

Notes in sequel to the abstract theory of the ε -algorithm

Notation 1. R denotes a prescribed ring assumed with zero element 0 and assumed to have a unit element 1; R_1 denotes the set of elements of R having a two sided inverse, $C\{R\}$ denotes the centre of R , i.e. the set of elements $c \in R$ such that $ac = ca$ for all $a \in R$ and $a(bc) = (ab)c, a(cb) = (ac)b, c(ab) = (ca)b$ for all $a, b \in R$.

Other elements of R are denoted by lower case

roman letters.

Definition 1 (i) To the class R we append a number z such that ~~the~~ the products az^m ($m=0,1,\dots$) are defined for all $a \in R$ and for which, in addition, $(az^m)(bz^n) = abz^{m+n}$ ($m, n=0,1,\dots$) for all $a, b \in R$. $az^m + bz^m = (a+b)z^m$ ($m=0,1,\dots$) for all $a, b \in R$.

(ii) The set of numbers a_0, a_1z, \dots, a_nz^n ($a_p \in R, p=0,1,\dots,n$) which we write in the form $\sum_{p=0}^n a_p z^p$ and denote by $A(z)$ is a member of a class of similar sets $\sum_{p=0}^{n'} a'_p z^p$ where $a'_p = a_p$ ($p=0,1,\dots,n$), $a'_p = 0$ ($p=n+1, n+2, \dots, n'$); this class is called a polynomial with coefficients over R and is denoted also by $A(z)$ a symbol such as $A(z)$. The system of all such polynomials is denoted by $P\{R\}$. Addition, subtraction, and multiplication of polynomials is defined with the help of the relationships of clause (i).

(ii) ~~The inverse of $A(z)^{-1}$ of the polynomial $A(z)$~~

(ix) ~~Arithmetic operation~~ Operations of addition, subtraction and multiplication involving elements of both R and $P\{R\}$ are defined by ~~writing~~ embedding R in $P\{R\}$ according to the scheme $a \rightarrow az^0$ for $(a \in R)$.

(iii) The reciprocal ~~of~~ $A(z)^{-1}$ of the polynomial $\sum_{i=0}^n a_i z^i \in P\{R\}$ ^{for which $a_0 \neq 0$ and $a_i \in R$} ~~is a class~~ ^{may also be represented as a} system of sets $a_0, a_1 z, \dots, a_n z^n$ for which relationships () hold. The system of all such reciprocals of polynomials is denoted by $IP\{R\}$. Multiplication ~~is~~ over $IP\{R\}$ is defined ^{in terms of that over $P\{R\}$} according to the relationship $\{A(z)A'(z)\}^{-1} = A(z)^{-1}A'(z)^{-1}$.

(v) ~~The product of $A(z) \in P\{R\}$ and $A'(z) \in IP\{R\}$ is defined, and~~

$$A(z)A'(z)^{-1} = A'(z)^{-1}A(z) = I z^0 (\in P\{R\}) = (I z^0)^{-1} (\in IP\{R\}) = I (\in R).$$

(iv) A finite set of ^{ordered elements of P} combinations of elements ordered elements of $IP\{R\}$ and $P\{R\}$ all have the form whose members consist of ordered elements of $IP\{R\}$ and $P\{R\}$ and have the form

$$D_1(z)^{-1} N_1(z) D_2(z)^{-1} N_2(z) \dots D_r(z)^{-1} N_r(z)$$
~~or the form~~
~~that obtained by suppressing $D_i(z)^{-1}$ or that obtained by sub $N_r(z)$ or both of these terms~~
~~of $IP\{R\}$ and $P\{R\}$ (the appearance of this expression may be changed by setting~~
 ~~$D_i(z) = I z^0$, i.e. omitting the term $D_i(z)^{-1}$, and also in a similar way by omitting~~
~~the term $N_r(z)$ is called a rational function with coefficients over R . (The~~
~~appearance of the expression~~ The system of all such rational functions is denoted by $R\{R\}$.

Addition ~~is~~ over $R\{R\}$ is defined by juxtaposition of ~~the~~ sets ~~sets~~ corresponding to constituent terms. The rational functions

$$D_1(z)^{-1} N_1(z) D_2(z)^{-1} N_2(z) \dots \hat{N}_{r'}(z) \dots D_r(z)^{-1} N_r(z) \\ + D_1(z)^{-1} N_1(z) D_2(z)^{-1} N_2(z) \dots \tilde{N}_{r'}(z) \dots D_r(z)^{-1} N_r(z)$$

and

$$D_1(z)^{-1} N_1(z) D_2(z)^{-1} N_2(z) \dots \{ \hat{N}_{r'} + \tilde{N}_{r'}(z) \} \dots D_r(z)^{-1} N_r(z)$$

are defined to be equal, the same holding true for the alternative forms described above.

Multiplication over $R\{R\}$ is defined in terms of the direct product of sets corresponding to constituent terms. The rational functions

$$\hat{D}_1(z)^{-1} \hat{N}_1(z) \hat{D}_2(z)^{-1} \hat{N}_2(z) \dots \hat{D}_r(z)^{-1} \hat{N}_r(z) \tilde{N}_1(z) \tilde{D}_2(z)^{-1} \tilde{N}_2(z) \dots \tilde{D}_r(z)^{-1}$$

and

$$D_1(z)^{-1} \cancel{N_1(z)} \cancel{D_2(z)} \cancel{N_2(z)} \dots D_r(z)^{-1} \{ \hat{N}_r(z) N_1(z) \} \tilde{D}_2(z)^{-1} \dots \tilde{D}_r(z)^{-1}$$

are defined to be equal. The same holds true for

$$\hat{D}_1(z)^{-1} \dots \hat{N}_{r-1}(z) \hat{D}_{r-1}(z)^{-1} \tilde{D}_1(z)^{-1} \tilde{D}_2(z)^{-1} \dots \tilde{N}_r(z)$$

and

$$\hat{D}_1(z)^{-1} \dots N_{r-1}(z) \{ \tilde{D}_1(z) \hat{D}_{r-1}(z) \}^{-1} \tilde{D}_2(z)^{-1} \dots \tilde{N}_r(z)$$

and also for

$$\hat{D}_1(z)^{-1} \dots \hat{N}_{r-1}(z) \hat{D}_{r-1}(z)^{-1} \tilde{D}_2(z)^{-1} \dots N_r(z)$$

and

$$\hat{D}_1(z)^{-1} \dots \hat{N}_{r-1}(z) \tilde{D}_2(z)^{-1} \dots N_r(z)$$

In all of the above cases in which rational functions are defined to be equal, analogous relationships ^{also} hold for the forms alternative to () described above.

(ig) Operations of addition, subtraction and multiplication involving elements of $\mathbb{R}\{R\}$ and $\mathbb{I}\mathbb{P}\{R\}$ are defined by embedding \mathbb{R} in $\mathbb{R}\{R\}$ according to

the scheme $a \rightarrow (Iz^0)^{-1}(az^0)$ ($a \in \mathbb{R}$) and adopting a similar artifice with respect to the other two systems.

It is clear that $\mathbb{R}\{R\}$ is a ring with inverse: that the rational functions of $\mathbb{R}\{R\}$ satisfy the axioms that define a ring is easily verified; the unit element of $\mathbb{R}\{R\}$ is $(Iz^0)^{-1}(Iz^0)$; the function represented by expression () has an inverse, namely

$$R(z)^{-1} = N_1(z)^{-1} D_1(z) \dots N_2(z)^{-1} D_2(z) N_1(z)^{-1} D_1(z),$$

(as is easily verified $R(z)R(z)^{-1} = R(z)^{-1}R(z) = I(z)$) and the same is true of the three alternative forms of this expression described above.

Although all manipulations given in this section concern elements of $\mathbb{R}\{R\}$, ~~we write~~ ^{we adopt, for the sake of conciseness, Notation 2.} All expressions derived by embedding elements of \mathbb{R} , ~~in $\mathbb{R}\{R\}$~~ $\mathbb{P}\{R\}$, and $\mathbb{I}\mathbb{P}\{R\}$ in $\mathbb{R}\{R\}$ ~~simply~~ ^{are written} as simple elements of these systems: thus ~~thus we use~~ the symbol $aA(z)$ ~~is~~ ^{denotes} $(Iz^0)^{-1}(az^0)$ $(Iz^0)^{-1}A(z)$, and so on.

In order to reassure the reader we remark that for most of the time the rational functions with which we shall be dealing are no more complex ^{in structure} than the simple fraction $D(z)^{-1}N(z)$.

Definition. Euclid's algorithm with respect to the unbounded sequence

of ~~the~~ finite prime integers τ_r ($r=1,2,\dots$) applied to the rational function $D_0(z)^{-1} N_0(z) \in \mathbb{R}\{z\}$ is the process of determining the ~~sequences~~ ^{three} sequences of polynomials $D_r(z), N_r(z)$ ($r=1,2,\dots$) ^{where} from

$$D_r(z) = \sum_{\nu=0}^{\tau_r} d_{r,\nu} z^\nu, \quad N_r(z) = \sum_{\nu=0}^{\tau_r} n_{r,\nu} z^\nu \quad (r=1,2,\dots)$$

$$D_r(z) = \sum_{\nu=0}^{\tau_{r-1}} d_{r,\nu} z^\nu \quad (r=1,2,\dots)$$

according to the scheme

$$d_{r,s} = n_{r-1,0}^{-1} \left\{ d_{r-1,s} - \sum_{\nu=0}^{s-1} n_{r-1,s-\nu} d_{r-1,\nu} \right\} \quad (s=0,1,\dots,\tau_r-1)$$

$$n_{r,s} = d_{r-1,\tau_r+s} - \sum_{\nu=0}^{\tau_r} n_{r-1,\tau_r+s-\nu} d_{r-1,\nu} \quad (s=0,1,\dots, \max_{r-1}(\sum_{\nu=0}^{\tau_r} n_{r-1,\tau_r-1-\nu} - \tau_r))$$

$$d_{r,s} = n_{r-1,s} \quad (s=0,1,\dots,\tau_{r-1})$$

for $r=1,2,\dots$. The process is said terminate if for some $r \geq 1$, the polynomial $N_r(z)$ produced by means of the above rules is identically $N_r(z) = 0(z)$.

If Euclid's algorithm can be applied to ~~the~~ if the above process can be applied to the rational function $D_0(z)^{-1} N_0(z)$ in the sense that ~~the~~ for these coefficients ~~the~~ $\{n_{r,\nu}\}$ that are produced before termination we have $n_{r,\nu} \in \mathbb{R}_I$ ($r=1,2,\dots$), this function is said to be T-regular.

~~Formulas~~
~~Relationships () which define Euclid's algorithm, are motivated by the~~

The polynomials $B_r(z)$, $D_r(z)$ and $N_r(z)$ derived during the application of Euclid's algorithm satisfy the relationships

$$D_{r-1}(z)^{-1} N_{r-1}(z) = \left\{ D_r(z) + D_r(z)^{-1} N_r(z) z^{\tau_r} \right\}^{-1} \quad (r=1, 2, \dots)$$

indeed, formulae () which define ~~Euclid's~~ the algorithm ~~is~~ derived from these relationships.

Theorem . If Euclid's algorithm can be applied to ~~a rational~~ the rational function $R(z) = D_0(z)^{-1} N_0(z)$, then it terminates. If $R(z)$ is regular with respect to the sequence $\{1, 1, \dots\}$ then, ^{using} the notation of Definition ...

and setting $\delta = \delta_0$, $\eta_0 = \eta$, ~~if $\eta \leq \delta - 1$,~~
~~if $\eta \leq \delta - 1$, we have $\delta_{2r+1} \leq \delta - r - 1$, $\eta_{2r+1} \leq \delta - r - 1$, $\delta_{2r+2} \leq \delta - r - 1$, $\eta_{2r+2} \leq \delta - r - 2$ ($r=0, 1, \dots$)~~
 (i) if ~~$\eta \leq \delta - 1$~~ ~~$\delta_{2r} \leq \eta - r$~~ ~~$\delta_{2r+1} \leq \eta - r + 1$~~ ~~$\eta_{2r+1} \leq \eta - r$~~ ~~$\delta_{2r+2} \leq \eta - r$~~ ($r=0, 1, \dots$)

(ii) if $\eta > \delta - 1$, we have $\delta_{2r+1} \leq \eta - r$, $\eta_{2r+1} \leq \eta - r - 1$, $\delta_{2r+2} \leq \eta - r - 1$, $\eta_{2r+2} \leq \eta - r - 1$ ($r=0, 1, \dots$)

for all polynomials $D_r(z)$, $N_r(z)$ produced.

If $R(z)$ is regular with respect to the sequence $\{2, 2, \dots\}$ then, using the

same notation, we have

(iii) if $\eta \leq \delta - 1$, we have $\delta_r \leq \delta - r$, $\eta_r \leq \delta - r - 1$ ($r=1, 2, \dots$)

(iv) if $\eta > \delta - 1$, we have $\delta_r \leq \eta - r + 1$, $\eta_r \leq \eta - r$ ($r=1, 2, \dots$).

Proof. Working through the first two stages of the algorithm, we find that when

$$\eta \leq \delta - 1, \eta_1 = \delta - \tau_1, \delta_1 = \eta_0, \eta_1 \leq \delta_0 - \tau_1, \delta_2 = \eta_1, \eta_2 \leq \eta_0 - \tau_2$$

$\eta_0 \leq \delta_0 - 1$ that we have $\delta_1 = \eta_0, \eta_1 \leq \delta_0 - \tau_1, \delta_2 = \eta_1 \leq \delta_0 - \tau_1, \eta_2 \leq \max(\eta_0 - \tau_2, \delta_0 - \tau_1 - 1)$

and hence that $\max(\delta_2, \eta_2) < \max(\delta_0, \eta_0)$. Continuing in this way, we

show that $\max(\delta_{2r}, \eta_{2r})$ ~~is strictly~~ $(r=1, 2, \dots)$ is a strictly decreasing

sequence ~~is~~. Since $\eta_{2r-1} = \delta_{2r}$ $(r=1, 2, \dots)$ it follows that at some stage ~~of~~

~~the algorithm~~ either ~~$N_{2r-1}(z) = 0$~~ the polynomial $N_{2r-1}(z)$ vanishes whilst

$D_{2r}(z)$ does not ~~or that this is true of the polynomials $N_{2r}(z), D_{2r}(z)$~~ . Hence

the algorithm terminates. The analysis of the case in which $\eta_0 > \delta_0 - 1$ is

conducted in a similar fashion. The same method suffices for the proof of

the remainder of the theorem.

For the sake of completeness, we remark that it can occur that ~~the~~

the coefficients of the highest powers of z in $D_r(z)$ and $B_r(z)$ are complementary

factors of zero; in this case the degree of $N_r(z)$ ^{occurring in formulae (1)} is less than ~~the~~ its

maximum possible maximum. This accounts for the occurrence of inequality

signs in the above theorem.

Noncommutative continued fractions

We consider two ~~types~~^{systems} of noncommutative continued fractions ~~expansions~~ — the pre- and post-continued fractions — and introduce the theory of the first with

Definition 1. The successive convergents $\{C_r\}$ of the continued fraction

$$\text{pre} \left\{ B_0 + \frac{A_1}{B_1 + \frac{A_2}{B_2 + \dots \frac{A_r}{B_r} \dots}} \right\} \quad (B_r \in \mathbb{R} (r=0,1,\dots), A_r \in \mathbb{R} (r=1,2,\dots))$$

where ~~$B_r \in \mathbb{R} (r=0,1,\dots)$~~ ~~$A_r \in \mathbb{R} (r=1,2,\dots)$~~ are defined by

$$\left. \begin{aligned} D_{r,0} &= B_r, & D_{r,s} &= B_{r-s} + D_{r,s-1}^{-1} A_{r-s+1} \quad (s=1,2,\dots,r) \\ C_r &= D_{r,r}. \end{aligned} \right\} (r=0,1,\dots)$$

where $r=0,1,\dots,r'$ if expansion () terminates with the coefficients A_r, B_r

~~It is clear that~~

and $r=0,1,\dots$ otherwise.

It is clear that ~~only under certain~~ the successive convergents of expansion () as defined in Definition by formulae () only exist if $D_{r,s} \in \mathbb{R}_I$ ($r=1,2,\dots; s=1,2,\dots,r$): we must at least have $B_r \in \mathbb{R}_I$. We shortly give ^{an} ~~another~~ alternative definition in which such stringent conditions are not imposed. In order to motivate this definition we give

Theorem . If the successive convergents of expansion () exist in the sense of Definition , they may also be ~~expected~~ ^{constructed} computed by determining the members of the two sequences ~~A_r, B_r~~ D_r, N_r ($r = -1, 0, \dots$) by means of the recursion

$$D_{-1} = 0, D_0 = I, N_{-1} = I, N_0 = B_0$$

$$D_r = B_r D_{r-1} + A_r D_{r-2}, N_r = B_r N_{r-1} + A_r N_{r-2} \quad \text{for } (r = 1, 2, \dots)$$

Then

$$C_r = D_r^{-1} N_r$$

for ~~all~~ ^{all} the ~~same~~ values of r as those for which the $\{C_r\}$ are defined.

Proof. The theorem is clearly true when ~~$r = 0, 1$~~ $r = 0, 1$. Assume it to be true for when r is replaced by $r-1$. ~~We have that is to~~ Applying this result to the continued fraction

$$B_1 + \frac{A_2}{B_2 + \dots \frac{A_r}{B_r}}$$

we have

$$C_{r,r-1} = \hat{D}_{r-1}^{-1} \hat{N}_{r-1}$$

where $C_{r,r-1}$ is as defined by formulae (), and

$$\hat{D}_{-1} = 0, \hat{D}_0 = I, \hat{N}_{-1} = I, \hat{N}_0 = B_1$$

$$\hat{D}_s = B_{s+1} \hat{D}_{s-1} + A_{s+1} \hat{D}_{s-2}, \hat{N}_s = B_{s+1} \hat{N}_{s-1} + A_{s+1} \hat{N}_{s-2} \quad (s = 1, 2, \dots, r-1)$$

It is clear that the equations

$$D_{s+1} = \hat{N}_s, \quad N_{s+1} = \hat{N}_s B_0 + \hat{D}_s A_1$$

hold when $s=0$, and it is easily shown that they hold when $s=0, 1, \dots, r-1$.

We then have

$$\begin{aligned} C_r = \hat{Q}_{r,r} &= B_0 + \hat{Q}_{r,r-1}^{-1} A_1 = B_0 + \hat{N}_{r-1}^{-1} \hat{D}_{r-1} A_1 = \hat{N}_{r-1}^{-1} \{ \hat{N}_{r-1} B_0 + \hat{D}_{r-1} A_1 \} \\ &= \hat{D}_r^{-1} N_r. \end{aligned}$$

The required result follows by induction.

It follows from the above theorem that ~~we may always compute~~ ^{they are} ~~defined in~~ ^{defined in} Definition ..., ~~if~~ ^{if} the convergents of expansion () ~~exist in the sense of~~ ^{exist in the sense of} ~~they may always be determined~~ ^{they may always be determined} by the use of the twin recursions (). However, it is clear that the sequences $\{\hat{A}_r\}, \{\hat{N}_r\}$ may be constructed, and the quotients $\{\hat{D}_r^{-1} N_r\}$ are well ~~determine~~ ^{can} exist even in cases in which the numbers $\{G_{r,s}\}$ cannot be determined (this ^{can} occurs, for example, when for some $r \geq 2$ $B_r \notin \mathbb{R}_I$). Thus, in order to define the convergents of expansion () under the widest possible conditions, we introduce

Definition. The successive convergents $\{C_r\}$ of the continued fraction () where $B_r \in \mathbb{R}$ are defined by formula () where $r=0, 1, \dots, r'$ if expansion terminates with the coefficients $A_{r'}, B_{r'}$ and $r=0, 1, \dots$

otherwise, and the numbers $\{D_r\}, \{N_r\}$ are so defined by the ~~two~~ ~~series~~ ~~recurrence~~ ().

~~We shall base all further theory upon Definition~~

All further theory will be based upon Definition, the above definition.

We shall use two transformations of continued fractions:

~~Theorem~~ ~~If~~ ~~Let~~ the successive convergents of ~~expansion~~ ^{expansion ()} ~~the continued~~ ~~fraction~~ ~~()~~ be ~~given~~ ~~by~~ ~~formulas~~ () and (), then the successive convergents of the continued fraction

(i) If $b_{2r} \in \mathbb{R}_I$ ($r=1,2,\dots$), then the successive convergents of the continued fraction

$$pre \left[b_0 + \frac{b_2 a_1}{b_2 b_1 + a_2} - \frac{b_4 a_3 b_2^{-1} a_2}{a_4 + b_4 (b_3 + a_3 b_2^{-1})} - \dots - \frac{b_{2r+2} a_{2r+1} b_{2r}^{-1} a_{2r}}{a_{2r+2} + b_{2r+2} (b_{2r+1} + a_{2r+1} b_{2r}^{-1})} - \dots \right]$$

are C_{2r} ($r=0,1,\dots$), ~~and, these of the continued fraction~~

(ii) If $b_{2r+1} \in \mathbb{R}_I$ ($r=0,1,\dots$) and either $b_1 \in \mathbb{C}\{\mathbb{R}\}$ or $b_0, a_1 \in \mathbb{C}\{\mathbb{R}\}$ then the successive

convergents of

$$pre \left[b_0 + \frac{b_1 a_1}{b_1 a_2 + b_1^{-1} a_1} - \frac{b_3 a_4 b_2^{-1} a_3}{a_3 + b_3 (b_2 + a_2 b_1^{-1})} - \frac{b_5 a_6 b_4^{-1} a_5}{a_5 + b_5 (b_4 + a_4 b_3^{-1})} - \dots - \frac{b_{2r+1} a_{2r} b_{2r-1}^{-1} a_{2r-1}}{a_{2r+1} + b_{2r+1} (b_{2r} + a_{2r} b_{2r-1}^{-1})} - \dots \right]$$

are C_{2r+1} ($r=0,1,\dots$).

Proof. Eliminating the numbers D_{2r-1}, D_{2r+1} ^(r,2) between the three ~~relations~~ ^{relationships} equations having the same form as the first of ~~relationships~~ () and expressing $D_{2r}, D_{2r-1}, D_{2r-2}, D_{2r-1}, D_{2r}$ in terms of the numbers $D_{2r+2}, D_{2r+1}, D_{2r}, D_{2r-1}$, we find that

$$D_{2r} = \{a_{2r} + b_{2r} (b_{2r-1} + a_{2r-1} D_{2r-2})\} D_{2r-2} - b_{2r} a_{2r-1} b_{2r-2}^{-1} a_{2r-2} D_{2r-4}$$

The numbers N_{2r} ~~satisfy~~ of the second of relationships () satisfy a similar equation. We have

Proof. The first two convergents of expansion () are, as is easily verified given b_1 in terms of the numbers D_0, D_2, N_0, N_2 of recursion () by means of the formulae $D_0^{-1} N_0, D_2^{-1} N_2$.

Proof. The recursion of the form () relating \hat{D}_r expansion q_r () may be written as are determined from the recursion

$$\hat{D}_0 = \underline{1} \quad \nearrow \quad \hat{D}_0 = \underline{1}$$

$$\hat{D}_1 = b_2 b_1 + a_2 \quad \nearrow \quad \hat{D}_1 = (b_2 b_1 + a_2) b_0 + b_2 a_1$$

$$\hat{D}_r = \{ a_{2r} + b_{2r} (b_{2r-1} + a_{2r-1} \hat{D}_{r-2}^{-1}) \} \hat{D}_{r-1} - b_{2r} a_{2r-1} b_{2r-2} a_{2r-2} \hat{D}_{r-2} \quad (r=2,3,\dots)$$

with a similar relationship involving the numbers $\{ \hat{D}_r \}$. It is clear that $\hat{D}_0 = D_0$ in terms of the numbers $\{ D_r \}$ of recursion (), $\hat{D}_0 = D_0, \hat{D}_1 = D_2, \dots$. We have $\hat{D}_r = D_{2r}$. Eliminating the numbers $D_{2r-1}, D_{2r-3} (r \geq 2)$ between the three equations having the same form as the first of relationships () which express $D_{2r-2}, D_{2r-1}, D_{2r}$ in terms of the numbers $D_{2r-4}, \dots, D_{2r-1}$, we find that the numbers $\{ D_{2r} \}$ satisfy a relationship of the form () in which D_{2r} replaces \hat{D}_r, D_{2r-2} replaces \hat{D}_{r-1} , and D_{2r-4} replaces \hat{D}_{r-2} and D_{2r-4} replaces \hat{D}_{r-1} and \hat{D}_{r-2} respectively. Hence $\hat{D}_r = D_{2r} (r=0,1,\dots)$. In a similar way we show

that $\hat{N}_r = N_{2r} (r=0,1,\dots)$, and hence $\hat{C}_r = C_{2r} (r=0,1,\dots)$.

By the use of similar methods, we show that the successive convergents $C_r (r=0,1,\dots)$ of expansion () are given by ~~by~~ in terms of the numbers $\{D_r\}, \{N_r\}$ of recursion () by means of the formula $C_r = \{b_1^{-1} D_{2r}\}^{-1} \{b_1^{-1} N_{2r}\}^{-1}$ ($r=0,1,\dots$).

~~Theorem... Let the successive convergents of expansion () be $C_r (r=0,1,\dots)$~~

Theorem... If $A_r \in \mathbb{C}[\mathbb{R}]$, $B_r \in \mathbb{R}_I (r=1,2,\dots)$ ~~the~~ and

~~$A_r \in \mathbb{C}[\mathbb{R}], B_r \in \mathbb{R}_I$~~ $f_r =$

$f_{-1} = f_0 = I, f_r = B_r B_{r-1} \dots B_1 (r=1,2,\dots)$

then the convergents of expansion () and ^{the} continued fraction

$$pc \left[b_0 + \frac{f_1^{-1} f_{-1} a_1}{I +} \frac{f_2^{-1} f_0 a_2}{I +} \dots \frac{f_r^{-1} f_{r-2} a_r}{I +} \dots \right]$$

are identical.

Proof. ~~The~~ The convergents $\{C_r\}$ of expansion () are ~~the~~ given in

terms of the numbers $\{d_r\}, \{n_r\}$ of recursion () by means of the

formulae $C_r = d_r^{-1} n_r$ (~~$r=0,1,\dots$~~), $d_r = f_r^{-1} d_r, n_r = f_r^{-1} n_r (r=0,1,\dots)$.

The theory of ~~the~~ post-continued fractions may be developed in a fashion analogous to that developed above. ~~The same~~ Provisionally, we define the successive convergents $\{\tilde{C}_r\}$ of the continued fraction

$$\text{post} \left\{ B_0 + \frac{A_1}{B_1 +} \frac{A_2}{B_2 +} \dots \frac{A_r}{B_r +} \dots \right\} \quad (B_r \in \mathbb{R} (r=0,1,\dots); A_r \in \mathbb{R} (r=1,2,\dots))$$

by means of the recursion

$$\left. \begin{aligned} \tilde{D}_{r,0} &= B_r, \quad \tilde{D}_{r,s} = B_{r-s} + A_{r-s+1} \tilde{D}_{r,s-1} \quad (s=1,2,\dots,r) \\ \tilde{C}_r &= \tilde{D}_{r,r} \end{aligned} \right\} (r=0,1,\dots)$$

and find that if they can be constructed, they also exist in the sense of ~~of the~~

Definition. The successive convergents $\{\tilde{C}_r\}$ of the continued fraction ()

are defined by the formulae

$$\tilde{D}_{-1} = 0 \quad \tilde{D}_0 = 1 \quad \tilde{N}_{-1} = 1 \quad \tilde{N}_0 = B_0$$

$$\tilde{D}_r = \tilde{D}_{r-1} B_r + \tilde{D}_{r-2} A_r, \quad \tilde{N}_r = \tilde{N}_{r-1} B_r + \tilde{N}_{r-2} A_r$$

$$\tilde{C}_r = \frac{\tilde{N}_r}{\tilde{D}_r}$$

where $r=0,1,\dots,r'$ if expansion () terminates with the coefficients $A_{r'}, B_{r'}$ and $r=0,1,\dots$ otherwise.

Comparison of recursions (), () and (), () reveals a principle ~~from which~~ by which the theory of post continued

fractions may be derived from that of the pre-system: equality, ~~addition~~ ~~subtraction~~ ~~and~~ ~~equality~~ ~~as~~ ~~we~~ ~~simply~~ ~~reverse~~ ~~the~~ ~~order~~ of all ~~arithmetic~~ ~~operations~~ (since addition in the formation of algebraic expressions, addition and subtraction are left unmodified unchanged, but the order in which all products are formed is reversed).

In general, the successive convergents of the ~~two~~ pre- and post-continued fractions ~~formed~~ from the same elements are ~~unequal~~: ~~if~~ ~~we~~ have, for example $B_i \in \mathbb{R}_i^+$ we have, for example,

$$\text{pre } \left[\frac{A_i}{B_i} \right] = B_i^{-1} A_i, \quad \text{post } \left[\frac{A_i}{B_i} \right] = A_i B_i^{-1}$$

Nevertheless, in certain cases they are equal:

Theorem $\left\{ \begin{array}{l} \text{if } A_i \in \mathbb{C} \setminus \mathbb{R}_i^+ \text{ (or } \mathbb{R}_i^+ \text{) and the} \\ \text{successive convergents} \end{array} \right\}$ $\left\{ \begin{array}{l} \text{the} \\ \text{continued fractions} \end{array} \right\}$ exist, then those of the

() in the senses as defined in Definition — and — are equal.

They also exist, and the two sequences of convergents are identical.

Proof. The theorem is clearly true when $r=1$. Assume that $C_{r-1} = \tilde{C}_{r-1}$ for some $r \geq 2$. Assume that it is true

for the convergents of order $r-1$, where $r \geq 2$. We apply this result to the continued fractions

$$\text{pre } \left[B_1 + \frac{A_2}{B_2 + \dots + \frac{A_r}{B_r}} \right]$$

and its post-equivalent, we find that if numbers N_s ($s=1, 2, \dots, r-1$) are computed by means of recursion (), and numbers D_s', N_s'

are computed by means of a similar recursion with the products reversed, then

$$\hat{D}_{r-1}^{-1} \hat{N}_{r-1} = N_{r-1}' D_{r-1}'^{-1}$$

However, ~~formulas~~ ^{relationships} () hold when $s = r-1$, and we also have

$$\tilde{D}_r = N_{r-1}', \quad \tilde{N}_r = B_0 N_{r-1}' + A_1 D_{r-1}'$$

Using these ~~three~~ ^{five} ~~sets~~ equations we find that $\tilde{D}^{-1} A_r = \tilde{N}_r \tilde{D}_r^{-1}$.

The proof of the theorem follows by induction.

For the sake of conciseness we have presented the theory of noncommutative continued fractions in terms of the ring R . But ~~it can also~~ the theory can also be developed in terms of any ring having properties similar to those of R , and, indeed, in terms of $R\{R\}$. It is then necessary to replace $\{A_r\}, \{B_r\}$ by $\{A_r(z)\}, \{B_r(z)\}$, ^{at every} ~~all expressions~~ involve rational functions but the derived results remain the same. In particular, the theory of continued ^{$R(z)$} fractions may be used to recover the original rational function from

the sequence of polynomials $\{B_r(z)\}$ produced by means of Euclid's algorithm, and to construct the rational function equivalent to this original $R(z)$.

Theorem . Let $B_r(z)$ ($r = 1, 2, \dots, r'$) be the sequence of polynomials determined by the application of Euclid's algorithm with respect to the sequence τ_1, τ_2, \dots (see Definition) to the rational function

$D_0(z)^{-1} N_0(z)$. Then

$$D_0(z)^{-1} N_0(z) = \text{pre} \left[\frac{1}{B_1(z) + \frac{z^{\tau_1}}{B_2(z) + \dots + \frac{z^{\tau_{r'-1}}}{B_{r'}(z)}}} \right]$$

(same as in original formulation)

Furthermore, the convergent

$$\tilde{N}(z) \tilde{D}(z)^{-1} = \text{post} \left[\frac{I}{B_1(z) + \frac{z^{\tau_1}}{B_2(z) + \frac{z^{\tau_2}}{\dots \frac{z^{\tau_{r-1}}}{B_r(z)}}}} \right]$$

also exists in the sense of Definition ..., and

$$D(z)^{-1} N(z) \equiv \tilde{N}(z) \tilde{D}(z)^{-1}.$$

Proof. Setting $G_{r-1,0}(z) = B_r(z)$,

$$G_{r-1,0}(z) = B_r(z), \quad G_{r-1,s}(z) = N_{r-s-1}(z)^{-1} D_{r-s-1}(z) \quad (s=1, 2, \dots, r-1)$$

we find that recursion () can be written in the form

$$G_{r-1,0}(z) = B_r(z), \quad G_{r-1,s}(z) = B_{r-s}(z) + G_{r-1,s-1}(z)^{-1} z^{\tau_{r-s}} \quad (s=1, 2, \dots, r-1)$$

Hence, in the sense of Definition ... and hence in that of Definition ...,

$$N(z)^{-1} D(z) = \text{pre} \left[B_1(z) + \frac{z^{\tau_1}}{B_2(z) + \frac{z^{\tau_2}}{B_3(z) + \dots \frac{z^{\tau_{r-1}}}{B_r(z)}}} \right]$$

Formula () follows immediately. The existence of the convergent () and

formula () are simple consequences of Theorem ...

The above theorem offers a method for determining the rational fraction $N(z)/D(z)$ of the form $\tilde{N}(z) \tilde{D}(z)^{-1}$ equivalent to a given function $D(z)^{-1} N(z)$. For a finite but sufficiently extensive sequence of integers τ_1, τ_2, \dots the Euclid's algorithm with respect to this sequence is

applied to the latter function, the recursions of the post-continued
fractions are applied to the polynomials $\{B_r(z)\}$ ^{in this way,} ~~to~~ derived, and
the former function is thereby constructed.

Euclid's algorithm for formal power series with coefficients over a ring

Definition. The unbounded set of numbers $a_\alpha z^\alpha, a_{\alpha+1} z^{\alpha+1}, \dots$ (α a finite integer, $a_\nu \in R$ ($\nu = \alpha, \alpha+1, \dots$)) which we write in the form $\sum_{\nu=\alpha}^{\infty} a_\nu z^\nu$ is a member of a class of similar sets $\sum_{\nu=\alpha'}^{\infty} a'_\nu z^\nu, \dots$

$\sum_{\nu=\alpha'}^{\infty} a'_\nu z^\nu$ for which $\alpha' \leq \alpha$ and

$$a'_\nu = 0 \quad (\nu = \alpha', \alpha'+1, \dots, \alpha-1), \quad a'_\nu = a_\nu; \quad (\nu = \alpha, \alpha+1, \dots)$$

this class is called a formal power series with coefficients over R and is denoted by ~~the~~ symbol $p\{\alpha; a_\nu\}$. ~~The system~~

Addition of ~~the~~ two series ~~$p\{\beta; b_\nu\}$~~ is defined by

$$p\{\alpha; a_\nu\} = p\{\beta; b_\nu\} + p\{\gamma; c_\nu\},$$

$$\alpha = \min(\beta, \gamma), \quad \alpha' = \max\{\beta, \gamma\}, \quad \longrightarrow$$

$$a_\nu = \begin{cases} b_\nu & \beta < \gamma \\ c_\nu & \gamma < \beta \end{cases} \quad (\nu = \alpha, \alpha+1, \dots, \alpha'-1) \quad a_\nu = b_\nu + c_\nu; \quad (\nu = \alpha', \alpha'+1, \dots)$$

Subtraction is defined similarly.

