



We start with a fixed complex number λ . If the modulus of λ is less than 1, λ lies inside the unit circle. If the modulus of λ is greater than 1, λ lies outside the unit circle.



Next, we form the successive powers $\lambda^0, \lambda^1, \lambda^2, \dots$ of λ . They lie on a spiral in the complex plane. The argument of λ^2 is twice $\text{arg}(\lambda)$, the argument of λ^3 is three times $\text{arg}(\lambda)$ and so on. If the modulus of λ is less than 1, the powers of $|\lambda|$ tend to zero and the spiral contracts toward the origin. If the modulus of λ is greater than 1, the spiral expands away from the origin.



Then we multiply each of the points $\lambda^0, \lambda^1, \lambda^2, \dots$ by a constant complex number b . This does no

more than rotate the spiral through $\arg(b)$ and magnify or diminish it by a factor of $|b|$. But a contracting spiral remains contracting and an expanding spiral remains expanding.



Lastly we add a constant C to each of the transformed points and produce points members

$$s_i = C + b\lambda^i \quad (i=0,1,\dots)$$

begin of a first order spiral sequence. In the case in which $|\lambda| < 1$, the sequence converges to C and the spiral upon which its members lie contracts towards C . In the case in which $|\lambda| > 1$, the spiral expands away from C . If $\lim_{i \rightarrow \infty} s_i$ is finite,

$$\lim_{i \rightarrow \infty} s_i = C.$$



Already we may pose a problem in connection with first order spiral sequences. It is known that

The successive members of S_k , S_{k+1} and S_{k+2} of a sequence have the forms

$$S_k = C + b\lambda^k, \quad S_{k+1} = C + b\lambda^{k+1}, \quad S_{k+2} = C + b\lambda^{k+2}$$

but the values of C , b , λ and k are unknown. Find the value of C alone. If we can solve this problem then what we are able to do is fit a first order spiral sequence to the subsequence S_k , S_{k+1} and S_{k+2} and find its centre.

One way of solving this problem is to work with six numbers $\varepsilon_0^{(k)}$, $\varepsilon_0^{(k+1)}$, $\varepsilon_0^{(k+2)}$, $\varepsilon_1^{(k)}$, $\varepsilon_1^{(k+1)}$ and $\varepsilon_2^{(k)}$ arranged as shown. We set

$$\varepsilon_0^{(k)} = S_k \quad \varepsilon_0^{(k+1)} = S_{k+1} \quad \varepsilon_0^{(k+2)} = S_{k+2}$$

We calculate the numbers $\varepsilon_1^{(k)}$ and $\varepsilon_1^{(k+1)}$ by use of the relationships

$$\varepsilon_1^{(k)} = \frac{1}{\varepsilon_0^{(k+1)} - \varepsilon_0^{(k)}}$$

$$\varepsilon_1^{(k+1)} = \frac{1}{\varepsilon_0^{(k+2)} - \varepsilon_0^{(k+1)}}$$

and the number $\varepsilon_2^{(k)}$ by use of the relationship

$$\varepsilon_2^{(k)} = \varepsilon_0^{(k+1)} + \frac{1}{\varepsilon_1^{(k+1)} - \varepsilon_1^{(k)}}$$

for then $\varepsilon_2^{(k)} = C$. We have

$$\varepsilon_1^{(k)} = \frac{1}{C + b\lambda^{k+1} - C - b\lambda^k} = \frac{1}{b\lambda^k(\lambda - 1)}, \quad \varepsilon_1^{(k+1)} = \frac{1}{b\lambda^{k+1}(\lambda - 1)}$$

and

$$\varepsilon_2^{(k)} = C + b\lambda^{k+1} + \frac{1}{\frac{1}{b\lambda^{k+1}(\lambda - 1)}(1 - \lambda)} = C + b\lambda^{k+1} - b\lambda^{k+1} = C$$

We repeat that we have found C alone. The determination of b and λ requires a little more work. But the process of fitting a first order spiral sequence to a subsequence and finding its centre can be done without the determination of b and λ .



We now pose another problem. It is known that the members s_{k+1} , s_{k+2} and s_{k+3} of a sequence have the forms

$$s_{k+1} = C' + b'\lambda'^{k+1}, \quad s_{k+2} = C' + b'\lambda'^{k+2}, \quad s_{k+3} = C' + b'\lambda'^{k+3}$$

but the values of c' , b' , λ' and k are unknown.
 Find the value of c' alone. (Before we dealt with S_k , S_{kr_1} and S_{kr_2} ; now we are dealing with S_{kr_1} , S_{kr_2} and S_{kr_3}). To solve this problem, we simply construct a further scheme of six numbers $\varepsilon_0^{(kr_1)}$, $\varepsilon_0^{(kr_2)}$, $\varepsilon_0^{(kr_3)}$, $\varepsilon_1^{(kr_1)}$, $\varepsilon_1^{(kr_2)}$ and $\varepsilon_2^{(kr_1)}$ in the same way and now $\varepsilon_2^{(kr_1)} = c'$. The two schemes obtained from S_k , S_{kr_1} , S_{kr_2} and S_{kr_1} , S_{kr_2} , S_{kr_3} overlap: they contain the numbers $\varepsilon_0^{(kr_1)}$, $\varepsilon_0^{(kr_2)}$ and $\varepsilon_1^{(kr_1)}$ in common. These numbers have the same values in both schemes.

In this way all schemes produced from $S_0, S_1, S_2; S_1, S_2, S_3; \dots$ may be combined. If S_0, S_1, S_2, \dots are successive members of the same first order spiral sequence, so that

$$S_i = C + b\lambda^i$$

for $i=0, 1, \dots$ then $\varepsilon_2^{(i)} = C$ for $i=0, 1, \dots$: the column

of numbers $\varepsilon_2^{(i)}$ consists of copies of the value of C, the centre of the first order spiral sequence to which the S_i belong.

As given earlier, the rules for computing the numbers $\varepsilon_1^{(m)}$ and $\varepsilon_2^{(m)}$ differ, but if we add the boundary values $\varepsilon_{-1}^{(m)} = 0$ ($m=1, 2, \dots$) the rules may be reduced to the same form. For

$$\varepsilon_1^{(m)} = \frac{1}{\varepsilon_0^{(m+1)} - \varepsilon_0^{(m)}} \text{ becomes } \varepsilon_1^{(m)} = \varepsilon_{-1}^{(m+1)} + \frac{1}{\varepsilon_0^{(m+1)} - \varepsilon_0^{(m)}}$$

and as before

$$\varepsilon_2^{(m)} = \varepsilon_0^{(m+1)} + \frac{1}{\varepsilon_1^{(m+1)} - \varepsilon_1^{(m)}}$$

They may be written in the single form

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + \frac{1}{\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)}}$$

and concern numbers occurring at the vertices of a rectangle

$$\begin{array}{ccccc} & \varepsilon_r^{(m)} & & \varepsilon_{r+1}^{(m)} & \\ \varepsilon_{r-1}^{(m+1)} & & \varepsilon_r^{(m)} & & \varepsilon_{r+1}^{(m+1)} \\ & \varepsilon_r^{(m+1)} & & \varepsilon_{r+1}^{(m)} & \end{array}$$

in the given array. The relationship is used with
 $n=0; m=0, 1, \dots$ to compute the numbers $\varepsilon_1^{(m)}$ and
 with $n=1; m=0, 1, \dots$ to compute the numbers $\varepsilon_2^{(m)}$.



