

Ch. 1. Theorems concerning determinants

Elements a, b, c, \dots \in a field. Enriched by ∞

$0/0$, ∞/∞ , $\infty \pm \infty$, ∞/∞ indeterminate; other numbers well determined.

$$\left| \begin{array}{cc} a_1 & b_1 & x_1 & y_1 \\ a_2 & b_2 & x_2 & y_2 \\ \vdots & \vdots & \ddots & \ddots \\ a_r & b_r & x_r & y_r \end{array} \right| \quad (1)$$

N. 1 (1) written as $|a_1, b_1, \dots, x_r, y_r|$

1.1. Compound determinants

$$\text{Th 1. } |a_1, b_1, \dots, x_r, | |a_1, b_2, \dots, x_r, y_r, z_{1r}| =$$

$$|a_1, b_1, \dots, x_r, y_r| |a_1, b_2, \dots, x_r, z_{1r}| - |a_1, b_2, \dots, x_r, z_r| |a_1, b_1, \dots, x_r, y_{rr}|$$

1.2. Schreins' Lemma

Th. 2.

$$|N_1, b_2, c_3, \dots, y_{r-1}, z_r| |D_1, b_2, c_3, \dots, y_{r-1}| - |D_1, b_2, c_3, \dots, y_{r-1}, z_r| |N_1, b_2, c_3, \dots, y_{r-1}|$$

$$= |N_1, b_2, c_3, \dots, y_r| |b_1, c_2, \dots, z_{r-1}|$$

C1. If either a) both $|D_1, b_2, c_3, \dots, y_{r-1}|$ and $|D_1, b_2, c_3, \dots, y_{r-1}, z_r|$ non zero, or

b) $|N_1, b_2, c_3, \dots, y_{r-1}, z_r|$ and $|D_1, b_2, c_3, \dots, y_{r-1}, z_r|$ not simultaneously zero and

$$|N_1, b_2, c_3, \dots, y_{r-1}| = |D_1, b_2, c_3, \dots, y_{r-1}| \quad \dots \quad \dots \quad \dots$$

$$\text{then } \frac{|N_1, b_2, c_3, \dots, y_{r-1}, z_r|}{|D_1, b_2, c_3, \dots, y_{r-1}, z_r|} - \frac{|N_1, b_2, c_3, \dots, y_{r-1}|}{|D_1, b_2, c_3, \dots, y_{r-1}|} = \frac{|b_1, c_2, \dots, z_{r-1}| |N_1, D_2, b_3, c_4, \dots, y_r|}{|D_1, b_2, c_3, \dots, y_{r-1}| |D_1, b_2, c_3, \dots, y_{r-1}, z_r|}$$

If we S. the $|D_1, b_2, c_3, \dots, y_{r-1}|$, $|D_1, b_2, c_3, \dots, y_{r-1}, z_r|$ is zero, then detl. quotient on rhs. infinite, otherwise finite.

C2. If none of $D_1, D_1 b_2, D_1 b_2 c_3, \dots, D_1 b_n c_n \dots y_{r+1}$ and

$|N_1 b_2 c_3 \dots y_{r+1} z_r|$ and $|D_1 b_2 c_3 \dots y_{r+1} z_r|$ not simultaneously zero. Then

$$\frac{|N_1 b_2 c_3 \dots y_{r+1} z_r|}{|D_1 b_2 c_3 \dots y_{r+1} z_r|} = \frac{N_1}{D_1} + \frac{b_1 |N_1 D_2|}{D_1 |D_1 b_2|} + \frac{|b_1 c_2| |N_1 D_2 b_3|}{|D_1 b_2| |D_1 b_2 c_3|} + \dots +$$

$$\frac{|b_1 c_2 \dots z_{r+1}| |N_1 D_2 b_3 c_4 \dots y_r|}{|D_1 b_2 \dots y_{r+1}| |D_1 b_2 c_3 \dots y_{r+1} z_r|}$$

$$\text{Th. 3. } |\alpha_1 b_2 c_3 \dots y_{r+1} z_r| |b_2 c_3 \dots y_{r+1}| - |\alpha_1 b_2 c_3 \dots y_{r+1} z_r| |\alpha_1 b_2 c_3 \dots y_{r+1}| =$$

$$- |\alpha_1 b_2 c_3 \dots z_{r+1}| |\alpha_2 b_3 c_4 \dots y_r|$$

C1. If either a) both $|\alpha_1 b_2 c_3 \dots y_{r+1}|, |\alpha_1 b_2 c_3 \dots y_{r+1} z_r|$ non-zero, or

b) $|\alpha_1 b_2 c_3 \dots y_{r+1} z_r|$ and $|\alpha_1 b_2 c_3 \dots y_{r+1}|$ not sim. zero and

$|\alpha_1 b_2 c_3 \dots y_{r+1}|$ and $|\alpha_2 b_3 c_4 \dots y_r|$ not sim. zero

then $\frac{|\alpha_1 b_2 c_3 \dots y_{r+1} z_r|}{|\alpha_1 b_2 c_3 \dots y_{r+1}|} - \frac{|\alpha_1 b_2 c_3 \dots y_{r+1}|}{|\alpha_1 b_2 c_3 \dots y_{r+1} z_r|} = \frac{(|\alpha_1 b_2 c_3 \dots z_{r+1}| |\alpha_2 b_3 c_4 \dots y_r|)}{|\alpha_1 b_2 c_3 \dots y_{r+1}| |\alpha_1 b_2 c_3 \dots y_{r+1} z_r|}$

If none of $|\alpha_1 b_2 c_3 \dots y_{r+1}|, |\alpha_2 b_3 c_4 \dots y_r|$ zero, quotient on r.h.s. is infinite
otherwise finite.

C2. If none of $|b_2|, |b_2 c_3|, \dots, |b_2 c_3 \dots y_{r+1} z_r|$ zero, and $|\alpha_1 b_2 c_3 \dots y_{r+1} z_r|$
and $|\alpha_2 b_3 c_4 \dots y_r|$ not sim. zero, then

$$\frac{|\alpha_1 b_2 c_3 \dots y_{r+1} z_r|}{|\alpha_1 b_2 c_3 \dots y_{r+1} z_r|} = \alpha_1 - \frac{b_1 a_2}{b_2} - \frac{|\alpha_1 b_2 c_3| |\alpha_2 b_3|}{b_2 |\alpha_1 b_2 c_3|} - \dots - \frac{|\alpha_1 b_2 c_3 \dots z_{r+1}| |\alpha_2 b_3 c_4 \dots y_r|}{|\alpha_1 b_2 c_3 \dots y_{r+1} z_r| |\alpha_2 b_3 c_4 \dots y_r|}$$

1.3. Simple and extended Hankel determinants

N2. $r \in \mathbb{J}$: $\begin{vmatrix} f_{m+1} & \dots & f_{mr} \\ f_{mr} & f_{m+2r} & \dots \end{vmatrix}$ denoted by $H[f_{mr}]_r$

set $H[f_{mr}]_{-2} = 0$ $H[f_{mr}]_{-1} = 1$

N3. $r \in \mathbb{J}$: $\begin{vmatrix} a & b & c_m & c_{mr} & c_{mr+2} \\ d_m & e_m & f_m & f_{mr} & f_{mr+2} \end{vmatrix}$ denoted by $H \begin{bmatrix} a & b \\ d_m & e_m \\ f_m & f_{mr} \end{bmatrix}_r$

$r=2: 0; r=-1: 1; r=0: a: r=1 \begin{vmatrix} a & b \\ d_m & e_m \end{vmatrix}$

$$\text{Th.4. } H[\alpha f_{rmm} \gamma^r]_r = \alpha^m \gamma^{r(r)} + [f_{rmm}]_r \quad (r \in \mathbb{J})$$

$$\text{N3 } \Delta_i g(\dots h, i, j, \dots) = g(\dots h, i+1, j, \dots) - g(\dots h, i, j, \dots)$$

$$\text{Th.5 } H[f_{rmn}]_r = H[\Delta_m^T f_m]_r$$

$$\text{Th.6. } \{H[t_{rmm+1}]_r\}^2 - H[t_{rmm}]_r, H[t_{rmm+2}]_r + H[t_{rmm}]_{r+1}, H[t_{rmm}]_{r-1}, \dots$$

$$\text{N4. } t_\omega = 0 \quad (\nu_2 = \mathfrak{I}_1) \quad H[t_{rmm}]_r = H_{m,r} \quad (r = \mathfrak{I}_{-2}; m = \hat{\mathfrak{I}})$$

$$H_{m+1,r}^2 - H_{m,r} H_{m+2,r} + H_{m,r+1} H_{m,r-1} = 0 \quad (r = \mathfrak{I}_{-1}, m = \hat{\mathfrak{I}})$$

Ch 2. Prediction based upon a linear model

2.1 Exponential extrapolation

Th 1 If $S_j = S + d\gamma^j$ ($j \in \mathbb{J}$) where S & γ are finite with $\gamma \neq 0, 1$

then $S = \varepsilon_2^{(m)}$ where $\varepsilon_2^{(m)} = \frac{S_m S_{mn} - S_{mn}^2}{S_m - 2S_{mn} + S_{mn}}$; if $|\gamma| < 1$,

$$\varepsilon_2^{(m)} = \lim_{n \rightarrow \infty} S_j \quad (m \in \mathbb{J})$$

2.2 Extrapolation using a linear recursion

N1 If $(c_\nu S_{mn})_0^r = G$ ($n \in \mathbb{J}$) ($r \in \mathbb{J}_1$) $\{c_\nu\}, G$ finite and δm , we write $\{S_\nu\} \in LR_r(c_\nu; G)$

N2 $E_i f(\dots, h, i, j, \dots) = f(\dots, h, i+1, j, \dots)$

Th 2 If $\{S_\nu\} \in LR_r(c_\nu; G)$ then $S = G / (c_\nu)_0^r$ well determined

Th 3. If $\{S_\nu\} \in LR_r(c_\nu; G)$ and roots (among diff.) $\gamma_1, \gamma_2, \dots, \gamma_r$ ($r \in \mathbb{J}_1$) are $\gamma_1, \gamma_2, \dots, \gamma_r$ with multiplicity h_1, h_2, \dots, h_r , resp. then when $(c_\nu)_0^r \neq 0$

$$S_\nu = S + (\gamma_{\nu 1} (d_{\nu 1} m^\nu)_0^r)$$

$$S_{\nu 1} = S + (\gamma_{\nu 1}^m (d_{\nu 1} m^\nu)_0^{h_{\nu 1}-1})_1^r$$

when $(c_\nu)_0^r = 0$, so that we may set $\gamma_1 = 1$

$$S_m = (d_{1, \nu} m^\nu)_0^{h_1} + (\gamma_{\nu 1}^m (d_{\nu 1} m^\nu)_0^{h_{\nu 1}-1})_2^r$$

$$\text{If } |\gamma_{\nu 1}| < 1 \quad (r \geq g_1) \quad S_m |_{M=\infty} = S$$

D.1 $S\{S_\nu\} = G / (c_\nu)_0^r$ associated with $\{S_\nu\} \in LR_r(c_\nu; G)$
the generalized limit of this sequence.

$$\text{Th 4. If } \{S_\nu\} \in LR, \{c_{\nu,j} G_j\} \text{ then } S\{S_\nu\} = \frac{H[S_{\nu m} [\Delta_m S_{\nu m}]]_r}{H[1 [\Delta_m S_{\nu m}]]_r} \quad (m=9)$$

Ans. Set $\varepsilon_{2r}^{(m)} = \frac{H[S_{\nu m} [\Delta_m S_{\nu m}]]_r}{H[1 [\Delta_m S_{\nu m}]]_r}$

$$\text{Th 5} \quad \varepsilon_{2r}^{(m)} = \frac{(W_{r,\nu}^{(m)} S_{\nu m})_0^r}{(W_{r,\nu}^{(m)})_0^r} \quad (r, m = 9)$$

$$\text{Th 6} \quad \varepsilon_{2r}^{(m)} = \frac{H[\Delta_m^\tau S_m]_r}{H[\Delta_m^{\tau+2} S_m]_{r+1}} \quad (r, m = 9)$$

Ch 3 The epsilon algorithm

3.1 The auxiliary numbers $\{\varepsilon_{2r+1}^{(m)}\}$

$$\varepsilon_{2r+1}^{(m)} = \frac{H [1 [\Delta_m^2 S_{2r+1}]_r]}{H [\Delta_m S_{2r+1} [\Delta_m^2 S_{2r+1}]_r]} \quad (r, m \in \mathbb{J})$$

Th 1 $\varepsilon_{2r+1}^{(m)} = \frac{H [\Delta_m^2 S_m]_r}{H [\Delta_m^2 S_m]_{r+1}}$

3.2. The fundamental numbers

Th 2 If $m, r \in \mathbb{J}$ fixed $\varepsilon_{r-1}^{(mn)}, \varepsilon_r^{(mn)}, \varepsilon_r^{(m)}$ well det. and

a) $\varepsilon_{r-1}^{(mn)} \neq \infty$ when $\varepsilon_r^{(mn)} = \varepsilon_r^{(m)}$ and also

b) $\varepsilon_r^{(mn)}$ and $\varepsilon_r^{(m)}$ not both infinite

then det equt for $\varepsilon_{r+1}^{(mn)}$ well det and $\varepsilon_{r+1}^{(mn)} = \varepsilon_{r-1}^{(mn)} + (\varepsilon_r^{(mn)} - \varepsilon_r^{(m)})^{-1}$

Th 3 If $m, r \in \mathbb{J}$ fixed $\varepsilon_r^{(m)}, \varepsilon_{r+1}^{(m)}, \varepsilon_{r-1}^{(mn)}$ well det and

a) $\varepsilon_r^{(m)}$ not infinite when $\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(mn)}$

b) $\varepsilon_{r+1}^{(m)}$ and $\varepsilon_{r-1}^{(mn)}$ not both infinite

then det equt for $\varepsilon_{r+1}^{(mn)}$ well det and $\varepsilon_{r+1}^{(mn)} = \varepsilon_r^{(m)} + (\varepsilon_{r+1}^{(m)} - \varepsilon_{r-1}^{(mn)})^{-1}$

Defl Contr. 3) ε -array from $\varepsilon_{-1}^{(m)} = 0, \varepsilon_0^{(m)} = S_m$ (~~$\varepsilon_{-1}^{(m)} = 0$~~) $m = \mathbb{J}_0^{m'}$ (3)

($m' \in \mathbb{J}$) where $\{S_m\}$ prescribed, is called forward application;

construction of infinite vertical strip by $2t+2$ columns from $\varepsilon_{-1}^{(m)} = 0$

$\varepsilon_r^{(m)} (r = \mathbb{J}_0^{m'}) \varepsilon_{2r+1}^{(m)} (m = \mathbb{J}_0)$ is called progressive application

D.2. If $\{\varepsilon_r^{(m)}\}$ can be contr. from (3) then we say these numbers can be contr. from the sequence $\{S_m\}$

3.3. The algebraic theory of the epsilon algorithm.

3.3.1 Invariant properties of the epsilon algorithm

Th 4. $\{\varepsilon_r^{(m)}\}$ from $\{S_\nu\}$, $\{\varepsilon_r^{(m)'}\}$ from $\{S_{\tau(\nu)}\}$ ($\tau \in \mathbb{T}$ fixed) then $\varepsilon_r^{(m)'} = \varepsilon_r^{(m+1)}$

Th 5. $\{\varepsilon_r^{(m)}\}$ from $\{S_\nu\}$, $\{\varepsilon_r^{(m)'}\}$ from $\{A + B S_\nu\}$

$$\varepsilon_{2r}^{(m)} = A + B \varepsilon_{2r}^{(m)} \quad \varepsilon_{2r+1}^{(m)} = B^{-1} \varepsilon_{2r+1}^{(m)}$$

Th 6. $\{\varepsilon_r^{(m)}\}$ from $\{S_\nu\}$, $\{\hat{\varepsilon}_r^{(m)}\}$ from $\hat{\varepsilon}_{-1}^{(m)} = 0$ $\hat{\varepsilon}_0^{(m)} = S_m$ by means of

$$\hat{\varepsilon}_{2r+1}^{(m)} = \hat{\varepsilon}_{2r-1}^{(m)} + B (\hat{\varepsilon}_r^{(m)} - \hat{\varepsilon}_1^{(m)})^{-1} \text{ then } \hat{\varepsilon}_{2r}^{(m)} = \varepsilon_{2r}^{(m)} \quad \hat{\varepsilon}_{2r+2}^{(m)} = B^{-1} \varepsilon_{2r+2}^{(m)}$$

Sequence:

3.3.2 Recursions for the numbers $\{\varepsilon_{2r}^{(m)}\}$ and $\{\varepsilon_{2r+1}^{(m)}\}$

Th 7. Assume $\varepsilon_{2r+2}^{(m)}, \varepsilon_{2r+1}^{(m)}, \varepsilon_{2r}^{(m)}, \varepsilon_{2r-1}^{(m)}, \varepsilon_{2r-2}^{(m)}, \varepsilon_{2r}^{(m+1)}, \varepsilon_{2r+1}^{(m+1)}, \varepsilon_{2r+2}^{(m+1)}$

well det. also four diff's having from $\varepsilon_r^{(m+1)} - \varepsilon_1^{(m+1)}$ and also four

diff's, $\varepsilon_{r+1}^{(m)} - \varepsilon_{r-1}^{(m+1)}$ also well det. Then

$$\text{det} \left(\varepsilon_{r+2}^{(m+1)} - \varepsilon_{r+1}^{(m+1)} \right)^{-1} \neq 0$$

$$(\varepsilon_{2r+2}^{(m+1)} - \varepsilon_{2r}^{(m+1)})^{-1} + (\varepsilon_{2r+1}^{(m+1)} - \varepsilon_{2r}^{(m+1)})^{-1} = (\varepsilon_{2r}^{(m+1)} - \varepsilon_{2r}^{(m)})^{-1} + (\varepsilon_{2r+1}^{(m+1)} - \varepsilon_{2r+1}^{(m)})^{-1}$$

Ind. assumption concerning $\varepsilon_{2r-3}^{(m)}, \varepsilon_{2r-2}^{(m)}, \dots, \varepsilon_{2r+1}^{(m+1)}$

Ch. 9. The Padé Table

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N1. = used to define polynomials... power series as sets of coefficients and to establish relationships between such sets of coeffs

N2. \Rightarrow used consistently for single summation: $p(z) = v_{j_1} z^{j_1} + v_{j_1+1} z^{j_1+1} + \dots + v_{j_k} z^{j_k}$

denoted by $\sum_{j_1}^{j_k} v_j z^j$
In $R(z) = \frac{\sum_{j_1}^{j_k} v_j z^j}{\sum_{i_1}^{i_k} t_i z^i}$

assumed that either $i_2 > i_1, j_2 > j_1; v_2, t_2$ not all zero

$f(z) = t_{i_0} z^{i_0} + t_{i_0+1} z^{i_0+1} + \dots \quad (i_0 \in \mathbb{J})$ written as $\sum_{i_0} t_i z^i$

~~If $t_{i_0} = 0$ we write it $i_0 = 0$, write $f(z) = \sum_{i_0} t_i z^i$~~

in $\sum_{i_0} t_i z^i$ assumed that $t_0 \neq 0$.

we set $t_0 = 0 \quad (\theta = -\mathbb{J}_1)$

Explanation of relationships between polynomial rat. fns. power series

Values denoted by : $R(z) = \frac{(v_j z^j)_{j_1}}{(t_i z^i)_{i_1}}$

4.1 The Padé quotient

Derivation of $\hat{v}_{j_1}^{(i,j)}$ and $\hat{t}_{i_1}^{(i,j)}$ from t_i by means \Rightarrow

$$(\hat{v}_{j_1}^{(i,j)} t_{i_1-j_1})_0^i = \hat{v}_{j_1}^{(i,j)} \quad (r = j_1); \quad (\hat{t}_{i_1}^{(i,j)} t_{i_1-j_1})_0^i = 0 \quad (r' = \mathbb{J}_{j_1+1}) \quad (i \neq 0)$$

Th. 1. $i, j \in \mathbb{J}$ fixed; set $\hat{v}_{j_1}^{(i,j)}, \hat{t}_{i_1}^{(i,j)}$ ~~can be det.~~ can be det. from (2) (3)

rat to $\hat{R}_{i,j}(z)$ unique.

Num, denom of $\hat{R}_{i,j}(z)$ may have common factor

Th. 2. To every $f(z) = \sum t_i z^i$ and every pair $i, j \in \mathbb{J}$, corresponds unique pair

of polynomials $p_{i,j}(z), q_{i,j}(z)$ not containing common factor with $t_0^{(i,j)} = t_0^{(i,j)} - 1$
s.t. $\deg p_{i,j}(z) \leq \deg q_{i,j}(z) \leq j, i$ resp.; formal expansion of
 $z^r \{q_{i,j}(z) f(z) - p_{i,j}(z)\}$ contains no power $\geq c \in \mathbb{N}^{j+1}$

Def $R_{i,j}(z) = \frac{P_{i,j}(z)}{Q_{i,j}(z)}$ with $\tau^{(i,j)}_0 = t_0$, $\eta^{(i,j)}_0 = 1$ called Padé quotient

order i,j derived from $f(t)$

Ex 1 $R_{1,1}(z)$ from $t_0 = t_1 = 1$, $t_2 = \frac{1}{2}$

Ex 2 $R_{2,1}(z)$ from $t_0 = 1$, $t_1 = \frac{1}{2}$, $t_2 = \frac{1}{4}$, $t_3 = \frac{1}{12}$

Ex 3 $R_{2,1}(z)$ from $t_0 = z^{-2}$, $t_1 = 0$

Th 3. If finite value of the argument, non-van. \Rightarrow Padé quotient well def

4.1.1. The primitive Padé quotient

Th 2 If $\tau^{(i,j)}_j$, $\eta^{(i,j)}_j$ in $R_{i,j}(z)$ det directly from linear equations, the $R_{i,j}(z)$ said to be primitive.

Th 4 $R_{i,j}(z)$ primitive: series expon. agrees with $f(z)$ as far as term in t^{i+j}

Th 5 Padé quotient of order i,j derived from $f(z) = \sum b_n z^n$ irredmble if

$$H_{j-i+1,i-1} \neq 0$$

Th 6 Num of primitive $R_{i,j}(z)$ given by

$$P_{i,j}(z) = \frac{H[L_{t_{j+i-j}}] \sum \tau^{(j-i)}_0 t_j z^{j+i-j}}{H_{j-i+1,i-1}}$$

$$T_{i,j}(z) = \frac{H[L_{\tau^{(j-i)}}] z^{i-j}}{H_{j-i+1,i-1}}$$

Th 7. Num values of num and det of primitive Padé quotient well def

Th 8. Assume $R_{i,j}(z)$ and its eight neighbours derived from $\sum b_n z^n$ are primitive; for any prescribed non-zero \neq value of $R_{i,j}(z)$ unequal to any of neighbours

4.1.2. The non-defective Padé quotient

D.3 Primitive $R_{i,j}(z)$ for which $\alpha_{i,i}^{(i,j)} \neq 0$ & $\omega_{j,j}^{(i,j)} \neq 0$ called non def.

Th. 9. $R_{i,j}(z)$ derived from $\sum t_i z^i$ non def. if. $H_{j-i+1, i-1}, H_{j-i+2, i-1}$
 $H_{j-i, i}$

4.2 The Padé table

Two dim array: bordered by $R_{i,-1}(z), R_{-1,j}(z)$

D4 i fixed $R_{i,j}(z); j \geq 0$ rows seq. j fixed $R_{i,j}(z); i \geq 0$ col seq.
 in front $R_{r,m+r}(z) R_{m+r,r}(z)$ forward diag seq. $R_{r,m-r}(z) = \tilde{J}_0$
 backward diag. $R_{r,r}(z)$ lie on prime diag $R_{m,r}(z)$ on sub prime
 diag.

D5 A Padé table exclusively composed of primitive Padé quot. is called
 a normal Padé table. Series generating such a table called normal

Th. 10 Padé table derived from $\sum t_i z^i$ is normal iff $H_{j-i+1, i}, i \geq 1, j \geq 1$

Th. 11 All quotients of normal Padé table are nondefective

Th. 12 $R_{i,j}(z)$ derived from normal $f(z)$ has series expansion agreeing
 with $f(z)$ as far as $i+j$ than t^{i+j}

Th. 13 Quotients $R_{i,j}(z)$ agreeing with $P_{m,n}(z)$ (non def.) form sq.

If formal series seqn. $\exists \tilde{\pi}_{i,j}(z) f(z) - \tilde{\pi}_{i,j}(z)$ in ascending powers. \exists
 z agrees with begin with $z^{\min(i,j)+1}$ sq. contains quotients D
 ndex. $(i, j) \in \mathbb{Z}_0^2$

Ex. 1 $R_{i,j}(z); i, j \in \mathbb{Z}_0^2$ from $\sum_{k=1}^{\infty} \frac{t_k}{(2k+1)!} z^k$ non values

$$\dots (i, j) \in \mathbb{Z}_0^2, i+j \leq 10, \frac{1}{2}(i!)^{-1} z^i$$

4.2.1 The Padé table associated with a rational function.

Th 14 If $f(z)$ generated by $R(z) = \frac{\sum_{j=0}^m v_j z^j}{\sum_{j=0}^n w_j z^j}$ the Padé table possesses infinite block bordered by row seq. $R_{i,j}(z) \quad i = 3_j$ and col seq $R_{i,j}(z) \quad (i = 3_j)$. If Padé table possesses such a block, then series from Stiunch it is derived in series expansion of $R(z)$

Ch.5 The epsilon algorithm and the Padé table

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5.1 Even order epsilon numbers and primitive Padé quotients

Th 1. $m, r \in \mathbb{J}$ fixed. $R_{j, m+r-1}(z)$ derived from $\sum t_i z^i$ primitive

$$\text{Set } S_r = \sum_{i=0}^{m-1} t_i z^i. \quad \varepsilon_{2r}^{(m)} = \frac{H[\bar{S}_{rm} \bar{[A_m]_{rm}} \bar{J}]_r}{H[\bar{I} \bar{[A_m]_{rm}} \bar{J}]_r}$$

$$\varepsilon_{2r+1}^{(m)} = \frac{H[I \bar{[A_m^2 S_{rm}]} \bar{J}]_r}{H[S_{rm} \bar{[A_m^2 S_{rm}]} \bar{J}]_r} \quad \text{represent well defined rat fun with}$$

finite coeffts; finite values $\Rightarrow z$ have well deft values

$$\varepsilon_{2r}^{(m)} = \frac{H[\bar{t}_{rm} \bar{j} \sum_{i=0}^{m-1} t_i z^{2r+i} \bar{J}]_r}{H[\bar{t}_{rm} \bar{J} z^{r-1}]_r}$$

$$\varepsilon_{2r+1}^{(m)} = \frac{H \left[\begin{matrix} 0 & z^{-r} \\ z^{-r} & [t_{rm}] \end{matrix} \right]_{r+1} z^m}{H_{mr}} \quad \text{and } \varepsilon_{2r}^{(m)} = R_{j, m+r-1}(z) \quad (6)$$

DI. $t_{m,r} \neq 0 (r, m \in \mathbb{J})$ $f(z) = \sum t_i z^i$ semi normal

N1. $z \in \rho_{ij}; \{f(z)\}$ means z is a pole $\Rightarrow R_{i,j}(z)$ derived from $f(z)$.

Th 2. $f(z)$ semi normal.

1) All rat fun $\varepsilon_{ij}^{(m)}$ can be produced from $\varepsilon_{-1}^{(m)} = 0$ $\varepsilon_0^{(m)} = \sum_{i=0}^{m-1} t_i z^i$ by means
 $\Rightarrow \varepsilon_{i,m}^{(m)} = \varepsilon_{k,m}^{(m)} + (\varepsilon_{i,m}^{(m)} - \varepsilon_{k,m}^{(m)})^{-1}$ and denoting Padé quot derived from
 $f(z)$ by $R_{ij}(z)$ (6) holds ($i, m \in \mathbb{J}$)

2) $z \neq 0$ further, $z \notin \rho_{ij}; \{f(z)\} (j \in \mathbb{I}, i \in I_0^{(m)})$, numerical values can
also be constructed as in 1)

$\varepsilon_{x_m}^{(m)} = \infty$ ($m \in \mathbb{P}$). If $z \in P_r^{(m)} \{f(z)\}$ ($r = \hat{j}_0^{i-1} m + I$) then num. vals. $\overset{(13)}{\exists}$

$\varepsilon_r^{(m)} r = \hat{j}_0^{i_1^*} m + \hat{P}$ can be const.; if z is pole of $R(z)$ then $\varepsilon_{x_{i-1}}^{(m)}$
($m \in \mathbb{I}$) all copies of finite numbers $\varepsilon_{x_m}^{(m)} = \infty$ ($m \in \mathbb{P}$), if z not pole

of $R(z)$, $\varepsilon_{x_m}^{(m)} = R(z)$, $\varepsilon_{x_m}^{(m)} = \infty$ ($m \in \mathbb{P}$)

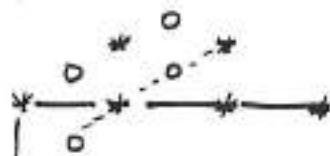
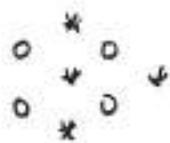
2) if $i < j+1$, $\varepsilon_r^{(m)} r = \hat{j}_0^{i-1}, m \in \mathbb{I}; m \in \hat{j}_0^{j-i}, r = \hat{j}_{i+j-m+2}^{i+j-m+2}$
can be const.; $\varepsilon_{x_{i+2}}^{(j-i+2)} = R(z)$, $\varepsilon_{x_m}^{(m)} = R(z)$, $\varepsilon_{x_m}^{(m)} = \infty m \in \hat{j}_{i+j-m+2}^{i+j-m+2}$

If $\exists \notin P_r^{(m)} \{f(z)\}$ ($r = \hat{j}_0^{i-1}, m \in \mathbb{I}, \mathbb{P}; m \in \hat{j}_0^{j-i}, r \in \hat{j}_{i+j-m+2}^{i+j-m+2}$) then
num. vals. of $\varepsilon_r^{(m)} r = \hat{j}_0^{i_1^*}, m \in \mathbb{P}; m \in \hat{j}_0^{j-i}, r = \hat{j}_{i+j-m+2}^{i+j-m+2}$ can be const;
if z is pole of $R(z)$ then numbers $\varepsilon_{x_{i-1}}^{(m)}, m \in \hat{j}_{i+j-m+2}^{i+j-m+2}$ = finite const

$\varepsilon_{x_m}^{(m)} = \infty m \in \hat{j}_{i+j-m+2}^{i+j-m+2}$; if z not pole of $R(z)$ then $\varepsilon_{x_m}^{(m)} (m \in \hat{j}_{i+j-m+2}^{i+j-m+2})$
can be const. and $\varepsilon_{x_m}^{(m)} = R(z)$, $\varepsilon_{x_{i+2}}^{(m)} = \infty m \in \hat{j}_{i+j-m+2}^{i+j-m+2}$; if, in
addition $\exists \in P_r^{(m)} \{f(z)\} m \in \hat{j}_0^{j-i}, r = (\hat{i}+\hat{j}-m+2)/2$ then numbers
 $\varepsilon_{x_m}^{(m)} (m \in \hat{j}_0^{j-i}, r = \hat{i}+\hat{j}-m+2)$ can also be const. and \exists these

$$\varepsilon_{x_{i+2}}^{(j-i)} = R(z)$$

Sketches



5.2 The epsilon array assoc. with a rational function

D2. If $\exists \hat{i}, \hat{j} \in I$ s.t. $H_{m,1} \neq 0$ $r = \hat{j}_{\hat{i}}^{2\hat{i}+1}$, $m = \hat{i}$ and

1) if $\hat{i} \geq \hat{j}+1$ $H_{m,r} = 0$ ($r = \hat{j}_{\hat{i}}^m$, $m \in I$)

2) if $\hat{i} < \hat{j}+1$ $H_{m,r} = 0$ ($r = \hat{j}_{\hat{i}}^{\hat{j}-r}$, $m \in I$)

whilst $H_{m,r} \neq 0$ ($r = \hat{j}_{\hat{i}}^{\hat{j}}$, $m = \hat{j}_{\hat{i}-r+1}$) and also $r = \hat{j}_{\hat{i}+1}^{\hat{i}}$, $m = \hat{i}$

then $\{t_j\}$ whose coeffs. on $\{t_j\}$ said to be degenerately semi-normal; we write $\{t_j\} \in \text{DSN}(\hat{i}, \hat{j})$

Th3. $\{t_j\} \in \text{DSN}(\hat{i}, \hat{j})$; $\sum t_j z^j$ series. expn. of $R(z)$.

1) if $\hat{i} \geq \hat{j}+1$, quotients $\varepsilon_r^{(m)}$ ($r = \hat{j}_{\hat{i}}^{2\hat{i}+1}$, $m \in I$) rep. well. det. rat. fns.

$\varepsilon_{2\hat{i}}^{(m)} \equiv R(z)$ $\varepsilon_{2\hat{i}+1}^{(m)} = \infty$. If finite values $\hat{j} \geq$ they represent well determined numbers and $\varepsilon_{2\hat{i}}^{(m)} = R(z)$ $\varepsilon_{2\hat{i}+1}^{(m)} = \infty$ $m \in I$

2) if $\hat{i} < \hat{j}+1$ $\varepsilon_r^{(m)}$ for $r = \hat{j}_{\hat{i}}^{2\hat{i}+1}$, $m \in I$ and also $m = \hat{j}_{\hat{i}}^{\hat{j}-\hat{i}}$, $r = \hat{j}_{\hat{i}+2}^{\hat{i}}$ rep. well. det. rat. fns.

$$\varepsilon_{2\hat{i}}^{(m)} \equiv R(z) \quad \varepsilon_{2\hat{i}+1}^{(m)} = \infty \quad m = \hat{j}_{\hat{i}}^{\hat{j}-\hat{i}} \quad r = \hat{j}_{\hat{i}+2}^{\hat{i}} \quad (10)$$

$$\varepsilon_{2\hat{i}}^{(m)} \equiv R(z) \quad \varepsilon_{2\hat{i}+1}^{(m)} = \infty \quad m = \hat{j}_{\hat{i}+2}^{\hat{i}}$$

for finite values of $\hat{j} \geq$ above quot. rep. well. det. numbers, eqns
corresp. to (10) also hold

N2 If for $r \in I_{\hat{i}}, m \in I$ $\varepsilon_r^{(m)}$ is pole of rat. fn $\varepsilon_{2\hat{i}}^{(m)}$ derived from $f(z) = \sum t_j z^j$

we write $\varepsilon_r^{(m)} \in P_r^{(m)} \{f(z)\}$

Th4 $\{t_j\} \in \text{DSN}(\hat{i}, \hat{j})$ $f(z)$ expn. of $R(z)$

1) if $\hat{i} \geq \hat{j}+1$, then rat. fns $\varepsilon_r^{(m)}$ $r = \hat{j}_{\hat{i}}^{2\hat{i}+1}$, $m \in I$ can be constructed

furthermore $\varepsilon_{2\hat{i}}^{(m)} \equiv R(z)$ ($m = \hat{j}_{\hat{i}-r+1}$). If $z \in P_r^{(m)} \{f(z)\}$, $r = \hat{j}_{\hat{i}}^{\hat{i}-1}$,
 $m = \hat{j}_{\hat{i}}^{\hat{j}-\hat{i}}$, $r = \hat{j}_{\hat{i}+1}^{\hat{i}}$ then min. values of $\varepsilon_r^{(m)}$ $r = \hat{j}_{\hat{i}}^{\hat{i}}$, $m \in I$

5.3 The extended epsilon array

introduction of $\varepsilon_r^{(m)}$ $r \in I$ $m > -\bar{J}_{(r \neq z)}$?

Th 5. $f(z)$ normal.

- 1) all rational functions $\varepsilon_r^{(m)}$ $r \in \bar{I}$ $m = \bar{J}_{(r \neq z)}$ can be produced using addl. init. values $\varepsilon_{\infty}^{(\infty)} = 0$ ($r \in \bar{I}_1$) and $\varepsilon_{\infty}^{(\infty)} = R_{r, \text{num}, 1}(z)$ ($r \in \bar{I}$, $m = \bar{J}_{(r \neq z)}$)
- 2) $z \neq 0$ finite $z \notin p_{i,j}(z)$ $\{f(z)\}$ ($i = \bar{J}$, $j = \bar{I}$), num values can be const. similarly

5.4 Recursions for primitive Padé quotients

Th 6. $i, j \in \bar{J}$ fixed: $R_{i,j}(z)$ $R_{i,j-1}(z)$ $R_{i-1,j}(z)$ $R_{i,j+1}(z)$ $R_{i-1,j+1}(z)$ primitive

- 1) these rational fns satisfy

$$\{R_{i,j}(z) - R_{i-1,j}(z)\}^{-1} + \{R_{i,j}(z) - R_{i-1,j+1}(z)\}^{-1} =$$

$$\{R_{i,j}(z) - R_{i,j-1}(z)\}^{-1} + \{R_{i,j}(z) - R_{i,j+1}(z)\}^{-1}$$

- 2) z non-zero finite: num values satisfy similar relationship

Th 7. $f(z)$ normal

- 1) $\{R_{i,j}(z)\}$ can be const. row by row in downward dir. from initial values

$$k_{-1,j}(z) = \infty \quad R_{0,j}(z) = \sum_0^j b_j z^j \quad (j \in \bar{J}) \quad R_{i,-1}(z) = 0 \quad (i \in \bar{I})$$

by systematic use of

$$R_{i-1,j}(z) = R_{i,j}(z) + \left[\{R_{i,j-1}(z) - R_{i,j}(z)\}^{-1} + \{R_{i,j+1}(z) - R_{i,j}(z)\}^{-1} - \{R_{i-1,j}(z) - R_{i,j}(z)\}^{-1} \right]^{-1}$$

- 2) for prescr. non-zero finite non-zero z num values simly const.: if $z \in p_{i,j}(z)$

$R_{i-1,j}(z) \neq R_{i,j-1}(z) + R_{i,j+1}(z) - R_{i-1,j}(z)$

$$R_{i-1,j}(z) = R_{i,j-1}(z) + R_{i,j+1}(z) - R_{i-1,j}(z)$$

N3. Replace (12) by $C \in N \cap S$

Th 8 If $E = \phi(W)$, $W = \phi(E)$ imply in $N \cap S$

$$\text{If } E = \phi(N, S) \quad E = \phi(S, N)$$

$$\text{Th 9} \quad (E+N-S)C^2 + 2(EW-NS)C + (EWJ + VN - ENJS - WNS) \geq 0$$

If irr prescr. \neq values \Rightarrow these quots are finite; these values also satisfy the eqn.

This real coeffs and sign.

$$\text{Either } E+N = N+S \text{ or } (E-N)(E-S)(W-N)(W-S) \geq 0$$

6.1 The general theory

D.1 $b_0 + \frac{a_1}{b_1 + \dots}$ (1) called c.f.: n^o $b_0 + \frac{a_1}{b_1 + \dots}$
 $\frac{a_r}{b_r + \dots}$ terminating c.f. $\frac{a_n}{b_n}$

{a_r}, {b₀} coeffs {a_n} part numer {b_n} part denomin

$$C_r = b_0 + \frac{a_1}{b_1 + \dots} \frac{a_r}{b_r} \text{, } r \text{th convergent}$$

$$C_r = G_{r,r} : G_{r,0} = b_r, G_{r,r+1} = b_{r+1} + \frac{a_{r+1}}{G_{r,r}} (r=I^{r-1})$$

(1) can be written as $b_0 + \frac{a_1}{b_1 + \dots} \frac{a_2}{b_2 + \dots} \dots$

N1 (1) written as $C[b_0; \frac{a_1}{b_1 + \dots}]$ (2) as $C_i = C[b_0; \frac{a_1}{b_1 + \dots} \dots]$

1) $b_0 = 0 : C[\frac{a_1}{b_1}]$

2) $b_0 + \frac{a_1}{b_1} \frac{a_2}{b_2} \dots C[b_0; \frac{a_2}{b_2}]$

3) $b_0 + \frac{a_1}{b_1} \frac{a_2}{b_2} \dots C[b_0; \frac{a_1}{b_1}, \frac{a_2}{b_2}] ; b_0 + \frac{a_1}{b_1} \frac{a_2}{b_2} \frac{a_3}{b_3} \dots C[b_0; \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}]$

4) a_r, b_r differ from $a_p, b_p, p=I^{r-1} : C_r [b_0; \frac{a_p}{b_p}, \frac{a_r}{b_r}]$

5) $b_0 + \frac{a'_1}{b'_1} \frac{a''_1}{b''_1} \frac{a'_2}{b'_2} \frac{a''_2}{b''_2} \dots C[b_0; \frac{a'_1}{b'_1}, \frac{a''_1}{b''_1}]$

$b_0 + \frac{a'_1}{b'_1} \frac{a''_1}{b''_1} \frac{a'_2}{b'_2} \dots \frac{a'_r}{b'_r} \frac{a''_r}{b''_r} : C[b_0; \frac{a'_1}{b'_1}, \frac{a''_1}{b''_1}]_{2r}$

$b_0 + \frac{a'_1}{b'_1} \frac{a''_1}{b''_1} \frac{a'_2}{b'_2} \dots \frac{a''_r}{b''_r} \frac{a_{r+1}}{b_{r+1}} : C[b_0; \frac{a'_1}{b'_1}, \frac{a''_1}{b''_1}]_{2r+1}$

$b_0 + \frac{a'_1}{b'_1} \frac{a''_1}{b''_1} \frac{a'_2}{b'_2} \dots \frac{a''_r}{b''_r} : C[b_0; \frac{a'_1}{b'_1}, \frac{a''_1}{b''_1}]$

$b_0 + \frac{a'_1}{b'_1} \frac{a''_1}{b''_1} \frac{a'_2}{b'_2} \dots : C[b_0; \frac{a'_1}{b'_1}, \frac{a''_1}{b''_1}, \frac{a'_2}{b'_2}, \dots]$

$$6) C \left[b_0; \frac{a_0}{b_0} \right]_0 \text{ reg. } C \left[b_0; \frac{a_0}{b_0} \right]_1 = C$$

D2. c.f. coeffs finite; point non-zero; is regular

Th1 converges of regular cf. well determined

6.1.1. The fundamental recursions

$$\text{Th2. } C \left[b_0; \frac{a_0}{b_0} \right] \text{ regular : } C \left[b_0; \frac{a_0}{b_0} \right]_r = \frac{N_r}{D_r} \quad (r=1)$$

$$N_{-1} = 1 \quad N_0 = b_0 \quad D_{-1} = 0 \quad D_0 = 1 \quad N_r = b_r N_{r-1} + a_r N_{r-2}$$

$$D_r = b_r D_{r-1} + a_r D_{r-2}$$

6.1.2. The equivalence transformation

$$\text{Th3 } \{y_r\} \text{ sequence of finite non-zero numbers } C \left[b_0; \frac{a_0}{b_0} \right] \text{ regular}$$

$$C \left[b_0; \frac{y_r a_r}{y_r b_r} \frac{y_0 y_{r-1} a_r}{y_r y_{r-1} b_r} \right]_r = C \left[b_0; \frac{a_0}{b_0} \right]_r$$

6.1.3. The contraction and extension of cont. fracto

$$\text{D3. } C \left[b_0; \frac{a_0}{b_0} \right]_r = C \left[b_0; \frac{a_0}{b_0} \right]_{2r}; C \left[b_0; \frac{a_0}{b_0} \right]_{even part} \text{ of } C \left[b_0; \frac{a_0}{b_0} \right]$$

$$C \left[b_0; \frac{a_0}{b_0} \right]_r = C \left[b_0; \frac{a_0}{b_0} \right]_{2r}; C \left[b_0; \frac{a_0}{b_0} \right]_{odd part} C \left[b_0; \frac{a_0}{b_0} \right]$$

Th4 $C \left[b_0; \frac{a_0}{b_0} \right]$ regular

$$C \left[b_0; \frac{a_0}{b_0} \right]_{2r} = C \left[b_0; \frac{b_2 a_1}{b_2 b_1 + a_2} - \frac{b_{2r} a_{2r-1} b_{2r-2}^{-1} a_{2r-2}}{a_{2r} + b_{2r}(b_{2r-1} + a_{2r-1} b_{2r-2}^{-1})} \right]_r$$

$$C \left[b_0; \frac{a_0}{b_0} \right]_{2r+1} = C \left[b_0; \frac{b_2}{b_2 + a_{2r} b_{2r}^{-1}} \right]_r$$

$$C \left[b_0; \frac{a_0}{b_0} \right]_{2r+1} = C \left[b_0 + b_1^{-1} a_1; \frac{-b_{2r+1} a_{2r} b_{2r-1}^{-1} a_{2r-1}}{a_{2r+1} + b_{2r+1} (b_{2r} + a_{2r} b_{2r-1}^{-1})} \right]_r$$

$$C \left[b_0; \frac{a_0}{b_0} \right]_{2r+2} = C \left[\dots \frac{-a_{2r+2} b_{2r+1}^{-1} a_{2r+1}}{b_{2r+2} + a_{2r+2} b_{2r+1}^{-1}} \right]_r$$

Th. 5. $C[b_0; \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_{r-1}}{b_{r-1}}, \frac{a_r}{b_r}] = \frac{H_r}{D_r}$ if finite non-zero number

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$$\text{ext. convgts} \ L \ b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots + \frac{a_{r-1}}{b_{r-1} + \frac{a_r}{b_r}}}} = \frac{\eta^{-1} a_m}{b_{m+1} + \eta^{-1} b_{m+2} + \frac{a_{m+3}}{b_{m+3}}} \dots$$

$$\text{one} \quad \frac{N_0}{D_0} \quad \frac{N_1}{D_1} \quad \frac{N_2}{D_2} \dots \quad \frac{N_{r-1}}{D_{r-1}} \quad \frac{N_r - \eta N_{r-1}}{D_r - \eta D_{r-1}} \quad \frac{N_r}{D_r} \quad \frac{N_{r+1}}{D_{r+1}} \quad \frac{N_{r+2}}{D_{r+2}} \dots$$

6.1.4 The Euler-Minding sum formulae

Th. $D_r \neq 0 \ (r=2)$ $C[b_0; \frac{a_1}{b_1}, \dots] = b_0 - (-1)^r \frac{a_1 a_2 \dots a_r}{D_{r-1} D_r}$

6.2. Continued fractions derived from power series

6.2.1 The corresponding continued fraction

Def $C\left[\frac{u_1}{1-}, \frac{u_2 z}{1-}, \dots\right]$ $\{u_n\}$ finite non-zero: regular \Leftrightarrow C-fraction

try for finite non-zero \Rightarrow converges \Rightarrow regular C-fraction well def.

Th. 8. $C_r(z) = C\left[\frac{u_1}{1-}, \frac{u_2 z}{1-}, \dots, \frac{p_r(z)}{t_r(z)}\right], r \geq 1$ (regular)

$$p_0(z) = 0 \quad p_1(z) = u_1 \quad \bar{t}_{1,0}(z) = \bar{t}_{1,1}(z) = 1$$

$$p_{2r}(z) = \sum_{j=0}^{r-1} \bar{t}_{2r,j}(z) z^j \quad p_{2r+1}(z) = \sum_{j=0}^r \bar{t}_{2r+1,j}(z) z^j \quad \bar{t}_{0,0}^{(r)} = u_1, \quad \bar{t}_{0,0}^{(r)} = 1$$

$$\bar{t}_{2r+1}(z) = \sum_{j=0}^r \bar{t}_{2r+1,j}(z) z^j \quad \bar{t}_{2r+2}(z) = \sum_{j=0}^r \bar{t}_{2r+2,j}(z) z^j$$

$$p_r(z) = p_{r-1}(z) - u_r z + p_{r-2}(z) \quad \bar{t}_r(z) = \bar{t}_{r-1}(z) - u_r z + \bar{t}_{r-2}(z)$$

rat. frns irreducible

Th. 9 (regular C-fraction) $C\left[\frac{u_1}{1-}, \frac{u_2 z}{1-}, \dots\right]_r = \sum t_{r,j} z^j : t_{0,0} = t_{1,0}, \quad j=0$

Th. 10 " $u_1 = t_{1,0} \quad u_{2,0} = \frac{H[-t_{1,0}, t_{1,1}]_{j-1} H[t_{1,1}]_{j-2}}{H[t_{1,0}]_{j-1} H[t_{1,0}, t_{1,1}]_{j-2}}$ $2 \leq j \leq r$

$$u_{2r+1} = \frac{H[t_{1,r}]_{j-1} H[t_{1,r+1}]_{j-1}}{H[t_{1,r}]_{j-1} H[t_{1,r+1}]_{j-1}}$$

Def C-fraction $C\left[\frac{u_1}{1}, \frac{u_2 z}{1}, \dots\right]$ denoted by imposing conditions that $t_{k,r} = t_k$
 is said to converge to series $f(z) = \sum t_k z^k$ $f(z)$ said to generate $C\left[\frac{u_1}{1}, \dots\right]$

Def. $H_{0,r} \neq H_{1,r} \neq 0$ said to be C-regular

Th. 11 $f(z)$ generates a regular C-fraction if it is C-regular

Th. 12 $f(z) = \sum t_k z^k$ generated by irreducible $R(z) = \frac{\sum_{j=0}^h t_j z^j}{\sum_{j=0}^{h-1} y_j z^j}$

$t_0 = t_0 \neq 0 \quad y_0 = 1 \quad y_r \neq 0 \quad \text{if } H_{0,h-1} \neq 0 \quad H_{0,r} = 0 \quad r = I_h \quad h = \max(i, j+1)$

c) $f(z)$ generated by $R(z)$ with $t_i - t_0 y_i \neq 0$ if $H_{0,h-1} \quad H_{0,r} = 0 \quad r = I_h^{(1)}$

$$h^{(1)} = \max(i, j).$$

Def R(z) generating $\sum t_k z^k$ for wh. $H_{0,r} \neq 0$ ($r = I_0^{h-1}$) $H_{0,r} = 0$ ($r = I_h$)

wh. $h = \max(i, j+1)$ and $H_{1,r} \neq 0$ ($r = I_0^{h-1}$) $H_{1,r} = 0$ ($r = I_h^{(1)}$) is called
 C-regular ct.

Def. power series whose coeffs. $\{t_k\}$ s.t. $\exists h \in \mathbb{N}$ for wh.

$$1) \quad H_{0,r} \neq 0 \quad H_{1,r} = 0 \quad (r = I_h^{h-1})$$

$$2) \quad H_{0,r} \neq 0 \quad (r = I_0^{h-1}) \quad H_{1,r} \neq 0 \quad (r = I_0^{h-2}) \\ = 0 \quad (r = I_h) \quad H_{1,r} = 0 \quad (r = I_{h-1})$$

is said to be degenerately C-regular. in case 1: $\{t_k\} \in DCR(h)$
 $\in DCR'(h)$

This $f(z) = \sum t_k z^k$ deg. C-reg. iff series exptn 1) C regular not for

if $\{t_k\} \in DCR(h) \quad h = i \geq j+1$; if $\{t_k\} \in DCR'(h) \quad h = j+1 \geq i$

if $\{t_k\} \subset DCR(h)$, $C\left[\frac{u_1}{1}, \frac{u_2 z}{1}, \dots\right]$ terminates with ~~at~~ ^{at} _{value}

N2 C-fraction generated by C-regular or deg. C-reg $f(z)$ denoted by
 $C\{f(u)\}$.

