^{s}[X_s, f^{(1)}(\lambda)X_{s-1}]_m$ and it is nonzero if condition (15) is satisfied

If $\{x, f\} \in E$, all numbers $\rho_r^{(m)}$, $N_r^{(m)}(\lambda)$ and $D_r^{(m)}(\lambda)$ $(r, m = 0, 1, \ldots)$ of formulae (2-6) can be constructed; since the determinant in condition (15) occurs in the denominator of $C_{2s}^{(m)}(\lambda)$ and is assumed to be nonzero, the quotients $C_{2s}^{(m)}(\lambda)$ $(s, m = 0, 1, \ldots)$ at least can be evaluated by use of formula (7).

A determinantal formula may be given for the numbers $\sigma_{2s}^{(m)}(\lambda)$ produced by use of formula (16);

it is obtained from formula (9b) by replacing x_i by $(\lambda - x_i)^{-1}$ consistently. Multiplying both numerator and denominator of the modified quotient by $(\Pi_{2s}^{(m)}(\lambda))^s$, the formula

$$\sigma_{2s}^{(m)}(\lambda) = \frac{[Y_{1,s}, f Y_s]_m}{[Y_s, f Y_{1,s}]_m} \qquad \sigma_{2s-1}^{(m)} = \frac{[Y_{0,s}, f Y_{2,s-1}]}{[Y_{0,s-1}, f Y_{1,0}]} \tag{20}$$

is obtained. Comparison with the second of formulae ([illegible]) yields the stated result of clause (i). The remaining results of the theorem are proved in the same way

For the sake of completeness, it is pointed out that the numbers $\sigma_{2s-1}^{(m)}(\lambda)$ produced by means of recursion (10) are values of rational functions of λ, with denominator $\Pi_{2s-1}^{(m)}(\lambda)$, they are well determined without the imposition of a further condition. The same considerations hold with regard to the numbers $\mu_{2s}^{(m)}(\lambda)$.

3. Interpolation and Extrapolation

The simple observation that if the argument values x_i are replaced by functions $(\lambda - x_i)^{-1}$ the reciprocal differences $\rho_{2s}^{(m)}$ become interpolating functions $C_{2s}^{(m)}(\lambda)$ has already produced the ϵ-algorithm of relationship (16), the simplest and most economical method, subject to the stated conditions, for evaluating these functions; the observation also leads directly, as will be shown below, to new interpolatory theory. Once made, the observation is trivial, and its implications are not difficult to work out; perhaps its most interesting feature is that it has not been made before. The simple relationship between reciprocal differences and rational interpolating functions was not so much discovered as forced upon the author's attention while working out the consequences of principles underlying the process of interpolation and the transformation of divergent series. These two subjects have recently become increasingly important in computational mathematics; new theory in what once might have been considered dead subjects is constantly being developed (mention may be made of recent generalisations of polynomial interpolation described in [6, 9-11]); it is highly probable that the principles concerned will find further application, and for this reason they are now outlined.

Suppose that an interpolatory function

(21) $\quad F_r(\lambda | x_m, \ldots, x_{m+r}; f_m, \ldots, f_{m+r})$

which assumes the value f_i when $\lambda = x_i$ $(i = m, \ldots, m+r)$ is available, and that this function possesses the property that if ϕ belongs to a certain class $\mathbb{C}_I$ of functions, $F_r(\lambda | x_m, \ldots, x_{m+r}; \phi(x_m), \ldots, \phi(x_{m+r})) = \phi(\lambda)$ for all distributions of arguments $x_m, \ldots, x_{m+r}$ and all λ for which this ~~eq~~ equation is meaningful. Suppose further that an estimate of the limit S of the sequence $S_0, S_1, \ldots$ is required, and that ψ is some function defined over the nonnegative integers for which $\lim\limits_{i=\infty} \psi(i) = \alpha$ exists. The number $F_r(\alpha | \psi(m), \ldots, \psi(m+r); S_m, \ldots, S_{m+r})$ offers an approximation to the value of S_i for that value of

i for which $\psi(i)$ tends to ∞: it offers an approximation to S. If $S_i = \phi\{\psi(i)\}$ $(i=m,\dots,m+r)$ where $\phi \in C_I$, and the estimate can be constructed, it yields the exact value of S. If, subject to appropriate conditions, a complete set of interpolatory functions (21) can be constructed by means of a recursive process, a corresponding extrapolation algorithm can be devised. It may even occur that a recursive process exists for constructing the values of the functions (21) with $\lambda=\infty$ that is simpler than the process which operates for general values of λ; in this case a correspondingly simpler extrapolation process exists.

Conversely, it is sometimes possible to obtain an interpolation process from an extrapolation algorithm. Suppose that an extrapolatory estimate

(22) $G_r(x_m, \ldots, x_{m+r}; S_m, \ldots, S_{m+r})$

of the limit S of the sequence $S_0, S_1, \ldots$ is available, that whenever the limit is well determined

$$\lim G_r(x_m, \ldots, x_{m+r}; S_m, \ldots, S_{m+r}) = S,$$

as x_i tends to infinity $(i = m, \ldots, m+r)$, and that G_r possesses the property that if θ belongs to a certain class $\mathbb{C}_E$ of functions for which $S_\theta = \lim_{x=\infty} \theta(x)$ exists, then $G_r(x_m, \ldots, x_{m+r}; \theta(x_m), \ldots, \theta(x_{m+r})) = S_\theta$ for all distributions of the $x_m, \ldots, x_{m+r}$ for which the left hand member of this equation can be evaluated. Assume further that for all λ belonging to a prescribed set Λ, $\xi(u, \lambda)$, regarded as a function of u, is a member of $\mathbb{C}_E$, that the value of the function $\eta(\lambda, y)$ becomes infinite as the value of y tends to λ, and that $\xi\{\eta(\lambda, y), \lambda\} = f(y)$ is independent of λ. (Pairs of functions ξ and η enjoying the latter property are easily obtained; for example, with τ^{-1} and τ two functions such that $\tau^{-1}\{\tau(\lambda)\} = \lambda$, $\eta(\lambda, y) = \tau(\lambda - y)$ and $\xi(u, \lambda) = f\{\lambda - \tau^{-1}(u)\}$ may be taken.) Replacing x_i by $\eta(\lambda, x_i)$ and S_i by $f_i = F(x_i)$ $(i = m, \ldots, m+r)$, where F is a prescribed function, expression (22) becomes

(23) $G_r(\eta(\lambda, x_m), \ldots, \eta(\lambda, x_{m+r}); f_m, \ldots, f_{m+r})$.

The latter offers an estimate of the value of $F(x_i)$ for that x_i for which $\eta(\lambda, x_i)$ tends to infinity: it offers an approximation to $F(\lambda)$. The first property of the extrapolation result described above yields the interpolatory result that the limit of the value of expression (23) as λ tends to x_i is f_i $(i=m,\dots,m+r)$. Replacing f_i by $\xi\{\eta(\lambda,x_i),\lambda\}$ $(i=m,\dots,m+r)$, the algebraic result is obtained that if $f_i=f(x_i)$ $(i=m,\dots,m+r)$ where $f\in\mathbb{C}_E$, the value of expression (23) is, when it can be determined, that of $f(\lambda)$ for all $\lambda\in\Lambda$. If, subject to appropriate conditions, an extrapolation algorithm can be applied to construct a set of estimates (22), a corresponding recursive ~~process~~ interpolatory process may be devised for the evaluation of the functions (23).

The Lagrange interpolation polynomial of degree r

in the variable λ, which assumes the value f_i when $\lambda = x_i$ $(i = m, \ldots, m+r)$ is the simplest example of an interpolating function of the form (21). These polynomials may be constructed by the use of divided differences (see, for example, [14] Ch. 1), or by use of the Aitken - Neville scheme [1, 12]. Taking $\psi(i) = (i+1)^{-1}$ in the above exposition, recursive schemes for estimating the limit of a sequence S_i $(i = 0, 1, \ldots)$ may be obtained; when S_i is an n^{th} order polynomial in $(i+1)^{-1}$, the schemes yield the exact value of the limit. (This method of extrapolation is violently unstable; a recursive extrapolation process based upon the use of divided differences yields a scheme which provides correction terms at each stage; the growth of error in the generation of these terms may be monitored, and the stage at which the terms no longer offer effective correction can be detected; see [24]).

Taking $\psi(i) = 2^{-i}$, the stable but computationally expensive Romberg scheme (see, for example, [18]) is obtained.