Multiplication of two series is defined by $p\{\alpha; a_\nu\} = p\{\beta; b_\nu\} p\{\gamma; c_\nu\}$,

$$\alpha = \beta + \gamma;$$

$$\downarrow \quad a_{\alpha+r} = \sum_{\nu=0}^r b_{\beta+\nu} c_{\gamma+r-\nu} \quad (r=0, 1, \dots)$$

The system of all such formal power series is denoted by $F\{R\}$.

It is easily shown that $\mathbb{F}\{\mathbb{R}\}$ is a ring with zero element $p\{\alpha; a_p\}$ for which $a_p = 0$ ($p = \alpha, \alpha+1, \dots$), unit element $p\{0; a_p\}$ for which $a_0 = I$, $a_p = 0$ ($p = 1, 2, \dots$), and centre consisting of all formal power series with coefficients over $\mathbb{C}\{\mathbb{R}\}$.

~~Definition~~.

Theorem. If $a_\alpha \in \mathbb{R}_I$, the formal power series $p\{\alpha; a_p\}$ has a two sided inverse.

Proof. The inverse in question has the form $p\{-\alpha; a'_p\}$. For the sake of conciseness we consider the inverse $f'(z) = p\{0; a'_p\}$ of the series $f(z) = p\{0; a_p\}$. The inverse of $p\{\alpha; a_p\}$ is then ~~given by~~ $z^{-\alpha} f'(z)$.

The left inverse $f'(z)$ of $f(z)$ satisfies the equation $f'(z)f(z) = I(z)$. We have

$$a'_0 a_0 = I, \quad \sum_{\nu=0}^r a'_\nu a_{r-\nu} = 0 \quad (r=1, 2, \dots)$$

These equations may be solved: we obtain

$$a'_0 = a_0^{-1} \quad \text{and} \quad a'_r = -a_0^{-1} \sum_{\nu=0}^{r-1} a'_\nu a_{r-\nu} \quad (r=1, 2, \dots)$$

The right inverse $f''(z) = p\{0; a''_p\}$ of $f(z)$ satisfies the equation $f(z)f''(z) = I(z)$. Its coefficients satisfy the equations $\| a_0 a''_0 = I, \sum_{\nu=0}^r a_\nu a''_{r-\nu} = 0$ ($r=1, 2, \dots$) $\|$
 $I(z)$. ~~For its coefficients~~ and we derive the formulae

$$a''_0 = a_0^{-1} \quad a''_r = -\sum_{\nu=0}^{r-1} a_\nu a''_{r-\nu} a_0^{-1} \quad (r=1, 2, \dots)$$

Since the left and right inverses of $f(z)$ may be constructed and hence of

$p\{\alpha; a_p\}$, may be constructed, and the system of such series constitutes a ring, it follows without further deliberation that these inverses are identical and unique. ⁹¹ Since, however, we require the formulae involved later, we prove that these two inverses are identical in another way. We have $a_0' = a_0^{-1} = a_0''$. Assume that $a_p' = a_p''$ ($p=0, 1, \dots, r-1$) ∞ Replacing a_p' by a_p'' in the second of formulae () and using

the appropriate equations of the set (), we have

$$a_r' = -a_0^{-1} a_r a_0^{-1} + \sum_{\nu=0}^{r-1} a_0^{-1} \sum_{\nu'=0}^{\nu-1} a_{\nu+\nu'-\nu'} a_{\nu'}'' a_{r-\nu-\nu'} a_0^{-1}$$

and, by rearrangement,

$$a_r' = -a_0^{-1} a_r a_0^{-1} + \sum_{\nu=0}^{r-2} \sum_{\nu'=1}^{\nu-1} a_0^{-1} a_{\nu'} a_{\nu'}'' a_{r-\nu-\nu'} a_0^{-1}$$

In the same way we derive an identical formula for a_r'' in terms of the coefficients a_p' ($p=0, 1, \dots, r-2$). Since $a_p' = a_p''$ ($p=0, 1, \dots, r-2$) we have $a_r' = a_r''$. It follows by induction that $a_r' = a_r''$ ($r=0, 1, \dots$).

Definition . Euclid's algorithm with respect to the under-sub unbounded sequence T of finite positive integers τ_r ($r=1, 2, \dots$) applied to the formal power series $f^{(0)}(z) \in \mathbb{F}\{\!\{R\}\!\}$ is the process of determining the sequence of polynomials ~~$B_r(z)$~~

$$B_r(z) = \sum_{\nu=0}^{\tau_r-1} b_{r,\nu} z^\nu \quad (r=1, 2, \dots)$$

and the sequence of formal power series $f^{(r)}(z) = p\left\{\frac{0}{\tau_r}; f^{(r-1)}\right\}$ ($r=1, 2, \dots$) according to the scheme ~~$\frac{0}{\tau_r}$~~

$$\frac{0}{\tau_r} \left[\frac{0}{\tau_r}; f^{(r-1)} \right] = f^{(r-1)}(z)^{-1}$$

$$b_{r,\nu} = \tilde{f}_\nu \quad (\nu=0, 1, \dots, \tau_r-1) \quad f_\nu^{(r)} = \tilde{f}_{\tau_r+\nu} \quad (\nu=0, 1, \dots)$$

for $r=1, 2, \dots$. The process is said to terminate if for some $r \geq 1$, $f^{(r)}(z) = 0(z)$. If the above process can be applied to the formal power series $f^{(0)}(z)$ in the sense that for those coefficients power series $f^{(r-1)}(z)$ that are produced before termination we have $f_0^{(r-1)} \in R_I$ ($r=1, 2, \dots$), this power series is said to be T -regular.

Definition . If $R(z) = D(z)^{-1} N(z) \in \mathbb{R}\{\mathbb{R}\}$ where

$$D(z) = \sum_{\nu=0}^{\delta} d_{\nu} z^{\nu}, \quad N(z) = \sum_{\nu=0}^{\eta} n_{\nu} z^{\nu}$$

$f(z) = p\{0; f_{\nu}\}$, and

$$\sum_{\nu=0}^r d_{\nu} f_{r-\nu} = \begin{cases} n_r & (r=0, 1, \dots, \eta) \\ 0 & (r = \overbrace{\delta+1, \delta+2, \dots}^{\eta}) \end{cases}$$

then the rational function $R(z)$ is said to generate the power series $f(z)$; we write $R(z) \sim f(z)$.

Theorem . Let $B_r(z)$ ($r=1, 2, \dots, r'$) be the sequence of polynomials produced by applying Euclid's algorithm with respect to the sequence T to the rational function $R(z) = D(z)^{-1} N(z)$, and let $R(z) \sim f(z)$. Then the formal power series $f(z)$ is T -regular, ~~and~~ Euclid's algorithm with respect to T applied to it also terminates and the same finite sequence of polynomials $\{B_r(z)\}$ is produced in the process.

Proof. We first prove that if, in the notation of Definition . . . , $D(z)^{-1} N(z) \sim f(z)$, and $N(z)^{-1} D(z) \sim \tilde{f}(z)$, then $\tilde{f}(z) \equiv f(z)^{-1}$.

Set $\tilde{f}(z) = p\{0; \tilde{f}_\nu\}$. Then

$$\sum_{\nu=0}^r n_\nu f_{r-\nu} = d_r \quad (r=0, 1, \dots, \delta)$$

so that

$$\sum_{\nu=0}^r d_\nu f_{r-\nu} = \sum_{\nu=0}^r d_\nu \sum_{\nu'=0}^{\nu} n_{\nu'} f_{\nu-\nu'}$$

We ~~evaluate~~ evaluate the numbers $\sum_{\nu=0}^r n_\nu f_{r-\nu}$, where $\sum_{\nu=0}^{\infty} f_\nu z^\nu = f(z)$ is the inverse of p from equation ()

$$\sum_{\nu=0}^r n_\nu f_{r-\nu} = \sum_{\nu=0}^r \sum_{\nu'=0}^{\nu} d_{\nu'} f_{\nu-\nu'} f_{r-\nu} \quad (r=0, 1, \dots)$$

The coefficient of $d_{\nu'}$ in this sum is

$$\sum_{\nu=\nu'}^r f_{\nu-\nu'} f_{r-\nu} = \sum_{\nu=0}^{r-\nu'} f_\nu f_{r-\nu-\nu'} = \begin{cases} 0 & (r \neq \nu') \\ 1 & (r = \nu') \end{cases} \uparrow$$

from the definition of f from equations (). Hence

$$\sum_{\nu=0}^r n_\nu f_{r-\nu} = \begin{cases} d_r & (r=0, 1, \dots, \delta) \\ 0 & (r=\delta+1, \delta+2, \dots) \end{cases}$$

and $N(z)^{-1} D(z) \sim f(z)^{-1}$.

It may also be shown by similar methods that if

$$D(z)^{-1} N(z) \sim \sum_{\nu=0}^{\infty} f_\nu z^\nu, \quad D(z)^{-1} N(z) \equiv B(z) + z^c D(z)^{-1} \hat{N}(z), \quad D(z)^{-1} \hat{N}(z) \sim \hat{f}(z)$$

where $D(z)$, $N(z)$, $B(z)$ are polynomials, c and

~~the~~

where

$$B(z) = \sum_{\nu=0}^{r-1} f_{\nu} z^{\nu}, \quad \hat{f}(z) = \sum_{\nu=0}^{\infty} \hat{f}_{\nu} z^{\nu}$$

then $\hat{f}_{\nu} = f_{r+\nu}$ ($\nu = 0, 1, \dots$).

With \hat{f} these two results in hand, it follows immediately that, in the notations of relationships () and ()

$$D_r(z)^{-1} N_r(z) \sim f^{(r)}(z) \quad (r=1, 2, \dots)$$

and hence that the polynomials $\{B_r(z)\}$ of Definition ... and ... are identical.

In particular, Euclid's algorithm applied to the rational function $D(z)^{-1} N(z)$ terminates with the production of \hat{f} , the zero rational function $D^{(r')}(z)^{-1} N^{(r')}(z)$, and hence the algorithm applied to the formal power series $f(z)$ terminates with the production of \hat{f} , the zero power series $f^{(r')}(z)$.

As in the case of the application of Euclid's algorithm to a rational function, so also the algorithm is also a process for deriving the partial denominators $\{B_r(z)\}$ and partial numerators $\{z^{r_r}\}$ of a continued fraction; from a formal power series the only difference between the two cases being that in the latter, the continued fraction may not terminate. The convergents of the continued fraction in question

are related to the power series by a distinctive property:

Theorem. Let $\{B_r(z)\}$ be the polynomials derived by application of Euclid's algorithm with respect to the sequence $\tau = \{\tau_r\}$ to the formal power series $\sum_{j=0}^{\infty} f_j z^j$. Set

$$C_r(z) = D_r(z)^{-1} R_r(z) = \text{pre} \left[\frac{1}{B_1(z) + \frac{z^{\tau_2}}{B_2(z) + \dots + \frac{z^{\tau_r}}{B_r(z)}}} \right] \quad (r=1, 2, \dots)$$

Then

$$C_r(z) \sim \sum_{j=0}^{\infty} f_{r,j}^{(\tau)} z^j$$

where

$$f_{r,j}^{(\tau)} = f_j \quad (j=0, 1, \dots, \tau_1 + \tau_2 + \dots + \tau_{r-1}) \quad (r=1, 2, \dots)$$

For ~~larger~~ larger values of j , higher values of r , we have
~~Proof. In the notation of Definition ..., we have~~

$$\{B_r(z) + z^{\tau_{r+1}} f^{(\tau)}(z)^{-1}\}^{-1} = f^{(r-1)}(z) \quad (r=1, 2, \dots)$$

~~Hence, in the sense of Definition ..., and hence in the sense~~
~~Since~~ Since $D_{r,0} \in \mathbb{R}_I$ for all partial denominators polynomials $\{B_r(z)\}$ constructed, all convergents of the continued fraction

$$\frac{z^{\tau_{r-1}}}{B_{r-1}(z) + \frac{z^{\tau_r}}{B_r(z) + \dots + \frac{z^{\tau_{r-1}}}{B_{r-1}(z) + z^{\tau_r} f^{(r)}(z)}}} \quad (r=1, 2, \dots, r-2)$$

evaluated over $\mathbb{F}\{\mathbb{R}\}$ exist ~~also~~ in the sense of Definition... Hence in the sense of that definition, and therefore in that of Definition...,

$$f(z) = p(z) \left[\frac{I}{B_1(z) + z^{\tau_1}} \dots \frac{z^{\tau_2}}{B_2(z) + z^{\tau_2}} \dots \frac{z^{\tau_{r-1}}}{B_{r-1}(z) + z^{\tau_{r-1}}} \right] \quad (r=2,3,\dots)$$

and, setting ~~$G_2(z) = I$~~

~~$$\hat{G}_2(z) = \hat{G}_3(z) \hat{G}_r(z) = D_{r-3}(z) = I(z), \quad \hat{G}_r(z) = D_{r-3}(z) G_{r-1}(z) G_{r-2}(z) \dots G_1(z)$$~~

$$\hat{G}_2(z) = \hat{G}_3(z) = I(z), \quad \hat{G}_r(z) = D_{r-3}(z) G_{r-1}(z) G_{r-2}(z) \dots G_1(z) \quad (r=2,3,\dots)$$

where

~~$$G_1(z) = D_1(z)^{-1} D_{r-2}(z), \quad G_r(z) = D_r(z)^{-1} D_{r-2}(z)$$~~

$$G_1(z) = B_1(z)^{-1} \quad G_2(z) = D_2(z)^{-1} \quad G_r(z) = D_r(z)^{-1} D_{r-2}(z) \quad (r=3,4,\dots)$$

we have, from Theorem ... ,

$$f(z) = \sum_{\nu=1}^{r-1} (-1)^{\nu} G_{\nu}(z) G_{\nu-1}(z) \dots G_1(z) + (-1)^r \{ B_{r-1}(z) + z^{\tau_{r-1}} \}^{-1} \hat{G}_r(z) z^{\tau_2 + \tau_3 + \dots + \tau_{r-1}}$$

We also have

$$G_r(z) = \sum_{\nu=1}^{r-1} (-1)^{\nu-1} G_{\nu}(z) G_{\nu-1}(z) \dots G_1(z) + (-1)^r \{ B_{r-1}(z) + z^{\tau_{r-1}} B_1(z)^{-1} \}^{-1} \hat{G}_r(z) z^{\tau_2 + \tau_3 + \dots + \tau_{r-1}}$$

Hence

$$\begin{aligned} f(z) - G_r(z) &= (-1)^r \left\{ B_{r-1}(z) + z^{\tau_{r-1}} \right\}^{-1} \left\{ B_{r-1}(z) + z^{\tau_{r-1}} B_1(z)^{-1} \right\}^{-1} \left\{ B_{r-1}(z) + z^{\tau_{r-1}} B_1(z)^{-1} \right\} \hat{G}_r(z) z^{\tau_2 + \tau_3 + \dots + \tau_{r-1}} \\ &= (-1)^r \left\{ B_{r-1}(z) + z^{\tau_{r-1}} \right\}^{-1} \left\{ B_{r-1}(z) + z^{\tau_{r-1}} B_1(z)^{-1} \right\}^{-1} \left\{ B_{r-1}(z) + z^{\tau_{r-1}} B_1(z)^{-1} \right\} \hat{G}_r(z) z^{\tau_2 + \tau_3 + \dots + \tau_{r-1}} \end{aligned}$$

With regard to the constituents of the expression on the right hand side

of this equation, $B_{r-1}(z) + z^{\tau_{r-1}} f^{(r-1)}(z)$ the first expression enclosed in braces is $f^{(r-1)}(z) = f^{(r)}(z)^{-1}$; the third such expression is a formal power series ^{it may be shown} of the form $p\{0; f_1^{(r)}\}$, where $f_1^{(r)} = b_{r-1,0} \in \mathbb{R}_I$, and τ_{r-1} in the proof of relationship (1) for $r=1$ given above, it may be shown

that the second such expression is equivalent to a formal power series of the form $p\{\tau_1; f_1^{(r)}\}$. In short, there exists a formal power series with coefficients $\{\tau_i\}$ such that

$$f(z) - C_r(z) = p\{\tau_2 + \tau_3 + \dots + \tau_{n_i}; f_1^{(r)}\}$$

Hence, if $C_r(z) = p\{0; f_1^{(r)}\}$, equations (1)_r hold.

Proof To prove the theorem for $r=1$ we have, using the notation of Definition (1)

$$f(z) = \{B_1(z) + z^{\tau_2} f^{(2)}(z)\}^{-1}$$

and hence

$$B_1(z)^{-1} - f(z) = B_1(z)^{-1} \sum_{i=1}^{\infty} \frac{f^{(2)}(z)}{f^{(1)}(z)} \{B_1(z) + z^{\tau_2} f^{(2)}(z)\}^{-1} z^{\tau_2} \\ = B_1(z)^{-1} f^{(2)}(z) z^{\tau_2}$$

$B_{1,0} \in \mathbb{R}_I$ and hence $B_1(z)^{-1}$ has a power series expansion of the form $p\{0; f_1^{(1)}\}$; the product $f^{(2)}(z)/f^{(1)}(z)$ is also $p\{0; f_1^{(2)}\}$; $B_1(z) + z^{\tau_2} f^{(2)}(z)$ also has a series expansion of the same form. Hence, if $C_1^{(1)}(z) = p\{0; f_1^{(1)}\}$, equation $f_{1,i}^{(1)} = f_{1,i}^{(2)}$ ($i = 0, 1, \dots, \tau_2 - 1$).

Definition . If Euclid's algorithm with respect to the ~~series~~ $p\{0; f_0\}$ sequence $\{1, 1, \dots\}$ ^{$\{z, z, \dots\}$} ~~can be~~ applied to the ^{formal power} series $p\{0; f_0\}$ can be prolonged indefinitely, the sequence $\{f_0\}$ is said to be \mathbb{C} -regular [A-regular]; if the algorithm terminates in the sense of Definition . . ., this sequence ~~$\{f_0\}$~~ is said to be degenerately \mathbb{C} -regular [degenerately A-regular].

Theorem . If the sequence $\{f_0\}$ is \mathbb{C} -regular [degenerately \mathbb{C} -regular] both it and the delayed sequence $\{f_{0+1}\}$ are A-regular [degenerately A-regular]. Furthermore, if ~~the series $p\{0; f_0\}$ generates the continued fraction~~ ~~the series $p\{0; f_0\}$ generates the pre \mathbb{C} -fraction~~ generated by the series $p\{0; f_0\}$ is

$$\frac{1}{b_{1,0} + \frac{z}{b_{2,0} + \frac{z}{\dots + \frac{z}{b_{r,0} + \dots}}}}$$

and the pre A-fraction generated by the same series is

$$\frac{1}{\hat{b}_{1,0} + \frac{z}{\hat{b}_{2,0} + \frac{z}{\dots + \frac{z}{\hat{b}_{r,0} + \hat{b}_{r,1}z + \dots}}}}$$

then the latter is the even part of the former. If the pre-A-fraction generated by the series $p\{0; f_{0+1}\}$ is

$$\frac{1}{\tilde{b}_{1,0} + \frac{z}{\tilde{b}_{2,0} + \frac{z}{\dots + \frac{z}{\tilde{b}_{r,0} + \tilde{b}_{r,1}z + \dots}}}}$$

then the continued fraction

$$f_0 + \frac{z}{b_{1,0} + b_{1,1}z} + \frac{z^2}{b_{2,0} + b_{2,1}z} + \dots + \frac{z^2}{b_{r,0} + b_{r,1}z} + \dots$$

is the odd part of \leftrightarrow expansion ().

Prop. If the sequence $\{f_r\}$ is \mathbb{C} -regular, the series produced by application of Euclid's algorithm with respect to the sequence $\{1, 1, \dots\}$ to the series $f^{(r)}(z) = \sum b_r z^r$ satisfy the recursion of the form

$$f^{(r-1)}(z) = z \{ b_r + z f^{(r)} \}^{-1} \quad (r=1, 2, \dots)$$

∃
Setting $\phi_0 = 1, \phi_1 = b_2$

$$\left. \begin{aligned} \phi_{2r} &= (-1)^r b_{4r} b_{4r-2}^{-1} b_{4r-4}^{-1} b_{4r-6}^{-1} \dots b_4 b_2^{-1} \\ \phi_{2r+1} &= (-1)^r b_{4r+2} b_{4r}^{-1} b_{4r-2}^{-1} b_{4r-4}^{-1} \dots b_4^{-2} b_2 \end{aligned} \right\} (r=1, 2, \dots)$$

and $\hat{b}_{1,0} = b_1, \hat{b}_{1,1} = b_2^{-1}$

$$\hat{b}_{r,0} = \phi_r^{-1} b_{2r} b_{2r-1} \phi_{r-1}, \quad \hat{b}_{r,1} = \phi_r^{-1} \phi_{r-1} + \phi_r^{-1} b_{2r} b_{2r-2} \phi_{r-1}, \quad (r=2, 3, \dots)$$

it is easily verified that the series

$$\hat{b}^{(r)}(z) = \hat{b}^{(r)}(z), \hat{b}^{(r)}(z) = \phi_r^{-1} \{ b_{2r} + z \hat{b}^{(r)}(z) \}^{-1} \phi_{r-1} \quad (r=1, 2, \dots)$$

satisfy the recursion

$$\hat{b}^{(r-1)}(z) = \{ \hat{b}_{r,0} + \hat{b}_{r,1} z + z^2 \hat{b}^{(r)}(z) \}^{-1} \quad (r=1, 2, \dots)$$

Hence Euclid's algorithm with respect to the sequence $\{z, z, \dots\}$ can also be applied to the series $f^{(r)}(z)$, and the sequence of polynomials $\{\hat{B}_r(z)\}$ so produced are given by $\hat{B}_r(z) = \hat{b}_{r,0} + \hat{b}_{r,1}z$ ($r=1, 2, \dots$).

If ~~the~~ the sequence of power series satisfying recursion () terminates with $f^{(2r')}(z) = 0(z)$ for some $r' \geq 1$, the recursion () also terminates with $\hat{f}^{(r')} = 0$; the polynomials $\hat{B}_r(z)$ ($r=1, 2, \dots, r'-1$) are given as above, but ~~$\hat{B}_{r'}(z) =$~~

$$\hat{B}_{r'}(z) = \hat{b}_{r',0} + \phi_{r'}^{-1} b_{2r'} \phi_{r'-1} + \hat{b}_{r',1}z$$

If termination ~~occurs with~~ of the sequence () occurs with $f^{(2r'+1)}(z) = 0(z)$ ($r' \geq 1$), the recursion () terminates with $\hat{f}^{(r'+1)} = 0$; the polynomials $\hat{B}_r(z)$ ($r=1, 2, \dots, r'$) are given as above, but

~~$$\hat{B}_{r'}(z) = \hat{b}_{r',0} + \phi_{r'}^{-1} b_{2r'} \phi_{r'-1} + \hat{b}_{r',1}z$$~~

~~$$\hat{b}_{r'+1,0} = \hat{\phi}_{r'+1}^{-1} b_{2r'+1} \phi_{r'} \quad \hat{b}_{r'+1,1} = \hat{\phi}_{r'+1}^{-1} \phi_{r'} + \hat{\phi}_{r'}^{-1} b_{2r'} \phi_{r'}$$~~

where

$$\hat{\phi}_{r'+1} = (-1)^{r'} b_{2r'}^{-1} b_{2r'-2}^{-1} b_{2r'-4}^{-1} \dots b_4^{-1} b_2^{-1}$$

if r' is odd and $r'' = [(r'+1)/2]$, and

$$\hat{\phi}_{r'+1} = (-1)^{r''} b_{2r'}^{-1} b_{2r'-2}^{-1} b_{2r'-4}^{-1} \dots b_4^{-1} b_2$$

if r' is even and $r'' = [r'/2]$.

Assuming again that the sequence $\{t_r\}$ is \mathbb{C} -regular, and
 setting $\tilde{\phi}_0 = I, \tilde{\phi}_1 = b_3 b_2^{-1}$

$$\left. \begin{aligned} \tilde{\phi}_{2r} &= (-1)^r b_{4r+1}^{-1} b_{4r-1}^{-1} b_{4r-3}^{-1} b_{4r-5}^{-1} \dots b_5^{-1} b_3^{-1} \\ \tilde{\phi}_{2r+1} &= (-1)^r b_{4r+2}^{-1} b_{4r}^{-1} b_{4r-2}^{-1} b_{4r-4}^{-1} \dots b_5^{-1} b_3 \end{aligned} \right\} (r=1,2,\dots)$$

and $\tilde{d}_{1,0} = b_1^2 b_2, \tilde{d}_{1,1} = b_1^2 b_3^{-1} + b_1$

$$\tilde{d}_{r,0} = \tilde{\phi}_r^{-1} b_{2r+1} b_{2r} \tilde{\phi}_{r-1}, \tilde{d}_{r,1} = \tilde{\phi}_r^{-1} \tilde{\phi}_{r-1} + \tilde{\phi}_r^{-1} b_{2r+1} b_{2r-1} \tilde{\phi}_{r-1}, (r=2,3,\dots)$$

it is easily verified that the series

$$\tilde{f}^{(0)}(z) = \tilde{\phi}^{(0)} z^{-1} \{ \tilde{f}^{(0)}(z) - f_0 \} = \sum_1 t_{r+1} z^r$$

$$\tilde{f}^{(r)}(z) = \tilde{\phi}_r^{-1} \{ b_{2r+1} + z \tilde{f}^{(2r)}(z) \} \tilde{\phi}_{r-1} \quad (r=1,2,\dots)$$

satisfy the recursion

$$\tilde{f}^{(r-1)}(z) = \{ \tilde{d}_{r,0} + \tilde{d}_{r,1} z + z^2 \tilde{f}^{(r)}(z) \}^{-1} \quad (r=1,2,\dots)$$

~~Hence the sequence $\{b_{r+1}\}$ is \mathbb{C} -regular, and~~

Hence Euclid's algorithm with respect to the sequence $\{2,2,\dots\}$ can also be applied to the series $\sum_1 t_{r+1} z^r$. The case in which $\{t_r\}$ is degenerately \mathbb{C} -regular is dealt with as above.

That expansion () the continued fraction () is the even part of () may be demonstrated with the aid of Theorems .. and It is, however, instructive to give a direct proof.

Setting, for some $r \geq 1$, for which the convergents exist,

$$\text{pre} \left[\frac{1}{b_1 + z} \frac{z}{b_2 + z} \dots \frac{z}{b_r + z} \right] = D_r(z)^{-1} N_r(z)$$

$$\text{post} \left[\frac{1}{\hat{b}_{1,0} + \hat{b}_{1,1}z} \frac{z}{\hat{b}_{2,0} + \hat{b}_{2,1}z} \dots \frac{z}{\hat{b}_{r,0} + \hat{b}_{r,1}z} \right] = \hat{N}_r(z) \hat{D}_r(z)^{-1}$$

From Theorem... , we have

$$D_r(z)^{-1} N_r(z) \sim \sum_{\nu=0}^{2r-1} f_\nu z^\nu + p\{2r; f'_\nu\}$$

$$\hat{N}_r(z) \hat{D}_r(z)^{-1} \sim \sum_{\nu=0}^{2r-1} \hat{f}_\nu z^\nu + p\{2r; \hat{f}'_\nu\}$$

where $\{f'_\nu\}, \{\hat{f}'_\nu\}$ are certain sequences well defined sequences whose precise nature does not at this point concern us. Then by subtraction

$$D_r(z)^{-1} N_r(z) - \hat{N}_r(z) \hat{D}_r(z)^{-1} = p\{2r; f''_\nu\}$$

where $f''_\nu = f'_\nu - \hat{f}'_\nu$ ($\nu=0,1,\dots$). Hence

$$N_r(z) \hat{D}_r(z) - D_r(z) \hat{N}_r(z) = \cancel{D_r(z)} p\{2r; f''_\nu\} D_r(z)$$

The polynomials $D_r(z), \hat{D}_r(z)$ contain no powers of z higher than the r^{th} with nonzero coefficients. All coefficients of z^ν with $\nu \geq 2r$ in the formal power series on the left hand side of this relationship are zero. The expression on the right hand side represents a formal power series commencing with a term involving z^{2r} ; this series

is therefore the zero series. Hence

$$D_{2r}(z)^{-1} M_{2r}(z) \equiv \hat{N}_r(z) \hat{D}_r(z)^{-1}$$

We note in passing that in the notation of relationships ()^{a1()?}

$$f'_v = f''_v \quad (v=0,1,\dots) \quad (v=2r, 2r+1, \dots)$$

It follows from Theorems... that the continued fractions () and () can be thrown into the forms

$$pe \left[\frac{a_1}{I+} \quad \frac{a_2 z}{I+} \quad \dots \quad \frac{a_r z}{I+} \quad \dots \right]$$

where

$$\phi_{-1} = \phi_0 = I, \quad \phi_r = b_r b_{r-1} \dots b_1, \quad a_r = \phi_r^{-1} \phi_{r-2} \quad (r=1, 2, \dots)$$

and

$$pe \left[\frac{v_1}{I+w_1 z+} \quad \frac{v_2 z^2}{I+w_2 z+} \quad \dots \quad \frac{v_r z^2}{I+w_r z+} \quad \dots \right]$$

where

$$\phi_{-1} = \phi_0 = I \quad \phi_r = \hat{b}_{r,0} \hat{b}_{r-1,0} \dots \hat{b}_{1,0} \quad v_r = \phi_r^{-1} \phi_{r-2} \quad w_r = \phi_r^{-1} \hat{b}_{r,1} \phi_{r-2} \quad (r=1, 2, \dots)$$

respectively. These transformations ~~lead to a simplification of~~ ~~much~~ simplify much of the ~~and~~ subsequent theory.

¹⁰¹ Although the derivation of expansions () and () from a \mathbb{C} -regular or \mathbb{A} regular formal power series by means of Euclid's algorithm involve

operations upon unbounded sequences of numbers, it is clear that the successive coefficients of these continued fractions are derived from a limited number only of the ^{generating} power series. Simple algorithms may be given for the derivation of the coefficients in ~~the~~ expansion () and ()

Theorem. Let the sequence $\{f_j\}$ be A -regular, ^{() be,} ~~the successive coefficients~~ and the associated of the continued fraction \Rightarrow generated by the formal power series $\sum f_j x^j$.

The coefficients $\{v_r\}$ $\{w_r\}$ of expansion () may be ~~determined~~ constructed by determining the numbers Θ_r, ψ_r ($r=0,1,\dots$) $d_{r,\nu}$ ($r=0,1,\dots; \nu=0,1,\dots,r$) as follows: initially ^{$\Theta_0=f_0, \psi_0=f_{1,0}$} $d_{0,0} = d_{1,0} = I, d_{0,1} = f_1 f_0^{-1}$; at stages r ^{determine} compute

$$d_{0,0} = d_{1,0} = I, \quad d_{0,1} = f_1 f_0^{-1}$$

$$\Theta_{r-1} = \sum_{\nu=0}^{r-1} d_{r-1,\nu} f_{2r-2-\nu} \quad \psi_{r-1} = \sum_{\nu=0}^{r-2} d_{r-1,\nu} f_{2r-2-\nu}$$

$$v_r = -\Theta_{r-1} \Theta_{r-2}^{-1} \quad w_r = -(\psi_{r-1} + v_r \psi_{r-2}) \Theta_{r-1}^{-1}$$

$$d_{r-1,0} = I \quad d_{r-1,1} = d_{r-2,1} + w_{r-1} d_{r-2,0}$$

$$d_{r-1,\nu} = d_{r-2,\nu} + w_{r-1} d_{r-2,\nu-1} + v_{r-1} d_{r-3,\nu-2}$$

$$d_{r,0} = I$$

$$d_{r,1} = d_{r-1,1} + w_r$$

$$d_{r,\nu} = w_r d_{r-1,\nu-1} + v_r d_{r-2,\nu-2}$$

$$d_{r,\nu} = d_{r-1,\nu} + w_r d_{r-1,\nu-1} + v_r d_{r-2,\nu-2} \quad (\nu=2,3,\dots,r-1)$$

for $r=2,3,\dots$, when the numbers $\{v_r\}, \{w_r\}$ given by formulae () are those occurring in expansion ().

~~If the~~
 If the sequence $\{f_n\}$ is degenerately $\#$ -regular, the above process terminates at stage $r+1$ for some $r \geq 1$ with $\Theta_r = 0$.

Proof. Denoting the successive convergents of expansion () by $D_r(z)^{-1} N_r(z)$

($r = 0, 1, \dots$), where

$$D_r(z) = \sum_{\nu=0}^r d_{r,\nu} z^\nu, \quad d_{r,0} = 1; \quad N_r(z) = \sum_{\nu=0}^{r-1} n_{r,\nu} z^\nu \quad (r = 0, 1, \dots)$$

we have

$$D_{r+1}(z)^{-1} N_r(z) \sim \sum_{\nu=0}^{2r-1} f_\nu z^\nu + p\{2r; f_r\} \quad (r = 0, 1, \dots)$$

Premultiplying this relationship throughout by the formal power series equivalent to $D_r(z)$, ~~we have~~ and equating coefficients of $z^{r+\nu}$

on both sides, we have

$$\sum_{\nu=0}^r d_{r,\nu} f_{r+r-\nu} = 0 \quad (r = 0, 1, \dots, r-1)$$

The polynomials $\{D_r(z)\}$ satisfy the recursion

$$D_r(z) \equiv (1 + w_r z) D_{r-1}(z) + v_r z^2 D_{r-2}(z)$$

Equating powers of z throughout this relationship, we derive ~~equation~~ formulae ().

Using these formulae, we have

$$\begin{aligned} \sum_{\nu=0}^r d_{r,\nu} f_{2r-\nu-2} &= \sum_{\nu=0}^{r-1} d_{r-1,\nu} f_{2r-\nu-2} + w_r \sum_{\nu=0}^{r-1} d_{r-1,\nu} f_{2r-\nu-3} + v_r \sum_{\nu=0}^{r-2} d_{r-2,\nu} f_{2r-\nu-4}, \\ \sum_{\nu=0}^r d_{r,\nu} f_{2r-\nu-1} &= \sum_{\nu=0}^{r-1} d_{r-1,\nu} f_{2r-\nu-1} + w_r \sum_{\nu=0}^{r-1} d_{r-1,\nu} f_{2r-\nu-2} + v_r \sum_{\nu=0}^{r-2} d_{r-2,\nu} f_{2r-\nu-3}, \end{aligned}$$

Using formulae () () and (), we have these relationships may be written

$$\Theta_{r-1} + v_r \Theta_{r-2} = 0 \quad \psi_{r-1} + w_r \Theta_{r-1} + v_r \psi_{r-2} = 0$$

From the first of these relationships, $\Theta_{r-1} = (-1)^{r-1} v_1 v_{r-1} \dots v_2 \Theta_0$. It is clear from formula () that $v_r \in \mathbb{R}_I$ ($r=2,3,\dots$), Hence $\Theta_{r-1} \in \mathbb{R}_I$ ($r=2,3,\dots$) and we derive formulae (). That the initial values for this algorithm are as stated is easily verified.

If the series $\sum_i f_i z^i$ is generated by the rational function $D_{r'}(z)^{-1} N_{r'}(z)$, we have in the notation of formula () $f_{i,r'} = f_i$ $i=2r', 2r'+1, \dots$. Since, in particular, $f_{r', 2r'} = f_{2r'}$ in this case it follows that in addition to formulae () we also have the further equation

$$\sum_{d=0}^{r'} d r'_d f_{2r'-d} = 0$$

i.e. $\Theta_{r'} = 0$.