6.2.2. The associated continued fraction

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$\exists \theta C \left[\frac{v_1}{1-w_1z} - \frac{v_2 z^2}{1-w_2z} \right]$ in wh. all w_i finite v_i non zero called reg. A-fact.

Th. 14 " reg. A /raction"

i) Successive evngts $H \left[\frac{v_1}{1-w_1z} - \frac{v_2 z^2}{1-w_2z} \right] = \hat{G}(z)$ have form $\frac{\hat{P}_r(z)}{\hat{Q}_r(z)}$

$$\text{wh. } \hat{P}_1(z) = 0 \quad \hat{P}_1(z) = 1 \quad \frac{1}{\hat{Q}_1(z)} = 1 - w_1 z$$

$$\hat{P}_r(z) = \sum_0^{r-1} \hat{q}_{r,j}^{(1)} z^j \quad \frac{1}{\hat{Q}_r(z)} = \sum_0^r \hat{q}_{r,j}^{(1)} z^j \quad r \geq 1, \quad v_0^{(r)} = v_1, \quad k_0^{(r)} = 1$$

$$\hat{P}_r(z) = (1 - w_1 z) \hat{P}_{r-1}(z) - v_1 z^2 \hat{P}_{r-2}(z) \quad \frac{1}{\hat{Q}_r(z)} = (1 - w_r z) \hat{Q}_{r-1}(z) - v_r z^2 \hat{Q}_{r-2}(z)$$

2) rat /ns $\hat{C}_r(z)$ irreducible

3) If $\hat{C}_r(z) = \sum t_{r,j} z^j$ then $t_{r,r} = t_{r,0} \quad r = I_0^{2r-1}$

$$v_1 = \hat{t}_{r,0} \quad w_1 = \frac{\hat{t}_{r,1}}{\hat{t}_{r,0}}$$

$$v_2 = \frac{H \left[\hat{t}_{r,0} \hat{J}_{r-3} - H \left[\hat{t}_{r,0} \hat{J}_{r-1} \right] \right]}{\left\{ H \left[\hat{t}_{r,0} \hat{J}_{r-2} \right]^2 \right\}}$$

$$w_2 = \frac{H \left[\hat{t}_{r,0} \hat{J}_{r-2} - H \left[\hat{t}_{r,0} \hat{J}_{r-1} - H \left[\hat{t}_{r,0} \hat{J}_r - H \left[\hat{t}_{r,0} \hat{J}_{r-3} \right] \right] \right]}{H \left[\hat{t}_{r,0} \hat{J}_{r-2} \right] H \left[\hat{t}_{r,0} \hat{J}_{r-1} \right]} \quad r = I_1^{r'} \quad (\hat{t}_{r-1} = 0)$$

$\exists \theta A$ -/raction derived from $C \left[\frac{v_1}{1-w_1z} - \frac{v_2 z^2}{1-w_2z} \right] = \sum t_{r,r} z^r$

$\hat{t}_{r,r} = t_r (r \geq I_0^{2r-1})$ said to be assoc. with $f(z) = \sum t_r z^r$ if $f(z)$ said to gen.

c.f.

$\exists \theta$ power series with coeffs $\{t_r\}$ s.t. A_{t_r, t_0} said to be A -regular

This $f(z)$ gen. reg A -fraction iff it's A -regular series

D12. Irreducible $R(z) = \sum b_r z^r$ for wh. $H_{0,r} \neq 0$ $r=0$ $H_{0,r}=0 \forall r > 0$ wh. $h =$ 22

max $(i, j+1)$ called A reg rat fr.

D13 power series with coeffs $\{t_i\}$ st. $\exists h$ s.t. for wh. $H_{0,r} \neq 0 \forall r=0$ $r=1, \dots, h-1$ call deg. $= 0 \forall r \geq h$

A reg. : $\{t_i\} \in DAR(h)$

Th 16 $f(z)$ deg A reg iff it is series expn. of deg A reg. rat. fr. $R(z)$; if $\{t_i\} \in DAR(h)$

\Leftrightarrow then $h = \max(i, j+1)$; $f(z)$ generates terminating A fraction

$$C \left[\frac{v_1}{1 - u_1 z} - \frac{u_2 z^2}{1 - u_2 z} \right]_h = R(z)$$

N3 A fraction generated by A reg or deg A reg series $f(z)$ written as $A\{f(z)\}$

6.2.3. The connection between corresponding and associated continued fractions

Th 17 If $f(z)$ C-reg. it is also A reg. $A\{f(z)\}$ is even part of $C\{f(z)\}$ also

$$C \left[\frac{v_1}{1 - u_1 z - v_1} - \frac{u_{2-1}}{1 -} \frac{v_{2-1}^{-1} u_2 z^2}{1 - u_2 z - v_{2-1}^{-1} u_2 z^2 - v_{2-1}} \right]_r = C \left[\frac{u_1}{1 -} \frac{u_2 z}{1 -} \right]_r$$

$$\text{where } v_0 = -u_{2-1} z$$

L1 Let $t'_{2r} = t_{2r}$ $t'_{2r+1} = 0$ $H'_{m,r} = H[t'_{2m}]_r$, then

$$H'_{2r} = H_{0,r}, H_{1,r-1}, H'_{0,2m} = H_{0,r}, H_{1,r-1}, H'_{1,2r} = 0, H'_{1,2m} = (-1)^r H_{0,r}$$

$$H'_{-1,2r} = 0, H'_{-1,2m} = (-1)^r H_{0,r}$$

Th 18 Let $\sum b_r z^r$ where $t'_{2r} = t_r$ $t'_{2r+1} = 0$ be A reg then $f(z) = \sum b_r z^r$

is C-reg. and if $C\{f(z)\} = C \left[\frac{u_1}{1 -} \frac{u_2 z}{1 -} \right]$ then $A\{f(z)\} = C \left[\frac{u_1}{1 -} \frac{u_2 z^2}{1 -} \right]$

6.2.4 The delayed series

D A series whose coeffs are $\{t_{m,r}\}$ is called delayed series derived from that's whose coeffs are $\{t_{0,r}\}$

Th 19 $f^{(m)}(z) = \sum t_{mn} z^m$ ~~regular~~:

$$1) A\text{-reg: } A\{f^{(m)}(z)\} = C \left[\frac{v_1^{(m)}}{1-w_1^{(m)}z} \frac{v_2^{(m)}z^2}{1-w_2^{(m)}z} \right] : v_1^{(m)} = t_m, w_1^{(m)} = \frac{t_{mn}}{t_m}$$

$$v_1^{(m)} = \frac{H_{m+2}-H_{m+1}}{H_{m+2}}, \quad w_1^{(m)} = \frac{H_{m+2}-H_{m+1}, H_{m+2}-H_{m+1}, H_{m+2}-H_{m+1}}{H_{m+2}-H_{m+1}}$$

$$\text{or } A\{f^{(m)}(z)\} = C \left[\sum_{i=0}^{m-1} t_i z^i; \frac{v_1^{(m)} z^m}{1-w_1^{(m)}z} \frac{v_2^{(m)} z^2}{1-w_2^{(m)}z} \right] = \sum t_{i,j} z^j$$

$$\hat{t}_{i,j}^{(m)} = t_j \quad (j \geq 1)$$

$$2) C\text{-regular: } C\{f^{(m)}(z)\} = C \left[\frac{u_1^{(m)}}{1-z} \frac{u_2^{(m)}z}{1-z} \right] : u_{2,j}^{(m)} = \frac{H_{m+1}-H_{m+2}}{H_{m+2}-H_{m+1}}$$

$$u_{2,j+1}^{(m)} = \frac{H_{m+2}H_{m+2}-2}{H_{m+2}-H_{m+1}} : C \left[\sum_{i=0}^{m-1} t_i z^i; \frac{u_1^{(m)} z^m}{1-z} \frac{u_2^{(m)} z^2}{1-z} \right] = \sum t_{i,j} z^j, \quad t_{i,j} = t_j \quad (j \geq 1)$$

$$\text{or } f^{(m)}(z) \text{ also } A\text{-reg.} \rightarrow C \left[\sum_{i=0}^{m-1} t_i z^i; \frac{v_1^{(m)} z^m}{1-w_1^{(m)}z} \frac{v_2^{(m)} z^2}{1-w_2^{(m)}z} \right] = C \left[\sum_{i=0}^{m-1} t_i z^i; u^{(m)} \right]$$

$$v_1^{(m)} = u_1^{(m)}, \quad v_2^{(m)} = u_2^{(m)} \quad v_1^{(m)} = u_{2,j-2}^{(m)} w_{j,j-1}^{(m)} \quad v_2^{(m)} = u_{2,j-1}^{(m)} + u_{2,j}^{(m)} \quad j \geq 2$$

6.2.4 The reciprocal series

D15 $\tilde{f}(z)$ from $f(z)$ $\tilde{f}(z) \equiv 1$ called reciprocal to $f(z)$

Th 20 The reciprocal series are:

Th 21 $m \in \mathbb{Z}$, $\tilde{f}^{(m)} = \sum \tilde{t}_{mn} z^m$ derived from recip series

$$1) A\text{-regular: if } A\{\tilde{f}^{(m)}(z)\} = C \left[\frac{\tilde{v}_1^{(m)}}{1-\tilde{w}_1^{(m)}z} \frac{\tilde{v}_2^{(m)}z^2}{1-\tilde{w}_2^{(m)}z} \right] \equiv 1$$

$$\left\{ C \left[\sum_{i=0}^{m-1} \tilde{t}_i z^i; \frac{\tilde{v}_1^{(m)} z^m}{1-\tilde{w}_1^{(m)}z} \frac{\tilde{v}_2^{(m)} z^2}{1-\tilde{w}_2^{(m)}z} \right] \right\}^{-1} \equiv \sum \tilde{t}_{i,j} z^j : \tilde{t}_{i,j}^{(m)} = t_j \quad (j \geq 1)$$

$$2) C\text{-regular: } C\{\tilde{f}^{(m)}(z)\} = C \left[\frac{u_1^{(m)}}{1-z} \frac{u_2^{(m)}z}{1-z} \right]$$

$$\left\{ C \left[\sum_{i=0}^{m-1} \tilde{t}_i z^i; \frac{u_1^{(m)} z^m}{1-z} \frac{u_2^{(m)} z^2}{1-z} \right] \right\}^{-1} \equiv \sum \tilde{t}_{i,j} z^j : \tilde{t}_{i,j}^{(m)} = t_j \quad (j \geq 1)$$

Th 22 $\sum \tilde{t}_j z^j$ recip to $\sum t_j z^j$ $H[\tilde{t}_{\sigma(j-i+1)}]_i = \tilde{H}_{j-i, i}$

$$\tilde{H}_{j-i+1, i} = (-1)^{i+1 + \frac{1}{2}(j-i)(j-i-1)} t_0^{-i-j-2} H_{j+1, j} \quad (i, j \in \mathbb{Z})$$

6.3. The continued fractions of the Padé Table

Th 23 $f^{(m)}(z)$ $\tilde{f}(z)$ $\tilde{f}^{(m)}(z)$ deriving from $f(z)$; also $R_{r,s}(z)$; $m \in \mathbb{Z}$;

1) $f^{(m)}(z)$ \mathbb{C} -regular $R_{m+r-1, r}(z)$ $R_{m+r, r}(z)$ $r \in \mathbb{Z}$ primitive

$R_{m+r-1, r}(z)$ non def. denoms $R_{m+r, r}(z)$ non def. numrs: if $\{f^{(m)}(z)\} = C \left[\frac{u}{z-r} \right]$

$$C \left[\sum_{j=0}^{m-1} \tilde{t}_j z^j; \frac{\tilde{u}_1 z^m}{1-w_1 z} \frac{\tilde{u}_2 z^2}{1-w_2 z} \right]_r = R_{m+r-1, r}(z)$$

$$C \left[\sum_{j=0}^{m-1} \tilde{t}_j z^j \dots \right]_r = R_{m+r, r}(z)$$

2) $f^{(m)}(z)$ A-reg.: $R_{r, m+r-1}(z)$ primitive $\nexists r \in \mathbb{Z}: \{f^{(m)}(z)\} = C \left[\frac{v_1^{(m)}}{1-w_1 z} \frac{\sqrt{w_2 z}}{1-w_2 z} \right]$

$$C \left[\sum_{j=0}^{m-1} \tilde{t}_j z^j; \frac{v_1^{(m)} z^m}{1-w_1 z} \frac{\sqrt{w_2 z^2}}{1-w_2 z} \right]_r = R_{r, m+r-1}(z) \quad (r \in \mathbb{Z})$$

3) $\tilde{f}^{(m)}(z)$ \mathbb{C} -reg. $R_{m+r-1, r}(z)$, $R_{m+r, r}(z) \quad (r \in \mathbb{Z})$ primitive $R_{m+r, r}(z)$

have non def. numrs. $\{R_{m+r, r}(z)\}$ non def. denoms: $\{f^{(m)}(z)\} = C \left[\frac{\tilde{u}_1^{(m)} \tilde{u}_2^{(m)}}{1-w_1 z} \right]$

$$\{C \left[\sum_{j=0}^{m-1} \tilde{t}_j z^j; \frac{\tilde{u}_1^{(m)} z^m}{1-w_1 z} \frac{\tilde{u}_2^{(m)} z^2}{1-w_2 z} \right]_r\}^{-1} = R_{m+r, r}(z)$$

$$\{C \left[\sum_{j=0}^{m-1} \tilde{t}_j z^j; \frac{\tilde{u}_1^{(m)} z^m}{1-w_1 z} \frac{\tilde{u}_2^{(m)} z^2}{1-w_2 z} \right]_{2m}\}^{-1} = R_{m+r, r}(z)$$

4) $\tilde{f}^{(m)}(z)$ A-reg.: $R_{m+r-1, r}(z)$ reg. primitive: $\{f^{(m)}(z)\} = C \left[\frac{v_1^{(m)}}{1-w_1 z} \right]$

$$\{C \left[\sum_{j=0}^{m-1} \tilde{t}_j z^j; \frac{v_1^{(m)} z^m}{1-w_1 z} \frac{v_2^{(m)} z^2}{1-w_2 z} \right]_r\}^{-1} = R_{m+r-1, r}(z)$$

$$\begin{array}{ccc}
 R_{0,m-1}(z) & R_{0,m}(z) & R_{0,m+1}(z) \\
 & R_{1,m} \cup R_{1,m+1}(z) & R_{1,m}(z) \\
 & R_{2,m}(z) & R_{2,m+1}(z)
 \end{array}$$

$$\begin{array}{ccc}
 R_{m-1,0}(z) & & R_{m-1,0}(z) \\
 R_{m,0}(z) & R_{m,1}(z) & R_{m,1}(z) \\
 & R_{m,n-1}(z) & R_{m,n-2}(z) \\
 & & R_{m,n}(z)
 \end{array}$$

Th. 24 Num + denom of $R_{i,j}(z)$ derived from $f(z)$ denoted by $\rho_{i,j}(z) \bar{\pi}_{i,j}(z)$ resp.

1) $i, j \in I$, fixed

a) if $R_{i-1,j-1}(z) R_{i-1,j}(z) R_{i,j}(z)$ primitive

$$\rho_{i,j}(z) = \rho_{i-1,j}(z) - u \frac{(j-i+1)}{z_i} z^2 R_{i-1,j-1}(z) \quad \bar{\pi}_{i,j}(z) = \bar{\pi}_{i-1,j}(z) - u \frac{(j-i+1)}{z_i} \bar{z}^2 \bar{\pi}_{i-1,j-1}(z)$$

$$\text{Sh. } u \frac{(j-i+1)}{z_i} = \frac{H_{j-i+1,i-2} H_{j-i+2,i-1}}{H_{j-i+1,i-1} H_{j-i+2,i-1}}$$

b) $R_{i-1,j-1}(z) R_{i,j-1}(z)$ and $R_{i,j}(z)$ primitive

$$\rho_{i,j}(z) = \rho_{i,j-1}(z) - u \frac{(j-i+1)}{z_{i-1}} z^2 \rho_{i-1,j-1}(z) \quad u \frac{(j-i+1)}{z_{i-1}} = \frac{H_{j-i+1,i-2} H_{j-i+1,i-1}}{H_{j-i+1,i-1} H_{j-i+2,i-1}}$$

$$\bar{\pi}_{i,j}(z) = \bar{\pi}_{i,j-1}(z) - u \frac{(j-i+1)}{z_{i-1}} \bar{z}^2 \bar{\pi}_{i-1,j-1}(z)$$

2) $R_{i-2,j-2}(z) R_{i-1,j-1}(z) R_{i,j}(z)$ primitive

$$\rho_{i,j}(z) = (1 - u \frac{(i-j+1)}{z_i} z^2) \rho_{i-1,j-1}(z) - v \frac{(j-i+1)}{z_i} z^2 \rho_{i-2,j-2}(z)$$

$$\bar{\pi}_{i,j}(z) = (1 - u \frac{(i-j+1)}{z_i} \bar{z}^2) \bar{\pi}_{i-1,j-1}(z) - v \frac{(j-i+1)}{z_i} \bar{z}^2 \bar{\pi}_{i-2,j-2}(z)$$

$$u \frac{(j-i+1)}{z_i} = \frac{H_{j-i+1,i-2} H_{j-i+2,i-1} - H_{j-i+1,i-1} H_{j-i+2,i-2}}{H_{j-i+1,i-2} H_{j-i+2,i-1}}$$

$$v \frac{(j-i+1)}{z_i} = \frac{H_{j-i+1,i-3} H_{j-i+1,i-1}}{H_{j-i+1,i-2}}$$

$R_{i-1,j-1} \ R_{i-1,j}$ $R_{i-1,j+1}$ $R_{i-2,j-1}$ $R_{i,j}$ $R_{i,j-1} \ R_{i,j}$ $R_{i-1,j-1}$ $R_{i,j}$

$f^{(m)}$ A-regular $R_{i,m}, \dots, R_{i,n}$ primitive \therefore differ from one another

If $R_{i-1,m}, \dots, R_{i,n}$ non primitive it is bounding member of block

of wh. $R_{i-1,m+1}, \dots, R_{i,n}$ is bounding member, in this case $R_{i-1,m}, \dots, R_{i,n}$

identical with $R_{i-1,m+1}, \dots, R_{i,n}$

$$\begin{array}{c} * \\ | \\ R_{i-1,m+1}, \dots, R_{i,n} \\ \hline * \\ \text{as} & R_{i,m+1}, \dots, R_{i,n} \end{array}$$

Th 25 $m \in I$: $f^{(m)}$ A-regular $\exists f^{(m)}(z) = C \left[\frac{v_1^{(m)}}{1-w_1^{(m)}z} \frac{v_2^{(m)}z^2}{1-w_2^{(m)}z} \dots \right]$

$r \in J$, fixed, $R_{i-1,m+1}, \dots, R_{i,n}$ primitive; set $w_{2r}^{(m)} = \frac{H_{m,r-2} H_{m+1,m+r-1}}{H_{m,r-1} H_{m+1,r-2}}$

successive coeffs of

$$\sum_{i,j} v_i^{(m)} z^i \frac{v_1^{(m)} z^m}{1-w_1^{(m)} z} \frac{v_2^{(m)} z^2}{1-w_2^{(m)} z} \dots \frac{v_{r-1}^{(m)} z^{r-1}}{1-w_{r-1}^{(m)} z + w_{2r}^{(m)} z^r} \frac{w_{2r}^{(m)} z^r}{1} \frac{w_{2r}^{(m)} v_r^{(m)} z^r}{1-w_{r+1}^{(m)} z} \dots$$

are $R_{0,m-1}(z), R_{1,m}(z), R_{2,m}(z), \dots, R_{n-1,m+n-2}(z), R_{n-1,m+n-1}(z), R_{n,m+n}(z)$

$R_{n+1,m+n}(z), \dots$

Th. 26 $R_{i,j}(z)$ from $f(z)$ $\check{R}_{i,j}^{(m)}(z)$ from $f^{(m)}(z)$, then

$$R_{i,j+m}(z) = \sum_{i,j} v_i^{(m)} z^i + z^m R_{i,j}^{(m)}(z) \quad i,j \in J$$

Th 27 $R_{i,j}(z)$ from $f(z)$ $\tilde{R}_{i,j}(z)$ from $\tilde{f}(z)$: $\tilde{R}_{i,j}(z) = R_{j,i}(z)^{-1} (i,j \in J)$

Th. 28 $m \in \mathbb{I}$ fixed then in $f(z) = \sum_{i,j} z^j$ non-zero: $R_{i,j}(z)/\text{non } f(z)$

$$\tilde{R}_{i,j}^{(m)}(z) \text{ from } f^{(m)}(z) \notin R_{j+m,i}(z) = \left\{ \sum_{i,j}^{m-1} b_j z^j + z^m R_{i,j}^{(m)}(z) \right\}^{-1}$$

Th. 29 $m \in \mathbb{I}$ fixed $f^{(m)}(z), f^{(mn)}(z) \subset$ -regular $\mathcal{C}\{f^{(m)}(z)\} = C \left[\frac{u_1^{(m)}}{1-z} \frac{u_2^{(mn)} z}{1-z} \right]$

$$\mathcal{C}\{f^{(mn)}(z)\} = C \left[\frac{u_1^{(mn)}}{1-z} \frac{u_2^{(mn)} z}{1-z} \right]$$

$$C \left[\frac{u_1^{(m)}}{1-z} \frac{u_2^{(m)} t}{1-z} \right]_{2r+1} = \frac{\rho_{2r+1}^{(m)}(z)}{\pi_{2r+1}^{(m)}(z)} \quad (r \in \mathbb{I})$$

$$\rho_{2r+1}^{(m)}(z) = \sum_{i=0}^r v_{2r+1,i}^{(m)} z^i \quad (v_{2r+1,0}^{(m)} = t_m) \quad \pi_{2r+1}^{(m)}(z) = \sum_{i=0}^r k_{2r+1,i}^{(m)} z^i \quad k_{2r+1,0}^{(m)} = 1$$

$$C \left[\frac{u_1^{(mn)}}{1-z} \frac{u_2^{(mn)} t}{1-z} \right]_{2r} = \frac{\rho_{2r}^{(mn)}(t)}{\pi_{2r}^{(mn)}(z)}$$

$$\rho_{2r}^{(mn)}(z) = \sum_{i=0}^{r-1} v_{2r,i}^{(mn)} z^i \quad (v_{2r,0}^{(mn)} = t_m) \quad \pi_{2r}^{(mn)}(z) = \sum_{i=0}^r k_{2r,i}^{(mn)} z^i \quad (k_{2r,0}^{(mn)} = 1)$$

$$C \left[\frac{u_1^{(m)}}{1-z} \frac{u_2^{(m)} t}{1-z} \right]_{2r} = t_m + z C \left[\frac{u_1^{(mn)}}{1-z} \frac{u_2^{(mn)} t}{1-z} \right]_{2r}$$

$$\rho_{2r+1}^{(m)}(z) = (\pi_{2r}^{(mn)} \pi_{2r+1}^{(mn)})^{-1}(z) + z \rho_{2r}^{(mn)}(z) \quad \pi_{2r+1}^{(m)}(z) = \pi_{2r}^{(mn)}(z)$$

$$v_{2r+1,i}^{(mn)} = t_m k_{2r,i}^{(mn)} + \rho_{2r,2r-i}^{(mn)} ((z)_1^r) \quad k_{2r,i}^{(m)} = k_{2r,i}^{(mn)} \quad (r = 1, 2, \dots)$$

6.4 The continued fractions of the even order epsilon array

Th. 30 $f(z)$ semi normal, $f^{(m)}(z)/\text{non } f(z)$: $\mathcal{C}\{f^{(m)}(z)\} = C \left[\frac{u_1^{(m)}}{1-z} \frac{u_2^{(m)} t}{1-z} \right]_{m \in \mathbb{I}}$

1) rational numbers $\epsilon_p^{(m)}(k, m \in \mathbb{I})$ can be produced from $\epsilon_{-1}^{(m)} = 0$ $\epsilon_0^{(m)} = \sum_{i=0}^{m-1} b_i z^i$

and $\epsilon_{2r}^{(m)} = C \left[\sum_{i=0}^{m-1} b_i z^i; \frac{u_1^{(m)} z^m u_2^{(m)} z}{1-z} \right]_{2r} \quad \epsilon_{2r+1}^{(m)} = C \left[\sum_{i=0}^{m-1} b_i z^i; \frac{u_1^{(m)} z^m u_2^{(m)} z}{1-z} \right]_{2r+1}$

2) $\pi \notin \mathcal{P}_r^{(m)} \{f(z)\}$ ($k, m \in \mathbb{I}$), numbers $\epsilon_r^{(m)}$ similarly produced.

6.5 Continued fractions derived from power series

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$$F(\lambda) = \sum t_\nu \lambda^{-\nu-1}, \quad f(z) = \sum t_\nu z^\nu; \quad F(\lambda) = \lambda^{-1} f(\lambda^{-1}) \quad f(z) = z^{-1} F(z')$$

$$\text{if } G\{\bar{F}(\lambda)\} \text{ has form } C\left[\frac{u_{n+1}}{\lambda - w_p}\right]_r \text{ then } G\{\bar{F}(\lambda)\}_r = C\left[\frac{v_p}{\lambda - w_p}\right]$$

Th 31) $\bar{F}(\lambda)$ generates $G\{\bar{F}(\lambda)\}_r = C\left[\frac{v_p}{\lambda - w_p}\right]$ if & only if

$$C\left[\frac{v_p}{\lambda - w_p}\right]_r = \sum t_{r,\nu} \lambda^{-\nu-1} \quad t_{r,\nu} = t_\nu (\nu \in J_0^{r-1}) \quad \text{iff } \bar{F}(\lambda) \text{ A regular;}$$

$$\text{unless } G\{\bar{F}(\lambda)\} \text{ irreducible}$$

$$C\left[\frac{v_p}{\lambda - w_p}\right]_r = \frac{q_r(\lambda)}{p_r(\lambda)} \quad (r \in J_0) \quad p_r(\lambda) = \sum_{i=0}^r k_{r,i} \lambda^i \quad q_r(\lambda) = \sum_{i=0}^{r-1} l_{r,i} \lambda^i$$

$$2) \bar{F}(\lambda) \text{ generates } G\{\bar{F}(\lambda)\}_r = C\left[\frac{u_{n+1}}{\lambda - w_p}\right]_r \text{ for some } r.$$

$$C\left[\frac{u_{n+1}}{\lambda - w_p}\right]_r = \sum t_{r,\nu} \lambda^{-\nu-1} \quad t_{r,\nu} = t_\nu \text{ or } l_{r,\nu} \quad \text{iff } \bar{F}(\lambda) \text{ C-regular}$$

in this case $\bar{F}(\lambda)$ also A-reg. and $C\left[\frac{v_p}{\lambda - w_p}\right]_r = C\left[\frac{u_{n+1}}{\lambda - w_p}\right]_r$

$$\text{if } C\left[\frac{v_1}{\lambda - w_1 - y_{1,r}} \frac{y_1^{-1} v_2}{\lambda - w_2 + y_2^{-1} y_1 - y_{1,r+1}} \frac{y_2}{1-}\right]_r = C\left[\frac{u_{n+1}}{\lambda - w_p}\right]_r$$

$y_j = -u_{j,p}$; contrary to $G\{\bar{F}(\lambda)\}$ irreducible

$$C\left[\frac{u_{n+1}}{\lambda - w_p}\right]_r = \frac{q_r(\lambda)}{p_r(\lambda)} \quad C\left[\frac{u_{n+1}}{\lambda - w_p}\right]_{r+1} = \lambda^{-1} + \lambda^{-1} \frac{q_{r+1}'(\lambda)}{p_{r+1}'(\lambda)}$$

$$p_r(\lambda) = \sum_{i=0}^r k_{r,i} \lambda^i \quad k_{r,0} = 1 \quad q_r'(\lambda) = \sum_{i=0}^{r-1} l_{r,i} \lambda^i \quad l_{r,r+1} = 1$$

$$k_{r,s} = \hat{k}_{r,r-s} = k_{2r-s,r-s} \quad (\nu \in J_0^r) \quad l_{r,s} = \hat{l}_{r,r-s} = l_{2r-s,r-s} \quad (\nu \in J_0^{r-1})$$

$$k_{r,p} = k_{2m+r-p}, \quad l_{r,p} = l_{2m+r-p} \quad (\nu \in J_0^r)$$

Th 32 $h \in J_0$ fixed

$$i) F(\lambda) = \frac{q(\lambda)}{p(\lambda)} \quad p(\lambda) = \sum_{i=0}^h k_i \lambda^i \quad k_h = 1 \quad k_0 \neq 0 \quad q(\lambda) = \sum_{i=0}^{h-1} l_i \lambda^i \quad l_{h-1} \neq 0$$

irreducible generator $\bar{F}(\lambda) = \sum t_\nu \lambda^{-\nu-1}$ deg. A regular: $\{t_\nu\} \in \text{DAR}(h)$

$A\{\bar{F}(\lambda)\}$ has form $C\left[\frac{v_p}{\lambda - w_p}\right]_h = F(\lambda)$

b) assume $\bar{F}(\lambda)$ also deg C-reg. then $\{t_\nu\} \in \text{DCR}(h) = G\{\bar{F}(\lambda)\} = C\left[\frac{u_{n+1}}{\lambda - w_p}\right]_h$

29) Assume $\hat{F}(z) = \frac{\hat{g}(z)}{\hat{p}(z)}$ where $\hat{p}(z) = \sum_{j=0}^{h-1} k_j z^j$, $k_{h-1} \neq 0$ and $\hat{g}(z) = \sum_{j=0}^{h-1} l_j z^j$

irreducible \Rightarrow gen. deg. & reg. $\hat{g}(z) = \sum_{j=0}^{h-1} l_j z^{j-h}$ then $\{l_j\} \in \text{DAR}(\hat{h})$

$$\Rightarrow \# \{\hat{g}(z)\} = C \left[\frac{V_h}{z - \lambda_h} - \frac{1}{z - \lambda_h} \right] = \hat{F}(z)$$

b) assume $\hat{g}(z)$ also deg C-reg, then $\{l_j\} \in \text{DCR}'(h)$ and

$$C \{\hat{g}(z)\} = C \left[\frac{u_{h-1}}{z - \lambda_{h-1}} - \frac{u_h}{z - \lambda_h} \right] = \hat{F}(z)$$

$$F(z) = \sum_{j=1}^h \frac{M_j}{z - \lambda_j} = \hat{F}(z) = \frac{M_1}{z} + \sum_{j=0}^{h-1} \frac{M_j}{z - \lambda_j}, M_1 \neq 0, M_j, \lambda_j / \text{intc non zero}$$

are two of above type

$$\text{Th 33 } \exists f(z) = \sum b_j z^{j-h} \text{ semi normal, each } \exists f^{(m)}(z) = \sum t_m z^{j-h-1}$$

generates reg A fact $C \left[\frac{V_r^{(m)}}{z - \lambda_r^{(m)}} \right] \rightarrow \text{reg C-intc } \not\subset C \left[\frac{u_{2h-1}^{(m)} u_h^{(m)}}{z - \lambda_r^{(m)}} \right]$

$$\text{Furthermore } C \left[\frac{V_r^{(m)}}{z - \lambda_r^{(m)}} \right] = \frac{q_r^{(m)}(z)}{p_r^{(m)}(z)}$$

$$p_r^{(m)}(z) = \sum_{j=0}^r k_{r,j} z^j, k_{r,0} \neq 0, k_{r,r} = 1, q_r^{(m)} = \sum_{j=0}^r l_{r,j} z^j, l_{r,0} \neq 0, l_{r,r-1} = t_m$$

$$C \left[\frac{u_{2h-1}^{(m)}}{z - \lambda_r^{(m)}} \right] = \frac{q_r^{(m)}(z)}{p_r^{(m)}(z)}$$

$$\therefore \exists_{2h-1} = t_m z^{-h} + z^{-h-1} \frac{q_r^{(m+1)}(z)}{p_r^{(m+1)}(z)}$$

$$\text{Th 34 } \exists f(z) = \sum b_j z^{j-h} \text{ semi normal}$$

$$\exists \text{ irreducible rat fm } \exists_{2h-1}^{(m)} (r, m \geq 1) \text{ prod fm } \exists_0^{(m)} = \sum_{j=0}^{m-1} t_j z^{-j-1}$$

$$\exists_{2h-1}^{(m)} = C \left[\sum_{j=0}^{m-1} t_j z^{-j-1}, \frac{u_1^{(m)} z^{-h}}{z - \lambda_1^{(m)}}, \frac{u_{2h-1}^{(m)}}{z - \lambda_{2h-1}^{(m)}} \right]_{2h-1}$$

$$= \sum_{j=0}^{m-1} (z^{-j-1} + z^{-m} \frac{q_r^{(m)}(z)}{p_r^{(m)}(z)})$$

$$\exists_{2h-1}^{(m+1)} = C \left[\sum_{j=0}^{m-1} t_j z^{-j-1}, \frac{u_1^{(m)} z^{-h}}{z - \lambda_1^{(m)}}, \frac{u_{2h-1}^{(m)}}{z - \lambda_{2h-1}^{(m)}} \right]_{2h-1}$$

$$= \sum_{j=0}^m (z^{-j-1} + z^{-m} \frac{q_r^{(m+1)}(z)}{p_r^{(m+1)}(z)})$$

2) λ fixed finite numbers & any zero of $f_r^{(m)}(\lambda)$. Numbers $\varepsilon_r^{(m)}$ can be prod as above |

Th 35 1) $F(\lambda)$ rat fn of Th 32 above gen deg. semi norm. s.t. $\sum b_r \lambda^{-r-1} \in \mathfrak{F}(\lambda)$

a) irreducible rat fns $\varepsilon_r^{(m)}$ ($r = \overline{I_0}, m = ?$) can be const. & $\varepsilon_{2h}^{(m)} = f(\lambda)$ $\varepsilon_{2h+1}^{(m)} = \infty$

b) λ fixed finite numbers not pole of any $\varepsilon_{2r}^{(m)}$ ($r = \overline{I_1}, m = ?$) numbers $\varepsilon_r^{(m)}$ ($r = \overline{I_0}, m \in \mathbb{P}$) can be const as above; if λ pole of $f(\lambda)$ $\varepsilon_{2h-1}^{(m)}$ all \in some finite constant $\varepsilon_{2h}^{(m)} = \infty$ identically; if not pole of $F(\lambda)$ $\varepsilon_{2h+1}^{(m)} = \infty$ can be const.

2) $\hat{F}(\lambda)$ rat fn of Th 32 above gen deg semi norm. $\hat{f}(\lambda) = \sum b_r \lambda^{-r-1}$

a) irreducible rat fns $\hat{\varepsilon}_r^{(m)}$ $m = \overline{I_0}, r = \overline{I_0}, r = \overline{I_0}, m = \overline{I_2}$ can be const.

$$\hat{\varepsilon}_{2h}^{(m)} = \hat{f}(\lambda) \quad \hat{\varepsilon}_{2h}^{(m)} = \hat{F}(\lambda) \quad m = \overline{I_1}$$

b) if given finite non zero value & any pole of $\hat{\varepsilon}_{2r}^{(m)}$ ($r = \overline{I_1}, m = ?$)

well def numbers $\hat{\varepsilon}_r^{(m)}$ ($m = \overline{I_0}, r = \overline{I_0}, r = \overline{I_0}, m = \overline{I_2}$) can be const as above; if λ pole of $\hat{F}(\lambda)$ $\hat{\varepsilon}_2^{(m)}, \hat{\varepsilon}_{2h-3}^{(m)}, m = \overline{I_1}$, consists of const finite number

$\hat{\varepsilon}_{2h-1}^{(m)}$ also has this value; $\hat{\varepsilon}_{2h-2}^{(m)} = \infty$ ($m = \overline{I_1}$). if λ not pole of $\hat{F}(\lambda)$ two further numbers $\hat{\varepsilon}_{2h}^{(m)}, \hat{\varepsilon}_{2h-1}^{(m)}$ and further column $\hat{\varepsilon}_{2h-1}^{(m)}$ can be const.

Th 36 $P_{ij,j}(\lambda) / \text{rat } \hat{f}(\lambda) = \sum b_r z^r \quad m \in \mathbb{S} \quad / \text{rat } \mathfrak{F}^{(m)}(\lambda) / \text{rat } \mathfrak{F}(\lambda), \mathfrak{F}^{(m)}(\lambda)$

/rat $\hat{\mathfrak{F}}(\lambda)$; set $z = \lambda^{-1}$:

) $\mathfrak{F}^{(m)}(\lambda)$ C-reg.

$$\lambda \in \left[\sum_{r=0}^{m-1} b_r \lambda^{-r-1}; \frac{w_1^{(m)-m}}{\lambda - w_1^{(m)}} \frac{w_2^{(m)}}{\lambda - w_2^{(m)}} \frac{w_m^{(m)}}{\lambda - w_m^{(m)}} \right] = R_{v,m,m-1}(\lambda) \quad (r = \overline{I})$$

$$\lambda \left[\frac{w_1^{(m)}}{\lambda - w_1^{(m)}} \frac{w_2^{(m)}}{\lambda - w_2^{(m)}} \dots \frac{w_m^{(m)}}{\lambda - w_m^{(m)}} \right] = R_{v,m,m-1}(\lambda)$$

$$2) \mathfrak{F}^{(m)} A \text{-reg}: \lambda \in \left[\sum_{r=0}^{m-1} b_r \lambda^{-r-1} \frac{v_1^{(m)-m}}{\lambda - v_1^{(m)}} \frac{v_2^{(m)}}{\lambda - v_2^{(m)}} \dots \frac{v_m^{(m)}}{\lambda - v_m^{(m)}} \right] = R_{v,m,m-1}(\lambda) \quad (r = \overline{I})$$

3) $\tilde{F}^{(m)}(\lambda)$ C-reg.

$$\{\lambda \in \sum_{\nu=0}^{n-1} t_\nu \lambda^{-\nu}; \frac{u_{1,\lambda}^{(m)}}{\lambda - u_{2,\lambda}^{(m)}} \frac{u_{2,\lambda}^{(m)}}{\lambda - u_{3,\lambda}^{(m)}} \dots \}_{\lambda}^{-1} = R_{mn-1,r}(\lambda) \quad (\text{e.g.})$$

$$\tilde{J}_{mn} \tilde{J}^{-1} = R_{mn-1,r}(\lambda)$$

4) $\tilde{F}^{(m)}(\lambda)$ A-reg

$$\{\lambda \in \sum_{\nu=0}^{n-1} t_\nu \lambda^{-\nu}; \frac{v_{1,\lambda}^{(m)}}{\lambda - u_{1,\lambda}^{(m)}} \frac{v_{2,\lambda}^{(m)}}{\lambda - u_{2,\lambda}^{(m)}} \dots \}_{\lambda}^{-1} = R_{mn-1,r}(\lambda) \quad (\text{e.g.})$$

6.6 Transformations of corresponding continued fractions

Th 37 $\tilde{F}(\lambda) = \sum t_\nu \lambda^{-\nu}$, $\tilde{F}'(\lambda) = \sum t'_\nu \lambda^{-\nu} = \tilde{F}(\lambda-\eta)$ if finite non zero be C-regular: $\mathcal{C}\{\tilde{F}(\lambda)\} = C\left[\frac{u_{1,\lambda}}{\lambda - u_{2,\lambda}} \frac{u_{2,\lambda}}{\lambda - u_{3,\lambda}} \dots\right]$ $\mathcal{C}\{\tilde{F}'(\lambda)\} = C\left[\frac{u'_{1,\lambda}}{\lambda - u'_{2,\lambda}} \frac{u'_{2,\lambda}}{\lambda - u'_{3,\lambda}} \dots\right]$
 $u'_1 = u_1$, $u'_2 = u_2 + u_1$, $u'_{j+2} = u_{j+2} - u_{j+1}^2 / u_{j+1}$, $u'_{j+1} = u_{j+1} + u_{j+2} / u_{j+1}$, $u'_{j+2} = u_{j+2} + u_{j+1}^2 / u_{j+1}$ ($j \in J_1$)

2) $\tilde{F}(\lambda)$ deg C-regular $\{t_\nu\} \in DCR(h)$ $\mathcal{C}\{\tilde{F}(\lambda)\} = C\left[\frac{u_{1,\lambda}}{\lambda - u_{2,\lambda}} \frac{u_{2,\lambda}}{\lambda - u_{3,\lambda}} \dots\right]_h$ and $\tilde{F}'(\lambda)$ deg. C-reg. (2.1) used to determine $\{u'_\nu\}$ ($\nu = j, h$) and either

a) $\{t'_\nu\} \in DCR(h)$, $\mathcal{C}\{\tilde{F}'(\lambda)\} = C\left[\frac{u'_{1,\lambda}}{\lambda - u'_{2,\lambda}} \frac{u'_{2,\lambda}}{\lambda - u'_{3,\lambda}} \dots\right]_{2h}$ or

b) $\{t'_\nu\} \in DCR(h^2)$: $u'_{2h} = 0$ $\mathcal{C}\{\tilde{F}'(\lambda)\} = C\left[\frac{u'_{1,\lambda}}{\lambda - u'_{2,\lambda}} \frac{u'_{2,\lambda}}{\lambda - u'_{3,\lambda}} \dots\right]_{2h-1}$

3) $\tilde{F}(\lambda)$ deg C-reg $\{t_\nu\} \in DCR(h')$: $\mathcal{C}\{\tilde{F}(\lambda)\} = C\left[\frac{u_{1,\lambda}}{\lambda - u_{2,\lambda}} \frac{u_{2,\lambda}}{\lambda - u_{3,\lambda}} \dots\right]_{2h-1}$

$\tilde{F}'(\lambda)$ deg C-reg.: $u_{2h} = 0$ (2.1) used and $\mathcal{C}\{\tilde{F}(\lambda)\}$ is as in a) and b) above

2) above: $\tilde{F}(\lambda)$ exp. of $F(\lambda)$. $F(\lambda)$ may have term in $M_F(\lambda+\eta)^{-\nu} (\tau \in J_1)$ if not $F(\lambda')$ where $\lambda' = \lambda - \eta$ is of same type as $F(\lambda)$; if it does $F(\lambda')$ now has term in $M_F(\lambda')^{-\nu} : \{t'_\nu\} \in DCR'(h)$

3) in same way

Th. 38, $\exists(\lambda) = \sum b_j \lambda^{-j-1}$ and $\exists'(\lambda) = \sum b'_j \lambda^{-j-1} = t\lambda^{-1}$; $\exists(\lambda)$ t finite number λ^{2h+1}

$$C\text{-reg } \mathcal{L}\{\exists(\lambda)\} = C\left[\frac{u_{2h+1}}{\lambda - t}, \frac{u_{2h}}{t}\right] \quad \mathcal{L}\{\exists'(\lambda)\} = C\left[\frac{u''_{2h+1}}{\lambda - t}, \frac{u''_{2h}}{t}\right]$$

$$u'_j = t + u_j; \quad u''_{2h+1}, u''_{2h} = u_{2h+1}, u_{2h} \quad u''_{2h+2h+1} = u_{2h+2h+1} \quad (\lambda = \bar{J}_1) \quad (128+107)$$

2) $\exists(\lambda)$ deg C-reg with $\{b_j\} \in DCR(h) \Rightarrow \exists'(\lambda)$ deg C-reg set $u''_{2h+1} = 0$

$$(128+5) \text{ need to det } u''_j, \quad \{b'_j\} \in DCR'(h) \quad \mathcal{L}\{\exists'(\lambda)\} = C\left[\frac{u''_{2h+1}}{\lambda - t}, \frac{u''_{2h}}{t}\right]_{2h+1}$$

3) $\exists(\lambda)$ deg C-reg with $\{b_j\} \in DCR'(h)$, $\exists'(\lambda)$ deg C-reg $(128+5) \text{ need}$

$$\text{with } \lambda = \bar{J}_0^{h+1}; \text{ either a) } \{b'_j\} \in DCR(h) \quad \mathcal{L}\{\exists'(\lambda)\} = C\left[\frac{u''_{2h+1}}{\lambda - t}, \frac{u''_{2h}}{t}\right]_{2h+1}$$

$$\text{b) } \{b'_j\} \in DCR(h-1) \quad (125) \text{ yields } u''_{2h+1} = 0: \quad \mathcal{L}\{\exists'(\lambda)\} = C\left[\frac{u''_{2h+1}}{\lambda - t}, \frac{u''_{2h}}{t}\right]_{2h+2}$$

2): $\exists(\lambda)$ from $F(\lambda)$ containing $M\lambda^{-1} F''(\lambda)$ does

3) " " " " " but $M \neq -t$ and a) if $M = -t$ b) holds

6.6.1 Rittishawar's addition theorem

Th. 39 Each of $\exists(\lambda)$, $\exists'(\lambda) = \exists(\lambda) (\lambda' = \lambda - t)$, $\hat{\exists}(\lambda') = t(\lambda'^{h+1} \exists'(\lambda'))$

$\exists''(\lambda) = \hat{\exists}(\lambda') = t(\lambda - \eta)^{-1} \exists(\lambda)$ ($\eta + t$ positive even numbers) be

$$C\text{-regular}. \quad \mathcal{L}\{\exists(\lambda)\} = C\left[\frac{u_{2h+1}}{\lambda - t}, \frac{u_{2h}}{t}\right] \quad \mathcal{L}\{\exists''(\lambda)\} = C\left[\frac{u''_{2h+1}}{\lambda - t}, \frac{u''_{2h}}{t}\right]$$

u''_j det from $\{u_{2h+1}\}$ as follows: u'_j as in Th. 37 in u_j from u'_j by means

$$j) \quad \hat{u}'_j = t_j + u'_j \quad \hat{u}_{2h+1}, \hat{u}_{2h} = u'_{2h+1}, u'_{2h} \quad \hat{u}_{2h} + \hat{u}_{2h+1} = u'_{2h} + u'_{2h+1} \quad (\lambda = \bar{J}_1) \quad (127+11)$$

$$u''_j / \text{from } \hat{u}'_j \text{ by means } \Rightarrow u''_j = \hat{u}'_j, \quad u''_2 = \hat{u}_2 - \eta$$

$$u''_{2h-2} u''_{2h-1} = \hat{u}_{2h-2} \hat{u}_{2h-1} \quad u''_{2h-1} + u''_{2h} = \hat{u}_{2h-1} + \hat{u}_{2h} - \eta \quad (127, 1) \quad (129+30)$$

2) if 4 series above deg C-reg $\{b_j\} \in DCR(h) \quad (120+121) \Rightarrow$ Th. 37 need

to det u'_j ($j = \bar{J}_1^{2h}$) with $\lambda = \bar{J}_1^h$; then found that either

a) $u''_{2h} \neq 0$, we set $u''_{2h+1} = 0 \Rightarrow$ det. \hat{u}'_j $\lambda = \bar{J}_1^{2h+1}$ using (127+8) with $(\lambda = \bar{J}_1^h)$

it is found that $\hat{u}'_{2h+1} \neq 0$, we set $\hat{u}'_{2h+2} = 0$, det. u''_j $\lambda = \bar{J}_1^{2h+2}$ using (129+30)

with $\lambda = \bar{J}_1^{h+1}$; it is found that $u''_{2h+2} \neq 0$ so that $\mathcal{L}\{\exists''(\lambda)\} = C\left[\frac{u''_{2h+1}}{\lambda - t}, \frac{u''_{2h}}{t}\right]_{2h+2}$

or $u'_{2h} = 0$, $\hat{u}_j (\lambda = \bar{\lambda})^{2h-1}$ using $(1_{20} + z_1) (\lambda \in I_1^{h-1})$ then found that either

i) $\hat{u}'_{2h-1} \neq 0$, we set $\hat{u}_{2h} = 0 \Rightarrow \det u''_j (\lambda = \bar{\lambda})^{2h}$ using $(1_{20} + z_0)$ with $\lambda = \bar{\lambda}_1^h$, then found that $u''_{2h} \neq 0 \Rightarrow$ that $C\{\mathcal{F}''(\lambda)\} = C\left[\frac{u''_{2h-1}}{\bar{\lambda} - \bar{\lambda}_1} \frac{u''_{2h}}{1 - 1}\right]_{2h}$

or ii) $\hat{u}'_{2h-1} = 0$, $u''_j (\lambda = \bar{\lambda})^{2h-1}$ det from $(1_{20} + z_0)$ with $(\lambda = \bar{\lambda}_1^{h-1})$, then found that $u''_{2h-2} \neq 0 \Rightarrow$ that $C\{\mathcal{F}'''(\lambda)\} = C\left[\frac{u'''_{2h-2}}{\bar{\lambda} - \bar{\lambda}_1} \frac{u''_{2h-2}}{1 - 1}\right]_{2h-2}$

3) if 9 series above deg C reg. $\{t_j\} \in DCR'(h)$, set $u_{20} = 0$, $(1_{20} + z_0)$ with $\lambda = \bar{\lambda}_1^h$ need to det $u'_j (\lambda = \bar{\lambda}_1^{2h})$, then found that either

a) $u'_{2h} \neq 0$, analysis then as in clause a) above except that at and $u'''_{2h-1} \neq 0$, $u'''_{2h-2} = 0$ and $C\{\mathcal{F}''(\lambda)\} = C\left[\frac{u'''_{2h-1}}{\bar{\lambda} - \bar{\lambda}_1} \frac{u'''_{2h}}{1 - 1}\right]_{2h-1}$ or

b) $u'_{2h} = 0$ analysis as in 2b) above except that in case i) $u'''_{2h-1} \neq 0$ $u'''_{2h} = 0$

a) $C\{\mathcal{F}(\lambda)\} = C\left[\frac{u''_{2h-1}}{\bar{\lambda} - \bar{\lambda}_1} \frac{u''_{2h}}{1 - 1}\right]_{2h-1}$, a) in clause ii) $u'''_{2h-3} \neq 0$, $u'''_{2h-2} = 0$

a) $C\{\mathcal{F}''(\lambda)\} = C\left[\frac{u'''_{2h-1}}{\bar{\lambda} - \bar{\lambda}_1} \frac{u''_{2h-2}}{1 - 1}\right]_{2h-2}$

... -

2) and 3) concern $\mathcal{F}(\lambda)$ from rational $F(\lambda)$. 2): $\mathcal{F}(\lambda)$ does not possess any term i) from $M_t \lambda^{-t} (t \in I_1)$; 2a) no term ii) from $M(\lambda - t)^{-1}$; 2b)i): is term i) thus form but $M \neq -t$; 2b)ii) is term ii) thus form and $M = -t$
 \therefore annihilated. 3) term $M_t \lambda^{-t}$ exists, analysis as before

... -

$\mathcal{F}_{p+1}^t, M_t (\lambda - \eta_p)^{-1}$ can be synthesised by arith.

6.7. A method 3) deriving an associated continued fraction

Th 40 $\mathcal{F}(\lambda) = \sum t_j \lambda^{-j-1} A$ reg.: seq. 3) series $\mathcal{F}_r(\lambda) = \sum r t_r \lambda^{-r-1} r \in I_1$ can

be det from $\mathcal{J}_r = \mathcal{F} - \mathcal{F}_r$ $\mathcal{F}_r(\lambda) = \frac{r t_0}{\lambda - \frac{r t_1}{r t_0} - \mathcal{F}_{r+1}(\lambda)}$ $r \in I_1$.

$C\{\mathcal{F}(\lambda)\} = C\left[\frac{V_r}{\lambda - \eta_{r+1}}\right]$ $V_r = r t_0$ $w_r = \frac{r t_1}{r t_0}$ ($r \in I_1$).