The members of a second order spiral sequence have the representation

$$s_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{l=0}^{z(j)} b_{j,l} i^l \right\}; \quad \sum_{j=1}^h \{z(j) + 1\} = 2$$

$(i=0, 1, \dots)$. This sequence has two forms: either
 (a) only one geometric term accompanied by a linear function of i is present and

$$s_i = C + \lambda_1^i (b_{1,0} + b_{1,1}i)$$

or
 (b) two geometric terms accompanied by constant coefficients are present and

$$s_i = C + b_{1,0} \lambda_1^i + b_{2,0} \lambda_2^i$$

In the latter case a second order spiral sequence may be regarded as a first order spiral sequence whose members are represented by $b_2 \lambda_2^i$ whose centre moves

on a further first order spiral upon which the points $C + b_1 \lambda_1^i$ lie. In the case in which both first order spirals are contracting the second order spiral sequence appears as shown.

The members of an n^{th} order spiral sequence have the representation

$$s_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{j=0}^{z(j)} b_{j,j} i^j \right\}; \quad \sum_{j=1}^h \{z(j)+1\} = n$$

Again various special forms are possible: one has only one geometric term accompanied by an $(n-1)^{\text{th}}$ degree polynomial in i ; another has n geometric terms accompanied by constant coefficients; other forms are intermediate to these two extremes.

If $\lim_{i \rightarrow \infty} s_i$ is finite, $\lim_{i \rightarrow \infty} s_i = C$.

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We may pose a problem in connection with n^{th} order spiral sequences. It is known that the members $s_{kn}, s_{kn+1}, \dots, s_{kn+2n}$ of a sequence have the form

$$S_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{v=0}^{z(j)} b_{j,v} i^v \right\} ; \sum_{j=1}^h \{z(j)+1\} = n$$

for $i = k, \dots, k+2n$, but the values of $C, h, \lambda_j^i, z(j)$, $b_{j,v}$ and k are unknown (in particular, the special form of the expression is unknown). Find the value of C alone. If we can solve this problem then what we are able to do is fit an n^{th} order spiral sequence to the subsequence S_k, \dots, S_{k+2n} and find its centre.

A systematic method for solving this problem concerns a scheme of numbers $\varepsilon_r^{(m)}$ as shown, with boundary values

$$\varepsilon_{-1}^{(i)} = 0 \quad (i = k+1, \dots, k+2n)$$

Starting from the initial values

$$\varepsilon_0^{(i)} = S_i \quad (i = k, \dots, k+2n)$$

the numbers in the array scheme are constructed column by column by use of the relationship

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + \frac{1}{\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)}}$$

which concerns numbers at the vertices of the square

$$\varepsilon_{r-1}^{(m+1)} \quad \varepsilon_r^{(m)} \quad \varepsilon_{r+1}^{(m)}$$

$$\varepsilon_r^{(m+1)}$$

in the scheme of numbers. The relationship is used firstly with $r=0$ and $m=k, k+1, \dots$ to construct the numbers $\varepsilon_1^{(m)}$, then with $r=1$ and $m=k, k+1, \dots$ to construct the numbers $\varepsilon_2^{(m)}$ and so on. Then $\varepsilon_{2n}^{(m)} = C$.

There is no time to give a proof of this result.



Just as in the case of first order spirals, we propose a second problem: it is known that the $2n+1$ members $S_{k+1}, S_{k+2}, \dots, S_{k+2n+1}$ of a sequence have the form

$$S_i = C' + \sum_{j=1}^{h'} \lambda'_j \left\{ \sum_{j=0}^{z'(j)} b'_{j,2} i^2 \right\}; \quad \sum_{j=1}^{h'} \{z'(j)+1\} = n$$

for $i = k+1, \dots, k+2n+1$ but the values of C' , h' , λ'_j , $z'(j)$, $b'_{j,2}$ and k are unknown. Find the value of C' alone. (Before the members were $S_k, S_{k+1}, \dots, S_{k+2n}$; now they are $S_{k+1}, \dots, S_{k+2n+1}$)

Again we simply construct a further scheme of numbers as shown and now $\varepsilon_{2n}^{(kn)} = C'$.

As before the two schemes of numbers derived from S_k, \dots, S_{k+2n} and $S_{k+1}, \dots, S_{k+2n+1}$ overlap and the numbers common to both schemes have the same values in both schemes: we put

$\varepsilon_0^{(kn)} = S_{k+1}, \varepsilon_0^{(kr+2)} = S_{kr+2}, \dots$ and so on. Thereafter $\varepsilon_1^{(kn)} = \frac{1}{\varepsilon_0^{(kr+2)} - \varepsilon_0^{(kn)}}$ ~~and~~ is the same in both schemes,

and so on. All schemes produced from $S_0, S_{2n}; S_1, \dots, S_{2n+1}$ may be combined. If S_0, S_1, \dots are successive members of the same n^{th} order spiral

sequence, so that

$$S_i = C + \sum_{j=1}^n \lambda_j^i \left\{ \sum_{j=0}^{2(j)} b_{j,j} \nu_j^{(i)} \right\}; \quad \sum_{j=1}^n \{z(j)\nu_j\} = n$$

for $i=0, 1, \dots$, then $\varepsilon_{2n}^{(i)} = C$ ($i=0, 1, \dots$): the column

containing the numbers $\varepsilon_{2n}^{(i)}$ consists of copies of C , the centre of the n^{th} order spiral sequence of which the S_i are successive members.

So far we have let the parameters C, b, λ etc. and the members s_i of the spiral sequences be complex numbers. It is, however, possible to take, as a special case, these numbers to be real.

For example, when C is real, b is real and positive and λ is real and negative (so that $\arg(\lambda) = \pi$) the members of the first order spiral sequence

$$s_i = C + b\lambda^i \quad (i=0, 1, \dots)$$

lie in a spiral in the complex plane, but the numbers s_i themselves lie on the real axis. Again, when C is real and b and λ are real and positive (so that we may take $\arg(\lambda) = 2\pi$) the spiral lies in the complex plane and the values of the s_i lie on the real axis again as shown.

We remark that if the s_i are real numbers, the numbers $e_r^{(m)}$ obtained from them are also real.

The processes for finding centres of spirals which we have described serve as the basis of a method for obtaining estimates of the limit of a sequence.