With $\psi(i) = x_i$ $(i = m, \ldots, m+r)$ the Lagrange interpolation polynomial process yields a general extrapolation algorithm producing estimates of the form (22). An interpolation process may be obtained from this extrapolation algorithm, as described above; but this path merely leads back to the Aitken–Neville process upon which the algorithm is based.

The quotient $C_{2s}^{(m)}(\lambda)$ of formula (7) is a further

example of an interpolatory function of the form (21).
As is easily verified from formulae (7, 9b-11b), $\lim_{\lambda=\infty} C_{2s}^{(m)}(\lambda) = \rho_{2s}^{(m)}$. Taking $\psi(i)$ $(i=0,1,\ldots)$ to be an appropriate sequence of increasing numbers and $\alpha=\infty$ in the above exposition, the algorithmic process $\rho_{-1}^{(m)}=0$ $(m=1,2,\ldots)$, $\rho_0^{(m)}=S_m$ $(m=0,1,\ldots)$

(24) $\rho_{r+1}^{(m)} = \{\psi(m+r+1)-\psi(m)\}(\rho_r^{(m+1)}-\rho_r^{(m)})^{-1}$

for $r, m = 0,1,\ldots$ for obtaining estimates $\rho_{2r}^{(m)}$ $(r,m=0,1,\ldots)$ of the limit of the sequence S_i $(i=0,1,\ldots)$ is devised. If S_i is the quotient of two n^{th} degree polynomials in the variable $\psi(i)$, with coefficients independent of i, and the numbers $\rho_{2n}^{(m)}$ $(m=0,1,\ldots)$ can all be constructed, each of them is equal to the limit in question. This algorithm with the ~~form~~ $\phi(i) = i$ was given by the author [22a].

With f_i replaced by S_i, the even order reciprocal difference $\rho_{2s}^{(m)}$ of formula (9b) offers an example of an extrapolatory estimate of the form (22). As is easily verified, its limiting value as x_i tends to infinity is S'_i $(i=m,\dots,m+r)$ as required. For the estimates $\rho_{2s}^{(m)}$, $\mathbb{C}_E$ is the class of quotients of s^{th} degree polynomials with nonzero coefficient of the s^{th} power of the variable in the denominator. An interpolatory function can be derived from formula (9b) by replacing x_i by $\eta(\lambda, x_i) = (\lambda - x_i)^{-1}$ $(i=m,\dots,m+r)$; it is the function $\sigma_{2n}^{(m)}(\lambda)$ of formula (21). An algebraic result may be deduced from the above general theory by taking $\xi(u,\lambda) = f_{2n}(\lambda - u^{-1})$; then $\xi\{\eta(\lambda,y),\lambda\} = f_{2n}(y)$ as required. If $f_{2n}(y)$ is the quotient of two n^{th} degree polynomials in the variable y, $f_{2n}(\lambda - u^{-1})$ is also such a quotient in the variable u and belongs to $\mathbb{C}_E$ for all λ such that $f_{2n}(\lambda)$ is finite. If all interpolatory estimates $\sigma_{2n}^{(m)}(\lambda)$ $(m=0,1,\ldots)$ can be evaluated, each of them has the value $f_{2n}(\lambda)$. A simple algorithm for the construction of the reciprocal differences $\rho_{2r}^{(m)}$ exists; a corresponding recursive process for the evaluation of the interpolatory functions $\sigma_{2r}^{(m)}(\lambda)$ may be devised; it is that of formula (16).

The above paragraphs offer an explanation of how the ϵ-algorithm was discovered; they have further applicatio[n]

Polynomials and rational functions are not the only functions to serve as a basis for interpolation and extrapolation schemes. The above explanation may serve as a principle for the development of further theory of interpolation and extrapolation processes based upon more general functions.

The extrapolation algorithm of relationship (24) is an example of an algorithm which functions by producing numbers $\varepsilon_r^{(m)}$ $(r=1,2,\ldots;\ m=0,1,\ldots)$ by use of the recursion

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + \gamma_r^{(m)} \left(\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)}\right)^{-1} \tag{25}$$

for $r, m = 0,1,\ldots$. Not all two-dimensional sequences have the form $\chi_r^{(m)} = \psi(m+r+1) - \psi(m)$ $(r, m = 0, 1, \ldots)$; the sequence $\chi_r^{(m)} = C$ $(r, m = 0, 1, \ldots)$ does not. The constant C in the resulting recursion is not of great significance: if the numbers produced by its use are written as $\varepsilon_r^{(m)}(C)$, then $\varepsilon_{2r}^{(m)}(C) = \varepsilon_{2r}^{(m)}(1)$, $\varepsilon_{2r+1}^{(m)}(C) = C\varepsilon_{2r+1}^{(m)}(1)$ $(r, m = 0, 1, \ldots)$. Without substantial loss, the special value $C = 1$ may be taken, and the resulting recursion has the form (25) with $\chi_r^{(m)}$ omitted. The theoretical basis for the associated algorithm, the ε-algorithm [23], is provided by the following result: let $P_{i,j}(\lambda)$ be that irreducible rational function whose numerator and denominator polynomials are of degrees $\leq j$ and $\leq i$ respectively, whose series expansion in ascending powers of λ agrees with a prescribed series $\sum_{k=0}^{\infty} t_k \lambda^k$ ($t_0 \neq 0$) for the largest number of initial terms. ($P_{i,j}$ is the approximating fraction considered in more general form by Jacobi [7] who derived various determinantal formulae; in its special form it was studied by Frobenius [5] and Padé [15]). Let the initial values to which the ε-algorithm is applied be provided by the partial sums $S_m = \sum_{k=0}^{m-1} t_k \lambda^k$ ($m = 0, 1, \ldots$). Subject to certain conditions involving the t_k, $\varepsilon_{2r}^{(m)} = P_{r,m+r-1}(\lambda)$ ($r, m = 0, 1, \ldots$). (Additional initial values, leading to the evaluation of the $P_{i,j}(\lambda)$ for $j < i-1$ may be introduced [25]). While idly investigating the formulae resulting from the choice $\chi_r^{(m)} = 1$ in relationship (25) (i.e. the most simple choice inconsistent with the general form (24)) the author noted that expressions obtained for the numbers $\varepsilon_{2r}^{(m)}$ were equivalent to extrapolatory determinantal expressions, simplified versions of those due to Jacobi and used by Frobenius, previously published by Schmidt [19] and republished by Shanks [20]. In this way the ε-algorithm was discovered.

A further form of the algorithm of relationship (25) has been introduced by Claessens [3]; it is that in which $\gamma_r^{(m)} = (\lambda - x_{m+r+1})^{-1}$ $(r,m = 0,1,\ldots)$. The theoretical basis for this algorithm, in simplified form, is as follows: let x_k and f_k $(k=0,1,\ldots)$ be prescribed argument and function value sequences, the x_k being distinct.

Subject to certain existence conditions, let $R_{i,j}(\lambda)$ be that interpolatory rational function whose numerator and denominator polynomials ~~of~~ are of degrees j and i respectively, which assumes the value f_k when $\lambda = x_k$ $(k=0,\ldots,i+j)$. If the initial values to which the extended ε-algorithm is applied are provided by the successive partial sums $S_m = R_{0,m}(\lambda)$ $(m=0,1,\ldots)$ of the Newton interpolation series derived from the sequences $\{x_k\}$, $\{f_k\}$ (the $R_{0,m}(\lambda)$ are Lagrange interpolating polynomials) then subject to certain conditions involving these sequences $\varepsilon_r^{(m)} = R_{r,m+r}(\lambda)$ $(r,m=0,1,\ldots)$. Again, further additional values permitting the evaluation of $R_{i,j}(\lambda)$ with $j<i$ may be introduced

Claessens was led to the discovery of the extended ε-algorithm by the consideration of interpolatory continued fractions not of Thiele form, but of a form introduced by Kronecker [8] in connection with a process initiated by Rosenhain [17] and Borchardt [2] for constructing the resolvent of two polynomials from systems of their numerical values.

The extended version of the ε-algorithm serves as a bridge between the original version and the extrapolati[on] algorithm of relationship (24). Letting all argument values x_i tend to a common value x, the initial values, expressed as partial sums of a Newton interpolation series, become partial sums of a power series; the interpolatory functions $R_{i,j}(\lambda)$ become approximating fractions $P_{i,j}(z)$; the terms $\gamma_r^{(m)} = (\lambda - x_{m+r+1})^{-1}$ in relationship (25) tend to the common value $C = (\lambda - x)^{-1}$ and in effect, as described above, the ε-algorithm is obtained. The extended ε-algorithm is an interpolatory process in the sense described at the beginning of this section. An extrapolation algorithm may be derived by setting $\lambda = 0$, $x_i = i^{-1}$ $(i = 1, 2, \ldots)$ when $\gamma_r^{(m)} = m+r+1$ $(r, m = 0, 1, \ldots)$; the extrapolatory algorithm of relationship (24), with $\psi(i) = i$, is then obtained.