4. uses $C\left[\frac{V_r}{\lambda - \eta_{r+1}}\right]$, in place of $\mathcal{F}(\lambda)$

7.1 Orthogonal polynomials

process $\mathcal{G}[\dots]$ operating on scalar s : $\mathcal{G}[s^2] = t_0 \quad (\nu=1)$

$$p(\lambda) = \sum_0^r k_{\nu, \nu} \lambda^\nu \quad \mathcal{G}[p(s)] = (k_0 t_0)_0^\nu \quad ; \quad \mathcal{G}\left[\frac{1}{\lambda-s}\right] = \sum_0^r t_\nu \lambda^{-\nu-1}$$

$$\text{D1 } p_r(\lambda) = \sum_0^r k_{\nu, \nu} \lambda^\nu \quad r=1 \text{ determined from } \mathcal{G}[s^r p_r(s)] = 0 \quad (\nu=0^{r+1})$$

$k_{\nu, \nu} = 1$ called orthog polys. gen by mon. seqn (1)

The 1 $\{p_r(\lambda)\}_{r=1}^{\infty}$ D1 uniquely det iff $\exists(\lambda) = \sum_0^r t_\nu \lambda^{-\nu-1}$ A regular

$$\text{then } p_r(\lambda) = \frac{H[\{t_\nu\} \lambda^\nu]_r}{H_{0, r-1}} \quad (r=1) \quad \mathcal{G}[p_r(s) p_{r-1}(s)] = 0$$

 $\exists(\lambda) = \sum_0^r t_\nu \lambda^{-\nu-1}$ deg $\deg \{t_\nu\} \in \text{DAR}(h)$ only $p_r(\lambda) \quad (r=I_0)$ det.

D2 $q_r(\lambda) = \mathcal{G}\left[\frac{p_r(\lambda) - p_r(s)}{\lambda - s}\right] \quad (r \neq 1)$ called assoc orthog polys. gen by $\{t_\nu\}$

$$\text{Th2 } q_r(\lambda) = \sum_0^{r-1} l_{\nu, \nu} \quad (r=1) \quad : l_{\nu, \nu} = (k_{\nu, \nu} t_{\nu-1, \nu-1})_{\nu=1}^{r-1}, \quad (\nu=I_1, \nu'=J_0^{r-1})$$

$$l_{\nu, r-1} = t_0 \quad r=1, \quad q_r(\lambda) = \frac{H[\{t_\nu\} \sum_0^{r-1} t_{\nu-1, \nu-1} \lambda^\nu]_r}{H_{0, r-1}} \quad (r=1)$$

$$\text{Th3 } p_r(\lambda) = (\lambda - w_r) p_{r-1}(\lambda) - v_r q_{r-2}(\lambda) \quad r=I_2 \quad p_0(\lambda) = 1 \quad p_1(\lambda) = \lambda - w_1 \quad w_1 = \frac{t_0}{t_1}$$

$$q_r(\lambda) = (\lambda - w_r) q_{r-1}(\lambda) - v_r q_{r-2}(\lambda) \quad q_0(\lambda) = 0 \quad q_1(\lambda) = w_1 \quad w_1 = t_0$$

$$\text{Th4 } \exists(\lambda) = \sum_0^r t_\nu \lambda^{-\nu-1} \text{ A reg } \mathcal{A}\{\exists(\lambda)\} = C\left[\frac{v_D}{\lambda - w_D}\right] : \mathcal{A}\{\exists(\lambda)\} = \frac{q_1(\lambda)}{p_1(\lambda)} \quad r=I$$

D3 we find $p_r^{(m)}(\lambda) = \sum_0^r k_{\nu, \nu}^{(m)} \lambda^\nu \quad (r=1)$ det from $\mathcal{G}[s^m p_r^{(m)}(s)] = 0$

($r=I_0^{r-1}$) $k_{\nu, \nu}^{(m)}$ called orthog polynoms to order m derived from $\{t_\nu\}$

$q_r^{(m)}(\lambda) = \mathcal{G}\left[\frac{p_r^{(m)}(\lambda) - p_r^{(m)}(s)}{\lambda - s}\right]$ called assoc orthog polys order m gen by $\{t_\nu\}$

or

Th 5 $m \in \mathbb{Z}$ fixed

i) orthog polys order m iff $\exists^{(m)}(\lambda) = \sum t_{mn} \lambda^{-n-1}$ A reg is assuming this

$$\text{ii) } p_r^{(m)}(\lambda) = \frac{H[\langle t_{mn} \rangle] \lambda^r}{t_{mn, r+1}}$$

iii) $p_r^{(m)}(\lambda)$ form orthog system: $\int [s^m p_r^{(m)}(\lambda) p_s^{(m)}(\lambda')] = 0 \quad r \neq s$

$$\text{iv) } q_r^{(m)}(\lambda) = \sum_0^{r-1} l_{r,s}^{(m)} \lambda^s \quad l_{r,s}^{(m)} = k_{r,s}^{(m)} t_{m+s-2-s-1} \quad r=1, 2 \dots, m-1$$

$$\text{v) } q_r^{(m)}(\lambda) = \frac{H[\langle t_{mn} \rangle] \sum_0^{r-1} t_{m+s-2-s-1} \lambda^s}{t_{mn, r+1}} \quad r=1$$

$$\text{vi) } p_r^{(m)}(\lambda) = (\lambda - w_r^{(m)}) p_{r-1}^{(m)}(\lambda) - v_r^{(m)} p_{r-2}^{(m)}(\lambda) \quad p_0^{(m)} = 1 \quad p_1^{(m)} = w_1^{(m)} \quad w_1^{(m)} = \frac{t_{mn}}{t_{mn}}$$

$$q_r^{(m)}(\lambda) = (\lambda - w_r^{(m)}) q_{r-1}^{(m)}(\lambda) - v_r^{(m)} q_{r-2}^{(m)}(\lambda) \quad q_0^{(m)} = 0 \quad q_1^{(m)}(\lambda) = v_1^{(m)} \quad v_1^{(m)} = t_{mn}$$

$$\text{vii) } A\{\exists^{(m)}(\lambda)\} = C \left[\frac{v_r^{(m)}}{\lambda - w_r^{(m)}} \right] \quad C \left[\frac{v_r^{(m)}}{\lambda - w_r^{(m)}} \right]_r = \frac{q_r^{(m)}(\lambda)}{p_r^{(m)}(\lambda)} \quad r \in \mathbb{Z}$$

Th 6 $m \in \mathbb{Z}$ fixed orthog polys $\{\}$ order $m, m+1$ generated in conjuncion

iff $\exists^{(m)}(\lambda) = \sum t_{mn} \lambda^{-n-1}$ C-reg.

$$p_r^{(m)}(\lambda) = \lambda p_{r-1}^{(m)}(\lambda) - u_{2r}^{(m)} p_{r-1}^{(m)}(\lambda) \quad p_r^{(m+1)}(\lambda) = \lambda p_r^{(m)}(\lambda) - u_{2m+1}^{(m)} p_{r-1}^{(m)}(\lambda)$$

$$q_r^{(m)}(\lambda) = q_{r-1}^{(m)}(\lambda) + u_{2r}^{(m)} q_{r-1}^{(m)}(\lambda) + t_m p_{r-1}^{(m+1)}(\lambda)$$

$$q_r^{(m+1)}(\lambda) = \lambda q_r^{(m)}(\lambda) + q_{r-1}^{(m+1)}(\lambda) - t_m p_r^{(m)}(\lambda) \quad r = 0 \dots, \infty \quad \{\exists^{(m)}(\lambda)\} = C \int \frac{u_{2r}^{(m)}}{\lambda - \bar{q}_r^{(m)}}$$

Th 7 $\{\}$ $\in \text{DSN}(h)$ such that $\exists(\lambda) = \sum t \lambda^{-n-1}$ given by $F(\lambda) = \frac{q(\lambda)}{p(\lambda)}$

wh. $p(\lambda) = \sum_0^h k_n \lambda^n$ ($k_0 \neq 0, k_h = 1$) $q(\lambda) = \sum_0^{h-1} l_n \lambda^n$ $l_{h-1} \neq 0$ then $p_r^{(m)}(\lambda) \neq 0$

\det as above, $\Rightarrow p_r^{(m)}(\lambda) = p(\lambda)$. for all $n \geq h$ \det as above $u_{2m+1}^{(m)} = 0$

$$C\{\exists^{(m)}(\lambda)\} = C \left[\frac{u_{2h-1}^{(m)}}{\lambda - \bar{q}_h^{(m)}} \right]_{2h} \quad m \in \mathbb{Z}$$

$$\text{also } C\left\{ \sum_0^{m-1} t \lambda^{-n-1} + \frac{u_{2m}^{(m)} \lambda^{-m}}{\lambda - \bar{q}_h^{(m)}} \frac{u_{2m+1}^{(m)}}{1 - \bar{q}_h^{(m)}} \right\}_{2h} = F(\lambda) \quad m \in \mathbb{Z}$$

7.2. The q-d algorithm

The $m \in \mathbb{Z}$ fixed.

$$1) \exists^{(m)}(\lambda), \exists^{(mn)}(\lambda) C\text{-regular: } \mathcal{C}\{\exists^{(m)}(\lambda)\} = C \left[\frac{u_{2m+1}^{(m)}}{\lambda -} \frac{u_{2m+1}^{(m)}}{\lambda -} \right]$$

$$\mathcal{C}\{\exists^{(mn)}(\lambda)\} = C \left[\frac{u_{2m+1}^{(mn)}}{\lambda -} \frac{u_{2m+1}^{(mn)}}{\lambda -} \right], \text{ then } u_1^{(m)} u_2^{(m)} = u_1^{(mn)}, u_2^{(m)} + u_3^{(m)} = u_2^{(mn)}$$

$$u_{2m+1}^{(m)} u_{2m+1}^{(m)} = u_{2m+2}^{(mn)} u_{2m+1}^{(mn)}, u_{2m+1}^{(m)} + u_{2m+1}^{(m)} = u_{2m+1}^{(mn)} + u_{2m+1}^{(mn)} (\lambda -)$$

$$2) \exists^{(m)}(\lambda), \exists^{(mn)}(\lambda) \text{ deg. } C\text{-reg. } \{t_{mn}\}, \{t_{mn}\} \in DCR(h)$$

$$\mathcal{C}\{\exists^{(m)}(\lambda)\} = C \left[\frac{u_{2m+1}^{(m)}}{\lambda -} \frac{u_{2m+1}^{(m)}}{\lambda -} \right]_{2h} \quad \mathcal{C}\{\exists^{(mn)}(\lambda)\} = C \left[\frac{u_{2m+1}^{(mn)}}{\lambda -} \frac{u_{2m+1}^{(mn)}}{\lambda -} \right]_{2h}$$

then (37+8) hold $\lambda = \lambda_2^h$ setting $u_{2m+1}^{(m)} = 0$

$\exists(\lambda)$ semi-normal: $u_\nu^{(m)}$ placed in

$$\begin{matrix} u_2^{(1)} & u_3^{(0)} & u_4^{(1)} \\ u_2^{(1)} & u_3^{(1)} & u_4^{(1)} \\ u_3^{(2)} & u_3^{(1)} & \vdots \\ u_2^{(2)} & \vdots & \end{matrix}$$

numbers in (37+8) occur as:

$$\begin{matrix} u_{2m+1}^{(m)} & u_{2m+1}^{(m)} & u_{2m+1}^{(m)} & u_{2m+1}^{(m)} & u_{2m+1}^{(m)} \\ u_{2m+2}^{(mn)} & u_{2m+2}^{(mn)} & u_{2m+2}^{(mn)} & u_{2m+2}^{(mn)} & u_{2m+2}^{(mn)} \\ u_{2m+1}^{(mn)} & u_{2m+1}^{(mn)} & u_{2m+1}^{(mn)} & u_{2m+1}^{(mn)} & u_{2m+1}^{(mn)} \end{matrix}$$

$\exists(\lambda)$ deg. semi-normal, $\{t_\nu\} \in DSN(h)$: array confined to $u_\nu^{(m)} \nu = \lambda_2^h m = 0$

$$u_{2m+1}^{(m)} = 0 \quad (m=1)$$

$$c1) \exists(\lambda) = \sum t_\nu \lambda^{-\nu-1} \text{ semi-normal } \mathcal{C}\{\exists t_{mn}\lambda^{-\nu-1}\} = C \left[\frac{u_{2m+1}^{(m)}}{\lambda -} \frac{u_{2m+1}^{(m)}}{\lambda -} \right]$$

$u_\nu^{(m)}$ can be computed by: $u_1^{(m)} = t_m \quad u_2^{(m)} = \frac{t_{m+2}}{t_{m+1}} : \quad u_3^{(m)} = u_2^{(mn)} - u_2^{(m)} \quad m = 1$

$$u_{2j}^{(m)} = u_{2j-1}^{(m)} u_{2j-2}^{(mn)} u_{2j-1}^{(mn)}, \quad u_{2j+1}^{(m)} = u_{2j-1}^{(mn)} + u_{2j-1}^{(mn)} - u_{2j}^{(m)} \quad (\nu \in \mathbb{Z}_2, m \in \mathbb{Z})$$

$$2) \exists(\lambda) \text{ deg. semi-normal } \{t_\nu\} \in DSN(h); \quad \mathcal{C}\{\exists^{(m)}(\lambda)\} = C \left[\frac{u_{2m+1}^{(m)}}{\lambda -} \frac{u_{2m+1}^{(m)}}{\lambda -} \right]_{2h}$$

then $u_\nu^{(m)}$ ~~$\nu = \lambda_2^h$~~ $\nu \in \mathbb{Z}_1^{2h}$ also computed as above ($\lambda = \lambda_2^h$ in 38+9) setting

$$u_{2m+1}^{(m)} = 0$$

(Forward app. of q-d alg.)

Polynomials $p_i^{(m)}(\lambda)$ set: $p_0^{(m)}(\lambda)$

computed by Th. 6

$$p_0^{(m)}(\lambda) \quad p_1^{(m)}(\lambda)$$

$$\text{ch. } p_{r-1}^{(m+1)}(\lambda) \quad p_r^{(m)}(\lambda)$$

$$p_0^{(m)}(\lambda) \quad p_1^{(m)}(\lambda) \quad p_2^{(m)}(\lambda)$$

$$p_{r-1}^{(m+1)}(\lambda) \quad p_r^{(m)}(\lambda)$$

$$p_r^{(m)}(\lambda)$$

$$\text{Th. 9 } \exists(\lambda) = \sum_{i=0}^{m-1} b_i \lambda^{-i-1} \text{ semi normal; } C \left\{ \sum_{i=0}^{m-1} b_i u_{2i+1} \lambda^{-i-1} \right\} = C_L \frac{u_{2m+1}^{(m)}}{\lambda -} \frac{u_{2m}^{(m)}}{\lambda -} \bar{}$$

$$\text{set } \varepsilon_{2r}^{(m)} = \sum_{i=0}^{m-1} b_i \lambda^{-i-1} + \lambda^{-m} C \left[\frac{u_{2m+1}^{(m)}}{\lambda -} \frac{u_{2m}^{(m)}}{\lambda -} \right]_{2m} \quad (r_m = 2), \text{ then}$$

$$\varepsilon_{2r}^{(m+1)} = \sum_{i=0}^{m-1} b_i \lambda^{-i-1} + \lambda^{-m} C \left[\frac{u_{2m+1}^{(m)}}{\lambda -} \frac{u_{2m}^{(m)}}{\lambda -} \right]_{2m} \quad (r_m = 3) \quad (\#)$$

$$2) \exists(\lambda) \text{ deg semi normal from } F(\lambda) \text{ if Th. 7; } C \left\{ \sum_{i=0}^{m-1} b_i u_{2i+1} \lambda^{-i-1} \right\} = C \left[\frac{u_{2m+1}^{(m)}}{\lambda -} \frac{u_{2m}^{(m)}}{\lambda -} \right]_{2m}$$

$$\text{rat/poly } \varepsilon_{2r}^{(m)} \text{ as above } r = \begin{cases} 3 & m=1, u_{2m+1}^{(m)} = 0 \\ 1 & \text{else} \end{cases} \quad (\#) \text{ with } r = \begin{cases} I & m=I \\ 1 & \text{else} \end{cases} \quad \varepsilon_{2r}^{(m)} = F(\lambda)$$

3) in both 2) above cases let $\{p_i^{(m)}(\lambda)\}$ be orthog poly's gen by $\{b_i u_{2i+1}\}$ (so that in case 2) $p_i^{(m)}(\lambda)$ terminated by $p_i^{(m+1)}(\lambda) = p_i(\lambda)$

$$\varepsilon_{2r}^{(m)} = \sum_{i=0}^{m-1} b_i \lambda^{-i-1} + \lambda^{-m} \left\{ \frac{t_m}{\lambda} + \frac{t_m u_2^{(m)} u_3^{(m)}}{\lambda p_1^{(m)}(\lambda)} + \frac{t_m u_2^{(m)} u_3^{(m)} u_4^{(m)}}{\lambda p_1^{(m)}(\lambda) p_2^{(m+1)}(\lambda)} + \dots + \frac{t_m u_2^{(m)} u_3^{(m)} \dots u_{2r}^{(m)}}{\lambda p_1^{(m)}(\lambda) p_2^{(m+1)}(\lambda) \dots p_r^{(m+1)}(\lambda)} \right\}$$

$$\varepsilon_{2r}^{(m+1)} = \sum_{i=0}^r \dots \frac{t_m u_2^{(m)} u_3^{(m)} \dots u_{2r}^{(m)}}{\lambda p_1^{(m+1)}(\lambda) p_2^{(m+1)}(\lambda) \dots p_r^{(m+1)}(\lambda)}$$

$$r=I \text{ in case 1) and taking } u_{2m+1}^{(m)} = 0 \text{ or } \begin{cases} I & \\ 1 & \text{else} \end{cases}$$

$$\text{Th. 10 } \exists(\lambda) \{p_i^{(m)}(\lambda)\} \text{ as in Th. 9; set, } u_{2r}^{(m)}(\lambda) = p_r^{(m)}(\lambda)^{-1} p_i^{(m+1)}(\lambda)$$

$$u_{2m+1}^{(m)}(\lambda) = p_r^{(m+1)}(\lambda)^{-1} p_{m+1}^{(m)}(\lambda) \text{ and } (r=I \text{ or } 1) \quad r=I \text{ in 1st form, } r=1 \text{ in 2nd}$$

$$\text{(or 2)): } \{1, u_r^{(m)}(\lambda)\} \text{ can be const from, } u_0^{(m)}(\lambda) = 1 \quad (u_r^{(m)}(\lambda) = p_r^{(m)}(\lambda) \text{ } m=I$$

$$\text{by, } u_{2r}^{(m)}(\lambda) = u_{2r-1}^{(m)}(\lambda)^{-1}, u_{2r-2}^{(m+1)}(\lambda), u_{2r-1}^{(m+1)}(\lambda)$$

$$, u_{2m+1}^{(m)}(\lambda) = \lambda - \{1, u_{2r}^{(m+1)}(\lambda)\} \{ \lambda - , u_{2r-1}^{(m+1)}(\lambda) \} \{1, u_{2r}^{(m)}(\lambda)\}^{-1}$$

$$r=I, m=1 \text{ (or 1), } r=I, m=I; m=I, \text{ in first form, } r=I, m=I \text{ in 2nd for 2)}$$

$$\text{in latter case, } u_{2r}^{(m)}(\lambda) = 1$$

$$\text{ratios of terms in Th. 9 are } u_{2r-1}^{(m)}(\lambda) = p_r^{(m)}(\lambda)^{-1} u_{2r}^{(m)} p_{r-1}^{(m)}(\lambda)$$

$$2u_{2r}^{(m)}(\lambda) = p_r^{(m)}(\lambda)^{-1} u_{2r+1}^{(m)} p_{r-1}^{(m)}(\lambda)$$

$$\text{Th. 11 } \exists(\lambda) \quad p_i^{(m)}(\lambda) \quad u_r^{(m)} \quad \frac{u_{2r+1}^{(m)}}{u_{2r+1}^{(m)}(\lambda)} \text{ as above, } u_r^{(m)}(\lambda) = 1 \text{ in case 1) } r=I, m=I$$

in case 2) can be contr. from $u_0^{(m)}(\lambda) = 0 \quad u_1^{(m)}(\lambda) = p_1^{(m)}(\lambda)^{-1} u_2^{(m)} \text{ (med)}$

$$z u_{2r}^{(m)}(\lambda) = \{1 + z u_{2r-1}^{(m)}(\lambda)\} \{1 + z u_{2r-2}^{(m)}(\lambda)\} \{1 + z u_{2r-3}^{(m)}(\lambda)\}^{-1} - 1$$

$$z u_{2r+1}^{(m)}(\lambda) = \left[\{1 + z u_{2r}^{(m)}(\lambda)\}^{-1} \right]^{-1} \left[\{1 + z u_{2r-1}^{(m)}(\lambda)\}^{-1} \right] \{1 + z u_{2r}^{(m)}(\lambda)\}^{-1} - 1$$

$r=I$, $m=I$ in case 1), $r=I^h$, $m=I$ first form. $r=I^h$, $m=I$ in second in case 2)

in latter $z u_{2r}^{(m)}(\lambda) = 0$ $m=I$

terms of series in Th. 9:

Th. 12 $\{z u_r^{(m)}(\lambda)\}$ are in Th. 9-11. set $z u_1^{(m)}(\lambda)^{-1} = 0$ $m=I$ $z u_0^{(m)}(\lambda) = t_m \lambda^{-1}$ $m=2$.

$z u_r^{(m)}(\lambda) = z u_r^{(m)}(\lambda) z u_{r-1}^{(m)}(\lambda)$ $r=I$, $m=I$ (or 1) $r=I^h$, $m=I$ (or 2) can be contr.

$$\text{from } z u_{2r-1}^{(m)}(\lambda) = [\lambda \{z u_{2r-3}^{(m+1)}(\lambda)^{-1} + z u_{2r-2}^{(m+1)}(\lambda)^{-1}\} - z u_{2r-2}^{(m)}(\lambda)^{-1}]^{-1}$$

$$z u_{2r}^{(m)}(\lambda) = \lambda^{-1} \{ z u_{2r-2}^{(m+1)}(\lambda) + z u_{2r-1}^{(m+1)}(\lambda) \} - z u_{2r-1}^{(m)}(\lambda)$$

$r=I$, $m=I$ (or 1), $r=I^h$, $m=I$ (or 2), in latter case $z u_{2h}^{(m)}(\lambda) = 0$ $m=I$
partial sum in Th. 9.

$$\varepsilon_{2r}^{(m)} = \sum_{j=0}^{m-1} t_j \lambda^{-j-1} + \lambda^{-m} \sum_{j=0}^{2r-1} z u_{2j}^{(m)}(\lambda), \quad \varepsilon_{2r}^{(m+1)} = \sum_{j=0}^{m-1} t_j \lambda^{-j-1} + \lambda^{-m} \sum_{j=0}^{2r} z u_{2j}^{(m)}(\lambda)$$

$$\text{set } \varepsilon_{2r+1}^{(m)} = \lambda^m \sum_{j=0}^{2r} z u_{2j}^{(m)}(\lambda)^{-1}$$

Th. 13. $\Xi(\lambda) \neq F(\lambda)$ as in Th. 9, $z u_r^{(m)}(\lambda)$ as in Th. 12; not into $\varepsilon_r^{(m)}$

with $r, m=I$ in case 1), $r=I^h$, $m=I$ in case 2) can be contr. from

$$\varepsilon_{-1}^{(m)} = 0 \quad (m=I), \quad \varepsilon_{-1}^{(m)} = \sum_{j=0}^{m-1} t_j \lambda^{-j-1} \quad (m=I) \text{ by use of } \varepsilon_{r+1}^{(m+1)} = \varepsilon_{r-1}^{(m+1)} + (\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)})^{-1}$$

with $r, m=I$ (or 1), $r=I^h$, $m=I$ (or 2), in latter case $\varepsilon_{2h}^{(m)} = F(\lambda)$ $\varepsilon_{2h+1}^{(m)} = \infty(m=I)$

Th. 14.) $\Xi(\lambda) = \sum_i t_i \lambda^{-i-1}$ as in 1) \Rightarrow Th. 9-12, ~~not into~~ $\neq \lambda$ st. $p_r^{(m)}(\lambda) \neq 0$ $r=I^h$, $m=I$

then $\text{Th. } \{z u_r^{(m)}(\lambda)\} - \{\varepsilon_r^{(m)}\}$ contr. as in 9-12

2) $\Xi(\lambda), F(\lambda)$ as in Th. 9-12, $\lambda \neq t_i^{(m)} p_r^{(m)}(\lambda) \neq 0$ $r=I^h$, $m=I$ also $p(\lambda) \neq 0$, numm

value of $\{z u_r^{(m)}(\lambda)\} - \{\varepsilon_r^{(m)}\}$ as in 2) \Rightarrow 9-12

b) if $p_r^{(m)}(\lambda) \neq 0$ $r=I^h$ but $p(\lambda) = 0$, then $\{z u_r^{(m)}(\lambda)\}$ terminates with $z u_{2h-1}^{(m)}(\lambda) = 0$

$\{z u_r^{(m)}(\lambda)\}$ with $z u_{2h-1}^{(m)}(\lambda) = \infty$, $\{z u_r^{(m)}(\lambda)\}$ with $z u_{2h-1}^{(m)}(\lambda) = \infty$, $\{\varepsilon_r^{(m)}\}$ with

$\varepsilon_{2h}^{(m)} = \infty$, $\{\varepsilon_{2h-1}^{(m)}\}$ finite const for $m=I$

Ch. 8. The convergence behaviour of the row and column sequences \Rightarrow the Padé table. 39 |

NI $S\{f(z)\}$ is sum the $f(z) = (t_j z^j)_0^\infty$ series $f(z) = \sum t_j z^j$ with $0 \notin$ converge if $f(z)-r$ is fin. obtained by anal. cont. outside the 0 inside \cup δ meromorphy.

$\lim_{n \rightarrow \infty} |\Gamma_n| = \omega$: $f(z)$ converges unif. inside $|z| < \omega$, diverges $|z| > \omega$

$$\text{use } R_{ij,lm}(z) - R_{ij,l}(z) = \frac{(-z)^i H_{j+1, l+1} z^{i+j+l}}{H_{j+1, l+1, l+1} \pi_{i,j}(z) \pi_{i,j,m}(z)} \quad (R_{ij,lm}, R_{ij,l} \text{ prim})$$

$$R_{ij,l^*}(z) \quad (l^* \in I_j, l \text{ prim}): \quad R_{ij,l^*}(z) = R_{ij,l-1}(z) + (-z)^i \sum_{j,l}^m \frac{H_{j+1, l+1} z^l}{H_{j+1, l+1, l+1} \pi_{i,j,m}(z)}$$

$$\text{i.e. } R_{ij,l^*}(z) \text{ part. sum of } R_{ij,l-1}(z) + (-z)^i \sum_{j,l}^m \frac{H_{j+1, l+1} z^l}{H_{j+1, l+1, l+1} \pi_{i,j,m}(z)}$$

Examine $\{R_{ij,l}(z)\}$ $S\{f(z)\}$ pole at $z = z^{(r)}$, otherwise reg. within $|z| < \tilde{\beta}' (\tilde{\beta}' > \beta_1 = k_1)$

$$B\{f(z)\} = \frac{A_1}{1-z^{m+1}} + g(z), \quad g(z) \text{ sum fn. of } g(z) = \sum g_k z^k \text{ converges unif. in } |z| < \tilde{\beta}'$$

$$\text{so } |g_k| < G \tilde{\beta}'^{-k-1} \quad \forall k \in \mathbb{Z}$$

$f(z)$, i.e. $\{R_{ij,l}(z)\}$ converges for $|z| < \tilde{\beta}_1$, diverges $|z| > \tilde{\beta}_1$

$$\text{Since } H_{j,0} = t_j, \forall A, z^{m+2-j}, \exists x' \in I \text{ s.t. } H_{j,0} \neq 0 \quad (\Rightarrow \tilde{\beta}_1), \quad R_{j,0}(z) \neq 0$$

$$\text{prim. denoms: } \pi_{j,0}(z) = 1 - \frac{t_j}{z}, \quad \text{i.e. } \pi_{j,0}(z)|_{z=0} = \pi_j(z) = 1 - z^{m+1}$$

$$\text{in any holod. domain from wh. } z^{m+1} \text{ excluded } \exists \alpha'' \in \mathbb{G} \text{ st } \left| \frac{1}{\pi_{j,0}(z) \pi_{j,1}(z)} - \frac{1}{\pi_j(z)} \right| < 1$$

$$\text{dans the real no.: } R_{j,1}(z) - \sum_{j,l}^m \frac{H_{j,l} z^l}{H_{j,0} \pi_{j,0}(z) \pi_{j,1}(z)} \text{ converges unif. with } \sum_j \frac{H_{j,l}}{H_{j,0}} z^l$$

$$|\sqrt{H_{j,0}}|_{z=0} = \tilde{\beta}_1^{-1}. \quad S\{\pi_j(z) f(z)\} \text{ reg. within } \omega \text{ upon } |z| = \tilde{\beta}'$$

$$\text{i.e. } t'_0 = t_0 + t_{0,1} = t_{0,1} - z^{m+1} t_{j,1} \quad \forall j \in I : \quad |\sqrt{t'}|_{z=0} = \tilde{\beta}'^{-1}$$

$$\underline{t_{j,1}} = |\sqrt{H_{j,1}}|_{z=0} \leq \tilde{\beta}_1^{-1} \tilde{\beta}'^{-1} \quad |\sqrt{\frac{H_{j,1}}{H_{j,0}}}|_{z=0} \leq \tilde{\beta}'^{-1}$$

$\{R_{ij}(z)\}$ converges for $|z| \leq 5'$ except at $z = z^{(r)}$. Since analytic $\lim_{j \rightarrow \infty} R_{ij}(z) = S\{f(z)\}^{40}$

$S\{f(z)\}$ has poles on $|z| = b$, otherwise reg within or upon $|z| = b'$ ($> b_r$)

poles $z = z^{(r)} r \in I_i^{i'}$ of orders $h_r^{(r)} (r \in I_i^{i'})$ set $i = (h_r^{(r)})_{i=1}^{i'}$

$|\sqrt{H_{2,i}}| / \sqrt{b_r} \rightarrow \infty = \omega_r$ re I then $\omega_r \leq b_r^{-r+1} (r \in I_i)$. Set $\bar{\pi}_i(z) = \prod_{r=1}^{i'} (1 - z^{(r)} / z)^{h_r^{(r)}}$

$f'(z) = \bar{\pi}_i(z) f(z) : S\{f'(z)\}$ reg in all upon $|z| = b'$... setting $\bar{\pi}_i(z) = \sum_0^i c_n z^n$

$c_0^{(i)} = 1 : t_{2,i}^{(i)} = t_{2,i} + (c_1^{(i)} t_{2,i-1})_0^{(i)} (z^{(i)}) | \sqrt{t_{2,i}} / \sqrt{b_r} \rightarrow \infty \leq b_r^{-r+1}$

$H_{2,i}$: last col replaced by $\{t_{2,i}^{(i)}\}$ First i columns of elements whose ratio's growth governed by powers of b_r^{-r+1} , last col by powers of b_r^{-r+1} : $w_i \leq b_r^{-r+1} b_r^{r-1}$

Sudden diminution of $\frac{w_r}{w_{r-1}}$ signals exhaustion of poles. Hadamard: $\sqrt{H_{2,i}} / \sqrt{b_r}$ exists if does not \rightarrow constant $\sqrt{H_{2,i}} / \sqrt{b_r} / \sqrt{H_{2,i-1}} / \sqrt{b_{r-1}}$ may not exist

However if $\frac{w_r}{w_{r-1}} < \frac{w_{r-1}}{w_{r-2}}$ then $\sqrt{H_{2,i-1}} / \sqrt{b_{r-1}}$ exists

$\exists \hat{D} \in \mathbb{J} H_{2,i-1} \neq 0 \text{ for } i = \hat{D} : R_{i,\hat{D}}(z) \rightarrow \bar{\pi}_i(z)$ i.e.

presence of $\{\bar{\pi}_{i,\hat{D}}(z) \bar{\pi}_{i-1}(z)\}^{-1}$ does not influence convergence. 2nd $|\sqrt{H_{2,i}} / \sqrt{H_{2,i-1}}| / \sqrt{b_r}$ =

$|\sqrt{H_{2,i}}| / \sqrt{b_r} : \text{if } \rightarrow S\{f(z)\}$ for $|z| < b'$ except at poles
 $|\sqrt{H_{2,i-1}}| / \sqrt{b_{r-1}}$

Absolute theory $\Rightarrow S\{f(z)\}$ poles $z_r^{(r)}$ orders $h_r^{(r)}$ $r \in I_i^{i'}$ upon \mathbb{C} ,

$|z| = b_r, r \in I_i^{i'}$ no other singls within or upon $|z| = b'$ $b' > b_r$

$i = ((h_r^{(r)})_{r=1}^{i'})_{i'=1}^{i'}$. diminution in $\frac{w_r}{w_{r-1}}$ when $i = i$, $\sqrt{H_{2,i-1}} / \sqrt{b_{r-1}}$

exists $\neq 0 \quad \bar{\pi}_{i,\hat{D}}(z) \rightarrow \bar{\pi}_i(z) \quad z_r^{(r)}$ roots of mult $h_r^{(r)}$, $R_{i,\hat{D}}(z) \rightarrow S\{f(z)\}$

$|z| \leq b'$ not containing poles as interior pt.

8.1 Hadamard's theory & the Taylor series

$$L.1 \quad \eta \in \mathbb{R}_0' \quad \exists \nu \in \mathbb{I}, \text{ st. } 1 - \eta^\nu > \eta, \quad 1 - \frac{\eta^{\nu-1}}{1-\eta} < \frac{1}{2} \quad (\nu = \hat{\nu})$$

$$L.2. \quad \{d_\nu\} \text{ seq. for wh. } |(d_{\nu-1}, d_{\nu+1}, -d_\nu^2)| \leq \eta^{6\nu} \quad (\nu = \hat{\nu}), \quad \sqrt{|d_\nu|} / \lim_{\nu \rightarrow \infty} = 1$$

$$\text{then } \exists \hat{\nu} \in \mathbb{I} \text{ st. } d_\nu \neq 0, \quad |\gamma_{\nu+1} - \gamma_\nu| < \eta^\nu \quad (\nu = \hat{\nu}) \text{ wh. } \gamma_\nu = \frac{d_\nu}{d_{\nu-1}} \quad (\nu = \hat{\nu})$$

$$L.3. \quad \eta \{d_\nu\}, \text{ as in L.2. } \exists D = \varphi \text{ with } |\varphi| = 1 \text{ st. } d_\nu = D \eta^\nu + O(\eta^\nu)$$

$$L.4. \quad \text{If } \sqrt{|H_{D,i}|} / \lim_{i \rightarrow \infty} = \omega_r = \beta_1^{-r-1} \quad r = \hat{r}, \quad \omega_i < \beta_1^{-i-1} \text{ then } \sqrt{|H_{D,i+1}|}$$

exists, its value is β_1^{-i-1} .

Th.1 Nec. & suff. cond. that $f(z) = \sum f(z_i) z^i$ ($f(z) = \sum c_i z^i$) should have at most i poles no other sing. on ∂D & $\{z_i\}$ converge $|z_i| = \beta_i$, & $f(z)$ is that $\omega_r = \beta_1^{-r-1} \quad r = \hat{r} \quad \omega_i < \beta_1^{-i-1}$

Decoding theory solves problem of counting poles upon ∂D or converge

If $f(z)$ represents for with poles on $|z| = \beta_1 < \beta_2 < \beta_3 < \dots$ was Th.1 necessarily: def. $\pi_i(z) = \pi_i(z)$ $f_i(z) = \pi_i f(z)$ apply Th.1 to $f_{i+1}(z)$ etc. alike

Th.2. Nec. & suff. condit. that $f(z) = \sum f(z_i) z^i$ ($f(z) = \sum c_i z^i$) should have in successive poles of orders $\alpha_i^{(r)}$ or orders $h_i^{(r)}$ resp. $r = \hat{r}$ upon $|z| = \beta_r$, ($\hat{r} = \hat{i}$) with $\beta_1 < \beta_2 < \dots < \beta_r$ no other sing. within

$|z| = \beta_r$ ($\beta_r > \beta_{\hat{r}}$) is that, defining

$$i_{r,i} = (h_{r,i}^{(r)})_{r=1}^{i-1} \quad i_{r,i} = (i_{r,i}^{(r)})_{r=1}^{i'-1} \quad (r = \hat{r})$$

$$w_{i_{r,i}+i} = \left\{ \prod_{r=1}^{i'-1} \beta_r^{-i_{r,i}^{(r)}} \right\} \beta_{r,i}^{i'-i-1} \quad (i = \hat{i}, \quad r = \hat{r})$$

and setting $i = ((h_{r,i}^{(r)})_{r=1}^{i-1})_{r=1}^{\hat{r}}, \quad \frac{w_i}{w_{i-1}} < \beta_r'^{-1}$ if $f(z)$ has no sing. on $|z| = \beta_r'$, $\frac{w_i}{w_{i-1}} \geq \beta_r'^{-1}$ if it has

$\left\{ \frac{w_r}{w_{r+1}} \right\}$ non-increasing: 1) $r \in J$ st. $\frac{w_r}{w_{r+1}} > 0$ $f(z)$ repr. fn with \tilde{r} poles in \mathbb{z} -plane ⁴²
 is integral fn / polynomial. 2) $\frac{w_r}{w_{r+1}} \rightarrow 0$ $f(z)$ repr. fn meromorphic in
 every finite region 3) $\frac{w_r}{w_{r+1}} \rightarrow \beta' \in \mathbb{R}_0$ $f(z)$ repr. fn meromorphic
 in every finite region $|z| \leq \beta' < \beta$ but has ∞ -itly many poles in
 neighborhood. 4) $0 < |z| = \beta$, fn has non-polar sing. on $|z| = \beta$ 4) $\tilde{r} \in J$ st
 $\frac{w_r}{w_{r+1}} = \beta^{-1}$ $\beta \in \mathbb{R}_0$, $r = \tilde{r}$ $f(z)$ repr. fn having finite no. of poles within $|z| = \beta$
 non-polar sing. on it.

8.2. The convergence of the row sequences of the Padé table

Th3. $f(z) = S\{f(z)\}$ has in successive poles $z_{r,i}^{(n)}$ of orders $h_{r,i}^{(n)}$ resp. $r = \tilde{r}, i$
 upon \mathcal{O}_0 $|z| = \beta_r$, ($r = \tilde{r}, \hat{r}$) with $\beta_1 < \beta_2 < \dots < \beta_{\tilde{r}}$ no other sing. within
 $|z| = \beta' > \beta_{\hat{r}}$, $i = ((h_{r,i}^{(n)})_{r=1}^{\tilde{r}})_{i=1}^{\hat{r}}$, then $\exists j \in J$ st. $R_{r,j}(z)$ derived from $f(z)$
 prim. $j = \tilde{j}_{r,j}$, then $\tilde{z} \rightarrow f(z)$ in any domain contained within $|z| = \beta'$
 not incl. any $\{z_{r,i}^{(n)}\}$ as interior pt. If $f(z)$ has sing. in $|z| = \beta'$ then \tilde{z}
 diverges for $|z| > \beta'$

Th4. $f(z) = S\{f(z)\}$ has one simple pole z_r , wpm each of \mathcal{O}_0 $|z| = \beta_r$,
 $r = \tilde{r}$, with $0 < \beta_1 < \beta_2 < \dots < \beta_{\tilde{r}}$, no other sing. within $|z| = \beta'$ wh. $\beta' > \beta_{\tilde{r}}$
 then \exists set $j_r \in J$ ($r = \tilde{r}, \hat{r}$) st. $R_{r,j_r}(z)$ derived from $f(z)$ prim. for $j = \tilde{j}_{r,j}$;
 furthermore for $r = \tilde{r}$, $R_{r,j_r}(z) \rightarrow$ unif to $f(z)$ in any domain contained
 within \mathcal{O}_0 $|z| = \beta_{r,j_r}$, not incl. any member of z_1, z_2, \dots, z_r as interior
 pt diverges for $|z| > \beta_{r,j_r}$

Th5 $\{S_n\}$ denom. set of non-zero numbers everywhere dense in \mathbb{z} -plane
 $\{\tilde{z}_n\}$ derived set $\tilde{z}_0, \tilde{z}_1, \tilde{z}_0, \tilde{z}_2, \tilde{z}_3, \dots$. Assume $\sum t_{3n} z^{3n} (t_{3n}) \neq 0$
 $n \geq 1$ converges for all finite z . Def t_{3n+1}, t_{3n+2} from $t_1 = \frac{t_0 \tilde{z}_0^{-1}}{t_0 \tilde{z}_0} \quad |\tilde{z}_0| \leq 1$
 $t_{3n+2} = \frac{t_0 \tilde{z}_0}{t_0 \tilde{z}_0} \quad |\tilde{z}_0| > 1$

$$t_{2j+1} = t_{2j+1} = \begin{cases} t_{2j} \xi_j & |\xi_j| \leq 1 \\ t_{2j} \xi_j^{-1} & |\xi_j| > 1 \end{cases}, \quad \text{Denote } R_{ij}(z) \text{ from } f(z) = \sum t_i z^i$$

Then $R_{ij}(z)$ converges for all finite z , $R_{ij}(z)$ diverges at all pts \bar{D} set lying anywhere dense in z -plane

8.3. The convergence of the column sequences of the Padé table

Th. 6. $f(z) = S\{f(z)\}$ has in succession $\tilde{\gamma}_r$ zeros $\tilde{z}_r^{(r)}$ of orders $\tilde{h}_r^{(r)}$ resp $(r = \tilde{I}_1^{(r)})$ upon ∂D $|z| = \tilde{b}_r$ ($r = \tilde{I}_1^{(r)}$) $0 < \tilde{b}_1 < \tilde{b}_2 < \dots < \tilde{b}_{\tilde{r}}$ and $\{f(z)\}^{-1}$ has no other sing. upon $|z| = \tilde{b}'$ where $\tilde{b}' > \tilde{b}_r$ the $\exists i' \in \tilde{I}$ st $R_{ij}(z)$, where $j = ((h_r^{(r)})_{r=1}^{j_r})_{r=1}^{j_r}$, generated by $f(z)$ irreducible for $i = I_{i'}$, this seq. converges uniformly in any domain contd. within $|z| = \tilde{b}'$ not incl. any $\tilde{z}_r^{(r)}$ as int. pt. If $f(z)$ is zero on $|z| = \tilde{b}'$ then $R_{ij}(z) : z \in \tilde{I}_{i'}$ diverges for $|z| > \tilde{b}'$

8.4. The construction and convergence of the even order epsilon-array

Th. 7. $f(z) = S\{f(z)\}$ ($f(z) = \sum b_i z^i$) has only one simple pole z_r upon each of $|z| = \tilde{b}_r$ ($r = \tilde{I}_1^{(r)}$) $0 < \tilde{b}_1 < \tilde{b}_2 < \dots < \tilde{b}_{\tilde{r}}$, no other sing. within $|z| = \tilde{b}' > \tilde{b}_r$
 $\exists m \in \tilde{I}$ st. numbers $\{\varepsilon_r^{(m)}\}$ ($r = \tilde{I}_0^{(m)}, m = \tilde{I}_m$) can be constr. from $z_0^{(m)} = (\tilde{b}_r^{(r)})^{(m)}$ ($m = \tilde{I}_m$)

- 2) for $i = \tilde{I}_0^{(m-1)}$ $\varepsilon_{2r}^{(m)}$ ($m \geq 2$) \rightarrow unif. to $f(z)$ in any domain within $|z| = \tilde{b}_{i+1}$ not incl. any member of z_1, z_2, \dots, z_r as int. pt, diverges for $|z| = \tilde{b}_{i+1}$
- 3) $\varepsilon_{2r}^{(m)} \rightarrow$ unif. to $f(z)$ in any domain contained within $|z| = \tilde{b}'$ not incl. $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_{\tilde{r}}$ as int. pt.

Th. 8. $f(z) = S\{f(z)\}$ has only one simple pole z_r upon each of ∂D $|z| = \tilde{b}_r$ ($r = \tilde{I}_1^{(r)}$), $0 < \tilde{b}_r < \tilde{b}_{r+1} < \dots < \tilde{b}_{\tilde{r}}$ For any finite non-zero $\varepsilon \neq \tilde{b}_r$ ($r = \tilde{I}_1^{(r)}$) $\exists m \in \tilde{I}$ st. $\varepsilon_r^{(m)}$ ($r = \tilde{I}_1^{(r)}, m = \tilde{I}_m$) can be constr. from partial sums of $f(z)$ and for such values of z

$$\text{then } \frac{\Delta_m \Sigma^{(mn)}_r}{\Delta_m \Sigma^{(m)}_{2r}} \Big|_{m=0} = z_m^{-1} z, \quad \frac{\Sigma^{(mn)}_{2r}}{\Sigma^{(m)}_{2r}} \Big|_{m=\infty} = z^{(H)^{-1}} z \quad |z^{(m)^{-1}} z| < 1$$

$$|z^{(m)^{-1}} z| < 1 \quad r = \int_0^{\hat{P}-1}$$

$$\frac{\Sigma^{(mn)}_{2rH}}{\Sigma^{(m)}_{2rH}} \Big|_{m=m} = z^{(m)^{-1}} z$$

C. If $\exists r \in \mathbb{J}_0^{h-1} (r = \int_0^{h-1} m \in \mathbb{J}_m)$ can be constr. by applying ε -alg to
(irreducibility)

$$S_m = S + (A_r \gamma_r^m)_{r=1}^h, \quad (m \geq 1) \quad |S| < \infty \quad |A_r| < \infty \quad r = \int_0^h / \gamma_m | < r < \infty \quad (r = \int_0^h)$$

$$\text{and } \frac{\Delta_m \Sigma^{(mn)}_{2r}}{\Delta_m \Sigma^{(m)}_{2r}} \Big|_{m=\infty} = \gamma_{mH}, \quad \frac{\Sigma^{(mn)}_{2r}}{\Sigma^{(m)}_{2r}} \Big|_{m=\infty} = 1 \quad |\gamma_{mH}| > 1, \quad \frac{\Sigma^{(mn)}_{2rH}}{\Sigma^{(m)}_{2rH}} \Big|_{m=\infty} = \gamma_{mH}^{-1} \\ (r = \int_0^{h-1})$$

Ch 9. The convergence of the sequence $\{z^{(n)}\}$

9.1 Samuel Lubkin's lemmata

L1. α is the series $\sum \alpha_n$ converges A; set $\phi_m = \frac{\alpha_{m+1}}{\alpha_m}$, $\psi_m = \phi_m \phi_{m+1} (1 + \phi_m)(1 + \phi_{m+1})$
 $A_m = (\alpha_n)_{n=1}^m$; β is $\sum \beta_n$; set $\gamma_m = \frac{\beta_m}{\alpha_m}$ $\tilde{\beta}_m = \frac{\beta_m + \beta_{m+1}}{\alpha_m + \alpha_{m+1}}$ $\tilde{B}_m = (\beta_n)_{n=1}^m$
If β converges to B we define $R_m(\alpha/\beta) = \frac{B - \tilde{B}_m}{A - A_m}$. We set $\{R_m(\alpha/\beta)\}_{m=\infty} = R(\alpha/\beta)$

D1 Assuming β converges: β converges faster than α is $R(\alpha/\beta) = 0$; β α converge at same rate if $\exists m' \in \mathbb{J}$ $K, K' \in \mathbb{R}$ st. $|R_m(\alpha/\beta)| < \frac{K'}{K}$: β converges no less rapidly than α if $\exists K'' \in \mathbb{P}$ st. $R(\alpha/\beta) < R^{K''}$

L1. Let $\phi_m > 0$ ($m = I_m$) ($m' \in \mathbb{I}$) then

- 1) β converges at same rate as α if $\exists m'' \in \mathbb{J}$, $K, K'' \in \mathbb{R}$ st. $KK' > 0$ & $R^{K''}_{K'}$ ($m = I_{m''}$)
- 2) β converges no less rapidly than α is γ_m bounded as $m \rightarrow \infty$
- 3) β converges faster than α if $\alpha_m \rightarrow 0$

L2. Let $\phi_m > 0$ ($m = I_m$) ($m' \in \mathbb{I}$) and let $\phi_m \rightarrow 0$

- 1) β converges at same rate as α if $\exists m'' \in \mathbb{J}$ & $K, K' \in \mathbb{R}$ with $KK' > 0$ st. $\tilde{\beta}_m \in R^{K'}_{K''}$ ($m = I_{m''}$)
- 2) β converges no less rapidly than α is $\tilde{\beta}_m$ bounded as $m \rightarrow \infty$
- 3) β converges faster than α is $\tilde{\beta}_m \rightarrow 0$

L3. Let $\phi_m > 0$ and $|1 + \phi_m| > K > 0$ ($m = I_m$) wh. $m' \in \mathbb{J}, K \in \mathbb{R}$

- 1) β converges no less rapidly than α is γ_m bounded as $m \rightarrow \infty$
- 2) β ... faster than α if $\gamma_m \rightarrow 0$

We write $\varepsilon_0^{(m)} = (\omega_m)_0^{m-1} \quad \varepsilon_2^{(m)} = (\rho_m)_0^m$ ($m = \mathbb{I}$) so that

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$$\omega_m - \Delta_m \varepsilon_0^{(m)} \quad (m = \mathbb{I}) ; \quad \beta_0 = \varepsilon_2^{(\mathbb{I})}, \quad \beta_m = \Delta_m \varepsilon_2^{(m-1)} \quad (m = \mathbb{I}_1)$$

$$\text{We also have } \phi_m = \frac{\Delta_m \varepsilon_0^{(m+1)}}{\Delta_m \varepsilon_0^{(m)}}$$

Define ϕ by $\phi_{m/m=\infty} = \phi$. Since $\varepsilon_2^{(m)} = \varepsilon_0^{(m)} \cdot \omega_m (1-\phi_m)^{-1}$, we have

$$\eta_m = (1-\phi_m)^{-1} - (1-\phi_{m+1})^{-1}$$

Th1 If $\{\varepsilon_0^{(m)}\}$ and $\{\varepsilon_2^{(m)}\}$ real converge, they have same limit

Th2. $\{\varepsilon_0^{(m)}\}$ real converg. $\exists m' \in \mathbb{I}$, $K \in \mathbb{R}$ st. $|1-\phi_m| > K > 0$ ($m = \mathbb{I}_{m'}$)

1) $\{\varepsilon_2^{(m)}\}$ converges

2) if $\exists m'' \in \mathbb{J}$, $K' \in \mathbb{R}$, st. $|1+\phi_m| > K'$, $\phi_m \phi_{m''} (1+\phi_m)(1+\phi_{m''}) > 0$ ($m = \mathbb{I}_{m''}$)

then $\{\varepsilon_2^{(m)}\}$ converges no less rapidly than $\{\varepsilon_0^{(m)}\}$

3) if, furthermore, $(\phi_{m+1} - \phi_m) \rightarrow 0$ then $\{\varepsilon_2^{(m)}\}$ converges faster than $\{\varepsilon_0^{(m)}\}$

Th3. If $\{\varepsilon_0^{(m)}\}$ real, converges; $\phi \neq -1, 0, 1$ then $\{\varepsilon_2^{(m)}\}$ converges faster than $\{\varepsilon_0^{(m)}\}$

E1 Th3 applied to $\varepsilon_0^{(m)} = \frac{1}{2\pi i} ((x+i)^{-1} x^2)_0^{m-1} \quad x \in \mathbb{R}'_+, \quad x \neq 0$ ($m = \mathbb{I}$)

Th4. $\{\varepsilon_0^{(m)}\}$ real, $\phi = 0$, so that $\{\varepsilon_0^{(m)}\}$ converges, and $\exists m' \in \mathbb{J}$ st. ϕ_m do not sign

for $m = \mathbb{I}_{m'}$, then $\{\varepsilon_2^{(m)}\}$ converges faster than $\varepsilon_0^{(m)}$.