We propose the following problem: given a sequence s_0, s_1, s_2, \dots of complex or real numbers, estimate its limit in terms of its initial members. A systematic solution proceeds as follows: fit a first order spiral sequence to s_0, s_1 and s_2 and find its centre—that is, determine $\varepsilon_2^{(0)}$ from s_0, s_1 and s_2 . This centre, $\varepsilon_2^{(0)}$, is the estimate of the limit derived from s_0, s_1 and s_2 . Fit a first ~~second~~ order spiral sequence to s_1, s_2 and s_3 and find its centre—that is, determine $\varepsilon_2^{(1)}$ from s_1, s_2 and s_3 . This centre, $\varepsilon_2^{(1)}$, is the estimate of the limit derived from s_1, s_2 and s_3 . Proceeding further: fit a second order spiral sequence to s_0, \dots, s_4 and find its centre—that is, determine $\varepsilon_4^{(0)}$ from

S_0, \dots, S_4 . This centre is the estimate of the limit derived from S_0, \dots, S_4 . Systematically: determine numbers $\varepsilon_r^{(m)}$ from S_0, S_1, \dots, S_{2r} . $\varepsilon_{2r}^{(m)}$ is the centre of the r th order spiral sequence fitted to S_m, \dots, S_{m+2r} : it is an estimate of the limit of the sequence S_0, S_1, \dots . This method of estimating the limit of a sequence is known as the ε -algorithm.

We know of course that if all S_i are members of the same n th order spiral sequence — they have the form

$$S_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{j=0}^{z(j)} b_{j,j+1} i^j \right\}; \sum_{j=1}^h \{z(j)+1\} = n$$

and the numbers $\varepsilon_{2n}^{(m)}$ can be constructed, then

$\varepsilon_{2n}^{(m)} = C$ ($m = 0, 1, \dots$): the method of estimation is exact.

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A question now arises: does this method work? To find out we consider some examples in which the S_m are partial the successive partial sums of an infinite series with known partial sum. We first

take S_m to be the sum of the first m terms of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

whose sum is $\ln(2) = 0.693\dots$. The ε -numbers derived from the first four partial sums are displayed.

The odd order numbers $\varepsilon_1^{(1)}, \dots, \varepsilon_1^{(3)}$ are written in because they are so simple and illustrate the relationships used ($\varepsilon_1^{(1)} = 0 + \frac{1}{1-0}$, $\varepsilon_2^{(2)} = \frac{2}{3} = 1 + \frac{1}{-2-1}$ and so on) but are otherwise replaced by stars.

The numbers $\varepsilon_2^{(0)}, \varepsilon_2^{(1)}$ and $\varepsilon_2^{(2)}$ are the estimates of $\ln(2)$ obtained from first order spiral fitting. $\varepsilon_4^{(0)} = 0.692$ is the centre of the second order spiral sequence fitted to S_0, \dots, S_4 ; it differs from $\ln(2)$ by 1 in the third decimal place.

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Actually we can do much better. The value of $\ln(2)$ to 11 decimals is 0.69314718056... We first construct the ε -array from the first eight terms

of the given series. $\epsilon_8^{(6)} = 0.693146\ldots$ is the centre of the fourth order spiral sequence fitted to s_0, \dots, s_8 ; it differs from $m(2)$ by 1 in the sixth decimal place.

We now take the numbers $\epsilon_0^{(6)}, \epsilon_0^{(1)}, \epsilon_2^{(5)}, \epsilon_2^{(1)}, \dots, \epsilon_8^{(6)}$ to be a new initial value sequence for the construction of another ϵ -array. In this new array $\epsilon_8^{(6)}$ differs from $m(2)$ by 3 in the eleventh decimal place

Approximately

$$\left| \ln(2) - \sum_{i=0}^{m-1} \frac{(-1)^i}{i+1} \right| = \frac{1}{2^m}$$

To calculate $\ln(2)$ by use of the series to an error of 3 in the eleventh decimal place requires about ten billion terms. Repeated use of the ϵ -algorithm yields the same accuracy from only eight terms.

For a second example we consider the partial sums $S_m = \sum_{i=0}^{m-1} \left(-\frac{1}{2} \right) \frac{1}{i+1}$. Now the sequence being transformed has the limit $2(\sqrt{2}-1) = 0.828427124743\dots$. We begin by forming the ε -array from the first six partial sums of the series. $\varepsilon_6^{(0)}$, the result obtained by fitting a third order spiral to S_0, \dots, S_6 differs from $2(\sqrt{2}-1)$ by 3 in the fifth decimal place. Once again we take a step-wise sequence from the ε -array and use its members as a new initial sequence. In the new array $\varepsilon_6^{(1)}$ differs from $2(\sqrt{2}-1)$ by 6 in the twelfth decimal place.

Again

$$\left| 2(\sqrt{2}-1) - \sum_{i=0}^{m-1} \left(-\frac{1}{2} \right) \frac{1}{i+1} \right| = \frac{1}{2^m}$$

approximately. To calculate $2(\sqrt{2}-1)$ by use of the series to an error of 6 in the twelfth decimal

place requires about a hundred billion terms.

Repeated use of the ε -algorithm yields the same accuracy from only six terms.

The two examples just given may be regarded as exercises in convergence acceleration. Instead of calculating and adding together a hundred billion terms of a series, or performing a hundred billion iterations of an iterative numerical process, we calculate only six members of the sequence involved, transform them and get the same result.

Alternatively, we charge the client for carrying out one hundred billion iterations at a reasonable rate of one dollar per iteration, but in fact only carry out six.

Perhaps a happier interpretation is to suggest that a great deal of information lies lurking in the first few members of a sequence: we have only to think of a way of getting it out. This interpretation serves us a little better when we come to consider

The transformation of divergent series, sequences.

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To show how divergent series arise and may be used we first consider the integrals

$$F(r, x) = r! e^x \int_x^\infty \frac{1}{t^{r+1}} e^{-t} dt \quad (x > 0; r=0, 1, \dots)$$

Because $x > 0$, the integrand is finite and the exponential factor ensures convergence of the integral: the integrals $F(r, x)$ exist. The integrand is the product of two factors

$\frac{1}{t^{r+1}}$ and e^{-t} . Integrals of products of two factors may,

subject to suitable convergence conditions, be transformed by integration by parts into integrals of products of two other terms, one being the derivative of one factor and the second being the indefinite integral of the other.

Thus $\int_x^\infty a(t) b(t) dt$ may be expressed in the form given.

In our case $a(t) = \frac{1}{t^{r+1}}$ and $b(t) = e^{-t}$ so that the

integral $\int_x^\infty \frac{1}{t^{r+1}} e^{-t} dt$ may be expressed in the form given and, multiplying throughout by $r! e^{-x}$, we see that the

functions $F(r, x)$ satisfy the simple recursion

$$F(r, x) = \frac{r!}{x^{r+1}} - F(r+1, x) \quad (r=0, 1, \dots)$$

All terms involved in $F(r, x)$ are positive: $F(r, x)$ is positive.

The recursion has the form

something positive = something positive - something positive

Thus

$$F(r, x) < \frac{r!}{x^{r+1}} \quad (r=0, 1, \dots)$$



We now consider the exponential integral $F(0, x)$.

We express $F(0, x)$ in terms of $F(1, x)$, then $F(1, x)$ in terms of $F(2, x)$, then $F(2, x)$ in terms of $F(3, x)$ and so on. We end up with a finite series and error term

$$F(0, x) = \sum_{i=0}^{m-1} (-1)^i \frac{i!}{x^{i+1}} + e_m(x)$$

where $e_m(x) = (-1)^m F(m, x)$. The corresponding infinite series is divergent. The ratio of the modulus of the i^{th} term to that of its predecessor is $\frac{i}{x}$. However large x may be, i always catches it up. When $i > x$, the terms of the infinite series increase in magnitude. Such series

diverge.