4. Lozenge algorithms

The function values $\sigma_r^{(m)}(\lambda)$ considered in Theorem 1 may be set in a two dimensional array, r denoting a column number and m the index associated with a forward diagonal. The function values in relationship (16) then occur at the vertices of a lozenge in this array. The algorithm of relationship (25) can be treated in the same way. The lozenge algorithm of relationship (16) possesses various properties shared by its more general counterpart. A useful purpose is served if these properties are derived for the more general form (25), and subsequently illustrated by reference to the function values $\sigma_r^{(m)}(\lambda)$.

In general terms the following results are derived for the ε-algorithm.

Firstly, a linear transformation of the function values f_i to which the ε-algorithm is applied results in a correspondingly simple modification of the function values $\varepsilon_r^{(m)}(\lambda)$, themselves.

Secondly a relationship connecting function values with even subscript alone

(26) $\varepsilon_{2r}^{(m+1)}(\lambda) = C,\ \varepsilon_{2r}^{(m)}(\lambda) = N,\ \varepsilon_{2r}^{(m+2)}(\lambda) = S,\ \varepsilon_{2r-2}^{(m+2)}(\lambda) = E,\ \varepsilon_{2r+2}^{(m)}(\lambda) = W$

may be derived. These values are disposed in the ε-array at the centre (C) and at the extremities (indicated by the compass points N, S, E and W) of a cross. For the ε-algorithm of relationship (25) with $\chi_r^{(m)} = 1$

$(r, m = 0, 1, \ldots)$ a similar relationship has already been derived [27]; it is

$$(27) \quad (C-N)^{-1} + (C-S)^{-1} = (C-E)^{-1} + (C-W)^{-1}.$$

Despite the author's pianissimo protests, this relationship and others of its kind have been referred to by various writers ~~as~~ [3,4,22] as Wynn identities; a general form for such identities is given.

Thirdly, the relationship connecting even order function values is satisfied if each value concerned is replaced by its image under a common fixed fractional linear transformation.

Lastly, the nature of relationship (16) itself and the manner in which its implementation may break down if the supplementary conditions (18) are not imposed, are considered. Relationship (16) may be interpreted in two senses. In the first, the $\sigma_r^{(m)}(\lambda)$ are regarded as general rational functions, or as ordered pairs of sequences of complex number polynomial coefficients; λ merely acts as a dummy variable to motivate arithmetic operations upon such sequences; it may be discarded. In this sense, all $\sigma_r^{(m)}(\lambda)$ may be determined as described in Theorem 1 without the imposition of condition (16). In the second sense, that of Theorem 1, λ is given a fixed numerical value and the $\sigma_r^{(m)}(\lambda)$ are single complex numbers. If condition (16) is not imposed, it may occur that for the λ in question $\sigma_r^{(m+1)}(\lambda) = \sigma_r^{(m)}(\lambda)$; $\sigma_{r+1}^{(m)}(\lambda)$ is then formally infinite; $\sigma_{r+2}^{(m-1)}(\lambda)$ and $\sigma_{r+2}^{(m)}(\lambda)$ are equal, and $\sigma_{r+3}^{(m-1)}(\lambda)$ is undetermined. After this misfortune, a subarray of numbers in the σ-array with vertex at $\sigma_{r+3}^{(m-1)}(\lambda)$ cannot be determined either. The disaster just described cannot occur when $r = 2s$ is even. The functions $\sigma_{2s+1}^{(m)}(\lambda)$ and $\sigma_{2s-1}^{(m+1)}(\lambda)$ are rational functions with denominators $\Pi_{2s+1}^{(m)}(\lambda)$ and $\Pi_{2s-1}^{(m+1)}(\lambda)$ respectively. Accordingly, the difference $\sigma_{2s}^{(m+1)}(\lambda) - \sigma_{2s}^{(m)}(\lambda)$ cannot possess a factor $\lambda - \lambda'$ where λ' is unequal to one of the x_i. The disaster can occur when $r = 2s+1$ is odd. The value of λ causing trouble is then one of the roots of the denominator of $\sigma_{2s+2}^{(m)}(\lambda)$, and

$\sigma^{(m-1)}_{2s+4}(\lambda)$ cannot be determined directly. Such values are excluded by the imposition of condition (16). It is however possible to consider the limiting form of the relationship connecting functions $\sigma^{(m)}_{r}(\lambda)$ with even suffix mentioned above, and in this way devise an emergency measure for the determination of $\sigma^{(m-1)}_{2s+4}(\lambda)$ in the circumstances just described.

⟹ Theorem 2. Let numbers $\varepsilon^{(m)}_{r}$ be derived from the initial values $\varepsilon^{(m)}_{-1}=0$ $(m=1,2,\ldots)$, $\varepsilon^{(m)}_{0}=S_m$ $(m=0,1,\ldots)$ by use of the relationship (25) for $r,m=0,1,\ldots$, where S_m $(m=0,1,\ldots)$ and $\chi^{(m)}_{r}$ are prescribed sequences of finite numbers.

i) Let numbers $\hat{\varepsilon}^{(m)}_{r}$ be derived from the initial values $\hat{\varepsilon}^{(m)}_{-1}=0$ $(m=1,2,\ldots)$, $\hat{\varepsilon}^{(m)}_{0}=\alpha S_m+\beta$ $(m=0,1,\ldots)$ $(\alpha\neq0$

by use of relationships similar to (25) with $\chi_{m,r}$ consistently replaced by $A\chi_{m,r}$ $(A \neq 0)$. Then

$$\hat{\varepsilon}^{(m)}_{2s} = \alpha \varepsilon^{(m)}_{2s} + \beta, \quad \hat{\varepsilon}^{(m)}_{2s+1} = A\alpha^{-1}\varepsilon^{(m)}_{2s+1}$$

for $s, m = 0, 1, \ldots$.

b) Let function values $\sigma^{(m)}_r(\lambda)$ be derived as described in clause (i) of Theorem 1. Function values $\hat{\sigma}^{(m)}_r(\zeta)$, where $\zeta = A\lambda + B$, may similarly be derived from the initial values $\hat{\sigma}^{(m)}_{-1}(\zeta) = 0$ $(m = 1, 2, \ldots)$, $\hat{\sigma}^{(m)}_0(\zeta) = \alpha f_m + \beta$ $(m = 0, 1, \ldots)$ with x_m replaced by $Ax_m + B$ in formulae (14, 16) $(m = 0, 1, \ldots)$ and

$$\hat{\sigma}^{(m)}_{2s}(\zeta) = \alpha \sigma^{(m)}_{2s}(\lambda) + \beta, \quad \hat{\sigma}^{(m)}_{2s+1}(\zeta) = A^{-1}\alpha^{-1}\sigma^{(m)}_{2s+1}(\lambda)$$

for $s, m = 0, 1, \ldots$

c) Let function values $\mu^{(m)}_r(\lambda)$ be derived as described in by use of relationships similar to (25) with $\chi_{m,r}$ consistently replaced by $A\chi_{m,r}$ $(A \neq 0)$. Then

$$\hat{\varepsilon}_{2s}^{(m)} = \alpha \varepsilon_{2s}^{(m)} + \beta, \quad \hat{\varepsilon}_{2s+1}^{(m)} = A\alpha^{-1}\varepsilon_{2s+1}^{(m)}$$

for $s, m = 0, 1, \ldots$ }.