E2 Th4 applied to $\varepsilon_0^{(m)} = ((x!)^{-1} x^2)_0^{m-1} \quad x \in \mathbb{R} \quad (x \neq 0)$ ($m = \mathbb{I}$)

Th5 $\{\varepsilon_0^{(m)}\}$ real converg, $\phi = -1$ and $\exists m' \in \mathbb{J}$ st. $\frac{1+\phi_{m+1}}{1+\phi_m} < 1$ ($m = \mathbb{I}_{m'}$)

then $\{\varepsilon_2^{(m)}\}$ converges faster than $\varepsilon_0^{(m)}$.

E3 Th5 applied to $\varepsilon_0^{(m)} = ((-1)^k (2k+1)^{-1})_0^{m-1}$ ($m = \mathbb{I}$)

Th6 $\{\varepsilon_0^{(m)}\}$ real converg $\exists m' \in \mathbb{J}$ st. $K \in \mathbb{R}$ st. $\phi_m > 0$ in $|1-\phi_m| > K$ ($m = \mathbb{I}_{m'}$), then

1) $\{\varepsilon_2^{(m)}\}$ converges 2) if $\exists m'' \in \mathbb{J}$, $K' \in \mathbb{R}$ st. $|\phi_{m+1} - \phi_m| < K'$ ($m = \mathbb{I}_{m''}$)
then $\{\varepsilon_2^{(m)}\}$ converges no less rapidly than $\{\varepsilon_0^{(m)}\}$ 3) if $\lim_{m \rightarrow \infty} (\phi_{m+1} - \phi_m) \rightarrow 0$ then $\{\varepsilon_2^{(m)}\}$ converges faster than $\{\varepsilon_0^{(m)}\}$

Th 7 $\{\varepsilon_0^{(m)}\}$ real convgt. If $m(1-\alpha_m)/m \rightarrow 0$ exists, then

1) $\varepsilon_2^{(m)}$ converges

2) if, in addition, condit. 1) of 2) of Th 6 holds, $\varepsilon_2^{(m)}$ converges no less rapidly than $\{\varepsilon_0^{(m)}\}$

3) if, furthermore, condit. 2) of 3) of Th 6 holds, then $\{\varepsilon_2^{(m)}\}$ converges faster than $\{\varepsilon_0^{(m)}\}$.

$$\text{Exe 4 } \Rightarrow 2) \Rightarrow \text{Th 7} \Rightarrow \varepsilon_0^{(m)} = ((\omega_0)^{-\gamma})_0^m \quad \gamma \in \mathbb{R},$$

$$3) \Rightarrow \varepsilon_0^{(m)} = \left(\prod_{r=0}^{m-1} \left(1 - \frac{1}{\theta_{r+1} X_{T+2}} \right) \right)_0^{m-1}$$

Ch. 10. Some analysis

10.1. Functions of a real variable

Concerning real valued $\epsilon(s)$ for $s \in \bar{\mathbb{R}}_a^b \subseteq \bar{\mathbb{R}}$ make following assumptions:

$s \in \mathbb{R}_a^b$: $\epsilon(s+\delta)_{\delta=0}(s \in P) = \epsilon(s+)$ $\epsilon(s-\delta)_{\delta=0}(s \in P) = \epsilon(s-)$ exist

When $s=a$ 1st min. when $s=b$ 2nd min. $\epsilon(s+) = \epsilon(s-) : s$ pt. of continuity of ϵ , otherwise discontin. $\epsilon(a+) = \epsilon(a)$ $\epsilon(b)$ cont. at $s=a$ otherwise discontin.

Similarly at $s=b$.

If $\epsilon(s,-) \geq \epsilon(s_0+)$ for all $R_{s_0}^s \subseteq \mathbb{R}_a^b$: ϵ nondecr in $\bar{\mathbb{R}}_a^b$; if in addn. $\exists t \in P$ st $\epsilon(t) - \epsilon(a) \leq t$ then ϵ bounded nondecr in $\bar{\mathbb{R}}_a^b$. If $\epsilon_0(s)$ decrable. as $\epsilon_1(s) - \epsilon_2(s)$ bounded nondecr in $\bar{\mathbb{R}}_a^b$, $\epsilon_0(s)$ is bounded var. in $\bar{\mathbb{R}}_a^b$. NI is bounded nondecr in $\bar{\mathbb{R}}_a^b$: $\epsilon \in D(\bar{\mathbb{R}}_a^b)$ or $\epsilon(s) \in D(\bar{\mathbb{R}}_a^b)$

10.2. Definition and properties of the Stieltjes integral

Th. 1. If $\omega(s)$ complex valued cont., $\epsilon(s)$ complex valued with re.+ imag. parts

of total. var. for $s \in \bar{\mathbb{R}}_a^b \subseteq \bar{\mathbb{R}}$ then the sum $A_p = (\omega(s_j) \{ \epsilon(s_{j+1}) - \epsilon(s_j) \}_{j=1}^{p-1})$ tends to a limit when every interval $\bar{\mathbb{R}}_{s_j}^{s_{j+1}}$ made arb. small

Th. 2. Limit $A_p / p \rightarrow \infty$ of prec. th. called Stieltjes integral of $\omega(s)$ wrt. $\epsilon(s)$

conventionally written $\int_a^b \omega(s) d\epsilon(s)$: we write $S[\omega; \epsilon]_a^b$. If limits exist

we set $S[\omega; \epsilon]_a^\infty = S[\omega; \epsilon]_a^b |_{b=\infty}$, $S[\omega; \epsilon]_{-\infty}^b = S[\omega; \epsilon]_a^b |_{a=-\infty}$ (def.)

If common limit exists $S[\omega; \epsilon]_{-\infty}^\infty = S[\omega; \epsilon]_{a=-\infty}^b |_{b=\infty} = S[\omega; \epsilon]_a^b |_{a=-\infty}$

Th. 2. If $\omega(s), D_g \epsilon(s)$ Riemann integrable over $s \in \bar{\mathbb{R}}_a^b$, and Stieltjes int. exists, then $S[\omega; \epsilon]_a^b = \int_a^b \omega(s) D_g \epsilon(s) ds$ (int on RHS = Riemann int.)
 $\epsilon \int_a^b \omega(s) ds$ the same in both Riemann-Stieltjes sense

Th 3 If for $s \in \bar{\mathbb{R}}_a^b$ $\alpha_2(s) = \alpha_1(s) + C$ at all pts of cont. of $s_1 + \alpha_2$ ($C \in \mathbb{R}$)⁹⁹
and $S[\omega; \epsilon]_a^b$ exists, $S[\alpha; \epsilon]_a^b$ exists $\Rightarrow S[\omega; \epsilon]_a^b = S[\omega; \epsilon]_a^b$

We assume that in $S[\omega; \epsilon]_a^b$ $\omega(a) = 0$

Th 4. 1) $S[1; \epsilon]_a^b = \epsilon(b) - \epsilon(a)$ 2) $S[\alpha; \epsilon]_a^{a'} + S[\alpha; \epsilon]_a^b = S[\alpha; \epsilon]_a^b$

3) $S[\omega_1(s); \epsilon]_a^b = C S[\omega_1; \epsilon]_a^b$ $|C| \in \mathbb{R}_0$

4) $S[\omega_1(a) \pm \omega_2(s)]_a^b = S[\omega_1; \epsilon]_a^b \pm S[\omega_2; \epsilon]_a^b$

5) If $\sum \omega_j(s)$ converges unif. for $s \in \bar{\mathbb{R}}_a^b$, $S[(\omega_j(s))]_a^b = (S[\omega_j; \epsilon]_a^b)_a^b$

6) $|S[\omega; \epsilon]_a^b| \leq S[\omega(s)]_a^b$

Integration by parts:

Th 5. If $\omega \in D(\bar{\mathbb{R}}_a^b)$, $S[\omega; \epsilon]_a^b$ exists then $S[\omega; \omega]_a^b$ exists and

$$S[\omega; \omega]_a^b = \omega(b)\omega(b) - \omega(a)\omega(a) - S[\omega; \epsilon]_a^b$$

Th 6 If $\omega_1(s) > \omega_2(s)$ ~~$s \in \bar{\mathbb{R}}_a^b$~~ and either

a) \exists pt of discontinuity of ω at wh. $\omega_1(s') > \omega_2(s')$ or

b) \exists non zero interval $\bar{\mathbb{R}}_{g''}^{g''} \subseteq \bar{\mathbb{R}}_a^b$ in wh ω increases w/ $\omega_1(s) > \omega_2(s)$ $s \in \bar{\mathbb{R}}_{g''}^{g''}$

then $S[\omega_1; \epsilon]_a^b > S[\omega_2; \epsilon]_a^b$

c1. If either

a) $\exists p, s' \in \bar{\mathbb{R}}_a^b$ 1) discontinuity of ω for wh. $\omega(s') > 0$ or

b) an interval $\bar{\mathbb{R}}_{g''}^{g''} \subseteq \bar{\mathbb{R}}_a^b$ over wh ω increases and $\omega(s) > 0$ ($s \in \bar{\mathbb{R}}_{g''}^{g''}$)

then $S[\omega; \epsilon]_a^b > 0$.

10.2.1 Sequences of Stieltjes integrals

Th 7. If $\epsilon_\nu \in D(\bar{\mathbb{R}}_a^b \subseteq \bar{\mathbb{R}})$ ($\nu \geq 1$) and $\epsilon_\nu(s)|_{\nu=\infty} = \epsilon(s)$ ($s \in \bar{\mathbb{R}}_a^b$) otherwise
and ω is continuous in $\bar{\mathbb{R}}_a^b$, then $S[\omega; \epsilon_\nu]_a^b|_{\nu=\infty} = S[\omega; \epsilon]_a^b$

10.3 Theorems of choice

Th. 8. If $\epsilon_i \in D(\bar{R}_a^b \subset \bar{R})$, and $\epsilon_r(s) \in \bar{R}_c^c \subseteq R_a$ ($b = \bar{R}_a^b$; $i = j$)
 $c \neq c$ being indep. of r then $\exists a \in R(\bar{R}_a^b)$ and an infinite sequence
 $\{j\}$ of integers $j_r, r \geq 1$ with $r_0 < r_1 < \dots$ can be found such that

$$\epsilon_{r_j}(s) |_{r \geq 0} = \epsilon(s) \text{ at all pts of continuity } D_s.$$

$$\epsilon_r(s) |_{r \geq 0} = \epsilon(s)$$

Th. 9. ϵ_i, ϵ /no/ prev. th.: either ϵ holds at all pts of continuity
 D_s in \bar{R}_a^b , or \exists another fn. $\hat{\epsilon}$ (besides ϵ) satisfying similar rels.

Th. 10. If $\{c_i\}$ is a seq. of no. of bdd. varn. in \bar{R}_a^b , and $\exists K \in P$ st
 $|c_i(a)| \leq K (r=1)$, then $\exists \{t_n\}$ of bdd. varn. in \bar{R}_a^b and an ortho seq. of
integers $\{j_r\}$ st $\epsilon_{r_j}(s) |_{r \geq 0} = \epsilon(s)$ at all pts of continuity of $\epsilon(s)$. Either
 $\epsilon_r(s) |_{r \geq 0} = \epsilon(s)$ at all pts of continuity D_s or \exists another fn. $\hat{\epsilon}$ satisfying
similar relationships to (10)

10.4 Orthogonal polynomials derived from positive distribution

11. If, when $\epsilon \in D(\bar{R}_a^b)$, $t_j = S[g^j : \epsilon]_a^b (j=1)$ exist, we write $\epsilon \in MD(\bar{R}_a^b)$ or
 $\epsilon(s) \in MD(\bar{R}_a^b)$, and $\{t_j\} \in MS(\epsilon | \bar{R}_a^b)$

$\bar{R}_a^b \subset R$ then $\epsilon \in MD(\bar{R}_a^b) \Leftrightarrow \epsilon \in MD(\bar{R}_a^b)$. Also $1 - e^{-x} \in MD(\bar{R}_0)$. However
 $1 - (1+x)^{-\frac{2}{3}} \notin MD(\bar{R}_0)$.

Th. 11 $\{t_j\} \in MS(\epsilon | \bar{R}_a^b)$ then either

$$\begin{aligned} \text{1) } & \exists r' \in \mathbb{J} \text{ st } H_{0,r'} > 0 \quad r = \overline{j_0 \dots j_{r'-1}} \\ & \quad \quad \quad = \overline{o \dots j_{r'}} \quad \text{or 2) } H_{0,r} > 0 \quad (r = 1) \end{aligned}$$

In 1st case ortho poly $\{p_r(x)\}$ defn from $\{t_j\}$ can be cond for $r = j_0$, 2nd case indefinitely
in both cases $p_r(x)$ is poly taken from all of form $p_r(x) = \sum_{k=0}^r k_{r,k} x^k$ st. $\hat{p}_r(s)$
 $k_{r,0} = 1$ st. $\hat{p}_r(s) = (\hat{k}_{r,1}, \hat{s}^{r'})_0^b$ minimises value $\int_a^b [\hat{p}_r(s)]^2 \cdot \epsilon(s) ds$

Th. 12. For $r \in J_1$, zeros $\lambda_{n,r}$ ($r = J_1^r$) of $p_r(\lambda)$ simple; furthermore $\lambda_{n,r} \in \bar{R}_n^b$ ($r = J_1^r$)

Th. 13. Subject to suitable enumeration $0 < \lambda_{m,1} < \lambda_{n,1} < \lambda_{m,2} < \dots < \lambda_{n,r} < \lambda_{m,m} < \dots$

Th. 14. $\Omega(s)$ poly. of degree $\leq 2k-1$ ($k \in J_1$) then $S[\Omega : s]_a^b = (M_{n,r} \Omega(\lambda_{n,r}))^r$,

$$\text{where } M_{n,r} = \frac{g_r(\lambda_{n,r})}{D_r p_r(\lambda) |_{\lambda=\lambda_{n,r}}} \quad (r = J_1^r)$$

Th. 15. Nt. factors $M_{n,r}$ of prev. th. true, summarized: $M_{n,r} > 0$ ($r = J_1^r$), $(M_{n,r})_r' = \alpha(b+r)$ ($r \in J_1$)

Th. 16. $s \in \bar{R}_n$ abs. small. If for prescribed λ' $\exists r' \in J$ st. $p_r(\lambda) \neq 0$ ($r = J_1^{r'}$) for $|r - r'| \leq \bar{R}_0^b$ for all suff. small $\delta \in P$, then λ belongs to anti pseudo-spectrum of s : $s \in \text{APS}(s)$. Set \mathcal{D} all such pts called anti pseudo spectrum of s . Complement wrt. whole of complex plane \mathcal{D} anti pseudo spectrum is pseudo-spectrum of s .

Th. 16. Let $\bar{R}_n^b, R_n^b, \bar{R}_{n'}^b \subseteq \bar{R}_n^b \subseteq \bar{R}_0$, suppose $s \in \text{MD}(\bar{R}_n^b)$ $s(a') < s(b')$

then $\exists r' \in J$ st. $p_r(\lambda)$ has at least one zero in $\bar{R}_{n'}^b$ ($r = J_1^{r'}$)

Not true that if s const over $\bar{R}_{n'}^b$, $p_r(\lambda)$ ultimately (near $\bar{R}_{n'}^b$) zero over $\bar{R}_{n'}^b$

N3. $s \in \text{MD}(\bar{R}_n^b)$: $ds^{(m)}(s) = s^m d(s)$ ($s = \bar{R}_n^b$), we write $s^{(m)} = s(m/\bar{R}_n^b)$ ($m \in \mathbb{Z}$)

Th. 17. $s \in \text{MD}(\bar{R}_n^b)$, then $s(2m/\bar{R}_n^b) \in \text{MD}(\bar{R}_n^b)$.

Th. 18. $s \in \text{MD}(\bar{R}_n^b \subseteq \bar{R}_0)$ then $s(m/\bar{R}_n^b) \in \text{MD}(\bar{R}_n^b)$

Th. 19. If $\{t_r\} \in \text{MS}(\subseteq \bar{R}_n^b \subseteq \bar{R}_0)$ then either

- i) $\exists r \in J$ st. $H_{0,r} > 0$ ($r = J_0^{r-1}$), $r \in J_0^r$ and either a) $H_{1,r} > 0$ ($r = J_0^{r-2}$) or b) $H_{1,r} = 0$ ($r = J_0^{r-1}$)
 or ii) $H_{0,r} > 0$ ($r = J_0^{r-1}$) or $H_{1,r} = 0$ ($r = J_0^r$)

or 2) $H_{0,r} > 0, H_{1,r} > 0$ ($r \in J$)

10.5 The problem of moments

Finding ϵ from $\{t_p\}$. Prob. find $\underline{\mathbb{R}}_n^b$ s.t. $\epsilon \in M(D(\underline{\mathbb{R}}_n^b))$ i.e. non prob sol:

... only one ϵ with $\epsilon(n)=0$, i.e. ... det.

Modest mom. prob.: ϵ : simple step fn: suffi. many M_s s.t. $\epsilon = \sum_s M_s$ over $t_p = S[s^b : \epsilon \cdot j_0^b (b=1, \dots, n)]$ Find $p_A(\lambda)$, λ_s roots solve $t_p = (M_s \lambda_s^b)_{b=1}^n$

for M_s . Also, $M_s = \frac{q(\lambda)}{D_s p(\lambda) b_s - \lambda_s}$

Connection between det. D mom prob. compl. converge $\epsilon \in \mathcal{T}(A)$

10.6. The Hausdorff moment problem

10.6.1 Totally monotone sequences

D2 $\{t_p\}$ ht. mon. if $(-\Delta_b)^T t_p \geq 0$ ($b, p \geq 1$)

Th.20 $\epsilon \in (\underline{\mathbb{R}}_0')$ satisfying $t_p = S[s^b : \epsilon \cdot j_0^b] (b=1)$ if $\{t_p\}$ totally monotone

Th.21 $\{t_p\}, \{t_p'\}$ ht. mon.: so are $\{t_p + t_p'\}, \{t_p t_p'\}$

10.6.2 The determinacy of the Hausdorff moment problem

Th.22 ϵ' & ϵ'' bdd var in $\underline{\mathbb{R}}_0'$ and $S[s^b : \epsilon' \cdot j_0^b] = 0 (b=1)$ then $\epsilon'(s) = \epsilon''(s) = \underline{\mathbb{R}}_0'$

Th.23 If $\epsilon', \epsilon'' \in \mathcal{D}(\underline{\mathbb{R}}_0')$ and $S[s^b : \epsilon' \cdot j_0^b] = S[s^b : \epsilon'' \cdot j_0^b] = 0 (b=1)$

then $\epsilon'(s) = \epsilon''(s) \quad \underline{\mathbb{R}}_0' \subseteq \mathcal{D}(s \cdot \underline{\mathbb{R}}_0')$

10.7 Completely monotonic functions

D3 Function θ said to be comp. mon. in $\underline{\mathbb{R}}_n^b$ if $(-\Delta_b^{\theta})^T \theta(s) \geq 0$ ($\frac{1}{b} = \underline{\mathbb{R}}_n^b, n \in \mathbb{N}$)

Th.24. θ comp. mon. over $\underline{\mathbb{R}}_0$ then $\{\theta(a + s)\}$ tot. mon ($s \in \mathbb{R}P$)

Th.25. θ comp. mon over $\underline{\mathbb{R}}_0$ iff it can be expressed in form $\theta(s) = S[e^{-bs} : \epsilon \cdot j_0]$
 $\epsilon \in \mathcal{D}(\underline{\mathbb{R}}_0')$ (int. converges for $\frac{1}{b} = \underline{\mathbb{R}}_0$)

Example: $M: \epsilon(0) = 0 \quad \epsilon(s) = M \quad s \in P; e^{-\frac{1}{b}} \epsilon(s) = 0 \quad s \in \underline{\mathbb{R}}_0' \quad \epsilon(s) = 1 \quad s \in \underline{\mathbb{R}},$
 $(\frac{1}{b} + w)^{-1} w \in P \quad \epsilon(s) = w^{-1}(1 - e^{-ws}) \quad s \in \underline{\mathbb{R}}_0$

Th 26.) Θ_0, Θ_1 comp. mon. over \bar{R}_a^b , then a) $(-\mathcal{D}_{\frac{1}{2}}^{r'})\Theta_0(\frac{1}{2})$ ($r' \in \mathbb{C}$ fixed)

b) $\Theta_0(\frac{1}{2}) + \Theta_1(\frac{1}{2})$ c) $\Theta_0(\frac{1}{2})\Theta_1(\frac{1}{2})$ also comp mon over same interval

2) $\psi_0(\frac{1}{2})$ be st. $\Theta_2(\frac{1}{2}) = \frac{\mathcal{D}_{\frac{1}{2}}\psi_0(\frac{1}{2})}{\psi_0(\frac{1}{2})}$ comp. mon. \bar{R}_a^b ; then $\Theta_3(\frac{1}{2}) = \{\psi_0(\frac{1}{2})\}^{-1}$

is also comp mon over same interval

3) $\psi_1(\frac{1}{2})$ be st. $\mathcal{D}_{\frac{1}{2}}\psi_1(\frac{1}{2})$ comp mon over \bar{R}_a^b ; then $\Theta_1(\frac{1}{2}) = e^{-\psi_1(\frac{1}{2})}$ also comp mon over same int.

converse not in gen true, int of comp mon fn. not comp mon., diff. w.r.t. of 2 comp

mon. fn. not comp mon; $\Theta_3(\frac{1}{2})$ comp mon, $\Theta_2(\frac{1}{2})$ not; $\Theta_4(\frac{1}{2})$ comp mon, $\Theta_1(\frac{1}{2})$ not

Examples: $\sum M_0 (\frac{1}{2} + \omega + \zeta)^{-1}, \sum M_0 (\frac{1}{2} + \omega)^{-2}, \sum_{r=0}^h M_r \prod_{r=0}^h (\frac{1}{2} + \omega + r)^{-1}$

converge to comp mon fn's $(M_r)_0^\infty \in \mathcal{P}$ $w \in R$

$$\prod_{j=0}^h \left\{ \frac{1 + \frac{z}{\zeta_j}}{1 + \frac{\omega}{\zeta_j}} \right\}^{d_j} R_{\frac{1}{2}, \omega}^{\frac{1}{2}} \in \bar{R}_a^b, d_j \in R, j = 0, 1, \dots, h, (d_j, \zeta_j) \in \bar{R}_a^b$$

if h infinite

$$\prod_{j=0}^h \left(1 + \frac{z}{\zeta_j} \right)^{-d_j} \zeta_j \in \bar{R}_0, (d_j = j_0^{h'}) \quad (z, \zeta_j) \in \bar{R}_0$$

should h' be infinite

10.8. Complex variable theory

10.8.1 The maximum modulus theorem and its consequences

L1. $\Theta(s)$ cont. and $\Theta(s) \leq M' \in \mathcal{P}$ ($s = \bar{R}_a^b$) and $(b-a)^{-1} \int_a^b \Theta(s) ds \geq M$, then $\Theta(s) = M'$ identically $s = \bar{R}_a^b$.

Th. 27. $f(z)$ anal., reg. within region $D \rightarrow$ upon its boundary C . If $|f(z)| \leq G$

for all z in C , either i) $f(z)$ const $\Rightarrow |f(z)| = G$ for all z , or

ii) $|f(z)| < G$ for all z within D .

10.8.2. Schwarz's lemma

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N.B. Denote max $|f(z)|$ for all z upon $|z|=5$ by $M(f, 5)$.

Th. 28 If $f(z)$ where $z = \xi e^{i\theta}$ regular for $|z| \leq 5$, $\in P$, and $f(0) = 0$, then

$$|f(5e^{i\theta})| \leq \frac{5}{5} M(f, 5) \quad (5 = \bar{R}_0^{\xi})$$

10.8.3. The Stieltjes Vitali theorem

Lc 2. If members of seq. of anal. fns. $\{f^{(r)}(z)\}$ satisfy $|f^{(r)}(z)| \leq G$
 ~~$0 < \xi < 5'$~~ ($|z| = \bar{R}_0^{\xi}, r \in I$) $G \in P$ ind. $\exists z \in I$, and the limits
 $f^{(r)}(z_i)|_{r \rightarrow \infty} = (z_i)$ exist where $\{z_i\}$ infinite sequence having
 limit point $z = 0$ then $f^{(r)}(z)|_{r \rightarrow \infty} = f(z)$ for all $|z| < 5'$, wh. $f(z)$ anal. fn.
 \exists for $|z| < 5$.

Th. 29 If members of seq. of anal. fns. $\{f^{(r)}(z)\}$ unif bd.ad. when
 z lies in and upon boundary of domain \mathbb{D} , and seq. converges at
 set of points $\{z_i\}$ having limit point z' in \mathbb{D} , then $f^{(r)}(z)$ converges
 unif in any subdomain \mathbb{D}' of \mathbb{D} having no boundary points in
 common with that of \mathbb{D} .

Ch 11 Integral transforms

$$\text{Consider } F(\lambda) = \Im \left[(\lambda - s)^{-1} : e^s \right]_a^b$$

N1. (i) written as $\Im T[\lambda : e^s]_a^b$

1.1 The Riesz-Herglotz theorem

Th1. If $g(z')$ is anal. has +ve real part throughout open disc $|z'| < 1$ iff it may be expressed as in the form $g(z') = \Im \left[\frac{e^{is} + z'}{e^{is} - z'} : e^{z'} \right]_0^{2\pi} + c$
 wh. $c' \in D(\bar{\mathbb{R}}_0)$ $g'(2\pi) > g'(0) = 0$ $c \in \mathbb{R}$

Th2. If $G(\lambda)$ is anal. $\Im \{G(\lambda)\} \leq 0$ in $\frac{1}{2}$ -plane $\Im(\lambda) > 0$ iff it may be expressed in the form $G(\lambda) = \lambda + S \int_{-\infty}^{1+2s} \frac{1+s}{\lambda-s} : e^{-s} ds + d$
 wh. $d \in D(\bar{\mathbb{R}})$ $d \in \bar{\mathbb{R}}^0$ $d \in \mathbb{R}$

N2. λ lies in any domain where pts p are s.t. $|p-s| \geq \delta$ ($s = \bar{\mathbb{R}}_a^b$) $\delta \in \mathbb{P}$ arb small and $\bar{\mathbb{R}}_a^b \subseteq \bar{\mathbb{R}}$ fixed, we write $\lambda \in E(\bar{\mathbb{R}}_a^b)$.

Th3. The fn. $G(\lambda)$ is anal. $\lambda \in E(\bar{\mathbb{R}}_a^b)$ and $\Im \{GG\} \leq 0$ (≥ 0) or $\Im(\lambda) > 0$ (< 0) if it may be expressed as in Th2

N3. When any $(\lambda) \in \bar{\mathbb{R}}_{0_0}^{0_1}$ we write $\lambda \in \Lambda_{-0_0}^{0_1}$

Th4. $G(\lambda)$ def. as in Th2.: $\lambda' G(\lambda) / |_{|\lambda|=0} \Rightarrow (\lambda = \Lambda_0^\pi \cup \Lambda_{-\pi}^{2\pi})$

1.2. Hamburger-Nevanlinna functions

D1. If $F(\lambda) = ST[\lambda : e^s]_a^b$ $\in D(\bar{\mathbb{R}}_a^b) \setminus \bar{\mathbb{R}}_a^b \subseteq \bar{\mathbb{R}}$ called an HN-fn.

Th5. $F(\lambda)$ analytic for $\lambda \in E(\bar{\mathbb{R}})$ and $\Im \{F(\lambda)\} \leq 0$ (≥ 0) for $\Im(\lambda) > 0$ (< 0) and $\lambda F(\lambda) / |_{|\lambda|=0} = t$ ($\lambda \in \mathbb{R} \equiv \Lambda_0^{2\pi} \cap \Lambda_{-\pi}^{2\pi}, t \in \mathbb{P}$) iff $F(\lambda)$ is HN fn. in \mathbb{D} from D with $t = \epsilon(0) - \epsilon(-\infty)$.

1.2.1 The Julia–Nevanlinna theorem

N3. If $\lambda \in E(\bar{\mathbb{R}}^b_\alpha)$ and \exists an $M \in \mathbb{P}$ st $|\lambda| \leq M$ then we write $\lambda \in BE(\bar{\mathbb{R}}^b_\alpha)$

~~that is,~~ Remark $G_t = \frac{t}{\alpha + \beta - \tilde{C}}$ ($t \in \mathbb{P}$, $\alpha, \beta \in \mathbb{R}$ ($\beta \neq 0$)) maps real axis in \tilde{G} plane

onto \mathbb{O} centre $\frac{-it}{2\beta}$ and radius $|\frac{t}{2\beta}|$ in G plane, lower half plane

$\operatorname{Im}(\tilde{G}) < 0$ into interior of the \mathbb{O} (K touches real axis of G -plane at origin)

The 6 If fn $G(\lambda)$ anal. with $\operatorname{Im}\{G(\lambda)\} \leq 0$ for $\operatorname{Im}(\lambda) > 0$, and admits representation

$G(\lambda) = \lambda^{-1} \{t + G_1(\lambda)\}$ in sector \mathbb{A}_0^π , where $t \in \mathbb{R}$ and $|G_1(\lambda)|_{|\lambda|=0} = 0$

($\lambda = \mathbb{A}_0^\pi$), then either

i) $t = 0$, $G(\lambda) = 0$ identically

ii) $t > 0$ and as λ ranges over $\operatorname{Im}(\lambda) \geq \beta_0 > 0$ value of $G(\lambda)$ remains

within \mathbb{O} K centre $\frac{-it}{2\beta_0}$ radius $|\frac{t}{2\beta_0}|$

When in ii) $\operatorname{Im}(\beta_0) = \beta_0$ either a) $G(\lambda) = \frac{t}{\lambda + w}$ ($w \in \mathbb{R}$), value of $G(\lambda)$

then lies on K or b) $G(\lambda)$ does not have this representation, value lies within K .

c.) $F(\lambda)$ HN fn st $S[s : \epsilon]^b$ exists; set $t_0 = \epsilon(s) - \epsilon(a)$ When $\lambda = \alpha + i\beta \in BE(\bar{\mathbb{R}}^b_\alpha)$ fixed, either

i) $t_0 = 0$, $F(\lambda)$ identically $\neq 0$,

ii) $\epsilon(s)$ simple step fn., 1 saltus at $s = w \in \bar{\mathbb{R}}^b_\alpha$ value of $F(\lambda)$ lies on 0

K_0 centre at $\frac{-it_0}{2\beta}$ radius $|\frac{t_0}{2\beta}|$, or

iii) ϵ value of $F(\lambda)$ lies in interior of K .

22. If $S(\lambda) = ST[\lambda/\epsilon \bar{J}_\alpha^b]$ or $\lambda \in D(\bar{R}_\alpha^b \subseteq \bar{R}_0)$ called an S-function

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Thy. $S(\lambda) = ST[\lambda/\epsilon \bar{J}_\alpha^b]$ is S-fn. Then $S(\lambda) = (\epsilon \lambda')^{-1} F'(\lambda')$ wh. $F'(\lambda')$ is the HN-fn $F'(\lambda') = ST[\lambda'/\epsilon' \bar{J}_{\alpha'}^{-b'}]$ $\lambda' = \lambda^{\frac{1}{2}}$ $b' = b^{\frac{1}{2}}$

$$d\epsilon'(-s') = d\epsilon'(s') = \begin{cases} 0 & (s' = s^{\frac{1}{2}}) \\ d\epsilon(s) & \end{cases} \quad \begin{array}{l} s = \bar{R}_0^\alpha \text{ (if } \alpha \in P) \\ s = \bar{R}_\alpha^b \end{array}$$

23 $F'(\lambda')$ called P' HN-fn companion to $S(\lambda)$

ϵ' symmetric anti-symmetric about $\epsilon(0) = \frac{1}{2}\{\epsilon'(0-) + \epsilon'(0+)\}$ so that

$$\epsilon'(-s') - \epsilon'(0-) = \epsilon'(s') - \epsilon(0+) \quad s' \in \bar{R}_0^b$$

If $S(\lambda)$ generates $\sum t_\lambda \lambda^{-\alpha-1}$, $F'(\lambda')$ geno. $\sum t'_\lambda \lambda'^{-\alpha-1}$, $t'_{2\alpha} = 2t_\alpha$, $t'_{2\alpha+1} = 0$

Thy. Fn $S(\lambda)$ anal. and $\operatorname{Im}\{\lambda^{\frac{1}{2}} S(\lambda)\} \leq 0$ for $\lambda \in E(\bar{R}_0)$ and $\lambda S(\lambda)|_{|\lambda|=0} = t$

($\lambda = \Lambda_0^{2\pi}, t \in \bar{R}_0$) iff $S(\lambda)$ is S-fn. $ST[\lambda/\epsilon \bar{J}_\alpha^b]$ with $t = \epsilon(b) - \epsilon(a)$

1) if $t_0 = 0$ $S(\lambda)$ identically zero

$$2) \lambda = \omega \in -P, \text{ then } S(\lambda) \in \bar{R}_{-\omega} \in -\bar{P} \quad \bar{R}_{-\omega}$$

3) assume $S[s \bar{J}_\alpha^b]$ exists, $\lambda = \omega + i\beta \in BE(\bar{R}_0)$ fixed either

a) ϵ simple step fn with but one values $s = \omega \in \bar{R}_\alpha^b$; value of $S(\lambda)$ lies on that arc of ∂K_0 whose centre is ω + $\frac{-it_0}{2\beta}$ radius $\frac{|t_0|}{2\beta}$ /

lies in $\frac{1}{2}$ plane H_0 defined by any $\{S(\lambda)\} \in \bar{R}_{-\arg(\lambda)}$ or

b) value of $S(\lambda)$ lies in interior of segment defined by intersection of H_0 and H_0' .

$F'(\lambda')$ gives 2nd inclusion domain, K_0' . centre at $-\frac{it_0}{2\lambda^{\frac{1}{2}} \operatorname{Im}(\lambda^{\frac{1}{2}})}$ radius

$\frac{t_0}{2\lambda^{\frac{1}{2}} \operatorname{Im}(\lambda^{\frac{1}{2}})}$ K_0' passes thru t_0^{-1} and origin. Centre of K' lies on K . Segment of ∂ def. by $K \cap H$ lies inside that def. by K' and H .

11.3.1. The iterated Laplace transform

$\text{Def } S'(\lambda) = \mathcal{S}[\lambda + s]_a^b$ ($s \in \mathbb{D}(\bar{R}_a^b \subseteq \bar{R}_0)$) is called an S -fn.

$$\underline{s}'(\lambda) = -s(-\lambda)$$

Theorem 9. If in Def $s_i = \mathcal{S}[e^{-\frac{1}{2}s} : s]_a^b$ ($\frac{1}{2} = \bar{R}_0$) then $S'(\lambda) = \int_a^b e^{-\lambda s} c_i(s) ds$.
 $(Re(\lambda) = \bar{R}_0)$

Def provides anal. cont. of Lap trans. for $\lambda = E(\bar{R}_0^b)$:

11.4. The Stieltjes inversion formula

Formula for $f(s)$: s complex select $\lambda' = \omega + ip' \in \text{Im}(s) = P$. Contour \mathcal{B} , oriented λ' formed from line $\text{Im}(s) = \eta$ $\eta \in R_{\lambda'}^{P'}$ and \circ $|s| = \frac{1}{2} > |\lambda'|$. $F(\lambda') = \frac{1}{2\pi i} \int_{\mathcal{B}} \frac{F(s)}{\lambda' - s} ds$

λ'' mirror image of λ' in $\text{Im}(s) = \eta$ $\frac{1}{2\pi i} \int_{\mathcal{B}} \frac{F(s)}{\lambda'' - s} ds = 0$ comp. conj. of this and added to \mathcal{B}

$$F(\lambda') = -\frac{1}{2\pi i} \int_{\mathcal{B}} \left\{ \frac{F(s)}{\lambda' - s} - \frac{\overline{F(s)}}{\bar{\lambda}' - \bar{s}} \right\} ds \quad \bar{s} \rightarrow \infty \quad F(\lambda') = \int_{\infty - ip}^{\infty + ip} \left\{ \frac{F(s)}{\lambda' - s} - \frac{\overline{F(s)}}{\bar{\lambda}' - \bar{s}} \right\} ds$$

On $\text{Im}(s) = \eta$ $\lambda' - s$ $\bar{\lambda}' - \bar{s}$ have conj. values $\bar{\lambda}' - \bar{s} = \bar{\lambda}'' - \bar{s}$: $F(\lambda') = \int_{\infty - ip}^{\infty + ip} \frac{\int_{-\pi}^{\pi} f(s) ds}{\lambda' - s} ds$

compare with $F(\lambda') = \int_{-\infty}^{\infty} \frac{d\omega(s)}{\lambda' - s}$

Theorem 10 $F(\lambda) = ST[\lambda/\omega]_a^b$ HN-fn. Then

$$\frac{1}{2} \{ \omega(s_2+0) + \omega(s_2-) - \omega(s_1+) - \omega(s_1-) \} = \lim_{\eta \rightarrow 0} \int_{s_1+ip}^{s_2+ip} F(\lambda) d\lambda / \eta = 0$$

$$s_1, s_2 \in \mathbb{R}$$

Theorem 11. 1) Values of HN fn $F(\lambda) = ST[\lambda/\omega]_a^b$ ($\lambda = R_{s_1}^{s_2} \subseteq \mathbb{R}$) obtained by anal. cont. from half planes $\text{Im } \lambda > 0$ and

2) $F(\lambda)$ takes real values in this interval

$$\text{if } \omega(s_1+) = \omega(s_1-) = \omega(s_2) \quad (s \in R_{s_1}^{s_2})$$

If σ cont over $R_{\delta_1}^{S_2}$, values of $F(\lambda)$ def. for $\operatorname{Im}(\lambda) \leq 0$ can be obtained by anal. cont. through gap. $\sigma(s)$ cont. incr. anal fn. real variable $s = \bar{R}_\alpha^b$ then $F(\lambda)$ may be cont. through gap. If $\sigma(s)$ cont. incr. non-anal in $s = \bar{R}_\alpha^b$ anal. cont impossible.

1.5 Riesz-Herglotz-Wall functions

D4. The fn. $W(z) = \sqrt{1-z} \cdot \Im \left[(1-z^2 \sin^2 \frac{1}{z})^{-\frac{1}{2}} : \zeta' \right]^\pi \quad \zeta' \in \partial(0, \pi)$

is called an HRW-fn

Th 11. Fn $W(z)$ is analytic has the real part $z = E(\bar{R}_{\delta_1}^{S_2})$, real for real z , iff $W(z)$ is an HRW-function.

$$W(z) \text{ HRW fn. } \Leftrightarrow W(z)^{-1}, W\left(\frac{z}{1-z}\right)$$

Th 12. $W(z)$ HRW-fn.: $\{\gamma(\gamma^{-1})\}^{\frac{1}{2}} W(\gamma^{-1})$ is an S-fn.

Ch 12. Power series

12.1. The series generated by Hamburger - Nevanlinna functions

Assumption concerning ϵ in $F(\lambda) = ST[\lambda/\epsilon]_n^b$: $t_{\lambda} = S[\lambda/\epsilon]^b \in \mathbb{C}_n^b$ ($\epsilon = 1$) exist

Th1. Let $\{t_{\lambda}\} \in MS(\epsilon/\bar{\mathbb{R}}_a^b)$, $S(\lambda) = \sum b_i \lambda^{-i-1}$, $F(\lambda) = ST[\lambda/\epsilon]_n^b$, $\eta = \max(|a_1|, |b_1|)$
 a, b not end pts of intervals over wh. ϵ const.

$S(\lambda)$ converges to $F(\lambda)$ $|A| = \bar{R}_y$, diverges for $|\lambda| = \bar{R}_y^2$

Th2. Let $\{t_{\lambda}\} \in MS(\epsilon/\bar{\mathbb{R}}_a^b)$, $F(\lambda) = ST[\lambda/\epsilon]_n^b$, $\lambda = \alpha + i\beta \in BE(\bar{\mathbb{R}})$
 fixed. Value of $F(\lambda)$ lies in each D circular regions $\{K'_{\lambda'}\}$: $K'_{\lambda'}$
 centre at $(t_{\lambda} \lambda^{-2-1})_0^{2+1} - i \frac{t_{\lambda'}}{2\pi \lambda^{2+1}}$, radius $1 \frac{|t_{\lambda'}|}{2\pi \lambda^{2+1}}$ ($\lambda' \in I$)

12.2. The series generated by Stieltjes functions

Th1 $\rightarrow S$ functions

Th3 Let $\{t_{\lambda}\} \in MS[\epsilon]_n^b$, $\alpha \in \mathbb{R}_0$, $S(\lambda) = ST[\lambda/\epsilon]_n^b$, $\lambda \overset{\text{arcs}}{\in} BE(\bar{\mathbb{R}}_0)$

fixed

1) $\lambda = \nu \in -P$, $S(\nu) \in -P$, and $(t_{\nu} \nu^{-2-1})_0^{2+1} < S(\nu) < (t_{\nu} \nu^{-2-1})_0^{2+1}$ ($\nu \in I$)

2) for other values of $\lambda \in BE(\bar{\mathbb{R}}_0)$, value of $S(\lambda)$ lies in each of regions of sequence $\{D_{\lambda'}\}$: $D_{\lambda'}$ is intersection of half-plane $H_{\lambda'}$ and circle $K'_{\lambda'}$:

$H_{\lambda'}$ is half plane $-(\nu' + i)\arg(\lambda) \leq \arg\{S(\lambda) - (t_{\lambda} \lambda^{-2-1})_0^{2+1}\} \leq -\arg(\lambda)$

$K'_{\lambda'}$ has centre at $(t_{\lambda} \lambda^{-2-1})_0^{2+1} - \frac{t_{\lambda'}}{2\pi \lambda^{2+1}}$, is of radius $1 \frac{|t_{\lambda'}|}{2\pi \lambda^{2+1}}$.

Further system of Oular regions from companion to

12.3.1. The definition and properties of an asymptotic series

$$\text{D.1. } g(\lambda) = \sum_i g_i \lambda^{d-i}, \quad \lambda^{d'} \{ G(\lambda) - (g_0 \lambda^{-d})_0^{d'} \} / | \lambda | \rightarrow 0 \quad (d' = I)$$

$\lambda \rightarrow \infty$ in prescribed domain M , $g(\lambda)$ resp. $G(\lambda)$ asymptotically in M , conversely $G(\lambda)$ generates $g(\lambda)$.

$$\text{D.2. } \lambda^{d'} \{ G(\lambda) - (g_0 \lambda^{-d})_0^{d'} \} / | \lambda | \rightarrow 0 \quad (\lambda \rightarrow \infty \text{ in } M)$$

If $G(\lambda)$ gen., only one series; converse untrue. e.g. $G(\lambda)$ gen. $f(\lambda)$ in $\text{Re}(\lambda) = \bar{R}_0 \subset \bar{R}$, then $G(\lambda) + A e^{-\alpha \lambda} \quad (|A|, \alpha \in \bar{P})$ generates $f(\lambda)$ in same domain.

Th. 3. $G(\lambda)$ regular for $| \lambda | = R_g$ ($\zeta \in \bar{P}$ fixed) $G(\lambda)$ gen. $g(\lambda) = \sum_i g_i \lambda^{-d-i}$ for all $\text{arg}(\lambda)$, then $g(\lambda)$ converges to $G(\lambda)$ for all sufficiently large $| \lambda |$.

$G(\lambda)$ gen. about λ_0 . Rel. of D.1 cannot hold for all $\text{arg}(\lambda)$.

Th. 4. $G'(\lambda), G''(\lambda)$ gen. $g'(\lambda) = \sum_i g'_i \lambda^{-d-i}, g''(\lambda) = \sum_i g''_i \lambda^{-d-i}$ resp. when

$\lambda \rightarrow \infty$ in M : in M

$$1) |A|G'(\lambda) \quad | \lambda | \in \bar{R}_0 \text{ gen. } \sum_i (A g'_i) \lambda^{-d}$$

$$2) G'(\lambda) + G''(\lambda) \text{ gen. } \sum_i (g'_i + g''_i) \lambda^{-d}$$

$$3) \hat{G}(\lambda) = G'(\lambda) G''(\lambda) \text{ gen. } \sum_i \hat{g}_i \lambda^{-d} \text{ wh. } \hat{g}_{d'} = (g'_i g''_{d-i})_0^{d'} \quad (d' = I)$$

$$4) \text{ if } g_0 \neq 0, \quad \tilde{G}(\lambda) = \{ G(\lambda) \}^{-1} \text{ gen. } \sum_i \tilde{g}'_i \lambda^{-d-i} \text{ where } \tilde{g}_0 = g_0^{-1} (g'_i g''_{d-i})_0^{d'} = 0$$

12.3.2. The asymptotic expansion of Hamburger-Nevanlinna function.

Th. 5. If $\{t_j\} \in \text{MS}_{\zeta} \subseteq \bar{j}_a^b$ then $F(\lambda) = \Im \int_{\gamma} \frac{1}{\lambda - t_j} \lambda^{\zeta} \text{ sat. } \lambda^{d'} \{ F(\lambda) - (t_j \lambda^{-d-1})_0^{d'} \} / | \lambda | \rightarrow 0$

$$\nu' = I \quad \ln \lambda \in \mathbb{E}(\bar{R})$$

C1. Above, rel. also sat. when $\lambda \rightarrow \infty$ in $\lambda = \lambda_{\pm}, \lambda_{\mp}^{4\pi}$

$\bar{R}_a^b \leq \bar{R}$, rel. sat. in all $\text{arg}(\lambda)$ is non-acc. inc. In all $\Im \lambda = \bar{R}$. Skier's phenomenon.

12.3.3. The asymptotic expansion of Stieltjes functions

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Th.6. $\{t_n\} \in M\delta$ [$\zeta_j a^j$] ($a \in \mathbb{R}_+$) then $S(a) = 3\pi [\bar{\gamma}/\alpha]_a^\infty$ satis.

$$\lambda^{2+\frac{1}{2}} \left\{ S(a) - (t_n \lambda^{-n-1})_0^{n-1} \right\} / |t_n| \xrightarrow{n \rightarrow \infty} 0 \quad (\nu' = 1) \quad \lambda \in \mathbb{C} \setminus (\bar{\mathbb{R}}_+)$$

C1 Above rel. also sat. when $\lambda \rightarrow \infty$ in $\lambda = \lambda_\infty^{2+\frac{1}{2}}$

12.4. The transformation of divergent power series

12.4.1 Borel's integrals

Suppose $f(z) = \sum t_n z^n$ converges in $|z| = \bar{R}_0 \leq R_0$. Set $f(z) = (t_n z^n)_0^\infty$

$|z| = R_0^{1/\nu}$. $\frac{1}{\nu} \in \mathbb{R}_+$ fixed $\sum \frac{t_n}{n!} z^n$ converges for $z = R_0$. set

$$\beta(\frac{1}{\nu}, z) = \left(\frac{t_n}{n!} z^n \right)_0^\infty \quad (\frac{1}{\nu}, |z| = R_0)$$

Borel's integral derived from $f(z)$ is $B[f(z)] = \int_0^\infty e^{-\frac{t}{z}} \beta(\frac{1}{\nu}; t) dt$

$B[f(z)]$ represents $f(z)$ within \mathbb{D}

N1. $L[0, z_0]$ line joining $z=0, z=z_0$

L1. If $B[f(z)]$ abs. converg. for $z = z_0 + t_0 e^{i\theta_0}$ ($t_0 \in \mathbb{R}$), also abs. converg. for $z = t_0 e^{i\theta_0}$. Furthermore converges for all z lying in \mathbb{D} with $t_0 e^{i\theta_0}$ as diam., provides anal.

cont. of $f(z)$.

L2. $\exists \epsilon$ arb. small, K' \mathbb{O} centre $\frac{1}{2}z_0$ ($|z_0| \in \mathbb{P}$) finite radius $> |\frac{1}{2}z_0| + \delta$

As z' describes K' , points z in wh. $\operatorname{Re}(\frac{z}{z'}) < 1 - \delta$ are found in \mathbb{C} upon

ellipses with focus at and points of $t_0 e^{i\theta_0}$

L3. $f(z)$ reg. in and upon $\mathbb{O} K$ diameter $t_0 e^{i\theta_0}$, Borel's int. $B[f(z)]$; abs. converg.

for $z = t_0 e^{i\theta_0}$.

D2. Through sing. pt. z_0 of $f(z)$ const. line $l_0 \perp$ to $t_0 e^{i\theta_0}$, \mathbb{H}_{z_0} half plane containing

origin bounded by l_{z_0} . Intersection all \mathbb{H}_{z_0} polygons of summability of $f(z)$.

Then $B[f(z)]$ converges uniformly in any domain lying in polygon of summability of $f(z)$.

Ex1 $f(z) = \sum n^2 B[f(z)] = \int_0^\infty e^{-\frac{t}{z}} \frac{1}{t} dt$ poly. of summ. $\operatorname{Re}(z) \geq R'$

Ex2 $f(z) = \sum 2z^n B[f(z)] = \int_0^\infty \left\{ e^{-\frac{t}{z}} + e^{-\frac{t}{2z}} \right\} dt$ poly. of summ. $\operatorname{Re}(z) \geq R'_-$

7.8 Let $\{t_i\} = M\mathbb{Z}[\zeta_{j_n}^b]$, $\bar{\mathcal{R}}_a^b \subseteq R$ $\bar{\mathcal{R}}_{\zeta_{j_n}^b}^{(b-1)} \subseteq \mathcal{R}_{\zeta_{j_n}^b}^{(b)}$: set $f(z) = \sum t_i z^i$ 63

$$f(z) = S [(-zs)^{-1} : \zeta]_n^b$$

Borel's polygon for $f(z)$ is a) $a \in \underline{\mathcal{R}}_0$, $Re(z) = \mathcal{R}_a^{b-1}$ b) $b \in -\bar{\mathcal{R}}_0$, $Re(z) = \mathcal{R}_{a-1}$

c) $0 \in \mathcal{R}_a^b$ $Re(z) = \mathcal{R}_{a-1}^{b-1}$. In all cases $\mathcal{B}[f(a)] = f(z)$ when z lies in Borel poly.

$$\rho(\zeta; z) = S [e^{-\frac{1}{2}\zeta z} : \zeta]_n^{-b}$$

12.4.2. Watson-Korantinna lemma

of all $G(\tau)$'s generating $f(\tau)$, one preserves prop. P. If transformation producing function with P, $G(\tau)$ being determined

Ch. B. The convergence of associated and corresponding continued fractions⁶⁴

13.1 The associated continued fractions generated by Hamburger-Nevanlinna functions

$$\mathcal{F}^{(1)}(\lambda) = \mathcal{F}(\lambda); \quad \mathcal{F}^{(d+1)}(\lambda) = \frac{\nu_d}{\lambda - w_d - \mathcal{F}^{(d)}(\lambda)} \quad d \in \mathbb{I},$$

$$A\{\mathcal{F}(\lambda)\} = C \left[\frac{\nu_d}{\lambda - w_d -} \right]$$

Th. 1. Let $\bar{\mathcal{E}}_1(\lambda) = \sum t_{1,\nu} \lambda^{-\nu-1}$ be asymptotic expn. in $\lambda \in \Lambda_0^{\bar{\pi}} \cup \Lambda_{-\bar{\pi}}$

\Rightarrow HN fn. $F_1(\lambda) = ST[\lambda/\varepsilon_1]_{a_1}^{b_1}$ ($\bar{R}_{a_1}^{b_1} \subseteq \bar{R}$). Set $v_1 = t_{1,0}$, $w_1 = \frac{t_{1,1}}{t_{1,0}}$.