The series offers an example of semi-convergence: the magnitude of the error term is less than that of the first term neglected: $|e_m(x)| < \frac{m!}{x^{m+1}}$.

Direct use of the series may be made when x is large enough and the accuracy requirements are not too severe. Suppose for example that we wish to compute the value of $e^{100} \int_{100}^{\infty} e^{-t} dt$ with an error less than 0.000 000 1. The error term $e_3(100)$ satisfies the inequalities

$$|e_3(100)| < \frac{3!}{100^4} = 0.000\ 000\ 06 < 0.000\ 000\ 1$$

The sum of the first three terms of the series yields the value of $e^{100} \int_{100}^{\infty} e^{-t} dt$ with an error less than 0.000 000 06 which is smaller than the permitted error of 0.000 000 1.

Thus

$$e^{100} \int_{100}^{\infty} e^{-t} dt = \frac{1}{100} - \frac{1}{10\ 000} + \frac{2}{100\ 000} = 0.009\ 902\ 0$$

to the required accuracy.

When $x=1$ the error terms satisfy the inequalities

$$|e_0(1)| < 0! = 1, |e_1(1)| < 1! = 1, |e_2(1)| < 2! = 2, |e_3(1)| < 3! = 6, \dots$$

and direct use of the series is not feasible.

When $x=1$ the successive partial sums are

$$S_0 = 0; S_1 = 0! = 1; S_2 = 0! - 1! = 0; S_3 = 0! - 1! + 2! = 2;$$

$$S_4 = 0! - 1! + 2! - 3! = -4; S_5 = 0! - 1! + 2! - 3! + 4! = 20; \dots$$

We propose to apply our method of fitting spirals. A spiral through the partial sums has the appearance indicated and expands rapidly. The value of $\int_1^{\infty} e^t dt$ is about 0.59635. Application of the ε -algorithm to the first ten terms of the series yields the estimate $e_{10}^{(0)} = 0.59509$ which is in error by 1 in the third decimal place. Repeated application to an initial sequence taken from a stepwise sequence in the first array yields the transformed estimate $e_{10}^{(0)} = 0.59639$ which is in error by 4 in the fifth place.

The reasons why the ε -algorithm works in the examples just considered are found in the classical theory of continued fractions. But in this short lecture we have no time for a detailed explanation. We shall describe instead another mode of application of the algorithm.

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Many problems in applied mathematics can be presented in terms of finding a fixed point of a function $f(x)$ being the real valued function of the real variable x : concerned, it is required to find the value C of x for which

$$x = f(x)$$

The form of this equation suggests a method of solution: guess a value S_0 of C , obtain another approximation S_1 to C by setting $S_1 = f(S_0)$, yet another by setting $S_2 = f(S_1)$ and so on. This iteration process may be represented graphically. Sketch the graphs of x and $f(x)$ as shown. Start

Start with the point S_0 on the graph of x . Draw a vertical line up to the graph of $f(x)$ and then a horizontal line across to the graph of x . The point of intersection gives us the value of S_1 . Then draw a vertical line down to the graph of $f(x)$ and then a horizontal line to the graph of x to obtain the value of S_2 . This process is continued: up to $f(x)$ and across to x to obtain S_3 , down to $f(x)$ and across to x to obtain S_4 , and so on.

All this is very pretty, but in practice convergence is often very slow. Naturally we are interested in producing a more rapidly convergent scheme, but must first say a few words about rates of convergence.

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A process which produces a sequence of numbers $C^{(0)}, C^{(1)}, \dots$ converging to C , for which

$$C^{(k+1)} - C = O(C^{(k)} - C)^r$$

is said to be r th order convergent. To illustrate this definition, suppose that $C = 2.0$ and $C^{(0)} = 2.1$.

We wish briefly to explain this definition.

$C^{(k)} - C$ indicates the error in $C^{(k)}$ as a measure of C . For an r^{th} order convergent process, we have

$$C^{(1)} - C = O \{ (C^{(0)} - C)^r \}$$

$$\begin{aligned} C^{(2)} - C &= O \{ (C^{(1)} - C)^r \} = O \{ O \{ C^{(0)} - C \}^r \} \\ &= O \{ (C^{(0)} - C)^{r^2} \} \end{aligned}$$

and more generally

$$C^{(k)} - C = O \{ (C^{(0)} - C)^{r^k} \}$$

To simplify the explanation suppose that the r^{th} order convergent process considered is such that the relationship $C^{(kn)} - C = O \{ (C^{(k)} - C)^r \}$ takes the special form $C^{(kn)} - C = (C^{(k)} - C)^r$ so that now

$$\cancel{(C^{(k)} - C)^r} \div (C^{(0)} - C)^{r^k}$$

Suppose that $C^{(0)} - C = 0.1 \times \dots$ where the crosses denote decimal figures which are of no interest to us. Successively

$$C^{(0)} - C = 0.1 \times \dots, (C^{(0)} - C)^2 = 0.01 \times \dots, (C^{(0)} - C)^3 = 0.001 \times \dots$$

and

$$(C^{(0)} - C)^n = 0.0 \dots 0 1 \times \dots \quad \leftarrow n-1 \rightarrow$$

so that for the r th order convergent process considered

$$C^{(k)} - C \approx 0.0 \dots 0 \underset{\leftarrow(r^k-1) \rightarrow}{1} \times \dots \quad (k=0,1,\dots)$$

What do these error terms look like? For a first order convergent process with $r=1$

$$C^{(0)} - C \approx 0.1 \times \dots$$

$$C^{(1)} - C \approx 0.01 \times \dots$$

$$C^{(2)} - C \approx 0.001 \times \dots$$

The errors are hardly decreasing. And this is the case for the iterative process for obtaining a fixed point of the function f by direct iteration as just described.

For a quadratically convergent process with $r=2$

$$C^{(0)} - C \approx 0.1 \times \dots$$

$$C^{(1)} - C \approx 0.01 \times \dots$$

$$C^{(2)} - C \approx 0.001 \times \dots$$

$$C^{(3)} - C \approx 0.0001 \times \dots$$

The errors are visibly decreasing to zero far more rapidly than in the linearly convergent case. For a cubically convergent process with $r=3$

$$C^{(0)} - C \approx 0.1 \times \dots$$

$$C^{(1)} - C \approx 0.001 \times \dots$$

$$C^{(2)} - C \approx 0.00001 \times \dots$$

It is possible to devise r^{th} order convergent schemes for finding a fixed point of a function that make use of the derivatives of f up to that of order $r-1$ (for example, Newton's method which is quadratically convergent and makes use of the first derivative of f). But in practice it is often difficult to form the required derivatives.

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It is, however, possible to devise an r^{th} order convergent scheme for finding a fixed point C of a function f that does not make use of derivatives. We start with an initial estimate $C^{(0)}$ of C . Taking $S_0 = C^{(0)}$ we perform the iteration $S_{i+1} = f(S_i)$ to obtain S_1, \dots, S_{2r} . We fit an r^{th} order spiral to S_0, \dots, S_{2r} and find its centre $C^{(1)}$, that is we determine $C^{(1)} = S_{2r}^{(0)}$ from S_0, \dots, S_{2r} . We now take $C^{(1)}$ to be a new initial estimate of C , iterate and find a new

centre $C^{(2)}$ of a spiral as before. Subject to suitable conditions

$$C^{(k+1)} - C = O \{ (C^{(k)} - C)^r \}$$

The iteration and spiral fitting performed at each stage are as illustrated.