Systems of continued fractions

In this section we consider systems of continued fractions derived

from a single power series.

Definition . If the sequences $\{f_m\}$ ($m=0,1,\dots$) are \mathbb{C} -regular, the sequence $\{f_n\}$ is said to be semi-normal. If these sequences are all degenerately \mathbb{C} -regular, a set of numbers $d_0 = I, d_p \in \mathbb{R}$ ($p=1,2,\dots, r' < \infty$) exists such that

$$\sum_{\nu=0}^{r'} d_\nu f_{m+r'-\nu} = 0 \quad (m=0,1,\dots)$$

and no ~~sub~~ infinite subsequence $\{f_{m' + r'}\}$ ($m' \geq 0$) satisfies a ^{similar} recursion of the form

$$\sum_{\nu=0}^{r'} d'_\nu f_{m'+r'-\nu} = 0 \quad (d'_0 = I; \{m=0,1,\dots\})$$

then the sequence $\{f_n\}$ is said to be degenerately semi-normal and in this case we write $\{f_n\} \in \mathcal{K}_{r'}^{\text{deg}}$.

Although we present the theory in terms of sequences of ^{of coefficients} sequences and their properties, in the ~~degenerate~~ degenerate cases considered ~~we use~~ the results derived depend more directly upon properties of sequences of rational functions. We therefore characterize these rational functions.

~~Theorem . $\{f_n\} \in \mathcal{K}_{r'}$ if and only if the series $\sum_{n=0}^{\infty} f_n z^n$ is generated by a rational function of the form~~

Theorem ... $\{f_p\} \in \mathcal{H}_r$, if and only if the series $\sum f_p z^p$ is generated by a rational function of the form

$$\left\{ \sum_{p=0}^{r'} d_p z^p \right\}^{-1} \left\{ \sum_{p=0}^{r'-1} n_p z^p \right\}$$

$$\left\{ \sum_{p=0}^{r'} d_p z^p \right\} \left\{ \sum_{p=0}^{r'-1} n_p z^p \right\}^{-1}$$

where $r'' \leq r'$.

If $\{f_p\} \in \mathcal{H}_r$, then each of the series $\sum f_p^{(m)} z^p$ is generated by a rational function of the form

$$\left\{ \sum_{p=0}^{r'} d_p z^p \right\}^{-1} \left\{ \sum_{p=0}^{r'-1} n_p^{(m)} z^p \right\}$$

($m=1, 2, \dots$)

in which $n_{r'-1}^{(m)} \neq 0$.

If ~~the~~ the series $\sum f_p z^p$ is generated by a function of the form

$$\left\{ \sum_{p=0}^{\tilde{r}} \tilde{d}_p z^p \right\}^{-1} \left\{ \sum_{p=0}^{\tilde{r}-1} \tilde{n}_p z^p \right\}$$

~~then the numbers~~ where $m' = r' - \tilde{r} > 0$, then the numbers $\{f_p^{(m')}\}$ satisfy

a recursion of the form

$$\sum_{p=0}^{\tilde{r}} \tilde{d}_p f_{m'+m+\tilde{p}-p}^{(m')} = 0$$

($m=0, 1, \dots$)

Proof. The first result of the theorem follows directly from Definition 6...

Assuming $r'' < r'$ we construct the polynomials $N_{\frac{r''}{r'}}^{(m)}(z) = \sum_{p=0}^{r'-1} n_p^{(m)} z^p$

~~recurring~~ occurring in expressions (). They are determined by the formulae

$$n_p^{(0)} = n_p \quad (p=0, 1, \dots, r'-1); \quad n_p^{(m)} = n_{p+1}^{(m-1)} - d_{p+1} t_{m-1} \quad (p=0, 1, \dots, r'-2), \quad n_{r'-1}^{(m)} = -d_{r'} t_{m-1} \quad (m=1, 2, \dots)$$

for, as is easily verified,

$$\{D(z)^{-1} N^{(m-1)}(z) - t_{m-1}\} z^{-1} = D(z)^{-1} N^{(m)}(z),$$

and i.e.

$$D(z)^{-1} N^{(m)}(z) \equiv \left\{ D(z)^{-1} N^{(1)}(z) - \sum_{\nu=0}^{m-1} t_{\nu} z^{\nu} \right\} z^{-m} \sim \sum_{\nu=0}^{m-1} t_{\nu} z^{\nu}$$

Since the sequences $\{t_{m-1}\}$ ~~is degenerate~~ are degenerately \mathcal{C} -regular,

t_{m-1} , in particular, is not a factor of zero ($m=1, 2, \dots$), and hence

$$n_{r-1}^{(m)} \neq 0 \quad (m=1, 2, \dots).$$

Turning to the various power series generated by the function (), we first ~~we~~ express this quotient in the form

$$\sum_{\nu=0}^{m-1} p_{\nu} z^{\nu} + z^m \left\{ \sum_{\nu=0}^{\tilde{r}-1} \tilde{d}_{\nu} z^{\nu} \right\}^{-1} \left\{ \sum_{\nu=0}^{\tilde{r}-1} \tilde{n}_{\nu} z^{\nu} \right\} R_{\tilde{r}}^{\sim}(z), \quad R_{\tilde{r}}^{\sim}(z) \equiv \left\{ \sum_{\nu=0}^{\tilde{r}-1} \tilde{d}_{\nu} z^{\nu} \right\}^{-1} \left\{ \sum_{\nu=0}^{\tilde{r}-1} \tilde{n}_{\nu} z^{\nu} \right\}$$

by setting

~~$$p_{\nu} = n_{\nu} - \sum_{\nu'=0}^{\nu-1} \tilde{d}_{\nu-\nu'} p_{\nu'}$$~~

$$p_{\nu} = n_{\nu} - \sum_{\nu'=0}^{\nu-1} \tilde{d}_{\nu-\nu'} p_{\nu'} \quad (\nu=0, 1, \dots, m-1)$$

~~$$\tilde{n}_{\nu} = n_{m+\nu} - \sum_{\nu'=0}^{m+\nu} p_{\nu'} \tilde{d}_{\nu-\nu'}$$~~

$$\tilde{n}_{\nu} = n_{m+\nu} - \sum_{\nu'=0}^{\nu-1} p_{\nu'} \tilde{d}_{\nu-\nu'} \quad (\nu=0, 1, \dots, \tilde{r}-1)$$

It is clear that $p_{\nu} = f_{\nu}$ ($\nu=0, 1, \dots, m-1$) and hence $R_{\tilde{r}}^{\sim}(z) \sim \sum_{\nu=0}^{m-1} f_{\nu} z^{\nu}$.

$$\left\{ \sum_{j=0}^{\infty} d_j z^j \right\}^{-1} \left\{ \sum_{j=0}^{\infty} n_j z^j \right\} = \Sigma'$$

Recursion () follows immediately from Definition 6 .

We have now established that if $\{f_j\} \in \mathcal{U}_1'$, each of the series $\{\Sigma(t_{m_j})z^j\}$ is generated by the same type of rational function as that which generates $\Sigma(t)z^j$. This ~~implies~~ will imply, in the theory given below, that the same type of algebraic invariance property ~~can be seen~~ is possessed by each of the continued fractions $\mathcal{C}\{\Sigma(t_{m_j})z^j\}$. If the series $\Sigma(t)z^j$ is generated by the rational function (), the same type of invariant property holds for all expansions $\Sigma(t_{m_j})z^j$ with $m_j \geq m'$. For the sake of conciseness in exposition, we confine our attention to series of the first ~~kind~~ kind.

Theorem . Let the series $\{\Sigma(t)z^j\}$ be semi normal, and let

$$\mathcal{C}\{\Sigma(t_{m_j})z^j\} = \text{pre} \left[\frac{a_1^{(m)}}{I+} \frac{a_2^{(m)}}{I+} \dots \frac{a_r^{(m)}}{I+} \dots \right] \quad (m=0,1,\dots)$$

and
$$\Sigma_0^{(m)}(z) = S_m(z) = \sum_{j=0}^{m-1} b_j z^j \quad (m=0,1,\dots)$$

$$\Sigma_{2r}^{(m)}(z) = \sum_{j=0}^{m-1} \text{pre} \left[\frac{a_1^{(m)} z^m}{I+} \frac{a_2^{(m)} z^m}{I+} \dots \frac{a_{2r}^{(m)} z^m}{I} \right]_{\substack{+m=0,1,\dots \\ (r=1,2,\dots) m=0,1,\dots}}$$

then
$$\text{pre} \mathcal{C}_{2r} \left[\mathcal{C}\{\Sigma(t_{m_j})z^j\} \right]$$

$$\Sigma_{2r}^{(m)}(z) = \text{pre} \left[S_m(z) + \frac{a_1^{(m)} z^m}{1+z} \frac{a_2^{(m)} z^m}{1+z} \dots \frac{a_{2r-1}^{(m)} z^m}{1+z} \right] \quad (r=1,2,\dots)$$

$$\text{pre} C_{2r} \left[\frac{S_m(z) + z^m \text{pre} C_{2r-1} \left[G\{f_m(z)\} \right]}{1+z} \right] \quad (r=1,2,\dots)$$

Furthermore, the ~~same~~ denominators of the successive

Furthermore, setting

$$\text{pre} C_{2r} \left[G\{f_m(z)\} \right] = D_r^{(m)}(z)^{-1} N_r^{(m)}(z) \quad (r,m=0,1,\dots)$$

the ~~denominators~~ denominators of the successive convergents of the

$G\{f_m(z)\}$ are, in order $D_0^{(m)}(z), D_1^{(m)}(z), \dots$ commencing with that of order

zero, $D_0^{(m)}(z), D_0^{(m+1)}(z), D_1^{(m)}(z), D_1^{(m+1)}(z), \dots, D_{r-1}^{(m)}(z), D_{r-1}^{(m+1)}(z), \dots$

the denominators of the successive convergents $\text{pre} C_r \left[G\{f_m(z)\} \right] (r=1,2,\dots)$

are

$$D_0^{(m)}(z), D_1^{(m)}(z), D_1^{(m+1)}(z), D_2^{(m)}(z), \dots, D_{r-1}^{(m+1)}(z), D_r^{(m)}(z), \dots \quad (m=0,1,\dots)$$

If ~~the series~~ the series $\sum t_r z^r \in \mathbb{H}_{r'}$, then relationships

() hold with ~~for~~ $r=1,2,\dots,r'$, the sequence () terminates with

$D_{r'}^{(m)}(z)$, and if the series $\sum t_r z^r$ is generated by the rational

function (), then $D_{r'}^{(m)}(z) = D(z) (m=0,1,\dots)$.

Proof. It follows from Theorem ... that for if the series $\sum |t_r|^2$ is semi-normal &

$$pe \left[z_m \rightarrow \frac{a_1^{(m)}}{I_+} \frac{a_2^{(m)}}{I_+} \dots \frac{a_{2r-1}^{(m)}}{I} \right] = pe \left[t_m + \frac{a_1^{(m)}}{I_+} + \frac{a_2^{(m)}}{I_+} \dots \frac{a_{2r-1}^{(m)}}{I} \right]$$

($r=1, 2, \dots; m=0, 1, \dots$)

Relationships () follow immediately. Since the denominators of

~~the successive convergents of the continued fraction~~

$$\frac{pe \left[t_m + \frac{a_1^{(m)}}{I_+} + \frac{a_2^{(m)}}{I_+} \dots \frac{a_{2r-1}^{(m)}}{I} \right]}{pe \left[t_m + \frac{a_1^{(m)}}{I_+} + \frac{a_2^{(m)}}{I_+} \dots \frac{a_{2r-1}^{(m)}}{I} \right]}$$

are also ~~these~~ identical with the denominators of those of ~~the~~

$$pe \left\{ f_m(z) \right\}$$

the convergent upon the right hand side of relationship () is also

the denominator of $pe C_{2r-1} \left[\left\{ f_m(z) \right\} \right]$. It follows that the successive

and, by virtue of relationship (), is also the denominator of

$pe C_{2r} \left[\left\{ f_m(z) \right\} \right]$, it follows that the ~~successive~~ denominators of $pe C_r \left[\left\{ f_m(z) \right\} \right]$

($r=1, 2, \dots$) are given by expressions ().

If $\{t_r\} \in \mathbb{N}_r$, all of the relationships proved above remain valid to the extent that the convergents referred to exist. Finally, in

this case $\left\{ f_m(z) \right\}$ is generated by the rational form series expansion of the rational function (), the denominator of the last convergent, $pe C_{2r} \left[\left\{ f_m(z) \right\} \right]$ of $\left\{ f_m(z) \right\}$ is the polynomial $D(z)$.

Theorem. Numbers $\{a_r^{(m)}\}$ ($r=1, 2, \dots; m=0, 1, \dots$) may be determined by use of the recursion

$$a_1^{(m)} = f_m, \quad a_2^{(m)} = -f_{m+1} f_m^{-1}, \quad a_3^{(m)} = a_2^{(m+1)} - a_2^{(m)} \quad (m=0, 1, \dots)$$

$$a_{2r}^{(m)} = a_{2r-1}^{(m+1)} a_{2r-2}^{(m+1)} a_{2r-1}^{(m)-1} \quad a_{2r+1}^{(m)} = a_{2r}^{(m+1)} + a_{2r-1}^{(m+1)} - a_{2r}^{(m)} \quad (r=2, 3, \dots; m=0, 1, \dots)$$

if and only if the series $\sum (t_r) z^r$ is semi-normal.

Numbers $a_r^{(m)}$ ($r=1, 2, \dots, 2l+1; m=0, 1, \dots$) may be determined by use of the above recursion, and $a_{2l+1}^{(m)} = 0$ ($m=0, 1, \dots$) if and only if

$$\{t_r\} \in \mathbb{L}_l$$

~~Proof~~ In both cases

$$\mathcal{C} \left\{ \sum (t_r) z^r \right\} = \text{pre} \left[\frac{a_1^{(m)}}{I+} \frac{a_2^{(m)}}{I+} \dots \frac{a_r^{(m)}}{I+} \dots \right] \quad (m=0, 1, \dots)$$

where these continued fractions are understood to terminate with the coefficient $a_{2l}^{(m)}$ in the second case.

Proof. Let the series $\sum (t_r) z^r$ be \mathbb{C} -regular and define the numbers $\{a_r^{(m)}\}$ by means of formulae (). These numbers may, of course, be constructed by means of Euclid's algorithm. For a fixed value of $m \geq 0$, the odd part of expansion () is

$$\text{pre} \left[a_1^{(m)} - \frac{a_2^{(m)} a_1^{(m)} z}{1 + (a_2^{(m)} + a_3^{(m)}) z} - \frac{a_4^{(m)} a_3^{(m)} z^2}{1 + (a_4^{(m)} + a_5^{(m)}) z} - \dots - \frac{a_{2r}^{(m)} a_{2r-1}^{(m)} z^r}{1 + (a_{2r}^{(m)} + a_{2r+1}^{(m)}) z} - \dots \right]$$

It follows from Theorem... that

$$A \left\{ \sum_1 t_{m+1} z^p \right\} = \text{pre} \left[\frac{-a_2^{(m)} a_1^{(m)}}{1 + (a_2^{(m)} + a_3^{(m)}) z} - \frac{a_4^{(m)} a_3^{(m)} z^2}{1 + (a_4^{(m)} + a_5^{(m)}) z} - \dots - \frac{a_{2r}^{(m)} a_{2r-1}^{(m)} z^r}{1 + (a_{2r}^{(m)} + a_{2r+1}^{(m)}) z} - \dots \right]$$

The even part of expansion () with m replaced by $m+1$ is

$$A \left\{ \sum_1 t_{m+1} z^p \right\} = \text{pre} \left[\frac{-a_1^{(m+1)}}{1 + a_2^{(m+1)} z} - \frac{a_3^{(m+1)} a_2^{(m+1)} z}{1 + (a_3^{(m+1)} + a_4^{(m+1)}) z} - \dots - \frac{a_{2r-1}^{(m+1)} a_{2r-2}^{(m+1)} z^r}{1 + (a_{2r-1}^{(m+1)} + a_{2r}^{(m+1)}) z} - \dots \right]$$

From Theorem... corresponding coefficients of these continued fractions

may be equated, and we derive

$$a_2^{(m)} a_1^{(m)} = a_1^{(m+1)}, \quad a_2^{(m)} + a_3^{(m)} = a_2^{(m+1)}$$

$$a_{2r}^{(m)} a_{2r-1}^{(m)} = a_{2r-1}^{(m+1)} a_{2r-2}^{(m+1)}, \quad a_{2r}^{(m)} + a_{2r+1}^{(m)} = a_{2r-1}^{(m+1)} + a_{2r}^{(m+1)} \quad (r=2,3,\dots; m=0,1,\dots)$$

Since $a_1^{(m)} = f_m$, and in each case $a_{2r-1}^{(m)} \in \mathbb{R}_1$ ($r=1,2,\dots; m=0,1,\dots$),

formulae () follow immediately.

The construction of the numbers $\{a_{2r}^{(m)}\}$ by ~~means~~ ^{the use} of the above ~~process~~ formulae () breaks down if ever $a_{2r-1}^{(m)} \notin \mathbb{R}_1$. ~~However,~~ ^{In this case,} Euclid's algorithm also fails to construct $b \left\{ \sum_1 t_{m+1} z^p \right\}$ for the value of m in question, $\sum_1 t_{m+1} z^p$ is no longer \mathbb{C} -regular, and $\sum_1 b_p z^p$

no longer semi-normal.

The above ~~method~~ ^{proof also} suffices for the second result of the theorem, ~~It needs~~ if we also take into account termination of the continued fractions involved. If $\{t_n\} \in \mathbb{L}_{r'}$, ~~Euclid~~ the denominator of $\{ \sum_{i=0}^r t_{ni} z^i \}$ ~~terminated~~ by means of Euclid's algorithm terminates in each case with the determination of the number $a_{2r+1}^{(m)} = 0$ ~~if~~ $(m=0,1,\dots)$. If, however, $a_{2r+1}^{(m)} = 0$ $(m=0,1,\dots)$, then setting

$$C_{2r} \left[\left\{ \sum_{i=0}^r t_{ni} z^i \right\} \right] = \frac{D_{r'}^{(m)}(z)^{-1} N_{r'}^{(m)}(z)}{d_{r'}^{(m)}} \uparrow$$

$$D_{r'}^{(m)} = \sum_{\nu=0}^{r'} d_{r'-\nu}^{(m)} z^\nu$$

It follows from Theorem... that $\sum_{\nu=0}^{r'} d_{r'-\nu}^{(m)} t_{n\nu} = 0$. Furthermore

$$C_{2r+1} \left[\left\{ \sum_{i=0}^r t_{ni} z^i \right\} \right] = C_{2r} \left[\left\{ \sum_{i=0}^r t_{n(i+1)} z^i \right\} \right]$$

and since the successive convergents of $C_{2r+1} \left\{ \left\{ \sum_{i=0}^r t_{ni} z^i \right\} \right\}$ and $C_{2r} \left\{ \left\{ \sum_{i=0}^r t_{n(i+1)} z^i \right\} \right\}$ have the same denominators, $d_{r'}^{(m)} = d_{r'}^{(0)}$ $(m=0,1,\dots; \nu=0,1,\dots)$ and $\sum_{\nu=0}^{r'} d_{r'-\nu}^{(0)} t_{n\nu} = 0$ $(m=0,1,\dots)$. Thus since all numbers $\{a_r^{(m)}\}$ preceding $a_{2r+1}^{(m)}$ can be constructed in each case, $\{t_n\} \in \mathbb{L}_{r'}$.

Theorem. If the series $\sum t_n z^n$ is semi-normal, ~~and~~ the numbers $\{a_r^{(m)}\}$ are those of Theorem ..., and let the denominators of the successive convergents of $C \text{ [pre } G \{f_m(z)\}]_r$ be (), then

$$\hookrightarrow D_0^{(m)}(z) = 1 \quad (m = J), \text{ and}$$

$$D_r^{(m)}(z) = D_{r-1}^{(m)}(z) + a_{2r}^{(m)} z D_{r-2}^{(m)}(z), \quad D_r^{(m+1)}(z) = D_{r-1}^{(m+1)}(z) + a_{2r+1}^{(m+1)} z D_r^{(m)}(z)$$

($r \leq J_1, m = J$)

If $\{t_n\} \in L_r$, then polynomials $D_r^{(m)}(z)$ ($r = J_1', m = J$) may be constructed by means of the first J recursions () and, if $D(z) = N(z)/K(z)$ is the rational function of Theorem ...,

$$D_{r_1}^{(m)}(z) = D(z) \quad (m = J)$$

Proof. ~~Recursion~~ Relationships () are simply the fundamental recursions ~~of Theorem ...~~ ~~for the~~ for the denominators of the successive convergents of the continued fraction $\text{pre } G \{f_m(z)\}$. Formulae () follow from the proof of Theorem ...

Theorem. Rational functions $\{E_r^{(m)}\}$ ($r=J_1, m=J$) may be determined

~~by use of~~ from the initial values

$$E_{-1}^{(m)} = 0 \quad (m=J) \quad E_0^{(m)} = \sum_0^{m-1} F_\nu z^\nu \quad (m=J)$$

by means of the recursion

$$E_{r+1}^{(m)} = E_{r-1}^{(m)} + (E_r^{(m)} - E_r^{(m)})^{-1} \quad (r, m=J)$$

if and only if the series $\sum_0^J F_\nu z^\nu$ is semi-normal

Numbers $E_r^{(m)}$ ($r=J_1, m=J$) may be determined by use of the above recursions and, if $D(z)^{-1}N(z)$ is the rational function of Theorem

$$\therefore E_{2r}^{(m)} = D(z)^{-1}N(z) \quad (m=J)$$

if and only if $\{F_\nu\} \in K_{r'}$.

All functions of the form $\{E_{2r}^{(m)}\}$ that ~~are~~ ^{can be} constructed are characterised by the property that if

$$E_{2r}^{(m)} \sim \sum F_{r,\nu}^{(m)} z^\nu \quad (r, m \in J)$$

then $F_{r,\nu}^{(m)} = F_\nu$ ($\nu = J_0^{m+r-1}$) ; furthermore, $F_{r,m+r} \neq F_{m+2r}$ also except when $\{F_\nu\} \in K_{r'}$ and $r=r'$ when $F_{r,m+r} = F_\nu$ ($\nu=J$).

Proof. Assuming the series $\sum_0^J F_\nu z^\nu$ to be semi-normal, the numbers

~~$\{a_r^{(m)}\}$ of Theorem ... can be constructed, the rational functions $\{E_{2r}^{(m)}\}$~~

of formulae () ^{of Theorem...} ~~may~~ can be constructed, and the numbers $\{a_r^{(m)}\}$

occurring therein satisfy recursions () of Theorem ... Using

Theorem ... , we may set

$$E_{2r}^{(m)} \equiv S_m(z) + z^m \sum_{\nu=2}^{2r} G_{\nu}^{(m)}(z) f_{m\nu}$$

where

$$H_{\nu}^{(m)}(z) = G_{\nu}^{(m)}(z) G_{\nu-1}^{(m)}(z) \dots G_2^{(m)}(z)$$

and

$$G_{2\nu}^{(m)}(z) = -D_{\nu}^{(m)}(z)^{-1} a_{2\nu}^{(m)} z D_{\nu-1}^{(m)}(z), \quad G_{2\nu+1}^{(m)}(z) = -D_{\nu}^{(m+1)}(z)^{-1} a_{2\nu+1}^{(m)} z D_{\nu-1}^{(m)}(z) \quad (\nu \in J, m \in J)$$

We shall set $G_1^{(m)}(z) = 0$ ($m \in J$).

From formula () of Theorem ... we have

$$E_{2r}^{(m+n)} \equiv S_m(z) + z^m \sum_{\nu=2}^{2r+1} H_{\nu}^{(m)}(z) f_{m\nu}$$

From the first of formulae () are recursions (), we have

$$I + G_{2\nu}^{(m)}(z) = D_{\nu}^{(m)}(z)^{-1} \{ D_{\nu}^{(m)}(z) - a_{2\nu}^{(m)} D_{\nu-1}^{(m)}(z) \} = D_{\nu}^{(m)}(z)^{-1} D_{\nu-1}^{(m+n)}(z) \quad (\nu \in J, m \in J)$$

Similarly

$$I + G_{2\nu+1}^{(m)}(z) = D_{\nu}^{(m+n)}(z)^{-1} D_{\nu}^{(m)}(z) \quad (\nu \in J, m \in J)$$

and, in consequence,

~~Equation ()~~

$$\{ I + G_{2\nu}^{(m+n)}(z) \} \{ I + G_{2\nu-1}^{(m)}(z) \} = \{ I + G_{2\nu+1}^{(m)}(z) \} \{ I + G_{2\nu}^{(m)}(z) \} \quad (\nu \in J, m \in J)$$

Similarly we have

$$\left. \begin{aligned} I + G_{2j}^{(m)}(z)^{-1} &= -D_{j-1}^{(m)}(z)^{-1} a_{2j}^{(m)-1} z^{-1} D_{j-1}^{(m)}(z) \\ I + G_{2j+1}^{(m)}(z)^{-1} &= -D_{j-1}^{(m)}(z)^{-1} a_{2j+1}^{(m)-1} z^{-1} D_j^{(m)}(z) \end{aligned} \right\} (\nu = J_1, m = J)$$

and, using the second of relationships (),

$$\{I + G_{2j-1}^{(m)}(z)^{-1}\} \{I + G_{2j}^{(m)}(z)^{-1}\} = \{I + G_{2j-2}^{(m)}(z)^{-1}\} \{I + G_{2j-1}^{(m)-1}(z)\} (\nu = J_2, m = J)$$

In terms of the functions $\{H_\nu^{(m)}(z)\}$ of formulae () and (), we have

~~$H_\nu^{(m)}(z) H_{\nu-1}^{(m)}(z)^{-1} = G_\nu^{(m)}(z) (\nu = J_1, m = J)$ we set $H_0^{(m)}(z)^{-1} = 0 (m = J)$.~~

~~Relationships () then yield~~

~~$\{I + H_{2j}^{(m)}(z) H_{2j-1}^{(m)}(z)^{-1}\} \{I + H_{2j+1}^{(m)}(z) H_{2j}^{(m)}(z)^{-1}\} =$~~

~~$\{I + H_{2j+1}^{(m)}(z) H_{2j}^{(m)}(z)^{-1}\} \{I + H_{2j}^{(m)}(z) H_{2j-1}^{(m)-1}(z)\}$~~

~~if we set $H_0^{(m)}(z)^{-1} = 0$, $H_1^{(m)}(z) = f_m z^m (m = J)$, $H_\nu^{(m)}(z) H_{\nu-1}^{(m)}(z)^{-1} = G_\nu^{(m)}(z)$~~

~~$\{H_{2j}^{(m)}(z) + H_{2j+1}^{(m)}(z)\} (\nu = J_1, m = J)$; relationships () then yield~~

$$\{H_{2j-1}^{(m)}(z) + H_{2j}^{(m)}(z)\} \{H_{2j-1}^{(m)}(z)^{-1} + H_{2j-2}^{(m)}(z)^{-1}\} = \{H_{2j}^{(m)}(z) + H_{2j+1}^{(m)}(z)\} \{H_{2j}^{(m)-1}(z) + H_{2j+1}^{(m)}(z)\} (\nu = J_1, m = J)$$

Similarly, from relationships ()

$$\{H_{2j-2}^{(m)}(z) + H_{2j-3}^{(m)}(z)\} \{H_{2j-2}^{(m)}(z)^{-1} + H_{2j-1}^{(m)}(z)^{-1}\} = \{H_{2j-1}^{(m)}(z) + H_{2j-2}^{(m)}(z)\} \{H_{2j-1}^{(m)-1}(z) + H_{2j}^{(m)}(z)\} (\nu = J_2, m = J_1)$$

A short inductive proof suffices to show that

$$H_{2j-1}^{(m+1)}(z) + H_{2j}^{(mn)}(z) = H_{2j}^{(m)}(z) + H_{2j+1}^{(m)}(z), \quad H_{2j-2}^{(m+1)}(z) + H_{2j-1}^{(mn)}(z) = H_{2j-1}^{(m)}(z) + H_{2j}^{(m)}(z) \quad (j=J, m=J)$$

In terms of the functions $\{H_r^{(m)}(z)\}$ we have

$$E_{2r}^{(m)}(z) = \sum_{i=0}^{m-1} H_{2i}^{(p)}(z) + \sum_{i=1}^{2r} H_{2i}^{(m)}(z) \quad (r, m=J)$$

and we introduce further functions $\{E_{2r+1}^{(m)}(z)\}$ given by

$$E_{-1}^{(m)}(z) = 0 \quad (m=J), \\ \downarrow \\ E_{2r+1}^{(m)}(z) = \sum_{i=1}^{2r+1} H_{2i}^{(m)}(z) \quad (r, m=J)$$

Using the second of relationships (), we find that

$$E_{2r}^{(m+1)}(z) - E_{2r}^{(m)}(z) = H_{2r+1}^{(m)}(z) \quad (r, m=J)$$

Similarly, using formula (),

$$E_{2r+2}^{(m)}(z) - E_{2r}^{(mn)}(z) = H_{2r+2}^{(m)}(z) \quad (r, m=J)$$

Furthermore

$$E_{2r+1}^{(m)}(z) - E_{2r-1}^{(m+1)}(z) = H_{2r+1}^{(m)}(z)^{-1}, \quad E_{2r+1}^{(mn)}(z) - E_{2r+1}^{(m)}(z) = H_{2r+2}^{(m)}(z)^{-1} \quad (r, m=J)$$

In short, the functions $\{E_r^{(m)}(z)\}$ ~~series~~ ~~recursion~~ may be constructed from the initial values () by means of recursion ().

Systematic

Implementation of recursion () breaks down when, ~~for~~ ~~or~~

some values of $r, m \in J$, $E_r^{(m)} - E_r^{(m)}(z)$ is not invertible. Suppose this to occur when $r = 2r''$ is even. We then have $H_{2r''+1}^{(m)}(z)$ is then not invertible, and in consequence, the same is true of $G_{2r''+1}^{(m)}(z)$.

Choose a pair of values $r', m' \in J$, $E_{r'}^{(m')} - E_{r'}^{(m')}(z)$ is not invertible. We may clearly suppose this to occur for some value of r' for which all similar differences with lesser value of r' are invertible. Assume, in the first instance, that the singular value of $r' = 2r''$ is even. It then follows that

$H_{2r''+1}^{(m')}(z)$ is not invertible, but that all functions $\{H_r^{(m')}(z)\}$ with $r \leq 2r''$ are invertible, and, in consequence, that the same is true with regard to the function $G_{2r''+1}^{(m')}(z)$ and the preceding functions $\{G_r^{(m')}(z)\}$. Referring to the second of formulae (), the functions $D_{r'}^{(m'+1)}(z)^{-1}$ and $D_{r'-1}^{(m'+1)}(z)$. Since, in particular, the function $G_{2r''}^{(m'+1)}(z)$ can be constructed, it follows from the first of formulae () that

$$G_{2r''+1}^{(m')} = -D_{r'}^{(m'+1)}(z)^{-1} a_{2r''+1}^{(m')} D_{r'-1}^{(m'+1)}(z).$$

The inverse of the function $D_{r'}^{(m'+1)}(z)^{-1}$, namely $D_{r'}^{(m'+1)}(z)$, certainly exists; since $D_{r'-1}^{(m'+1)}(z)^{-1}$ occurs in the formula for $G_{2r''}^{(m'+1)}$, the inverse of $D_{r'-1}^{(m'+1)}(z)$ is invertible. If $a_{2r''+1}^{(m')} \in \mathbb{R}_2$, $G_{2r''+1}^{(m')}$ can be

constructed. We have assumed this to be impossible, hence $a_{2^{m+1}}^{(m+1)} \in \mathbb{R}_J$.

~~This implies that $f_m(z)$ is not C-regular~~

Since this number ~~is~~ is produced during the construction of $\mathcal{B}\{f_m(z)\}$ by means of Euclid's algorithm, it follows that the series $f_m(z)$ is not C-regular, and that ~~$\sum_{i=0}^{\infty} f_i(z)$ is not semi-normal.~~

~~in consequence that if application of the ϵ -algorithm breaks down in the manner described, $\sum_{i=0}^{\infty} f_i(z)$ is not a semi-normal series.~~

We now ~~consider the case~~ show that if the ~~functions~~ ~~of~~ the form $E_{2^{m+1}}^{(m+1)}(z)$ and $E_{2^{m+1}}^{(m+1)}(z)$ can be constructed, their difference is always invertible. ~~If~~ If $E_{2^{m+1}}^{(m+1)}(z) - E_{2^{m+1}}^{(m+1)}(z)$ can be constructed but is not invertible, the function

$$G_{2^{m+2}}^{(m)}(z) = -D_{2^{m+1}}^{(m)}(z)^{-1} a_{2^{m+2}}^{(m)} z D_{2^{m+1}}^{(m)}(z)$$

can be constructed but is not invertible. However, the inverse of $D_{2^{m+1}}^{(m)}(z)^{-1}$ exists, ~~and~~ that of $D_{2^{m+1}}^{(m)}(z)$ occurs in the formula for ~~$G_{2^{m+1}}^{(m)}(z)$~~ $G_{2^{m+1}}^{(m)}(z)$ which has been assumed to be well determined.

Since all ~~series~~ functions $G_{2^{m+2}}^{(m)}(z)$ ($m \equiv J$) can be constructed the all numbers $a_{2^{m+2}}^{(m+1)}$, in particular, ~~is~~ ~~are~~ well determined, and from the

first of formulae (), $a_{2^{m+1}}^{(m+1)} \in \mathbb{R}_J$. ~~It also~~ Furthermore, it also follows

from this formula that, $a_{2r+2}^{(m)} = a_{2r+1}^{(m+1)} a_{2r}^{(m+1)} a_{2r+1}^{(m)-1} \in R_{\Gamma}$. Thus

$G_{2r+2}^{(m)}(z)$ is invertible, as is the difference $E_{2r+1}^{(m+1)}(z) - E_{2r+1}^{(m)}(z)$.

We have now shown that implementation of the ε -algorithm can only ~~occur~~ ^{break down} in one way, and that this breakdown implies that the series $\sum_1 F_j z^j$ is not semi-normal.

If $\{F_j\} \in L_{\Gamma}$, then in succession $G_{2r+1}^{(m)}(z) = 0$ ($m = J$), $H_{2r+1}^{(m)}(z) = 0$ ($m = J$), and the functions $\{E_{2r}^{(m)}(z)\}$ are given by formula (). Again, breakdown in the implementation of the ε -algorithm before the ^{functions} ~~numbers~~ $\{E_{2r}^{(m)}(z)\}$ are produced implies that $\{F_j\} \notin L_{\Gamma}$.

The stated results concerning the power series $\sum_1 F_{2r}^{(m)} z^j$ of formula () are direct consequences of Theorem ..., and those concerning the functions $\{E_{2r}^{(m)}(z)\}$ follow from Theorem ...

The ~~sequence of terms~~ rational functions produced by application of the ε -algorithm to the sequence of partial sums of a semi-normal series are simply related to those produced by application ~~to the sequence of terms~~ to the sequence of accumulated sums of these partial sums and to the sequence of successive terms of the series. For simplicity in exposition, we present ~~these theorems describing~~ relationships in terms of elements of a ring, rather than of rational functions.