Let $\bar{\mathcal{E}}_2(\lambda) = \sum t_{2,\nu} \lambda^{-\nu-1}$ be derived from $\mathcal{F}_1(\lambda) = \frac{v_1}{\lambda - w_1 - \mathcal{F}_2(\lambda)}$,

$F_2(\lambda)$ be derived from $\frac{F_1(\lambda)}{\lambda - w_1 - F_2(\lambda)} = \frac{v_1}{\lambda - w_1 - F_2(\lambda)}$. Then $F_2(\lambda)$ is HN fn. $ST[\lambda/\varepsilon_2]_{a_2}^{b_2}$

$\bar{R}_{a_2}^{b_2} \subseteq \bar{R}_{a_1}^{b_1}$, $\bar{\mathcal{E}}_2(\lambda)$ is its asympt. expn. in $\lambda \in \Lambda_0^{\bar{\pi}} \cup \Lambda_{-\bar{\pi}}$

D1. If in $F(\lambda) = ST[\lambda/\varepsilon]_{a_\varepsilon}^{b_\varepsilon}$, ε simple step fn., $F(\lambda)$ is degen. HN-fn.; ε not such that $F(\lambda)$ non-degen. HN fn.

Th. 2. $\bar{\mathcal{E}}_1(\lambda)$ asympt. expn. of $F_1(\lambda)$ as in prev. th. Constr. seq. of series $\bar{\mathcal{F}}_T(\lambda)$

$$= \sum t_{T,\nu} \lambda^{-\nu-1} \quad \text{as } T \rightarrow \infty$$

$$\bar{\mathcal{F}}_T(\lambda) = \frac{v_T}{\lambda - w_T - \bar{\mathcal{F}}_{T+1}(\lambda)} \quad F_T(\lambda) = \frac{v_T}{\lambda - w_T - F_{T+1}(\lambda)} \quad v_T = t_{T,0} \quad w_T = \frac{t_{T,1}}{t_{T,0}}$$

for all T st. $\bar{\mathcal{F}}_T(\lambda), F_T(\lambda)$ def. Either

- $F_1(\lambda)$ is deg. HN fn., $\exists T \in \mathbb{I}$ st above process terminates with $F_T(\lambda), F_{T+1}(\lambda)$ both id. zero $v_{T+1} = 0$

- $F_1(\lambda)$ non-degen; above process prolonged identically.

In both cases: $\bar{\mathcal{F}}_T(\lambda) = ST[\lambda/\varepsilon_T]_{a_T}^{b_T}$, $\bar{R}_{a_m}^{b_m} \subseteq \bar{R}_{a_T}^{b_T}$, $\bar{\mathcal{F}}_T(\lambda)$ asympt. expn. of $F_T(\lambda)$ in $\Lambda_0 \cup \Lambda_{-\bar{\pi}}$, $v_T > 0$

B.2. The sequence of nested circular value regions

Th. 3. $\left\{ \frac{q_r(\lambda)}{p_r(\lambda)} \right\}$ successive coeffs. of assoc. c.f. gen by $F(\lambda)$ HN fm; for $\operatorname{Im}(\lambda) > 0$ value λ $F(\lambda)$ lies within $\odot K_r$, centre at point

$$\frac{\operatorname{Im} \{ p_{r+1}(\lambda) \overline{q_r(\lambda)} + p_r(\lambda) q_{r+1}(\lambda) \}}{2 \operatorname{Im} \{ p_{r+1}(\lambda) \overline{p_r(\lambda)} \}}, \quad \frac{\operatorname{Re} \{ p_{r+1}(\lambda) \overline{q_r(\lambda)} - p_r(\lambda) \overline{q_{r+1}(\lambda)} \}}{2 \operatorname{Im} \{ p_{r+1}(\lambda) \overline{p_r(\lambda)} \}}$$

radius $\frac{|p_r(\lambda) q_{r+1}(\lambda) - p_{r+1}(\lambda) q_r(\lambda)|}{2 \operatorname{Im} \{ p_{r+1}(\lambda) \overline{p_r(\lambda)} \}}$

Th. 4. $\{K_r\}$ and $\left\{ \frac{q_r(\lambda)}{p_r(\lambda)} \right\}$ as above: K_r lies inside K_{r+1} , K_r passes through $\frac{q_{r+1}(\lambda)}{p_{r+1}(\lambda)}, \frac{q_r(\lambda)}{p_r(\lambda)}$, touches K_{r+1} at $\frac{q_{r+1}(\lambda)}{p_{r+1}(\lambda)}$.

Th. 5. If HN-fn. $F(\lambda)$ goes. converg. at. 2th corrugt $C \left[\frac{n_m}{\lambda - i\omega} \frac{n_m}{\lambda - i\omega} \right]_x$, then when $\operatorname{Re} \operatorname{Im}(\lambda) > 0$ then corrugt + two poles/roots lie on K_r

Th. 6. Radius $\odot K_r$ is $\{ 2 \operatorname{Im}(\lambda) \left(\frac{|\lambda|^{2+1}}{\prod_{r'=1}^r V_{r'}} \right)^{1/(2+1)} \}^{-1}$.

Series $\sum_{r=1}^{\infty} \frac{|\lambda|^{2+1}}{\prod_{r'=1}^r V_{r'}}$ are real numbers. Series diverges $\operatorname{Im}(\lambda) \neq 0$ radii

$\odot K_r \rightarrow 0$ limit pt. case. $A \{ \mathcal{T}(n) \} \rightarrow F(\lambda)$ if exists $\rightarrow F(\lambda)$

series converges $\{K_r\} \rightarrow$ hard position limit \odot case.

~~limit pt. case \leftrightarrow recovered from $F(\lambda)$~~

Starting with $\{t_n\}$: limit pt. case $A \{ \mathcal{T}(\lambda) \} \rightarrow F(\lambda)$, \leftrightarrow recovered from $F(\lambda)$.

Limit \odot case: $p_{2k} p_{2m} q_{2k} q_{2m} \rightarrow p q p' q'$; $F'(\lambda)$ any HN fm

value of $\frac{q(\lambda) - q'(\lambda) F'(\lambda)}{p(\lambda) - p'(\lambda) F'(\lambda)}$ lies within upon unit \odot . Expression

rep. HN fm generating $\{t_n\}$

Limit \odot case: mon prob. det.; Limit \odot case: mon prob. indept.

D2. If λ lies within any domain within finite dist. of origin, belongs to anti pr. spectrum of $\sigma \in M\bar{D}(\bar{\mathbb{R}}_a^b)$, we write $\lambda \in BEPS[\epsilon]$

Th. 7. Let set $\{ST[\lambda/\epsilon]_a^b\}$ converge for $\lambda \in BE(\bar{\mathbb{R}})$, then the c.f. also converges for $\lambda \in BEPS[\epsilon]$

B.2. The corresponding continued fractions generated by Stieltjes functions

D3. If in $S(\lambda) = ST[\lambda/\epsilon]_a^b$ defining S-fn. S, ϵ is simple step fn., S is degen. S-fn., if $\sigma \in M\bar{D}(\bar{\mathbb{R}}_a^b)$ not such fn., then S is nondegen. S-fn.

Th. 8. $S(\lambda)$ gen. $C\left[\frac{w_{2r-1}}{\lambda -} \frac{w_{2r}}{1 -}\right]$. Either

- 1) S is degen.; if $r \in I_1$, c.f. terminates with w_{2r} $w_{2r} \in \bar{P}(\lambda =)$,
- 2) S nondegen., c.f. non terminating $w_{2r} \in \bar{P}(\lambda = I_1)$

One of λ_r in D3 zero: r odd; if none + (even)

Circles derived from complement of $\{S(\lambda)\}$ denoted by K_{2r}' . Those derived from even part of $S[\lambda/\epsilon]_a^b$ K_{2r}''. Let $K_{2r}'' = \tan^{-1} \lambda^{-1} K_{2r+1}'$. K_{2r+1} lies within K_{2r}'' touching it at C_{2r+1}, K_{2r}' K_{2r}'' intersect at arg(λ). Superimpose two systems obtain nested convex regions

Th. 9. Set $C_r = [C\{S(\lambda)\}]_r$; $\lambda \in BE(\bar{\mathbb{R}}_0)$. Os K_{2r}' K_{2r}'' line through origin making $\angle \frac{1}{2}\arg(\lambda)$ with the real axis. K_{2r+1} lies within K_{2r}'' touching it at C_{2r+1}. Value of S(λ) lies within each K_{2r}''

Th. 10. Given $\sigma \in M\bar{D}(\bar{\mathbb{R}}_a^b \subseteq \bar{\mathbb{R}}_0)$, set $\epsilon'' = \epsilon b/a$. If $b/\lambda \geq s > 0$ (s ab small)

λ belongs to anti pseudo spectrum of $\sigma + \epsilon''$ with $\lambda \in BEPS(0, \epsilon, \epsilon'')$

Th. 10. Let $\{ST[\lambda/\epsilon]_a^b\}$ converge. uniformly for $\lambda \in BE(\bar{\mathbb{R}}_0)$, then the

c.f. also converges uniformly for $\lambda \in BE(0, \epsilon, \epsilon'')$

13.3. Convergence criteria expressed in terms of the coefficients of the continued fraction

$$\text{Th 11. Non deg. HN fn } F(\lambda) = ST[\lambda/\epsilon]_a^b, A\{F(\lambda)\} = C\left[\frac{v_2}{\lambda - v_2} - \bar{J}\right]. \text{ If } \sum_{i=1}^{\infty} v_i^{-\frac{1}{2}}$$

diverges, c.f. converges uniformly to $F(\lambda)$ for $\lambda \in \overline{BEPs}[\epsilon]$

$$\text{Th 12. Non deg. S fn } S(\lambda) = ST[\lambda/\epsilon]_a^b \text{ given } C\left[\frac{u_{2j+1}}{\lambda - u_{2j}} - \bar{J}\right]. \text{ If}$$

$$\sum_{i=1}^{\infty} u_i^{-\frac{1}{2}} \text{ diverges, } C\{S(\lambda)\} \rightarrow S(\lambda) \text{ for } \lambda \in \overline{BEPs}[0, \epsilon, \epsilon'')$$

13.4. Convergence criteria expressed in terms of the coefficients of the series

L1. If $\sum_{i=1}^{\infty} s_i$ ($s_i \in \mathbb{P}, i=1, \dots$) converges, series $\sum_{i=1}^{\infty} (s_1 s_2 \dots s_i)^{\frac{1}{i}}$ also converges

$$\text{Th 13. Non deg. HN fn. } F(\lambda) = ST[\lambda/\epsilon]_a^b \text{ given series } \Xi(\lambda) = \sum t_i \lambda^{-i-1}.$$

If $\sum_{i=1}^{\infty} t_i^{-\frac{1}{2}}$ diverges, $A\{F(\lambda)\} \rightarrow F(\lambda)$ $\lambda \in \overline{BEPs}[\epsilon]$.

c1. Let $\{t_i\} \in M\mathbb{S}[\epsilon]_a^b$ $\bar{R}_a^b \subset \mathbb{R}$. ~~$A\{F(\lambda)\} = ST[\lambda/\epsilon]_a^b \rightarrow F(\lambda)$~~ $\lambda \in \overline{BEPs}[\epsilon]$.

c2. Let $\{t_i\} \in M\mathbb{S}[\epsilon]_a^b$. Suppose that $\exists G, \eta \in \mathbb{P}$ $h \in \mathbb{J}$, st. $t_i \leq G(2h)! \eta^{-2h}$ $i=1$. $A\{F(\lambda) = ST[\lambda/\epsilon]_a^b\} \rightarrow F(\lambda)$ $\lambda \in \overline{BEPs}[\epsilon]$

$$\text{Th 14. Non deg. S fn } S(\lambda) = ST[\lambda/\epsilon]_a^b \text{ given. } \Xi(\lambda) = \sum t_i \lambda^{-i-1} \text{ If}$$

$\sum_{i=1}^{\infty} t_i^{-\frac{1}{2}}$ diverges, $C\{\Xi(\lambda)\} \rightarrow S(\lambda)$ $\lambda \in \overline{BEPs}[0, \epsilon, \epsilon'')$

c1. Let $\{t_i\} \in M\mathbb{S}[\epsilon]_a^b$ where $\bar{R}_a^b \subset \mathbb{R}_0$. $C\{S(\lambda) = ST[\lambda/\epsilon]_a^b\} \rightarrow S(\lambda)$ $\lambda \in \overline{BEPs}[0, \epsilon, \epsilon'')$

c2. Let $\{t_i\} \in M\mathbb{S}[\epsilon]_a^b$, so $\bar{R}_a^b \subseteq \bar{R}_0$. Suppose $\exists G, \eta \in \mathbb{P}$, $h \in \mathbb{J}$, st. $t_i \leq G(2h+1)! \eta^{-2h-1}$ $C\{S(\lambda) = ST[\lambda/\epsilon]_a^b\} \rightarrow S(\lambda)$ $\lambda \in \overline{BEPs}[0, \epsilon, \epsilon'')$

Th. 15 Let $\{t_j\} \in MS[\zeta]_n^b$, $\{\hat{t}_j\} \in MS[\hat{\zeta}]_n^b$ $\bar{R}_n^b \subseteq \bar{R}$. Suppose at ⁶⁸
 pts. of discont. $\eta = c$, $\epsilon(s+) - \epsilon(s-) \geq 0$ [$\hat{\epsilon}(s+) - \hat{\epsilon}(s-)$], at pts where
 $\epsilon(s)$ cont. $\frac{d\epsilon(s)}{ds} \geq 0$ $\frac{d\hat{\epsilon}(s)}{ds}$ ($\lambda \in P$). If $\sum_{j=1}^{\infty} t_{2j}^{-\frac{1}{2}}$ diverges, so does
 $\sum_{j=1}^{\infty} \hat{t}_{2j}^{-\frac{1}{2}}$. If $\bar{R}_n^b \subseteq \bar{R}_0$, when $\sum_{j=1}^{\infty} t_j^{-\frac{1}{2}}$ diverges $\sum_{j=1}^{\infty} \hat{t}_j^{-\frac{1}{2}}$ also.
 diverges

13.5 Theorem D occurs

13.5.1. The logarithmic derivatives of certain functions

Th. 16. Let $d_\nu, z_\nu \in \mathbb{R}$ ($\nu = 1$) $|z_\nu| \in \mathbb{R}_S \subseteq \mathbb{R}_{\text{hyp}}$ ($\nu = 1$), $\exists h \in \mathbb{I}$, such

that $\sum d_\nu z_\nu^{-h-1}$ converges absolutely. Let $g(z) =$ Define $g(z)$ by

$$g_1(z) = \prod_{\nu=0}^{\infty} \left(1 - \frac{z}{z_\nu}\right)^{d_\nu} e^{d_\nu \ln \left(\frac{z}{z_\nu} + \frac{1}{2} \frac{z^2}{z_\nu^2} + \dots + \frac{1}{h} \frac{z^h}{z_\nu^h}\right)}$$

$$g(z) = -z^{1-h} \frac{D_z g_1(z)}{g_1(z)} \quad \text{Then } g(z) = \left(\frac{d_\nu z_\nu^{-h-1}}{1-z_\nu^{-1}} \right)_0^{\infty} \quad z = BE(\bar{R}_{\text{hyp}}^{+})$$

i) even: a) $z_\nu > 0$ /n $d_\nu > 0$, $z_\nu < 0$ /n $d_\nu < 0$ $\nu = 1$ then $F(\lambda) = g(\lambda')$

$\$$ is HN - h

b) $z_\nu > 0$ /n $d_\nu < 0$ $z_\nu < 0$ /n $d_\nu > 0$ ($\nu = 1$) then $\bar{F}(\lambda) = -g(\lambda')$ HN /n

2) odd: a) $d_\nu > 0$ ($\nu = 1$) then $F(\lambda) = g(\lambda')$ is HN /n

b) $d_\nu < 0$ ($\nu = 1$) $F(\lambda) = -g(\lambda')$ is HN /n.

In all cases $\{F(\lambda)\}$ converges for $\lambda = BE[n \bar{R}_{\text{hyp}}^{-1}]$

If even all $\{d_\nu\}$ same sign, $\{z_\nu\}$ both signs: form $\hat{g}_1(z) = e^{\frac{A z}{h+1}} g_1(z)$

$A = (d_\nu z_\nu^{-h-1})_0^{\infty}$, applies Theorem (h even odd).

Th. 17 $g(z)$ as in Th. 16, Define $S(\lambda)$ as follows

- i) even, for $\nu = 1$ a) $d_\nu > 0$ $z_\nu > 0$ $S(\lambda) = g(\lambda')$ b) $d_\nu < 0$ $z_\nu \neq 0$, $S(\lambda) = g(-\lambda')$
- c) $d_\nu < 0$ $z_\nu > 0$ $S(\lambda) = -g(\lambda')$ d) $d_\nu > 0$ $z_\nu < 0$ $S(\lambda) = -g(-\lambda')$
- ii) odd a) $S(\lambda) = g(\lambda')$ b) $S(\lambda) = -g(-\lambda')$ c) $S(\lambda) = -g(\lambda')$ d) $S(\lambda) = g(-\lambda')$

In all cases $S(\lambda)$ is an S-function, generating S-function converges for $\lambda \in BE(\bar{R}_0^{\delta})$

$$\text{Ex. 1 } g_1(z) = z^{-\frac{1}{2}} \sin(z^{\frac{1}{2}}) = \prod_{j=0}^{\infty} \left\{ 1 - \frac{z}{(2j+1)^{\frac{1}{2}}} \right\}$$

$$g(z) = \frac{1}{2} \left\{ z^{\frac{1}{2}} \cot(z^{\frac{1}{2}}) - 1 \right\} \quad g(\lambda^{-1}) = \left(\frac{-(\lambda+1)^{-\frac{1}{2}} \pi^{-2}}{1 - (\lambda+1)^{\frac{1}{2}} \pi^{-2}} \right)^{\frac{1}{2}}$$

$S(\lambda) = -g(\lambda^{-1})$ is an S-function & $\{S(\lambda)\}$ converges for $\lambda \in BE(\bar{R}_0^{\delta-2})$

$$\text{Ex. 2. } g_1(z) = z e^{z^2} \Gamma(z) = \prod_{j=0}^{\infty} \left\{ 1 + \frac{z}{2j+1} \right\}^{-1} e^{\frac{z^2}{2j+1}}$$

$$g(z) = \frac{\partial_z \Gamma(z)}{\Gamma(z)} + z^{-1} + \gamma \quad g(\lambda^{-1}) = \left(\frac{(\lambda+1)^{-2}}{\lambda+1} \right)^{\frac{1}{2}} \quad S(\lambda) = -g(-\lambda^{-1})$$

J-th. $\{S(\lambda)\}$ converges uniformly for $\lambda \in BE(\bar{R}_0')$

13.5.2. Laplace transforms of totally monotone functions

$$\text{Th 18. Let } S(\lambda) = \int_1^{\infty} \frac{\psi_0(\frac{1}{z})}{\psi_1(\frac{1}{z})} e^{\lambda \frac{1}{z}} dz \quad (\operatorname{Re}(\lambda) = \bar{R}_0) \text{ where}$$

$$1) \psi_0(\frac{1}{z}) = \prod_{j=0}^{h'} \left\{ \frac{1 + \frac{1}{z_j}}{1 + \frac{1}{z_j''}} \right\}^{d_j} \bar{R}_{\frac{1}{z_j''}} \subseteq \bar{R}_{\delta}, \quad (\delta' \in \mathbb{P}) \quad d_j \in \bar{R}_0 \quad (j=0^{h'})$$

with, if $h=\infty$, $\sum d_j \frac{1}{z_j''}^{-1}$ converges.

$$2) \psi_1(\frac{1}{z}) = \prod_{j=0}^{h'} \left(1 + \frac{1}{z_j''} \right)^{y_j}, \quad z_j'' \in \bar{R}_{\delta''}, \quad (\delta'' \in \mathbb{P}) \quad y_j \in \bar{R}_0 \quad (j=0^{h'})$$

with if $h'=\infty$ $\sum y_j \frac{1}{z_j''}^{-1}$ converges

$$3) \psi_2(z) \text{ analytic for } |z| \in \bar{R}_0^{\delta'''}, \quad (-1)^j D_z^j \psi_2(\frac{1}{z}) \in \bar{R}_0 \quad (\frac{1}{z} \in \bar{R}_0 \setminus \{0\})$$

then $S'(\lambda)$ is an S'-fn. $\{S'(\lambda)\} \rightarrow S'(\lambda) \quad (\lambda \in BE(\bar{R}_0))$

$$\text{Ex. } \psi_0(\frac{1}{z}) = 1 \quad \psi_1(\frac{1}{z}) = 1 + \frac{1}{z} \quad \psi_2(\frac{1}{z}) = 0 \quad S'(\lambda) = \int_0^{\infty} \frac{e^{-\lambda \frac{1}{z}}}{1 + \frac{1}{z}} dz$$

13.6. The equivalence between the method of Boole and the use of continued fractions

N1 Suppose $\bar{R}_a^b \subset R$. If when $a > 0 = b > 0 \in BE(\bar{R}_{a-1}^{b-1})$ or when $a < 0, b > 0 \in BE(\bar{R}^{b-1}) \cap BE(\bar{R}_{b-1})$ then we write $z \in CD(a, b)$.

Th 19. Let $f(z) = \sum b_j z^j$ $\{b_j\} = M\bar{R}_a^b \in MD(\bar{R}_a^b)$ $\bar{R}_a^b \subset R$

i) Boole's polygon derived from $f(z)$ is included in convex domain of $f(z)$
and $\{f(z)\}$ if it exists

ii) Within Boole polygon $B\{f(z)\} \supset f(z) = \Re \left[(1 - z)^{-1} : z \right]$

iii) $A\{f(z)\} \rightarrow f(z) \rightarrow \{f(z)\}$ (if finite) $\rightarrow f(z)$

$$\text{Dir. series } \frac{g_r(\lambda)}{p_r(\lambda)} = \left(\frac{\mu_{n,r}}{\lambda - \lambda_{n,r}} \right)_r, \quad \frac{1}{\lambda - \lambda_{n,r}} = \int_0^{-\infty} e^{-\lambda s} (\lambda - \lambda_{n,r}) ds \quad (r=1, 2)$$

convg. abs. & unif. for $\Im(\lambda) = R_S$ ($S \in \mathbb{P}$ and small)

$$\frac{g_r(\lambda)}{p_r(\lambda)} = \left(\mu_{n,r} \int_0^{-\infty} e^{-\lambda s} (\lambda - \lambda_{n,r}) ds \right)_r = \int_0^{-\infty} v_r(s) e^{-\lambda s} ds$$

$$v_r(s) = (\mu_{n,r} e^{\lambda_{n,r} s})_r$$

L.2. Let $S^{(2r+1)}(\lambda_1, \lambda_2, \dots, \lambda_r)$ ($\nu, \nu' \in \mathbb{I}$) be symm. homog. sum of terms of

form $\lambda_1^{\nu_1} \lambda_2^{\nu_2} \lambda_r^{\nu_r}$ in wh. $(\nu_r)_r = 2r+1$ all pos. dist. comb. of $\{\nu_r\}$ being

taken into account coefft of each term being 1. If $\lambda_r \in R$ ($r \in \mathbb{I}_1$), then

$$S^{(2r+1)}(\lambda_1, \lambda_2, \dots, \lambda_r) \geq 0$$

L.3. Let $\hat{p}_{2r+1}(\lambda)$ be polyam. degr. $2r+1$ coefft of λ^{2r+1} , real roots. If

$\sum t_{2r+1,\nu} (\lambda^{-1})^\nu$ is series expan of $\lambda^{2r+1} p_{2r+1}(\lambda)^{-1}$ in inverse powers of

λ , then $t_{2r+1,\nu} \geq 0$ ($\nu \in \mathbb{I}$)

L4. re \mathcal{G} , fixed. $\sum t_{r,s} \lambda^{-s-1}$ series expn. \mathcal{G} in inverse powers of λ \Rightarrow

if $\frac{g_r(\lambda)}{p_r(\lambda)} = C \left[A \left\{ \sum t_{r,s} \lambda^{-s-1} \right\} \right]_r = \sum b_s \lambda^{-s-1}$ given by HN then

$$0 < t_{r,s+1} \leq t_{r,s}, \quad (s=0)$$

L5. In above notation $|t_{r,s+1}| \leq \sqrt{t_{r,s} t_{r,s+2}} \quad (s=0)$

L6. Suppose coeffs of $\mathcal{F}(\lambda) = \sum b_s \lambda^{-s-1}$ given by HN to satisfy

$t_{r,s} \leq G (s!) \eta^{-s-1} \quad (s=0)$, $G, \eta \in \mathbb{R}$ fixed. Let

$$C \left[A \left\{ \mathcal{F}(\lambda) \right\} \right]_r = \int_0^{-i\infty} v_r(\frac{1}{z}) e^{-z\lambda} \quad \text{Im } \lambda = R_\delta \quad (\delta \in \text{Part small})$$

$$= \left(\frac{M_{r,s}}{\lambda - \alpha_{r,s}} \right)_r \quad (\lambda_{r,s} \in \mathbb{R}, M_{r,s} \in \mathbb{R}, \delta = \delta_r)$$

$$v_r(\frac{1}{z}) = (M_{r,s} e^{\lambda_{r,s} \frac{1}{z}})$$

$$\text{Then } v_r(\frac{1}{z})|_{z=0} = v(\frac{1}{z}) \quad (|\frac{1}{z}| \in \mathbb{R}_0^{s+1}) \quad v(\frac{1}{z}) = \left(\frac{t_{r,s}}{z!} \frac{1}{z} \right)_0^{\infty} |\frac{1}{z}| = \bar{R}_0^{s+1}$$

Must show $v_r(\frac{1}{z})|_{z=0} \rightarrow v(\frac{1}{z})$ in all $\frac{1}{z}$ in imag axis.

L7. $\{v_r(\frac{1}{z})\}$ as above, $\frac{1}{z} = i\gamma$ pure imag., $r \in \mathcal{I}$ fixed

$$|\partial_{\frac{1}{z}}^{2r'} v_r(\frac{1}{z})| \leq G \frac{(2r')!}{(\eta - \frac{\gamma}{4})^{2r}}, \quad |\partial_{\frac{1}{z}}^{2r'+1} v_r(\frac{1}{z})| \leq G \sqrt{2} \frac{(2r+1)!}{(\eta - \frac{\gamma}{4})^{2r+1}} \quad (s=0)$$

L8. For $v_r(\frac{1}{z})$ above \rightarrow limit when $\frac{1}{z}$ lies in rectangle Q given by

$$R(\frac{1}{z}) = \bar{R} \eta^{-\frac{1}{2}\delta}, \quad \text{Im } \frac{1}{z} = \bar{R}_{-\gamma}^{\gamma} \quad (\gamma \in \mathbb{R} \text{ part large}); \quad v(\frac{1}{z}) \text{ is analytic}$$

bounded in Q .

L9. If $\{t_{r,s}\}$, coeffs of series gen by HN for $F(\lambda)$ in inverse powers of λ ,

sat. inequalities of L6, then $F(\lambda)$ has integral repn. of form

$$F(\lambda) = \int_0^{-i\infty} v(\frac{1}{z}) e^{-\lambda \frac{1}{z}} dz \quad (\text{Im } \lambda = R_\delta), \quad F(\lambda) = \int_0^{i\infty} v(\frac{1}{z}) e^{-\lambda \frac{1}{z}} (\bar{J}_n(\lambda)) e^{-\bar{R}_{\frac{1}{z}}} dz$$

$\delta \in \text{Part large}$.

Th. 21. Let $\{t_{ij}\}$, coefft. of series expn. of $S'/m S(\gamma)$ in decr. power γ > 72

whchly $t_{ij} \in G'(2), \gamma \mapsto (\gamma + i) \quad G', \gamma' \in P$ fixed, then $S(\gamma)$ has int. rep.

$$S(\gamma) = \lambda^{-\frac{1}{2}} \int_0^{-\infty} v'(\frac{1}{3}) e^{-\gamma^{\frac{1}{2}} \frac{v}{3}} dv \quad (\lambda = \text{BE}(\bar{R}_0))$$

13.7 A characterisation of Stieltjes fractions

Th. 22. $C \left[\frac{w_{2r-1}}{\gamma +} \frac{w_{2r}}{1+} \right]_{\gamma r} = \text{const. comp. c.f. given by } S'/m \quad S'(\gamma) =$

$$S[(\gamma + 3)^{-1}; \epsilon]_0^\infty, \quad \gamma \in P$$

$$S'(\gamma) = C \left[\frac{w_{2r-1}}{\gamma +} \frac{w_{2r}}{1+} \right]_{\gamma r} = \min_{x_1, \dots, x_r} S \left[\frac{\{1 + (x_0(\gamma + s)^r),\}^r}{\gamma + s} : \epsilon \int_0^\infty \right]^{1/r} \quad (r = 1,)$$

$$C \left[\frac{w_{2r-1}}{\gamma +} \frac{w_{2r}}{1+} \right]_{\gamma r H} - S'(\gamma) = \min_{x_1, \dots, x_r} S \left[\frac{\{1 + (x_0'(\gamma + s)^r),\}^r}{\gamma + s} : \epsilon \int_0^\infty \right]^{1/r},$$

Th. 23 $\exists'(\gamma) = \sum (-1)^r t_r \gamma^{-r-1}$ series expn. of $S'/m \quad S'(\gamma) = S[(\gamma + 3)^{-1}; \epsilon]$

let $b \{ \exists'(\gamma) \} = C \left[\frac{w_{2r-1}}{\gamma +} \frac{w_{2r}}{1+} \right]. \quad \text{When } \gamma \in P$

$$((-1)^r t_r \gamma^{-r-1})_0^{2r-1} < C \left[\frac{w_{2r-1}}{\gamma +} \frac{w_{2r}}{1+} \right]_{\gamma r} < S'(\gamma) \quad (r = 2)$$

$$((-1)^r t_r \gamma^{-r-1})_0^{2r} > C \left[\frac{w_{2r-1}}{\gamma +} \frac{w_{2r}}{1+} \right]_{\gamma r H} > S'(\gamma)$$

majorant prop. $\exists T[\gamma / \epsilon]_0^\infty$ gen. comp. const/cont. enough $\gamma = \text{BE}(\bar{R}_0^b)$

$\hat{S}(\gamma) = \exists T[\gamma / \hat{\epsilon}]_0^\infty$ another J-fraction st. at pts. of discontin. $\hat{\epsilon} \in$

$\epsilon(s+) - \epsilon(s-) > 0 \quad \{ \hat{\epsilon}(s+) - \hat{\epsilon}(s-) \}$, at pts. of cont. $\hat{\epsilon} \in$

$\frac{d\hat{\epsilon}(s)}{ds} \geq \frac{d\epsilon(s)}{ds} \quad \hat{\epsilon} \in P$. \hat{S} also gen. comp. cf. const. unit
for $\gamma = \text{BE}(\bar{R}_0^b)$.

Ch 14. The diagonal sequences of the Padé table and the even order e-array. 73

$$f(z) = \sum b_j z^j \rightarrow f^{(m)}(z) = \sum t_{m,j} z^j \quad (m=2). \quad H_{m,r} \neq H_{m+1,r} \neq 0 \quad r = \mathbb{J}: \mathcal{L}\{f^{(m)}(z)\} = C \left[\frac{\bar{u}_1^{(m)} z^m}{1 - \bar{w}_1^{(m)} z} \right]$$

$$C \left[\sum_{i=0}^{m-1} b_i z^i; \frac{\bar{u}_1^{(m)} z^m}{1 - \bar{w}_1^{(m)} z} \right]_{2r} = R_{1,m+r-1}(z), \quad C \left[\sum_{i=0}^{m-1} b_i z^i; \frac{\bar{u}_1^{(m)} z^m}{1 - \bar{w}_1^{(m)} z} \right]_{2m} = R_{1,m+r}(z)$$

$$H_{m,r} \neq 0 \quad (r = \mathbb{J}) \quad \mathcal{L}\{f^{(m)}(z)\} = C \left[\frac{\bar{v}_1^{(m)}}{1 - \bar{w}_1^{(m)} z} - \frac{\bar{v}_2^{(m)} z}{1 - \bar{w}_2^{(m)} z} \right]$$

$$C \left[\sum_{i=0}^{m-1} b_i z^i; \frac{\bar{v}_1^{(m)} z^m}{1 - \bar{w}_1^{(m)} z} - \frac{\bar{v}_2^{(m)} z}{1 - \bar{w}_2^{(m)} z} \right]_r = R_{1,m+r-1}(z) \quad (r, m > \mathbb{J})$$

$$\tilde{f}(z) = \sum b_j z^j \text{ from } \tilde{f}(z) \tilde{f}'(z) = 1. \quad H_{m,r} \neq 0 \quad H_{m+r} \neq 0 \quad (r = \mathbb{J})$$

$$C \left[\sum_{i=0}^{m-1} b_i z^i; \frac{\bar{u}_1^{(m)} z^m}{1 - \bar{w}_1^{(m)} z} \right]_{2r}^{-1} = R_{m+r-1,r}(z), \quad \{C \left[\sum_{i=0}^{m-1} b_i z^i; \frac{\bar{u}_1^{(m)} z^m}{1 - \bar{w}_1^{(m)} z} \right]_{2m}\}^{-1}$$

$$\tilde{F}(z) = \sum b_i z^i \tilde{\lambda}^{-j-1} \rightarrow \tilde{F}^{(m)}(z) = \sum t_{m,j} \lambda^{-j-1} \quad (m=2). \quad H_{m,r}, H_{m+r} \neq 0 \quad \mathcal{L}\{F^{(m)}(z)\} = C \left[\frac{\bar{u}_1^{(m)} z^m}{\lambda - \bar{w}_1^{(m)} z} \right]$$

$$H_{m,r} \neq 0 \quad (r = \mathbb{J}) \quad \mathcal{L}\{\tilde{F}^{(m)}(z)\} = C \left[\frac{\bar{v}_1^{(m)}}{\lambda - \bar{w}_1^{(m)}} \right]$$

$$C \left[\sum_{i=0}^{m-1} b_i z^i; \frac{\bar{v}_1^{(m)} z^m}{1 - \bar{w}_1^{(m)} z} - \frac{\bar{v}_2^{(m)} z}{1 - \bar{w}_2^{(m)} z} \right]_r = \lambda C \left[\sum_{i=0}^{m-1} b_i \tilde{\lambda}^{-j-1}; \frac{\bar{v}_1 \tilde{\lambda}^{-m}}{\lambda - \bar{w}_1^{(m)}} - \frac{\bar{v}_2}{\lambda - \bar{w}_2^{(m)}} \right]_r$$

$$\xi_1^{(m)} = 0 \quad \xi_0^{(m)} = \sum_{i=0}^{m-1} b_i \tilde{\lambda}^{-j-1} \quad C \left[\sum_{i=0}^{m-1} b_i \tilde{\lambda}^{-j-1}; \frac{\bar{v}_1 \tilde{\lambda}^{-m}}{\lambda - \bar{w}_1^{(m)}} - \frac{\bar{v}_2}{\lambda - \bar{w}_2^{(m)}} \right]_{2r} = \xi_{2r}^{(m)}$$

14.1 The structure of the Padé table associated with a Hammerer-Hurwitz function

Th1. Let HAI in $F(z) = \frac{P(z)}{Q(z)}$ give $\tilde{F}(z) = \sum b_i z^i \tilde{\lambda}^{-j-1} \cdot H_{m,r}$ from coeffs of

$\sum b_i z^i, \tilde{H}_{m,r}$ from recip series

$$1) F(z) \text{ degen HM. /n: } \exists r' \in \mathbb{J}, \text{ s.t. either } \begin{cases} r' > 0 & r = \mathbb{J}_0^{r'-1}, \\ H_{2m,r} = 0 & r = \mathbb{J}_{r'}, \\ H_{0,r} > 0 & (r = \mathbb{J}_0^{r'}) \end{cases} \quad H_{0,r} > 0 \quad (r = \mathbb{J}_0^{r'})$$

$$H_{m,r} > 0 \quad (r = \mathbb{J}_0^{r'-1}) \quad \begin{cases} (m=2) & \text{or } b \\ r' & = 0 \quad H_{0,r} > 0 \quad (r = \mathbb{J}_0^{r'-1}) \\ r' & = 0 \quad H_{0,r} = 0 \quad (r = \mathbb{J}_{r'}) \end{cases} \quad H_{2m,r}, H_{m,r} > 0 \quad (r = \mathbb{J}_0^{r'})$$

2) if $F(z)$ is a non-deg. HN fn then $H_{2m,r}, H_{m,r} > 0 \quad (r, m \geq 2)$

$\sum b_i z^i \mapsto \{R_{i,j}(z)\}$. 1a) $R_{i,r-1}(z) \quad (i = \mathbb{J}_{r-1}) \quad R_{r,j}(z) \quad (j = \mathbb{J}_{r-1})$ /nm boundary on the block ident. quota. 1b) $R_{i,r-1} \quad (i = \mathbb{J}_{r-1}) \quad R_{r-1,j}(z) \quad (j = \mathbb{J}_{r-1})$ /nm boundary of the block

Th. 2 If Padé table gen by HN/m contains blocks each block contains 4^{th}
members; possible block composed of $R_{2r, 2mrr}(\varepsilon)$ normal, non-defective
 $R_{2r, 2mrr+1}(\varepsilon)$ normal defective, $R_{2r+1, 2m+r+1}(\varepsilon)$ over-normal.

Th. 3 $\{R_{ij}(\varepsilon)\}$ from $f(\varepsilon) = S[\zeta(\varepsilon)]_n^b \in M(\bar{R}_n^b)$ $f(\varepsilon)$ non degen
 HN/m . Value of $f(\varepsilon)$ lies in interiors of 0) two sequences $\{\hat{K}_{j, 2m-1}\}$
 $\sim \hat{K}_{2m-1, r}$: ($m=J$) $\hat{K}_{0, 2m-1}$ is line through $R_{0, 2m-1}(\varepsilon)$ marking $\angle \delta$ $(2m-1)_m$
with one real axis $\hat{K}_{m, 2m-1}$ lies inside $\hat{K}_{0, 2m-1}$ touching latter at
 $R_{0, 2m-1, 1}(\varepsilon)$ passing through $R_{m, 2m-1, 1}(\varepsilon)$. 2) $m=J$, $\hat{K}_{2m-1, 0} \sim 0$ passing
through origin having no tangent there line marking $\angle \delta$ $(2m-1)_m$ with one
real axis, and passing through $R_{2m-1, 0}(\varepsilon)$. $\hat{K}_{2m-1, 1, 1}$ lies inside $\hat{K}_{2m-1, 0}$
touching latter at $R_{2m-1, 0}(\varepsilon)$ passing through $R_{2m-1, 1, 1}(\varepsilon)$ (red)

Th. 4. In above notation $R_{0, 2mrr}(\varepsilon)$ lies on $K_r^{(2m-1)}(1, m=J)$, value 0
 $R_{2mrr, r}(\varepsilon)$ lies on $K_r^{(2m-1)}(1, m=J)$

19.2. The structure of the Padé table generated by a Shabtai function

Th. 5. Let S/n $S(\varepsilon) = ST[\varepsilon]_n^b$ generate $\Xi(\varepsilon) = \sum b_i \varepsilon^{i-1}$ then from

either a) $H_{mr} > 0$ ($r=J_0^{(r-1)}$) $\int_{(m=J)}^{(m=J)} H_{0,r} > 0$ ($r=J_0^{(r-1)}$)
 $H_{mr} = 0$ ($r=J_1^{(r-1)}$) $H_{0,r} = 0$ ($r=J_1^{(r-1)}$) $H_{mr} = 0$ ($r=J_1^{(r-1)}$)

or b) $H_{0,r}, H_{0,r} > 0$ ($r=J_0^{(r-1)}$) $H_{mr}, H_{mr} > 0$ ($r=J_0^{(r-2)}$) $m=1$,
 $H_{0,r}, H_{0,r} = 0$ ($r=J_1^{(r-1)}$) $H_{mr}, H_{mr} = 0$ ($r=J_1^{(r-1)}$)

2) $S(\varepsilon)$ non degen S/n . $H_{mr} > 0$ $\hat{H}_{mr} > 0$ $b_m = 0$.

Th. 6. Padé table gen by $\sum b_i \varepsilon^i$ when $\sum b_i \varepsilon^{i-1}$ gen by non degen
 S/n is normal

Th. 7. $\{R_{ij}(z)\}$ gen by $f(z) = S \left[(1-z)^{-1} : \infty \right]_a^b$ $\in MD(\bar{R}_a^b)$ & $g(z) = ST \left[\gamma / z \right]_a^b$

non degen S-fraction $\geq \in BE(\bar{R}_{b-1}^{a-1})$.

$\Im(z) \neq 0$ value of $f(z)$ lies in interior $\Rightarrow \Theta, \{\hat{K}_r^{(m-1)} \tilde{J}^{(n-1)} K_r\}$
constr as in Th 3 with $\Im z$ consistently replacing m .

14.3 The convergence of the diagonal sequences of the Padé table associated with a Hamburger - Nevanlinna function

$t_j = S \left[z^j : \in \right]_{-\infty}^{\infty}$ det: $\Re \{ \tilde{f}(z) = \sum t_j z^{j-1} \}$ converges completely; indet.: limit 0 case. Limit pt, limit 0 case w.r.t. $\Re \{ \tilde{f}^{(2m)}(z) \}$ occurs according as to whether $t_{2m+1} = S \left[z^j : \in^{(2m)} \right]_{-\infty}^{+\infty}$ det or indet. Any soln $\hat{s}^{(2)}(z) \rightarrow$ soln $\hat{s}^{(2m)}$ of (1): set $\hat{s}^{(2m)} = \hat{s}^{(2m)} \left[2m \right]_{-\infty}^{+\infty}$. Possible soln to (2) exists if and only if (1)
soln of (2). General pattern of converge behaviour $\{R_{mn+1}(z)\} (m \in I_0^{2m+1})$
 $\{R_{mn+1}(z)\} (m \in I_0^{2m+1})$ $m, m' \in I$, $\rightarrow f(z) \in CD(\bar{R}_a^b)$ diagonal seqs. outside
domain.

Th. 8. $\{R_{ij}(z)\}$ from $\tilde{f}(z) = \sum t_j z^j$ gen by $f(z) = S \left[(1-z)^{-1} : \in \right]_a^b$ $\tilde{f}(z) =$

$ST \left[\gamma / z \right]_a^b$ non deg HN fn. $\tilde{f}(z)$ series resp to $f(z)$. If when $\lambda \in BE(\bar{R})$
 $\Re \{ \sum t_{mn} \lambda^{-j} \}$, $\Re \{ \sum \tilde{t}_{mn+1} \lambda^{-j-1} \}$ are comp convgt for $m \in I$, then
all forward diag seqs. of Padé quota converge wif $h \in \Theta(\bar{R}_a^b) \rightarrow f(z)$
c1 $f(z), \tilde{f}(z), f(z)$ as above. If $\sum_{j=1}^{\infty} t_{2m+j}$ and $\sum_{j=1}^{\infty} \tilde{t}_{2m+j}$ diverge ($m \in I$)
all forward diagonal seqs. $\rightarrow f(z)$

c2 $f(z), \tilde{f}(z), f(z)$ as above. If $\tilde{G}, \tilde{\eta}, \tilde{G}, \tilde{\eta} \in P$, $h, \tilde{h} \in I$ s.t. $t_{2j} < G \eta^j (j+h)!$
 $\tilde{t}_{2j+1} < \tilde{G} \tilde{\eta}^j (j+h)!$ ($j \in I$), all forward diag seqs. $\rightarrow f(z) \in CD(\bar{R}_a^b)$

c3 $\{R_{ij}(z)\}$ from $f(z)$ above, $\bar{R}_a^b \subset R$. If all forward diag seqs. $\rightarrow f(z)$, $z \in CD(\bar{R}_a^b)$
c4. Let $d_p, z_p \in R$ ($p \in I$) with $1/z_p \in R$, $\delta \in P, p \in I$, $\exists h \in I$, such. $\sum d_p z_p^{-h-1}$ converges absolutely, and i) if $h > 0$: a) $z_p d_p > 0$ ($p \in I$) b) $z_p d_p < 0$ ($p \in I$) 2) h odd: $d_p > 0$ $\forall p$ or $d_p < 0$ ($p \in I$). Let $g_i(z) = \prod_{j=0}^{\infty} \left(1 - \frac{z}{z_p} \right)^{d_p} e^{d_p \ln \left(\frac{z}{z_p} \right) + \dots + h \frac{d_p}{z_p}}$; $f(z) = z^{-h} \frac{D_p g_i(z)}{g_i'(z)}$
all forward diag seqs. $\rightarrow f(z) \in BA(\bar{R}_{-z}^{z_p})$

D1 Infin. seg. of quots. for which successive $R_{i,j}(z)$ to $R_{i,j}(z)$ is such that no either $i' > i$, $j' > j$ or $i' < i$, $j' > j$ is called a progressive seg.

The. Let $\{R_{ij}(z)\}$ from $f(z) = S[(1-z)^{-1}] \in \bar{\mathbb{R}}_a^b \subset R$. Consider if any progressive seg. converge w.r.t. $z < \min(1/a^{-1}, 1/b^{-1})$ to $f(z)$

#4. The convergence of the diagonal sequences of the Padé table associated with a Stieltjes function.

General picture $\forall m, n \in \mathbb{J}$ s.t. $\{R_{m,n-1}(z)\} (m = \bar{J}_0^{n'}) R_{m,n+1}(z) (m = \bar{J}_0^{n''})$
 $\rightarrow f(z)$; thereafter, $R_{m,n-1}(z)$, $R_{m,n+1}(z)$ converge but not to common limit

$\{R_{m,n+1}(z)\}$ behave in same way

The. Let $\{R_{ij}(z)\}$ from $\sum t_i z^i$ from $f(z) = S[(1-z)^{-1}] \in \bar{\mathbb{R}}_a^b \subset R$

w.r.t. deg $S - 1/n$. $\tilde{f}(z) = \sum b_i z^i$ recip series

If when $\lambda \in BE(\bar{\mathbb{R}}_0)$ $b\left\{\sum t_{mn}\lambda^{-j-1}\right\} = 1$ & $b\left\{\sum t_{mn+1}\lambda^{-j-1}\right\}$
 converge for $m = 3$ then all forward diag seq. $\rightarrow z \in BE(\bar{\mathbb{R}}_0^{z-1})$

to $f(z)$

c1 If $\sum_{i=0}^{\infty} t_{mn}^{-\frac{1}{2}} \sum_{i=m+1}^{n-\frac{1}{2}}$ diag for $m \in \mathbb{J}$ converge above

c2 If $\exists G_2, \tilde{G}, \tilde{\eta} \in \bar{\mathbb{P}}$ h.c. \exists , s.t. $b_i < G_2^2(2i+1)$, $t_{mn} < \tilde{G}\eta^2(2i+\tilde{h})$

then converge

(C3 to Th 8. applies as much to this th. as to Th 8. since $C3(R_n^b) = BE(\bar{\mathbb{R}}_0^b)$)

c3 ~~$d_b \in \bar{\mathbb{P}}$~~ Let $d_b, z_b \in \mathbb{R}$ with either $d_b \in \bar{\mathbb{P}}$ or $-d_b \in \bar{\mathbb{P}}$ \Rightarrow 9
 and with $z_b \in \bar{\mathbb{P}}$ or $-z_b \in \bar{\mathbb{P}}$, with $|z_b| \in \bar{\mathbb{R}}_0$ and $S \in \bar{\mathbb{P}}$. Suppose that

st $\sum d_j z_j^{-h-1} \cdot \det g(z)$ as in C4 to Th 8. Then $\Rightarrow 0 \Leftrightarrow f(z) = g(z)$

$z_b < 0$ $f(z) = -g(z)$. All diag seq. $\rightarrow f(z) z \in BE(\bar{\mathbb{R}}_0)$

c4. $\{t_b\}$ cst. mon. $f(z) = (t_b z^b)_0^\infty$ for $|z| < 1$ and const for $z \in BE(1, \infty)$

All diag seq. $\rightarrow f(z) z \in BE(1, \infty)$

C5 All forward diag seqs of Padé table gen by HRW-in $f(z) \rightarrow f(z)$
 $\text{BE}(\bar{\mathbb{R}}_1)$. 77

4.5 The construction and convergence of the diagonal sequences & the even order epsilon array.

Th 11 All numbers $\{\varepsilon_i^{(m)}\}$ can be const. by applying s-ally to $\{(t_i z)^{-2+i}\}_0^{\infty}$
 all diag seqs $\rightarrow S(\lambda)$ if $1+a$ or $1+b$ or $1+c+2$ hold:

$$1) S(\lambda) = ST[\lambda]_n \Rightarrow \bar{\mathbb{R}}_a^b \subseteq \bar{\mathbb{R}}_0 \Leftrightarrow M(\bar{\mathbb{R}}_a^b) \lambda \in \text{BE}(\bar{\mathbb{R}}_0^b)$$

$$\sim \sum_{i=1}^{\infty} t_i z^{-2+i} \quad (m-2) \text{ diag } \Leftrightarrow \exists \alpha, \beta \in \mathbb{P} \text{ s.t. } t_i \leq \alpha \gamma^i (2+i)! \quad i=1$$

$$c) b < 0 \Leftrightarrow S(\lambda) = -S'(\lambda), S'(\lambda) \text{ is th 10.}$$

All s-numbers can be const. by applying s-ally to $\{(t_i z^2)\}_0^{\infty}$

all diag seqs $\rightarrow f(z)$

$$3) f(z) \text{ is ln. of C3 to Th 10 } z = \text{BE}(\bar{\mathbb{R}}_1)$$

$$4) \{t_i\} \text{ tot max } f(z) = \lim_{n \rightarrow \infty} (t_n z^n)_0^\infty \text{ in } |z| = \bar{\mathbb{R}}_0^b \text{ def by anal cont in } \\ z = \text{BE}(\bar{\mathbb{R}}_1) \Leftrightarrow \text{BE}(\bar{\mathbb{R}}_1)$$

$$5) \sum_i t_i z^i \text{ is series expn of HRW-in } f(z). z = \text{BE}(\bar{\mathbb{R}}_1)$$

4.6. The Padé table derived from a Stieltjes series

$$f(z) = \sum_i t_i z^i : \{t_i\} \in S \text{ LS } \varepsilon_i^{(b)} \stackrel{(1)}{\sim} \bar{\mathbb{R}}_a^b \subseteq \bar{\mathbb{R}}_0 \Leftrightarrow M(\bar{\mathbb{R}}_a^b) \ni z$$

$f(z)$ gen by $G\{f(z)\} = S[(1+z)^{-1}]_0^b$. booke: mon poly index $G\{f(z)\}$

$$\text{taken to be any one of poly. form } G\{f(z)\} = \frac{t_0}{1+z} + G\{f''(z)\}$$

$$\text{expnsible in } G\{f''(z)\} = S[(1+z)^{-1}]_0^b \quad z^{(b)} \in M(\bar{\mathbb{R}}_a^b) \quad \text{if } f(z) \rightarrow R_{ij}(z) \quad f''(z) \rightarrow \tilde{R}_{ij}^{(b)}(z)$$

$$R_{i+j,j}(z) = \frac{t_0}{1+z} \quad (i,j = 0)$$

Th 12. $\{R_{i,j}(z)\}$ from $f(z)$ with coeffs (14) $z \in \mathbb{P}$:

$$R_{0,2m-1}(z) < G\{f(z)\} \quad R_{0,2m}(z) > G\{f(z)\} \quad 0 < R_{2m-1,0}(z) < G\{f(z)\}_{m+1}$$

If $R_{2m,0}(z) > 0$ then $R_{2m,0}(z) > G\{f(z)\}$

Th 13. $R_{i,j}(z)$ as above $z \in \mathbb{P}$, $m \in \mathbb{J}$ fixed

- 1) $\{R_{r,2m+r-1}(z)\}$ increases monotonically and $R_{r,2m+r-1}(z) < G\{f(z)\}_{r+1}$
- 2) $\{R_{r,2m+r}(z)\}$ decr. mon. + $R_{r,2m+r}(z) > G\{f(z)\}_{r+1}$
- 3) $\{R_{2m+r,r}(z)\}$ incr. mon., $R_{2m+r,r}(z) < G\{f(z)\}_{r+1}$
- 4) $\{R_{2m+r+2,r}(z)\}$ in general composed of three constituents: negative numbers increasing in mag; an infinite number; monoton decr seq. of positive numbers if there are $R_{2m+r+2,r}(z) \quad l = l_r$, then $R_{2m+r+2,r}(z) > G\{f(z)\}_{r+1}$.