21

The examples which we have used to illustrate application of the ε -algorithm concern sequences of real numbers. But iterative processes in applied mathematics produce sequences of many other kinds: sequences of functions, of vectors, of graphs and so on.

The relationship of the ε -algorithm

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + \frac{1}{\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)}}$$

applied to members of a field may be written as

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + (\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)})^{-1}$$

and in this form may be applied to any mathematical system over which addition, subtraction and the formation of an inverse are defined. Theories of continued fractions

and, as part of them, theories of the ε -algorithm, over many such systems have been developed. We mention as examples of such systems, rings over which an inverse is defined (for example, square matrices over a field), inner product spaces (for example, vectors over a field) and a number of nonassociative algebras. In this short talk we do not have time to consider such theories, but consider instead another type of spiral fitting.

22

The ε -algorithm is concerned with the limit of a sequence S_0, S_1, \dots . It works by fitting an r^{th} order spiral sequence to a subsequence S_m, \dots, S_{m+2r} by use of relationships holding for $i=m, \dots, m+2r$

$$S_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{j=0}^{\tau(j)} b_{ij} i^j \right\}; \sum_{j=1}^h \{\tau(j)+1\} = r$$

and taking $C = \varepsilon_{2r}^{(m)}$ to be an estimate of $\lim_{i \rightarrow \infty} S_i$.

The method is as illustrated.

We now consider a continuous analogue of this method.

We are concerned with a continuous function $S(t)$ and wish to estimate its limit $\lim_{t \rightarrow \infty} S(t)$. We effect a replacement of variables: i by t and λ_j^i by $e^{-\alpha_j t}$ and say that the function

$$S(t) = C + \sum_{j=1}^h e^{-\alpha_j t} \left\{ \sum_{p=0}^{r(j)} b_{j,p} t^p \right\}; \sum_{j=1}^h \{r(j)+1\} = r$$

is an r^{th} order spiral function. We propose to estimate $\lim_{t \rightarrow \infty} S(t)$ by fitting a spiral function $s(t)$ to $S(t)$ at one value x_0 of t , so that s and S have $2r^{\text{th}}$ degree contact at x_0 , and taking the centre of this spiral function to be the estimate. The process works like this.

First we must be able to find the centre C of an r^{th} order spiral function. We pose the following problem in connection with n^{th} order spiral functions:

- a) it is known that s is an n^{th} order spiral function of the above form, but the values of C, h, α_j , $r(j)$ and $b_{j,p}$ are unknown
- b) corresponding to some point x_0 in the range of t

the value

$$S^{(0)} = \lim_{t \rightarrow \infty} s(t)$$

and the $2n$ derivative values

$$S^{(i)} = \lim_{t \rightarrow \infty} \frac{d^i S(t)}{dt^i} \quad (i=1, \dots, 2n)$$

are given, but the value of ∞ itself is unknown.

Find the centre C of the ~~continuous~~ spiral function s alone.

23

A solution to this problem proceeds as follows.

Construct the indicated array of numbers from the initial values

$$\omega_{-1}^{(m)} = 0 \quad (m=1, \dots, 2n) \quad \omega_0^{(m)} = S^{(m)} \quad (m=0, \dots, 2n)$$

by use of the relationships

$$\omega_{2r+1}^{(m)} = \omega_{2r-1}^{(m+1)} + \frac{\omega_{2r}^{(m)}}{\omega_{2r}^{(m+1)}}, \quad \omega_{2r+2}^{(m)} = \omega_{2r}^{(m+1)} (\omega_{2r+1}^{(m)} - \omega_{2r+1}^{(m+1)})$$

These relationships concern numbers situated at the vertices of 2x2-edges in the given array. In the case of the ε -algorithm, one relationship was used; now

we have two.

Just to go through the motions let us calculate $\omega_2^{(0)}$. To calculate $\omega_1^{(0)}$ we use the first formula

$$\omega_1^{(0)} = 0 + \frac{S^{(0)}}{S^{(1)}} = \frac{S^{(0)}}{S^{(1)}}$$

Similarly $\omega_1^{(1)} = \frac{S^{(1)}}{S^{(2)}}$. To calculate $\omega_2^{(0)}$ we use the second formula:

$$\omega_2^{(0)} = S^{(1)} \left\{ \frac{S^{(0)}}{S^{(1)}} - \frac{S^{(1)}}{S^{(2)}} \right\} = S^{(1)} - \frac{S^{(1)}}{S^{(2)}}^2$$

If the numbers $\omega_r^{(n)}$ can be constructed as described, $\omega_{2n}^{(0)} = C$. Again we do not prove this result, but indicate that it might be true by considering the case, with $n=1$, in which $s(t) = C + be^{-\alpha t}$. We have

$$S^{(0)} = C + be^{-\alpha x}, S^{(1)} = \frac{dS^{(0)}(x)}{dx} = -b\alpha e^{-\alpha x}, S^{(2)} = \frac{d^2S^{(0)}(x)}{dx^2} = b\alpha^2 e^{-\alpha x}$$

Then $\omega_2^{(0)} = C + be^{-\alpha x} - \frac{b^2 \alpha^2 e^{-2\alpha x}}{b\alpha^2 e^{-\alpha x}} = C$

If $\lim_{t \rightarrow \infty} s(t)$ is finite, $\lim_{t \rightarrow \infty} s(t) = C$.

Having found out how to determine the centre of an n^{th} order spiral function, we are in a position to describe our method for estimating or defining $\lim_{t \rightarrow \infty} S(t)$ in terms of derivative values at a finite argument value.

Construct numbers $\omega_r^{(m)}$ from the initial values $\omega_0^{(m)} = \frac{S^{(m)}}{\cancel{1, 2, 3, \dots}}$ ($m=0, 1, \dots$) where

$$S^{(i)} = \lim_{t \rightarrow \infty} \frac{d^i S(t)}{dt^i} \quad (i=0, 1, \dots)$$

$\omega_{2r}^{(0)}$ is the centre of the r^{th} order spiral function $s(t)$ fitted to $S(t)$ at $t=\infty$ in the sense that

$$\lim_{t \rightarrow \infty} \frac{d^i s(t)}{dt^i} = \lim_{t \rightarrow \infty} \frac{d^i S(t)}{dt^i} \quad (i=0, \dots, 2r)$$

and is an estimate of $\lim_{t \rightarrow \infty} S(t)$

This method of estimating the limit of a function is known as the ω -algorithm.

We are concerned with a continuous function $S(t)$ and wish to estimate its limit $\lim_{t \rightarrow \infty} S(t)$. We effect a replacement of variables: i by t and λ_j^i by $e^{-\alpha_j t}$ and say that the function

$$s(t) = C + \sum_{j=1}^h e^{-\alpha_j t} \left\{ \sum_{p=0}^{\tau(j)} b_{j,p} t^p \right\}; \sum_{j=1}^h \{\tau(j)+1\} = r$$

is an r^{th} order spiral function. We propose to estimate $\lim_{t \rightarrow \infty} S(t)$ by fitting a spiral function $s(t)$ to $S(t)$ at one value x_0 of t , so that s and S have $2r^{\text{th}}$ degree contact at x_0 , and taking the centre of this spiral function to be the estimate. The process works like this.