b) Let function values $\sigma_r^{(m)}(\lambda)$ be derived as described in clause (i) of Theorem 1. Function values $\hat{\sigma}_r^{(m)}(\mathfrak{f})$, where $\mathfrak{f} = A\lambda + B$, may similarly be derived from the initial values $\hat{\sigma}_{-1}^{(m)}(\mathfrak{f}) = 0$ $(m = 1, 2, \ldots)$, $\hat{\sigma}_0^{(m)}(\mathfrak{f}) = \alpha f_m + \beta$ $(m = 0, 1, \ldots)$ with x_m replaced by $Ax_m + B$ in formulae (14, 16) $(m = 0, 1, \ldots)$ and

$$\hat{\sigma}_{2s}^{(m)}(\mathfrak{f}) = \alpha \sigma_{2s}^{(m)}(\lambda) + \beta, \quad \hat{\sigma}_{2s+1}^{(m)}(\mathfrak{f}) = A^{-1}\alpha^{-1}\sigma_{2s+1}^{(m)}(\lambda)$$

for $s, m = 0, 1, \ldots$

c) Let function values $\mu_r^{(m)}(\lambda)$ be derived as described i clause (ii) of Theorem 1. Function values $\hat{\mu}_r^{(m)}(\zeta)$, where ζ is as above, may similarly be derived from ~~the~~ the initial values $\hat{\mu}_{-1}^{(m)} = 0$ $(m = 1, 2, \ldots)$, $\hat{\mu}_0^{(m)}(\lambda) = A^{-1}\alpha(\zeta - x_m)^{-1} f_m$ $(m = 0, 1, \ldots)$ ~~initial values similar to (,) with x_m replaced by $Ax_m + B$ and f_m by αf_m $(m = 0, 1, \ldots)$~~, by means of a relationship similar to (16) with λ and x_m transformed as above, and

$$\hat{\mu}_{2s}^{(m)}(\zeta) = A^{-1}\alpha\,\mu_{2s}^{(m)}(\lambda), \quad \hat{\mu}_{2s+1}^{(m)}(\zeta) = \alpha^{-1}\mu_{2s+1}^{(m)}(\lambda)$$

iia) The numbers $\varepsilon_r^{(m)}$ with even suffix values may be derived from the initial values $\varepsilon_0^{(m)} = S_m$ $(m = 0, 1, \ldots)$ alone by use of the relationship

$$(28)\quad \gamma_{2s}^{(m)}\left(\varepsilon_{2s}^{(m+1)} - \varepsilon_{2s}^{(m)}\right)^{-1} + \gamma_{2s}^{(m+1)}\left(\varepsilon_{2s}^{(m+1)} - \varepsilon_{2s}^{(m+2)}\right)^{-1}$$
$$= \gamma_{2s-1}^{(m+1)}\left(\varepsilon_{2s}^{(m+1)} - \varepsilon_{2s-2}^{(m+2)}\right)^{-1} + \gamma_{2s+1}^{(m)}\left(\varepsilon_{2s}^{(m+1)} - \varepsilon_{2s+2}^{(m)}\right)^{-1}$$

with $s, m = 0, 1, \ldots$ (when $s = 0$ the first term on the right hand side of this equation is to be omitted). Those with odd suffix values may be derived from the initial values $\varepsilon_{-1}^{(m)}=0$ $(m=1,2,\ldots)$ and $\varepsilon_{1}^{(m)}=\gamma_1^{(m)}(S_{m+1}-S_m)^{-1}$ $(m=0,1,\ldots)$ by use of a relationship obtained from (28) by replacing ~~§~~ $2s$ consistently by $2s+1$ (without omission of terms).

b) Subject to the various assumptions concerning the sequences $\underset{\sim}{x}, \underset{\sim}{f}$ made in clauses ~~(i)~~ (i, iii.) of Theorem 1, the function values concerned with even suffix values may be derived from the initial values $\sigma_0^{(m)}(\lambda)=f_m$ $(m=0,1,\ldots)$ by use of the relationship

$$(29)\quad \left\{z_{m+2s+1}(\lambda)-z_m(\lambda)\right\}\left\{\sigma_{2s}^{(m+1)}(\lambda)-\sigma_{2s}^{(m)}(\lambda)\right\}^{-1}$$

$$+\left\{z_{m+2s+2}(\lambda)-z_{m+1}(\lambda)\right\}\left\{\sigma_{2s}^{(m+1)}(\lambda)-\sigma_{2s}^{(m+2)}(\lambda)\right\}^{-1}$$

$$= \{z_{m+2s+1}(\lambda) - z_{m+1}(\lambda)\}\{\sigma_{2s}^{(m+1)}(\lambda) - \sigma_{2s-2}^{(m+2)}(\lambda)\}^{-1}$$

$$+ \{z_{m+2s+2}(\lambda) - z_m(\lambda)\}\{\sigma_{2s}^{(m+1)}(\lambda) - \sigma_{2s+2}^{(m)}(\lambda)\}^{-1}$$

with $s,m = 0,1,\ldots$} (again the first term on the right hand side of this equation is to be omitted when $s=0$). Function values of the form $\sigma_{2s+1}^{(m)}(\lambda)$ may be derived from the initial values $\sigma_{-1}^{(m)}(\lambda) = 0$ $(m=1,2,\ldots)$ and $\sigma_1^{(m)}(\lambda) = \{z_{m+1}(\lambda) - z_m(\lambda)\}(f_{m+1} - f_m)^{-1}$ $(m=0,1,\ldots)$ by use of a relationship obtained from (29) by replacing $2s$ consistently by $(2s+1)$ without omission of terms).

c) Subject to the stated assumptions, the function values $\mu_r^{(m)}(\lambda)$ with even suffix values of clause (ii) of Theorem 1 may be derived from the initial values $\mu_0^{(m)}(\lambda) = z_m(\lambda) f_m$ $(m=0,1,\ldots)$ by use of a relationship obtained from (29) as described by replacement of ε by μ. The function values of the form $\mu_{2s+1}^{(m)}(\lambda)$ may be derived from the initial values $\mu_{-1}^{(m)}(\lambda)=0$ $(m=1,2,\dots)$ and $\mu_{1}^{(m)}(\lambda)=\{z_{m+1}(\lambda)-z_m(\lambda)\}\{z_{m+1}(\lambda)f_{m+1}-z_m(\lambda)f_m\}$ $(m=0, 1,\dots)$ by use of a relationship obtained from (29) by replacing ε and $2s$ consistently by μ and $2s+1$.

iii) Let $\alpha,\dots,\delta$ be fixed complex numbers with $\alpha\delta-\beta\gamma \neq 0$.

aa) Let the relationship

$$\gamma_{2s}^{(m)}+\gamma_{2s}^{(m+1)}=\gamma_{2s-1}^{(m+1)}+\gamma_{2s+1}^{(m)} \tag{30}$$

be satisfied for certain values of $s, m \geq 0$ ($\gamma_{2s-1}^{(m+1)}$ is to be omitted if $2s=0$). For these values of s and m, relationship (28) also holds as described with ε

replaced by $(\alpha\varepsilon+\beta)(\gamma\varepsilon+\delta)^{-1}$, ε being in turn $\varepsilon_{2s}^{(m+1)}$, $\varepsilon_{2s}^{(m)}$, $\varepsilon_{2s}^{(m+2)}$, $\varepsilon_{2s+2}^{(m)}$ and, if $2s>0$, $\varepsilon_{2s-2}^{(m+2)}$. If relationship (20) holds as described consistently for $s,m=0,1,\ldots$ and numbers $\hat{\varepsilon}_{2s}^{(m)}$ are generated as described in clause iia) from the initial values $\hat{\varepsilon}_0^{(m)}=(\alpha S_m+\beta)(\gamma S_m+\delta)^{-1}$ $(m=0,1,\ldots)$, then

$$\hat{\varepsilon}_{2s}^{(m)}=(\alpha\varepsilon_{2s}^{(m)}+\beta)(\gamma\varepsilon_{2s}^{(m)}+\delta)^{-1}$$

for $s,m=0,1,\ldots$ (the numbers $\hat{\varepsilon}_{2s}^{(m)}$ may also be generated by use of relationships similar to (25) from the initial values $\hat{\varepsilon}_{-1}^{(m)}=0$ $(m=1,2,\ldots)$ and $\hat{\varepsilon}_0^{(m)}$ as just described).

β) If the above conditions hold with $2s$ replaced consistently by $2s+1$, then the above results, modified in the same way, also hold (the initial conditions for the determination of numbers $\tilde{\varepsilon}_{2s+1}^{(m)}$ with odd suffix in the second result are to be replaced by $\tilde{\varepsilon}_{-1}^{(m)} = \beta/\delta$ $(m = 1, 2, \ldots)$ and $\tilde{\varepsilon}_{1}^{(m)} = (\alpha \varepsilon_1^{(m)} + \beta)(\gamma \varepsilon_1^{(m)} + \delta)^{-1}$ $(m = 0, 1, \ldots)$ ~~and~~ the parenthetic remark at the end of the preceding clause is to be discarded).