~~Theorem~~ . Let $S_m \in R_1$ ($m=J$) and let $\varepsilon_r^{(m)}$ be ~~the~~
 Theorem . Let $\sum F_j z^j$ be a semi-normal series, and the rational functions $E_r^{(m)}(z)$ ($r=J, m=J$) ~~be~~ be the rational functions produced from ~~the~~ initial values () by means of recursion (); Then rational functions $^{(1)}E_r^{(m)}(z)$ ($r=J, m=J$) ~~may~~ be produced from the initial values

$$^{(1)}E_{-1}^{(m)}(z) = 0 \quad (m=J), \quad ^{(1)}E_{00}^{(m)} = \sum_0^m E_0^{(2)}(z) \quad (m=J)$$

by means of a recursion similar to (), and for $r, m=J$

$${}^{(1)}E_{2r}^{(m)}(z) = \sum_0^{(m+r)} E_0^{(j)}(z) + \sum_0^{r-1} \{ E_{2(r-j)}^{(m+j+2)}(z)^{-1} - E_{2(r-j)}^{(m+j+1)}(z)^{-1} \}^{-1}$$

$${}^{(1)}E_{2r+1}^{(m)}(z) = E_{2r}^{(m)}(z)^{-1}$$

$$E_{2r}^{(m)}(z) = {}^{(1)}E_{2r+1}^{(m)}(z)^{-1}$$

$$E_{2r+1}^{(m)}(z) = \sum_0^r \{ {}^{(1)}E_{2(r-j)+1}^{(m+j+1)}(z)^{-1} - {}^{(1)}E_{2(r-j)+1}^{(m+j)}(z)^{-1} \}^{-1}$$

If, in addition $E_{2r+1}^{(m)}(z)$ is invertible ($r, m \in \mathbb{J}$), then rational functions

${}^{(-1)}E_r^{(m)}(z)$ ($r \in \mathbb{J}, m \in \mathbb{J}$) can be produced from the initial values

$${}^{(-1)}E_{-1}^{(m)}(z) = 0, \quad (m \in \mathbb{J}), \quad {}^{(-1)}E_0^{(m)} = T_m z^m \quad (m \in \mathbb{J})$$

by means of a recursion similar to (), and for $m \in \mathbb{J}$

$${}^{(-1)}E_{2r}^{(m)}(z) = E_{2r+1}^{(m)}(z)^{-1}$$

$${}^{(-1)}E_{2r+1}^{(m)}(z) = \sum_0^r \{ E_{2(r-j)+1}^{(m+j+1)}(z)^{-1} - E_{2(r-j)+1}^{(m+j)}(z)^{-1} \}^{-1}$$

$$E_{2r}^{(m)}(z) = \cancel{E_0^{(0)}(z)} + \sum_0^{m+r-1} {}^{(-1)}E_0^{(j)}(z) + \sum_0^{r-1} \{ E_{2(r-j)}^{(m+j+2)}(z)^{-1} - {}^{(-1)}E_{2(r-j)}^{(m+j+1)}(z)^{-1} \}^{-1}$$

$$E_{2r+1}^{(m)}(z) = {}^{(-1)}E_{2r}^{(m)}(z)^{-1}$$

If, ~~in addition~~ in the above, the condition that $\sum F_j z^j$ be semi-normal be replaced by the condition that $\{F_j\} \in \mathbb{L}_r$, then ~~number~~ functions ${}^{(1)}E_r^{(m)}(z)$ ($r \in \mathbb{J}, m \in \mathbb{J}$) can be constructed as above,

relationships ~~of the~~ () hold for these functions and for the functions $E_r^{(m)}(z)$ ($r \in \mathbb{J}_0^{2r}, m \in \mathbb{J}$), and if $D(z)^{-1}N(z)$ is the rational function

$\left[\begin{matrix} \{i\} \\ \{j\} \in \mathbb{L}, \{R(z)\} \end{matrix} \right]$ } Theorem ..

$${}^{(1)}E_{2r+1}^{(m)}(z) = N(z)^{-1}D(z) \quad (m \in \mathbb{J})$$

Furthermore, functions ${}^{(-1)}E_r^{(m)}(z)$ ($r \in \mathbb{J}_1^{2r}, m \in \mathbb{J}$) can also be constructed as above, they are related to the functions $E_r^{(m)}(z)$ ($r \in \mathbb{J}_1^{2r}, m \in \mathbb{J}$) by formulae of the form (), and

$${}^{(-1)}E_{2r}^{(m)}(z) = 0. \quad (m \in \mathbb{J})$$

Proof. We begin by adopting two notations: for \forall suitably defined function $f(m, r)$ of the two integer arguments m and r , we set $\Delta f(m, r) = f(m+1, r)$, $\nabla f(m, r) = f(m, r+1) - f(m+1, r)$. Rearranging relationships () and replacing r by $2r+1$, we have, for example,

$$\nabla E_{2r}^{(m)} = (\Delta E_{2r+1}^{(m)})^{-1} \quad \nabla E_{2r}^{(m)}(z) = \left\{ \Delta E_{2r+1}^{(m)}(z) \right\}^{-1}$$

We first prove formula () in conjunction with

$$\Delta^{(1)} E_{2r}^{(m)}(z) = \left[\nabla \left\{ E_{2r-2}^{(m+1)}(z) \right\}^{-1} \right]^{-1}$$

It is easy to verify that formula () holds for $r=0$ and $r=1$, and that formula () is correct when $r=1$. We assume that for a fixed integer pair $r \in \mathbb{J}_2, m \in \mathbb{J}$ the numbers these formulae

~~have been proved for all values of r up to and including $r-1$~~
 these formulae have been proved correct with r replaced by $1, 2, \dots, r-1$ and for
 $m \in J$ in each case. ~~We then have~~ Since $\{\Delta E_{2r+2}^{(m)}(z)\}^{-1}$ is constructed
 during application of the ~~epitaxial~~ algorithm to the sequence $\{\sum_{i=0}^{m-1} E_i(z)\}^{-1}$
 and $E_{2r+2}^{(m)}(z)^{-1}$, ~~both~~ $E_{2r+2}^{(m+2)}(z)^{-1}$ both exist, $[\Delta \{E_{2r+2}^{(m)}(z)^{-1}\}]^{-1}$ exists, and

we have

$$\begin{aligned} {}^{(1)}E_{2r}^{(m)}(z) &= {}^{(1)}E_{2r-2}^{(m+2)}(z) + \{\Delta {}^{(1)}E_{2r-1}^{(m)}(z)\}^{-1} \quad \leftarrow \text{crossed out} \\ &= {}^{(1)}E_{2r-2}^{(m+2)}(z) + [\Delta \{E_{2r-2}^{(m)}(z)^{-1}\}]^{-1} \quad (m \in J) \quad () \end{aligned}$$

Replacing m by $m+1$ in ~~this~~ formula (), ~~and~~ subtracting formula ()
 as it stands from the modified version, and using formula () with
 r replaced by $r-1$ and m by $m+1$, we derive

$$\Delta {}^{(1)}E_{2r}^{(m)}(z) = [\Delta \{E_{2r-4}^{(m+2)}(z)^{-1}\}]^{-1} + \Delta \left\{ [\Delta \{E_{2r-2}^{(m+2)}(z)^{-1}\}]^{-1} \right\}$$

The first term on the right hand side of this equation is

$$E_{2r-2}^{(m+2)}(z) \left\{ [E_{2r-4}^{(m+3)}(z) - E_{2r-2}^{(m+2)}(z)]^{-1} E_{2r-4}^{(m+3)}(z) \right\}$$

However,

$$E_{2r-4}^{(m+3)}(z) = E_{2r-2}^{(m+2)}(z) - \{E_{2r-3}^{(m+3)}(z) - E_{2r-3}^{(m+2)}(z)\}^{-1}$$

Using this formula, and that for the difference

So... concerning the ^{93, 94, 95} ~~recurrence~~ ~~function~~ $\{E_r^{(m)}(z)\}$ are proved in the same way.

Using this formula, and that for the difference $(i-1)E_{2r-4}^{(m+3)} - (i-1)E_{2r-2}^{(m+2)}$ derived from it, expression (80) becomes

$$\begin{aligned} & (i-1)E_{2r-2}^{(m+2)} \left\{ (i-1)E_{2r-3}^{(m+3)} - (i-1)E_{2r-3}^{(m+2)} \right\} \left[(i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-3}^{(m+3)} - (i-1)E_{2r-3}^{(m+2)} \right]^{-1} \\ & = (i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+2)} \left[(i-1)E_{2r-3}^{(m+3)} - (i-1)E_{2r-3}^{(m+2)} \right] \end{aligned}$$

But

$$(i-1)E_{2r-3}^{(m+2)} = (i-1)E_{2r-1}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+1)} \quad (m=0,1,\dots)$$

and, using this formula together with the version of it obtained by replacing m by $m+1$, we find that expression (80) becomes

$$\begin{aligned} & (i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+2)} \left\{ \left[(i-1)E_{2r-1}^{(m+2)} - (i-1)E_{2r-2}^{(m+3)} - (i-1)E_{2r-2}^{(m+2)} \right]^{-1} \right. \\ & \left. - (i-1)E_{2r-1}^{(m+1)} + \left[(i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+1)} \right]^{-1} \right\} (i-1)E_{2r-2}^{(m+2)} \end{aligned} \quad (81)$$

The second and third terms upon the right hand side of equation (79) taken together can be written in the form

$$\begin{aligned} & (i-1)E_{2r-2}^{(m+3)} \left[(i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+3)} \right]^{-1} (i-1)E_{2r-2}^{(m+2)} \\ & - (i-1)E_{2r-2}^{(m+2)} \left[(i-1)E_{2r-2}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} \right]^{-1} (i-1)E_{2r-2}^{(m+1)} \end{aligned} \quad (82)$$

Using expressions (81) and (82), the right hand side of equation (79) may be rewritten as

$$\begin{aligned}
 & (i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+2)} \left[\left\{ (i-1)E_{2r-1}^{(m+2)} - (i-1)E_{2r-1}^{(m+1)} \right\} (i-1)E_{2r-2}^{(m+2)} \right] \\
 & + \left\{ (i-1)E_{2r-2}^{(m+3)} - (i-1)E_{2r-2}^{(m+2)} \right\} \left[\left\{ (i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+3)} \right\}^{-1} (i-1)E_{2r-2}^{(m+2)} \right] \\
 & - (i-1)E_{2r-2}^{(m+2)} \left[\left\{ (i-1)E_{2r-2}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} \right\}^{-1} \left\{ (i-1)E_{2r-2}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} \right\} \right] \\
 & = - (i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+2)} \left[\left\{ (i-1)E_{2r-1}^{(m+2)} - (i-1)E_{2r-1}^{(m+1)} \right\} (i-1)E_{2r-2}^{(m+2)} \right] .
 \end{aligned}$$

However

$$(i-1)E_{2r-1}^{(m+2)} - (i-1)E_{2r-1}^{(m+1)} = \left\{ (i-1)E_{2r}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} \right\}^{-1} .$$

Thus equation (79) becomes

$$\begin{aligned}
 (i)E_{2r}^{(m+1)} - (i)E_{2r}^{(m)} &= (i-1)E_{2r-2}^{(m+2)} - (i-1)E_{2r-2}^{(m+2)} \left[\left\{ (i-1)E_{2r}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} \right\}^{-1} (i-1)E_{2r-2}^{(m+2)} \right] \\
 &= (i-1)E_{2r-2}^{(m+2)} \left\{ (i-1)E_{2r}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} \right\}^{-1} \left[- \left\{ (i-1)E_{2r}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} \right\} - (i-1)E_{2r-2}^{(m+2)} \right] \\
 &= - (i-1)E_{2r-2}^{(m+2)} \left[\left\{ (i-1)E_{2r}^{(m+1)} - (i-1)E_{2r-2}^{(m+2)} \right\}^{-1} (i-1)E_{2r}^{(m+1)} \right] \\
 &= - \left\{ (i-1)E_{2r-2}^{(m+2)}^{-1} - (i-1)E_{2r}^{(m+1)}^{-1} \right\}^{-1} \\
 &= \left\{ (i-1)E_{2r}^{(m+1)}^{-1} - (i-1)E_{2r-2}^{(m+2)}^{-1} \right\}^{-1} .
 \end{aligned}$$

In short, assuming that formulae (69) and (76) hold with r replaced by $r-1$, we have proved that formula (76) is true as it stands.

We now use the relationship

$$(i)_{E_{2r}^{(m+1)}} - (i)_{E_{2r}^{(m)}} = \left\{ (i)_{E_{2r+1}^{(m)}} - (i)_{E_{2r-1}^{(m+1)}} \right\}^{-1} \quad (m=0,1,\dots)$$

and formula (76) to derive

$$(i-1)_{E_{2r}^{(m+1)}}^{-1} - (i-1)_{E_{2r-2}^{(m+2)}}^{-1} = (i)_{E_{2r+1}^{(m)}} - (i)_{E_{2r-1}^{(m+1)}} \quad (m=0,1,\dots)$$

Using this relationship, and formula (69) with r replaced by $r-1$ and m by $m+1$, we derive formula (69) as it stands.

We now turn to formula (68). For a fixed value of $i \geq 1$ we have, using formula (76)

$$(i)_{E_{2(r-v)}^{(m+v)}} - (i)_{E_{2(r-v-1)}^{(m+v+1)}} = \left\{ (i-1)_{E_{2(r-v-1)}^{(m+v+2)}}^{-1} - (i-1)_{E_{2(r-v-1)}^{(m+v+1)}}^{-1} \right\}^{-1} \\ (r, m=0,1,\dots; v=0,1,\dots,r-1) .$$

By summation, we obtain

$$\sum_{v=0}^{r-1} \left\{ (i)_{E_{2(r-v)+1}^{(m+v)}} - (i)_{E_{2(r-v-1)}^{(m+v+1)}} \right\} = (i)_{E_{2r}^{(m)}} - (i)_{E_0^{(m+r)}} \\ = \sum_{v=0}^{r-1} \left\{ (i-1)_{E_{2(r-v-1)}^{(m+v+2)}}^{-1} - (i-1)_{E_{2(r-v-1)}^{(m+v+1)}}^{-1} \right\}^{-1} \quad (r, m=0,1,\dots)$$

which is equivalent to formula (68).

Formula (72) is, of course, equivalent to formula (69).

Turning to formula (73) we have, for a fixed value of $i \geq 1$,

$$(i-1)E_{2(r-v)+1}^{(m+v+1)} - (i-1)E_{2(r-v)+1}^{(m+v+2)} = \{ (i)E_{2(r-v)+1}^{(m+v+1)-1} - (i)E_{2(r-v)+1}^{(m+v)-1} \}^{-1}$$

$$(r, m=0, 1, \dots; v=0, 1, \dots, r)$$

using formula (69) again and hence, by summation,

$$\sum_{v=0}^r \{ (i-1)E_{2(r-v)+1}^{(m+v+1)} - (i-1)E_{2(r-v)+1}^{(m+v+2)} \} = (i-1)E_{2r+1}^{(m+1)}$$

$$= \sum_{v=0}^r \{ (i)E_{2(r-v)+1}^{(m+v+1)-1} - (i)E_{2(r-v)+1}^{(m+v)-1} \}^{-1}, \quad (r, m=0, 1, \dots)$$

or formula (73).

The remaining results of the theorem concerning the case in which $i \leq 0$ are proved in the same way.

If the arithmetic operations used in the execution of the ϵ -algorithm are consistently restricted to be the simple arithmetic operations associated with such numbers, Theorem 11 can be related to application of the algorithm to the numbers of clauses 1. - 10. of § 3.

Theorem 11 does not apply directly to extended Cayley numbers or to vectors: numbers of the first type are not known to be members of a distributive ring with associating inverse, whilst those of the second are not closed with respect to binary multiplication. Nevertheless, as we shall see in § 8, the result of the theorem also holds true for these numbers.

The Padé quotient.

If $f_0 \in \mathbb{R}_1$ and $f_0 \in \mathbb{R}$ ($\nu = J_0$), the series $\sum f_0 z^\nu$ possesses a two-sided inverse $\sum \tilde{f}_0 z^\nu$, to which all of the theory of §5 may be applied. The rational functions occurring in this theory are ^{simply} related not only to the series $\sum \tilde{f}_0 z^\nu$ but also to the series $\sum f_0 z^\nu$ itself. As a preliminary, we give

Theorem. Let $\sum \hat{f}_0 z^\nu$ be the series ^{inverse} reciprocal to $\sum f_0 z^\nu$, and $\tilde{D}(z)^{-1} \tilde{N}(z)$ be a rational function generating a series expansion _{for some $\nu \in \mathbb{J}_1$}

$\sum \tilde{f}_0 z^\nu$ for which $\tilde{f}_{\nu} = f_{\nu}$ ($\nu = J_0^{-1}$); then the rational function if

$$\tilde{N}(z)^{-1} \tilde{D}(z) = \sum \hat{f}_0 z^\nu, \text{ we have } \hat{f}_0 = f_0 \text{ } (\nu = J_0^{-1})$$

Proof. It follows from the first part of the proof of Theorem.. that

~~$\sum \hat{f}_0 z^\nu$ is the ~~in~~ series inverse to inverse series of $\sum f_0 z^\nu$.~~

Hence

Proof. Since $\tilde{f}_{\nu} = f_0 = f_0^{-1}$ and ~~$f_0 \in \mathbb{R}_1$~~ $\in \mathbb{R}_1$, the series $\sum \tilde{f}_0 z^\nu$ has an inverse, and, ~~this inverse is~~ from the first part of the proof of Theorem.., this inverse is $\sum \hat{f}_0 z^\nu$. Hence $\hat{f}_0 = \tilde{f}_0^{-1} = f_0^{-1}$

= f_0 , and

$$\begin{aligned} \hat{f}_{\nu} &= -f_{\nu}^{-1} \sum_{\nu=0}^{\nu-1} \hat{f}_0 \hat{f}_{\nu-\nu} \\ &= -f_0^{-1} \sum_{\nu=0}^{\nu-1} \hat{f}_0 \hat{f}_{\nu} \end{aligned} \quad \begin{aligned} &\text{---} \\ &(\nu = J_0^{-1}) \end{aligned}$$

Since these formulae which determine $\hat{f}_{\nu} (\nu \geq J_0^{r-1})$ are ~~the~~ the same as those which determine the first r coefficients of the series inverse to $\sum_{\nu=0}^{\infty} \tilde{f}_{\nu} z^{\nu}$, which is ~~the~~ ~~series~~ ~~inverse~~ ~~to~~ ~~$\sum_{\nu=0}^{\infty} \tilde{f}_{\nu} z^{\nu}$~~ , and we have ~~$\hat{f}_{\nu} = \tilde{f}_{\nu} (\nu \geq J_0^{r-1})$~~ .

~~Definition~~

Let ~~$f \in R (\nu \geq J)$~~ $\sum_{\nu=0}^{\infty} f_{\nu} z^{\nu}$, with $f_{\nu} \in R (\nu \geq J)$, be a prescribed ^{formal} power series. Definition. Assume that for \blacklozenge fixed $r, m \in \mathbb{J}$, the orthogonalisation process described by formulae () of Theorem is sufficient to determine the numbers $v_{\nu}^{(m)}, w_{\nu}^{(m)} (\nu \geq J_{r,1}^m)$ from the coefficients $f_{m+\nu} (\nu \geq J_0^{r-1})$ of the series $\sum_{\nu=0}^{\infty} f_{\nu} z^{\nu}$; ~~the~~ ^{the} Padé quotient $P_{r,m+r-1}(z)$ derived from the series $\sum_{\nu=0}^{\infty} f_{\nu} z^{\nu}$ is then said to exist and is defined by the formula

$$P_{r,m+r-1}(z) \equiv \sum_{\nu=0}^{m-1} f_{\nu} z^{\nu} + z^m C \left[\text{pre} \left\{ \frac{v_{\nu}^{(m)}}{1+w_{\nu}^{(m)} z} \frac{v_{\nu}^{(m)} z^{\nu}}{1+w_{\nu}^{(m)} z} \right\} \right]_r$$

Assume that for fixed $r, m \in \mathbb{J}$ with $r+m > 0$, the orthogonalisation process mentioned ~~Assume that for fixed $r, m \in \mathbb{J}$~~

~~above is sufficient~~ to determine numbers $\tilde{v}_{\nu}^{(m)}, \tilde{w}_{\nu}^{(m)} (\nu \geq J_1^r)$ by a similar process from the ~~series~~ coefficients $\tilde{f}_{m+\nu} (\nu \geq J_0^{r-1})$ of the series $\sum_{\nu=0}^{\infty} \tilde{f}_{\nu} z^{\nu}$ inverse to $\sum_{\nu=0}^{\infty} f_{\nu} z^{\nu}$ (it being assumed that the latter exists); the Padé quotient $P_{m+r-1,r}(z)$ derived from the series $\sum_{\nu=0}^{\infty} \tilde{f}_{\nu} z^{\nu}$ is then said to exist and is defined by the formula

$$P_{m+r-1,r}(z) = \left[\sum_{l=0}^{m-1} f_l z^l + z^m C \left[\text{pre} \left\{ \frac{v_1^{(m)}}{1+w_1^{(m)}z} \frac{v_r^{(m)} z^2}{1+w_r^{(m)}z} \right\} \right]_r \right]^{-1}$$

If ~~the~~ additional assumptions concerning the series $\sum f_l z^l$ are made, it is clear from the theory of § that the Padé quotients $\{P_{r,mr-1}(z)\}$ and $\{P_{m+r-1,r}(z)\}$ may be constructed by processes alternative to those occurring in the above definition. We have chosen to ~~give a constructive~~ ^{based the} definition of the Padé quotient upon a construction which requires the least restrictive auxiliary conditions. ~~As per Definition serves to~~ The Padé quotients $\{P_{r,mr-1}(z)\}$ and $\{P_{m+r-1,r}(z)\}$ of Definition ... belong, of course to the ensemble $P_{i,j}(z)$ ($i,j = J$) to which, for convenience, ~~has been~~ the quotient $P_{0,-1}(z) \equiv 0$ has been adjoined. Of these quotients, those of the form $P_{r,r-1}(z), P_{r,r}(z), P_{r,r+1}(z)$ ($r = J$) are defined in two ways which, are, from Theorem ... , equivalent ???

Let $\sum_1 f_n z^n$ be a prescribed formal power series. As
 Theorem. ~~Let $m' \in J$ and, for $m, i, j \in J$, let the Padé quotient~~
 $P_{i,j}^{(m')}(z)$ derived from the series $\sum_1 f_n(z) z^n$ exist; then ~~the~~ if
 $m' \in J_0$, the ~~Padé~~ quotient $P_{i,j+m'}^{(m')}(z)$ derived from
 the series $\sum_1 f_{n+j} z^n$ also exists, and

$$\cancel{P_{i,j+m'}^{(m')}(z) = \sum_0^{m'-m} \frac{f_{j+m'+n}}{f_{j+m'+n}} z^n + z}$$

$$P_{i,j+m'}(z) = \sum_0^{m'-1} f_{j+m'+n} z^n + z^{m'} P_{i,j}^{(m')}(z)$$

Assume that the series $\sum_1 \tilde{f}_n z^n$ ~~exists~~ inverse to $\sum_1 f_n z^n$ exists
 and that for $i, j \in J$, the Padé quotient $\tilde{P}_{i,j}(z)$ derived
 from $\sum_1 \tilde{f}_n z^n$ also exists; then the quotient $P_{j,i}(z)$
 derived from the series $\sum_1 f_n z^n$ also exists, and we have

$$P_{j,i}(z) = P_{i,j}(z)^{-1}$$

Proof. Relationship () is a simple consequence of the Definition
 the first part of Definition () ... ; and the formula ()
 follows from Theorem .

Definition... If the series $\sum f_n z^n$ and its inverse series $\sum \tilde{f}_n z^n$ are both seminormal, then we say that the series $\sum f_n z^n$ is normal if both it and its inverse are semi-normal.

It follows from Theorem... that we ^{can} construct the complete ensemble of Padé quotients $P_{i,j}(z)$ derived from a normal series $\sum f_n z^n$ by applying the ε -algorithm to the initial values $\{\sum_{i=0}^{m-1} f_n z^n\}$ to produce quotients of the form $P_{i,m-r-1}(z)$ ($i, m \geq 1$), and, ~~to the~~ after constructing the inverse series $\sum \tilde{f}_n z^n$, to the initial values $\{\sum_{i=0}^{m-1} \tilde{f}_n z^n\}$ to produce rational functions $\tilde{F}_{m,r}^{(m)}(z)$ ($m, r \geq 1$), after which $\tilde{P}_{m,r}(z) = P_{m,r-1}(z) \tilde{F}_{m,r}^{(m)}(z) = \tilde{F}_{2r}^{(m)}(z)^{-1}$ ($m, r \geq 1$ ($m+r > 0$)). However, it is possible to ~~construct~~ construct this complete ensemble of quotients at one blow by means of the ε -algorithm, without the intermediate construction of the inverse series.

Theorem . Rational functions $E_r^{(m)}(z)$ ($r = J_1, \hat{r} = r+2, m = J - \hat{r}$) may be determined from the initial values

$$E_{-1}^{(m)}(z) = E_{2m}^{(-m)}(z) = 0 \quad (m = J_1) \quad E_0^{(m)}(z) = \sum_0^{m-1} f_j z^j \quad (m = J)$$

by means of the recursion

$$E_{r+1}^{(m)}(z) = E_{r-1}^{(m)}(z) + \{E_r^{(m)}(z) - E_r^{(m)}(z)\}^{-1} \quad (r = J; \hat{r} = r+2, m = J - \hat{r})$$

if and only if the series $\sum_0^{\infty} f_j z^j$ is normal. We then have

$$E_{2r}^{(m)}(z) = P_{r, m-r-1}(z) \quad (r = J, m = J - r)$$

where $P_{i,j}(z)$ is the Padé quotient of order i,j derived from the series $\sum_0^{\infty} f_j z^j$.

Proof. That the functions $E_r^{(m)}(z)$ ($r = J$) can be constructed in the manner described if and only if $\sum_0^{\infty} f_j z^j$ is a semi normal series has been proved in Theorem...; that ~~the~~ formulae () hold for the functions of this set is also known (see formula ()).

We now turn to the remaining functions $\{E_r^{(m)}(z)\}$ of the theorem. If both ~~series~~ the series $\sum_0^{\infty} f_j z^j$ and its reciprocal $\sum_0^{\infty} \tilde{f}_j z^j$ are semi-normal, the quotients $P_{r,r}(z), P_{r+1,r}(z)$ ($r = J$) can be

constructed in two ways. The first is quite simply by applying the relationships () for $r, m = J$ to the ^{sequence} initial values $\{ \sum_{i=0}^{m-1} f_i z^i \}$

When $E_{2r}^{(1)}(z) = P_{r,r}(z)$, $E_{2r+2}^{(0)}(z) = P_{r,r}(z)$ ($r = J$). ~~Now~~ To consider the second, we remark that if functions $\tilde{E}_r^{(m)}(z)$ ($r, m = J$) are produced

by means of application of the ~~same~~ applying relationships of the form () for $r, m = J$ to the ^{sequence} initial values $\{ \sum_{i=0}^{m-1} f_i z^i \}$, then

$$\tilde{E}_{2r}^{(m)}(z) = P_{m+r-1, r}^{-1}(z) \quad (m, r = J \text{ (} m+r > 0 \text{)}).$$

If functions $\tilde{E}_r^{(m)}(z)$ ($r, m = J$) are ~~also~~ similarly produced from the sequence

$$\left\{ \sum_{i=0}^m \sum_{j=0}^{i-1} f_j z^j \right\} \quad m = J \text{ then, from Theorem}$$

$$\dots, \quad {}^{(1)}\tilde{E}_{2r+1}^{(m)}(z) = \tilde{E}_{2r}^{(m+1)}(z)^{-1} = P_{m+r, r}(z) \quad (r, m = J).$$

$${}^{(1)}\tilde{E}_{2r+1}^{(0)}(z) = P_{r, r}(z) = E_{2r}^{(1)}(z), \quad {}^{(1)}\tilde{E}_{2r+1}^{(1)}(z) = P_{r+1, r}(z) = E_{2r+2}^{(0)}(z) \quad (r = J)$$

We now show that not only are alternate members of the sequences $\{ {}^{(1)}\tilde{E}_r^{(0)}(z) \}$, $\{ E_r^{(1)}(z) \}$ and $\{ {}^{(1)}\tilde{E}_r^{(1)}(z) \}$, $\{ E_r^{(0)}(z) \}$ equal, but that these two sequences are equivalent. We have, of course, ${}^{(1)}\tilde{E}_0^{(0)}(z) = 0 = E_{-1}^{(1)}(z)$, ${}^{(1)}\tilde{E}_0^{(1)}(z) = f_0 = f_0^{-1} = E_1^{(0)}(z)$. Assume it to have been shown that ${}^{(1)}\tilde{E}_{2r}^{(0)}(z) = E_{2r-1}^{(1)}(z)$, ${}^{(1)}\tilde{E}_{2r+1}^{(1)}(z) = E_{2r+1}^{(0)}(z)$ ($r = J_0$).

Then, ~~using~~ using relationships of the form (),

$$\begin{aligned} {}^{(1)}\tilde{E}_{2r}^{(0)}(z) &= {}^{(1)}\tilde{E}_{2r-2}^{(1)}(z) + \left\{ {}^{(1)}\tilde{E}_{2r-1}^{(1)}(z) - {}^{(1)}\tilde{E}_{2r-1}^{(0)}(z) \right\}^{-1} \\ &= E_{2r-1}^{(0)}(z) + \left\{ E_{2r}^{(0)}(z) - E_{2r-2}^{(1)}(z) \right\}^{-1} = E_{2r-1}^{(1)}(z) \end{aligned}$$

and, using the same relationships,

$$\begin{aligned} {}^{(1)}\tilde{E}_{2r}^{(1)}(z) &= {}^{(1)}\tilde{E}_{2r}^{(0)}(z) + \left\{ {}^{(1)}\tilde{E}_{2r+1}^{(0)}(z) - {}^{(1)}\tilde{E}_{2r-1}^{(1)}(z) \right\}^{-1} \\ &= E_{2r-1}^{(1)}(z) + \left\{ E_{2r}^{(1)}(z) - E_{2r}^{(0)}(z) \right\}^{-1} = E_{2r+1}^{(0)}(z). \end{aligned}$$

Hence, by induction, we have ${}^{(1)}\tilde{E}_{2r}^{(0)}(z) = E_{2r-1}^{(1)}(z)$, ${}^{(1)}\tilde{E}_{2r}^{(1)}(z) = E_{2r+1}^{(0)}(z)$

($r \geq J$), i.e. in view of relationships (), ${}^{(1)}\tilde{E}_r^{(0)}(z) = E_{r-1}^{(1)}(z)$, ${}^{(1)}\tilde{E}_r^{(1)}(z) = E_{r+1}^{(0)}(z)$ ($r \geq J$).

The array of ~~numbers~~ ^{functions} $\{ {}^{(1)}\tilde{E}_r^{(m)}(z) \}$ may be transposed about the leading diagonal $\{ {}^{(1)}\tilde{E}_r^{(0)}(z) \}$, so that diagonals remain diagonals and columns become rows in the transposed array. It follows from the results of the preceding paragraph that this transposed array may be joined to the array of functions $\{ E_r^{(m)}(z) \}$ in such a way that if the functions $\{ {}^{(1)}\tilde{E}_r^{(0)}(z) \}$ and $\{ {}^{(1)}\tilde{E}_r^{(1)}(z) \}$ are placed over the functions $\{ E_{r-1}^{(1)}(z) \}$ and $\{ E_{r+1}^{(0)}(z) \}$ respectively,

the pairs of superposed ~~matrix~~ functions are identical. In this way we obtain an infinite square array of functions: furthermore, subsuming the transposed array of functions $\{ {}^{(1)}\tilde{E}_r^{(m)}(z) \}$ within the extended array of functions $\{ E_r^{(m)}(z) \}$, we have, in particular,

$$E_{2r}^{(m)}(z) = P_{r, m+r-1}(z) \quad (r=1, u=J-r).$$

The relationships of the ε -algorithm concern four numbers occurring at the ~~vert~~ vertices of a lozenge in the array of functions $\{ E_r^{(m)}(z) \}$; these relationships express the fact that the product of the differences of the vertically and horizontally opposed functions are equal to the unit rational function unity. If this array is transposed about a diagonal, ~~the~~ these two pairs of functions are interchanged but the relationship between them is unaltered. Thus the transposed array of functions $\{ {}^{(1)}\tilde{E}_r^{(m)}(z) \}$ referred to in the preceding paragraph can be constructed by extending the determination of the array of functions $\{ E_r^{(m)}(z) \}$ so as to concern all members of the joint square array. It is, of course, necessary to adjoin additional boundary values. These

are obtained from the conditions $(1) \sum_{-1}^{\infty} E_{-1}^{(mn)}(z) = 0 \ (m \equiv J)_4$ or, in the notation of the extended array of functions $\{E_r^{(m)}(z)\}$, $E_{2m}^{(-m)}(z) = 0 \ (m \equiv J)$. (For the sake of completeness we remark that it is unnecessary to determine the coefficients and partial sums of the inverse series: the functions $\{\sum_{10}^{m-1} f_p z^p\}^{-1} = P_{m-1,0}(z) = E_{2m}^{(-m)}(z) \ (m \equiv J_1)$ are ^{determined} ~~constructed~~ automatically during the construction of the extended array of functions $\{E_r^{(m)}(z)\}$.)

If at any stage in the above manipulations, it proves impossible to determine one of the functions $\{E_r^{(m)}(z)\}$, it ~~must~~ then follow, as in the proof of Theorem ..., that either the series $\sum f_p z^p$ or its inverse is not semi-normal, and hence that $\sum f_p z^p$ is not a normal series.

Theorem . ~~Let~~ Assuming that the Padé quotients, derived from a prescribed formal power series and occurring in the following formulae exist, we have

$$\{P_{i+1,j}(z) - P_{i,j}(z)\}^{-1} + \{P_{i+1,j}(z) - P_{i,j}(z)\}^{-1} =$$

$$\{P_{i,j-1}(z) - P_{i,j}(z)\}^{-1} + \{P_{i,j+1}(z) - P_{i,j}(z)\}^{-1} \quad \leftarrow$$

and

$$\{P_{i-1,j}^{-1}(z) - P_{i,j}^{-1}(z)\}^{-1} + \{P_{i+1,j}^{-1}(z) - P_{i,j}^{-1}(z)\}^{-1} =$$

$$\{P_{i,j-1}^{-1}(z) - P_{i,j}^{-1}(z)\}^{-1} + \{P_{i,j+1}^{-1}(z) - P_{i,j}^{-1}(z)\}^{-1}$$

where $i, j \in \mathbb{J}$, in both cases.