First we must be able to find the centre C of an r^{th} order spiral function. We pose the following problem in connection with n^{th} order spiral functions:

- it is known that s is an n^{th} order spiral function of the above form, but the values of $C, h, \alpha_j, \tau(j)$ and $b_{j,p}$ are unknown
- corresponding to some point x_0 in the range of t

the value

$$S^{(0)} = \lim_{t \rightarrow \infty} s(t)$$

and the $2n$ derivative values

$$S^{(i)} = \lim_{t \rightarrow \infty} \frac{d^i s(t)}{dt^i} \quad (i=1, \dots, 2n)$$

are given, but the value of ∞ itself is unknown.

Find the centre C of the ~~continuous~~ spiral function s alone.

23

A solution to this problem proceeds as follows.

Construct the indicated array of numbers from the initial values

$$\omega_{-1}^{(m)} = 0 \quad (m=1, \dots, 2n) \quad \omega_0^{(m)} = S^{(m)} \quad (m=0, \dots, 2n)$$

by use of the relationships

$$\omega_{2r+1}^{(m)} = \omega_{2r-1}^{(m+1)} + \frac{\omega_{2r}^{(m)}}{\omega_{2r}^{(m+1)}}, \quad \omega_{2r+2}^{(m)} = \omega_{2r}^{(m+1)} (\omega_{2r+1}^{(m)} - \omega_{2r+1}^{(m+1)})$$

These relationships concern numbers situated at the vertices of zig-zags in the given array. In the case of the ε -algorithm, one relationship was used; now

we have two.

Just to go through the motions let us calculate $\omega_2^{(0)}$. To calculate $\omega_1^{(0)}$ we use the first formula

$$\omega_1^{(0)} = 0 + \frac{S^{(0)}}{S^{(1)}} = \frac{S^{(0)}}{S}$$

Similarly $\omega_1^{(1)} = \frac{S^{(1)}}{S^{(2)}}$. To calculate $\omega_2^{(0)}$ we use the second formula:

$$\omega_2^{(0)} = S^{(1)} \left\{ \frac{S^{(0)}}{S^{(1)}} - \frac{S^{(1)}}{S^{(2)}} \right\} = S^{(1)} - \frac{S^{(1)} S^{(1)}}{S^{(2)}}$$

If the numbers $\omega_r^{(n)}$ can be constructed as described, $\omega_{2n}^{(0)} = C$. Again we do not prove this result, but indicate that it might be true by considering the case, with $n=1$, in which $s(t) = C + be^{-\alpha t}$. We have

$$S^{(0)} = C + be^{-\alpha x}, S^{(1)} = \frac{dS(x)}{dx} = -b\alpha e^{-\alpha x}, S^{(2)} = \frac{d^2S(x)}{dx^2} = b\alpha^2 e^{-\alpha x}$$

$$\text{Then } \omega_2^{(0)} = C + be^{-\alpha x} - \frac{b^2 \alpha^2 e^{-2\alpha x}}{b\alpha^2 e^{-\alpha x}} = C$$

If $\lim_{t \rightarrow \infty} s(t)$ is finite, $\lim_{t \rightarrow \infty} s'(t) = C$.

Having found out how to determine the centre of an n^{th} order spiral function, we are in a position to describe our method for estimating or defining $\lim_{t \rightarrow \infty} S(t)$ in terms of derivative values at a finite argument value.

Construct numbers $\omega_r^{(m)}$ from the initial values $\omega_0^{(m)} = \frac{S^{(m)}}{\cancel{S^{(m)}}}$ ($m=0, 1, \dots$) where

$$S^{(i)} = \lim_{t \rightarrow \infty} \frac{d^i S(t)}{dt^i} \quad (i=0, 1, \dots)$$

$\omega_{2r}^{(0)}$ is the centre of the r^{th} order spiral function $s(t)$ fitted to $S(t)$ at $t=\infty$ in the sense that

$$\lim_{t \rightarrow \infty} \frac{d^i s(t)}{dt^i} = \lim_{t \rightarrow \infty} \frac{d^i S(t)}{dt^i} \quad (i=0, \dots, 2r)$$

and is an estimate of $\lim_{t \rightarrow \infty} S(t)$

This method of estimating the limit of a function is known as the ω -algorithm.

If $S(b)$ is an n^{th} degree spiral function and has the form

$$C + \sum_{j=1}^h e^{-\alpha_j t} \left\{ \sum_{j=0}^{z(j)} b_{j,2j} t^{2j} \right\}, \quad \sum_{j=1}^h \{z(j)+1\} = n$$

The formation of the numbers $\omega_r^{(m)}$ terminates with

$$\omega_{2n}^{(0)} = C$$

This method of estimating the limiting value of a function has application in estimating the end-point of the trajectory of an aerodynamic vehicle, given its position, velocity components, etc. We consider an application of another kind.

25

We may naturally take the function S being treated to be a function of two variables, so the point at which spiral fitting takes place and t . The derivative values used then have the form

$$S^{(i)} = \lim_{t \rightarrow \infty} \frac{\partial^i}{\partial t^i} S(x, t) \quad (i=0,1,\dots)$$

In particular, we may let S have the form

$$S(x, t) = \int_x^t \psi(t') dt'$$

where ψ is a suitable function. Now

$$\lim_{t \rightarrow \infty} S(x, t) = \int_x^\infty \psi(t') dt'$$

Also

$$S^{(0)} = \lim_{t=x} \int_x^t \psi(t') dt' = 0$$

and

$$S^{(i)} = \lim_{t=x} \frac{\partial^i \int_x^t \psi(t') dt'}{\partial t^i} = \frac{d^{i-1} \psi(x)}{dx^{i-1}} = \psi^{(i-1)}(x)$$

If numbers $\omega_r^{(m)}$ are constructed from the initial values

$$\omega_0^{(0)} = 0, \quad \omega_0^{(m)} = \psi^{(m-1)}(x) \quad (m=1, 2, \dots)$$

$\omega_{2r}^{(0)}$ is an estimate of the continued fraction integral

$$(CF) \int_x^\infty \psi(t') dt'$$

Setting $r=1$, the first order continued fraction integral approximation is

$$\omega_2^{(0)} = S^{(0)} - \frac{S^{(1)^2}}{S^{(2)}} = -\frac{\psi(x)^2}{\psi'(x)}$$

$$\text{If } \psi(t) = \sum_{j=1}^h e^{-\alpha_j t} \left\{ \sum_{j=0}^{z(j)} b_{j,j+1} t^j \right\}; \sum_{j=1}^h \{z(j)+1\} = n$$

then formation of the numbers $\omega_r^{(m)}$ terminates

with $\omega_{2n}^{(0)} = (\text{cf}) \int_0^\infty \psi(t') dt'$. For example, when $n=1$,

$$\omega_2^{(0)} = -\frac{(b_1 e^{-\alpha_1 x})^2}{-b_1 e^{-\alpha_1 x}} = \frac{b_1 e^{-\alpha_1 x}}{\alpha_1} = \int_x^\infty b_1 e^{-\alpha_1 t'} dt'.$$

26

Since Texas is an oil-producing state the last problem we consider is that of the owner of a well producing oil with cost price c per barrel who wishes to determine

- a) the price P per barrel which maximizes his profit and
- b) his profit at the optimal price P

We now describe a solution and hasten to say that the economic part of the argument is taken directly from the mid nineteenth century French economist Cournot who considered the problem of

the owner of a spring producing mineral water at zero cost per bottle.