Th 14. $\{R_{i,j}(z)\} z = m$ as above

- 1) $\{R_{r,2m-r-1}(z)\}$ increases monotonically until $R_{m,m-1}(z)$ encountered, thereafter seq. decr. monotonically terminating with $R_{2m,0}(z)$; $R_{r,2m-r-1}(z) < G\{f(z)\}_{r+1}$
- 2) $\{R_{r,2m-r}(z)\}$ positive numbers decr. monoton. until $R_{m,m}(z)$ encountered; then increases monotonically until $R_{r',2m-r'-1}(z) = \infty$ ($r' > m$), then -ve numbers decr. mon. in mag. until $R_{2m,0}(z)$. For quota having the values $R_{r,2m-r}(z) > G\{f(z)\}_{r+1}$ ($r = g_0$)

Th 15. $\{R_{i,j}(z)\}$ derived from $f(z)$ as above $z \in \mathbb{P}$ fixed $m, r \in \mathbb{J}$ fixed

- 1) $R_{r,2m+r-1}(z)$ lower bound for all Padé quotients in semi-infinite band bordered by $\{R_{r+2,2m+r+2-1}(z)\} (\omega = \mathcal{G})$ $\{R_{r+2,2m+r-2-1}(z)\} (\omega = \mathcal{G}_0)$ $R_{m+r, m+r-1}(z)$
- 2) $R_{r,2m+r}(z)$ upper bound for all PQu in band bordered by $\{R_{r+2,2m+r+2}(z)\}$ ($\omega = \mathcal{G}$) $\{R_{r+2,2m+r-2}(z)\} (\omega = \mathcal{G}_0)$ $\{R_{m+r, m+r}(z)\} \omega = \mathcal{G}$

Th 16. If $\{R_{i,j}(z)\}$ & $f(z)$ as above $z \in P$ we have, find

- 1) all Poles derived from $(t_0 z^*)_0^{2m+1}$ entry $R_{m,m}(z)$ gives sharpest lower bound to $\Re\{f(z)\}$
- 2) 1) all Poles derived from $(t_0 z^*)_0^{2m}$, $R_{m,m}(z)$ gives sharpest upper bound.

Th 17. $\{R_{i,j}(z)\}$ & $f(z)$ as above $\bar{\mathbb{R}}_a^b \subset \mathbb{R} z = b^{-1}$

i) value of any $\Re R_{i,j}(z)$ real +ve

e) values of quots $\{R_{i,j}(z)\}$ if any progressive seq. $\rightarrow f(z) = 3 \left[(1+z_3)^{-1} \right]_n^b$

Th 18. Suppose $m f(z) = 3 \left[(1+z_3)^{-1} : \delta \right]_a^b$, ϵ const in $z = \bar{\mathbb{R}}_{b,h}^b \subset \bar{\mathbb{R}}_a^b$ except

for values of sing. $M_{b,h} \in P$ at $z = b_h$, $\nu = \mathbb{J}_0^{h-1}$ then

$$i) f(z) = 3 \left[(1+z_3)^{-1} : \delta \right]_a^{b_h} + (M_h (1+z_{b_h})^{-1})_0^{h-1} (M_h)_0^{h-1} \in P$$

and if $f(z) = 3 \left[(1+z_3)^{-1} : \delta \right]_a^b$

$$\hat{f}_{(0)}(z) = 3 \left[(1+z_3)^{-1} : \tilde{\delta} \right]_a^{\tilde{b}} + (\tilde{M}_h (1+z_{\tilde{b}_h})^{-1})^{h-1} M_h \in P \Rightarrow \mathbb{J}_0^{h-1}$$

assume $b_0 > b_1 > \dots > b_{h-1}$, $\tilde{b}_0 > \tilde{b}_1 > \dots > \tilde{b}_{h-1}$, $\nu' \in \mathbb{J}_0^{h-1}$, $\delta' \in P$ fixed and small

2) $R_{b,h}(z)$ ($m=9$) converges wif for z in any open domain contained in

$|z| < b_{h+1}^{-1} - \delta'$ not incl. any b_j^{-1} as interior pt. $R_{b,h}(z)$ $m=7$

converges in any open domain contained in $|z| < \tilde{b}_{h+1}^{-1} - \delta'$ not incl. any \tilde{b}_j^{-1} as int. pt.

3) z lies in disc $|z| = \bar{\mathbb{R}}_0^{-b_{h+1}-\delta'}$ cut along real axis. $\bar{\mathbb{R}}_{-b^{-1}}^{-b_{h+1}-\delta'}$

values of quots on any prog. seq. bounded by subprinc. diag -

$R_{b,h}(z)$ ($m=9$) $\rightarrow f(z)$; z lies in cut disc $|z| = \bar{\mathbb{R}}_0^{-\tilde{b}_{h+1}-\delta'}$ cut

along $\bar{\mathbb{R}}_{-b^{-1}}^{-b_{h+1}-\delta'}$ values of quots on any prog. seq. bounded by
princ. diag + col. $R_{m,m}(z)$ ($m=9$) $\rightarrow f(z)$.

Example $\sum_i (-1)^{j_i} z^{j_i}$ $z = 1$ $\epsilon_{13} = 1 - e^{-3} \approx \frac{1}{e^3}$ $\tilde{S}_1^{(13)} = 1 - e^{-3} \int_0^1$
 ≈ 0.6969

$$\sum_i (-1)^{j_i} (2^{j_i})^{-1} z^{j_i} \quad z = 1.0 \quad (\text{from } \omega_{13} = 5) \quad ((-1)^0 (2^0)^{-1})_0 = 0.693197$$

14.17. A comparison between the epsilon algorithm and the generalised Euler transformation

Consider $S^{(r,m)} = (W_{\nu}^{(r,m)} S_{m,r})_0^{r'} \quad (r,m=9)$ $S_m = (t_r z^r)_0^{m-1} \quad m=9$

If regular $(W_{\nu}^{(r,m)})_0^{r'} = 1 \quad (r,m=9)$

$$S[(1+z_3)^{-1}; \sigma]_0^{20} = S^{(r,m)} = (-z)^m S \left[\frac{z^m (W_{\nu}^{(r,m)} (-z)^r)^{r'}}{1+z_3} \right] \quad (r,m=9)$$

$$r=2, \quad \sum_{\nu} W_{\nu}^{(2,m)} (-z_3)^{\nu} = \left\{ \sum_{\nu} Y_{\nu}^{(2,m)} (-z_3)^{\nu} \right\}^2$$

$$S[(1+z_3)^{-1}; \sigma]_0^{20} = S^{(r,m)} = (-z)^m S \left[\frac{z^m \{ Y_{\nu}^{(2,m)} (-z_3)^{\nu} \}_0^{r'} \}^2}{1+z_3} \right] \quad (r,m=9)$$

The 19. Values of $S_{\nu} \quad (\nu = j_m^{m+2r})$ used in two ways to obtain approx. to $G\{f(z)\}$: first $\epsilon_{2r}^{(m)}$ from $\epsilon_0^{(m)} = S_{\nu} \quad (\nu = j_m^{m+2r})$ 2nd as above

$$(-1)^m [G\{f(z)\} - \epsilon_{2r}^{(m)}] < (-1)^m [G\{f(z)\} - S^{(2r,m)}]$$

Euler: $\sum T_{\nu} : \quad T_{\nu} = \xi_{\nu} \eta^{\nu} \quad (\nu = j_m)$ $E^{(r,m)}(\eta) = (T_{\nu})_0^{m-1} + \frac{\eta^m}{1-\eta} \left(\frac{1}{1-\eta} \Delta_m S_m \right)_0^{r'}$

The 20.) $E^{(2r,m)}$ of above type 2) if $f(z) = \sum t_r z^r$ shall give same the results

b) $\epsilon_{2r}^{(m)}$ from $\epsilon_0^{(r)} = (t_r z^r)_0^{r-1} \quad (r = j_m^{m+2r})$

c) $E^{(2r,m)}(\eta)$ from $T_{\nu} = t_{\nu} z^{\nu}$

$$(-1)^m [G\{f(z)\} - \epsilon_{2r}^{(m)}] < (-1)^m [G\{f(z)\} - E^{(2r,m)}(\eta)] \quad ?_{m=2}$$

$$(-1)^m [G\{f(z)\} - \epsilon_{2r}^{(m)}] < (-1)^m [G\{f(z)\} - E^{(2r,m)}(\eta)] \quad r=1, 1$$

Not possible for favorable comparison for $E^{(2r,m)}$, consider $w_1 S_m + w_2 J_m$ instead

14.8 The epsilon algorithm and the transformation of trigonometric series

$$\text{Abel: } (A) \sum_i \{ t'_i \cos i\theta + t''_i \sin i\theta \} = A(\beta) /_{\beta=1-\theta}$$

$$A(\beta) = \sum_i \{ t'_i \cos i\theta + t''_i \sin i\theta \} \beta^i$$

transf/mn $\sum_i t'_i \cos i\theta$ as $\sum_i t'_i z^i$

$$\sum_i t''_i \sin i\theta \text{ by } \sum_{j=1}^m t'_j z^j$$

$$\text{Ex: } \sum_i (i+1)^{-1} \cos i\theta \quad \text{then } \theta = \frac{\pi}{4}$$

Compare Euler

$$\text{Cauchy form } \sum_{i=0}^{(m)} S_m \quad \sum_{i=r+1}^{(m)} = (\sum_{i=0}^{(r)})_0^m \quad (r, m = 0)$$

$$C_r^{(m)} = \frac{\sum_{i=r}^{(m)}}{\binom{m+r}{r}}; \quad \text{applied to } ((i+1)^{-1} \sin (2\pi i))_0^{m-1}$$

14.9 The epsilon algorithm and operational formulae of numerical analysis

If when $t'_j = t'_j \sum_{n=1}^j \phi(n) (j \geq 1)$ $\sum_i t'_i z^i$ amenable to transformation

by ε -alg. & apply it to $\{ (t'_j z^j \sum_{n=1}^j \phi(n))_0^{m-1} \}$

The 21) If for $n \in \mathbb{N}_0$ fixed, $\phi_1(n) = \int_0^\infty e^{-nt} dt \sim \infty (\bar{R}_0')$

$\phi_2(n) = \int_0^\infty e^{-nt} dt \sim \infty MD(\bar{R}_0)$ then $\{ (-\partial_n)^j \phi_1(n) \} \{ (-\Delta n)^j \phi_2(n) \}$

and $\{ E_n^j \phi_1(n) \}$ are totally monotone

2) $\{ t'_j \}$ totally monotone: $\sum_i t'_i (-\partial_n)^j \phi_1(n) z^j, \sum_i t'_i (-\Delta n)^j \phi_2(n) z^j$

$\sum_i E_n^j \phi_1(n) z^j$ converge for $|z| \leq \bar{R}_0'$ defining now $f_T(z)$ regular in

$BE(\bar{R}_1)$

3) when $z \in BE(\bar{R}_1)$ ε -array can be built up from $\{ S[f_T(z)]_m \}$ and in each case along $\varepsilon_{\geq r}, r = 1 \rightarrow f_T(z)$

$$\text{Ex: } \phi(n) = \hat{\phi}(n) - \hat{\phi}(n) \quad \hat{\phi}(n) = e^{0.6n} \quad n = 0$$

Non-commutative continued fractions

Scalar c.f.s.: compute $C_r \in C[b_0; \frac{a_r}{b_r}]$: $C_r = G_{r,r}$

$$G_{r,0} = b_r, \quad G_{r,r+1} = b_{r+1} + \frac{a_{r+1}}{G_{r,r}} \quad (r = I_0^{r-1})$$

Provisionally $\text{pre } C[B_0; \frac{A_r}{B_r}]$: $\text{pre } C_r = G_{r,r}'$

$$G_{r,0}' = B_r, \quad G_{r,r+1}' = B_{r+1} + G_{r,r}^{-1} A_{r+1} \quad (r = I_0^{r-1})$$

$$\text{post } C_r = G_{r,r}'' \quad G_{r,0}'' = B_r \quad G_{r,r+1}'' = B_{r+1} + A_{r+1} G_{r,r}^{-1} \quad (r = I_0^{r-1})$$

Th. 1 $B_0 \in N$ $A_r \in N$ $r = I_1$, $B_r \in N_I$ ($r = I_0$) $G_{r,r}'$ above $\in N_I$

If $\text{pre } N_{r-1} = \text{pre } D_0 = 1$ $\text{pre } N_r = B_0$ $\text{pre } D_r = 0$

$$\begin{aligned} \text{pre } N_r &= B_r, \quad \text{pre } N_{r-1} + A_r \text{ pre } N_{r-2} \quad r = I_1, \\ \text{pre } D_r &= B_r, \quad \text{pre } D_{r-1} + A_r \text{ pre } D_{r-2} \end{aligned}$$

then $\text{pre } D_r \in N_I$ $r = I_1$ and $\text{pre } C_r = \text{pre } D_r^{-1} \text{ pre } N_r \quad r = I_1$

$G_{r,r}'' \in N_I$ above similar initial conditions

$$\text{post } N_r = \text{post } N_{r-1} D_r + \text{post } N_{r-2} A_r \quad \text{post } D_r = \text{post } D_{r-1} B_r + \text{post } D_{r-2} A_r \quad (r = I_1)$$

$$\text{post } C_r = \text{post } N_r \text{ post } D_r^{-1}$$

Consider $\text{pre } C_2 = \text{pre } C[B_0 + \frac{A_1}{B_1 + \frac{A_2}{B_2}}]$. Scalar with $B_2 = 0$ well def.

non-operator $B_2 \notin N_I$, first def. no model Th. 1 can be used. F.:

D1 Given $A_r \in N$ ($r = I_1$), $B_r \in N$ ($r = I_0$) assumed that the seq.

$\{\text{pre } N_r\}, \{\text{pre } D_r\}$ of Th. 1 well def. Furthermore $\text{pre } D_r^{-1} \text{ pre } N_r$ $r = I_1$ well

def. $\text{pre } C_r = \text{pre } [B_0 + \frac{A_1}{B_1 + \frac{A_2}{B_2 + \dots}}]_{r=I_1}$ def as in Th. 1, $\text{post } C_r$ similarly

If $\text{pre } H + \text{pre } B \text{ pre } C + \text{pre } D \text{ pre } E \text{ pre } F + \dots = \text{pre } U + \text{pre } V \text{ pre } W + \text{pre } X \text{ pre } Y \text{ pre } Z$
 then correspond. $\text{post } A + \text{post } B + \text{post } C + \text{post } D + \text{post } E + \text{post } F + \text{post } G + \text{post } H + \text{post } I + \text{post } J + \text{post } K + \text{post } L + \text{post } M + \text{post } N + \text{post } O + \text{post } P + \text{post } Q + \text{post } R + \text{post } S + \text{post } T + \text{post } U + \dots + \text{post } Z$

Th 2. $\{\text{pre } N_r\} \{\text{pre } D_r\}$ from $\text{pre } C[B_0; \frac{A_1}{B_1+}]$ $\eta_0 \neq 0, \infty \in N_S$ $\Rightarrow I$, then

$$\text{pre } C[B_0; \frac{\eta_1 A_1}{\eta_1 B_1+} \frac{\eta_2 \eta_{2-1} A_2}{\eta_2 B_2+}] \rightarrow \left\{ \left(\prod_{r=1}^r \eta_r \right) N_r \right\} \left\{ \left(\prod_{r=1}^r \eta_r \right) D_r \right\}$$

$$\text{pre } C[B_0; \frac{A_1}{B_1+}]^r = \text{pre } C[B_0; \frac{\eta_1 A_1}{\eta_1 B_1+} \frac{\eta_2 \eta_{2-1} A_2}{\eta_2 B_2+}]^r$$

Otherwise if $\eta_0 \in N_S$

Th 3. $\{\text{pre } N_r\} \{\text{pre } D_r\}$ from $\text{pre } C[B_0; \frac{A_1}{B_1+}]$

$$1) \text{ pre } [B_0, \frac{B_2 A_1}{B_2 + A_2 -} \frac{B_{2+2} A_{2+1} B_{2+1}^{-1} A_{2+2}}{A_{2+2} + B_{2+2} (B_{2+1} + A_{2+1} B_{2+1}^{-1}) -}]_{2+2} \rightarrow \text{pre } N_{2+} \text{ pre } D_{2+}$$

2) if $B_1 \in N_S$ or $B_0 A_1 \in N_S$ then

$$\text{pre } [B_0 + B_1^{-1} A_1, \frac{B_{2+1} A_{2+2} B_{2+1}^{-1} A_{2+2-1}}{A_{2+2+} + B_{2+2+} (B_{2+2} + A_{2+2} B_{2+2-1}) -}]_{2+1} \rightarrow \{B_1^{-1} \text{ pre } N_{2+1}\} \{B_1^{-1} \text{ pre } N_{2+1}\}$$

Th 4. $\{\text{pre } C_r\}$ from $\text{pre } C[B_0; \frac{A_1}{B_1+}]$ successive part sums of

$$B_0 + \sum_{r=1}^{\infty} \left\{ \prod_{T=0}^{r-2} (-\text{pre } D_{T+1}^{-1} A_T \text{ pre } D_{T+2}) \right\} B_1^{-1} A_1 \quad (\text{D as in Th 1})$$

Non commutative orthogonal polynomials

Given $\{T_\nu\} \in N$: process $\text{pre } \mathcal{T}[L..]$: $\text{pre } [A B^\rho] = A T_\rho$ ($\rho = I$)

Elements of N and S (scalars) undergo normal arithmetic inside brackets, elements of N premultiply numbers $\{T_\nu\}$.

Consider determination of $\text{pre } P_r(\lambda) = \sum_{s=0}^r \text{pre } K_{r,s} \lambda^s$ ($r = I$)

$\text{pre } K_{r,s} \in N$ ($I = \mathbb{Z}_{\geq 0}^{r+1}$), $\text{pre } K_{r,r} = I$ ($r = I$) $\lambda \in N_S$ from $\text{pre } \mathcal{T}[\text{pre } P_r(s) s^\rho] = 0 \quad \forall s \in N$

First: $\text{pre } P_r(\lambda) = (\lambda - \text{pre } W_r) \text{pre } P_{r-1}(\lambda) - \text{pre } V_r \text{pre } P_{r-2}(\lambda)$ ($r = I$)

Th5 Given $\{ \text{pre } V_j \} \in N$ $\text{pre } V_0 \in N$; pre. constraint $\text{pre } P_r(\lambda)$ from $\text{pre } P_r(\lambda) = 0$
 $\text{pre } P_0(\lambda) = I$ by means of (25), furthermore $\{ \bar{V}_j \}$ can be found st. (26) hold.

Th6. Given $\{ \bar{V}_j \} \in N$, $\{ \text{pre } P_r(\lambda) \}$ can be const. from (24) if in case 7 const.
 $\{ \text{pre } K_{r,s} \}$ by means of $\text{pre } K_{0,0} = \text{pre } K_{0,1} = I$ $\text{pre } K_{1,0} = -T_1 T_0^{-1}$

$$\text{pre } V_r = \{ (\text{pre } K_{r,1,0} T_{r+2,1})_0^{k-1} \} \{ (\text{pre } K_{r-2,0} T_{r+2,2})_0^{r-2} \}^{-1}$$

$$\text{pre } W_r = \{ \text{pre } V_r (\text{pre } K_{r-2,0} T_{r+2,1})_0^{k-2} - (\text{pre } K_{r-1,0} T_{r+2})_0^{k-1} \} \{ (\text{pre } K_{r-1,0} T_{r+2,1})_0^{k-1} \}^{-1}$$

and (22) (25) $(\text{pre } K_{r,0} T_{r+2})_0^r \in N$

Th7. If $\{ \text{pre } P_r(\lambda) \}$ with $\text{pre } K_{r,r} = I (r=1)$ exists, (24) satisfied, then

$$\text{pre } Q_r(\lambda) = \overline{\text{pre } J} [(\lambda - s)^{-1} \{ \text{pre } P_r(\lambda) - \text{pre } P_r(s) \}] \quad \text{pre } Q_r(\lambda) = \sum_{s=0}^{k-1} \text{pre } P_r(s)$$

(r=0) also exists: $\text{pre } Q_0(\lambda) = 0 \quad \text{pre } Q_1(\lambda) = T_0$

$$\text{pre } Q_r(\lambda) = (\lambda - \text{pre } W_r) \text{pre } Q_{r-1}(\lambda) - \text{pre } V_r \text{pre } Q_{r-2}(\lambda) \quad (r=I_2)$$

$$\text{post system: post } \overline{\text{pre } J} A s^0 = T_0 A (Q = I) \quad \text{post } P_0(\lambda) = I \quad \text{post } P_r(\lambda) = \lambda - T_0^{-1}$$

$$\text{post } P_r(\lambda) = (\lambda - \text{post } P_{r-1}(\lambda)) (\lambda - \text{post } W_r) - \text{post } P_{r-2}(\lambda) \text{post } V_r \quad r=I_2$$

$$\text{det from post } \overline{\text{pre } J} [\text{post } P_r(s) s^0] = 0 \quad (s=I_0^{k-1}) \quad \text{post } P_r(\lambda) = \sum_{s=0}^{k-1} \text{post } K_{r,r} s^0$$

$K_{r,r} = I$. $\{ \text{post } Q_r(\lambda) \}$ also exist

$$\{ \text{pre } P_r^{(m)}(\lambda) \} \text{ from } \{ \bar{T}_{max} \}: \text{pre } \overline{P}_{r,1}^{(m)}(\lambda) = 0 \quad \text{pre } \overline{P}_0^{(m)}(\lambda) = I$$

$$\text{pre } P_r^{(m)}(\lambda) = \{ \lambda - \text{pre } W_r^{(m)} \} \text{pre } \overline{P}_{r-1}^{(m)}(\lambda) - \text{pre } V_r^{(m)} \text{pre } \overline{P}_{r-2}^{(m)}(\lambda) \quad (r=I_2)$$

Th8. $m \in J$ fixed $\{ \text{pre } P_r^{(m)}(\lambda) \}, \{ \text{pre } P_{r+1}^{(m+1)}(\lambda) \}$ can be const. Then $\exists \{ \text{pre } U_r^{(m)} \}$

$$\text{pre } P_r^{(m)}(\lambda) = \lambda \text{pre } \overline{P}_{r,1}^{(m+1)}(\lambda) - \text{pre } U_{0r}^{(m)} \text{pre } \overline{P}_{r-1}^{(m)}(\lambda), \text{pre } P_{r+1}^{(m)} = \text{pre } P_r^{(m)} - \text{pre } U_{2r+1}^{(m)} \text{pre } \overline{P}_{r-1}^{(m)}$$

$$\text{set } \text{pre } U_1^{(m)} = \text{pre } V_1^{(m)}: \text{pre } U_{2r}^{(m)} = \text{pre } W_1^{(m)}, \text{pre } U_{2r+1}^{(m)} \text{pre } U_{2r+2}^{(m)} = \text{pre } V_r^{(m)}$$

$$\text{pre } U_{2r+1}^{(m)} + \text{pre } U_{2r}^{(m)} = \text{pre } W_r^{(m)}$$

$$\{ \text{pre } Q_r^{(m)}(\lambda) \} \text{ from } \text{pre } Q_r^{(m)}(\lambda) = \text{pre } \mathcal{I} [(\lambda - s)^{-1} s^m] \text{ pre } P_r^{(m)}(\lambda) - \text{pre } P_r^{(m)}(s) \mathcal{J}_r = \mathcal{I}$$

$$\text{pre } Q_r^{(m)}(\lambda) = \sum_{i=0}^{r-1} \text{pre } L_{r,i}(\lambda) \quad \text{pre } Q_0^{(m)} = 0 \quad \text{pre } Q_1^{(m)}(\lambda) = T_m$$

$$\text{pre } Q_r^{(m)}(\lambda) = \{ \lambda - \text{pre } W_r^{(m)} \} \text{ pre } Q_{r-1}^{(m)}(\lambda) - \text{pre } V_r^{(m)} \text{ pre } Q_{r-2}^{(m)}(\lambda) \quad (r \neq 1)$$

If $\{\text{pre } P_r^{(m)}(\lambda)\}, \{\text{pre } P_r^{(m)}(\lambda)\}$ exist, also have

$$\begin{aligned} \text{pre } Q_r^{(m)}(\lambda) + \text{pre } U_r \text{ pre } Q_{r-1}^{(m)}(\lambda) - \text{pre } Q_{r-1}^{(m+1)}(\lambda) &= T_m \text{ pre } P_{r-1}^{(m+1)}(\lambda) \\ \lambda \text{ pre } Q_r^{(m)}(\lambda) - \text{pre } Q_r^{(m+1)}(\lambda) - \text{pre } U_{r+1} \text{ pre } Q_{r-1}^{(m+1)}(\lambda) &= T_m \text{ pre } P_r^{(m)}(\lambda) \end{aligned} \quad ? \quad (r \neq 1)$$

Non-commutative associated and corresponding continued fractions

\exists certain fns involving elements of $N \oplus \lambda$ having power series repns.

$$(\lambda - w)^{-1} v = \sum_i w^i v \lambda^{-i-1}$$

$$\text{pre } \mathcal{I} [\dots] \text{ used to represent such fns. } \text{pre } \mathcal{I} [(\lambda - s)^{-1}] = \sum_i \tilde{T}_i \lambda^{-i-1}$$

$$\text{Rational fn. with operator valued coeff: } P^{(r)}(\lambda) = \sum_{i=0}^{r-1} K_i \lambda^i$$

$$Q^{(r)}(\lambda) = \sum_{i=0}^{r-1} L_i \lambda^i \quad (r \in \mathbb{Z}_0^r) \quad K, L \in N \quad \lambda \in \mathbb{K}$$

$$\text{Rat fn. linear comb. of } P^{(0)}(\lambda)^{-1} Q^{(0)}(\lambda) P^{(1)}(\lambda)^{-1} Q^{(1)}(\lambda) \dots P^{(n)}(\lambda)^{-1} Q^{(n)}(\lambda)$$

$$\text{inverse } Q^{(n)}(\lambda)^{-1} P^{(n)}(\lambda) \dots Q^{(1)}(\lambda)^{-1} P^{(1)}(\lambda) Q^{(0)}(\lambda)^{-1} P^{(0)}(\lambda)$$

$$\text{Most rat. fns. with obs. we deal have form } P(\lambda)^{-1} Q(\lambda)$$

$$\text{One rat fn may have many reps.: } (\lambda - \tau_0 \tau_0^{-1})^{-1} T_0 = T_0 (\lambda - \tau_0 \tau_0^{-1} \tau_1)^{-1}$$

$$= (\lambda \tau_0^{-1} - \tau_0^{-1} \tau_1 \tau_0^{-1})^{-1}$$

$$\text{If } K_{\tau_0}^{(n)} \in N, \quad P^{(n)}(\lambda)^{-1} = K_{\tau_0}^{(n)} \lambda^{-n} \sum_{i=0}^n K_{\tau_{i-1}} K_{\tau_i}^{(n)} \lambda^{-i} + K_0^{(n)} K_{\tau_0}^{(n)}$$

Each factor has well defined unique series expn in λ !

Theo. $\{\text{pre } P_r(\lambda)\}, \{\text{pre } Q_r(\lambda)\}$ def. from $\{\tilde{T}_0\} \in N$ as above; then

$$\text{pre } P_r(\lambda)^{-1} \text{ pre } Q_r(\lambda) = \text{pre } C \left[\frac{\text{pre } V_r}{\lambda - \text{pre } W_r} \right]_r \quad (r \neq 1)$$

$$\text{and } \text{pre } C \left[\frac{\text{pre } V_r}{\lambda - \text{pre } W_r} \right]_r = \sum_{i=0}^{2r-1} \tilde{T}_i \lambda^{-i-1} + \sum_{i=2r}^r \tilde{T}_{2r+i} \lambda^{-i-1} \quad (r \geq 2)$$

$$\text{post } \text{asove c.f. post } Q_r(\lambda) \text{ post } P_r(\lambda)^{-1} = \text{post } C \left[\frac{\text{post } V_D}{\lambda - \text{post } W_D} \right]_{r=1}^{\infty}$$

$$\text{pre } P_r(\lambda)^{-1} \text{ pre } Q_r(\lambda) = T_0 \lambda^{-1} + T_1 \lambda^{-2} + \text{terms in } \lambda^{-3}, \lambda^{-4}, \dots$$

$$\text{post } Q_r(\lambda) \text{ post } P_r(\lambda)^{-1} = \dots$$

22. If $\text{pre } P_r^{(m)}(\lambda)$ can be written from $\sum T_{mrj} \lambda^{-j-1}$, $\sum T_{mr+1,j} \lambda^{-j-1}$ said to be pre-A-regular; post A regular def. similarly.

If for me g fixed, $\sum T_{mrj} \lambda^{-j-1}$, $\sum T_{mr+1,j} \lambda^{-j-1}$ both pre A-regular

$$\text{pre } C \left[\frac{\text{pre } U_j^{(m)}}{\lambda - \text{pre } W_j^{(m)}} \right] \text{ can be written as } \text{pre } C \left[\frac{\text{pre } U_1^{(m)}}{\lambda - \text{pre } U_2^{(m)}} \frac{\text{pre } U_2^{(m)}}{\lambda - \text{pre } U_{2+1}^{(m)} - \text{pre } U_{2+2}^{(m)}} \right]$$

even part of $\text{pre } C \left[\frac{\text{pre } U_{2j-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2j}^{(m)}}{I -} \right]$ $\{U_i^{(m)}\}$ numbers of Th8

$$\text{pre } C \left[\frac{\text{pre } U_{2j-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2j}^{(m)}}{I -} \right] = \sum_{r=0}^{2j-1} T_{mrj} \lambda^{-j-1} + \sum_{r=1} T_{mr+1,j} \lambda^{-j-1}$$

$$\text{pre } C \left[\frac{\text{pre } U_{2j-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2j}^{(m)}}{I -} \frac{\text{pre } U_{2j+1}^{(m)}}{\lambda -} \right] = \sum_{r=0}^{2j} T_{mrj} \lambda^{-j-1} + \sum_{r=1} T_{mr+1,j} \lambda^{-j-1}$$

$$\text{pre } C \left[\frac{\text{pre } U_{2j-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2j}^{(m)}}{I -} \right] = \text{pre } P_r^{(m)}(\lambda)^{-1} \text{ pre } Q_r^{(m)}(\lambda)$$

$$\text{pre } C \left[\frac{\text{pre } U_{2j-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2j}^{(m)}}{I -} \frac{\text{pre } U_{2j+1}^{(m)}}{\lambda -} \right] = T_0 \lambda^{-1} + \lambda^{-1} \text{pre } P_r^{(m)}(\lambda)^{-1} \text{pre } Q_r^{(m)}(\lambda)$$

The operator q-d algorithms

Th10 $m \in \mathbb{Z}$ if $\sum T_{mrj} \lambda^{-j-1}$, $\sum T_{mr+1,j} \lambda^{-j-1}$ generate (53) and

$$\text{pre } C \left[\frac{\text{pre } U_{2j-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2j}^{(m)}}{I -} \right]. \text{then } \text{pre } U_2 \text{pre } U_1 = \text{pre } U_1, \text{pre } U_2 + \text{pre } U_3 = \text{pre } U_2$$

$$\text{pre } U_{2r+2} \text{pre } U_{2r+1} = \text{pre } U_{2r+1} \text{pre } U_{2r} \quad \left. \right\} (r=j_1)$$

$$\text{pre } U_{2r+3} + \text{pre } U_{2r+2} = \text{pre } U_{2r+2} + \text{pre } U_{2r+1} \quad \left. \right\} (r=j_1)$$

$$\text{post } q-\alpha: \text{post } U_1^{(m)} \text{ post } U_2^{(m)} = \text{post } U_1^{(m+n)} \quad \text{post } U_2^{(m)} + \text{post } U_3^{(m)} = \text{post } U_2^{(m+n)} \quad 87$$

$$\text{post } U_{2r+1}^{(m)} \text{ post } U_{2r+2}^{(m)} - \text{post } U_{2r}^{(m)} \text{ post } U_{2r+1}^{(m+n)}, \text{post } U_{2r+3}^{(m)} + \text{post } U_{2r+2}^{(m)} = \text{post } U_{2r+2}^{(m+n)} + \text{post } U_{2r+1}^{(m+n)} \quad (i = l_1)$$

The noncommutative version of Baner's bridge

As in scalar case we may write

$$\sum_{l=0}^{m-1} T_\lambda \lambda^{-l-1} + \lambda^{-m} \text{pre } C \left[\frac{\text{pre } U_{2r-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2r}^{(m)}}{\lambda -} \right]_r$$

$$= \sum_{l=0}^{m-1} T_\lambda \lambda^{-l-1} + \lambda^{-m-1} \text{pre } C \left[\frac{\text{pre } U_{2r-1}^{(m+n)}}{\lambda -} \frac{\text{pre } U_{2r}^{(m+n)}}{\lambda -} \frac{\text{pre } U_{2r+1}^{(m+n)}}{\lambda -} \right]_{r-1}$$

$$= E_{2r}^{(m)}$$

$$\sum_{l=0}^{m-1} T_\lambda \lambda^{-l-1} + \lambda^{-m} \text{pre } C \left[\frac{\text{pre } U_{2r-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2r}^{(m)}}{\lambda -} \frac{\text{pre } U_{2r+1}^{(m)}}{\lambda -} \right]_r$$

$$= \sum_{l=0}^{m-1} T_\lambda \lambda^{-l-1} + \lambda^{-m-1} \text{pre } C \left[\frac{\text{pre } U_{2r-1}^{(m+n)}}{\lambda -} \frac{\text{pre } U_{2r}^{(m+n)}}{\lambda -} \right]$$

$$= E_{2r}^{(m+n)}$$

$$\dots \text{pre } C \left[\frac{\text{pre } U_{2r-1}^{(m)}}{\lambda -} \frac{\text{pre } U_{2r}^{(m)}}{\lambda -} \right] \text{ and } T_m \lambda^{-1} + \{ \text{pre } P_r^{(m)}(\lambda)^{-1} \text{pre } U_2 \text{ pre } P_0^{(m)}(\lambda) \} T_m \lambda^{-1}$$

$$+ \{ \text{pre } P_1^{(m+n)}(\lambda)^{-1} \text{pre } U_3 \text{ pre } P_0^{(m)}(\lambda) \} \{ \text{pre } P_r^{(m)}(\lambda)^{-1} \text{pre } U_2 \text{ pre } P_0^{(m)}(\lambda) \} T_m \lambda^{-1}$$

$$+ \{ \text{pre } P_r^{(m)}(\lambda)^{-1} \text{pre } U_{2r} \text{ pre } P_{r-1}^{(m)}(\lambda) \} \dots \{ \text{pre } P_r^{(m)}(\lambda)^{-1} \text{pre } U_2 \text{ pre } P_0^{(m)}(\lambda) \} T_m \lambda^{-1}$$

$$+ \{ \text{pre } P_r^{(m)}(\lambda)^{-1} \text{pre } U_{2r+1} \text{ pre } P_{r-1}^{(m+n)}(\lambda) \} \dots \{ \text{pre } P_r^{(m)}(\lambda)^{-1} \text{pre } U_2 \text{ pre } P_0^{(m)}(\lambda) \} T_m \lambda^{-1}, \dots$$

have converg. converge. \Rightarrow point. summa.

Th II Let $\{T_{mn}\}$ and $\{T_{m+n}\}$ gen. $\{\text{pre } P_r^{(m)}(\lambda)\}$ $\{\text{pre } P_r^{(m+n)}(\lambda)\}$ $m \in J$ fixed

$$1) \text{pre } U_{2r}^{(m)} = \text{pre } P_r^{(m)}(\lambda)^{-1} \text{pre } P_r^{(m+n)}(\lambda) \quad \text{pre } U_{2r+1}^{(m)} = \text{pre } P_r^{(m+n)}(\lambda)^{-1} \text{pre } P_{r+1}^{(m)}(\lambda)$$

$$\text{pre } U_0^{(m)} = I \quad \text{pre } U_1^{(m)} = \lambda - T_{mn} T_m^{-1}$$

$$\text{pre } U_{2r}^{(m+n)} = \text{pre } U_{2r+1}^{(m+n)} \in \text{pre } U_{2r+1}^{(m)} \text{ pre } U_{2r+2}^{(m)}$$

$$(\lambda - \text{pre } U_{2r+1}^{(m)}) (I - \text{pre } U_{2r}^{(m)}) = (I - \text{pre } U_{2r}^{(m+n)}) (\lambda - \text{pre } U_{2r+1}^{(m+n)}) \quad r \in I$$

$$2) \text{pre}_2 U_{2r}^{(m)} = \text{pre}_2 U_{2r}^{(m)-1} (\bar{I} - \text{pre}_1 U_{2r}^{(m)}) \quad \text{pre}_1 U_{2r+1}^{(m)} = \text{pre}_1 U_{2r+1}^{(m)} (\bar{A} - \text{pre}_1 U_{2r+1}^{(m)}) \quad 28$$

$$\text{pre}_2 U_0^{(m)} = 0 \quad \text{pre}_2 U_1^{(m)} = (\lambda T_m T_m^{-1} - \bar{I})^{-1}$$

$$(\bar{I} + \text{pre}_2 U_{2r}^{(m)}) (\bar{I} + \text{pre}_2 U_{2r-1}^{(m)}) = (\bar{I} + \text{pre}_2 U_{2r-1}^{(m)}) (\bar{I} + \text{pre}_2 U_{2r-2}^{(m)})$$

$$(\bar{I} + \text{pre}_2 U_{2r}^{(m)-1}) (\bar{I} + \text{pre}_2 U_{2r+1}^{(m)-1}) = (\bar{I} + \text{pre}_2 U_{2r-1}^{(m)-1}) (\bar{I} + \text{pre}_2 U_{2r}^{(m)-1})$$

$$3) \text{pre}_3 U_r^{(m)} = \text{pre}_2 U_r^{(m)} \text{pre}_3 U_{r-1}^{(m)} \quad (r=1, \dots) \quad \text{pre}_3 U_{-1}^{(m)} = 0 \quad \text{pre}_3 U_0^{(m)} = T_m \gamma^{-1}$$

$$\text{pre}_3 U_{2r}^{(m)-1} + \text{pre}_3 U_{2r+1}^{(m)-1} = \gamma \left\{ \text{pre}_3 U_{2r-1}^{(m)-1} + \text{pre}_3 U_{2r}^{(m)-1} \right\} \quad (r=1)$$

$$\Rightarrow \left\{ \text{pre}_3 U_{2r+3}^{(m)} + \text{pre}_3 U_{2r+1}^{(m)} \right\} = \text{pre}_3 U_{2r+1}^{(m)} + \text{pre}_3 U_{2r}^{(m)}$$

$$\Rightarrow E_{2r}^{(m)} = \sum_{j=0}^{m-1} \text{pre}_3 U_0^{(m)} \gamma^{-j-1} + \gamma^{-m} \sum_{j=0}^{2r-1} \text{pre}_3 U_j^{(m)}$$

$$E_{2r+1}^{(m)} = \gamma^m \sum_{j=0}^r \text{pre}_3 U_j^{(m)-1}$$

$$E_{-1}^{(m)} = 0 \quad E_0^{(m)} = \sum_{j=0}^{m-1} T_m \gamma^{-j-1} \quad E_{r+1}^{(m)} = E_{r-1}^{(m)} + (E_r^{(m)} - E_r^{(m)})^{-1} \quad (= J)$$

post form: $\text{post}_1 U_{2r}^{(m)} = \text{post} P_r^{(m)}(\lambda) \text{post} P_r^{(m)}(\lambda)^{-1}$

post $U_{2r+1}^{(m)} = \text{post} P_{r+1}^{(m)}(\lambda) \text{post} P_{r+1}^{(m)}(\lambda)^{-1}$

A fundamental theorem concerning operator valued orthogonal polynomials

Th3. $\{\tau_\nu\} \in N \quad \sum \tau_\nu m \lambda^{-j-1}$ pre & regular ($m=J$) then $\sum \tau_\nu \lambda^{-j-1}$ pre semi-normal; post semi-normal series defined in same way

Th4. $\sum \tau_\nu \lambda^{-j-1}$ both pre & post semi normal then

$$\text{pre} P_r^{(m)}(\lambda)^{-1} \text{pre} Q_r^{(m)}(\lambda) = \text{post} Q_r^{(m)}(\lambda) \text{post} P_r^{(m)}(\lambda)^{-1} \quad (m=J)$$

$$\text{pre} P_r^{(m)}(\lambda)^{-1} \text{pre} Q_r^{(m)}(\lambda) = (T_m - T_m T_m^{-1})^{-1} T_m = (T_m^{-1} \lambda - T_m^{-1} T_m T_m^{-1})^{-1}$$

$$\text{post} Q_r^{(m)}(\lambda) \text{post} P_r^{(m)}(\lambda) = T_m (\lambda - T_m^{-1} T_m)^{-1} = (T_m^{-1} \lambda - T_m^{-1} T_m T_m^{-1})^{-1}$$

Ch 16. The vector epsilon algorithm

N1. Lower case bold face letters denote row vectors; $\underline{\underline{z}}^T$ col vector; \bar{z} complex conjugate; $\underline{\underline{z}}^* = \bar{\underline{\underline{z}}}^T$.

The inverse of a vector

$$\underline{\underline{z}} = (z_1, z_2, \dots, z_n) : \underline{\underline{z}}^{-1} = ((z_j \bar{z}_j)^T)_1^n = (zz^*)^{-1} \underline{\underline{z}}$$

D1 If $0 < z \underline{\underline{z}}^* < \infty$, $\underline{\underline{z}} \neq 0$ we write $\underline{\underline{z}} \in \mathbb{F}$, $\underline{\underline{z}}$ is called an \mathbb{F} -vector

The vector epsilon algorithm

$$\begin{aligned} \{\underline{\underline{\varepsilon}}_r^{(m)}\} &\text{ produced from } \underline{\underline{\varepsilon}}_{r-1}^{(m)} = \underline{\underline{0}} \quad (m = J_1) \quad \underline{\underline{\varepsilon}}_0^{(m)} = \underline{\underline{s}}_m \quad (m = J) \\ \text{by } \underline{\underline{\varepsilon}}_r^{(m)} &= \underline{\underline{\varepsilon}}_{r-1}^{(m)} + (\underline{\underline{\varepsilon}}_{r-1}^{(m+1)} - \underline{\underline{\varepsilon}}_r^{(m)})^{-1} \quad (m, r \in J) \end{aligned} \quad (1)$$

Th1 Let $\{\underline{\underline{s}}_m\} \in \mathbb{F}$. Only cases in wh. breakdown can occur vector ε -alg. break down occurs when $\underline{\underline{\varepsilon}}_r^{(m+1)} = \underline{\underline{\varepsilon}}_r^{(m)}$. In those vectors $\underline{\underline{s}}_k$ are produced $\underline{\underline{\varepsilon}}_r^{(m)} \in \overline{\mathbb{F}}$.

The application of the vector epsilon algorithm

s(x) def. on interval of real axis; discrete; seek $\underline{\underline{s}}$ vector of solution values; solve iteratively so: $s_{m+1} = \phi^{(m)}(s_m)$

McLeod's Isomorphisms

Indirect representation: $\underline{\underline{z}} \mapsto \underline{\underline{\varepsilon}}$ produced during reduction \rightarrow inversion
 $\underline{\underline{s}} \mapsto \underline{\underline{s}}_m \quad \underline{\underline{\varepsilon}}_r^{(m)}$ from $\underline{\underline{s}}_m \quad \underline{\underline{\varepsilon}}_r^{(m)}$ from $\underline{\underline{s}}$ then $\underline{\underline{s}}_r^{(m)} \leftrightarrow \underline{\underline{\varepsilon}}_r^{(m)}$

N2. Set $AB + BH = \langle A, B \rangle$

Matrices of $\langle \cdot, \cdot \rangle$: $\langle \underline{\underline{r}}_x^{(n)}, \underline{\underline{r}}_y^{(n)} \rangle = I \quad (D = I^n) \quad \langle \underline{\underline{r}}_y^{(n)}, \underline{\underline{r}}_{x'}^{(n)} \rangle = Q \quad (x = J_1^n, x' = J_{D-1})$ (2)

$$\underline{\underline{Z}} = (\underline{\underline{z}}, \underline{\underline{r}}_x^{(n)})_1^n : z' \pm z'' \Leftrightarrow \underline{\underline{z}}' \pm \underline{\underline{z}}''$$

$$\underline{\underline{Z}}^2 = (\underline{\underline{z}}^2, \underline{\underline{r}}_x^{(n)})_1^n + ((\underline{\underline{z}} \underline{\underline{z}}_1, \langle \underline{\underline{r}}_x^{(n)}, \underline{\underline{r}}_x^{(n)} \rangle)_{n+1}^n)_1^{n-1} = (\underline{\underline{z}}^2)_1^n \quad (3)$$

$$\underline{\underline{Z}}^{-1} = \{(\underline{\underline{z}}^2)_1^n\}^{-1} \underline{\underline{Z}}$$

McLeod's matrices $M_1^{(n)} = \begin{pmatrix} I^{(n-1)} & O^{(n-1)} \\ O^{(n-1)} & -I^{(n-1)} \end{pmatrix}$ $2^n \times 2^n$ matrix with units along diagonal, zero elsewhere by $I^{(n)}$

$$M_2^{(n)} = \begin{pmatrix} O^{(n-1)} & I^{(n-2)} & O^{(n-1)} \\ & O^{(n-2)} & -I^{(n-2)} \\ I^{(n-2)} & O^{(n-2)} & O^{(n-1)} \\ O^{(n-2)} & -I^{(n-2)} & O^{(n-1)} \end{pmatrix} M_2^{(n)} \text{ obtained from } M_1^{(n)} \text{ by repl. } O^{(n-2+1)} \text{ by } \begin{pmatrix} I^{(n-2)} & O^{(n-2)} \\ \tilde{O}^{(n-2)} & -I^{(n-2)} \end{pmatrix}$$

A further system of anticommuting matrices

McLeod only for real vectors of finite dim

Theorem 2. $2^n \times 2^n$ matrix with units along backward diag., zero elsewhere by $\tilde{I}^{(n)}$

Define r matrices $\tilde{\Xi}_{ij}^{(r)} (\nu = I_i)$ of dim. $2^r \times 2^r$ by $\tilde{\Xi}_{11}^{(r)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\tilde{\Xi}_{ij}^{(r)} = \begin{pmatrix} \tilde{\Xi}_{i,j}^{(r-1)} & O^{(r-1)} \\ O^{(r-1)} & \tilde{\Xi}_{i,j}^{(r-1)} \end{pmatrix} (i = I_2, j = I_1), \quad \tilde{\Xi}_{i,j}^{(r)} = \begin{pmatrix} O^{(r-1)} & \tilde{I}^{(r-1)} \\ -\tilde{I}^{(r-1)} & O^{(r-1)} \end{pmatrix} i = I_2$$

$$\text{They sat. } \tilde{\Xi}_{ij}^{(r)} \tilde{\Xi}_{i,j}^{(r)} = -\tilde{I}^{(r)} (\nu = I_i), \quad \langle \tilde{\Xi}_{i,j}^{(r)}, \tilde{\Xi}_{i,j'}^{(r)} \rangle = O^{(r)} (i = I_2, j = I_{2m+1})$$

The 3. The matrices $P_{ij}^{(n)} = i \tilde{\Xi}_{ij}^{(n)} (\nu = I_i)$ and $P_j^{(n)} = \tilde{\Xi}_{11}^{(n)}, \tilde{\Xi}_{1,j+1}^{(n)} (\nu = I_j)$

$n \geq n$ satisfy (8)

A complex number extension to the Clifford algebra

$$z \leftrightarrow \bar{z} \quad z = (x_\nu \tilde{x}_\nu)_i + (y_\nu \tilde{y}_\nu)_i \quad \bar{z}_j = x_{j,i} y_{j,i}$$

$$\text{if } \tilde{z} = (x_\nu \tilde{x}_\nu)_i - (y_\nu \tilde{y}_\nu)_i \leftrightarrow \bar{z} \text{ then } z^{-1} \leftrightarrow \{(x_\nu^2 + y_\nu^2)\}^{-1} \tilde{z}$$

$$\text{must have } \tilde{z} \tilde{\bar{z}} = \{(x_\nu^2 + y_\nu^2)\} \tilde{I} \text{ then so if}$$

$$\tilde{x}_{\nu}^2 = 1 (\nu = I_i) \quad \langle x_\nu, x_{\nu'} \rangle = 0 (\nu = I_1, \nu' = I_{2m+1})$$

$$x_\nu y_{\nu'} = y_{\nu'} x_\nu \quad (\nu, \nu' = I_i) \quad (1.0)$$

$$\tilde{y}_\nu^2 = -\tilde{I} (\nu = I_i) \quad \langle y_\nu, y_{\nu'} \rangle = 0 \quad (\nu = I_1, \nu' = I_{2m+1})$$

The 4. If $P_{ij}^{(n')} \nu = I_i (n' \in I_{2m+1})$ then $x_\nu = P_{1j}^{(n')} Y_\nu = P_{1j}^{(n')} P_{2j+1}^{(n')} (n' \in I_i)$

Show (1.0)

\mathbb{Z} -numbers

$zz'z + z z' z'' + z'' z' z'$ have same form - (6)

Def. A number of the form $\mathbb{Z} = (x_\nu x_\nu)^n + (y_\nu y_\nu)^{n'}$, where (21)

$$x_\nu^2 = \begin{cases} (x_\nu)^n & (\nu = 1, 2, \dots, n) \\ 0 & (\nu = n+1, n+2, \dots) \end{cases}$$

$$x_\nu y_{\nu'} = y_\nu x_{\nu'} \quad (\nu = 1, \dots, n, \nu' = 1, \dots, n')$$

$$y_\nu^2 = \begin{cases} (y_\nu)^{n'} & (\nu = 1, 2, \dots, n') \\ 0 & (\nu = n'+1, n'+2, \dots) \end{cases}$$

$$\overline{x_\nu y_{\nu'}} = y_{\nu'} \overline{x_\nu} \quad (\nu = 1, \dots, n, \nu' = 1, \dots, n')$$

and $\{x_\nu\}, \{y_\nu\}$ finite real or comp. numbers for wh. $\sum_n |x_\nu|^2 \leq M_1^2$ $\sum_{n'} |y_\nu|^2 \leq M_2^2$

converge, called \mathbb{Z} numbers if $\mathbb{Z}, \mathbb{Z}', \mathbb{Z}'', \dots$ all have same rep. with same x_ν, y_ν with $\mathbb{Z}, \mathbb{Z}', \mathbb{Z}'', \dots \in \mathbb{Z}$. Conj. numbers $\tilde{\mathbb{Z}} = (x_\nu x_\nu)^n - (y_\nu y_\nu)^{n'}$

$$((\mathbb{Z})) = (x_\nu^2)^n + (y_\nu^2)^{n'}$$

Th 5. $\mathbb{Z}, \mathbb{Z}' \in \mathbb{Z}$ with coefft. $\{x_\nu\}, \{y_\nu\}$ $\{x'_\nu\}, \{y'_\nu\}$ resp. then

$$\{\mathbb{Z}, \mathbb{Z}'\} = \mathbb{Z} \tilde{\mathbb{Z}}' + \mathbb{Z}' \tilde{\mathbb{Z}} = 2 \left\{ (x_\nu x'_\nu)^n + (y_\nu y'_\nu)^{n'} \right\}$$

NB If $\{\mathbb{Z}, \mathbb{Z}'\} = 0$, we write $\mathbb{Z} \perp \mathbb{Z}'$.