Proof. For conciseness in exposition, we have exhibited the ε -algorithm as a device for constructing ^{the} Padé quotients derived from a normal series

dealt only with the

cases in which the q -d algorithm of Theorem... and the equivalent orthogonalisation processes, and the recursive algorithms of Theorems

... - ... were applied to series which were either semi-normal or for which all algorithms terminated at the same point in a consistent manner; we ^{have} also exhibited the ε -algorithm as a device for constructing the Padé quotients ^{derived} from a normal series, i.e. in the case in which all such quotients exist. Nevertheless, it is clear that if the Padé quotients corresponding to the functions $E_{2r+2}^{(m)}(z)$, $E_{2r}^{(m+1)}(z)$, $E_{2r}^{(m+2)}(z)$ exist, so that those corresponding to $E_{2r}^{(m)}(z)$, $E_{2r+2}^{(m+1)}(z)$ also exist ($r \in J_1, m \in J_r$) then, at the cost of complicating the exposition, relationships of the form

$$\begin{array}{l}
 E_{2r}^{(m+1)}(z) = \\
 E_{2r+1}^{(m)}(z) = \\
 E_{2r+1}^{(m+1)}(z) = \\
 E_{2r+2}^{(m)}(z) = 1
 \end{array}
 \left\{
 \begin{array}{l}
 \{E_{2r+2}^{(m+2)}(z) - E_{2r}^{(m+1)}(z)\}^{-1} + E_{2r+1}^{(m+2)}(z) - E_{2r+1}^{(m+1)}(z) = 0 \\
 E_{2r+1}^{(m+1)}(z) - E_{2r+1}^{(m+2)}(z) = \{E_{2r}^{(m+2)}(z) - E_{2r}^{(m+1)}(z)\}^{-1} \\
 E_{2r+1}^{(m+1)}(z) - E_{2r+1}^{(m)}(z) = \{E_{2r}^{(m)}(z) - E_{2r}^{(m+1)}(z)\}^{-1} \\
 \{E_{2r+2}^{(m)}(z) - E_{2r}^{(m+1)}(z)\}^{-1} + E_{2r+1}^{(m)}(z) - E_{2r+1}^{(m+1)}(z) = 0
 \end{array}
 \right.$$

by an extension of the methods used in the proofs of the preceding theorems, may ~~also~~ be shown to hold, the functions $E_{2r+1}^{(m+1)}(z)$, $E_{2r}^{(m+2)}(z)$, $\{E_r^{(m)}(z)\}$ with

odd suffix in these formulae being well defined; the existence or otherwise of the Padé quotients corresponding to junctions $E_{2r}^{(m)}(z)$ with other values of m and higher values of r does not affect the validity of these relationships. Adding these four formulae together, we obtain a recursion involving junctions $\{E_r^{(m)}(z)\}$ with even suffix alone. Setting $i=r, j=m+r$ and using formula (), we obtain ~~the first result of the theorem~~ relationship (). ~~Recall~~

Relationship () may be obtained by rearranging relationship () or, independently, by considering the inverse of the series from which the quotients $\{P_{i,j}(z)\}$ in question are derived.

~~If the series in question is normal, the recursion of the above theorem~~ ~~the remark that~~ may be used, ~~to construct~~ in conjunction with ~~appropriate~~ ~~initial values~~, to construct the entire array of Padé quotients ~~derived from a normal series~~. It may also be shown by methods similar to those used above, that,

~~the~~ assuming the polynomials in question to exist,

$$\{E_{2r+3}^{(m)}(z) - E_{2r+1}^{(m)}(z)\}^{-1} + \{E_{2r-1}^{(m+2)}(z) - E_{2r+1}^{(m)}(z)\}^{-1} = \{E_{2r+1}^{(m+2)}(z) - E_{2r+1}^{(m)}(z)\}^{-1} + \{E_{2r+1}^{(m)}(z) - E_{2r+1}^{(m)}(z)\}^{-1};$$

furthermore, a similar relationship with ~~each~~ ~~each~~ polynomial $E_r^{(m)}$ replaced by its inverse also holds.

Vector continued fractions

We now consider the continued fractions derived from formal power series whose coefficients are vectors with finite complex number components, and whose argument functions are a scalar.

Arithmetic operations upon vectors

^{The sets}
Definition. If the vectors $\underline{z}, \underline{z}'$ are defined by

$\underline{z} = (z_1, z_2, \dots, z_p)$ ($z_j = x_j + iy_j$, $j \in J_1^p$), $\underline{z}' = (z'_1, z'_2, \dots, z'_p)$ ($z'_j = x'_j + iy'_j$, $j \in J_1^p$),
are vectors with complex where the $\{z_j\}, \{z'_j\}$ are finite complex numbers, are vectors with complex
then $\underline{z} \pm \underline{z}' = (z_1 \pm z'_1, z_2 \pm z'_2, \dots, z_p \pm z'_p)$ and if $c = c_1 + ic_2$ is a complex
number $c\underline{z} = (cz_1, cz_2, \dots, cz_p)$. ~~in the sense that \underline{z} is at least one of the components of \underline{z} is~~

~~defined by the formula~~ $\underline{z}^{-1} = \left\{ \sum_{j=1}^p (x_j^2 + y_j^2) \right\}^{-1} \underline{\bar{z}}$
if \underline{z} is non-zero, ~~that is at least one of the components of \underline{z} is~~ \underline{z}^{-1} is
non-zero, and $\underline{z} = x_j + iy_j$, ($j \in J_1^p$), ~~the~~ inverse of \underline{z} , ~~is~~ \underline{z}^{-1} is defined by the formula

$\underline{\bar{z}} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_p)$ and
where the bar denotes the complex conjugate.

We remark that if \underline{z} is ~~a $1 \times p$ matrix~~ treated as a $1 \times p$ matrix,
 $\underline{z}^{-1} = (\underline{z} \underline{z}^*)^{-1} \underline{\bar{z}}$ where ~~the~~ the asterisk denotes the complex
conjugate transpose.

The generalised inverse A^+ of the $p \times q$ ($p \times q \in J_1$) matrix
 A is uniquely determined L^{-1} by the conditions $AA^+A = A$,
 $A^+AA^+ = A^+$, $(AA^+)^* = AA^+$, $(A^+A)^* = A^+A$. In this sense \underline{z}^{-1} is

the transpose of \vec{z}^T .

p. 27
f(z) = f^T

Arithmetic operations upon

Formal power series with vector valued coefficients

Definition. The unbarred sets

$$\vec{f}(z) = \sum \vec{f}_p z^p \quad \vec{f}'(z) = \sum \vec{f}'_p z^p$$

where the $\{\vec{f}_p\}, \{\vec{f}'_p\}$ are vectors, with complex number are formal power series with vector valued coefficients; Furthermore $\vec{f}(z) \pm \vec{f}'(z) = \sum (\vec{f}_p \pm \vec{f}'_p) z^p$

and if ξ is a complex number $\xi \vec{f}(z) = \sum \xi \vec{f}_p z^p$.

If \vec{f}_0 is non-zero, the inverse of $\vec{f}(z)$ is defined by treating the coefficients of the latter as $1 \times p$ matrices, and setting $z_0 = 1/(\vec{f}_0 \vec{f}_0^*)$, $\vec{f}_0 = z_0 \vec{f}_0$

$$\vec{f}(z)^{-1} = \sum \vec{f}_p z^p$$

$$\vec{b}_r = \sum_{j=0}^{r-1} (\vec{f}_j \vec{f}_{r-j}^*) z_0^r, \quad \vec{z}_r = -\sum_{j=0}^{r-1} \vec{z}_j \vec{b}_{r-j}, \quad \vec{f}_r = \sum_{j=0}^{r-1} \vec{z}_j \vec{f}_{r-j} \quad (r=1, 2, \dots)$$

We remark that the numbers $\{b_r\}, \{z_r\}$ recurring in the above definition of an inverse are real numbers

The generalised inverse of the formal power series $\vec{f}(z)$ whose coefficients $\{\vec{f}_p\}$ are treated as $1 \times p$ matrices is uniquely determined by the four equations (treated as

relationships ^{for between} ~~among~~ formal power series) $f(z) \hat{f}(z) f(z) = f(z), \hat{f}(z) f(z) \hat{f}(z) = \hat{f}(z), f(z) \hat{f}(z) = \sum_1 G_p z^p$ ($G_p^* = G_p, p = J$), $\hat{f}(z) f(z) = \sum_1 H_p z^p$ ($H_p^* = H_p, p = J$). In this sense $\hat{f}(z)^{-1}$ is the series whose coefficients are those of $f(z)^+$ transposed.

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An isomorphism and its uses

A connection between processes involving ~~the~~ ~~operational~~ arithmetic operations upon vectors and formal power series with vector valued coefficients ^{on the one hand} and corresponding processes involving these operations upon elements of a ring and formal power series with coefficients over a ring on the other may be established by the use of an appropriate vector-matrix isomorphism.

Definition . Denote the $2^r \times 2^r$ zero matrix by $O^{(r)}$ and the $2^r \times 2^r$ matrix with components i ($= \sqrt{-1}$) along the principal backward diagonal left and zeros elsewhere by $\tilde{I}^{(r)}$ ($r = J_1$); define ^{recursively} the set of r matrices $\Gamma_p^{(r)}$ ($p = J_1^r$), each of dimension $2^r \times 2^r$, recurs as follows:

$$\Gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

Vector valued rational functions

Definition . The vector $\underset{\sim}{f}(z)$ whose τ^{th} component is the rational function

$$f_{\tau}(z) = \left\{ \sum_0^{\delta_{\tau}} a_{\tau, \nu} z^{\nu} \right\}^{-1} \left\{ \sum_0^{\eta_{\tau}} n_{\tau, \nu} z^{\nu} \right\} \quad (\tau = \mathcal{J}_1^P)$$

where the ~~$\{ \delta_{\tau} \} \{ \eta_{\tau} \}$~~ $\delta_{\tau}, \eta_{\tau} \in \mathcal{J}$ ($\tau = \mathcal{J}_1^P$) and the $\{ a_{\tau, \nu} \} \{ n_{\tau, \nu} \}$ are complex numbers, is called a vector valued rational function. Addition and subtraction of ~~vector values such~~ vector valued rational functions is defined in terms of ~~the~~ ^{these} corresponding ~~the~~ operations upon corresponding components. The inverse $\underset{\sim}{f}(z)^{-1}$ of $\underset{\sim}{f}(z)$ (assumed not identically equal to zero) is the vector whose τ^{th} component is the rational function

$$\hat{f}_{\tau}(z) = \left\{ \sum_0^{\delta_{\tau}} a_{\tau, \nu} z^{\nu} \right\}^{-1} \left\{ \sum_0^{\eta_{\tau}} \bar{n}_{\tau, \nu} z^{\nu} \right\} \quad (\tau = \mathcal{J}_1^P)$$

where

$$D(z) = \sum_1^P f_{\nu}(z) \hat{f}_{\nu}(z).$$

Definition . If in the notation of Definition ..., $f_{\tau}(z) = \sum f_{\tau, \nu} z^{\nu}$ ($\tau = \mathcal{J}_1^P$) for sufficiently small values of z , ~~the vector value~~ and ~~f_{ν} in the~~ $f_{\nu} = (f_{1, \nu}, f_{2, \nu}, \dots, f_{\mathcal{J}, \nu})$ ($\nu = \mathcal{J}$), the vector valued function $\underset{\sim}{f}(z)$ is said to generate the series $f(z) = \sum f_{\nu} z^{\nu}$; we write $\underset{\sim}{f}(z) \approx f(z)$

Theorem . If, in the sense of Definition ..., $f_1(z) \sim f_1'(z)$, $f_2(z) \sim f_2'(z)$
 then $f_1(z) \pm f_2(z) \sim f_1'(z) \pm f_2'(z)$; furthermore, in the senses of Definition
 .. and ..., $f(z)^{-1} \sim f'(z)^{-1}$.

Proof. The first part of result of the theorem is obvious enough.

With regard to the second part, we remark that, by the definition

~~that if $f(z)^{-1} \sim \sum_{\nu=0}^{\infty} \hat{f}_{\nu} z^{\nu}$, then, by definition~~

~~$\sum_{\nu=0}^{\infty} \hat{f}_{\nu} z^{\nu} \sim D(z)^{-1} \sum_{\nu=0}^{\infty} \bar{f}_{\nu} z^{\nu}$~~

using the

notation of Definition ..., if $D(z)^{-1} = \sum_{\nu=0}^{\infty} \bar{f}_{\nu} z^{\nu}$ for sufficiently
 small values of z , and $f(z)^{-1} \sim \sum_{\nu=0}^{\infty} \hat{f}_{\nu} z^{\nu}$, then

$$\hat{f}_r = \sum_{\nu=0}^r \bar{f}_{\nu} \bar{f}_{r-\nu} \quad (r=0, 1, \dots)$$

As is easily verified $D(z) = \sum_{\nu=0}^{\infty} \hat{b}_{\nu} z^{\nu}$ where $\hat{b}_r = \sum_{\nu=0}^r \bar{f}_{\nu} \bar{f}_{r-\nu}^*$
 ($r=0, 1, \dots$). The numbers $\{\bar{f}_{\nu}\}$ and $\{\hat{b}_{\nu}\}$ are related by means of

the equations $\sum_{\nu=0}^r \bar{f}_{\nu} \hat{b}_{r-\nu} = 1$, $\sum_{\nu=0}^r \bar{f}_{\nu} \hat{b}_{r-\nu} = 1$, etc. In
 conclusion,

if we set $\bar{b}_r = \hat{b}_r \bar{f}_0$ ($r=0, 1, \dots$), the numbers \bar{b}_r are

defined by the first of formulae (), the numbers $\{\bar{f}_r\}$
 can be extracted by means of the second of formulae (),

and, with these numbers available, the coefficients $\{\hat{f}_n\}$ are given by the third of formulae ().

of the series generated by the function $f(z)^{-1}$

~~subscript~~

$$\Gamma_{\nu}^{(\nu)} = I \quad (\nu = J_1^r), \quad \Gamma_{\nu}^{(\nu)} \Gamma_{\nu'}^{(\nu)} + \Gamma_{\nu'}^{(\nu)} \Gamma_{\nu}^{(\nu)} = O^{\mathbb{K}} \quad (\nu = J_1^{r-1}, \nu' = J_{p+1}^r)$$

If \tilde{Z} is the matrix isomorphic to \tilde{z} , we have $\tilde{Z}\tilde{Z} = \sum_1^p (x_{\nu}^2 + y_{\nu}^2) I$; hence, if \tilde{z} is non-zero,

$$\sqrt{\tilde{Z}^{-1}} = \left\{ \sum_1^p (x_{\nu}^2 + y_{\nu}^2) \right\}^{-1/2} \tilde{Z} \text{ and } \tilde{Z}^{-1} \leftrightarrow \tilde{Z}^{-1}.$$

As is easily verified, if \tilde{z} and \tilde{z}' are treated as row vectors

$$\tilde{z}\tilde{z}' + \tilde{z}'\tilde{z} = \sum_1^p (\tilde{z}\tilde{z}'^* + \tilde{z}'\tilde{z}^*) I.$$

Hence

$$\tilde{z}\tilde{z}'\tilde{z} = \tilde{z}(\tilde{z}'\tilde{z} + \tilde{z}\tilde{z}') - \tilde{z}\tilde{z}'\tilde{z}' = \left(\sum_{\nu} \tilde{z}_{\nu} \tilde{z}'_{\nu}^* + \sum_{\nu} \tilde{z}'_{\nu} \tilde{z}_{\nu}^* \right) \tilde{z} - (\tilde{z}\tilde{z}^*) \tilde{z}'$$

where \tilde{z}'^T is the transpose of \tilde{z}' .

We now show that the above isomorphism is preserved during the inversion of formal power series.

$\sim \text{ for comp. envs, } \wedge \text{ for inverse}$

Theorem... Let $\hat{f}(z) = \sum_0^{\infty} \hat{f}_{\nu} z^{\nu}$ be a formal power series with vector valued coefficients, with \hat{f}_0 non-zero, and let $\hat{f}(z)^{-1} = \sum_0^{\infty} \hat{f}_{\nu}^{-1} z^{\nu}$ be its inverse in the sense of Definition... . Let $\hat{f}_{\nu} \leftrightarrow F_{\nu} \quad (\nu = J)$ and $\sum_0^{\infty} \hat{F}_{\nu} z^{\nu}$ be the inverse of the series $\sum_0^{\infty} F_{\nu} z^{\nu}$ in the sense of Definition... . Then $\hat{f}_{\nu} \leftrightarrow F_{\nu} \quad (\nu = J)$.

Proof. It is clear to begin with that $\hat{f}_0 \leftrightarrow F_0$.

~~Eliminating the numbers $\{\hat{f}_{\nu}\}$ from the first and second of formulas~~

~~we find that and setting $\Theta_{i,j} = (\hat{f}_i \hat{f}_j^*) / (\hat{f}_0 \hat{f}_0^*) \quad (i, j = J)$, we~~

find that

~~$$\sum_0^p \sum_0^{p-\nu} z_{\nu}^i \Theta_{\nu, p-\nu} = 0 \quad (i = J)$$~~

we find that

$$\sum_{\nu=0}^r \sum_{\nu'=0}^{r-\nu} f_{\nu} f_{r-\nu-\nu'}^* = 0 \quad (r=J_1)$$

As a preliminary to proving that this relationship also holds for the remaining coefficients, we remark that

$$\sum_{\nu=0}^r \sum_{\nu'=0}^{r-\nu} F_{\nu} \tilde{F}_{r-\nu-\nu'} = \sum_{\nu=0}^r f_{\nu} f_{r-\nu}^* \quad (r=J)$$

and that if the numbers $\{b_{\nu}\}$ are eliminated from the first and second of formulae (), then

$$\sum_{\nu=0}^r z_{\nu} \sum_{\nu'=0}^{r-\nu} f_{\nu} f_{r-\nu-\nu'}^* = 0 \quad (r=J_1)$$

Having ~~constructed~~ ^{determined} the vectors \hat{f}_{ν} ($\nu=J_2$), we ~~do~~ construct the matrices \hat{F}_{ν} ($\nu=J_2$) isomorphic to them; ~~we have, of course.~~ From the third of formulae (), we ~~then~~ have

$$\hat{F}_r = \sum_{\nu=0}^r z_{\nu} \tilde{F}_{r-\nu} \quad (r=J)$$

Using ~~the~~ formulae ()-(), we then find that

$$\begin{aligned} \sum_{\nu=0}^r F_{r-\nu} \hat{F}_{\nu} &= \sum_{\nu=0}^r F_{r-\nu} \sum_{\nu'=0}^{\nu} z_{\nu'} \tilde{F}_{\nu-\nu'} \\ &= \sum_{\nu=0}^r z_{\nu} \sum_{\nu'=0}^{r-\nu} \cancel{F_{\nu} f_{r-\nu-\nu'}} \tilde{F}_{r-\nu-\nu'} \\ &= \left\{ \sum_{\nu=0}^r z_{\nu} \sum_{\nu'=0}^{r-\nu} f_{\nu} f_{r-\nu-\nu'}^* \right\} I = 0 \quad (r=J_1) \end{aligned}$$

Hence the $\{\hat{F}_{\nu}\}$ are identified as the coefficients of the series inverse to $\sum_{\nu=0}^{\infty} F_{\nu} z^{\nu}$.

establish a connection between

We now show how a rational function with matrix valued ^{finite dimensional} coefficients ~~vector~~ valued rational functions and rational functions with matrix valued coefficients.

Definition.... Let the ~~vector~~ vector valued ~~for~~ rational function $f(z)$ generate the series $\sum_{j=0}^{\infty} f_j z^j$, ~~and~~ ^{let} the rational function with matrix valued coefficients (in the sense of Definition...) $f(z)$ generate the series $\sum_{j=0}^{\infty} f_j z^j$, ~~we say that these two functions correspond to each other and~~ and let $f_j \leftrightarrow f_j$ ($j=0, 1, \dots$). We say that these two rational functions correspond, and write $f(z) \leftrightarrow f(z)$.

Theorem . Let $f(z)$ be a ^{finite dimensional ($p < \infty$)} vector valued rational function. Express its components as fractions with a common denominator, which by further rationalisation if necessary, may be taken to have real coefficients only, so that $f(z)$ is the vector whose α th component is

$$f_{\alpha}(z) = \left\{ \sum_{j=0}^{\infty} d_j z^j \right\}^{-1} \left\{ \sum_{j=0}^{\infty} n_{\alpha, j} z^j \right\} \quad (\alpha = 1, \dots, p)$$

Let $n = \max(\eta_\tau) (\tau \in J_1^p)$, and set $n_{\tau, \nu} = 0$ ($\nu \in J_{\eta_\tau+1}^p$), and with this convention, let n_ν be the vector ~~($n_{1, \nu}, n_{2, \nu}, \dots, n_{p, \nu}$)~~ ^{matrix isomorphic to} ~~($n_{1, \nu}, n_{2, \nu}, \dots, n_{p, \nu}$)~~ ($\nu \in J_1^p$). ~~Let $n_\nu \leftrightarrow n_\nu$ ($\nu \in J_1^p$)~~

Then

$$f(z) \leftrightarrow \left\{ \sum_0^r (d_\nu I) z^\nu \right\}^{-1} \sum_0^r n_\nu z^\nu.$$

Proof. Setting $n_\nu = (n_{1, \nu}, n_{2, \nu}, \dots, n_{p, \nu})$ ($\nu \in J_1^p$), the coefficients of the series $\sum_0^r f_\nu z^\nu$ generated by $f(z)$ may be obtained recursively by use of the formulae

$$d_0^{-1} \{ n_r - \sum_1^r d_\nu f_{r-\nu} \} \quad (r \in J_0^p)$$

$$f_r = -d_0^{-1} \sum_1^r d_\nu f_{r-\nu} \quad (r \in J_{\eta_1+1}^p) \quad (d_\nu = 0, \nu \in J_{r+1}^p)$$

(It has been assumed that the series $\sum_0^r f_\nu z^\nu$ can be constructed by expanding each component of $f(z)$ and regrouping coefficients of corresponding powers of z in the form of vectors; use of the above scheme is equivalent to this process.)

The coefficients of the series $\sum F_r z^r$ generated by the rational function upon the right hand side of relationship () may be obtained by use of the recursion

$$f_r = \begin{cases} d_0^{-1} \left\{ N_r - \sum_{i=1}^r d_i f_{r-i} \right\} & (r = 0, 1, \dots, \gamma) \\ -d_0^{-1} \sum_{i=1}^r d_i f_{r-i} & (r = \gamma+1, \dots) \end{cases} \quad (d_0 = 0, d_i \in \mathbb{R}^n)$$

Formulae () involve addition and subtraction of vectors, and multiplication by a real scalar. From Theorem ..., the isomorphism being used is preserved during these operations, and hence $f_D \leftrightarrow f_D$ ($D \equiv \mathbb{R}$).

(It should be pointed out that we have not characterised the class of rational functions with matrix valued coefficients whose series expansions have coefficients all isomorphic to vectors.)

Euclid's algorithm for formal power series with vector valued coefficients
~~arithmetic operations upon formal power series with vector valued~~
~~coefficients, and~~

The arithmetic operations needed for the ^{application} ~~implementation~~ of Euclid's algorithm to formal power series with vector valued power series have ~~been~~ now been defined; ~~and~~ the isomorphism relating to these operations can be used to derive an algebraic result for this mode of application of the algorithm. We give

~~We say that~~
 Definition. Euclid's algorithm with respect to the unbounded sequence τ of finite positive integers τ_r ($r \in \mathbb{J}$) applied to the formal power series $f(z) = \sum_{\mathbb{N}} f_p z^p$ with vector valued coefficients ~~is~~ the process of determining the sequence of polynomials with vector valued coefficients

$$\tilde{f}_r(z) = \sum_{\mathbb{N}} \tilde{f}_{r,p} z^p$$

and the sequence of power series $f^{(r)}(z) = \sum_{\mathbb{N}} f_p^{(r)} z^p$ according to the scheme

$$\sum_{\mathbb{N}} \tilde{f}_{r,p} z^p = f^{(r-1)}(z)^{-1}$$

$$\tilde{f}_{r,p} = f_p^{(r)} \quad (p \in \mathbb{J}_0^{\tau_r-1}) \quad f_p^{(r)} = f_{\tau_r+p}^{(r)} \quad (p \in \mathbb{J})$$

for $r \in \mathbb{J}$. The process is said to terminate if, in the above ~~series~~ ~~for all series~~ ~~that are produced~~, ~~we say that~~ ~~if, for some $r \in \mathbb{J}$~~

the series $f^{(r)}(z)$

; the process is said to terminate if, for some $r \in J$, the series $f^{(r)}(z)$ has identically zero coefficients all coefficients of the series $f^{(r)}(z)$ are zero;

if the process can be continued indefinitely, in the sense that $f^{(r)} \neq 0$ for all series $f^{(r)}(z)$ or terminates, in the sense that $f^{(r)} = 0$ for all series $f^{(r)}(z)$ series that are produced, then we say that Euclid's algorithm with respect to the sequence T can be applied to the series $f(z)$ and that this series is T -regular.

~~THEOREM~~
 Theorem. Let $f(z) = \sum_{p \in J} f_p z^p$ be a formal power series with vector valued coefficients, and $f_p \leftrightarrow F_p$ ($p \in J$) in the sense of the isomorphism of Definition ... $f(z)$ is T -regular in the sense of Definition ... if and only if the series $\sum_{p \in J} F_p z^p$ with matrix valued coefficients is T -regular in the sense of Definition ...; furthermore, in the notation of formulae () and (), $f_p \leftrightarrow b_p$ ($p \in J_0^{T-1}$) for all polynomials $b_r(z)$ and $b_s(z)$ that are produced. ~~If the vectors $\{f_p\}$ satisfy a recursion of the form~~
~~If Euclid's algorithm may be applied to the series generated by a vector~~
 ~~$\sum_{p \in J} f_p z^p = 0$~~ (11 = J)
 valued rational function, then the
 where the $\{c_p\}$ are real numbers, then Euclid's algorithm applied to
 this series terminates.

Proof. The vector-matrix isomorphism of Definition... is preserved during all the operations described in Definition E... and ...; in particular,

~~the matrices produced in the first p during the implementation~~
 in the notations of these two definitions, $\tilde{f}_\nu^{(r)}(z) \leftrightarrow \tilde{F}_\nu^{(r)}(z)$ in the sense that $\tilde{f}_\nu^{(r)} \leftrightarrow \tilde{F}_\nu^{(r)}$ ($\nu = J$), and furthermore $F_0^{(r-1)}$ is in for all series that

are produced. Furthermore $F_0^{(r-1)}$ is invertible if and only if $f_0^{(r-1)}$ is at each stage, ~~the~~ or breaks down together.

ignoring (for all matrices and corresponding vectors, the implementation) of Euclid's algorithm either may be continued, ~~the~~

If the vectors $\{f_\nu\}$ satisfy the recursion (), $f(z)$ is the series expansion of the rational function () with $d_\nu = d_{r-\nu}$ ($\nu = J_0'$)

$$s(z) = \left\{ \sum_0^r d_\nu z^\nu \right\}^{-1} \sum_0^{r-1} n_\nu z^\nu$$

where $d_\nu = d_{r-\nu}$ ($\nu = J_0'$) and

$$\sum_0^r d_\nu f_{r-\nu} = n_\nu$$

Lastly $f_\nu \leftrightarrow F_\nu^{(r)}$ ($\nu = J_0'$), i.e. $d_{r,\nu} \leftrightarrow d_{r,\nu}$ ($\nu = J_0'^{r-1}$)

(if $f_\nu = (f_{\nu,0}, f_{\nu,1}, \dots, f_{\nu,p})$ $f_\nu = (f_{\nu,1}, f_{\nu,2}, \dots, f_{\nu,p})$ ($\nu = J$) and

$n_\nu = (n_{\nu,1}, n_{\nu,2}, \dots, n_{\nu,p})$ then $\sum_0^r f_{\nu,r} z^\nu$ is the series expansion

of the rational function. The corresponding matrix series is

$\sum_0^r F_\nu z^\nu$ is the series expansion of the rational function $\left\{ \sum_0^r D_\nu z^\nu \right\}^{-1} \left\{ \sum_0^{r-1} H_\nu z^\nu \right\}$ where $D_\nu \leftrightarrow D_\nu = d_\nu I$ ($\nu = J_0'$) $N_\nu \leftrightarrow N_\nu$ ($\nu = J_0'^{r-1}$) and

Theorem . If Euclid's algorithm with respect to the sequence T may be applied to the series generated by ~~the~~ ^{the} finite-dimensional vector valued rational function $f(z)$, then the algorithm terminates. ^{if/then app.}
 Furthermore, ^(in the notation of Definition ...) $\{d_r(z)\}_{r=1}^{\infty}$ is the sequence of polynomials produced by the algorithm, then the function $f(z)$ may be reconstructed by implementing the recursion

~~$$d_{r+1}(z) = d_r(z), \quad d_{r+1}(z) = \frac{d_r(z)}{d_{r+1}(z) + z}$$~~

$$d_1(z) = b_{r_1}(z), \quad d_{r+1}(z) = \frac{d_{r-1}(z)}{d_{r+1}(z) + z^{r_1-r} d_r(z)^{-1}} \quad (r = \underline{J}_1^{r-1})$$

when $f(z) = d_{r_1}(z)^{-1}$, the $\{d_r(z)\}$ being ~~treated as~~ vector valued rational functions.

~~Proof.~~ The components of a finite dimensional vector valued rational function may be expressed as fractions with a common denominator, ~~where the coefficients of~~
~~If this common denominator polynomial has complex roots~~
~~complex numbers, multiplication of~~
 furthermore, by further rationalisation if necessary, this denominator may be taken to be a polynomial with real coefficients.
 Hence, the function $f(z)$ in question ~~is~~ is a vector whose r^{th}

Proof. It follows from Theorem ... that if Euclid's algorithm with respect to the sequence T can be applied to the series $f(z)$ generated by $\underline{f}(z)$, it can also be applied to the series $F(z)$ generated by the rational function $\underline{f}(z)$ with matrix valued coefficients isomorphic to $\underline{f}(z)$, and that if the for the ~~two~~ polynomials $\{b_r(z)\}, \{b_r(z)\}$ so produced $b_r(z) \leftrightarrow b_r(z)$ ($r = \underline{J}_2'$) in the sense of Definition The rational function $f(z)$ may be reconstructed by means of a recursion similar to () involving the polynomials $b_r(z)$. Hence the function constructed by use of a recursion () ~~which involves~~ ^{three dimensional} The isomorphism between vector valued ~~and~~ rational matrix valued functions and rational functions with matrix valued coefficients is preserved during the operations ~~used~~ ^{occurring} in recursion (). Hence the function constructed by use of this recursion is isomorphic to $f(z)$ and is, in consequence, $\underline{f}(z)$.

The vector q -d algorithm

Equipped with the isomorphisms of Definitions ... - ..., we may proceed in a quite mechanical manner to develop ^{the} theory of the continued fractions ~~as~~ and the Padé table derived from ^{a formal} power series with vector valued coefficients. Certain of the numbers ~~occurring~~ occurring in this theory are not isomorphic to vectors; for example, the matrices $\{A_r^{(m)}\}$ of the q -d algorithm produced from matrices $\{f_m\}$ isomorphic to vectors $\{f_m\}$ are not themselves isomorphic to vectors. Again, the polynomials $\{d_r^{(m)}(z)\}$, $\{n_r^{(m)}(z)\}$ ~~produced~~ of Theorems ... - ... produced from a power series $\sum_i f_i z^i$ whose coefficients are isomorphic to vectors are not themselves isomorphic to vector valued rational functions. Against this, the convergents of the continued fractions considered, and the Padé quotients defined with their help, are vector valued rational functions. We do not propose to give ^{an independent exposition} ~~a complete~~ account of the theory of vector valued continued fractions; ~~since~~ such an account would merely be a recapitulation, in a notation suited to vectors, of results already dealt with at adequate length in previous sections.

There are, however, some points of independent interest, and at

at this juncture we derive a vector form of the q-d algorithm which, when applied to a series with vector valued coefficients, involves vectors alone.

Definition.... Let $f(z) = \sum_{n=J} f_n z^n$ be a formal power series with vector valued coefficients. If Euclid's algorithm with respect to the sequence $\{1, 1, \dots\}$ can be applied ^(without terminating) to each of the series $\sum_{n=J} f_n z^n$ ($n=J$)

then the series $f(z)$ is said to be semi-normal.

~~Theorem. Let the sequence of numbers $\{b_r^{(m)}\}$ (i.e. polynomials of zero degree) produced by application of Euclid's algorithm with respect to the sequence $\{1, 1, \dots\}$ to the series $\sum_{n=J} f_n z^n$ ($n=J$).~~

~~These~~

~~Theorem. Vectors $\{b_r^{(m)}\}$ ($r=J_2, m=J$) may be determined by use~~

Theorem. Define the ternary product $\underline{a} \underline{b} \underline{a}$ involving the two vectors $\underline{a}, \underline{b}$ by means of the formula

$$\underline{a} \underline{b} \underline{a} = (\underline{a} \bar{b}^* + \bar{b} \underline{a}^*) \underline{a} - (\underline{a} \underline{a}^*) \bar{b}$$

Adopting this convention, vectors $b_r^{(m)}$ ($r=J_2, m=J$) may be determined ~~by~~ from the initial values $b_{J_1}^{(m)} = f_m^{-1}$ ($m=J$) by use of the formulae

$$b_{2r+2}^{(m)} = \eta_r^{(m)-1} \epsilon_{r-1}^{(m+1)} b_{2r+1}^{(m+1)} \wedge^{(m+1)-1} \wedge^{(m)-1}$$

$$b_{2r+3}^{(m)} = \eta_r^{(m)} \left\{ \epsilon_{r-1}^{(m+1)} b_{2r+2}^{(m+1)} \wedge^{(m+1)} + \epsilon_{r-2}^{(m+1)} b_{2r}^{(m+1)} \wedge^{(m+1)} - \eta_{r-1}^{(m)-1} b_{2r+1}^{(m)} \wedge^{(m)-1} \right\} \eta_r^{(m)}$$

Where $\epsilon_{-2}^{(m)-1} = 0, \epsilon_{-1}^{(m)} = \eta_{-1}^{(m)} = 1 (m \geq J)$, and we adopt the additional convention that

$$\begin{cases} b_1^{(m)} b_3^{(m)-1} b_5^{(m)} \dots b_{4r+1}^{(m)} = \eta_{2r}^{(m)}, & b_1^{(m)-1} b_3^{(m)} b_5^{(m)-1} \dots b_{4r+3}^{(m)} = \eta_{2r+1}^{(m)} \\ b_2^{(m)} b_4^{(m)-1} b_6^{(m)} \dots b_{4r+2}^{(m)} = \epsilon_{2r}^{(m)}, & b_2^{(m)-1} b_4^{(m)} b_6^{(m)-1} \dots b_{4r+4}^{(m)} = \epsilon_{2r+1}^{(m)} \end{cases} \quad (r, m \in J)$$

the numbers $\{\eta_r^{(m)}\}, \{\epsilon_r^{(m)}\}$ are defined by reversing the order of these products (so that $b_5^{(m)} b_3^{(m)-1} b_1^{(m)} = \eta_2^{(m)}$, and so on), and the numbers $\{\eta_r^{(m)-1}\}, \dots$ are defined by the formulae $b_{4r+1}^{(m)-1} \dots b_5^{(m)-1} b_3^{(m)} b_1^{(m)-1} = \eta_{2r}^{(m)-1} (r, m \in J)$,

if and only if ~~the series~~ $\sum f_r z^r$ is a semi-normal series with vector valued coefficients. These vectors $\{b_r^{(m)}\}$ that can be produced constructed in this way are the vectors (i.e. vector valued polynomials of $b_r^{(m)}(z)$ of zero degree) produced by application of Euclid's algorithm with respect to the sequence $\{1, 1, \dots\}$ to each of the series $\sum f_{m+1} z^m (m \in J)$.

If, in the construction of the vectors $\{b_r^{(m)}\}$, ~~by means of~~ the above recursion terminates systematically in the sense that for some $r' \in J$, ~~$b_{2r'+1}^{(m)} = 0 (m \geq 3)$~~ , then $\sum f_r z^r$ is the ~~lowest~~ order series expansion of a vector valued rational function, having the form (\cdot) with $\nu > \eta$.