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b) Let function values $\sigma_r^{(m)}(\lambda)$ with even suffix values be derived as described in clause ib); let corresponding function values $\hat{\sigma}_r^{(m)}(\lambda)$ be obtained from the initial values $\sigma_0^{(m)}(\lambda) = (\alpha f_m + \beta)(\gamma f_m + \delta)^{-1}$ $(m = 0, 1, \ldots)$ by use of a recursion similar to (29); then $\hat{\sigma}_{2r}^{(m)}(\lambda) = (\alpha \sigma_{2r}^{(m)}(\lambda) + \beta)(\gamma \sigma_{2r}^{(m)}(\lambda) + \delta)^{-1}$ for all $\sigma_{2r}^{(m)}(\lambda)$ derived. A similar result holds with regard to function values $\sigma_r^{(m)}(\lambda)$ and $\tilde{\sigma}_r^{(m)}(\lambda)$ with odd suffix values; now the initial values for the construction of the latter are $\tilde{\sigma}_{-1}^{(m)}(\lambda) = \beta\gamma^{-1}$ $(m = 1, 2, \ldots)$ and $\tilde{\sigma}_1^{(m)}(\lambda) =$ $\{\alpha(x_{m+1} - x_m) + \beta(f_{m+1} - f_m)\}\{\gamma(x_{m+1} - x_m) + \delta(f_{m+1} - f_m)\}^{-1}$ $(m = 0, 1, \ldots)$. (The function values $\hat{\sigma}_{2s}^{(m)}(\lambda)$ with even suffix values may also be derived by use of a relationship similar to (16) with the initial values $\hat{\sigma}_{-1}^{(m)}(\lambda) = 0$ $(m = 1, 2, \ldots)$ and $\hat{\sigma}_0^{(m)}(\lambda)$ as above.)

c) Results similar to those of the preceding clause hold with regard to the function values $\mu_r^{(m)}(\lambda)$; the function values involving $\hat{\sigma}_0^{(m)}(\lambda)$ above are to be replaced by $\hat{\mu}_0^{(m)}(\lambda) = \{\alpha f_m + \beta(\lambda - x_m)\}\{\gamma f_m + \delta(\lambda - x_m)\}^{-1}$ $(m = 0, 1, \ldots)$ and those involving $\tilde{\sigma}_1^{(m)}(\lambda)$ are to be replaced by $\tilde{\mu}_1^{(m)}(\lambda)$

$=\{\alpha\Delta_m+\beta f_1(\lambda,x_m)\}\{\gamma\Delta_m+\delta f_1(\lambda,x_m)\}$ where $\Delta_m=x_{m+1}-x_m$

and $f_1(\lambda,x_m)=f_{m+1}(\lambda-x_m)+f_m(\lambda-x_{m+1})$ $(m=0,1,\ldots)$.

iv a) Assume that functions $\varepsilon_r^{(m)}(z)$ of a variable z may be constructed by replacing numbers S_m, $\varepsilon_r^{(m)}$ by functions $S_m(z)$, $\varepsilon_r^{(m)}(z)$ consistently in the statement at the beginning of the theorem and in addition that in the neighbourhood of every point in the complex plane either $S_m(z)$ or $S_m(z)^{-1}$ is continuous $(m=0,1,\ldots)$.

a ~~x~~) For fixed values λ of z, and of $s, m \geq 0$, let

$$\cancel{\varepsilon^{(m+1)}_{2s-1}(\lambda)=}$$

$$(31)\quad \varepsilon^{(m+1)}_{2s-1}(\lambda)=\varepsilon^{(m+2)}_{2s-1}(\lambda),\ \varepsilon^{(m)}_{2s-1}(\lambda)\neq\varepsilon^{(m+1)}_{2s-1}(\lambda),\ \varepsilon^{(m+2)}_{2s-1}(\lambda)\neq\varepsilon^{(m+3)}_{2s-1}(\lambda);$$

assume further that relationship (30) holds as described, and $\gamma^{(m)}_{2s+1}\neq 0$. The value of $\varepsilon^{(m)}_{2s+2}(\lambda)$ may be obtained by use of the singular rule

$$(32)\quad \varepsilon^{(m)}_{2s+2}(\lambda) = \gamma^{(m)^{-1}}_{2s+1}\left\{\gamma^{(m)}_{2s}\,\varepsilon^{(m)}_{2s}(\lambda) + \gamma^{(m+1)}_{2s}\,\varepsilon^{(m+2)}_{2s}(\lambda) - \gamma^{(m+1)}_{2s-1}\,\varepsilon^{(m)}_{2s-2}(\lambda)\right\}$$

(the term involving $\varepsilon^{(m)}_{2s-2}(\lambda)$ is to be omitted if $s=0$).

β) Let the conditions of the preceding subclause, with $2s$ replaced consistently by $2s+1$, hold. With this modification, the stated results also hold.

b) If function values $\sigma^{(m)}_{r}(\lambda)$ are constructed as described in clause (i) of Theorem 1, without the imposition of condition (15), it can occur that conditions of the form (31) with ε replaced by σ hold. In this case the value of $\sigma^{(m)}_{2s+2}(\lambda)$ may be determined by use of the singular rule

(33) $\sigma^{(m)}_{2s+2}(\lambda) = \{z_{m+2s+2}(\lambda) - z_m(\lambda)\}^{-1} \Big[\{z_{m+2s+1}(\lambda) - z_m(\lambda)\} \sigma^{(m)}_{2s}(\lambda)$

$+ \{z_{m+2s+2}(\lambda) - z_{m+1}(\lambda)\} \sigma^{(m+2)}_{2s}(\lambda) - \{z_{m+2s+1}(\lambda) - z_{m+1}(\lambda)\} \sigma^{(m)}_{2s-1}(\lambda)\Big]$

(the term involving $\sigma^{(m)}_{2s-2}(\lambda)$ is to be omitted if $s=0$).

c) If function values $\mu^{(m)}_r(\lambda)$ are constructed as described in clause (ii) of Theorem 1, without the imposition of condition (15), it can occur that conditions of the form (31) with ε and $2s$ replaced by μ and $2s+1$ consistently hold. In this case the value of $\mu^{(m)}_{2s+3}(\lambda)$ may be determined by use of a singular rule obtained from formula (33) with σ and $2s$ replaced by μ and $2s+1$ consistently

Proof. The results immediately following clauses ia), iia), iiia) and iva) are simple corollaries to these clauses; attention will therefore be confined to the latter.

Evidently the results of clause ia) are correct when $s=0$; they may, by use of relationship (25) with r replaced by $2s$ and $2s+1$ in succession, easily be established by induction.

Replacing r in succession by $2s$ and $2s+1$ $(s>0)$ in relationship (25) the two equations

$$\varepsilon^{(m)}_{2s+1} - \varepsilon^{(m+1)}_{2s-1} = \gamma^{(m)}_{2s}\left(\varepsilon^{(m+1)}_{2s} - \varepsilon^{(m)}_{2s}\right)^{-1}, \quad \varepsilon^{(m+1)}_{2s+1} - \varepsilon^{(m)}_{2s+1} = \gamma^{(m)}_{2s+1}\left(\varepsilon^{(m)}_{2s+2} - \varepsilon^{(m+1)}_{2s}\right)^{-1}$$

are obtained; replacing m by $m+1$ in the first, and m by $m+1$ and s by $s-1$ in the second, two further equations are obtained. The sum of the left hand side expressions of the first pair of equations is equal to that of the second. The same is therefore true with regard to the right hand side expressions, and equation (28) is derived by rearrangement. The special form obtaining when $s=0$ is derived in the same way. The stated result concerning the generation of the numbers $\varepsilon_r^{(m)}$ with even suffix values follows. That concerning the corresponding numbers with odd suffix values is obtained similarly.

Adopting a nomenclature similar to that stated as (26) above, relationship (28) becomes, when $s>0$,

$$(34)\quad \gamma_{2s}^{(m)}(C-N)^{-1} + \gamma_{2s}^{(m+1)}(C-S)^{-1} = \gamma_{2s-1}^{(m+1)}(C-E)^{-1} + \gamma_{2s+1}^{(m)}(C-W)^{-1}$$

The fractional linear transformation of ε described in clause (iii) may be achieved by implementing a sequence of linear transformations and an inversion, i.e. with regard to the latter, the replacement of ε by ε^{-1}. The required results concerning linear transformation are those of the first clause; it remains to show that if, subject to the condition (30) with $s > 0$ and with C being unequal to N, S, E or W, relationship (34) holds, then the relationship

$$(35)\quad \chi_{2s}^{(m)}(C^{-1}-N^{-1})^{-1} + \chi_{2s}^{(m+1)}(C^{-1}-S^{-1})^{-1} = \chi_{2s-1}^{(m+1)}(C^{-1}-E^{-1})^{-1} + \chi_{2s+1}^{(m)}(C^{-1}-W^{-1})$$

also holds. When the value of C is zero, each term ~~becomes~~ becomes zero in this new relationship, which is accordingly satisfied. When C is nonzero, $(C^{-1}-N^{-1})^{-1} = C - C^2(C-N)^{-1}$. Thus; and multiplying relationship (34) throughout by $-C^2$ and adding $(\chi_{2s}^{(m)} + \chi_{2s}^{(m+1)})C$ to the left hand side and

$(\gamma_{2s-1}^{(m+1)} + \gamma_{2s+1}^{(m)})C$ to the right hand side of the result, relationship (35) is obtained. The case in which $s=0$ is dealt with in the same way. If relationship (30) holds consistently, the further stated results also follow directly. The subsequent result concerning numbers $\varepsilon_r^{(m)}$ with odd suffix values is obtained similarly.