The owner determines the number $n(p)$ of barrels sold weekly at price p by observation. For example, during the first week he sells at cost price c ; many people say thank you and buy, but he makes no profit. He then asks an astronomical price, he makes no sale and again no profit. He then settles down and after several weeks produces a graph of $n(p)$ against p that looks like this. He then reasons in the following way. At price p his weekly profit is $n(p)(p-c)$. This attains a maximum when p takes the value P for which

$$\frac{dn(p)}{dp}(p-c) + n(p) = 0$$

So he solves the equation

$$p = c - \frac{n(p)}{\left\{ \frac{dn(p)}{dp} \right\}} \Leftrightarrow p = f(p) \text{ where } f(p) = c - \frac{n(p)}{\left\{ \frac{dn(p)}{dp} \right\}}$$

In other words he finds a fixed point of the function f . We have encountered this problem before: he solves it by direct iteration or acceleration. The

first part of his problem, finding the optimal price P , has been solved.

27

At the optimal price P

$$\frac{dn(P)}{dP}(P-c) + n(P) = 0$$

The profit at this price is

$$n(P)(P-c) = \frac{-n(P)^2}{\left\{ \frac{dn(P)}{dP} \right\}}$$

We have also met the right hand side expression:
it is the first continued fraction integral
approximation to

$$\int_P^\infty n(p) dp$$

It is equal to this integral if the market response function has the exponential form $n(p) = be^{-\alpha p}$

The mid nineteenth century economic theory that we have used describes a theoretically perfect OPEC situation: all oil producers have the same cost

price; they act together as one man; they dictate the price to the market; It appears to be not quite the case that these assumptions hold.

In a slightly different formulation of the theory, the condition that determines P is not

$$n(P)(P-c) = - \frac{n(P)^2}{\left\{ \frac{dn(P)}{dP} \right\}}$$

but $n(P)(P-c) = \int_P^\infty n(p)dp$

The original solution of the problem now appears as a first order approximation in the new formulation.

$n(p)$ is the number of barrels sold per week at price p . This function determines an inverse function $p(n)$ of n . If the owner for some reason wishes to sell n barrels during a certain week, he offers the oil at $p(n)$ per barrel and the market responds by buying n barrels. In terms of the inverse function

$$\int_P^\infty n(p)dp = \int_0^{n(P)} p(n)dn$$

The optimality condition can now be expressed as

$$P_c - c = \frac{1}{n(P)} \int_0^{n(P)} p(n) dn = \frac{\int_0^{n(P)} p(n) dn}{\int_0^{n(P)} dn}$$

P_c is now expressed as a ratio involving the integral of a distribution and its first moment over a variable n which is determined by the market. In the new formulation the market determines the price. The optimality condition may be decomposed in terms of individual cost prices c_k and shares n_k of the market relating to a group of producers. If only one producer is present and $p(n)$ is a logarithmic function, the solution to the newly formulated problem reduces to that of the original problem. Unfortunately there is no time to go into further details.

28

In this talk we have touched superficially upon many subjects. Just in case any of you are interested in studying the matter further, we give some references.

For general background, notes of some lectures given first at a Summer School in Brugia and then at the University of Wisconsin

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For a recent book

Cuyt A. and L. Wuytack, Nonlinear methods in numerical analysis, North Holland (1987)

The numerical examples of the ε -algorithm

were taken from

W - Transformation de séries à l'aide de l'^e-
algorithme, Comptes Rendus Acad Sci. (Paris)
275 A (1972) 1351-1353

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The use of the ε -algorithm in finding the
fixed point of a function is described in
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Zeit. Ang. Math. Mech., 51 (1971) 145-148

Brezinski C., Sur un algorithme de résolution
des systèmes non linéaires, Comptes Rendus
Acad Sci. (Paris) 272 A (1971) 145-148

A theory of continued fractions over a ring with
inverse, together with a corresponding theory of the
 ε -algorithm is given in

W - Continued fractions whose coefficients obey
a noncommutative law of multiplication, Arch.
Rat. Mech. Anal., 12 (1963) 273-312

Theories of estimating the limiting value of a sum function by fitting spiral functions are given in

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28 (1968) 83-118

W — A convergence theory of some methods of integration, Jour. reine ang. Math., (Gelle's Journal)
285 (1976) 181-208

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W — The evaluation of singular and highly oscillatory integrals by use of the anti-derivative, Calcolo 15 (1978) fasc. IV bis, 1-103