Proof. If Euclid's algorithm can be applied in the manner described to the series $\sum_{n=0}^{\infty} f_n(z) z^n$, then the series ~~is among~~ $\sum_{n=0}^{\infty} f_n(z) z^n$, the series isomorphic to it is \mathbb{C} -regular \rightarrow to produce vectors $\tilde{b}_r^{(m)}$ ($r \in \mathcal{J}_A$),

then $\sum_{n=0}^{\infty} f_n(z) z^n$, the series isomorphic to ~~it~~ $\sum_{n=0}^{\infty} f_n(z) z^n$ is \mathbb{C} -regular and $\mathcal{C}\{f_n(z)\} = [I: b_1^{(m)} + ; z: b_2^{(m)} +]$, where $\tilde{b}_r^{(m)} \leftrightarrow b_r^{(m)}$

($r \in \mathcal{J}_1$) for $m \in \mathcal{J}_3$. Setting ~~the~~ $\phi_0^{(m)} = I, \phi_r^{(m)} = b_r^{(m)} b_{r-1}^{(m)} \dots b_1^{(m)}$ ($r \in \mathcal{J}_1, m \in \mathcal{J}$), $a_r^{(m)} = \phi_r^{(m)-1} \phi_{r-2}^{(m)}$ ($r \in \mathcal{J}_2, m \in \mathcal{J}$)

we have ~~$\mathcal{C}\{f_n(z)\} = \text{pre} [f_n: I + ; a_r^{(m)} z: I +]$~~

$$\phi_0^{(m)} = I, \phi_r^{(m)} = b_r^{(m)} b_{r-1}^{(m)} \dots b_1^{(m)} \quad (r \in \mathcal{J}_1, m \in \mathcal{J}), a_r^{(m)} = \phi_r^{(m)-1} \phi_{r-2}^{(m)} \quad (r \in \mathcal{J}_2, m \in \mathcal{J})$$

it follows from Theorem ... that $\mathcal{C}\{f_n(z)\} = \text{pre} [f_n: I + ; a_r^{(m)} z: I +]$ and, from Theorem ... , that the ~~numbers~~ ^{matrices} $\{a_r^{(m)}\}$ satisfy relationships (). Hence, for the matrices $\{\phi_r^{(m)}\}$, we have

$$\left. \begin{aligned} \phi_1^{(m+1)-1} &= \phi_2^{(m)-1} \phi_1^{(m)-1}, \quad \phi_2^{(m+1)-1} = \phi_3^{(m)-1} \phi_1^{(m)} + \phi_2^{(m)-1} \\ \phi_{2r+1}^{(m+1)-1} \phi_{2r-1}^{(m+1)} \phi_{2r}^{(m+1)-1} \phi_{2r-2}^{(m+1)} &= \phi_{2r+2}^{(m)-1} \phi_{2r}^{(m)} \phi_{2r+1}^{(m)-1} \phi_{2r-1}^{(m)} \\ \phi_{2r+2}^{(m+1)-1} \phi_{2r}^{(m+1)} + \phi_{2r+1}^{(m+1)-1} \phi_{2r-1}^{(m+1)} &= \phi_{2r+3}^{(m)-1} \phi_{2r+1}^{(m)} + \phi_{2r+2}^{(m)-1} \phi_{2r}^{(m)} \end{aligned} \right\} (r \in \mathcal{J}_1, m \in \mathcal{J})$$

We now show how these relationships may be rearranged in a form in which ~~matrix-matrix~~ matrix expressions isomorphic to vectors only occur. For convenience in exposition, we assume that $r \in \mathbb{Z}_2$. Relationship () may be written as ~~is equivalent to~~

$$\textcircled{A} \quad \phi_{2r+1}^{(m+1)^{-1}} \delta_{2r}^{(m+1)^{-1}} \phi_{2r-2}^{(m)} = \phi_{2r+2}^{(m)^{-1}} \delta_{2r+1}^{(m)^{-1}} \phi_{2r-1}^{(m)}$$

~~is~~ Taking the inverse of both sides, ~~this relationship may also be written as~~ we derive

~~$$\phi_{2r-1}^{(m)^{-1}} \delta_{2r}^{(m)^{-1}} \phi_{2r+1}^{(m+1)^{-1}} \delta_{2r}^{(m+1)^{-1}} \phi_{2r-2}^{(m)}$$~~

$$\textcircled{A'} \quad \phi_{2r-2}^{(m+1)^{-1}} \delta_{2r}^{(m+1)} \phi_{2r+1}^{(m)} = \phi_{2r-1}^{(m)^{-1}} \delta_{2r+1}^{(m)} \phi_{2r+2}^{(m)}$$

or, equivalently,

$$\textcircled{A''} \quad \phi_{2r-2}^{(m+1)^{-1}} \delta_{2r+1}^{(m+1)} \phi_{2r+1}^{(m)} \delta_{2r}^{(m)} \phi_{2r-1}^{(m)} = \phi_{2r-1}^{(m)^{-1}} \delta_{2r+1}^{(m)} \delta_{2r+2}^{(m)} \delta_{2r+1}^{(m)} \phi_{2r+1}^{(m)}$$

Premultiplying the expressions on each side of relationship () ~~by~~ $\textcircled{A''}$ in which r has been replaced by $r-1$, by the expressions on the corresponding sides of relationship () ~~in which~~ $\textcircled{A''}$ r has been replaced by $r-1$ we derive

$$\phi_{2r-3}^{(m+1)^{-1}} \delta_{2r-2}^{(m+1)^{-1}} \phi_{2r}^{(m+1)} \delta_{2r+1}^{(m+1)} \phi_{2r}^{(m)} \delta_{2r}^{(m)} \phi_{2r-2}^{(m)} =$$

$$\phi_{2r-2}^{(m)^{-1}} \delta_{2r-1}^{(m)^{-1}} \phi_{2r+1}^{(m)} \delta_{2r+2}^{(m)} \phi_{2r+1}^{(m)} \delta_{2r-1}^{(m)} \phi_{2r-2}^{(m)}$$

~~Pre~~ ~~Pre~~

Premultiplying the components of this relationship ~~by~~ in a similar manner by those of relationship () in which r has been replaced

Matrices isomorphic to vectors belong to a well determined class; products of pairs of such matrices ~~also belong~~ ^{belong} to another such class. We now show that the coefficients of certain continued fractions derived from ~~series~~ formal power series whose coefficients are matrices of the first class are matrices of the second class.

Theorem. Let the coefficients $\{B_r\}$ of the ^{terminating or nonterminating} continued fraction $[I; B_r +]$ be matrices isomorphic to ^{vectors} vectors in the sense of Definition..., and let this continued fraction be equivalent to $[A_1; A_2 +]$ in the sense of Theorem...; then ~~the~~ ^{matrix} coefficient A_1 is isomorphic to a vector, and the remaining matrices A_p ($p = 2, 3, \dots$) that are defined are products of pairs of matrices isomorphic to matrices. The partial numerators ~~are~~ $A_{2p-1} A_{2p-2}$ ($p = 2, 3, \dots$) of the even part of the latter continued fraction, and those $A_{2p} A_{2p-1}$ ($p = 1, 2, 3, \dots$) of the odd part of this expansion are also products of pairs of such matrices.

Proof. We have $A_1 = B_1^{-1}$; thus A_1 is isomorphic to a vector. Thereafter

~~set~~ $A_2 = b_1^{-1} b_2^{-1}, A_p = b_1^{-1} b_2^{-1} \dots b_{p-1}^{-1} b_p^{-1} b_{p-2} b_{p-3} \dots b_1$ $p = 3, 4, \dots$

A_2 is clearly as described. If A, B, C are matrices isomorphic to vectors

with A nonzero, then

$A^{-1}BCA = (A^{-1}BA^{-1})(ACA)$ is clearly the product of two such matrices. ~~By~~ By inspection of formula (), it follows that A_p is a nested product of this form, and may therefore be represented as the product of two matrices isomorphic to vectors.

The remaining results of the theorem may be proved by the use of expressions similar to those occurring in relationship ().

~~It follows from the above~~

The results of the above theorem may clearly be extended to continued fractions of the form $[1:B_1+; z:B_2+]$ derived from power series. Hence, if the ~~coeff~~ coefficient sequence $\{f_j\}$ involved is composed of matrices isomorphic to vectors, it follows that the matrices $\{u_p\}, \{v_p\}$ ~~($p \geq J_2$)~~ u_p, v_p ($p \geq J_2$) derived ~~in the~~ ^{during} by means of the orthogonalisation process of Theorem ..., and that the matrices $\{a_r^{(m)}\}$ ($r \geq J_2, m \in J$) constructed during the implementation of the $q-d$ algorithm as described in Theorem ..., are all products of pairs of matrices isomorphic to vectors.

The vector ε -algorithm

We restrict ourselves in this section to giving a theorem which finds application in a later section concerning the transformation of vector sequences. We confine ourselves to giving an algebraic result in the theory of the ε -algorithm applied to vector valued rational functions; this result finds application in a later section concerning the transformation of vector sequences. Theorem. If the ^{use} ~~finite dimensional~~ vectors of the sequence $\{f_n\}$ satisfy a system of relationships of the form

$$\sum_{\nu=0}^r d_{\nu} f_{n+\nu} = 0 \quad (n = J)$$

where the $\{d_{\nu}\}$ are real numbers, then the series $\sum_{\nu=0}^{\infty} f_{\nu} z^{\nu}$ is generated by a vector valued rational function $f(z)$. If ~~the~~ vector valued rational functions ~~$\{z_{\nu}^{(m)}(z)\}$ can be~~ $z_{\nu}^{(m)}(z)$ ($\nu = J_1, m = J$) can be constructed from the initial ~~initial~~ values

$$z_{\nu}^{(m)}(z) = 0 \quad (m = J_1), \quad z_{\nu}^{(m)}(z) = \sum_{\nu=0}^{n-1} f_{\nu} z^{\nu} \quad (m = J)$$

by means of the recursion

$$z_{\nu+1}^{(m)}(z) = z_{\nu-1}^{(m)} + \{z_{\nu}^{(m)}(z) - z_{\nu}^{(m)}(z)\}^{-1}$$

for $\nu = J_0^{-1}, m = J$, then $z_{\nu}^{(m)}(z) = f(z)$ ($m = J$)

Proof. That the series $\sum_{\nu=0}^{\infty} f_{\nu} z^{\nu}$ is generated by the rational function

$$\underline{f}(z) = \left(\sum_{j=0}^{r-1} d_j z^j \right)^{-1} \sum_{j=0}^{r-1} n_j z^j$$

where

$$\sum_{j=0}^r d_j f_{r-j} = n_r \quad (r = j_0^{r-1})$$

if ~~and only if~~ the vectors $\{f_j\}$ ~~is from the~~ satisfy recursion () is already known from theorem. The ^{remaining} further result of the theorem is a simple consequence of the last part of theorem... applied to the series $\sum_{j=0}^r f_j z^j$ isomorphic to $\sum_{j=0}^r f_j z^j$.

The vector valued Padé table

Definition . Let $\sum_{j=0}^{\infty} f_j(z) = J_1^r$ be the vectors of

Definition . Let $\sum_{j=0}^{\infty} f_j(z)$ be a given series with finite dimensional vector coefficients, and let $\hat{r} = r+2$. Let $\hat{m}, \hat{n} \in \mathbb{J}$ and let $\hat{r} = r+2$. If vectors $b_{\nu}^{(m)}$ ($\nu = J_1^{\hat{r}}$) can be determined by setting $f_0^{(m)}(z) = \sum_{j=0}^{\hat{r}-1} f_{j+1} z^j$, $f_{\nu}^{(m)}(z)^{-1} = \sum_{j=0}^{\hat{r}-1} \tilde{f}_{j+1}^{(m)} z^j$, $b_{\nu+1}^{(m)} = f_0$, $b_{\nu+1}^{(m)}(z) = \sum_{j=0}^{\hat{r}-1} \tilde{f}_{j+1}^{(m)} z^j$ ($\nu = J_0^{\hat{r}-1}$) then the Padé quotient of

Definition . Let $\sum_{j=0}^{\infty} f_j(z)$ be a given series with finite dimensional vector coefficients, and let $m, r \in \mathbb{J}$. If linear functions $b_{\nu+1}^{(m)} + b_{\nu}^{(m)} z^2$ ($\nu = J_1^r$) can be determined by setting $f_0^{(m)}(z) = \sum_{j=0}^r f_{j+1} z^j$, $f_{\nu}^{(m)}(z)^{-1} = \sum_{j=0}^r \tilde{f}_{j+1}^{(m)} z^j$, $b_{\nu+1,0}^{(m)} = f_0$, $b_{\nu+1,1}^{(m)} = f_1$, $f_{\nu+1}^{(m)}(z) = \sum_{j=0}^r \tilde{f}_{j+1}^{(m)} z^j$ ($\nu = J_0^{r-1}$) then the Padé quotient of order $m, r-1$ derived from the series $\sum_{j=0}^{\infty} f_j(z)$ is said to exist, and we set

$$P_{[m/r-1]}(z) = \sum_{j=0}^{m-1} b_j z^j + z^m C [I + b_{r+1} z]$$

$$P_{[m/r-1]}(z) = \sum_{j=0}^{m-1} b_j z^j + z^m C [I: \sum_{j=0}^r b_{j0}^{(m)} + b_{j1}^{(m)} z; z: \sum_{j=0}^r b_{j0}^{(m)} + b_{j1}^{(m)} z]$$

If $f_0 \neq 0$, and the sequence of linear functions $\tilde{b}_{\tau,0}^{(m)} + \tilde{b}_{\tau,1}^{(m)} z$ ($\tau = J_1^r$) can ~~be~~ similarly be determined from the series $\sum_i \tilde{f}_{m+r} z^i$, where $\sum_i \tilde{f}_i z^i$ is the inverse series derived from $f(z)$, then the Padé quotient of order ~~is~~ $m+r-1, r$ derived from $f(z)$ is said to exist, and we set

$$P_{m+r-1, r}(z) = \left[\sum_{i=0}^{m-1} \tilde{f}_i z^i + z^m C \left[I: \tilde{b}_{1,0}^{(m)} + \tilde{b}_{1,1}^{(m)} z + ; z: \tilde{b}_{r,0}^{(m)} + \tilde{b}_{r,1}^{(m)} z + \right]_r \right]^{-1}$$

Theorem. Let $i, j \in J$. If the Padé quotients of orders i, j ~~and~~ $i+1, j+1$ respectively derived from the formal power series $\sum_i f_i z^i$ with vector valued coefficients exists ~~and~~ it generates the series $\sum_i f_{i,j}^{(i,j)} z^i$, then $f_{i,j}^{(i,j)} = f_j$ ($i = J_0^{i+j}$). Furthermore, if the quotient of order $i+1, j+1$ also exists, then $f_{i,j+1}^{(i,j)} \neq f_{i,j+1}$.

Proof. The above result is a direct consequence of Theorem.

Let $f(z) \in F[[z]]$, and $f(z)^{-1}$ exist; if both $f(z)$ and $f(z)^{-1}$ are semi-normal. **Definition.** If the formal power series $\sum_i f_i z^i$ with vector valued coefficients possesses an inverse ~~series~~, and both these series are semi-normal, then $f(z)$ is said to be normal.

Theorem . Rational functions $\xi_r^{(m)}(z)$ ($r=J, m=J$) can be constructed from the initial values ~~and~~ values () by means of relationships () for $m=J$ if and only if the series $\sum_{j=0}^{\infty} f_j z^j$ is semi-normal.

Rational functions $\xi_r^{(m)}(z)$ ($r=J, \hat{r}=r+2, m=J-\hat{r}$) can be constructed from the initial values () in conjunction with ~~()~~

$\sum_{m=0}^{\infty} \xi_m^{(-m)}(z) = 0$ ($m=J$) if and only if the series ~~()~~ $f(z)$ is semi-normal.

Proof. The above result is only an extension to the domain of vector valued rational functions of Theorem . .

Sequence to sequence transformations

is used in

The ε -algorithm finds ~~applications in~~ applied mathematics ~~in certain cases~~ it is often found ~~to~~ as a convergence acceleration device: ~~interest~~ subsequences

of numbers ~~derived~~ ^{taken} from the transformed set $\{\varepsilon_{2r}^{(m)}\}$ ~~in~~ derived from an initial value sequence $\{S_m\}$ in certain circumstances converge more rapidly to the limit or formal limit ~~of a prescribed sequence~~ $\{S_m\}$ than this sequence itself. The significance of the numbers $\{\varepsilon_{2r}^{(m)}\}$ produced by means of the ε -algorithm derives from ~~the~~ theory ~~of~~ which we have given ~~earlier~~ in this paper, and this theory ^{may} can also be used to derive algebraic properties of the ~~derived~~ ^{transformed} sequences.

We first give some elementary results in the theory of the ε -algorithm as applied to sequences of elements of a ring.

Theorem. Let $S_\nu \in \mathcal{R}$ ($\nu = \overline{J}$), and $\{\varepsilon_r^{(m)}\}$ be ~~a~~ ^{the} set of numbers that can be produced from the initial values

$$\varepsilon_{-1}^{(m)} = 0 \quad (m = \overline{J})$$

$$\varepsilon_0^{(m)} = S_m \quad (m = \overline{J})$$

(i) ~~If $A \in \mathcal{R}$, $B \in \mathcal{R}$, then~~

by use of the relationships

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m)} - (\varepsilon_r^{(m)} - \varepsilon_r^{(m)})^{-1}$$

for all $m, r \in \mathbb{J}$ such that $(z_r^{(m+1)} - z_r^{(m)}) \in \mathbb{R}_I$.

(i) Let $m' \in \mathbb{J}$; ~~can~~ corresponding to all numbers $\{z_r^{(m'+m)}\}$ that can be produced, numbers $\{\hat{z}_r^{(m)}\}$ can be produced from the initial values $\hat{z}_{-1}^{(m)} = 0$ ($m \geq \mathbb{J}_1$), $\hat{z}_0^{(m)} = S_{m'+m}$ ($m \geq \mathbb{J}$), by use of relationships similar to (), and for these corresponding numbers $\hat{z}_r^{(m)} = z_r^{(m'+m)}$.

(ii) Let $a \in \mathbb{R}$, $b, c \in \mathbb{R}_I$; corresponding to the numbers $\{z_r^{(m)}\}$

that can be produced, numbers $\{\tilde{z}_r^{(m)}\}$ can be produced from the initial values

$$\tilde{z}_{-1}^{(m)} = 0 \quad (m \geq \mathbb{J}_1) \quad \tilde{z}_0^{(m)} = a + b S_m c \quad (m \geq \mathbb{J})$$

by use of relationships similar to (), and for these corresponding numbers

$$\tilde{z}_{2r}^{(m)} = a + b z_{2r}^{(m)} c \quad \tilde{z}_{2r+1}^{(m)} = c^{-1} z_{2r+1}^{(m)} b^{-1}$$

(iii) Let $d_r \in \mathbb{R}$ ($r \geq \mathbb{J}$); corresponding to the numbers $\{z_r^{(m)}\}$ that can be produced, ~~for~~ numbers $\{\hat{z}_r^{(m)}\}$ can be produced from initial values similar to () by use of the relationships

$$\hat{z}_{r+1}^{(m)} = d_r + \hat{z}_{r-1}^{(m+1)} + (\hat{z}_r^{(m+1)} - \hat{z}_r^{(m)})^{-1}$$

and for these corresponding numbers

$$\hat{z}_{2r}^{(m)} = \sum_{i=1}^r d_{2i+1} + z_{2r}^{(m)}, \quad \hat{z}_{2r+1}^{(m)} = \sum_{i=0}^r d_{2i} + z_{2r+1}^{(m)}$$

(iv) Let $g, h \in \mathbb{R}_I$; corresponding to the numbers $\{\varepsilon_r^{(m)}\}$ that can be produced, numbers $\{\varepsilon_r^{(m)}\}$ can be produced from initial values similar to () by use of the relationships

$$\varepsilon_{2r+1}^{(m)} = \varepsilon_{2r-1}^{(m)} + g(\varepsilon_{2r}^{(m)} - \varepsilon_{2r}^{(m)})^{-1} h$$

$$\varepsilon_{2r+2}^{(m)} = \varepsilon_{2r}^{(m)} + h(\varepsilon_{2r+1}^{(m)} - \varepsilon_{2r+1}^{(m)})^{-1} g$$

and for these corresponding numbers $\varepsilon_{2r}^{(m)} = \varepsilon_{2r}^{(m)}$, $\varepsilon_{2r+1}^{(m)} = g\varepsilon_{2r+1}^{(m)} h$.

(v) The numbers $\{\varepsilon_{2r}^{(m)}\}$ that can be produced satisfy the relationships

$$\begin{aligned} & \{\varepsilon_0^{(m)} - \varepsilon_0^{(m)}\}^{-1} + \{\varepsilon_0^{(m+1)} - \varepsilon_0^{(m)}\}^{-1} = \{\varepsilon_2^{(m-1)} - \varepsilon_0^{(m)}\}^{-1} \\ & \{\varepsilon_{2r}^{(m)} - \varepsilon_{2r}^{(m)}\}^{-1} + \{\varepsilon_{2r}^{(m+1)} - \varepsilon_{2r}^{(m)}\}^{-1} = \{\varepsilon_{2r+2}^{(m-1)} - \varepsilon_{2r}^{(m)}\}^{-1} + \{\varepsilon_{2r-2}^{(m+1)} - \varepsilon_{2r}^{(m)}\}^{-1}; \end{aligned}$$

if, in addition for these numbers $\{\varepsilon_{2r}^{(m)}\}$ if the numbers occurring in these formulae are also invertible, similar formulae in which each number is replaced by its inverse also hold; similar results also hold for numbers of the form $\{\varepsilon_{2r+1}^{(m)}\}$.

Proof. All of the above results are easily proved by induction.

We note, in particular, from the above theorem that the initial values $\varepsilon_{-1}^{(m)} = 0$ ($m \geq J_1$) may be replaced by $\varepsilon_{-1}^{(m)} = d_0$ ($m \geq J_1$) ($d_0 \in \mathbb{R}$) without affecting the numbers $\{\varepsilon_{2r}^{(m)}\}$; furthermore, the term $(\varepsilon_r^{(m)} - \varepsilon_r^{(m)})^{-1}$

occurring in the ε -algorithm relationships, may be replaced by $g(z_{2r}^{(m+1)} - \varepsilon_{2r}^{(m)})^{-1} h$, $h(\varepsilon_{2r+1}^{(m)} - z_{2r+1}^{(m)})g$ respectively, also without

affecting ~~the~~ the numbers $\{z_{2r}^{(m)}\}$. The above results may also be extended: for example; if ~~let~~ ^{let} the S_m be square matrices, ~~and~~ ^{and} a, b be ~~non-singular~~ ^{non-singular} and ~~be~~ ^{be} c and the ~~S_m be~~ ^{S_m be} square matrices, and ~~be~~ ^{be} a is a square matrix of suitable dimension; then if

matrices $\{z_r^{(m)}\}$ are produced from the initial values $\tilde{z}_{-1}^{(m)} = 0$ ($m = J$),

$\tilde{z}_0^{(m)} = a + b \times S_m \times c$ ($m = J$) where the direct product is used, we

have $\tilde{z}_{2r}^{(m)} = a + b \times \varepsilon_{2r}^{(m)} \times c$, $\tilde{z}_{2r+1}^{(m)} = b^{-1} \times \varepsilon_{2r+1}^{(m)} \times c^{-1}$.

Recursions involving values of rational functions

Definition. The set $C\{R\}$ of elements $\xi \in R$ for which $\xi\alpha = \alpha\xi$ for every $\alpha \in R$ and $\xi(\alpha\beta) = (\xi\alpha)\beta$, $\alpha(\xi\beta) = (\alpha\xi)\beta$, $\alpha(\beta\xi) = (\alpha\beta)\xi$ for every $\alpha, \beta \in R$ is called the centre of the ring R .

In earlier sections of this paper ~~defined~~ considered rational functions ~~and~~ ^{to be} sets of coefficients together with a formal variable z which obeyed certain rules when ~~was~~ ^{is} considered ~~to~~ ^{to} be in a well defined manner during the transformations considered but is otherwise undefined; indeed the variable z is ~~not~~ indeed, the sole purpose in introducing the variable z is to motivate the operations upon the coefficients of the rational functions. It is, however, clear from the above definition that the formal variable z ~~behaves as if it were~~ associated with a rational function with coefficients over a ring R behaves as if it were a member ^{is given} of the ring $C\{R\}$. If z is ~~not~~ given a value in $C\{R\}$, ~~then~~ ^a values in $C\{R\}$, each rational function is a mapping of $C\{R\}$ onto R . The recursion takes a value in R , should this value be defined for the z in question. The recursions involving

rational functions regarded as sets of coefficients which we have so far considered become recursions involving ~~the~~ values of rational functions.

In many cases the theory of recursion involving values is obtained by mere verbal transposition from that of recursions involving functions. For example, a system of linear recursions between polynomials ~~$\{D(z)\}$~~ regarded as sets of coefficients becomes quite trivially a system of linear recursions between values of polynomials. Difficulties arise whenever the functions ~~concerned~~ ~~involved~~ are nonlinear in powers of the argument z or the operations involved are nonlinear. It may occur, for example, that the rational function $D(z)^{-1}N(z)$ is well defined in the sense which we have ~~the~~ hitherto considered but, for the value of z in question, ~~the~~ the values of $D(z)$ in question does not belong to \mathbb{R}_I . Again, the values of two such functions may be well defined, but their difference may not be invertible. In particular, ~~it may occur that~~ although the rational functions $\{E_r^{(m)}(z)\}$ of Theorem ... can be constructed from the partial sums ~~of the series~~ $\{\sum_{j=0}^{m-1} b_j z^j\}$ of a semi-normal series, it may occur that the recursive construction of the values of these rational

functions by means of these ε -algorithm breaks down, either because for the value of z in question the value of one or more of these rational functions is ~~not~~ undefined, or because one or more of the differences $\{E_r^{(m)}(z) - E_r^{(m-1)}(z)\}$ is not invertible. Nevertheless, if the values of the functions $\{E_r^{(m)}(z)\}$ ~~may~~ ^{can} be constructed by means of the ε -algorithm, we may say ~~the~~ something about the series from whose partial sums they are constructed.

Let $S_p \in \mathbb{R}$ ($p \geq J$) and $f(z) = \sum_{v=J}^{\infty} f_v z^v$ ~~where $f_v = S_{2v+1} - S_{2v}$ ($2v \geq J$)~~ where $f_v = S_{2v+1} - S_{2v}$ ($2v \geq J$).

Theorem (i) If numbers $\{E_r^{(m)}\}$ ($r, m \geq J$) can be constructed from the initial values

$$E_r^{(m)} = 0 \quad (m \geq J_1), \quad \varepsilon_0^{(m)} = S_m \quad (m \geq J)$$

where $S_m \in \mathbb{R}$ ($m \geq J$) and 0 is the zero element of \mathbb{R} , by means of the relationships

$$E_{r+1}^{(m)} = E_r^{(m)} + (E_r^{(m)} - E_r^{(m-1)})$$

for $r, m \geq J$, then the series $\sum_{v=J}^{\infty} f_v z^v$, where $f_v = S_{2v+1} - S_{2v}$ ($2v \geq J$), is semi-normal.

(ii) If numbers $E_r^{(m)}$ ($r = J_1, \hat{r} = r+2, m \geq J - \hat{r} + 1$) can be constructed from ^{the} initial values $(\)$ in conjunction with $E_{2m}^{(m)} = 0$ ($m \geq J$) by means of relationships $(\)$ for $r \geq J, m \hat{r} = r+2, m \geq J - \hat{r}$, then the

series of ~~f_r~~ above series $\rightarrow \sum f_r z^r$ of relationships

series $f(z)$ is ~~semi~~ normal.

(iii). If numbers $F_r^{(m)}$ ($r=1, \dots, m=J$) can be cons

Proof. If numbers $\varepsilon_r^{(m)}$ ($r, m=J$) can be produced under the circumstances adumbrated in clause (i), then corresponding numbers $\{\hat{\varepsilon}_r^{(m)}\}$ can be produced from the initial values of

$$\hat{\varepsilon}_{-1}^{(m)} = 0 \quad (m=J), \quad \hat{\varepsilon}_0^{(m)} = S_m - S_0 \quad (m=J)$$

by means of relationships similar to () and $\hat{\varepsilon}_{2r}^{(m)} = S_0 + \varepsilon_{2r}^{(m)}$ ($r, m=J$) (clause (i) of Theorem); we now have $f(z) = \sum_{r=0}^{\infty} \hat{\varepsilon}_r^{(m)} z^r - \varepsilon_0$

($r=J$), and $\hat{\varepsilon}_0^{(m)} = \sum_{i=0}^{m-1} f_i z^i$ when $z=1$.

We now define numbers $\{h_r^{(m)}\}$ in terms of the numbers $\{\varepsilon_r^{(m)}\}$ with $z=1$

$\{\hat{\varepsilon}_r^{(m)}\}$ by means of formulae similar to () (in particular, we

now have $h_{-1}^{(m)} = 0$, $h_0^{(m)} = ?$ ($m=J$) and find that they among elements of \mathbb{R} hold for rational functions with coefficients over \mathbb{R} .

Obey relationships, similar to defining numbers $\{g_r^{(m)}\}$ in terms

of the numbers $\{h_r^{(m)}\}$ by means of formulae similar to () and,

in turn, numbers $\{p_r^{(m)}\}$ in terms of the numbers $\{g_r^{(m)}\}$ by means

of formulae similar to () and, lastly numbers $\{a_r^{(m)}\}$ in

terms of the numbers $\{d_r^{(m)}\}$ by formulae similar to (), we

find that ~~numbers~~ the numbers $\{a_r^{(m)}\}$ can be produced from the initial values () by means of relationships (). This is only possible if the series $\sum_{r=0}^{\infty} f_r z^r$ is seminormal.

Clause (ii) of the theorem is proved in the same way.

We now consider application of the ε -algorithm to numbers satisfying an inhomogeneous linear difference equation, and, as a preliminary, prove

Theorem. Let the numbers $S_n \in \mathbb{R}$ ($n \equiv J$) satisfy the recursion

$$\sum_{i=0}^{r'} d_i S_{m+i} = H \quad (m \equiv J)$$

where $r' \in \mathbb{N}$, $d_i \in \mathbb{R}$ ($i \equiv 0, \dots, r'$), $H \in \mathbb{R}$ and no recursion of a similar form with r' replaced by a smaller integer; set $\sum_{i=0}^{r'} d_i = D$. The it is not possible that both D and H are both factors of zero

Proof. Assume that $\alpha D = 0$, $\beta H = 0$ where $\alpha, \beta \neq 0$. Then by multiplying relationships () throughout by α and β respectively we obtain the find that

$$\sum_{i=0}^{r'} \hat{d}_i S_{m+i} = 0 \quad (m \equiv J)$$

where $\hat{d}_j = \alpha d_j$ ($j \in J_0'$) and $\sum_{j \in J_0'} \hat{d}_j = 0$. Setting $d_j' = \sum_{j \in J_0'} \hat{d}_j$

we find that equations () may be written in the form

$$\sum_{j \in J_0'} d_j' S_{m+j} - \sum_{j \in J_0'} d_j' S_{m+j-1} = 0, \quad (m \in J)$$

i.e.

$$\sum_{j \in J_0'} d_j' S_{m+j} = \sum_{j \in J_0'} d_j' S_j = H'. \quad (m \in J)$$

Hence, under the stated assumptions, recursion () has been reduced to a form in which r is replaced by $r-1$.

Theorem (i) ^{Let $r \in J$} . If, numbers $\varepsilon_r^{(m)}$ ($r \in J_0', m \in J$) can be produced from

the initial values () ~~from~~ by use of ~~the~~ relationships () with $r \in J_0', \{m \in J\}$, and the numbers $\{S_j\}$ satisfy an irreducible recursion of the form () ^{with $\sum_{j \in J_0'} d_j' \in \mathbb{R}_+$} ~~with $\sum_{j \in J_0'} d_j' \in \mathbb{R}_+$~~ ^{and} ~~is not a matrix~~, then

$$\varepsilon_{2r}^{(m)} = D^{-1} H; \quad (m \in J)$$

conversely, if ^{for some $r \in J$} numbers $\varepsilon_r^{(m)}$ ($r \in J_0', m \in J$) can be produced from the initial values () by use of relationships () with $r \in J_0', \{m \in J\}$, and $\varepsilon_{2r}^{(m)} = S \in \mathbb{R}$ ($m \in J$), then the numbers $\{S_j\}$ satisfy an irreducible recursion of the form () with $\sum_{j \in J_0'} d_j' \in \mathbb{R}_+$

(ii) If, for some $r \in J$, numbers $z_{2r}^{(m)}$ ($r = J_0^{2r-1}, m = J$) can be produced from the initial values () by use of relationships () with $\{r = J_0^{2r-2}, m = J\}$, and the numbers $\{S_{\nu}\}$ satisfy an irreducible recursion of the form (), ~~and~~ $\sum_{\nu=0}^{\infty} z_{\nu}^{(m)} = 0$ ~~is a factor of zero, then if~~ $\hat{z}_{\nu} = 0$ ($\nu \neq 0$), $\hat{z}_{\nu} = \omega_{\nu}$ ($\nu = J_0^{2r-1}$) then

$$z_{2r}^{(m)} = N^{-1} \sum_{\nu=0}^{r-1} \nu \omega_{\nu}; \quad (m = J)$$

Conversely, if for some $r \in J$, numbers $z_{2r}^{(m)}$ ($r = J_0^{2r-1}, m = J$) can be produced from the initial values () by use of relationships () with $r = J_0^{2r-2}, m = J$ and $z_{2r}^{(m)} = S' \in \mathbb{R}$ ($m = J$), then the numbers $\{S_{\nu}\}$ satisfy an irreducible recursion of the form () with

$$\sum_{\nu=0}^{\infty} z_{\nu}^{(m)} = 0$$

Proof. As in the proof of Theorem... we show that under the conditions of clause (i), ~~we have, in the notation of Definition...~~ ~~the series~~ $\{f_{\nu}\} \in \mathbb{L}_r$??, and that if $S(z) = \sum_{\nu=0}^{\infty} z_{\nu}^{(m)} z^{\nu}$ is the rational function which generates the series $\sum_{\nu=0}^{\infty} f_{\nu} z^{\nu}$, then $z_{2r}^{(m)} = S(\frac{1}{2r})$ ($m = J$). Furthermore from Theorem..., if $S_m = \sum_{\nu=0}^{m-1} f_{\nu}$ ($m = J$), then the numbers $\{S_m\}$ satisfy the recursion $\sum_{\nu=0}^{r-1} S_{m+\nu} = N$ ($m = J$)

~~and the numbers~~

where $\hat{d}_p = d_{r-p}$ ($p=1$) and $N \equiv N(I)$ and no other recursion of

similar form in which r' can be replaced by a smaller integer.