Subject to the conditions of the last clause, C in relationship (34) is the value of a function which is continuous for argument values z in the neighbourhood of λ; as z tends to λ, C becomes infinite but N, S, ~~W remain~~ and E remain finite. For suitable values of $z \neq \lambda$, $(C-N)^{-1} - C^{-1} = NC^{-1}(C-N)^{-1}$, and similar relationships hold with regard to the other three terms in equation (34). Subtracting $(\gamma_{2s}^{(m)} + \gamma_{2s}^{(m+1)})C^{-1}$ from the left hand side of equation (34), and $(\gamma_{2s-1}^{(m+1)} + \gamma_{2s+1}^{(m)})C^{-1}$ from the right hand side, multiplying both sides of the resulting equation by C^2, and letting z tend to λ, the equation is a version of formula (32), namely

$$\gamma_{2s}^{(m)} N + \gamma_{2s}^{(m+1)} S = \gamma_{2s-1}^{(m+1)} E + \gamma_{2s+1}^{(m)} W,$$

which is a version of formula (32), is obtained. The special relationship holding when $s=0$ is derived in the same way. The stated result concerning numbers $\varepsilon_r^{(m)}$ with odd suffix values is obtained similarly.

Special interest attaches to algorithms of the form (25) for which relationships (30) hold as described, and also with $2s$ replaced by $2s+1$, for all $s, m \geq 0$, i.e. for which

$$(36) \quad \chi_r^{(m)} + \chi_r^{(m+1)} = \chi_{r-1}^{(m+1)} + \chi_{r+1}^{(m)}$$

for $r, m = 0, 1, \ldots$, where $\chi_{-1}^{(m)} = 0$ $(m = 1, 2, \ldots)$. In particular, if the initial values $\varepsilon_0^{(m)}$ $(m = 0, 1, \ldots)$ are subjected to a fractional linear transformation as described in clause (iii), the resulting numbers $\varepsilon_r^{(m)}$ with even suffix values are obtainable from their original counterparts by use of the same transformation. Furthermore, breakdown of the algorithm in the circumstance described in clause (iv) may be repaired by use of the derived singular rules wherever it occurs in the array of numbers $\varepsilon_r^{(m)}$ concerned. There is, however, only one algorithm of this algebraic type, namely that of

Thiele's reciprocal differences (taking this algorithm to include the extensions described in Theorem 1): fixing x_0 arbitrarily, and determining subsequent x_m by use of the formula $x_{m+1} = x_m + \gamma_0^{(m)}$ $(m = 0, 1, \ldots)$ it then follows that $\gamma_r^{(m)} = x_{m+r+1} - x_m$ $(r, m = 0, 1, \ldots)$ and relationships (2,25) allowing for a change in notation, are the same. It may be noted in passing that the singular rule described in clause (ivb) of the above theorem now allows the determination of reciprocal differences when certain of the function values f_m become infinite. When this occurs in isolation with $m > 0$, $\rho_1^{(m-1)} = \rho_1^{(m)} = 0$, and $\rho_2^{(m-1)}$ takes the singular value

$$(x_{m+1}-x_{m-1})^{-1}\{(x_m-x_{m-1})f_{m-1}+(x_{m+1}-x_m)f_{m+1}\}.$$

Similar modification of the algorithms of Theorem 1 is possible. Infinity may be accepted as a function value like any other.

For the ε-algorithm of formula (25) with $\gamma_r^{(m)}=1$ $(r,m=0,1,\ldots)$, relationships (26) are satisfied for $r=1,2,\ldots;$ $m=0,1,\ldots$ but not as described for $r,m=0,1,\ldots$. For any constellation C, N, S, E and W taken from the numbers $\varepsilon_{2s}^{(m)}$ $(s,m\geq 0)$ relationship (27), with the numbers $\gamma_r^{(m)}$ discarded, is satisfied; it is also satisfied when N, C and S represent three consecutive members of the sequence $\varepsilon_0^{(m)}=S_m$ $(m=0,1,\ldots)$ of initial values, if E is taken to be infinite. For such values of C, N, S, E and W, relationship (27) is still satisfied if every member of this set is replaced by its image under a fractional linear transformation as described in clause (iiia) (the image of E, when infinite, must be obtained by a limiting process). It is not, however, true that if relationship (25) with $\chi_{r,r}^{(m)} = 1$ is applied to the initial values obtained by applying a linear fractional transformation to previously used initial values S_m $(m = 0, 1, \ldots)$, the new ~~even order~~ ε-numbers with even suffix are obtainable from their old counterparts by use of the same transformation. Furthermore, breakdown of the ε-algorithm of formula $(25)^{\gamma}$ with $\chi_r^{(m)} = 1$, which occurs when $C = \varepsilon_r^{(m+1)}$ for some $r \geq 1$ and $m \geq 0$ becomes infinite in isolation, can be retrieved by use of the singular rule $W = N + S - E$. This rule cannot be applied when $r = 0$. For the ε-algorithm itself, infinity is not an initial value like any other.

There is a converse problem associated with the first result of clause (ii): given a system of relationships of the form

$$(29)\quad \kappa_{s,1}^{(m)}\left(\varepsilon_{2s}^{(m+1)} - \varepsilon_{2s}^{(m)}\right)^{-1} + \kappa_{s,2}^{(m)}\left(\varepsilon_{2s}^{(m+1)} - \varepsilon_{2s}^{(m+2)}\right)^{-1}$$
$$= \kappa_{s,3}^{(m)}\left(\varepsilon_{2s}^{(m+1)} - \varepsilon_{2s-2}^{(m+2)}\right)^{-1} + \kappa_{s,4}^{(m)}\left(\varepsilon_{2s}^{(m+1)} - \varepsilon_{2s+2}^{(m)}\right)^{-1}$$

where $\kappa_{s,j}^{(m)} \neq 0$ $(j = 1, \ldots, 4)$ for $s, m = 0, 1, \ldots$ (the term involving $\kappa_{s,3}^{(m)}$ being omitted when $s = 0$) and a set of initial values $\varepsilon_0^{(m)} = S_m$ $(m = 0, 1, \ldots)$ determine, if possible, a system of relationships of the form (25) leading to the construction of the same numbers $\varepsilon_{2s}^{(m)}$ $(s, m = 0, 1, \ldots)$. It is required that multiplication of relationship (37) throughout by a number $\omega_s^{(m)}$ should produce an identity of the form (28), so that $\omega_s^{(m)} \kappa_{s,1}^{(m)} = \gamma_{2s}^{(m)}$, $\omega_s^{(m)} \kappa_{s,2}^{(m)} = \gamma_{2s}^{(m+1)}$ and so on. In both identities in which it ~~occurs~~ appears, $\gamma_{2s}^{(m+1)}$ must have the same value; the same holds true for $\gamma_{2s-1}^{(m+1)}$:

$$\omega_s^{(m)} \kappa_{s,2}^{(m)} = \omega_s^{(m+1)} \kappa_{s,1}^{(m+1)}, \quad \omega_{s+1}^{(m)} \kappa_{s+1,3}^{(m)} = \omega_s^{(m+1)} \kappa_{s,4}^{(m+1)}.$$

Eliminating the ω-numbers from these relationships, the condition that the required algorithm can be found is that

$$\kappa_{s,4}^{(m+1)} \kappa_{s,1}^{(m+2)} \kappa_{s+1,2}^{(m)} \kappa_{s+1,3}^{(m+1)} = \kappa_{s,2}^{(m+1)} \kappa_{s,4}^{(m+2)} \kappa_{s+1,3}^{(m)} \kappa_{s+1,1}^{(m+1)}$$