Th 6. If $\mathbb{Z}, \mathbb{Z}' \in \mathbb{Z}$, then $\mathbb{Z} \mathbb{Z}' \in \mathbb{Z}$,

$$\mathbb{Z} \mathbb{Z}' \mathbb{Z} = \{\mathbb{Z}, \tilde{\mathbb{Z}}'\} \mathbb{Z} - ((\mathbb{Z})) \tilde{\mathbb{Z}}'$$

C If $\mathbb{Z} \perp \tilde{\mathbb{Z}}'$, then $\mathbb{Z} \mathbb{Z}' \mathbb{Z} = -((\mathbb{Z})) \tilde{\mathbb{Z}}'$

Th 7 If $\mathbb{Z}, \mathbb{Z}', \mathbb{Z}'' \in \mathbb{Z}$ then $\mathbb{Z} \mathbb{Z}' \mathbb{Z}'' + \mathbb{Z}'' \mathbb{Z}' \mathbb{Z}' \in \mathbb{Z}$:

$$\mathbb{Z} \mathbb{Z}' \mathbb{Z}'' + \mathbb{Z}' \mathbb{Z}'' \mathbb{Z} = \{\mathbb{Z}, \tilde{\mathbb{Z}}''\} \mathbb{Z} + \{\mathbb{Z}', \tilde{\mathbb{Z}}''\} \mathbb{Z} - \{\mathbb{Z}, \mathbb{Z}''\} \tilde{\mathbb{Z}}'$$

Th 8. \mathbb{Z} is \mathbb{Z} number (21) $x = \{(x_\nu^2)\}_1^n \frac{1}{2}$ $y = \{(y_\nu^2)\}_1^{n'} \frac{1}{2}$

1) if $n=0$ \mathbb{Z} sats. $\mathbb{Z}^2 + y^2 \underset{n}{=} 0$ 2) if $n'=0$ $\mathbb{Z}^2 - x^2 I = 0$

3) if $n, n' = 0$ $\mathbb{Z}^4 - 2(x^2 - y^2) \mathbb{Z}^2 + (x^2 + y^2)^2 I = 0$

Ω matrices and some results concerning norms

Th 9. Ω matrix has form $\tilde{\Omega} = (x_{ij} \tilde{X}_{ij})_i + (y_{ij} \tilde{Y}_{ij})_i$, wh. $\tilde{X}_{ij} = \sum_{h_j}^{(r)} \Gamma_{h_j}^{(r)} \tilde{X}_{ij} = \Gamma_{j_1}^{(r)} \Gamma_{h_j}^{(r)} (\tilde{X}_{ij} = \tilde{I}_j)$ wh. dist h_1, h_2, h_3, \dots make up integer set \mathbb{Z}_+^{m+n} , $r \leq (m+n)$, $\Gamma_j^{(r)} (j=1, \dots, r)$ given by (1), $\{x_{ij}\}, \{y_{ij}\}$ finite real comp numbers for wh. should $m+n$ be infinite $\sum_i^n |x_{ij}|^2 \sum_i^{n'} |y_{ij}|^2$ converge.

Th 9. x and y as in Th 8

a) If $r < \infty$, then if either $n=0$ or $n'=0$ $\tilde{\Omega}$ has two eigenvalues (first case $\pm iy$, second $\pm x$) each of mult. 2^{r-1} ; $nn' \neq 0$, $\tilde{\Omega}$ has four eigenvalues $\pm x \pm iy$, each of mult

b) $r=\infty$: infinite matrix $\tilde{\Omega}$ has pl. spectrum. If either $n=0$ or $n'=0$, two such pts: first case $\pm iy$, second $\pm \infty$; $nn' \neq 0$, four such points $\pm x \pm iy$.

Th 10. Accompanied by factor $\pm i, \pm 1$ each element of $\{x_{ij}\}, \{y_{ij}\}$ occurs once only over in each row & each col. of Ω matrix

Th 11. $\{x_{ij}\}, \{y_{ij}\}$ real: $\tilde{\Omega} = \tilde{\Omega}^*$

$$\text{Vector norms } v = (v_1, v_2, \dots, v_m) : \|v\|_1 = (|v_1|)^m, \|v\|_2 = \sqrt{(|v_1|^2 + \dots + |v_m|^2)}^{\frac{1}{2}}$$

$$\|\tilde{v}\|_\infty = \max_{i \in \mathbb{Z}_+^m} |v_i|$$

$$\tilde{A} = (a_{ij}) \text{ max norm: } \|A\|_1 = \max_{j \in \mathbb{Z}_+^m} (|a_{1j}|, \dots, |a_{mj}|), \|A\|_2 = (\max \text{ eigenvalue } AA^*)^{\frac{1}{2}}$$

$$\|A\|_\infty = \max_{i \in \mathbb{Z}_+^m} (|a_{i1}|, \dots, |a_{im}|), \|A\|_E = \left\{ \left(\max_{i,j} |a_{ij}| \right) \tilde{\Omega}_1 \right\}^{\frac{1}{2}}, \|A\|_2 \leq \|A\|_E \leq m^{\frac{1}{2}} \|A\|_2$$

Th 12. $\|\Omega\|_1 = \|\tilde{\Omega}\|_\infty = (|x_{11}|)^m + (|y_{11}|)^{n'}$. r finite: $\|\Omega\|_E = m^{\frac{1}{2}} (x^2 + y^2)^{\frac{1}{2}}$

$m = 2^m$. $\{x_{ij}\}, \{y_{ij}\}$ real: $\|\Omega\|_2 = (x^2 + y^2)^{\frac{1}{2}}$, r finite $\|\tilde{\Omega}\|_E = m^{\frac{1}{2}} \|\tilde{\Omega}\|_2$

Th 13. $\tilde{z} \Leftrightarrow \tilde{Z}$ (Ω matrix): $\|\tilde{z}\|_1 \geq \|\tilde{Z}\|_1 = \|\tilde{\Omega}\|_\infty \quad \|\tilde{z}\|_2 = \|\tilde{Z}\|_2$

\tilde{Z} pure real pure imag. (29) strict equality.

Th 14. $\{t_j\} \in F$ for finite non zero scalar

$$i) \text{Prescribe } \tilde{\varepsilon}_{-1}^{(m)} = 0 \quad (m=J) \quad \tilde{\varepsilon}_{-m}^{(m-1)} = 0 \quad (m=J) \quad \tilde{\varepsilon}_0^{(m)} = (t_0 \mu^2)_0^{(m)} \quad (m=J)$$

only reason for wh. progressive iteration of $\{\tilde{\varepsilon}_r^{(m)}\}$ by way of (4) with $r=J \neq m+J$ -

breaks down is that now some pairs $\tilde{\varepsilon}_r^{(m)}, \tilde{\varepsilon}_r^{(m)}$ equal. Resulting $\tilde{\varepsilon}_r^{(m)} \in F$

$$ii) \{t_r\} \rightarrow \{T_r\} \quad (r=J) \quad E_r^{(m)} = 0 \quad (m=J) \quad E_{-m}^{(m-1)} = 0 \quad (m=J) \quad E_0^{(m)} = (T_0 \mu^2)_0^{(m)}$$

$$\tilde{\varepsilon}_r^{(m)} \leftrightarrow E_r^{(m)} \quad \text{for } \tilde{\varepsilon}_r^{(m)} \text{ contr., also } \| \tilde{\varepsilon}_r^{(m)} \|_1 \geq \| E_r^{(m)} \|_1 = \| E_r^{(m)} \|_\infty \quad \| \tilde{\varepsilon}_r^{(m)} \|_2 = \| E_r^{(m)} \|_2$$

vector \mathbb{z} -array = vector Padé valued Padé table

The functional Padé table

$$z_0 = z(\sqrt{s}) \quad \text{if } 0 \neq 0 \quad (0=J) \quad \tilde{z} = (z(0), z(s), z(2s), \dots) \quad (\tilde{z})^{-1} = \{(\delta z(0s) \overline{z(0s)})_0^{(0)}\}^{-1} \tilde{z}$$

$$\text{functional inverse: } \tilde{z}(s)^{-1} = \overline{\tilde{z}(s)} \left\{ \int_0^\infty \tilde{z}(s) \overline{\tilde{z}(s)} ds \right\}^{-1} \quad \frac{1}{s} = \underline{R}_0$$

D4. If $\int_0^\infty z(s) \overline{z(s)} ds < \infty$ \tilde{z} is called an f / h .

$$\tilde{\varepsilon}_r^{(m)} \rightarrow \tilde{\varepsilon}_r^{(m)} \quad \tilde{\varepsilon}_{2r}^{(m)} = \tilde{\varepsilon}_{2r}^{(m)} \quad \tilde{\varepsilon}_{2r+1}^{(m)} = \delta \tilde{\varepsilon}_{2r+1}^{(m)} \quad \tilde{\varepsilon}_{r+1}^{(m)} = \tilde{\varepsilon}_{r+1}^{(m)} + \{ \delta (\tilde{\varepsilon}_r^{(m)} - \tilde{\varepsilon}_r^{(m)}) \}^{-1}$$

$$\delta \rightarrow 0 : \tilde{\varepsilon}_{rr+1}^{(m)}(\frac{1}{s}) = \tilde{\varepsilon}_{rr+1}^{(m)}(\frac{1}{s}) + \{ \tilde{\varepsilon}_r^{(m)}(\frac{1}{s}) - \tilde{\varepsilon}_r^{(m)}(\frac{1}{s}) \}^{-1} \quad (r=J \quad m=J-(k+1)-1)$$

Th 15 $t_p(\frac{1}{s}) \quad (p=I) \in F$ for finite non zero complex no. Breakdown in $\tilde{\varepsilon}_r^{(m)}(\frac{1}{s})$ if

$$\text{then } \tilde{\varepsilon}_{-1}^{(m)}(\frac{1}{s}) = 0 \quad \text{and} \quad \tilde{\varepsilon}_{-m}^{(m-1)}(\frac{1}{s}) = 0 \quad (m=J) \quad \tilde{\varepsilon}_0^{(m)} = (t_0(\frac{1}{s}) \mu^2)_0^{(m)} \quad m=J$$

occurs solely in presence of identically equal $\tilde{\varepsilon}_{rr+1}^{(m)}(\frac{1}{s})$ and $\tilde{\varepsilon}_r^{(m)}(\frac{1}{s})$
 $\tilde{\varepsilon}_r^{(m)}(\frac{1}{s}) \in F$ for constructed f .

A conjecture

$$\tilde{\varepsilon}_r^{(m)} \text{ from } \{\tilde{\varepsilon}_r^{(m)}\} \text{ has vectors satisfying } (c_{rh} s_m)_{0}^{(h)} = (c_{rh})_0^{(h)} \quad (h \in I, m=J) \quad \tilde{\varepsilon}_{2h+3}^{(m)} = 0$$

length of vectors unrestricted, $h=1$ covered by Th 15?

Th 16. $\{\tilde{\varepsilon}_r^{(m)}\}$ from 2×1 real vectors $\{\tilde{s}_m\}$ ((48) holds) satisfying (47) -

Th17 If $\{\varepsilon_i^{(m)}\}$ form complex vector $\{s_m\}$ satisfying (47) $\{c_j\}$ real, $\{s_m\} \leftrightarrow \{\tilde{s}_m\}$
 $\{s_m\}$ either pre- or post-totally definite sequence, (48) holds

Ch 17. The first confluent form of the epsilon algorithm: the rational function limit and the continued fraction limit 95

17.1 The first confluent form of the epsilon algorithm

$$\mu = \mu_0 + m\Delta\mu \quad S_m = S(\mu) \quad \varepsilon_{2r}^{(m)} = \varepsilon_{2r}'(\mu) \quad \varepsilon_{2r+1}^{(m)} = \Delta\mu \varepsilon_{2r+1}'(\mu) \quad (r=1)$$

$$\text{epsilon alg} \Rightarrow \varepsilon_1'(\mu) = 0 \quad \varepsilon_0'(\mu) = S(\mu) \quad \overset{(1)}{\varepsilon_{r+1}'(\mu)} = \varepsilon_{r+1}'(\mu + \Delta\mu) + \left\{ \frac{\varepsilon_r'(\mu + \Delta\mu) - \varepsilon_r'(\mu)}{\Delta\mu} \right\}$$

$$\Delta\mu \rightarrow 0 \quad \varepsilon_{r+1}'(\mu) = \varepsilon_{r+1}'(\mu) + \left\{ \frac{\partial_\mu \varepsilon_r'(\mu)}{\Delta\mu} \right\} \overset{(2)}{J'} \quad (r=1)$$

1) If $\{\varepsilon_r'(\mu)\}$ can be prod. from (1) by means of (2), we say $\{\varepsilon_r'(\mu)\}$ can be

prod. $\overset{\text{by means of } T}{\text{from}}$ first confluent form of ε -alg from $S(\mu)$

$$\text{N1. Set } t_{\nu}' = \overset{\text{D}_\mu}{\mathcal{D}} S(\mu) \quad (r=1) \quad H_{\nu, r} = H[t_{\nu, m}]_r \quad (r=1, \nu=1)$$

Th1. Assumed $\overset{\text{D}_\mu}{\mathcal{D}} S(\mu)$ ($\nu=1$) exist

a) all $\varepsilon_r'(\mu)$ ($r \in J_0^{(1)}$) can be prod. by means of first confluent form of ε -alg from $S(\mu)$

or b) recursive cond. terminates: $r \in J_0^{(1)}$ st $\varepsilon_{r+1}'(\mu)$ ($r \in J_0^{(1)}$) not

identically const., $\varepsilon_{r+1}'(\mu)$ is and $\varepsilon_{r+1}'(\mu) = \infty$

$$\text{For these } \mu \text{ which can be const. } \varepsilon_{2r}'(\mu) = \frac{H_{0,r}}{H_{2,r-1}'} \quad \varepsilon_{2r+1}'(\mu) = \frac{H_{1,r}}{H_{2,r-1}'}$$

Th2. If for $r \in J$ fixed $\varepsilon_r'(\mu)$ can be prod. from $S(\mu)$, derivative $\overset{\text{D}_\mu^r}{\mathcal{D}} S(\mu)$

cont. for $\mu' \in \mathbb{R}_{\mu''}^{\mu+\delta} \quad \sigma \in \mathbb{P}$ then $\hat{\varepsilon}_r'(\mu)$ ($r \in J_0^{(2)}$) can be prod. from $S(\mu+\sigma)$ and $\hat{\varepsilon}_r'(\mu) = \varepsilon_r'(\mu+\sigma)$ ($r \in J_0^{(2)}$)

Th3. If $\{\varepsilon_r'(\mu)\}$ can be prod. from $S(\mu)$, $\tilde{S}(\mu) = A + B S(\mu)$, A, B finite const. B non-zero, then two $\{\tilde{\varepsilon}_r'(\mu)\}$ correspond to $\{\varepsilon_r'(\mu)\}$ can be prod. from $\tilde{S}(\mu)$ with $\varepsilon_r'(\mu) = A + B \varepsilon_{2r}'(\mu) \quad \tilde{\varepsilon}_{2r+1}'(\mu) = B^{-1} \varepsilon_{2r+1}'(\mu)$.

Th 4 if $\{\varepsilon_r(n)\}$ can be prod from $S(n)$, const. fns $\{\tilde{\varepsilon}_r(n)\}$ can be prod from $\tilde{\varepsilon}_{r+1}(n) = 0$ $\tilde{\varepsilon}_0(n) = S(n)$ by means of $\tilde{S}_{rr}(n) = \tilde{\varepsilon}_{r+1}(n) + \sum \tilde{D}_n \tilde{\varepsilon}_r(n)$ of finite non-zero comp numbers: $\tilde{\varepsilon}_{rr}'(n) = \varepsilon_{rr}'(n)$ $\tilde{\varepsilon}_{2r+1}'(n) = \eta^{-1} \varepsilon_{2r+1}'(n)$

Th 5 $\varepsilon_r'(n)$ from $S(n)$: even order fns so produced satisfy

$$\{\varepsilon_{0r+2}'(n) - \varepsilon_{2r}'(n)\} + \{\varepsilon_{2r+2}'(n) - \varepsilon_{2r}'(n)\}^{-1} = D_n \{\tilde{D}_n \varepsilon_{2r}'(n)\}^{-1}$$

NQ. If $S(n)$ satis. irreducible inhomog. lin diff eqns $(c_0 \tilde{D}_n S(n))_0^h = c_0 S$
 S and $\{c_h\}$ const finite complex numbers $h \in \mathbb{I}$ we write $S(n) \in \mathcal{LD}_h(S)$.

If $(c_0 \tilde{D}_n S(n))_0^h = H$ irreducible, $H \neq c_0$ not sim. zero: $S = \frac{H}{c_0}$ well def.

Th 6. $\varepsilon_r'(n)$ ($r = I_0^{2h}$) can be prod from $S(n) \in \mathcal{LD}_h(S)$, and $\varepsilon_{2h}'(n) = S$

Th 7 If $(c_0 \tilde{D}_n S(n))_0^h = 0$ ($h \in \mathbb{I}$) $\{\varepsilon_h\}$ finite comp numbers with $c_0 \neq 0$, $c_r = 0$ $c_h = 0$, and $\{\varepsilon_r'(n)\}$ prod from $S(n)$ then

$$\varepsilon_0'(n) \varepsilon_1'(n) - \varepsilon_1'(n) \varepsilon_2'(n) + \varepsilon_2'(n) \varepsilon_3'(n) - \dots + \varepsilon_{2h-2}'(n) \varepsilon_{2h-1}'(n) = 0$$

identically

If $c_0 c_1 S(n) + c_1 \tilde{D}_n S(n) = 0$ ($c_0, c_1 \neq 0$) then $\varepsilon_0'(n) \varepsilon_1'(n) = -\frac{c_1}{c_0}$

identically

17.2 The rational function limit and the continued fraction limit

D2. If $\{\varepsilon_{2r}'(n)\}$ prod from $S(n)$ and for some $n \in \mathbb{I}$ a) $\{\varepsilon_{2r}'(n)\}$ terminates in sense that $\varepsilon_{2n}'(n)$ is const. ind. of n or b) $\{\varepsilon_{2r}'(n)\}$ ($r = I_0^{h-1}$) does not \rightarrow limit as $n \rightarrow \infty$ but $\varepsilon_{2n}'(n)$ does ($n=0$ included), then we say (RF) $S'(n)|_{n=0}$ exists. Both cases we may take

$$(\text{RF}) S(n)|_{n=\infty} = \varepsilon_{2n}'(n)|_{n=\infty}$$

D3 If $\{\varepsilon'_{2r}(n)\}$ goes from $S(n)$, n fixed finite, then a) termination or
in D2a) or $\varepsilon'_{2r}(n)|_{n=\infty}$ exists (finite) we say (CF) $S(n)|_{n=\infty}$ exists
first case $(CF) S(n)|_{n=\infty} = \varepsilon'_{2n}(n)$, second $(CF) S(n)|_{n=\infty} = \varepsilon'_{2r}(n)|_{n=\infty}$

Th 8. Both rat for limit & cont limit of $S(n) \in L\mathcal{D}_h(S)$ exist, and
 $(RF) S(n)|_{n=\infty} = (CF) S(n)|_{n=\infty} = S$

If $S(n) \in L\mathcal{D}_h(S)$ has limit, $S(n)|_{n=\infty} = S$

Ch 18. The second confluent form of the epsilon algorithm: the definition of an integral as the limit of a continued fraction

Consider $\int_{\mu}^{\infty} \phi(s) ds$ $\phi(s)$ comp valued in, comp number, path of integration // the real axis. limit $\sum_i \phi(\mu + s_i \Delta \mu) \Delta \mu$ $\Delta \mu \rightarrow 0$. Sufficient for $\int_{\mu}^{\infty} e^{-s} ds$ not to $\int_{\mu}^{\infty} e^{-s} ds$ or $\int_{\mu}^{\infty} \sin s ds$. $\sum_i \phi(\mu + s_i \Delta \mu) \Delta \mu \pi^{i-2-1}$
 $\rightarrow \left(\sum_i \phi(\mu + s_i \Delta \mu) \Delta \mu \pi^{i-2-1} \right)^{1/\pi}$. Seq. of hrs. $C \left[\frac{\psi(\mu; \Delta \mu)}{\pi - \psi(\mu; \Delta \mu)} \right]_r$ $I_r = \lim_{\Delta \mu \rightarrow 0}$

$$\text{NI Set } t_{-1}'' = 0 \quad t_0'' = D_{\mu} \phi(\mu) \quad \text{then } H_{m,r}'' = H[t_m'' \dots t_r''] \quad I_r$$

$$\text{Th 1. (3)} = \frac{H_{-1,r}''}{H_{0,r-1}''} \text{ formally}$$

Note that $D_{\mu}^{\alpha} \phi(\mu)$ ($\alpha \geq \mathbb{I}_0^{2+1}$) exist but already when $\phi(\mu) = e^{\mu}$, already $H_{-1,2}''$ meaningless
 $\frac{H_{-1,2}''}{H_{0,1}''}$

18.1 The second confluent form of the epsilon algorithm

Th 2. Let $D_{\mu}^{\alpha} \phi(\mu)$ ($\alpha \geq \mathbb{I}_0^{2+1}$) exist. Constr. of $\sum \epsilon_r''(\mu)$ from $\epsilon_{-1}''(\mu) = \epsilon_0''(\mu) = 0$

by means of $\{ \epsilon_{2r+1}''(\mu) - \epsilon_{2r-1}''(\mu) \} \{ \phi(\mu) + D_{\mu} \epsilon_{2r}''(\mu) \} = 1$

$$\{ \epsilon_{2r+2}''(\mu) - \epsilon_{2r}''(\mu) \} D_{\mu} \epsilon_{2r+1}''(\mu) = 1$$

either 1) may be cont. so as to produce $\epsilon_r''(\mu)$ ($r \geq \mathbb{I}_0^{2+1}$) or
2) terminates in sense that either a) $\phi(\mu) + D_{\mu} \epsilon_{2r}''(\mu) = 0$ identically and $\epsilon_{2r+1}''(\mu) = \infty$ for some r s.t. $2r \in \mathbb{I}_0^{2+1}$ or b) $\epsilon_{2r-1}''(\mu)$ is finite const and $\epsilon_{2r}''(\mu) = \infty$ identically for some r s.t. $2r+1 \in \mathbb{I}_0^{2+1}$. For those hrs. produced $\epsilon_{2r}''(\mu) = \frac{H_{-1,r}''}{H_{0,r-1}''} \quad \epsilon_{2r+1}''(\mu) = \frac{H_{2,r}''}{H_{0,r}}$

2) If $\{\varepsilon_r'(n)\}$ can be prod. from (6) by means of (7) we say $\{\varepsilon_r''(n)\}$ ^{yy} can be prod. by means of second confluent form of ε -alg. from $\phi(n)$.

18.2 The connection between the first and second confluent forms of the epsilon algorithm

Th.3. If $\{\varepsilon_r'(n)\}$ can be prod. by first conf. form of ε -alg. from $S(n)$, then correspond $\{\varepsilon_r''(n)\}$ can be prod. by means of second conf. form from $D_0 S(n)$ and $\varepsilon_{2r}''(n) = \varepsilon_{2r}'(n) - S(n)$, $\varepsilon_{2r+1}''(n) = \varepsilon_{2r+1}'(n)$ ($r=0$)

18.1.1 The theory of the second confluent form of the epsilon algorithm

Th.4. If for fixed $r \in \mathbb{J}$, $\varepsilon_r''(n)$ can be prod. by second confluent form of ε -alg. from $\phi(n)$

$D_n^{(r+1)} \phi(n)$ continuous for $\mu' \in \bar{R}_{\mu}^{(r+1)} \subset \mathbb{P}$ then $\hat{\varepsilon}_r''(n)$ ($r=0$) can be prod. from $\phi(n+\omega)$ and $\hat{\varepsilon}_r''(n) = \varepsilon_r''(n') / (\mu' - \mu_{r+1})$ ($r=0$)

2) if $\{\varepsilon_r''(n)\}$ from $\phi(n)$ then a) if B finite non zero complex number, correspond.

$\{\tilde{\varepsilon}_r''(n)\}$ can be prod. from $B\phi(n)$ and $\tilde{\varepsilon}_{2r}''(n) = B\varepsilon_{2r}'(n)$, $\varepsilon_{2r+1}''(n) = B^{-1}\varepsilon_{2r+1}'(n)$

b) $\{\varepsilon_r''(n)\}$ can be prod. from $\hat{\varepsilon}_{-1}''(n) = \hat{\varepsilon}_0''(n) = 0$ by means of

$$\{\hat{\varepsilon}_{2r+1}''(n) - \varepsilon_{2r+1}''(n)\} \{ \phi(n) + D_n \varepsilon_{2r}'(n) \} = h$$

$$\{\hat{\varepsilon}_{2r+2}''(n) - \hat{\varepsilon}_{2r}''(n)\} D_n \varepsilon_{2r+1}'(n) = h \quad h \text{ finite non zero comp. number}$$

and for this prod. $\hat{\varepsilon}_{2r}''(n) = \varepsilon_{2r}''(n)$, $\hat{\varepsilon}_{2r+1}''(n) = h \varepsilon_{2r+1}''(n)$

$$c) \{\varepsilon_{2r+2}''(n) - \varepsilon_{2r}''(n)\}^{-1} - \{\varepsilon_{2r}''(n) - \varepsilon_{2r+2}''(n)\}^{-1} = D_n \{\phi(n) + D_n \varepsilon_{2r}''(n)\}^{-1}$$

$$\{\varepsilon_{2r+3}''(n) - \varepsilon_{2r+1}''(n)\}^{-1} - \{\varepsilon_{2r+1}''(n) - \varepsilon_{2r+3}''(n)\}^{-1} = D_n \{D_n \varepsilon_{2r}''(n)\}^{-1}$$

d) if $\phi(n) = L D_h(S)$ then $\{\varepsilon_{2r}''(n)\}$ terminates with $\varepsilon_{2h+2}''(n)$; $S=0$:

$\phi(n) + D_n \varepsilon_{2h+2}''(n) = 0$ identically; $S \neq 0$: $\varepsilon_{2h+1}''(n) = S$ and

$\varepsilon_{2h+2}''(n)$ identically infinite.

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18.2. The rational function integral and the continued fraction integral

D2 If $\{\varepsilon_{2n}''(\mu)\}$ prod from $\phi(n)$ and for some $n \in \mathbb{J}$

i) $\{\varepsilon_{2n}''(\mu)\}$ terminates with $\phi(\mu') + D_{\mu'} \varepsilon_{2n}''(\mu') = 0$ ident. or $\varepsilon_{2n}''(\mu')$

identically infinite or ii) the two $\int_{\mu}^{\mu'} \phi(\frac{1}{z}) dz + \varepsilon_{2n}''(\mu') (r = \mathcal{I}_0^{n-1})$ do not tend to finite limit as $\mu' \rightarrow \infty$ but with $r=n$ does ($n=0$ included)
then $\phi(n)$ is r.f.i.

$$(RF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz = \left\{ \int_{\mu}^{\mu'} \phi(\frac{1}{z}) dz + \varepsilon_{2n}''(\mu') \right\} \Big|_{\mu'=\infty}$$

D3. If $\{\varepsilon_{2n}''(\mu)\}$ prod from $\phi(n)$ and a) seq terminates as in a) above or b)
converges to finite limit, we say $\phi(n)$ is c.f.i.; in first case $(CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz =$

$$\varepsilon_{2n}''(\mu), \text{ in second case } (CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz = \varepsilon_{2n}''(\mu) \Big|_{\mu=\infty}$$

Examples $\phi(n) = e^{-\mu} \quad \varepsilon_2(\mu) = e^{-\mu} = (RF) \int_{\mu}^{\infty} e^{-\frac{1}{z}} dz = (CF) \int_{\mu}^{\infty} e^{-\frac{1}{z}} dz$

$$\dots \quad e^{\mu} - -e^{\mu} = \dots e^{\frac{1}{z}} dz \quad e^{\frac{1}{z}} dz$$

$$\# \quad \sin \mu \quad \varepsilon_4(\mu) = \cos \mu = \dots \sin \frac{1}{z} \quad \sin \frac{1}{z}$$

$$(RF) \int_{\mu}^{\infty} D_{\mu} S(\frac{1}{z}) dz = \left\{ (RF) S(\mu) \Big|_{\mu=\infty} \right\} - S(\mu)$$

$$(CF) \quad \dots \quad = (CF) \quad \dots$$

Exercises 18.4 Euler integration

$$\sum_i t_i z^i \rightarrow f(z) \text{ suff small } z \quad (E) \sum_i t_i = f(z) \Big|_{z=1}$$

$$\text{Integrals } E(\gamma) = \int_{\gamma}^{\infty} e^{-z^{\frac{1}{2}}} \phi(\frac{1}{z}) dz \text{ corrct for } \operatorname{Re}(\gamma) = R_0 \quad \omega \in \overline{\mathbb{R}}, \quad (E) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz = E(\gamma) \Big|_{\gamma=\infty}$$

$$\text{Examples } (E) \int_{\mu}^{\infty} e^{-z^{\frac{1}{2}}} dz = -e^{\mu} \quad (E) \int_{\mu}^{\infty} \sin z dz = \cos \mu$$

(CF) \equiv (E) in above examples.

Ch 19 The rational function integral

- If second confluent term of z-adj. terminates with $\varepsilon_{\mu_1}(\mu)$: $(RF) \int_{\mu_1}^{\infty} \phi(z) dz$ otherwise
 if $\int_{\mu_1}^{\mu_2} \phi(z) dz + \varepsilon_{\mu_1}(\mu)$ (\hat{I}_0^{h-1}) does not \rightarrow limit, but $\exists. r=r$ does, then this limit = $(RF) \int_{\mu_1}^{\mu_2} \phi(z) dz$
- D1. If $\phi(\mu)$ is r.f.i. $\int_{\mu_1}^{\infty} \phi(z) dz$ exists, = $(RF) \int_{\mu_1}^{\infty} \phi(z) dz$ then $\phi(\mu)$ is reg. rat. fn. integrable (r.r.f.i.)
- D2. If $\phi(\mu)$ is r.f.i. and $\partial_{\mu} (RF) \int_{\mu_1}^{\infty} \phi(z) dz = -\phi(\mu)$, $\phi(\mu)$ is consistently rat. fn. integrable (c.r.f.i.)
- D3. If $\phi(\mu)$ is r.f.i. $\mu \in R_{\mu_1}^{\mu_2}$ and $(RF) \int_{\mu_1}^{\mu_2} \phi(z) dz = \int_{\mu_1}^{\mu_2} \phi(z) dz + (RF) \int_{\mu_2}^{\infty} \phi(z) dz$
 $\phi(\mu)$ is transitionally rational function integrable t.r.f.i. (μ_1, μ_2)
 $\phi(\mu)$ (c.r.f.i. (μ_1, μ_2)) \rightarrow $\phi(\mu)$ c.r.f.i. over this range
 we always have $(RF) \int_{\mu_1}^{\infty} B \phi(z) dz = B (RF) \int_{\mu_1}^{\infty} \phi(z) dz / B < \infty$

19.1 The degenerate theory.

$\phi(\mu) \in L\mathbb{D}_h(S) : (c_s \partial_{\mu}^s \phi(\mu))_0^{h'} = c_s S$ (irreducible). [$c_s S$ replaced by G ,
 c_s, G not sim. zero].

$G=0 : \lambda_{\tau} (\tau \in \hat{I}_0^{h'})$ roots of mult. h_{τ} , $\tau \in \hat{I}_0^{h'}$ resp. $\partial (c_s \lambda^s)_0^{h'}$, then

$$\phi(\mu) = (e^{-\lambda_{\tau'} \mu} (d_{\tau'}, \mu')_{r=0}^{h_{\tau'}-1})_{\tau'=1}^{h'}, \quad (d_{\tau'}, h_{\tau'}, -1 \neq 0 \tau' \in \hat{I}_0^{h'})$$

$G \neq 0, c_s \neq 0$ (S finite non-zero)

$$\phi(\mu) = S + (e^{-\lambda_{\tau'} \mu} (d_{\tau'}, \mu')_{r=0}^{h_{\tau'}-1})_1^{h'}, \quad (d_{\tau'}, h_{\tau'}, -1 \neq 0 \tau' \in \hat{I}_0^{h'})$$

$G \neq 0, c_s = 0$ ($\tau = \hat{I}_0^{h'-1}$) $S = \infty, \lambda_{\tau} (\tau = \hat{I}_{\tau'})$ roots of mult. h_{τ}
 $(\hat{I} = \hat{I}_{\tau'})$ resp. $\partial (c_s \lambda^s)_0^{h'-1}$ then

$$\phi(\mu) = (v'' c_{s'})^{-1} G \mu^{s'} + (d_s \mu')_{r=0}^{s'-1} + (e^{-\lambda_{\tau'} \mu} (d_{\tau'}, \mu')_{r=0}^{h_{\tau'}-1})_{\tau'=\tau'}^{h'} \quad (d_{\tau'}, h_{\tau'}, -1 \neq 0 \tau' = \hat{I}_{\tau'})$$

Th1. The fn. $\phi(\mu) \in \mathcal{LD}_h(S)$ is r.f.i. in particular when $S \neq 0$ (RF) $\int_{\mu}^{\infty} \phi(t_3) dt_3 = \infty$. 102

Th2 If $\phi(\mu) \in \mathcal{LD}_h(0)$ and $\int_{\mu}^{\infty} \phi(t_3) dt_3$ exists, then $\phi(\mu)$ is r.r.f.i.

Th3. The fn. $\phi(\mu) \in \mathcal{LD}_h(0)$ is c.t.i.

Th4. The fn. $\phi(\mu) \in \mathcal{LD}_h(S)$ is t.r.f.i. $(-\infty, \infty)$

Th5. If $\phi(\mu) \in \mathcal{LD}_h(S)$, then (RF) $\int_{\mu}^{\infty} \phi(t_3) dt_3 = (\text{E}) \int_{\mu}^{\infty} \phi(t_3) dt_3$

19.2. The general theory

Th6. The process of rational function integration is regular

Th6.7. If (RF) $\int_{\mu}^{\infty} \phi(t_3) dt_3$ exists and is finite, then $\phi(\mu)$ is c.r.f.i. (deriv. in \mathcal{D} right hand deriv.)

Th8. If $\phi(\mu_1)$ is r.f.i. Then $\phi(\mu)$ is loc. l. (μ_2, μ_3) where $\bar{R}_{\mu_2} \subseteq \bar{R}_{\mu_1}$.

19.3 A special convergence result.

D4. $r \in \mathbb{J}$ fixed, $\omega_0, \omega_1, \dots, \omega_r$ ($\omega = \mathbb{J}_0$) the segs. of finite complex numbers,
 $\text{Re}(\omega_r) > \text{Re}(\omega_{r-1})$ ($\omega = \mathbb{J}_1$). If $g(\mu')$ sat. $e^{(\omega_r)_0 \mu'} \{ g(\mu') - (\omega_r e^{-\omega_r \mu'})_0 \}_{\mu' \in \mathbb{J}_0} = 0$ ($r = \mathbb{J}_0$), we write $g(\mu') \approx (\omega_r e^{-\omega_r \mu'})_0$.

L1. If for $m, r \in \mathbb{J}$ fixed, $(-\mathcal{D}_{\mu'})^r g(\mu') \approx (\omega_r \omega_r^r e^{-\omega_r \mu'})_0$
($r = \mathbb{J}_m^{m+2r}$), the for large μ'

$$H \left[\mathcal{D}_{\mu'}^{m+r} g(\mu') \right]_r \sim (-1)^{m+r} \omega_0 \omega_1 \dots \omega_r (\omega_0 \omega_1 \dots \omega_r)^m e^{-\frac{-(\omega_0 + \dots + \omega_r) \mu'}{V(\omega_0, \omega_1, \dots, \omega_r)}}^2$$

where $V(\omega_0, \omega_1, \dots, \omega_r)$ is the Vandermonde det formed from $\omega_0, \dots, \omega_r$

Th9. Assume for $r \in \mathbb{J}$ fixed $\int_{\mu}^{\mu'} \phi(\frac{1}{z}) dz - g(\mu) \cong -\left(\frac{\alpha_0}{\alpha_0} e^{-\alpha_0/\mu'}\right)_0^r$

$g(\mu)$ well def in $\mathbb{R} \setminus \mu$, $(-\partial_{\mu}^r \phi(\mu)) \cong (\alpha_0 \alpha_0' e^{-\alpha_0/\mu'})_0^r$ ($r = \mathbb{J}_0^{2r-1}$)

$$\text{then } \int_{\mu}^{\mu'} \phi(\frac{1}{z}) dz + \varepsilon''_{2r}(\mu) \sim g(\mu) - \frac{\alpha_0 \sum_{j=0}^{r-1} (\alpha_j - \alpha_0)^2}{\prod_{j=0}^{r-1} \alpha_j^2} e^{-\alpha_0/\mu'}$$

where $\varepsilon''_{2r}(\mu)$ from $\phi(\mu)$; if, in addn, $\operatorname{Re}(\omega_r, r) \in \mathbb{P}$ for some $r \in \mathbb{J}_0^r$ then $\phi(\mu)$ is cr.f.n. and $(RF) \int_{\mu}^{\mu'} \phi(\frac{1}{z}) dz = g(\mu)$

c1. If $\phi(\mu) = (\alpha_0 e^{-\alpha_0/\mu})_0^{\infty}$ $\{\alpha_0, \{\alpha_j\}_{j \in \mathbb{J}}\} \subset \mathbb{R}$ and real and

$$\alpha_0 \in -\mathbb{R}_0 \quad (j = \mathbb{J}_0^{r-1}) \quad \alpha_j \in \mathbb{R} \quad (j = \mathbb{J}_r) \quad \alpha_0 \in \mathbb{R}_0 \quad (j = \mathbb{J})$$

the infinite sum assumed to exist for prescr. value $\Omega(\mu)$, then $\phi(\mu)$ is c.r.f.i.

c2. $r \in \mathbb{J}$ fixed, $\varepsilon'_{2r}(\mu)$ prod by first cont. form ε -alg from $S(\mu)$

$$\text{if } S(\mu) - S \cong (\alpha_0 e^{-\alpha_0/\mu})_0^r \quad (-\partial_{\mu}^r) S(\mu) \cong (\alpha_0 \alpha_0' e^{-\alpha_0/\mu})_0^r \quad (r = 2, \mathbb{J})$$

$$\text{then for large } \mu \quad \varepsilon'_{2r}(\mu) \sim S + \alpha_0 \sum_{j=0}^{r-1} \left(1 - \frac{\alpha_j}{\alpha_0}\right)^2 e^{-\alpha_0/\mu};$$

if in addn. $\operatorname{Re}(\omega_r, r) > 0$ for some $r \in \mathbb{J}_0^r$ then $(RF) S(\mu)/\mu = \infty$ exists and equal to S .

Ch 20 The continued fraction integrals

$(CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz = \varepsilon''_{2r}(\mu)$ if $\{\varepsilon''_{2r}(\mu)\}$ terminates with $\varepsilon''_{2r}(\mu)$; otherwise

$(CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz = \varepsilon''_{2r}(\mu)/r=\infty$. Values of $\phi(\frac{1}{z})$ in \bar{R}_{μ} not used

In many cases these values can be obtained by anal. cont. in many cases

$\int_{\mu}^{\infty} \phi(z) dz$ does not exist for $\mu \in \bar{R}_{\mu}$.

D1. If $\phi(\mu)$ is c.f.i. $\int_{\mu}^{\infty} \phi(\frac{1}{z}) dz$ exists = $(CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz$, $\phi(\mu)$ is r.c.f.i.

$$\dots \quad \text{if } \phi(\mu) \in \bar{R}_{\mu}, \text{ then } (CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz = -\phi(\mu), \quad \phi(\mu) \text{ is c.c.f.i.}$$

$$\dots \quad \text{for } \mu \in \bar{R}_{\mu_1}^{(k_2)}, \text{ and } (CF) \int_{\mu_1}^{\infty} \phi(\frac{1}{z}) dz = \int_{\mu_1}^{\mu_2} \phi(\frac{1}{z}) dz + (CF) \int_{\mu_2}^{\infty} \phi(\frac{1}{z}) dz \\ \phi(\mu) \text{ is c.f. } (\mu_1, \mu_2)$$

As for n.f. integration $(CF) \int_{\mu}^{\infty} B \phi(\frac{1}{z}) dz = B (CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz \quad (B \in R_0)$

if $\phi(\mu)$ is c.f.i.

20.1 The degenerate theory

Th 1. The fn. $\phi(\mu) \in \mathcal{L}\mathcal{D}_h(S)$ is c.f.i.; in part, when $S \neq 0$ $(CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz = S$

If $\phi(\mu) = \mathcal{L}\mathcal{D}_h(0)$, $\int_{\mu}^{\infty} \phi(\frac{1}{z}) dz$ exists, $\phi(\mu)$ is r.c.f.i.

The fn. $\phi(\mu) \in \mathcal{L}\mathcal{D}_h(0)$ is c.f.i.

" " " " " i.e. t.c.f.i. $(-\infty, \infty)$

If $\phi(\mu) \in \mathcal{L}\mathcal{D}_h(S)$ then $(CF) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz = (E) \int_{\mu}^{\infty} \phi(\frac{1}{z}) dz$

20.2 A remarkable equivalence

Th 2. Assumed that 1) $\phi(\mu)$ possesses derivs. all orders at μ

2) $\varepsilon''_{2r}(\mu)$ from $\phi(\mu)$, 3) $\phi(\mu) = e^{-\lambda\mu} \Theta(\mu)$ $\lambda \in R_0$ λ ind. of μ

$\Rightarrow \sum_i \left\{ \mathcal{D}_{\mu}^i \Theta(\mu) \right\} \lambda^{-i-1}$ A regular, assoc. c.f. $C \left[\frac{V_0(\mu)}{\lambda - v_0(\mu)} \right]$

Then $\varepsilon''_{2r}(\mu) = -\lambda^n C \left[\frac{V_0(\mu)}{\lambda - v_0(\mu)} \right]$

20.2.1. The integrand expressed as an integral transform

Th. 3. If $\Theta(\mu') = S[e^{\mu' s}; \epsilon]_a^b$, $\mu' = \bar{R}_{\mu-s}^{n+3}$, $s \in D(\bar{R}_a^b)$, $\delta \in \text{Parb. small}$

then $D_{\mu'}^p \Theta(\mu')$ ($p \geq 0$) exist and $D_{\mu'}^p \Theta(\mu') = S[g^p; \epsilon]_a^b$, $\epsilon \in MD(\bar{R}_a^b)$

subject to some assumptions concerning $\Theta(\mu') = S[e^{-\mu' s}; \epsilon]$, $\epsilon \in D(\bar{R}_a^b)$

we have $D_{\mu'}^p \Theta(\mu') = (-1)^p S[g^p; \epsilon']$, $(\epsilon' \in MD(\bar{R}_a^b))$

1) Assume Stieltjes integrals to be defined we set $I(\mu; \epsilon, g) = S[e^{\mu s}; \epsilon]_a^b$

($g = \bar{R}_a^b$) so that $dI(\mu; \epsilon, g) = e^{\mu s} d\epsilon(s)$ ($s = \bar{R}_a^b$). In Stieltjes

integral expansion involving $\Theta(\mu; \epsilon, g)$ we omit any s .

e.g. (3) becomes $D_{\mu'}^p \Theta(\mu) = S[g^p I(\mu; \epsilon)]_a^b$ ($p \geq 0$) and

$$S\left[\frac{e^{\mu s}}{s-\delta}; \epsilon\right]_a^b = S\pi[\gamma] I(\mu; \epsilon)]_a^b$$

2) It is assumed that $\phi(\mu') = e^{-\lambda \mu'} \Theta(\mu')$, $\Theta(\mu') = S[e^{\mu s}; \epsilon]_a^b$

$\mu' = \bar{R}_{\mu-s}^{n+3}$, $\mu \in R$, $\delta \in \text{Parb small}$, $\epsilon \in D(\bar{R}_a^b)$ and $\lambda \in BE(PS[I(\mu; \epsilon)])$

We set $\Xi(\lambda) = \sum_i \{D_{\mu'}^i \Theta(\mu)\} \lambda^{-i-1}$. Further assumed that either

1) if $\bar{R}_a^b \subseteq \bar{R}$ $\{ \Xi(\lambda) \}$ completely convergent or

2) $\bar{R}_a^b \subseteq \bar{R}_0$ $\{ \Xi(\lambda) \}$ converges for $\lambda \in BE(\bar{R}_0)$

We then write $\phi \in JT[\mu; \epsilon, \lambda]_a^b$

3) It is assumed that $\phi(\mu') = e^{-\lambda \mu'} \Theta(\mu')$, $\Theta(\mu') = S[e^{-\mu s}; \epsilon]_a^b$, $\mu' = \bar{R}_{\mu-s}^{n+3}$

where μ, δ and ϵ as in prev. def. $-\lambda \in BE(PS[-I(\mu; \epsilon)]_a^b)$

Set $\Xi'(\lambda) = \sum_i \{(-D_{\mu'})^i \Theta(\mu)\} \lambda^{-i-1}$. Further assumed that either

1) if $\bar{R}_a^b \subseteq \bar{R}$ $\{ \Xi'(\lambda) \}$ completely convergent or

2) if $\bar{R}_a^b \subseteq \bar{R}_0$ $\{ \Xi'(\lambda) \}$ converges for $-\lambda \in BE(\bar{R}_0)$

We then write $\phi \in JT'[\mu; \epsilon, \lambda]_a^b$

ζ has pts 0 increase in \bar{R}_{μ_1} , λ fixed $\phi(\zeta)$ of 22 remains finite
for $\zeta = \bar{R}_\mu$, $|\phi(\zeta)| \leq e^{c\frac{1}{\zeta}}$ for $\zeta = \bar{R}_\mu$. Thus 0 23 poss. outside at
certain pts $\zeta \in \bar{R}_\mu$

Th 4. $\phi \in \mathcal{IT}[\mu; \epsilon, \lambda]_a^b$: $\phi(\mu)$ is c.f.i.: $(cc) \int_{\mu}^{\infty} \phi(\zeta) d\zeta = e^{-\lambda \mu} S[\lambda / (\mu - \cdot)]$.

Th 5. $\phi \in \mathcal{IT}[\mu; \epsilon, \lambda]_a^b$: $\operatorname{Re}(\lambda) \in R_b$, $\phi(\mu)$ is r.c.f.i.

Th 6. $\phi \in \mathcal{IT}[\mu'; \epsilon, \lambda]_a^b$ $\mu' = \bar{R}_{\mu-\delta}'$, δ is small, $\phi(\mu)$ is
c.c.f.i.

Th 7. If $\phi \in \mathcal{IT}[\mu; \epsilon, \lambda]_a^b$, $\mu \in R_\mu$, then $\phi(\mu)$ is t.c.f.i. (μ_1, μ_2)

Th 8. $\phi \in \mathcal{IT}[\mu; \epsilon, \lambda]_a^b$ $b \in R$ then $(cc) \int_{\mu}^{\infty} \phi(\zeta) d\zeta = (E) \int_{\mu}^{\infty} \phi(\zeta) d\zeta$.

Th 9. If $\phi(\mu') = e^{-\lambda \mu'} \Theta(\mu')$, $\Theta(\mu') = S[e^{-\lambda \zeta}; \epsilon]_a^b$ ($\epsilon \in D(\bar{R})$) where

$\lambda \in BE(Ps[I(\mu'; \epsilon)]_a^b)$ $\Im = \bar{R}$ $\mu' = \bar{R}_{\mu-\delta}'$, and the series

$\sum_{n=1}^{\infty} \{D_{\mu'}^n \Theta(\mu')\}^{-\frac{1}{2n}}$ diverges, then $\phi(\mu)$ is c.f.i.

Th 10. If, for $\mu' = \bar{R}_{\mu-\delta}'$ where $S \in R$ is small, $\phi(\mu') = e^{-\lambda \mu'} \Theta(\mu')$
and either 1) $\Theta(\mu') = S[e^{-\lambda \zeta}; \epsilon]_0^{\infty}$ $\epsilon \in D(\bar{R}_0)$ where $\lambda \in BE(Ps[I(\mu'; \epsilon)]_0^{\infty})$
or 2) $\Theta(\mu') = S[e^{-\lambda \zeta}; \epsilon_2]_0^{\infty}$ $\epsilon_2 \in D(\bar{R}_0)$ where $-\lambda \in BE(Ps[I(\mu'; \epsilon_2)]_0^{\infty})$
and the series $\sum_{n=1}^{\infty} \{D_{\mu'}^n \Theta(\mu')\}^{-\frac{1}{2n}}$ diverges, then $\phi(\mu)$ is c.f.i.