To prove clause (ii), we remark that ~~under the conditions of this clause~~ if $D=0$

$S_m^{(-1)} = S_{m+1} - S_m$ ($m=1$) satisfy the recursion

$$\sum_{p=0}^{r'-1} c_p^{(-1)} S_{m+p}^{(-1)} = N \quad (m=1)$$

where $c_p^{(-1)} = \sum_{r=1}^{m+p} c_r$ ($r=1$ to $r'-1$). ~~The result of clause (i) or~~
~~under the assumptions of the first part of clause (i), it follows from theorem ... that~~
~~extension of theorem, it follows that furthermore, the rational~~

functions ${}^{(-1)}E_{2r'}^{(m)}(z)$ ($r=1$ to r' , $m=1$) can be produced by means of relationships similar to () to the initial values

~~$f_m z^m = S_{m+1}(z) - S_m(z)$ ($m=1$) and, in particular,~~

~~$E_{2r'-1}^{(m)}(z) = {}^{(-1)}E_{2r'}^{(m)}(z)$ ($m=1$).~~ Applying the result of clause (i) to

the set of numbers $\{ {}^{(-1)}E_r^{(m)} \}$, we find that ~~${}^{(-1)}E_{2r'}^{(m)}$~~ corresponding to

the functions $\{ {}^{(-1)}E_r^{(m)}(z) \}$, we find that ~~${}^{(-1)}E_{2r'}^{(m)} = E_{2r'-1}^{(m-1)} = \frac{1}{D} \frac{d}{dz} E_{2r'}^{(m)}$~~

The proof of the ~~second~~ ~~part~~ result of clause (ii) is ~~proved as above~~,
 similar to carried out as for that of the second part of clause (i).

Theorem . Let $\{\xi_r^{(m)}\}$ be a set of numbers produced ~~from~~
~~initial values of the form ()~~ by application of relationships
to initial values of the form () and, for those numbers
 $\{\xi_r^{(m)}\}$ that are so defined, set

$$\phi_r^{(m)} = \sum_{j=0}^{r-1} \xi_{2j+1}^{(m)} (\xi_{2j+2}^{(m)} - \xi_{2j}^{(m)})$$

$$\psi_r^{(m)} = \sum_{j=0}^{r-1} (\xi_{2j+2}^{(m)} - \xi_{2j}^{(m)}) \xi_{2j+1}^{(m)} ;$$

then

$$\phi_r^{(mn)} - \phi_r^{(m)} = \xi_{2r-1}^{(mn)} (\xi_{2r}^{(m)} - \xi_{2r}^{(mn)})$$

$$\psi_r^{(mn)} - \psi_r^{(m)} = (\xi_{2r}^{(m)} - \xi_{2r}^{(mn)}) \xi_{2r-1}^{(mn)} .$$

If, for some $r' \in J$, numbers $\xi_r^{(m)}$ ($r \in J_0^{2r'}, m \in J$) can be
produced from the initial values () by use of relationships ()
with $r \in J_0^{2r'-1}, m \in J$ and the numbers $\{d_j\}$ satisfy an irreducible
recursion of the form () and $\sum_{j=0}^{r'} d_j \in \mathbb{R}_J$, then the
numbers $\phi_{r'}^{(m)}, \psi_{r'}^{(m)}$ of formulae () and () are constants
and independent of m .

Proof. Assuming the numbers concerned to be defined, we have

$$\left. \begin{aligned} \varepsilon_{2r+1}^{(m)} \varepsilon_{2r}^{(m)} - \varepsilon_{2r-1}^{(m+1)} \varepsilon_{2r}^{(m)} - \varepsilon_{2r+1}^{(m)} \varepsilon_{2r}^{(m+1)} + \varepsilon_{2r-1}^{(m+1)} \varepsilon_{2r}^{(m+1)} &= -I \\ \varepsilon_{2r+1}^{(m+1)} \varepsilon_{2r+2}^{(m)} - \varepsilon_{2r+1}^{(m+1)} \varepsilon_{2r}^{(m+1)} - \varepsilon_{2r+1}^{(m)} \varepsilon_{2r+2}^{(m)} + \varepsilon_{2r+1}^{(m)} \varepsilon_{2r}^{(m+1)} &= I \end{aligned} \right\} (i = J_0^{m'})$$

adding these equations, and $\varepsilon_{2r-1}^{(m+1)} \varepsilon_{2r}^{(m+1)}$ to both sides of the resulting relationship, we obtain formula (). Formula () is derived in the same way. Under the conditions of the last part of the theorem

$\varepsilon_{2r}^{(m)}$ is a constant, independent of m , $\phi_{r'}^{(m+1)} - \phi_{r'}^{(m)} = 0$, and ~~hence~~

$\phi_{r'}^{(m)}$ is therefore also independent of m .

We note in passing that if the initial value sequence $\{S_m\}$ from

which the numbers $\{\varepsilon_r^{(m)}\}$ are derived is replaced by $\{a + b S_m c\}$

$a \in \mathbb{R}$, $b, c \in \mathbb{R}_I$, then $\phi_r^{(m)}$ becomes $c^{-1} \phi_r^{(m)} c$ and $\psi_r^{(m)}$ becomes

$b \psi_r^{(m)} b^{-1}$.

Theorem... may also be extended so as to concern ~~the~~ values of functions; ~~we may~~ also at the same time the ^{processes of} summation and differencing applied to the initial value sequences ^{of that theorem} may be prolonged indefinitely and results concerning the sets of numbers derived from these sequences may be obtained.

Theorem. Let $S_m \in \mathbb{R}$ ($m = \mathbb{J}$), and $\{S_m^{(i)}\}$ ($i = \hat{\mathbb{J}}$) be the sequences derived from the relationships

$$S_m^{(i)} = \sum_0^m S_\nu^{(i-1)} \quad (i = \mathbb{J}_1), \quad S_m^{(i-1)} = S_{m+1}^{(i)} - S_m \quad (i = \mathbb{J}) \quad (\text{---})$$

for $m = \mathbb{J}$, and let $\{z_r^{(i)}\}$ be the numbers that can be constructed from the initial values

$$(i) z_{-1}^{(m)} = 0 \quad (m = \mathbb{J}_1), \quad (ii) z_0^{(m)} = S_m^{(i)} \quad (m = \mathbb{J})$$

by means of relationships similar to () ~~with $i = \hat{\mathbb{J}}$~~ with $i = \mathbb{J}$; then the numbers $\{z_r^{(i)}\}$ of one array are related to those of the preceding array by means of the formulae

$$(i) z_{2r}^{(m)} = \sum_0^{m+r} z_0^{(i-1)} + \sum_0^{r-1} \left\{ z_{2(r-\nu-1)}^{(i-1)} - z_{2(r-\nu-1)}^{(i-1)} \right\}^{-1} \quad (i = \mathbb{J}_1)$$

$$(ii) z_{2r+1}^{(m)} = z_{2r}^{(i-1)}$$

$${}^{(i)}\Sigma_{2r}^{(m)} = {}^{(i)}\Sigma_0^{(0)} + \sum_{i=0}^{m/r-1} {}^{(i-1)}\Sigma_0^{(r)} + \sum_{i=0}^{r-1} \left\{ {}^{(i-1)}\Sigma_{2(r-i)}^{(m+i)} - {}^{(i-1)}\Sigma_{2(r-i-1)}^{(m+i)} \right\}^{-1} \quad (i \equiv \mathbb{J})$$

$${}^{(i)}\Sigma_{2r+1}^{(m)} = {}^{(i-1)}\Sigma_{2r}^{(m)-1}$$

and the numbers $\{ {}^{(i-1)}\Sigma_r^{(m)} \}$ are related to the numbers of the succeeding array by means of the formulae

$${}^{(i-1)}\Sigma_{2r}^{(m+i)} = {}^{(i)}\Sigma_{2r+1}^{(m)-1} \quad \left. \vphantom{{}^{(i-1)}\Sigma_{2r}^{(m+i)}} \right\} (i \equiv \mathbb{J}_1)$$

$${}^{(i-1)}\Sigma_{2r+1}^{(m+i)} = \sum_{i=0}^r \left\{ {}^{(i)}\Sigma_{2(r-i)+1}^{(m+i)} - {}^{(i)}\Sigma_{2(r-i)}^{(m+i)} \right\}^{-1}$$

$${}^{(i-1)}\Sigma_{2r}^{(m)} = {}^{(i)}\Sigma_{2r+1}^{(m)-1} \quad \left. \vphantom{{}^{(i-1)}\Sigma_{2r}^{(m)}} \right\} (i \equiv \mathbb{J})$$

$${}^{(i-1)}\Sigma_{2r+1}^{(m)} = \sum_{i=0}^r \left\{ {}^{(i)}\Sigma_{2(r-i)+1}^{(m)} - {}^{(i)}\Sigma_{2(r-i)}^{(m)} \right\}^{-1}$$

for all $m, r \in \mathbb{J}$ such that all numbers occurring in each formula can be constructed.

Let the members of the sequence $\{ S_m^{(0)} \}$ satisfy the irreducible recursion

$$\sum_{i=0}^h d_i S_{\nu+i}^{(0)} = N \quad (m \equiv \mathbb{J})$$

and set $\sum_{i=0}^h d_i = D$. Assuming that the sequences concerned can be constructed

a) if $D, N \neq 0$ then

$$(0) \sum_{2h}^{(m)} = D^{-1} N \quad (m = J)$$

$$(1) \sum_{2h+1}^{(m)} = D N^{-1} \quad (m = J)$$

$$(i) \sum_{2h+2i-1}^{(m)} = 0 \quad (i = J_2, m = J) \quad (1)$$

$$(i) \sum_{2h}^{(m)} = 0 \quad (i = J_1, J, m = J) \quad (1')$$

b) if $N=0$ there exists an $\hat{i} \in J_0^{h-1}$ such that

$$(i) \sum_{2h}^{(m)} = 0 \quad (i = J_0^{\hat{i}})$$

the numbers $\sum_{2h}^{(m)} (i+1)$ ($m = J$) are identical finite nonzero constants

and

$$(i) \sum_{2(h+i-\hat{i})-1}^{(m)} = 0 \quad (i = J_{i+1}^{\hat{i}}, m = J)$$

whereas the numbers $\sum_{2h}^{(m)} (i)$ with $i = -J$, satisfy conditions (1)';

c) if $D=0$, then the numbers $\sum_{2h}^{(m)} (i)$ with $i \in J$, satisfy conditions

(1) with $i = J_1, m = J$; for nonpositive values of i there exists an $\hat{i} \in J_0^{n-1}$ such that

$$(i) \sum_{2h+2i-1}^{(m)} = 0 \quad (i = -J_0^{\hat{i}-1}, m = J)$$

$$(-i) \sum_{2h-2i-1}^{(m)} = N^{-1} D^{(-i-1)} \quad (m = J)$$

$$(-i-1) \sum_{2h-2i-2}^{(m)} = D^{(-i-1)-1} N \quad (m = J)$$

where $C^{(-i-1)} = \sum_{k=i+1}^h \binom{h}{k} C_k$ and $\binom{h}{i}$ denotes the binomial coefficient $\prod_{j=1}^i \{(h-j+1)/j\}$, while

$$(i) \sum_{2n-2i-2}^{(m)} = 0 \quad (i = -J_{i+2}, m = J)$$

In all of the above cases the construction of each ~~array~~ ε -array terminates with the formation of the last named sequence belonging to the array.

Proof. Formulae () - () are proved ^{as} an extension of the results of Theorem ... in the same way ^{that the results of} ~~as~~ Theorem ... ^{proved} ~~is deduced from~~ are deduced from Theorem

It follows from Theorem ... that the extent to which the progressive construction of a set of numbers $\{\varepsilon_r^{(m)}\}$ can be continued is, in the case in which this is consistently impossible after a certain value of the suffix r has been reached, determined by the linear recursion which the members of the initial value sequence satisfy. Thus ^{a part of the} ~~the~~ remaining results of the theorem can be upon an investigation of the recursions satisfied by the numbers $\{S_m^{(i)}$

The behaviour of the numbers $\{\epsilon_r^{(i)}\}$ under the conditions of clause a) has already been dealt with in Theorem...

To consider the numbers $\{S_r^{(i)}\}$ with $i \in J_1$ under the same conditions, we first remark that $S_0^{(i)} = S_0^{(0)}$, $S_{m+1}^{(i)} - S_m^{(i)} = S_m^{(i)}$ ($m \in J$). Thus, determining

the numbers $\{c_\nu^{(i)}\}$ from the relationships $c_0^{(i)} = -c_0$, $c_\nu^{(i)} = (c_{\nu-1} - c_\nu)$

($\nu \in J_1^n$), $c_{n+1}^{(i)} = c_n$ we have

$$\sum_{\nu=0}^{n+1} c_\nu^{(i)} S_{\nu+m}^{(i)} = N. \quad (m \in J) \quad (**)$$

If the numbers $\{S_m^{(i)}\}$ satisfy a relationship of the form

$$\sum_{\nu=0}^{\hat{n}} \hat{c}_\nu S_{\nu+m}^{(i)} = \hat{N} \quad (m \in J) \quad (**')$$

with $\hat{n} < n+1$, then forming the difference of two successive members of this set of equations, we have

$$\sum_{\nu=0}^{\hat{n}} \hat{c}_\nu S_{\nu+m}^{(i)} = 0. \quad (m \in J)$$

IF

$$\sum_{v=0}^{\hat{n}} \hat{c}_v S_{v+m}^{(1)} = \hat{G} \quad (m \equiv \mathfrak{J}) \quad (32)$$

with $\hat{n} < n + 1$ then, forming the difference of two successive members of this set, we have

$$\sum_{v=0}^{\hat{n}} \hat{c}_v S_{v+m}^{(0)} = 0. \quad (m \equiv \mathfrak{J})$$

If $\hat{n} < n$, then recursion (24) is not irreducible, contrary to assumption; if $\hat{n} = n$, then $G = 0$, also contrary to assumption. Hence if a recursion of the form (32) holds, $\hat{n} = n + 1$, and recursion (32) is obtained merely by multiplying relationship (31) throughout by a

suitable constant. We have, of course, $\sum_{v=0}^{n+1} c_v^{(1)} = 0$, $\sum_{v=1}^{n+1} v c_v^{(1)} = \sum_{v=0}^n c_v = C$.

The numbers $\{c_v^{(1)}\}$ play the same rôle with regard to the sequence $\{S_m^{(1)}\}$ as do the numbers $\{c_v\}$ with regard to the sequence $\{S_m\}$ in formulae (6), of which relationships (25) are consequences.

The above process may be continued: we have recursions of the form

$$\sum_{v=0}^{n+i} c_v^{(i)} S_{v+m}^{(i)} = G \quad (i \equiv \mathfrak{J}_2, m \equiv \mathfrak{J})$$

where now $\sum_{v=0}^{n+i} c_v^{(i)} = \sum_{v=1}^{n+i} v c_v^{(i)} = 0$ ($i \equiv \mathfrak{J}_2$). Hence we derive

relationships (26).

With regard to the numbers $\binom{i}{r} c_r^{(m)}$ with $i \in -\mathfrak{J}$, we have first of all $S_m^{(-1)} = S_{m+1}^{(0)} - S_m^{(0)}$ ($m \in \mathfrak{J}$). Hence, from formulae (24),

$$\sum_{v=0}^n c_v S_{v+m}^{(-1)} = 0. \quad (m \in \mathfrak{J}) \quad (33)$$

If the numbers $\{S_m^{(-1)}\}$ satisfy a recursion of the form

$$\sum_{v=0}^{n'} \tilde{c}_v S_{v+m}^{(-1)} = \tilde{G} \quad (m \in \mathfrak{J}) \quad (34)$$

with $n' < n$, then it may be shown, as in the construction of recursion (31), that

$$\sum_{v=0}^{n'+1} c''_v S_{v+m}^{(0)} = \tilde{G} \quad (m \in \mathfrak{J}) \quad (35)$$

with $\sum_{v=0}^{n'+1} c''_v = 0$. If $n' < n - 1$, it would then follow that recursion (24) is reducible. If $n' = n - 1$, then comparison of recursions (24) and (35) reveals that $c''_v = k c_v$ ($v \in \mathfrak{J}_0^n$) for some nonzero element k , and hence, in this case, $\sum_{v=0}^n c_v = 0$, contrary to assumption. Thus the numbers $\{S_m^{(-1)}\}$ do not satisfy a recursion of the form (34) with $n' = n$, $\tilde{G} = 0$, and recursion (34) is simply obtained from (33) by multiplication throughout by a suitable constant.

In a similar way we show that for the members of all sequences $\{S_m^{(i)}\}$ with $i \in \mathcal{J}_1$ we have

$$\sum_{v=0}^n c_v S_{v+m}^{(i)} = 0, \quad (i \in \mathcal{J}_1, m \in \mathcal{J})$$

Relationships (27) then follow directly from formulae (5).

We now turn to clause b) and have, as in the derivation of recursion (31)

$$\sum_{v=0}^{n+1} c_v^{(1)} S_{v+m}^{(1)} = 0, \quad (m \in \mathcal{J})$$

Since $\sum_{v=0}^{k+1} c_v^{(1)} = 0$, this recursion is reducible, and we obtain

$$\sum_{v=0}^n c_v S_{v+m}^{(1)} = G^{(1)}, \quad (m \in \mathcal{J})$$

an irreducible recursion in which the value of $G^{(1)}$ may be obtained by setting $m = 0$, for example (the value of $G^{(1)}$ depends upon the initial members of the sequence $\{S_m^{(0)}\}$, and is not determined by the values of the $\{c_v\}$ and G alone). If $G^{(1)} \neq 0$, we have conditions relating to the sequences $\{S_m^{(i+1)}\}$ similar to those holding for the sequences $\{S_m^{(i)}\}$ in clause a), and we derive relationships

(28) and (29) with $\hat{i} = 0$. It may, however, occur that $G^{(1)} = 0$, and we then derive an irreducible recursion of the form

$$\sum_{v=0}^n c_v S_{v+m}^{(2)} = G^{(2)}, \quad (m \equiv \mathfrak{J})$$

We now show that it is impossible for the system of recursions

$$\sum_{v=0}^n c_v S_{v+m}^{(i)} = 0 \quad (i \equiv \mathfrak{J}_0^n, m \equiv \mathfrak{J}) \quad (36)$$

to hold.

Let us assume that they do hold. We set

$$\phi^{(i)} = \sum_{v=0}^n c_v S_v^{(i)}; \quad (i \equiv \mathfrak{J}_0^n) \quad (37)$$

As is easily verified, $\Delta_1^r S_v^{(i)} = 0$ ($i = 0, v \equiv \mathfrak{J}_0^{r-1}$), $\Delta_1^r S_v^{(i)} = S_{v-r}^{(r)}$ ($i = 0, v \equiv \mathfrak{J}_r^n$) ($r \in \mathfrak{J}_0^n$). Since, from formulae (36) and (37), $\Delta_1^r \phi^{(i)} = 0$ ($i = 0, r \equiv \mathfrak{J}_0^n$), we have

$$\sum_{v=0}^n c_v S_{v-r}^{(r)} = 0. \quad (r \equiv \mathfrak{J}_0^n) \quad (38)$$

The last equation of this set implies that $S_0^{(n)} = 0$ (and hence, since $S_0^{(i)} = S_0^{(0)}$ ($i \equiv \mathfrak{J}$), $S_0^{(r)} = 0$ ($r \equiv \mathfrak{J}_0^n$)). Assume that we have shown that

$S_v^{(r+1)} = 0$ ($v \in \mathfrak{J}_1^{n-r-1}$). Then we have $S_0^{(r)} = 0$, $S_v^{(r)} = S_v^{(r+1)}$ -
 $S_{v-1}^{(r+1)} = 0$ ($v \in \mathfrak{J}_1^{n-r-1}$) and, by substituting these values in
 relationship (38), $S_{n-r}^{(r)} = 0$ also. Hence, in particular, $S_v^{(0)} = 0$
 ($v \in \mathfrak{J}_0^i$). It then follows from recursion (36) with $i = 0$ that
 $S_v^{(0)} = 0$ ($v \in \mathfrak{J}$), and in this case recursion (24) is certainly not
 irreducible.

Hence, for some $\hat{i} \in \mathfrak{J}_0^{n-1}$, we have the irreducible recursion

$$\sum_{v=0}^n c_v S_{v+m}^{(\hat{i}+1)} = G^{(\hat{i}+1)} \neq 0. \quad (m \in \mathfrak{J})$$

The stated results of clause b) concerning the sequences of numbers
 $\{(i)_{\epsilon_r}^{(m)}\}$ for $i \in \mathfrak{J}_{\hat{i}}$ are now derived as in the proof of clause
 a); the analysis concerning these numbers for $i \in -\mathfrak{J}_1$ is precisely
 the same as for that clause.

We now turn to clause c). The analysis of the sequences of
 numbers $\{(i)_{\epsilon_r}^{(m)}\}$ with $i \in \mathfrak{J}_1$ is as in the proof of clause a).

Setting $c_v^{(-1)} = - \sum_{\tau=0}^v c_{\tau}$ ($v \in \mathfrak{J}_0^{n-1}$), recursion (24) may in this case
 be rewritten as

$$\sum_{v=0}^{n-1} c_v^{(-1)} S_{v+m}^{(-1)} = G. \quad (m \in \mathfrak{J})$$

If this recursion is reducible, it can be shown that recursion (24) is not irreducible. If $C^{(-1)} = \sum_{v=0}^{n-1} c_v^{(-1)} \neq 0$, then the analysis of the sequences of numbers $\{({}^{(i)}\epsilon_r^{(m)})\}$ for $i \equiv -\mathfrak{J}_1$ is conducted as in the proof of clause a). If $C^{(-1)} = 0$, we derive the irreducible recursion

$$\sum_{v=0}^{n-2} c_v^{(-2)} S_{v+m}^{(-2)} = G \quad (m \equiv \mathfrak{J})$$

and proceed as above. It has only to be mentioned that, as is easily

verified, $\sum_{v=0}^{n-i'} c_v^{(-i')} = \sum_{v=i'}^n \binom{v}{i'} c_v$ ($i' \equiv \mathfrak{J}_0^{n-1}$). It may, of course, occur that in this analysis $\sum_{v=0}^{n-i'} c_v^{(-i')} = 0$ ($i' \equiv \mathfrak{J}_0^{n-1}$); it is then found that the sequence $\{S_m^{(-n)}\}$ consists of constant nonzero members, while the numbers $\{S_m^{(i)}\}$ ($i \equiv -\mathfrak{J}_{n+1}$) are all zero.

A hierarchy of function sequences produced by means of the first confluent form of the epsilon algorithm

By applying relationships similar to (8) to the successive integrals and derivatives of the function $S(\mu)$, one obtains a hierarchy of sequences of functions $\{({}^{(i)}\epsilon_r^{(\mu)})\}$ in analogy with the hierarchy of ϵ -arrays discussed in the previous section. It will be recalled

Recursions involving vectors

~~With the theory of the preceding section in hand~~

With ~~both~~ the ~~isom~~ vector-matrix isomorphism studied in §... and ~~the theory of the preceding section~~ in hand, it is a relatively simple matter to derive results analogous to those of ~~the~~ ~~theorems~~ the preceding section.

Theorem . Let \underline{z}_r ($r \in J$), ~~and~~ be a sequence of vectors, and $\{\underline{z}_r^{(m)}\}$ be the set of vectors that can be produced from the initial values

$$\underline{z}_{-1}^{(m)} = \underline{0} \quad (m \in J), \quad \underline{z}_0^{(m)} = \underline{z}_m \quad (m \in J)$$

by use of the relationships

$$\underline{z}_{r+1}^{(m)} = \underline{z}_{r-1}^{(mn)} + (\underline{z}_r^{(mn)} - \underline{z}_r^{(m)})^{-1}$$

for all $m, r \in J$ such that $(\underline{z}_r^{(mn)} - \underline{z}_r^{(m)}) \in \mathbb{R}J$

(i) Let $m' \in J$; corresponding to all vectors $\{\underline{z}_r^{(m'+m)}\}$ that can be produced, vectors $\{\hat{\underline{z}}_r^{(m)}\}$ can be produced from the initial values

$$\hat{\underline{z}}_{-1}^{(m)} = \underline{0} \quad (m \in J), \quad \hat{\underline{z}}_0^{(m)} = \underline{z}_{m'+m} \quad (m \in J)$$

by use of relationships similar to (), and for these corresponding vectors

$$\hat{\underline{z}}_r^{(m)} = \underline{z}_r^{(m'+m)}$$

(ii) Let \underline{g} be a vector and b a ^{finite} non-zero complex number, corresponding

to the vectors $\{\underline{z}_r^{(m)}\}$ that can be produced, vectors $\{\underline{z}_r^{(m)}\}$ can be produced from the initial values

$$\underline{z}_{-1}^{(m)} = 0 \quad (m \rightarrow \infty), \quad \underline{z}_0^{(m)} = a + b S_m \quad (m \rightarrow \infty)$$

by use of relationships similar to (), and for these corresponding vectors

$$\underline{z}_{2r}^{(m)} = a + b \underline{z}_{2r}^{(m)} \quad \underline{z}_{2r+1}^{(m)} = b^{-1} \underline{z}_{2r}^{(m)}$$

(ii) Let $\underline{z}_r^{(m)}$ be a sequence of vectors; corresponding to the vectors $\{\underline{z}_r^{(m)}\}$

that can be produced, some number vectors $\{\underline{z}_r^{(m)}\}$ can be produced from initial values similar to () by use of relationships analogous to () and, for these corresponding vectors, formulae analogous to () hold

(iv) Let g be a finite nonzero complex number; corresponding to the vectors $\{\underline{z}_r^{(m)}\}$ that can be produced, some number vectors $\{\underline{z}_r^{(m)}\}$ can be produced from initial values similar to () by use of the relationships

$$\underline{z}_{r+1}^{(m)} = \underline{z}_r^{(m)} + g (\underline{z}_r^{(m)} - \underline{z}_{r-1}^{(m)})^{-1}$$

and for these corresponding vectors $\underline{z}_{2r}^{(m)} = \underline{z}_{2r}^{(m)}, \underline{z}_{2r+1}^{(m)} = g \underline{z}_{2r}^{(m)}$.

(v) The vectors $\{\underline{z}_r^{(m)}\}$ that can be produced satisfy relationships analogous to () and (), and if the vectors concerned are also nonzero,

analogous similar formulae in which each vector \underline{z}_r is replaced by its inverse also hold; similar analogous results also hold for vectors of the form $\{\underline{z}_{2r}^{(m)}\}$

Proof. These results may either be proved anew by induction or treated, with the aid of Theorem..., as corollaries to Theorem.

Theorem. Let s_p ($p=J$) be vectors and $f(z) = \sum_{p=0}^{\infty} f_p z^p$ where $f_p = s_{p+1} - s_p$ ($p=J$).

(i) If vectors $\xi_r^{(m)}$ ($r=J, m=J$) can be constructed from the initial values

$$\xi_{-1}^{(m)} = 0 \quad (m=J_1), \quad \xi_0^{(m)} = s_m$$

by means of the relationships

$$\xi_{r+1}^{(m)} = \xi_{r-1}^{(mn)} + (\xi_r^{(mn)} - \xi_r^{(m)})$$

for $r, m=J$, then the series $f(z)$ is, in the sense of Definition..., semi-normal.

(ii) If vectors $\xi_r^{(m)}$ ($r=J, \hat{r}=r \div 2, m=J_{-\hat{r}}$) can be constructed from the

initial values () in conjunction with $\xi_{2m}^{(-m)} = 0$ ($m=J_1$) by means of

relationships () for $r=J, \hat{r}=r \div 2, m=J_{-\hat{r}}$, then the series $f(z)$ is, in the sense of Definition..., normal.

Proof. The above results may be obtained by applying the isomorphism of Def. ... investigated in Theorem... to the results of Theorem....

Theorem. (i) If, for some $r' \in J_1$, vectors $\xi_r^{(m)}$ ($r=J_0, m=J$) can be produced from the initial values () by use of relationships

() with $r = J_0^{2l-1}$, $m = J$, and the vectors $\{S_m\}$ satisfy a recursion of the form

$$\sum_{j=0}^{r'} c_j S_{m+j} = \eta \quad (m = J)$$

in which the $\{c_j\}$ are real numbers and η a vector, and no other recursion of similar form in which r' can be replaced by a smaller integer, and ~~the~~ $c = \sum_{j=0}^{r'} c_j \neq 0$, then

$$\xi_{2r'}^{(m)} = d^{-1} \eta; \quad (m = J)$$

conversely, if, for some $r' \in J$, vectors $\xi_{2r'}^{(m)}$ ($r = J_0^{2r'-1}$, $m = J$) can be produced from the initial values () by use of relationships () with $r = J_0^{2r'-2}$, $m = J$ and $\xi_{2r'}^{(m)} = \eta$ ($m = J$), then the vectors $\{S_m\}$ satisfy an irreducible recursion of the form () with $\sum_{j=0}^{r'} c_j \neq 0$.

(ii) If, for some $r' \in J$, vectors $\xi_{2r'}^{(m)}$ ($r = J_0^{2r'-1}$, $m = J$) can be produced from the initial values () by use of relationships () with $r = J_0^{2r'-2}$, $m = J$ and the vectors $\{S_m\}$ satisfy ~~an irreducible recursion~~ a recursion of the form () as described, and now ~~to~~ $\sum_{j=0}^{r'} c_j = 0$, then

$$\xi_{2r'-1}^{(m)} = \left\{ \sum_{j=1}^{r'} v_j c_j \right\} \eta^{-1}; \quad (m = J)$$

conversely, if for some $r' \in J$, vectors $\xi_{2r'-1}^{(m)}$ ($r = J_0^{2r'-1}$, $m = J$) can be produced from the initial values () by use of relationships () with $r = J_0^{2r'-2}$, $m = J$ and $\xi_{2r'-1}^{(m)} = \eta^{-1}$ ($m = J$) then the vectors $\{S_m\}$

satisfy an irreducible recursion of the form () with $\sum_{i=0}^{\infty} c_i = 0$.

Proof. Again, the above results may be deduced from Theorems ... and ...

~~We remark that~~ Results analogous to those of Theorem ... also hold with regard to the transformation of vector sequences; it has only to be remarked that the coefficients $\{c_i\}, \{c_{ij}^{(k)}\}, \dots$ are now real numbers.

Further notes

Not. 1 R prescribed ring

Def 1 (i) to R append \mathbb{Z} (ii) polynomials $\mathbb{P}\{R\}$ (iii) inverse of polynomials $\mathbb{I}\mathbb{P}\{R\}$
(iv) rational fns. $R\{R\}$. addn. mult (v) addn subtn mult over R ~~$R\{R\}$~~ $\mathbb{I}\mathbb{P}\{R\}$
defined by embedding in \mathbb{R} .

Not. 2. Expressions derived by embedding in \mathbb{R} written as if they were elements of \mathbb{R} ~~$R\{R\}$~~
 $\mathbb{I}\mathbb{P}\{R\}$

Def. Euclid's alg wrt. $T = \tau_1, \tau_2, \dots$ applied to rat. fn. defined via coefficients
termination. T -regular
rat fns sat. recursion

Th. If Euclid's alg. can be applied to rat fn. it terminates
consideration of case with $\{1, 1, \dots\}$ $\{2, 2, \dots\}$

Noncommutative continued fractions

Def ...'. $Q_{r,s} = B_r$ $Q_{r,s} = B_{r-s} - Q_{r,s-1}^{-1} A_{r-s}$ ($s \geq 1$)

Th If convrts. exist in sense of Def ' also may be computed by $D_{r-1} = I$
 $D_0 = \dots$ two term recursion for $\{D_i\} \{N_i\}$

Def. Convergts defined by $D_r^{-1} N_r$ of Th ↑

Th. even part, odd part

Th. $A_1 \in \mathbb{C}\{\mathbb{R}\}$ $B_1 \in \mathbb{R}_T$ $b_0 + \frac{a_1}{b_1 t} \dots \equiv b_0 + \frac{f_1^{-1} f_{-1} a_1}{I_+} \frac{f_2^{-1} f_0 a_2}{I_+} \dots$

Def. ~~Post~~ Convgts of post cont fract def. by trin recursion and $\tilde{C}_r = \tilde{N}_r \tilde{D}_r^{-1}$

Th. if $A_1 \in \mathbb{C}\{\mathbb{R}\}$ pre $[B_1 + \frac{A_2}{B_2 t} \dots] \equiv$ post $[\dots]$

For conciseness have presented c.f. then in terms of \mathbb{R} . Also poss. in terms of $\mathbb{R}\{\mathbb{R}\}$.

Th. $\dots \{B_i(z)\}$ polynomials det from Euclid's alg. w.r.t. τ_1, τ_2, \dots from $D(z)^{-1} N(z)$

then $D(z)^{-1} N(z) = \text{pre} \left[\frac{I}{B_1(z)t} \frac{z^{\tau_1}}{B_2(z)t} \dots \frac{z^{\tau_{r-1}}}{B_r(z)} \right]$

const $\tilde{N}(z) \tilde{D}(z)^{-1} \equiv \text{post} [\dots]$ exists in case of def.

or $D(z)^{-1} N(z) \equiv \tilde{N}(z) \tilde{D}(z)^{-1}$

Euclid's algorithm for formal power series with coefficients over a ring

Def. formal power series as class: addn. subtr mult. system $\mathbb{F}\{\mathbb{R}\}$

$\mathbb{F}\{\mathbb{R}\}$ is ring

Th. if $\alpha_0 \in \mathbb{R}_T$, $p\{\alpha; \alpha_0\}$ has two sided inverse

Def. Euclid's alg. w.r.t. τ_1, τ_2, \dots applied to $f(z) = p\{0; \beta_0^{(0)}\}$

to give $B_r(z)$. termination. τ -regular

Def. $D(z)^{-1} N(z) \sim f(z)$

Th. $\{B_i(z)\}$ from Euclid's alg wrt. T to rat fr. $D(z)^{-1}N(z) = R(z)$. $R(z) \sim f(z)$
 then $f(z)$ T -reg and Euc alg. also terminates and produces same p/q's

Th. $\{B_i(z)\}$ from Euc. alg wrt. from $\sum_i f_i z^i$. Set $C_i(z) = \text{pre} \left[\frac{1}{B_i(z)} \frac{z^i}{B_i(z)} \right]$

$C_i(z) \sim \sum_j f_{i,j}^{(m)} z^j$ where $f_{i,j}^{(m)} = f_j \lambda = \prod_0^{m-1} (c_{i,1} z + \dots + c_{i,m-1} z^{m-1})$

Def \mathbb{C} -regular A -regular ~~pre~~ ^{sequence $\{f_i\}$} deg. \mathbb{A} -reg. deg A reg

Th. If $\{f_i\}$ \mathbb{C} reg. [deg \mathbb{C} -reg], both it and $\{f_{i,r}\}$ A reg.

if pre \mathbb{C} -fact from $\sum_i f_i z^i$ is $\frac{1}{b_{1,r}} \frac{z}{b_{2,r}} \dots \frac{z}{b_{r,r}} \dots$ pre A fact is $\frac{1}{\hat{b}_{1,0} + \hat{b}_{1,1} z} \frac{z^2}{\dots}$

latter is even part of former:

if pre A -fact from $p\{0; b_{r,i}\}$ is $\frac{1}{\hat{b}_{1,0} + \hat{b}_{1,1} z} \frac{z^2}{I_r}$ then $\hat{b}_{1,0} + \hat{b}_{1,1} z + \dots$

odd part of $\textcircled{2}$

\mathbb{C} -fact can be thrown into form $\text{pre} \left[\frac{a_1}{I_r} \frac{a_2 z}{I_r} \dots \frac{a_{r-1} z^{r-1}}{I_r} \right]$
 A - $\text{pre} \left[\frac{v_1}{I_{r+1,0} z} \frac{v_2 z^2}{I_{r+1,0} z} \dots \right]$

Th. $\{f_i\}$ A regular v_r and w_r by orthogonalization. also deg. A reg.