$(s, m = 0, 1, \ldots)$. If this condition is satisfied, set $\omega_0^{(0)} = 1$

and $\omega_0^{(m+1)} = \omega_0^{(m)} \kappa_{0,2}^{(m)} / \kappa_{0,1}^{(m+1)}$ $(m=0,1,\ldots)$ and ~~thereof~~ thereafter $\omega_s^{(m)} = \omega_{s-1}^{(m+1)} \kappa_{s-1,4}^{(m+1)} / \kappa_{s,3}^{(m)}$ for $s=1,2,\ldots;\ m=0,1,\ldots$; $\chi_{2s}^{(m)}$ and $\chi_{2s+1}^{(m)}$ are then given by the formula

$$\chi_{2s+k}^{(m)} = \omega_s^{(m)} \kappa_{s,2k+1}^{(m)} \quad (s,m=0,1,\ldots;\ k=0,1).$$

5. Invariance properties

corollaries relating to the functions [illegible]

The results of clauses (ib,c) and (iiib,c) of Theorem 2 serve

of the results of clauses (i) and (iii) of Theorem 2 serve

as pointers to properties of invariance of the functions $C_{2r}^{(m)}(\lambda)$ and $C_{2r+1}^{(m)}(\lambda)$. In his final presentation of the use of continued fractions for the purposes of rational function interpolation, Thiele [21] noted the existence of the computationally more economical inverse differences, but rejected their use in favour of reciprocal differences in view of the algebraic invariant properties enjoyed by the latter. These properties may be described as follows. Denote the reciprocal differences $\rho_r^{(m)}$ of formulae (9), derived from argument and function values x_i, f_i $(i=0,1,\ldots)$, by $\rho_r^{(m)}(x,f)$. With $A, B, \cancel{C, D},$ $\alpha, \beta, \gamma,$ and δ fixed complex numbers such that $A\cancel{D-BC} \neq 0$, $\alpha\delta - \beta\gamma \neq 0$, set

$$(37) \qquad y_i = \frac{Ax_i + B}{\cancel{Cx_i + D}}, \qquad g_i = \frac{\alpha f_i + \beta}{\gamma f_i + \delta}$$

for $i = 0,1,\ldots$, and denote the reciprocal differences derived from the argument and function values y_i, g_i $(i=0,1,\ldots)$ by $\rho_r^{(m)}(y,g)$. The relationship

$$\rho_{2s}^{(m)}(y,g) = \frac{\alpha\rho_{2s}^{(m)}(x,f) + \beta}{\gamma\rho_{2s}^{(m)}(x,f) + \delta}$$

holds in all cases in which ~~both~~ the determinantal expressions ~~involved~~ for $\rho^{(m)}_{rs}(x,f)$ ~~are~~ is well-determined. In the special case in which ~~$C\neq$~~ $\gamma = 0$, the further relationship

$$\rho^{(m)}_{2s+1}(y,g) = \left(\frac{A\delta}{\alpha}\right)\rho^{(m)}_{2s+1}(x,f)$$

holds, again when the determinantal expression for $\rho^{(m)}_{2s+1}(x,f)$ is ~~both expressions are well~~ determined.

The above invariant properties have counterparts concerning the rational functions $C^{(m)}_r(\lambda)$.

Theorem 3. Let the two sequences of argument and function values x_i, f_i $(i=0,1,\ldots)$ and y_i, g_i $(i=0,1,\ldots)$ be related by means of formulae (33) where $A,\ldots,\delta$ are fixed complex numbers with $A\cancel{D-BC}\neq 0$, $\alpha\delta-\beta\gamma \neq 0$, and let

$$\zeta = \frac{A\lambda+B}{\cancel{C\lambda+D}}.$$

Denote the determinantal quotients (13) derived from x_i, f_i $(i = 0, 1, \ldots)$ and λ by $C_r^{(m)}(\lambda|x,f)$ and let the symbol $C_r^{(m)}(\mathfrak{z}|y,g)$ have a similar meaning.

The relationship

$$(39)\quad C_{2s}^{(m)}(\mathfrak{z}|y,g) = \frac{\alpha C_{2s}^{(m)}(\lambda|x,f) + \beta}{\gamma C_{2s}^{(m)}(\lambda|x,f) + \delta}$$

holds in all cases in which ~~both~~ the determinantal expression~~s~~ for the function~~s~~ $C_{2s}^{(m)}(\lambda|x,f)$ is ~~are~~ well determined. When $\cancel{\alpha}\gamma = 0$, the further relationship

$$(40)\quad C_{2s+1}^{(m)}(\mathfrak{z}|y,g) = \left(\frac{A\delta}{\alpha}\right) C_{2s+1}^{(m)}(\lambda|x,f)$$

holds when ~~the right~~ hand quotient the determinantal expressions for the functions $C_{2s+1}^{(m)}(\lambda|x,f)$ ~~is~~ are well-determined.

Proof Set ~~$A=1, B=0$~~ $\alpha = 1$, $\beta = \gamma = \delta = 0$. As is easily verified directly from formula (13b), relationship (39) holds

in this special case. set $A=1$, $B=0$; it [illegible] [illegible] [illegible]
(39) holds

$\beta=\gamma=0 \quad \delta=1 \quad \alpha$

in the three special cases ~~$B=C=0, D=1$, A~~ unrestricted;

$\alpha=\beta=1 \quad \gamma=0 \quad \beta$

~~$A=D=1, C=0, B$~~ unrestricted; $\alpha=\delta=0 \quad \beta=1 \quad \gamma$

~~$A=D=0, B=1$, C~~ unrestricted.

Any general transformation of the form (38b) can be compounded from a succession of special transformations of this type, and hence, with $A, \ldots, D \to A$ and B) ~~$\alpha, \ldots, \delta$ as described~~, relationship (39) holds for general fractional linear sequence f_i. Combining the two ~~transformations of the argument. Set $A=1, B=C=D=0$~~ results obtained for transformations of x_i and f_i, and ~~repeat the above process with $A, \ldots, D$ replaced by $\alpha, \ldots, \delta$.~~ (Relationship (39) in its general form as described has been verified. Relationship (40) is demonstrated in the same way.

6. Confluent algorithms

Throughout the above exposition it has been assumed that the argument sequence x_i $(i=0,1,\ldots)$ consists of mutually distinct numbers. It is possible to regard the sequence f_i $(i=0,1,\ldots)$ as being values of a function f for corresponding argument values x_i, to let groups of these values tend to common values, to make suitable assumptions concerning the differentiability of f at the points of confluence, to replace certain entries in the determinants occurring in the above formulae by differences formed from neighbouring elements and in this way obtain expressions in which derivatives of f occur. The confluent theory takes a particularly simple form when all arguments are assumed to lead to a common value; the distribution $x_i = x + ih$ $(i=0,1,\ldots)$ is adopted, all derivatives of f at the point x are assumed to exist, and h is allowed to tend to zero. The functions $C_r^{(m)}(\lambda)$ evolve, in particular, to a confluent form $C_r(\lambda, x)$ and become successive convergents of the continued fraction corresponding to (see [16]) the Taylor series

$$(41) \qquad \sum_{i=0}^{\infty} \frac{\mathcal{D}^i f(x)}{i!} (\lambda - x)^i$$

where $\mathcal{D} \equiv d/dx$. The coefficients of this continued fraction may be expressed in terms of Thiele's reciprocal derivatives. The theory is described in Ch. 15 of [14]. Confluent forms of the algorithms of formulae (16) may be obtained by setting

$$\varepsilon_{2r}^{(m)}(\lambda) = \varepsilon_{2r}(\lambda, x+mh), \quad \varepsilon_{2r+1}^{(m)}(\lambda) = h^{-1}\varepsilon_{2r+1}(\lambda, x+mh)$$

$(r, m = 0, 1, \ldots)$ in conjunction with the above substitution for x_i given above, and letting h tend to zero. The algorithm then yields a difference-differential process for constructing successive convergents $\varepsilon_{2r}(\lambda, x)$ $(r = 0, 1, \ldots)$ of the continued fraction associated with the series (41) (i.e. the even order convergents of the continued fraction corresponding to this series (41)). The algorithm of (lower (ii) of Theorem) formulae ~~(,)~~ may be treated in the same way, and yields a difference-differential process for constructing functions $\mu_r(\lambda, x)$ $(r = 0, 1, \ldots)$; from these, the odd-order convergents $C_{2r+1}(\lambda, x) = \mu_{2r+1}(\lambda, x)^{-1}$ of the continued fraction corresponding to the series (41) may be obtained.