Th 11. If, for $\mu' = \bar{R}_{\mu-\delta}'$, $\phi(\mu') = e^{-\lambda \mu'} \Theta(\mu')$ and

1) $\Theta(\mu') = S[e^{-\lambda \zeta}; \epsilon]_0^{\infty}$ ($\epsilon \in D(\bar{R}_0)$), $\lambda \in BE(\bar{R}_0)$ and the series

$\sum_{n=1}^{\infty} \{D_{\mu'}^n \Theta(\mu')\}^{-\frac{1}{2n}}$ diverges then $\phi(\mu)$ is c.c.f.i. for all $\mu \in \bar{R}_{\mu_1}^{\mu_2}$ and is
t.c.f.i. (μ_2, μ_3) for all $\bar{R}_{\mu_1}^{\mu_2} \subseteq \bar{R}_{\mu_3}^{\mu_2}$

2) $\Theta(\mu') = S[e^{-\lambda \zeta}; \epsilon_2]_0^{\infty}$ ($\epsilon_2 \in D(\bar{R}_0)$), $\lambda \in BE(\bar{R}_0)$ and $\sum_{n=1}^{\infty} \{D_{\mu'}^n \Theta(\mu')\}^{-\frac{1}{2n}}$
diverges then $\phi(\mu)$ is c.c.f.i. for $\mu = \bar{R}_{\mu_1}$ and is t.c.f.i. (μ_2, μ_3) for all $\bar{R}_{\mu_2}^{\mu_3} \subseteq \bar{R}_{\mu_1}^{\mu_2}$

clause i): $\phi(\frac{z}{\lambda})$ may have sing. in the range $\frac{z}{\lambda} = \underline{R}_{\mu_1, \gamma}$, in second case $\phi(\frac{z}{\lambda})$ regular in this range.

$$\text{Consider } \phi(\mu) = \omega e^{-\gamma \mu} \int_0^\infty e^{\mu s} e^{-\gamma s} ds \quad \lambda \in BE(\bar{R}_0)$$

$\omega \in \mathbb{P}$, $\eta \in \mathbb{R}$, $\mu < \eta$. $\phi(\mu)$ satis. cond. of clause i) Th 10.:

$$\alpha_s(s) = \omega \eta^{-s} (1 - e^{-\gamma s}) \quad (s \geq \bar{R}_0) \quad D_\mu^\rho \Theta(\mu) = \omega! \omega (\eta - \mu)^{-\rho-1} \quad (\rho = 1)$$

series $\sum_{\mu=0}^\infty \{D_\mu^\rho \Theta(\mu)\}^{-\frac{1}{2\rho}}$ diverges for $\mu' \in \bar{R}^{2-\delta}$: $\phi(\mu)$ c.c.f.i. for $\mu \in \bar{R}^{2-\delta}$

$\mu = \bar{R}^{2-\delta}$. $\phi(\frac{z}{\lambda}) = \omega e^{-\gamma \frac{z}{\lambda}} (\eta - \frac{z}{\lambda})^{-1}$ pole at $\frac{z}{\lambda} = \eta$ on path D integration

$$\text{Extension: } \phi(\mu) = \omega e^{-\gamma \mu} \int_0^\infty e^{\mu s} (\omega_s e^{-\gamma s})_{s=0}^\infty ds \quad \lambda \in BE(\bar{R}_0)$$

$\omega_\tau \in \mathbb{R} \setminus \mathbb{P}$, $\eta_\tau > \mu + \delta > \mu$ ($\tau = 1$) $\sum_{\tau=0}^\infty \omega_\tau$ converges. Again cond. of

clause i) D Th 10. satis. $D_\mu^\rho \Theta(\mu') = \omega! (\omega_\tau (\eta_\tau - \mu)^{-\rho-1})_{\tau=0}^\infty$

$$\leq \omega! \delta^{-\rho-1} (\omega_\tau)_{\tau=0}^\infty \quad (\text{as } \exists), \quad \phi(\mu) \text{ c.c.f.i.}, \mu = \bar{R}^{2-\delta} (\eta = \min_{\tau \in \mathbb{N}} \eta_\tau)$$

$\phi(\frac{z}{\lambda})$ simple poles at denumerably infinite set of pts η_τ in $\frac{z}{\lambda} = \bar{R}_{\mu_1, \gamma}$

Extension to multiple poles alg sing. possible

Th 12. If $\phi(\mu) = e^{-\gamma \mu} \Theta(\mu)$ and

1) $D_\mu^\rho \Theta(\mu) \geq 0$, $\mu' = \bar{R}^{\mu_1+\delta}$ ($\rho = 1$), $\lambda \in BE(\bar{R}_0)$ and the series

$\sum_{\mu=0}^\infty \{D_\mu^\rho \Theta(\mu)\}^{-\frac{1}{2\rho}}$ diverges, then $\phi(\mu)$ is c.c.f.i. $\mu = \bar{R}^{\mu_1}$ and is

t.c.f.i. (μ_2, μ_3) for all $\bar{R}_{\mu_2}^{\mu_3} \subseteq \bar{R}^{\mu_1}$

2) $(-D_\mu^\rho \Theta(\mu)) \geq 0$ ($\mu' = \bar{R}_{\mu_1-\delta}$, $\rho = 1$), $\lambda \in BE(\bar{R}_0)$ and the

series $\sum_{\mu=0}^\infty \{D_\mu^\rho \Theta(\mu)\}^{-\frac{1}{2\rho}}$ diverges then $\phi(\mu)$ is c.c.f.i. $\mu = \bar{R}_{\mu_1}$

and is c.c.f.i. (μ_2, μ_3) for all $\bar{R}_{\mu_2}^{\mu_3} \subseteq \bar{R}_{\mu_1}$.

Th. 13. Let $\alpha_j, \beta_j, \gamma_j \in R$ ($j \in I$) $\eta_j, \chi_j \in \underline{R}_0$ ($j \in J$) be such that the series

$$\sum_i (\beta_j^{-1} - \alpha_j^{-1}) \eta_j \text{ and } \sum_i (\gamma_j^{-1} \chi_j \text{ converge; set } \tilde{\mu} = \min_{j \in J} (\beta_j, \gamma_j))$$

$$1) \text{ The function } \phi(\mu) = e^{-\tilde{\mu}\mu} \left\{ \prod_{j=0}^{\infty} \left(\frac{1 - \frac{\mu}{\alpha_j}}{1 - \frac{\mu}{\beta_j}} \right) \right\} \left\{ \prod_{j=0}^{\infty} \left(1 - \frac{\mu}{\gamma_j} \right)^{\chi_j} \right\}^{-1}$$

$\lambda \in BE(\underline{R}_0)$ is c.c.f.i. for $\mu = \underline{R}_{\tilde{\mu}-\delta}$ and is t.c.f.i. (μ_2, μ_3)

$$\text{for all } \underline{R}_{\mu_2} \leq \underline{R}_{\tilde{\mu}-\delta}$$

$$2) \text{ The function } \phi(\mu) = e^{-\tilde{\mu}\mu} \left\{ \prod_{j=0}^{\infty} \left(\frac{1 + \frac{\mu}{\alpha_j}}{1 + \frac{\mu}{\beta_j}} \right)^{\eta_j} \right\} \left\{ \prod_{j=0}^{\infty} \left(1 + \frac{\mu}{\gamma_j} \right)^{\chi_j} \right\}^{-1} (\lambda \in BE(\underline{R}))$$

is c.c.f.i. for $\mu = \underline{R}_{\tilde{\mu}+\delta}$ and t.c.f.i. (μ_2, μ_3) for all $\underline{R}_{\mu_2} \leq \underline{R}_{\tilde{\mu}+\delta}$

Th. 14. If $\phi(\mu) = e^{-\tilde{\mu}\mu} S [e^{\mu^2/\sigma}]_n^b$ ($\lambda \in BE(\underline{R}_a^b)$) $\sigma \in (\underline{R}_a^b)$ $\underline{R}_a^b \subset R$, then

1) $\phi(\mu)$ is c.c.f.i. for $\mu = R \Rightarrow \phi(\mu)$ is t.c.f.i. (μ_1, μ_2) for all $\underline{R}_{\mu_1} \leq R$

$$3) (c_F) \int_{\mu}^{\infty} \phi(\frac{t}{\mu}) dt^{\frac{1}{2}} = (E) \int_{\mu}^{\infty} \phi(\frac{t}{\mu}) dt^{\frac{1}{2}} (\mu \neq R)$$

4) if $Re(\lambda) \notin R_b$ then $\phi(\mu)$ is n.e.f.u. for $\mu = R$.

Th. 15. If $\phi(\mu) \in e^{-\tilde{\mu}\mu} \Theta(\mu)$ $\lambda \in BE(\underline{R}_0')$, $D_{\mu'}^{\tilde{\mu}} \Theta(\mu') \geq 0$ ($\mu' = \underline{R}^0$ $\Rightarrow \exists$)

and for one value δ $\mu \in R$ the sequence $\{D_{\mu'}^{\tilde{\mu}} \Theta(\mu)\}$ is totally monotone

then 1)-3) of Th. 14 hold, and if $Re(\lambda) \in R$, $\phi(\mu)$ is ref.i. for $\mu = R$

Th. 16. If $\phi(\mu) = e^{-\tilde{\mu}\mu} \Theta(\mu)$ $\lambda \in BE(\underline{R}_0')$, $D_{\mu'}^{\tilde{\mu}} \Theta(\mu') \geq 0$ ($\mu' = \underline{R}^0$ $\Rightarrow \exists$)

and for one finite real value δ for the series $\sum_i \{D_{\mu'}^{\tilde{\mu}} \Theta(\mu)\} z^i$ converges for $|z| \leq \underline{R}_0'$ to an R.H.W fn, then conclusion 1) over the holds.

$\phi \in \mathcal{G}T[\mu, \sigma, \lambda]_a^b$ $\underline{R}_a^b \equiv \underline{R}$, then $\phi(\mu)$ c.f.i. at one pt; when $\underline{R}_a^b \not\equiv \underline{R}_0$ $\phi(\mu)$ c.f.i. for all $\mu \in$ semi infinite interval $\underline{R}_a^b \subset R$, c.f.i. for all $\mu \in R$.

Ch. 21. The third and fourth confluent forms of the epsilon algorithm

21.1. The fourth confluent form of the epsilon algorithm

Th.1. Assumed that deriv. $D_\mu^2 \Theta(\mu)$ ($\mu = 0$) of $\Theta(\mu)$ exist. If not then $\{\varepsilon_{-1}^{(m)}(\mu; \lambda)\}$

can be proved by applying ε -rule to $\varepsilon_{-1}^{(m)}(\mu; \lambda) = O(m=2)$

$\varepsilon_0^{(m)}(\mu; \lambda) = (\{\varepsilon_{-1}^{(m)}(\mu; \lambda)\} \lambda^{-2-1})_0^{m-1}$ ($m=2$) then same rule for $m=3$ fixed

from $\varepsilon_{-1}^{(m)}(\mu; \lambda) = 0$ $\varepsilon_0^{(m)}(\mu; \lambda) = (\{\varepsilon_{-1}^{(m)}(\mu; \lambda)\} \lambda^{-2-1})_0^{m-1}$ by means of

$$\{\varepsilon_{2r+1}^{(m)}(\mu; \lambda) - \varepsilon_{2r-1}^{(m)}(\mu; \lambda)\} \{(\Theta/\mu) - \lambda \varepsilon_{2r}^{(m)}(\mu; \lambda) + D_\mu \varepsilon_{2r}^{(m)}(\mu; \lambda)\}_j^2 = 1$$

$$\{\varepsilon_{2r+2}^{(m)}(\mu; \lambda) - \varepsilon_{2r}^{(m)}(\mu; \lambda)\} \{\lambda \varepsilon_{2r+1}^{(m)}(\mu; \lambda) + D_\mu \varepsilon_{2r+1}^{(m)}(\mu; \lambda)\}_j^2 = 1$$

— · —

Letting $\varepsilon_{-2}^{(m)}(\mu; \lambda) = \infty$ have

$$\{\varepsilon_{2r+2}^{(m)}(\mu; \lambda) - \varepsilon_{2r}^{(m)}(\mu; \lambda)\}_j^{-1} + \{\varepsilon_{2r+2}^{(m)}(\mu; \lambda) - \varepsilon_{2r}^{(m)}(\mu; \lambda)\}_j^{-1} = \varepsilon$$

$$(\lambda + D_\mu) \{(\Theta/\mu) - (\lambda - D_\mu) \varepsilon_{2r}^{(m)}(\mu; \lambda)\}_j^{-1} \quad (r=2)$$

Th.2. Assumed that deriv. $D_\mu^2 \Theta(\mu)$ ($\mu = 2$) of $\Theta(\mu)$ exist $\sum_i \{\varepsilon_{-1}^{(m)}(\mu; \lambda)\}_j z^j$ is normal series. Padé quota $\{R_{i,j}(\mu; z)\}$ gen. by this series satisfy

$$R_{-1,j}(\mu; z) = \infty \quad R_{0,j}(\mu; z) = (\{\varepsilon_{-1}^{(m)}(\mu; \lambda)\} z^2)_0^j \quad (j=2)$$

$$R_{i,-1}(\mu; z) = 0, \quad R_{i,0}(\mu; z) = [(\{\varepsilon_{-1}^{(m)}(\mu; \lambda)\} z^2)_0^i]^{-1} \quad (i=2)$$

$$\begin{aligned} & \{R_{i+1,j+1}(\mu; z) - R_{i,j}(\mu; z)\}_j^{-1} + \{R_{i-1,j-1}(\mu; z) - R_{i,j}(\mu; z)\}_j^{-1} \\ & = z \{(1 + z D_\mu) \{(\Theta/\mu) - (1 - z D_\mu) R_{i,j}(\mu; z)\}\}_j^{-1} \quad (i,j=2) \end{aligned}$$

$\tilde{\Theta}(\mu)$ obtained by $\Theta(\mu') \tilde{\Theta}(\mu') = 1$ ($\mu = \overline{R}_{\mu-1}^{k+1}$).

Ch. 22. A partial differential equation associated with the epsilon algorithm¹⁰

22.1 The ϕ -array

Numbers $\{\phi_r^{(m)}\}$ in two dimensional array: m diag., r col. Int.

$\Theta_r^{(m)} \{ \phi_{r-1}^{(mn)}, \phi_r^{(m)}, \phi_{r+1}^{(mn)}, \phi_{r+2}^{(m)} \} = 0$ numbers at vertices of triangle in ϕ -array

$$(\text{cases in wh. } \Theta_r^{(m)} \text{ ind. } \delta) \text{ i.e. } \phi_{r-1}^{(mn)} - \phi_r^{(m)} + \phi_{r+1}^{(mn)} - \phi_{r+2}^{(m)} = 0 \quad (1)$$

also z-alg.

redraft ϕ -array in terms of numbers with odd and even suffix

dual triangle alg:

$$\begin{matrix} \circ \phi_0^{(m)} \\ \circ \phi_0^{(m)}, \phi_0^{(m)} \\ \circ \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)} \\ \circ \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)} \\ \circ \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)} \\ \circ \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)}, \phi_0^{(m)} \end{matrix}$$

$$\begin{matrix} \circ \phi_r^{(m)} \\ \circ \phi_r^{(m)}, \phi_r^{(m)} \\ \circ \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)} \\ \circ \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)} \\ \circ \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)} \\ \circ \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)}, \phi_r^{(m)} \end{matrix}$$

$$\circ \Theta_r^{(m)} \{ \circ \phi_{r-1}^{(m)}, \phi_r^{(m)}, \circ \phi_{r+1}^{(m)}, \phi_{r+2}^{(m)} \} = 0$$

$$\Theta_r^{(m)} \{ \circ \phi_{r-1}^{(m)}, \phi_r^{(m)}, \circ \phi_{r+1}^{(m)}, \phi_{r+2}^{(m)} \} = 0$$

$$\begin{matrix} \circ \phi_{r-1}^{(mn)}, \phi_{r-1}^{(m)} \\ \circ \phi_{r-1}^{(mn)}, \phi_{r-1}^{(m)}, \phi_r^{(m)} \\ \circ \phi_{r-1}^{(mn)}, \phi_{r-1}^{(m)}, \phi_r^{(m)}, \phi_r^{(m)} \\ \circ \phi_r^{(m)} \end{matrix}$$

$$\text{e.g. } \circ \phi_r^{(m)} - \circ \phi_{r-1}^{(mn)} = \phi_{r-1}^{(mn)} - \phi_r^{(m)}, \quad \phi_r^{(m)} - \circ \phi_{r-1}^{(mn)} = - \{ \circ \phi_r^{(mn)} - \circ \phi_r^{(m)} \} \quad (4)$$

$$(2) \rightarrow \circ \phi_r^{(m)} - \circ \phi_{r-1}^{(mn)} = \phi_{r-1}^{(mn)} - \phi_{r-1}^{(m)}, \quad \phi_r^{(m)} - \circ \phi_{r-1}^{(mn)} = \circ \phi_r^{(mn)} - \circ \phi_r^{(m)} \quad (5)$$

$$\text{z-alg} \rightarrow \circ \varepsilon_r^{(m)} - \circ \varepsilon_{r-1}^{(mn)} = (\varepsilon_{r-1}^{(mn)} - \varepsilon_{r-1}^{(m)})^{-1}, \quad \varepsilon_r^{(m)} - \circ \varepsilon_{r-1}^{(mn)} = (\circ \varepsilon_r^{(mn)} - \circ \varepsilon_r^{(m)})^{-1}$$

$$\text{numbers } \circ \phi_r^{(m)}, \phi_r^{(m)}, \circ \phi_r^{(mn)} \text{ alone: } (4) \rightarrow \circ \phi_{r+1}^{(mn)} + \circ \phi_{r-1}^{(mn)} + \circ \phi_r^{(m-1)} + \circ \phi_r^{(m)} = 4 \circ \phi_r^{(m)} \quad (\phi \text{ sim})$$

$$(5) \rightarrow \circ \phi_{r+1}^{(m-1)} - \circ \phi_r^{(mn)} + \circ \phi_{r-1}^{(mn)} - \circ \phi_r^{(m)} = 0 \quad (\phi \text{ sim})$$

$$\{ \circ \varepsilon_r^{(m)} - \circ \varepsilon_{r-1}^{(mn)} \}^{-1} + \{ \circ \varepsilon_r^{(m)} - \circ \varepsilon_{r+1}^{(mn)} \}^{-1} = \{ \circ \varepsilon_r^{(m)} - \circ \varepsilon_r^{(m-1)} \}^{-1} + \{ \circ \varepsilon_r^{(m)} - \circ \varepsilon_r^{(m)} \}^{-1} \quad (\varepsilon \text{ sim})$$

22.2 The derivation of partial differential equations

$$\begin{matrix} \circ \phi_r^{(m)} \\ \circ \phi_{r-1}^{(m)}, \phi_r^{(m)} \\ \circ \phi_{r-1}^{(m)}, \phi_r^{(m)}, \phi_{r+1}^{(m)} \\ \circ \phi_{r-1}^{(m)}, \phi_r^{(m)}, \phi_{r+1}^{(m)}, \phi_{r+2}^{(m)} \\ \circ \phi_r^{(m)} \end{matrix}$$

coordinates x, y

interval $\delta, \Delta t$, Number

lying in triangle

do diff at pts in $x-y$ plane:

$x, y-2\delta$

$x-\delta, y-\delta$

$x+\delta, y-\delta$

$x-2\delta, y$

x, y

$x+2\delta, y$

$x-\delta, y+\delta$

$x+\delta, y+\delta$

$x, y+2\delta$

$$x = x' + 2\delta S \quad y = y' + 2(mn)\delta$$

$$(4) \rightarrow \phi_o(x, y) - \phi_o(x-2\delta, y) = \phi_i(x-\delta, y+\delta) - \phi_i(x-\delta, y-\delta) \quad \text{8232}$$

$$\phi_o(x, y+2\delta) - \phi_o(x, y) = - \{ \phi_i(x+\delta, y+\delta) - \phi_i(x-\delta, y+\delta) \}$$

$$\delta \rightarrow 0 : \frac{\partial \phi_0}{\partial x} = \frac{\partial \phi_1}{\partial y}, \quad \frac{\partial \phi_0}{\partial y} = -\frac{\partial \phi_1}{\partial x} \quad \text{Cauchy Riemann (9)}$$

$$(5) \rightarrow \frac{\partial \phi_0}{\partial x} = \frac{\partial \phi_1}{\partial y}, \quad \frac{\partial \phi_0}{\partial y} = \frac{\partial \phi_1}{\partial x} \quad (10)$$

22.3 The partial differential equation associated with the epsilon algorithm

$$\varepsilon_r^{(n)} = \delta^{-1} \varepsilon_0(x,y), \quad \varepsilon_r^{(n)} = \delta^{-1} \varepsilon_1(x,y); \quad \frac{\partial \varepsilon_0}{\partial x} \frac{\partial \varepsilon_1}{\partial y} = 1 \quad \frac{\partial \varepsilon_1}{\partial x} \frac{\partial \varepsilon_0}{\partial y} = 1 \quad (11)$$

Th1 $\varepsilon_0(x,y), \varepsilon_1(x,y)$ sat (11): functional relationship $f\{\varepsilon_0(x,y), \varepsilon_1(x,y)\} = 0$ holds

22.4 The partial differential equation of the Padé surface

$$\text{Eliminate } \phi_1 \text{ from (9): } \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (\phi_0 \text{ same}). \quad (10) \rightarrow \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \Leftrightarrow \text{PDEs}$$

$$\Leftrightarrow (11) \rightarrow \frac{\partial \left\{ \frac{\partial \varepsilon}{\partial x} \right\}^{-1}}{\partial x} = \frac{\partial \left\{ \frac{\partial \varepsilon}{\partial y} \right\}^{-1}}{\partial y} \quad \text{for both } \varepsilon_0, \varepsilon_1: \text{part diff. eqn. of Padé surface}$$

22.5 Symmetric algorithms: self conjugate systems \Rightarrow partial differential equations

D1 Suppose that each of Θ_T ($T=0,1$) is dual alg concerning A_T, B_T, C_T, D_T

$$(T=0,1): \begin{matrix} B_T \\ A_T \\ C_T \\ D_T \end{matrix} \quad \begin{matrix} A_T \\ B_T \\ C_T \\ D_T \end{matrix} \quad \text{can be written in form } \Theta_T(A_T, B_T, C_T, D_T) = 0 \quad (15)$$

If A_T and B_T , and C_T and D_T ($T=0,1$) interchanged: $\Theta_T(B_T, A_T, D_T, C_T) = 0$

$T=0,1$ identical with (15), algorithm called symmetric

Θ_T : $D-A = C-B$: $C-B = D-A$ unchanged. Alg of (5) \Rightarrow alg symmetric

D2. Two first order part diff eqns in ind vars. $x+y$, dep vars. ϕ_0, ϕ_1 ; if system unchanged when x and y and, sim. to ϕ_0, ϕ_1 interchanged, system is called self-conjugate

Th2. System \Rightarrow part diff eqns deriving from symmetric dual alg.
alg. is self-conjugate

(9) \rightarrow (10) and (11) self-conj

22.6 Adjoint partial differential equations

D.3 Given part diff eqn in dep var $\phi_0(x,y)$, another in $\phi_1(x,y)$, we say these systems are adjoint if they may be der. from two torsim first order eqns by elimination

Th. 3. A part diff. eqn. deriving in first case from single Lorange alg is self-adjoint

part diff. eqn. ① Padé surface is self adjoint

22.7 Special solutions ② the partial differential equation of the Padé surface

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ has gen soln } \phi(x,y) = f_0(x+iy) + f_0'(x-iy) : f_0, f_0' \text{ twice}$$

$f_0(z) + f'(z)$ ^{2nd} order diff. able. bcs ① \Rightarrow

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial x^2} = 0 \dots \phi(x,y) = f_1(x+iy) + f_1'(x-iy) \text{ similarly} \quad (21)$$

Th. 4. Eqn (14) has solns $\varepsilon(x,y) = f(x+y) - z(x,y) = f'(x-y) : f(z) f'(z)$
twice diff. able (no ③) \Rightarrow

hyp. part. diff. eqn. (15) and first order finite diff. approx.

$\phi(x+2\delta, y) + \phi(x-2\delta, y) = \phi(x, y+2\delta) + \phi(x, y-2\delta)$ have same soln (21)

ind. size of δ

Th. 5. Part diff. eqn (10) and its first order finite diff. approx.

$$\left\{ \varepsilon(x,y) - \varepsilon(x-2\delta, y) \right\}^{-1} + \left\{ \varepsilon(x,y) - \varepsilon(x+2\delta, y) \right\}^{-1} =$$

$$\left\{ \varepsilon(x,y) - \varepsilon(x, y-2\delta) \right\}^{-1} + \left\{ \varepsilon(x,y) - \varepsilon(x, y+2\delta) \right\}^{-1}$$

both possess solns of Th. 4.

Ch 23 Error analyses of the epsilon algorithm

23.1 A perturbation analysis

Th1. Assumed errors in $\{\varepsilon_r^{(m)}\}$ due solely to presence of abs errors $S(m)$ in $\{S_m\}$

Abs error $S(m')$ in value of S_m , ($m' \in \mathbb{J}$ fixed) violates numbers $\varepsilon_r^{(m)} (r \in \mathbb{J}_1)$,
 $m = g_{m'-r}^{(m')} (m \in \mathbb{J})$; first approx: abs error $\delta_r^{(m)}(m')$ in each of these numbers det.
 by applying $(\varepsilon_{r+1}^{(m)} - \varepsilon_r^{(m)}) \{ \delta_r^{(m)}(m') - \delta_{r+1}^{(m)}(m') \} + (\varepsilon_r^{(m)} - \varepsilon_{r+1}^{(m)}) \{ \delta_{r+1}^{(m)}(m') - \delta_{r+2}^{(m)}(m') \} = 0$
 $(r = g_{m-g_{m'-r}}^{(m)}, (m \in \mathbb{J}))$ to initial values $\delta_0^{(m)}(m') = S(m')$, $\delta_r^{(m+1)}(m') = 0 (r = \mathbb{J}_1)$,
 $\delta_r^{(m'-r+1)}(m') = 0 (r = \mathbb{J}_0^{m'-1})$. Abs error $\delta_r^{(m)}$ in $\varepsilon_r^{(m)}$ due to influence of
 $S(m')$ in S_m , ($m = \tilde{g}_{m'}^{(m')}$) satisfies $|\delta_r^{(m)}| \leq (|\delta_r^{(m)}(m')|)_{m' \leq m}^{\text{max}}$.

— — —

Grossly conservative: assume errors $\{\delta_r^{(m)}(m')\}$ combine in worst poss way, unlikely important
 in the case to say nothing of many.

23.2 The convergence and stability of the epsilon algorithm

Derive estimates of orders of magnitude of numbers produced; obtain approx
 values of factors wh. magnify or diminish small errors at each stage:
 stability analysis

23.2.1 Totally monotone and totally oscillating initial sequences

L1. $\{S_\tau\}$ totally monotone, $\nu \in \mathbb{J}$ fixed: $\{(-1)^\nu \Delta_\tau^\nu S_\tau\}$ totally monotone

L2 $\{S_\tau\}$ totally monotone; $H_{m,r}^{(\nu)} = (-1)^\nu H [\Delta_\tau^\nu S_{\tau+m}]_r$ ($\nu, m, r \in \mathbb{J}$):
 either $H_{m,r}^{(\nu)} > 0 (r = \mathbb{J}_0^{r-1})$ ($r \in \mathbb{J}$) or $H_{m,r}^{(\nu)} > 0 (r = \mathbb{J})$
 $= 0 (r = r')$

Th2. $\{\varepsilon_r^{(m)}\}$ from totally monotone $\{S_m\}$: $\varepsilon_{2r}^{(m)} > 0 \quad \varepsilon_{2m+1}^{(m)} < 0 (r, m \in \mathbb{J})$ (5)

D1 if $S_m = S + \left(\sum_{\tau=0}^{\nu} \frac{1}{(m+\omega+\tau)^{-1}} \right), (m \in \mathbb{J}) \quad S \in \mathbb{R} \quad \omega \in \mathbb{P} \quad \omega_j \in \mathbb{R}_0 (\tau \in \mathbb{J}_1)$

series for S_0 converges, we write $\{S_m\} \in M\bar{I} \{S/\omega_j; \omega\}$

c1. If $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in M \cap \{S/a_\nu; \omega\}$ $S \in \underline{R}_0$ then (s) hold

D2 If $S_m = S + (b_\nu \lambda_\nu^m)_0^\infty$ ($m \in \mathbb{J}$) $S \in R$ $b_\nu \in \underline{R}_0$ $\lambda_\nu \in \underline{R}'_0$ $(\nu = I)$

series for S_0 converges, we write $\{S_m\} \in O \cap \{S/b_\nu; \lambda_\nu\}$

c2. If $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in M \cap \{S/b_\nu; \lambda_\nu\}$ where $S \in \underline{R}_0$, (s) hold

D3 If $\{(-1)^r S_r\}$ is totally monotone, $\{S_r\}$ is totally oscillating

L3 $\{S_r\}$ totally oscillating, $\nu' \in \mathbb{J}$ fixed: $\{(-1)^{\nu'} \Delta_{\nu'}^0 S_r\}$ totally oscillating

L4 $\{S_r\}$ totally oscillating: $\hat{H}_{m,r}^{(\nu')} = (-1)^{r+m} H [\Delta_{\nu'}^0 S_{m+r}]_r$ ($r, m \in \mathbb{J}$)

for $r, m \in \mathbb{J}$ either $\hat{H}_{m,r}^{(\nu')} > 0$ ($r = \exists_0^{\nu'-1}$) ($r \in \mathbb{J}$) or $\hat{H}_{m,r}^{(\nu')} < 0$ ($r = \mathbb{J}$).

Th 3. $\{\varepsilon_r^{(m)}\}$ from totally oscillating sequence $\{S_r\}$: $(-1)^m \varepsilon_r^{(m)} \geq 0$ $(-1)^{m+1} \varepsilon_r^{(m+1)} \leq 0$ ($m \in \mathbb{J}$)

D4 $S \in R$, $\{S_r\}$ totally oscillating: $\{S + S_r\}$ substantially totally oscillating

Th 4 $\{\varepsilon_r^{(m)}\}$ from subst. tot. osc $\{S_m\}$: for a fixed $r \in \mathbb{J}$ $\varepsilon_r^{(m)} (m \in \mathbb{J})$

oscillating sequence, $\varepsilon_{2r+1}^{(m)} (m \in \mathbb{J})$ ($m = I$) alternating

D5. $S_m = S + (-1)^m (a_\nu \prod_{\eta=0}^p (m+\omega+\eta))_0^\infty$ ($m \in \mathbb{J}$) $S \in R$, $a_\nu \in \overline{R}$, $\omega \in \underline{R}_0$ ($\nu = I$), series for S_0 converges, we write $\{S_m\} \in O \cap \{S/a_\nu; \omega\}$

c1 If $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in O \cap \{S/a_\nu; \omega\}$ conclusion of Th 4 hold

D6 $S_m = S + (-1)^m (b_\nu \lambda_\nu^m)_0^\infty$ ($m \in \mathbb{J}$) $S \in R$ $b_\nu \in \underline{R}_0$ $\lambda_\nu \in \underline{R}'_0$ ($\nu = I$)

series for S_0 converges, we write $S_m \in O \cap \{S/b_\nu; \lambda_\nu\}$.

c2. $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in O \cap \{S/b_\nu; \lambda_\nu\}$; conclusion 1) Th 4 hold

23.2.2 Orders of magnitude of the numbers $\{\varepsilon_r^{(m)}\}$

$$\text{L5. For } r=3 \quad \frac{H[\tau!]_r}{H[(r+2)!]_{r-1}} = \frac{1}{r+1} \quad \frac{H[(r+3)!]_{r-1}}{H[(r+1)!]_r} = \frac{(r+1)(r+2)}{2} \quad \frac{H[\tau!]_r}{H[\tau!]_{r-1}} = (r!)^2$$

Def If small $\delta \in \mathbb{P}$ prescribed, an $m' \in \mathbb{N}$ exists such that $|f_1(m) - f_2(m)| \leq \delta (m \geq m')$. we write $f_1(m) \doteq f_2(m)$.

$$\frac{f_1(m)}{f_2(m)} \Big|_{m=\infty} = 1 : f_1(m) \doteq f_2(m). \quad m' \text{ in Def need not be large. } \delta \text{ prescr., is suff}$$

$$\text{large } S_m = S + a_1(\omega m)^{-1}, \quad / \frac{S_m - S}{S} \leq \delta (m \geq m') \quad \text{then } m' = 0$$

Th 5 $\{\varepsilon_r^{(m)}\}$ from $\{S_m\}$, $S_m = S + a(\omega m)^{-1} (-1)^r \Delta_m^r S_m \doteq a r! (\omega m)^{-(r+1)}$
 $\exists r \in \mathbb{R}, \omega, a \in \mathbb{P}$; then $\varepsilon_{2r}^{(m)} = S + a(\omega m)^{-1} \varepsilon_{2r+1}^{(m)} = \frac{(r+1)(r+2)(\omega m)^{-(r+2)}}{2a} \quad (r \geq 1) \quad (1)$

C1 If $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in MI\{S/a_0; \omega\}$ $a_0 = a > 0$, (1) holds.

Th 6. $\{\varepsilon_r^{(m)}\}$ from $S_m = S + a(\omega m)^{-1} (m \geq 1)$

$$\varepsilon_{2r}^{(m)} = S + \frac{a}{(r+1)(\omega m)^{r+1}} \quad \varepsilon_{2r+1}^{(m)} = - \frac{(r+1)(r+2)(\omega m)^{r+2}(\omega m+r+1)}{2a} \quad (m \geq 1)$$

no improvement in convergence

Th 7. $\{\varepsilon_r^{(m)}\}$ from $\{S_m\}$ s.t. $S_m = S + (-1)^m a(\omega m)^{-1}$

$$(-1)^{r+m} \Delta_m^r \{(-1)^m S_m\} \doteq a r! (\omega m)^{-r-1} \quad (r \geq 1, \quad a \neq 0 \text{ then})$$

$$\varepsilon_{2r}^{(m)} \doteq S + \frac{(-1)^m (r!)^2 a}{2^{2r} (\omega m)^{2r+1}} \quad \varepsilon_{2r+1}^{(m)} \doteq \frac{(-1)^{m+1} 2^{2r+1} (m'm)^{2r+1}}{(r!)^2 a} \quad (r \geq 1) \quad (1\#)$$

C1. $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in OI\{S/a_0; \omega\}$ $a_0 = a > 0$: relationships (1<#>) hold.

Substantial improvement in convergence.

Th 8. $\{\varepsilon_r^{(m)}\}$ from $S_m = S + (c_r \lambda_r)^m \quad (m \geq 1) \quad \exists r \in \mathbb{R} \quad |c_r| \in \mathbb{P}$

$|\lambda_r| > |\lambda_{r+1}|, \quad r = 1, \dots, n$, series converges for $m = 0$, then

$$\varepsilon_{2r}^{(m)} \doteq S + \frac{c_r \prod_{j=0}^{r-1} (\lambda_r - \lambda_j)^2 \{ \lambda_r^m \}}{\prod_{j=0}^{r-1} (1 - \lambda_j)^2}, \quad \varepsilon_{2r+1}^{(m)} \doteq \frac{\prod_{j=0}^{r-1} (1 - \lambda_j)^2}{\{ \prod_{j=0}^{r-1} (\lambda_r - \lambda_j)^2 \} (\lambda_r - 1) \lambda_r^m}$$

c1. $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in M\bar{D}\{S/b_\nu; \lambda_\nu\}$ $b_\nu \in \mathbb{R}$ $\lambda_\nu > \lambda_{\nu+1}$, ($\nu = \bar{J}$)

$$\varepsilon_{2r}^{(m)} = S + \frac{b_r \sum_{\nu=0}^{r-1} (\lambda_\nu - \lambda_\nu)^2 \} \lambda_\nu^m}{\prod_{\nu=0}^{r-1} (1 - \lambda_\nu)^2} \quad \varepsilon_{2r+1}^{(m)} = \frac{\frac{r-1}{\nu=0} (1 - \lambda_\nu)^2}{b_r \sum_{\nu=0}^{r-1} (\lambda_\nu - \lambda_\nu)^2 \} (\lambda_r - 1) \lambda_r^m} \quad (\nu = \bar{J})$$

c2 $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in \bar{M}\bar{D}\{S/b_\nu; \lambda_\nu\}$ s.t. $b_\nu > 0$ $\lambda_\nu > \lambda_{\nu+1}$, ($\nu = \bar{J}$)

$$\varepsilon_{2r}^{(m)} = S + \frac{b_r \sum_{\nu=0}^{r-1} (\lambda_\nu - \lambda_\nu)^2 \} (-\lambda_r)^m}{\prod_{\nu=0}^{r-1} (1 + \lambda_\nu)^2} \quad \varepsilon_{2r+1}^{(m)} = - \frac{\frac{r-1}{\nu=0} (1 + \lambda_\nu)^2}{b_r \sum_{\nu=0}^{r-1} (\lambda_\nu - \lambda_\nu)^2 \} (\lambda_{r+1} - \lambda_r)^m}$$

Since $\left\{ \frac{r-1}{\nu=0} (1 + \lambda_\nu)^2 \right\} > \left\{ \frac{r-1}{\nu=0} (1 - \lambda_\nu)^2 \right\}$ $r = \bar{J}$, $\lambda_\nu \in R'_0$ ($\nu = \bar{J}$) coefft

① except term in oscillating case less than coefft in monotonic case,
but differ slightly.

23.2.3 The amplification formulae

$$\begin{array}{cccc} \varepsilon_{r-1}^{(m)} & \varepsilon_r^{(m-1)} & \varepsilon_{r+1}^{(m-2)} & \varepsilon_{r+2}^{(m-2)} (1 + \varepsilon_{r+2}^{(m-2)}) \\ & & \varepsilon_{r-1}^{(m-1)} (1 + \varepsilon_{r+1}^{(m-1)}) & \varepsilon_{r+2}^{(m-1)} (1 + \varepsilon_{r+2}^{(m-1)}) \\ \varepsilon_{r-1}^{(m-1)} & \varepsilon_r^{(m)} (1 + \varepsilon_r^{(m)}) & \varepsilon_{r+1}^{(m)} (1 + \varepsilon_{r+1}^{(m)}) & \\ & \varepsilon_r^{(m-1)} & \varepsilon_{r+1}^{(m-1)} & \varepsilon_{r+2}^{(m)} (1 + \varepsilon_{r+2}^{(m)}) \\ & & \varepsilon_{r+1}^{(m-1)} & \end{array}$$

Th 9. Accurate values of $\{\varepsilon_r^{(m)}\}$ derived from precex. initial sequence, that

introduc. of rel. error $\varepsilon_r^{(m)}$ in $\varepsilon_r^{(m)}$ causes rel. errors $\delta \varepsilon_{r+1}^{(m-1)}$ in $\varepsilon_{r+1}^{(m-1)}$, $\varepsilon_{r+2}^{(m)}$ in

$\varepsilon_{r+2}^{(m-1)}, \dots$ to appear, abs. error $\alpha_r^{(m)}$ in $\varepsilon_r^{(m)}$ leads to abs. error $\alpha_{r+1}^{(m-1)}$ in $\varepsilon_{r+1}^{(m-1)}, \dots$. Let $\eta_r^{(m)} = (\varepsilon_r^{(m)} - \varepsilon_r^{(m-1)})$. Ignoring second & higher powers of $\alpha_r^{(m)}$

$$\varepsilon_{r+1}^{(m-1)} = - \frac{\varepsilon_r^{(m)} \eta_r^{(m)} \varepsilon_r^{(m)}}{\varepsilon_{r+1}^{(m-1)}} \quad \zeta_{r+1}^{(m)} = \frac{\varepsilon_r^{(m)} \eta_r^{(m)} \varepsilon_r^{(m)}}{\varepsilon_{r+1}^{(m)}} \quad \alpha_{r+2}^{(m-2)} = \frac{\varepsilon_r^{(m)} \eta_r^{(m)} \eta_{r+1}^{(m)} \varepsilon_r^{(m)}}{\varepsilon_{r+2}^{(m-2)}}$$

$$\varepsilon_{r+2}^{(m)} = - \frac{\varepsilon_r^{(m)} \eta_r^{(m)} \eta_{r+1}^{(m)} \varepsilon_r^{(m)}}{\varepsilon_{r+2}^{(m)}} \quad \varepsilon_{r+2}^{(m-1)} = \frac{\varepsilon_r^{(m)}}{\varepsilon_{r+2}^{(m-1)}} \quad \left\{ 1 - (\eta_r^{(m)} + \eta_{r+1}^{(m)}) \eta_{r+1}^{(m)} \right\} \varepsilon_r^{(m)}$$

Similarly, to first order of approxm. $\omega_{r+1}^{(m-1)} = -\gamma_r^{(m)} \omega_r^{(m)}$ $\omega_{r+2}^{(m)} = \gamma_r^{(m)} \omega_r^{(m)}$

$$\omega_{r+2}^{(m-2)} = \gamma_r^{(m)} \gamma_{r+1}^{(m-1)} \omega_r^{(m)} \quad \omega_{r+2}^{(m)} = -\gamma_r^{(m-1)} \gamma_{r+2}^{(m-2)} \omega_r^{(m)}$$

$$\omega_{r+2}^{(m-1)} = \left\{ 1 - (\gamma_r^{(m)} + \gamma_{r+1}^{(m-1)}) \right\} \omega_r^{(m)}$$

23.2.4 An analysis of stability

D8. If for certain pair $m, r \in \mathbb{J}$, presented $\frac{\text{small } \delta \in P}{\frac{\varepsilon_{2r+1}^{(m)}}{\varepsilon_{2r}^{(m)}} - S} / \leq \delta$

($r' = r, m' = m; r' = r+1, m' = \overline{I}_{m-2}^m$), we say that S is relatively large; if, however, $| \frac{S}{\varepsilon_{2r'}^{(m)}} | \leq \delta$ ($r' = r, m' = m; r' = r+1, m' = \overline{I}_{m-2}^m$) S rel. small

Deal with $\varepsilon_{2r}^{(m)} \doteq S + g_r(m)$ $\varepsilon_{2r+2}^{(m)} \doteq S + g_{r+1}(m)$ $g_r(m), g_{r+1}(m) \rightarrow 0, n \rightarrow \infty$

vals. of $m + r$ can always be found s.t. S always either rel. small or large

Th. 10. $\{\varepsilon_r^{(m)}\}$ from $\{S_m\}$ of th. 5.; propagation of error as follows:

S rel. large

S rel. small

$$\varepsilon_{2r+1}^{(m-1)} \doteq \varepsilon_{2r+1}^{(m)} \doteq \frac{2S(rm)(w+rm)}{(r+2)a} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+1}^{(m-1)} \doteq \varepsilon_{2r+1}^{(m)} \doteq \frac{2(w+rm)}{(r+2)} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-1)} \doteq \varepsilon_{2r+2}^{(m)} \doteq \frac{(w+rm)^2}{(r+2)^2} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-2)} \doteq \varepsilon_{2r+2}^{(m)} \doteq \frac{(w+rm)^2}{(r+1)(r+2)} \doteq \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-1)} \doteq \left\{ 1 - \frac{2(w+rm)^2}{(r+2)^2} \right\} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-1)} \doteq \frac{r+2}{r+1} \left\{ 1 - \frac{2(w+rm)^2}{(r+2)^2} \right\} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r}^{(m-1)} \doteq \varepsilon_{2r}^{(m)} \doteq -\frac{a}{2r(r+1)S} \varepsilon_{2r-1}^{(m)}$$

$$\varepsilon_{2r}^{(m-1)} \doteq \varepsilon_{2r}^{(m)} \doteq -\frac{(w+rm)^2}{2r} \varepsilon_{2r-1}^{(m)}$$

Growth of error from one even order col to next: S large & small
error increased by factor of order of $(w+rm)^2$; S large this takes
place from even order col to next odd, S small two factors $\delta(w+rm)$
Abs error propagation same. Illustrated by $\{1 + (w+rm)^{-1}\}$

Th. 11. $\{\varepsilon_r^{(m)}\}$ from $\{S_m\}$ of Th. 7; propagation of error as follows:

S rel. large

$$\varepsilon_{2r+1}^{(m-1)} \div \varepsilon_{2r+1}^{(m)} \div \frac{(-1)^{m+1} 2^{2r+1} (\omega_m)^{2r+1}}{(r!)^2 a} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-2)} \div \varepsilon_{2r+2}^{(m)} \div \frac{1}{4} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-1)} \div \frac{1}{2} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-1)} \div \varepsilon_{2r}^{(m)} \div -\frac{a}{2^{2m} \{(-1)!\}^2 (\omega_m)^{2r+1}} \varepsilon_{2r-1}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-2)} \div \varepsilon_{2r+2}^{(m)} \div \frac{4(\omega_m)^2}{(r+1)^2} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m-1)} \div \frac{2(\omega_m)^2}{(r+1)^2} \varepsilon_{2r}^{(m)}$$

S both rel. large and small:

$$\alpha_{2r+1}^{(m-1)} \div \alpha_{2r+1}^{(m)} \div \frac{2^{4r+2} (\omega_m)^{4r+2}}{(r!)^4 a^2} \alpha_{2r}^{(m)}$$

$$\alpha_{2r+2}^{(m-2)} \div \alpha_{2r+2}^{(m)} \div \frac{1}{4} \alpha_{2r}^{(m)}, \alpha_{2r+2}^{(m-1)} \div \frac{1}{2} \alpha_{2r}^{(m)}$$

$$\alpha_{2r}^{(m-1)} \div \alpha_{2r}^{(m)} \div \frac{\{(-1)!\}^4 a^2}{2^{4r+4} (\omega_m)^{4r+2}} \alpha_{2r-2}^{(m)}$$

S large: loss of accuracy from even order est to odd, by from even to even rel error slightly attenuated

S rel small: little loss of accuracy from even to odd, but error mult by factor of order $(\omega_m)^2$ from odd to even. Abs error small shows abs error slightly attenuated. S small, loss of rel acc. due to decrease in estimates of limit $(\omega_m)^2$ at each stage

Example $\{1 + (-1)^m (\omega_m)^{-1}\}$

Th. 12 $\{\varepsilon_r^{(m)}\}$ from $\{S_m\} \in MD\{S/b_\nu; \lambda_\nu\}$ $b_\nu > 0, \lambda_\nu > \lambda_{\nu+1}, (\nu=1)$; propagation of error as follows

S rel. large

$$\varepsilon_{2r+1}^{(m-1)} \div \frac{S \lambda_\nu \prod_{j=0}^{r-1} (1-\lambda_{\nu+j})^2}{(\lambda_{\nu-1} \prod_{j=0}^{r-1} (\lambda_{\nu+j}-\lambda_\nu)^2) b_\nu \lambda_\nu} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m)} \div -\left(\frac{\lambda_\nu}{1-\lambda_\nu}\right)^2 \varepsilon_{2r}^{(m)}$$

S rel. small

$$\varepsilon_{2r+1}^{(m-1)} \div \frac{\lambda_\nu}{1-\lambda_\nu} \varepsilon_{2r}^{(m)}$$

$$\varepsilon_{2r+2}^{(m)} \div -\frac{b_\nu \prod_{j=0}^{r-1} (\lambda_{\nu+j}-\lambda_\nu)^2}{b_{\nu+1} \prod_{j=0}^{r-1} (\lambda_{\nu+j}-\lambda_\nu)^2 \lambda_{\nu+1}^m} \varepsilon_{2r}^{(m)}$$

S both rel large and small:

$$\alpha_{2r+1}^{(m)} \doteq \frac{\sum_{n=0}^{r-1} (1-\lambda_r)^n}{(1-\lambda_r)^2 \left\{ \prod_{n=0}^{r-1} (\lambda_r - \lambda_{r-n})^2 \right\} b_r \lambda_r} \alpha_{2r}^{(m)}$$

$$\alpha_{2r+2}^{(m)} \doteq - \left(\frac{\lambda_r}{1-\lambda_r} \right)^2 \alpha_{2r}^{(m)}$$

— —

λ_r close to unity abs error considerably increased. S large rel error increased by same factor, S small further increase due to decr. in size of estimate. $\lambda_r \leq \frac{1}{2}$ ε -only stable

Theorem /s or m oscillating seqs. Plot. by reversing signs of all $\{\lambda_i\}$

Large and small S $\alpha_{2r+2}^{(m)} \doteq \left(\frac{\lambda_r}{1+\lambda_r} \right)^2 \alpha_{2r}^{(m)}$. abs error attenuated.

Whether rel error also depends on whether S large small. ε -only unconditionally stable

23.2.5 The smoothing effect of the epsilon algorithm

Th 13. $\{\varepsilon_r^{(m)}\}$ from $\{S_m\}$ If S rel. large, one number $\varepsilon_{2r}^{(m)}$ infected by (rel or abs) error of S then resulting errors $A_{2r}^{(m)}$ in $\varepsilon_{2r}^{(m)}$ given by

$$A_{2r}^{(m)} = \frac{(2(1-r))}{4^{(r-r)}} \quad r = 0, \pm 1, \quad m = \sum_{n=2(r-r')}^{m'} (m \in \mathbb{Z})$$

$$\begin{array}{ccc} & & \frac{1}{16} \\ \text{---} & & \frac{1}{16} \\ \delta & \frac{1}{4} & \frac{1}{4} \\ \delta & \frac{1}{2} & \frac{1}{4} \\ \delta & \frac{1}{4} & \frac{1}{4} \\ & & \frac{1}{16} \end{array}$$

23.5 A singular rule for the epsilon algorithm

ε -array from 2nd minimal zeros: $\{\varepsilon_{2r}^{(m)}\}$ rat pts.; $\{\varepsilon_{2r+1}^{(m)}\}$ polymers in ε^{-1} poss $\varepsilon_{2r+1}^{(m)}$ and $\varepsilon_{2r+2}^{(m)}$ have eq values, $\varepsilon_{2r+2}^{(m)}$ has pole at this pt. Phenomenon isolated: Padé quot. cannot have eq. values from $m=2n+2$

Consider

$$\begin{array}{cccc}
 & \Sigma_{r-1}^{(m)} & & \\
 (\dagger) \Sigma_{r-2}^{(m+1)} & & (\dagger) \Sigma_r^{(m)} & \\
 \Sigma_{r-3}^{(m+2)} & \Sigma_{r-1}^{(m+1)} & & \Sigma_{r+1}^{(m)} \\
 (\dagger) \Sigma_{r-2}^{(m+2)} & & (\dagger) \Sigma_r^{(m+1)} & \\
 & \Sigma_{r-1}^{(m+2)} & &
 \end{array}$$

The 14 Numbers displayed in above fig can be det by appl. e-alg to
precn. seq.: Let

$$C = \Sigma_{r-1}^{(m+2)} \left\{ 1 - \Sigma_{r-1}^{(m+2)} \Sigma_{r-1}^{(m+1)} \right\}^{-1} + \Sigma_r^{(m)} \left\{ 1 - \Sigma_{r-1}^{(m)} \Sigma_{r-1}^{(m+1)} \right\}^{-1} - \Sigma_{r-3}^{(m+2)} \left\{ 1 - \Sigma_{r-3}^{(m+2)} \Sigma_{r-1}^{(m+1)} \right\}^{-1}$$

$$\text{then } \Sigma_{r+1}^{(m)} = C \left\{ 1 + C \Sigma_{r-1}^{(m+1)} \right\}^{-1}$$

Suppose $\Sigma_{r-2}^{(m+1)}$, $\Sigma_{r-2}^{(m+2)}$ nearly eq.: $\Sigma_{r-1}^{(m+1)}$ large but each \dagger

$(1 - \Sigma_{r-1}^{(m+2)} \Sigma_{r-1}^{(m+1)})$, $(1 - \Sigma_{r-1}^{(m)} \Sigma_{r-1}^{(m+1)})$, $(1 - \Sigma_{r-3}^{(m+2)} \Sigma_{r-1}^{(m+1)})$ well det,

C well det, $\Sigma_{r+1}^{(m)}$ well det

$$\Sigma_{r-2}^{(m+1)} = \Sigma_{r-2}^{(m+2)} : \Sigma_{r+1}^{(m)} = \Sigma_{r-1}^{(m+2)} + \Sigma_{r-1}^{(m)} - \Sigma_{r-3}^{(m+2)} \quad (47)$$

Example $((x!)^{-1} 2^x)_0^{m-1}$ as mat in array: $x=2$; number array
const with help of (47)