Systems of continued fractions

Let $\{f_{i,r}\}$ ($m = \infty$) \mathbb{C} -regular $\{f_i\}$ semi normal. If these seqs. all deg. \mathbb{C} -regular, $d_0 = I$, $d_1 \in \mathbb{R}$ ($\mathbb{R} = \mathbb{J}_1'$) exists such that

$\sum_{i=0}^n d_i f_{n-i} z^{-1} = 0$, no subseq $\{f_{n-i}\}$ $n \geq 0$ into sim recursion

$\sum_{i=0}^n d_i f_{n-i} z^{-1} = 0$ ($d_0 = 1$) $n \geq 0$ $\{f_i\}$ deg. semi-normal, $\{f_i\} \in \mathcal{L}_r$

Th. $\{f_i\} \in \mathcal{L}_r$ iff $\sum_{i=0}^n f_i z^i$ gen by $\{\sum_{i=0}^{r-1} d_i z^i\}^{-1} \{\sum_{i=0}^{r-1} n_i z^i\}$ $r \leq r'$

If $\{f_i\} \in \mathcal{L}_r$, each η series $\sum_{i=0}^n f_{n-i} z^{-1}$ gen by rat (n. η) form $\{\sum_{i=0}^{r-1} d_i z^i\}^{-1} \sum_{i=0}^{r-1} n_i z^i$

when $n_i^{(m)} \neq 0$

If $\sum_{i=0}^n f_i z^i$ gen by $\{\sum_{i=0}^{r-1} \tilde{d}_i z^i\}^{-1} \{\sum_{i=0}^{r-1} n_i z^i\}$ when $n_i = 1 - i > 0$

then numbers $\{f_{n-i}\}$ sat recursion $\sum_{i=0}^n \tilde{d}_i f_{n-i} z^{-1} = 0$ ($n \geq 0$)

Th. $\sum_{i=0}^n f_i z^i$ semi-normal

has rat. without explanation $\mathcal{C} \{ \sum_{i=0}^n f_{n-i} z^{-1} \} = \text{pre} \left[\frac{a_1^{(m)}}{I_r} \frac{a_2^{(m)}}{I_r} \dots \right]$ ($n \geq 0$) $\Sigma_0^{(m)} = S_m(z) = \sum_{i=0}^{m-1} f_i z^i$

$$\Sigma_{2r}^{(m)} = \text{pre } \mathcal{C}_{2r} [\mathcal{C} \{ \sum_{i=0}^n f_{n-i} z^{-1} \}]^{\otimes}$$

then

$$\Sigma_{2r}^{(m)} = S_m(z) + z^m \text{pre } \mathcal{C}_{2r-1} [\mathcal{C} \{ f_m(z) \}]$$

Setting $\text{pre } \mathcal{C}_{2r} [\mathcal{C} \{ f_m(z) \}] = \mathcal{D}_r^{(m)}(z)^{-1} N_r^{(m)}(z)$ denotes η successive convergents η

$\text{pre } \mathcal{C}_r [\mathcal{C} \{ f_m(z) \}]$ are $\mathcal{D}_0^{(m)}(z) \mathcal{D}_1^{(m)}(z) \mathcal{D}_2^{(m)}(z) \mathcal{D}_3^{(m)}(z) \dots \oplus$

$\{f_i\} \in \mathcal{L}_r$, rels \oplus hold for $r \geq r'$ $n \geq 0$, seq \oplus terminates with $\mathcal{D}_r^{(m)}(z)$

if $\sum_{i=0}^n f_i z^i$ gen by ~~rat~~ rat (n. η), $\mathcal{D}_r^{(m)}(z) = \mathcal{D}(z)$. ($n \geq 0$)

Th. numbers $\{a_{2i+1}^{(m)}\}$ gen. by qd-alg iff $\sum_{i=0}^n f_i z^i$ semi-normal

for $r \geq r'$ $a_{2i+1}^{(m)} = 0$ if $\{f_i\} \in \mathcal{L}_r$

both cases $\mathcal{C} \{ \sum_{i=0}^n f_{n-i} z^{-1} \} = \text{pre} \left[\frac{a_1^{(m)}}{I_r} \frac{a_2^{(m)}}{I_r} \dots \frac{a_r^{(m)}}{I_r} \right]$

Th. $\sum_i f_i z^i$ semi normal, $\{a_{r,i}^{(m)}\}$ numbers of Th., denoms of succ convts $C[\text{pre } \{b_i\} \{d_i\}]$

$$\text{Let } (\cdot): D_0^{(m)}(z) = I \quad D_1^{(m)}(z) = D_0^{(m)}(z) + a_{1,0}^{(m)} + D_{1,1}^{(m)}(z), \dots \quad (\cdot)$$

$\{f_i\} \in L_{r'}$: $D_i^{(m)}(z)$ ($r = J_1', m = J$) const by $T^{i+1} \eta(\cdot)$; $D(z)^{-1} N(z)$ rat fun
of Th., $D_i^{(m)}(z) = D(z)$ ($m = J$)

Th. Rat fun $E_i^{(m)}(z)$ ($r = J_1, m = J$) by s-alg to $E_0^{(m)} = \sum_0^{n-1} F_\nu z^\nu$ iff $\sum_i f_i z^i$ semi norm

Fun $E_i^{(m)}(z)$ ($r = J_1', m = J$) det. fun \uparrow and $E_{r'}^{(m)}(z) = D(z)^{-1} N(z)$ where $D(z) = \sum_0^r d_\nu z^\nu$

$\sum_0^{r-1} n_\nu z^\nu \in \mathcal{P}\{R\}$ iff $\sum_0^{n-1} f_i z^i$ sat. lin. recursion $\sum_0^{r-1} d_{r-\nu} z^{-\nu} S_{m,r}(z) =$

$\sum_0^{r-1} z^{-\nu} n_{r-\nu}$ no recursion of similar form in Sh. 1' repl. by smaller integer
iff $\{f_i\} \in L_{r'}$.

In All hrs $\{E_{r'}^{(m)}(z)\} \sum_{r'}^{(m)} E_{r'}^{(m)}(z) = \sum_1 F_{r',\nu}^{(m)} z^\nu$ $F_{r',\nu} = F_\nu$ ($\nu = J_0^{m+r-1}$); also

$F_{r',m+n}^{(m)} \neq f_{m+n}$ except when $f_\nu \in L_{r'}$ $r = r'$ when $F_{r',\nu}^{(m)} = F_\nu$ ($\nu = J$)

Th. $\sum_i f_i z^i$ semi normal $E_i^{(m)}(z)$ as above $\{E_i^{(m)}(z)\}$ ($r = J_1, m = J$) ~~fun~~ can be

prod from $\{E_0^{(m)}(z) = \sum_0^m E_0^{(p)}(z)\}$ and $\{E_{r'}^{(m)}(z) = \dots\}$

If in addn. $E_{r',m}^{(m)}(z)$ invertible then $\{E_i^{(m)}(z)\}$ can be prod from $\{E_0^{(m)} = f_m z^m\}$
and $\{E_{r'}^{(m)}(z) = \dots\}$

If, in above, condition that $\sum_i f_i z^i$ be semi-normal replaced by $\{f_i\} \in L_{r'}$
then $\{E_i^{(m)}(z)\}$ ($r = J_1^{r'}$, $m = J$) const as above and if $D(z)^{-1} N(z)$ is rat fun Th

$$\{E_{r'+i}^{(m)}(z) = N(z)^{-1} D(z) \quad (m \geq 0)$$

Also for $\{E_i^{(m)}(z)\}$ and $\{E_{r'+i}^{(m)}(z) = 0$

The Padé quotient

Th. $\sum \tilde{f}_r z^r$ series inverse to $\sum f_r z^r$, $\tilde{D}(z) \tilde{N}(z)^{-1} \sim \sum \tilde{f}_{r,s}(z)$ for which for some $r \in \mathbb{J}$, $\tilde{f}_{r,s} = \tilde{f}_r$ ($r = \mathbb{J}_0^{-1}$) then $\tilde{N}(z)^{-1} \tilde{D}(z) = \sum \hat{f}_{r,s} z^s$ $\hat{f}_{r,s} = f_r$ ($r = \mathbb{J}_0^{-1}$)

Def $\sum f_r z^r$ $f_r \in \mathbb{R}$ ($r \in \mathbb{J}$) prescribed Assume ^{for fixed $r, m \in \mathbb{J}$} orthog process described in Th. suff to det. $v_r^{(m)}, w_r^{(m)}$ $r \in \mathbb{J}_0^v$ from $f_{m,r}$ ($r \in \mathbb{J}_0^v$). Padé quot

$P_{r,m,r-1}(z)$ said to exist given by $P_{r,m,r-1}(z) = \sum_{i=0}^{m-1} f_i z^i + \dots$]_r

Assume for fixed $r, m \in \mathbb{J}$ ($r, m > 0$) orthog. process suff to det $\tilde{v}_r^{(m)}, \tilde{w}_r^{(m)}$ ($r \in \mathbb{J}_0^v$) from $\tilde{f}_{m,r}$ ($r \in \mathbb{J}_0^v$) of series $\sum \tilde{f}_r z^r$. Padé quot $P_{r,m,r-1}(z)$ derived from $\sum \tilde{f}_r z^r$ said to exist given by Y^{-1}

Th. $\sum f_r z^r$ prescribed. For $n, i \in \mathbb{J}$, $j \in \mathbb{J}_{i-1}$ let $P_{i,j,i}^{(n)}(z)$ from $\sum f_{i+r} z^r$ exist, then $P_{i,j,m}(z)$ from $\sum f_r z^r$ also exists and $P_{i,j,m}(z) = \dots$

Assume $\sum \tilde{f}_r z^r$ inverse to $\sum f_r z^r$ exists, $\tilde{P}_{i,j,i}(z)$ from $\sum \tilde{f}_r z^r$ also exists then $P_{i,i}(z)$ from $\sum f_r z^r$ also exists and $P_{i,i}(z) = P_{i,i}(z)^{-1}$

Def. $\sum f_r z^r$ normal if both it and inverse semi normal

Th. Rat frac $E_i^{(m)}(z)$ ($r \in \mathbb{J}_1, i = r-2, m \in \mathbb{J}_{-i}$) may be det from $E_{-i}^{(m)}(z) = E_{2m}^{(-m)}(z) = 0$ ($m \in \mathbb{J}_1$) $E_0^{(m)}(z) = \sum_{i=0}^{m-1} f_i z^i$ ($m \in \mathbb{J}$) by... if $\sum f_r z^r$ semi normal. We then have $E_{2r}^{(m)}(z) = P_{r,m,r-1}(z)$ ($r \in \mathbb{J}, m \in \mathbb{J}_{-r}$).

Theorem. Assuming Padé quotients $\{P_{i-1,j}(z) - P_{i,j}(z)\}^{-1} + \dots, \{P_{i-1,j}(z)^{-1} - P_{i,j}(z)^{-1}\}$

Vector continued fractions

Arithmetic operations upon vectors

Def. vector addn subn scalar mult inverse

$$\text{Gen inverse: } \underset{\sim}{z}^{-1} = (\underset{\sim}{z}^+)^T$$

Arithmetic operations upon formal power series with vector valued ~~power~~ coefficients

Definition. series addn subn scalar mult inverse

Gen inverse

Vector valued rational functions

Definition. $f(z)$ n^{th} component rat. fn of z . Inverse

Definition $f(z)$ gen by $f(z)$

Th. $f_1(z) \sim f_1(z)$ $f_2(z) \sim f_2(z)$ $f_1(z) \pm f_2(z) \sim \dots \pm f_1(z)^{-1} \sim f_1(z)^{-1}$

An isomorphism and its uses

Def. $\Gamma_{\nu}^{(\nu)}$... $\underset{\sim}{z} = \sum_{\nu} (\alpha_{\nu} \Gamma_{\nu}^{(\nu)} + \gamma_{\nu} \Gamma_{\nu}^{(\nu)} \Gamma_{\nu+1}^{(\nu)})$. $c = \alpha \text{ rips? wrong}$

Th. Vector matrix isomorphism preserved during addn, subn, scalar mult and inversion

Theorem $f(z) = \sum f_j z^j$ f_0 nonzero $f(z)^{-1} = \sum \hat{f}_j z^j$ $f \leftrightarrow \hat{f}$
 $(\sum f_j z^j)^{-1} = \sum \hat{f}_j z^j, \hat{f}_j \leftrightarrow \hat{f}_j$

Definition $\forall f(z)$ generates $\sum f_j z^j, f(z) \in \mathbb{R}\{\mathbb{R}\}$ generates $\sum b_j z^j$
 $f \leftrightarrow \hat{f} (j \geq 1)$. two rat fns correspond. we write $f(z) \leftrightarrow \hat{f}(z)$

Th. $f(z) =$ vect. valued rat fn. express components with common denom real coeffs
 $f(z) = \left\{ \sum_0^r d_j z^j \right\}^{-1} \sum_0^p n_j z^j. f(z) \leftrightarrow \left\{ \sum_0^r (d_j I) z^j \right\}^{-1} \sum_0^p n_j z^j$

Euclid's algorithm for formal power series with vector valued coefficients

Definition. Euclid's alg w/ T applied to $f(z) = \sum f_j z^j$ is process of det. $b_r(z) = \sum_0^{r-1} b_{r,j} z^j \dots$ termination. T -regular

Th $f(z) = \sum f_j z^j$ $f \leftrightarrow \hat{f} \iff (j \geq 1) f(z) T$ reg. iff $\sum b_j z^j T$ -reg.
 $b_1(z) \leftrightarrow \hat{b}_1(z)$

Th. ~~if~~ If Euclid's alg w/ T can be applied to $f(z)$ gen by rat fn $f(z)$, it terminates. reconstruction of $f(z)$ from $\{b_r(z)\}$.

The vector q-d algorithm

Def. semi normal $\sum f_j z^j$

Th. \underline{aba} defined. ~~and~~ vect. q-d alg for semi-normal vect series

Th In $[I: \mathbb{C}_+]$ let $\{B_j\}$ be isomorphic to vectors, c.f. equivalent to
 the $[A_j: I+]$ then $A_j \geq J_2$ is a proof of pair of matrices isomorphic to
 vector vectors, even part odd part part name
 extension to orthogonalization, $q-d$ alg

The vector z -algorithm

Th. If $\{f_j\}$ sat $\sum_{j=0}^m c_j f_j \rightarrow 0$ ($m \geq 0$), then $\sum_j f_j z^j$ gen. by rat.

In $f(z)$. If $\sum_{j=0}^m c_j^{(m)}(z)$ ($r \in J_1, m \geq 0$) from $\sum_{j=0}^{m-1} f_j z^j$, then $\sum_{j=0}^m c_j^{(m)}(z) = f_j(z)$

Def. Let $\sum_j f_j z^j$ have coeffs finite dim. vectors, det. $b_{r,0}^{(m)} + b_{r,1}^{(m)} z$ ($r \in J_1$)

by Euclid proc: then Padé quot. of order $r, m-r-1$ derived from $f(z)$

exists and $P_{r,m-r-1}(z) = \sum \dots$

Similarly for det. of $P_{m-r-1,r}(z)$ from $f(z)$

Th. Let $i, j \in J$ If $p_{i,j}(z)$ exists, then $\sum_j f_j^{(i,j)} z^j$ then $f_j^{(i,j)} = f_j$ $j \in J_0^{(i,j)}$

if quot. of order $i+1, j+1$ also exists, then $f_{i+1, j+1}^{(i,j)} \neq f_{i,j}$

Def if both $f(z)$ & $f(z)^{-1}$ semi normal, $f(z)$ is normal

Th rat fun $\sum_{j=0}^m c_j^{(m)}(z)$ $r \in J, m \geq 0$ can be const from $\sum_{j=0}^{m-1} f_j z^j$ if $f(z)$

semi normal

Sequence to sequence transformations

Th. $S_p \in \mathbb{R} (p \in \mathbb{J}) \{z_v^{(m)}\}$ from $\{S_m\}$.

(i) $\hat{z}_v^{(m)}$ from S_m then $\hat{z}_v^{(m+1)} = \hat{z}_v^{(m)}$ (ii) $\hat{z}_v^{(m)}$ from $a + b S_m c$

(iii) $\hat{z}_{2M}^{(m)} = d_1 + \hat{z}_{2M-1}^{(m+1)} + (\hat{z}_1^{(m+1)} - \hat{z}_1^{(m)})^{-1}$ (iv) $\hat{z}_{2M}^{(m)} = \hat{z}_{2M-1}^{(m+1)} + g (\hat{z}_{2M}^{(m+1)} - \hat{z}_{2M}^{(m)})^{-1} h$

(v) recursion for $\{\hat{z}_{2M}^{(m)}\}$ from $\{\hat{z}_{2M}^{(m-1)}\} \in \{\hat{z}_{2M}^{(m)}\}$

Mention $\hat{z}_0^{(m)} = a + b x S_m v c$

Recursions involving values of rational functions

Def. Centre of \mathbb{R}

Describe z behaves as element of $\mathbb{C}(\mathbb{R})$

Th. $S_p \in \mathbb{R} (p \in \mathbb{J}) f(z) = \sum_{i=0}^{\infty} A_i z^i f_p = S_{2M-1} - S_V$

(i) if $\hat{z}_v^{(m)}$ from $\hat{z}_0^{(m)} = S_m$, $\sum_{i=0}^{\infty} A_i f(z)$ semi normal

(ii) if $\hat{z}_v^{(m)}$ from $\hat{z}_{2M}^{(m)} = 0$ also $f(z)$ normal

Th. $S_p \in \mathbb{R} (p \in \mathbb{J})$ sat. $\sum_{i=0}^{\infty} d_i S_{m+i} = H (m \in \mathbb{J}) \forall i \in \mathbb{J} d_i \in \mathbb{R} H \in \mathbb{R}$

no recursion sim. form with i' replaced by smaller integers; set $\sum_{i=0}^{\infty} d_i = D$

not pos for $D \in H$ both factors of zero

Th. (i) $i' \in \mathbb{J}$, numbers $\hat{z}_i^{(m)} (i = \mathbb{J}_0^{i'}, m \in \mathbb{J})$ from S_m sat \uparrow , then

with $\sum_{i=0}^{\infty} d_i \in \mathbb{R}_{\neq 0}$, then $\hat{z}_{2M}^{(m)} = D^{-1} H$. Conversely if this result holds

Sim sat. \uparrow with $\sum_{i=0}^{\infty} d_i \in \mathbb{R}_{\neq 0}$

(ii) if as in (i) with $\sum_{i=0}^{\infty} d_i = 0$, then $\hat{z}_{2M}^{(m)} = N^{-1} \sum_{i=1}^{\infty} d_i$, conversely...

Th. Assuming numbers defined, set $\phi_i^{(m)} = \sum_{j=0}^{i-1} \varepsilon_{2j+1}^{(m)} (\varepsilon_{2j+2}^{(m)} - \varepsilon_{2j}^{(m)}) \psi_j^{(m)} = \dots$
 then $\phi_{i'}^{(m)} - \phi_{i''}^{(m)} = \dots$. If for some $i' \in \mathcal{J}$ numbers $\varepsilon_{i'}^{(m)}$ for $m \in \mathbb{N}$ sat. ↑
 with $\sum_{i=0}^{i'} d_i \in \mathbb{R}_i^+$, then $\phi_{i'}^{(m)}$ $\psi_{i'}^{(m)}$ const.

Th. $S_m \in \mathbb{R}$, $S_m^{(i)}$ by sum. + diff. relationships $\varepsilon_{2i}^{(m)} = \dots$
 numbers $\sum_{i=0}^h d_i$, $S_m^{(i)} = \mathbb{N}$, then description of sequences $\{\varepsilon_{i'}^{(m)}\}$

Recursions involving vectors

Th. Results (i)-(v) in analogy with Th...

Th. $S_j \in (\mathbb{R}^+)^j$ be vectors $f(z) = \sum_{j=0}^{\infty} f_j z^j$, $f_j = S_{2j+1} - S_{2j}$ ($j=0$)

(i) if $\{\varepsilon_{i'}^{(m)}\}$ from $\{S_m\}$ then $\sum_{j=0}^{\infty} f_j z^j$ semi normal

(ii) " " " " in conj. with $\sum_{i=0}^{i'} \varepsilon_{2i}^{(m)} = 0$ $f(z)$ normal

Th. if $\varepsilon_{i'}^{(m)}$ ($i \in \mathcal{J}_0^{\mathbb{R}^+}$, $m \in \mathbb{N}$) and $\{S_j\}$ sat $\sum_{i=0}^{\infty} c_i S_{2i} = \mathbb{N}$, $\sum_{i=0}^{\infty} c_i = c \neq 0$

then $\varepsilon_{2i}^{(m)} = d + n$, conversely ...

(ii) if as in (i) by $c = 0$ then $\varepsilon_{2i-1}^{(m)} = \left\{ \sum_{j=0}^{i-1} c_j \right\} i^{m-1}$ ($m \in \mathbb{N}$), conversely.

Results analogous to Th with $\{c_{i'}^{(i)}\}$ real numbers.

The q-d algorithm is a process for determining the coefficients in the continued fractions $\frac{c_m}{1 - \frac{q_1^{(m)} z}{1 - \frac{e_1^{(m)}}{1 - \frac{q_2^{(m)} z}{1 - \frac{e_2^{(m)}}{1 - \dots}}}}$ corresponding to the successive series $\sum_{j=0}^{\infty} c_{m+j} z^j$ for $m=0, 1, \dots$.

From the initial values $e_0^{(m)} = 0$ ($m=1, 2, \dots$), $q_1^{(m)} = c_{m+1}/c_m$ ($m=0, 1, \dots$)

the coefficients are constructed recursively by use of the relationships (1a) $e_r^{(m)} = e_{r-1}^{(m+1)} + q_r^{(m+1)} - q_r^{(m)}$, (1b) $q_{r+1}^{(m)} = \frac{e_r^{(m+1)} e_r^{(m+1)}}{e_r^{(m)}}$ with $r=1, 2, \dots; m=0, 1, \dots$. The numbers $q_r^{(m)}, e_r^{(m)}$ may be placed in a two-dimensional array in which r corresponds to a column and m to a forward diagonal;

relationships (1a, b) then concern numbers occurring at the vertices of lozenges in this array. Letting $H_r^{(m)}$ be the r th order Hankel determinant with element c_{m+i+j} in the i th row and j th column ($i, j=1, \dots, r$) with $H_0^{(m)} = 1$ (a similar notation and convention are used below)

(2a) $q_r^{(m)} = H_{r-1}^{(m)} H_r^{(m+1)} / (H_{r-1}^{(m+1)} H_r^{(m)})$, (2b) $e_r^{(m)} = H_{r-1}^{(m+1)} H_{r+1}^{(m)} / (H_r^{(m)} H_r^{(m+1)})$

($r=1, 2, \dots; m=0, 1, \dots$). The determinants $H_r^{(m)}$ satisfy the relationship

(3) $H_r^{(m)} H_r^{(m+2)} - (H_r^{(m+1)})^2 = H_{r-1}^{(m+2)} H_{r+1}^{(m)}$, which can be used to

verify formulae (1a; 2a, b). Formulae (1a, b) were obtained in essence by Stieltjes, but appear first to have been

stated ~~by~~ explicitly on p. 382 of the book: Wall H.S.,
 Analytic ²⁷ theory of continued fractions, Van Nostrand, New
 York (1948).

When $c_m = \sum_{r=1}^n a_r (\lambda_r)^m$ the $\{a_r, \lambda_r\}$ being complex
 numbers with $|\lambda_r| > |\lambda_{r+1}| > 0$ ($r=1, \dots, n-1$) and construction of
 the numbers $q_r^{(m)}$ ($r=1, \dots, n; m=0, 1, \dots$) does not break down
 (i.e. $H_r^{(m)} \neq 0$ ($r=1, \dots, n; m=0, 1, \dots$)), $H_r^{(m)} \sim B(r) \Delta(r)^m$ for large
 m , where $B(r)$ is independent of m , and $\Delta(r) = \prod_{z=1}^r \lambda_z$,
 and in consequence $q_r^{(m)} \sim \lambda_r$ ($r=1, \dots, n$), $e_r^{(m)} = O((\lambda_{r+1}/\lambda_r)^m)$
 ($r=1, \dots, n-1$) for large m . In particular, if c_m is the k^{th}
 component $\underset{(4)}{v} \underset{(4)}{g}_m(k) v^{\text{tr}}$ (k fixed in the range $1 \leq k \leq n$) of the vector
 $\underset{(4)}{g}_m$ produced by use of the recursion $\underset{(5)}{g}_{m+1} = A \underset{(5)}{g}_m$ ($m=0, 1, \dots$),
 A being an suitable $n \times n$ matrix with eigenvalues $\{\lambda_r\}$
 as above and $\underset{(4)}{g}_0$ is suitably chosen, the $\{\lambda_r\}$ may be
 determined approximately by use of the above result.

This aspect of q -d algorithm theory was developed by
 H. Rutishauser (Der Quotienten-Differenzen-Algorithmus,
 Birkhäuser, Basel (1957)).

In the paper under review, the author begins by presenting a summary of q -d algorithm theory in which superscripts and suffices are consistently interchanged in the above presentation (which is that adopted by Rutishauser and retained by subsequent writers). He then gives a three-dimensional version of the q -d algorithm for the determination of the eigenvalues $\{\lambda_r\}$ of A . For $r=1, \dots, n$;

$k=1, \dots, n-r+1$; $m=0, 1, \dots$, let $G_r^{(m)}(k)$ be the r th order determinant with element $g_{m+1}^{(k+j-1)}$ in the i th row and j th column ($i, j=1, \dots, r$), with $G_0^{(m)}(k) = 1$, so that

$$(b) G_r^{(m)}(k) G_r^{(m+1)}(k+1) - G_r^{(m+1)}(k) G_{r+1}^{(m)}(k+1) = G_{r-1}^{(m+1)}(k+1) G_{r+1}^{(m)}(k)$$

($r=1, \dots, n-1$; $k=1, \dots, n-r$; $m=0, 1, \dots$) and set (7a) $q_r^{(m)}(k) =$

$$G_{r-1}^{(m)}(k) G_{r+1}^{(m+1)}(k) / \{G_{r-1}^{(m+1)}(k+1) G_r^{(m)}(k)\}, (7b) e_r^{(m)}(k) =$$

$$G_{r-1}^{(m)}(k+1) G_{r+1}^{(m)}(k) / \{G_r^{(m)}(k) G_r^{(m)}(k+1)\}. \text{ Then, with } e_0^{(m)}(k) = 0$$

($m=1, 2, \dots$), $q_1^{(m)}(k) = g_{m+1}^{(k)} / g_m^{(k)}$ ($m=0, 1, \dots$) both for $k=1, \dots, n$,

$$(7a) e_r^{(m)}(k) = q_{r+1}^{(m)}(k+1) + e_{r-1}^{(m+1)}(k+1) - e_r^{(m)}(k), (7b) q_{r+1}^{(m)}(k) =$$

$$q_r^{(m)}(k) e_r^{(m+1)}(k) / e_r^{(m)}(k) \text{ with } r=1, \dots, n-1; k=1, \dots, n-r; m=0, 1, \dots$$

The numbers $q_r^{(m)}(k), e_r^{(m)}(k)$ may be placed in n superposed

q -e arrays, the construction of the k^{th} array extending as far as
 the column containing $q_{n-k+1}^{(m)}(k)$ ($k=1, \dots, n$). Subject to the
 absence of breakdown, as described above, $G_r^{(m)}(k) \sim B(r, k) \lambda_r^m$
 for large m , where $B(r, k)$ is independent of m , and
 $q_r^{(m)}(k) \sim \lambda_r$ ($k=1, \dots, n; r=1, \dots, n-k+1$), $e_r^{(m)}(k) = O((\lambda_{r+1}/\lambda_r)^m)$
 ($k=1, \dots, n-1; r=1, \dots, n-k$) for increasing m . It is shown that
 if A is totally positive and the initial vector g_0 is suitably
 chosen, the above algorithm does not break down.

The author gives a second scheme of relationships which
 are incorrect. (The functions Q_r, E_r defined on p. 60 are the
 same as q_r, e_r defined on p. 59; the reviewer suggests that
 either a misprint occurs in the definitions of Q_r, E_r or the
 author's proof (not provided) of the relationships involving
 these functions is faulty.)

Prompted by a referee's suggestion, the author remarks
 that the above algorithms ^{of formulae (8a, b) is} are clearly less efficient for the
 determination of eigenvalues than the original q -d algorithm,
 in the sense that the ~~new~~ former is three-dimensional
 whereas the latter is two-dimensional. The reviewer submits
 that this is not so clear. Use of all components of the

vectors g_m permits the construction of $q_n^{(0)}$, an estimate²⁺ of λ_n , from the vectors g_m ($m=0, \dots, n$). Application of the original q -d algorithm to a fixed component ^{$g_m^{(k)}$} of the vectors g_m requires the use of g_m ($m=0, \dots, 2n$) for the construction of the first estimate $q_n^{(0)}$ of λ_n . In cases in which A is a complicated linear operator but the k th component $g_m^{(k)}$ of g_m is approximated very closely by ~~an exp~~ formula (4), the additional computational complexity of the new version ~~of the new version~~ of the q -d algorithm may be far outweighed by the requirement, on the part of the original q -d algorithm, that n additional vectors g_m must be evaluated to obtain the first estimate of λ_n . Naturally, practical use requires the ~~evaluation~~ evaluation of more than one estimate of λ_n , but the same considerations apply.

It may also be remarked that a special choice of the initial vector sequence g_m ($m=0, 1, \dots$) leads to a two-dimensional non-lozenge form of the q -d algorithm:

taking $g_m^{(k)} = c_{m+k-1}$ ($m=0, 1, \dots; k=1, \dots, n$), $G_r^{(m)}$ becomes $H_r^{(m)k-1}$ above, relationship (6) becomes (3), and, setting

$$\hat{q}_r^{(m)} = H_{r-1}^{(m+1)} H_r^{(m)} / (H_{r-1}^{(m+2)} H_r^{(m)}) \text{ with } e_r^{(m)} \text{ as defined by (2b),}$$

and $e_0^{(m)} = 0$ ($m=1, 2, \dots$), $\hat{q}_1^{(m)} = c_{m+1}/c_m$ ($m=0, 1, \dots$), relationships

$$(9a, b) \text{ become } e_r^{(m)} = \hat{q}_r^{(m+1)} + e_{r-1}^{(m+2)} - \hat{q}_r^{(m)}, \quad \hat{q}_{r+1}^{(m)} = \hat{q}_r^{(m)} e_r^{(m+1)} / e_r^{(m)}.$$

Under conditions analogous to those holding for the q -d algorithm, $\hat{q}_r^{(m)} \sim \lambda_r$ ($r=1, \dots, n$) and $e_r^{(m)} = O((\lambda_{r+1}/\lambda_r)^m)$ ($r=1, \dots, n-1$) for large m .

Furthermore the algorithm of relationships (10a, b) has a confluent version, obtained by setting $g_m(k) = \gamma_m(t_k)$,

$$e_r^{(m)}(k) = E_r^{(m)}(t_k) \Delta t, \quad q_r^{(m)}(k) = Q_r^{(m)}(t_k), \text{ where } t_k = t + (k-1)\Delta t,$$

and letting Δt tend to zero. Letting $\Gamma_r^{(m)}(t)$ be the r th order determinant with element $\Delta^{j+i} \gamma_{m+i-1}(t)$ in the i th row

and j th column ($i, j=1, \dots, r$) where $\Delta \equiv d/dt$, so that

$$P_r^{(m)}(t) \Delta \Gamma_r^{(m+1)}(t) - \Gamma_r^{(m+1)}(t) \Delta P_r^{(m)}(t) = \Gamma_{r-1}^{(m+1)}(t) \Gamma_{r+1}^{(m)}(t)$$

and setting $Q_r^{(m)}(t) = \Gamma_{r-1}^{(m)}(t) \Gamma_r^{(m+1)}(t) / \{\Gamma_{r-1}^{(m+1)}(t) \Gamma_r^{(m)}(t)\}$, $E_r^{(m)}(t) =$

$$P_{r-1}^{(m)}(t) \Gamma_{r+1}^{(m)}(t) / \{\Gamma_r^{(m)}(t)\}^2, \text{ with } E_0^{(m)}(t) = 0, \quad Q_0^{(m)}(t) = \gamma_{m+1}(t) / \gamma_m(t)$$

($m=0, 1, \dots$), relationships (9a, b) become $E_r^{(m)}(t) = E_{r-1}^{(m+1)}(t) + \Delta Q_r^{(m)}(t)$,

$$Q_{r+1}^{(m)}(t) = Q_r^{(m)}(t) E_r^{(m+1)}(t) / E_r^{(m)}(t) \quad (r=1, 2, \dots; m=0, 1, \dots).$$

In the confluent case the matrix vector relationship (5) takes the form

$$\gamma_{m+1}(t) = \int_0^\infty \gamma_m(u) d\Omega(t, u). \text{ If } \gamma_m(t) = \sum_{r=1}^n a_r(t) \lambda_r^m \text{ then, under}$$

appropriate conditions, $Q_r^{(m)}(k) \sim \lambda_r$ ($r=1, \dots, n$), $E_r^{(m)}(k) = O((A_{r+1}/A_r)^m)$ ($r=1, \dots, n-1$) for large m .

It should also be pointed out that numerous multi-component versions of the q-d algorithm exist. For example, letting $F_r^{(m)}(k)$ be the r th order determinant with element $g_{m+i, j-2} (k+i-1)$ in the i th row and j th column ($i, j=1, \dots, r$) with $F_0^{(m)}(k) = 1$, so that

$$F_r^{(m)}(k) F_r^{(m+z)}(k+1) - F_r^{(m+1)}(k) F_r^{(m+1)}(k+1) = F_{r-1}^{(m+z)}(k+1) F_{r+1}^{(m)}(k)$$

($r=1, \dots, n-1; k=1, \dots, n-r; m=0, 1, \dots$) set, with $z=0, 1$,

$$q_{z,r}^{(m)}(k) = F_{r-1}^{(m)}(k) F_r^{(m+1)}(k+z) / \{F_r^{(m)}(k) F_{r-1}^{(m+1)}(k+z)\}$$

$$e_{z,r}^{(m)}(k) = F_{r+1}^{(m)}(k) F_{r-1}^{(m+1)}(k+z) / \{F_r^{(m)}(k) F_r^{(m+1)}(k+z)\}.$$

Then, with $e_{z,0}^{(m)}(k) = 0$ ($z=0, 1; m=1, 2, \dots; k=1, \dots, n$) $q_{z,1}^{(m)} = g_{m+1}(k+z) / g_m(k)$.

($z=0, 1; m=0, 1, \dots; k=1, \dots, n-z$), $e_{z,r}^{(m)}(k) = q_{1-z,r}^{(m+1)}(k+z) - q_{1-z,r}^{(m)}(k) + e_{z,r-1}^{(m+1)}(k+1-z)$.

$q_{z,r+1}^{(m)}(k) = q_{z,r}^{(m+1)}(k+1) e_{1,r}^{(m+1)}(k+z) / e_{1,r}^{(m)}(k)$ with $z=0, 1; r=1, \dots, n-z-1;$

$k=1, \dots, n-r; m=0, 1, \dots$. When the components $g_m(k)$ result from

recursion (5) as described a relationship of the form $F_r^{(m)}(k) \sim$

$B(r, k) \Lambda_{-}(r)^m$ holds for large m and, in particular, $q_{0,r}^{(m)}(k) \sim \lambda_r$

for $r=1, \dots, n; k=1, \dots, n-r+1$.