The following notations are adopted: with

$$f_i = \frac{1}{i!}\partial^i f(x)$$

for $i=0,1,\dots$, $F_{m,n,r}$ is the array composed of $n-m+1$ rows and $r-m+1$ columns, the $(j+1)^{th}$ element in the $(i+1)^{th}$ row being f_{m+i+j} $(i=0,\dots,n-m;\ j=0,\dots,r-m)$; Λ_m is a column array containing $m+1$ elements, that in position $(i+1)$ being $(\lambda-x)^{m-i}$ $(i=0,\dots,m)$; Λ'_m is the column array containing $m+1$ elements obtained from the corresponding elements of Λ_m by partial differentiation with respect to λ, thus $(m-i)(\lambda-x)^{m-i-1}$ in position $(i+1)$ $(i=0,\dots,m-1)$ and zero in position

$(m+1)$; $\Sigma_{m,n}$ is a column array containing $n-m+1$ elements, that in position $(i+1)$ being $\sum_{r=0}^{i+m} (\lambda - x)^{n-m+r-i} f_r$ $(i=0,\ldots,n-m)$; $\Sigma'_{m,n}$ is the column array containing $n-m+1$ elements obtained by differentiating the corresponding elements of $\Sigma_{m,n}$ partially with respect to λ. Determinants composed of elements taken from the above arrays are denoted by square brackets as following formulae indicate.

Theorem 4. Let all derivatives of the function f exist at the point x.

(i) Let functions $\sigma_r(\lambda, x)$ be obtained from the initial values $\sigma_{-1}(\lambda,x)=0$, $\sigma_0(\lambda,x)=f(x)$ by use of the difference-differential recursion

(42) $\epsilon_{r+1}(\lambda,x) = \epsilon_{r-1}(\lambda,x) + (r+1)(\lambda-x)^{-2}\{2\epsilon_r(\lambda,x)\}^{-1}$

with $r=0,1,\ldots$. If $f(x)$ is the irreducible quotient of two n^{th} degree polynomials in x, recursion (42) terminates with the production of the function $\epsilon_{2n}(\lambda,x) = f(\lambda)$; otherwise unboundedly many functions $\epsilon_r(\lambda,x)$ may be produced. For the functions that may be produced

(43) $$\epsilon_{2s-1}(\lambda,x) = \frac{[\Lambda'_s, \Lambda_s, \bar{F}_{1,s+1,s-1}]}{[\bar{F}_{1,s,s}](\lambda-\mu)^{2s}}, \quad \epsilon_{2s}(\lambda,x) = \frac{[\Sigma_{0,s,s} F_{0,s,s-1}]}{[\Lambda_s, F_{0,s,s-1}]}$$

for $s=1,2,\ldots$ and of these $\epsilon_{2s}(\lambda,x) = C_{2s}(\lambda,x)$, where $C_{2s}(\lambda,x)$ is the s^{th} convergent of the continued fraction associated with the series (41) in ascending powers of $\lambda-x$.

ii) Let functions $\mu_r(\lambda,x)$ be obtained from the initial values $\mu_{-1}(\lambda,x)=0$, $\mu_0(\lambda,x) = (\lambda-x)^{-1}f(x)$ by use of the difference - differential recursion obtained from formula (42) by replacing σ by μ. If $f(x)$ is an irreducible rational function of x, the degrees of the polynomials in the numerator and denominator being $n+1$ and n respectively, the above recursion terminates with the production of the function $\mu_{2n+1}(\lambda, x) = f(\lambda)^{-1}$; otherwise unboundedly many functions $\mu_r(\lambda, x)$ may be produced. For the functions that can be constructed

$$\mu_{2s-1}(\lambda, x) = \frac{[\Lambda_{s-1}, F_{1,s,s-1}]}{[\Sigma_{1,s}, F_{2,s+1,s}]}, \quad \mu_{2s}(\lambda, x) = \frac{[\Sigma^{*}_{1,s+1}, \Sigma'_{1,s+1}, F_{2,s+2,s}]}{[F_{2,s+1,s+1}]}$$

for $s = 1, 2, \ldots$ and of these $\mu_{2s+1}(\lambda, x) = C_{2s+1}(\lambda, x)^{-1}$, where $C_{2s+1}(\lambda, x)$ is the $(2s+1)^{th}$ convergent of the continued fraction corresponding to the series (41) in ascending powers of $\lambda - x$.

Proof. Formula (42) is the limiting form of relationship (16) obtained by setting $x_m = x + mh$, $\sigma_{2s-1}(\lambda, x+mh) = h\sigma^{(m)}_{2s-1}(\lambda)$, $\sigma_{2s}(\lambda, x+mh) = \sigma^{(m)}_{2s}(\lambda)$ $(m=0,1,\dots)$, and letting h tend to zero. Formulae (43), ~~…~~ may be obtained by setting, in addition $f_m = f(x+mh)$ $(m=0,1,\dots)$ and carrying out appropriate manipulations upon formulae (30), ~~…~~. These manipulations involve the formulae

$$(44)\qquad \sum_{\tau=0}^{j} \binom{j}{\tau} (x-\lambda)^{\tau} \partial^k \{(\lambda - x)^{j-\tau} f(x)\} = (-1)^j k! f_{k-j} \quad (j=0,\dots,k)$$

for $j=0,\dots,k$ and [illegible] and

$$(45)\qquad \partial^k \{(\lambda - x)^{-1} f(x)\} = k! \sum_{\tau=0}^{k} (\lambda - x)^{-\tau-1} f_{k-\tau}$$

$$(46)\qquad \partial^k \{(\lambda - x)^{-2} f(x)\} = -k! \sum_{\tau=0}^{k} (\tau+1)(\lambda - x)^{-\tau-2} f_{k-\tau}$$

~~The deviation of fo~~

The derivation of formula (13a) is briefly indicated; it sufficiently exemplifies the derivation of the remaining results. ~~Formula ( ) yields the quotient of two $(2s)^{th}$ order determinants~~

$$\sigma^{(m)}_{2s-1}(\lambda) = \frac{[Y_{0,s}, f\, Y_{2,s}]_m}{[Y_{0,s-1}, f\, Y_{0,s-1}]_m}.$$

$\sigma^{(m)}_{2s-1}(\lambda)$

In each of these two determinants occurring in formula (20a) fr. the elements in row $(i+1)$ are replaced by the symbolic sum $\sum_{\tau=0}^{i} \binom{i}{\tau} (-1)^{\tau}$ row $(i-\tau+1)$, $(i=1,\dots,2s-1)$. Adopting the above substitution for x_i, f_i and neglecting numerical constants and powers of h that appear in the transformation of both numerator and denominator, the elements in row $(i+1)$ of the numerator become the derivatives of order i with respect to

x of the elements in the first row $(i=1,\dots,2s-1)$. The elements in the first $(s+1)$ even-numbered columns lying below the principal diagonal are annihilated by the above operation. The value of the numerator determinant becomes equal to the product of $\prod_{z=1}^{s} z!$ and that of a determinant with element $\Delta^{s+i+1}\{(\lambda-x)^{j}f(x)\}$ in the $(i+1)^{th}$ row and $(j-1)^{th}$ column $(i=0,\dots,s-2;$ $j=2,\dots,s)$. This determinant is extended to order $s+1$ by the addition of a first row, the elements of which are $\cancel{1}$ $0,1$ followed by $\cancel{s}$ $s-1$ zeros, of a second row with elements $\cancel{0,1}$ 1 followed by $\cancel{s-1}$ s zeros, and of remainders of two columns containing the elements $\Delta^{s+i+1}\{(\lambda-x)^{j}f(x)\}$ for $j=0,1$ and $i=0,\dots,s-2$. The

$(j+1)^{th}$ column of this extended determinant is modified by the addition of multiples of previous columns, as suggested by formula (44), with $j = s, s-1, \ldots, 1$; this operation isolates terms of the form ~~$(-1)^j (s+i+1)!$~~ $(-1)^j (s+i+1)!\, f_{s+i-j+1}$. Products of factorials and powers of -1 are separated out, columns are rearranged, and what remains is the numerator determinant in formula (43a). The required treatment of the denominator determinant in formula (20a) is slightly simpler; formula (43a) is easily obtained. Formula (43b) is derived in the same way, it being necessary to remark that certain of the terms in sums of the form (45, 46) may be removed by the subtraction of suitable multiples of columns of the array $F_{0,s,s-1}$.

It is easily demonstrated that the determinant $[F_{1,n+1,n+1}]$ is identically zero for all x if and only if f is the irreducible quotient of two n^{th} degree polynomials in x, and that the determinantal quotient $\Theta_{2s}(\lambda, x)$ of formula (43 b) is then equal to $f(\lambda)$. That this quotient is subject to the stated existence conditions equivalent to the s^{th} convergent of the continued fraction associated with the series (41) follows from known formulae relating to such continued fractions (see [15] §).

The remaining results of the theorem are obtained in the same way.

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