30

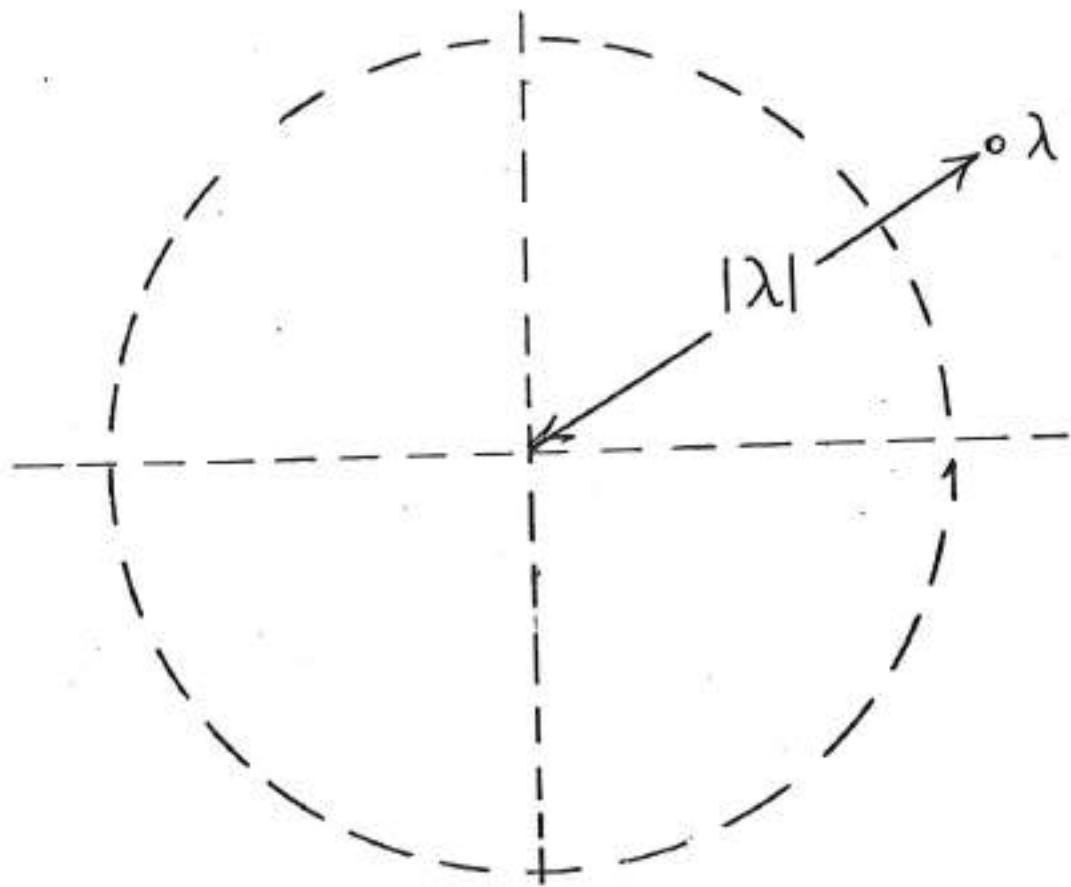
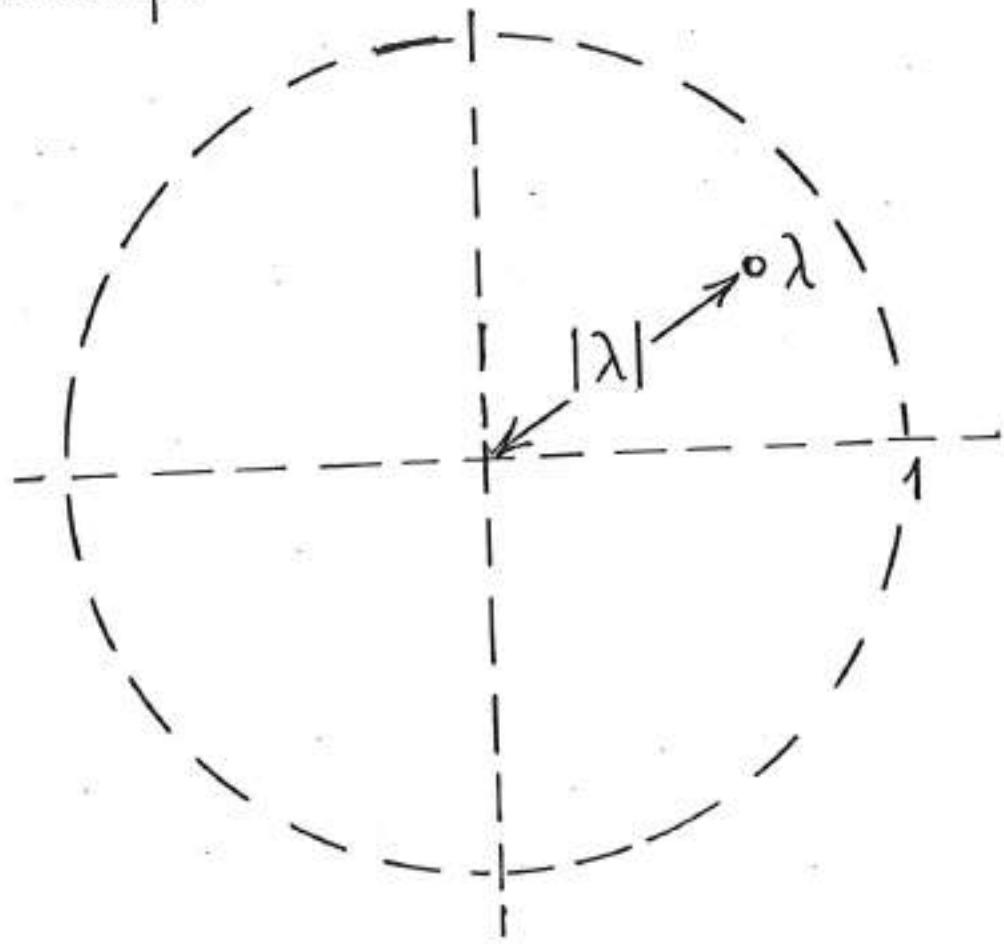
The economic part of the theory of optimal price determination is taken from

Cournot A.A., Recherches sur les principes mathématiques de la théorie des richesses, M. Rivière, Paris (1838)

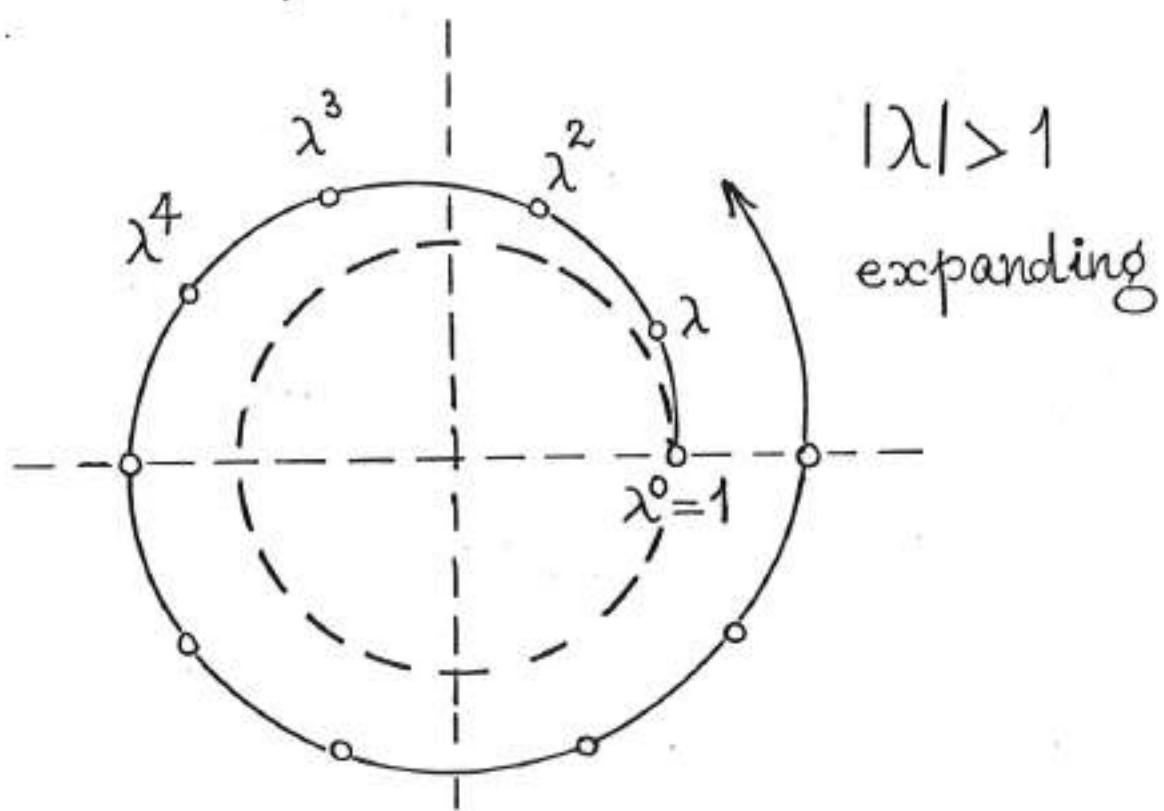
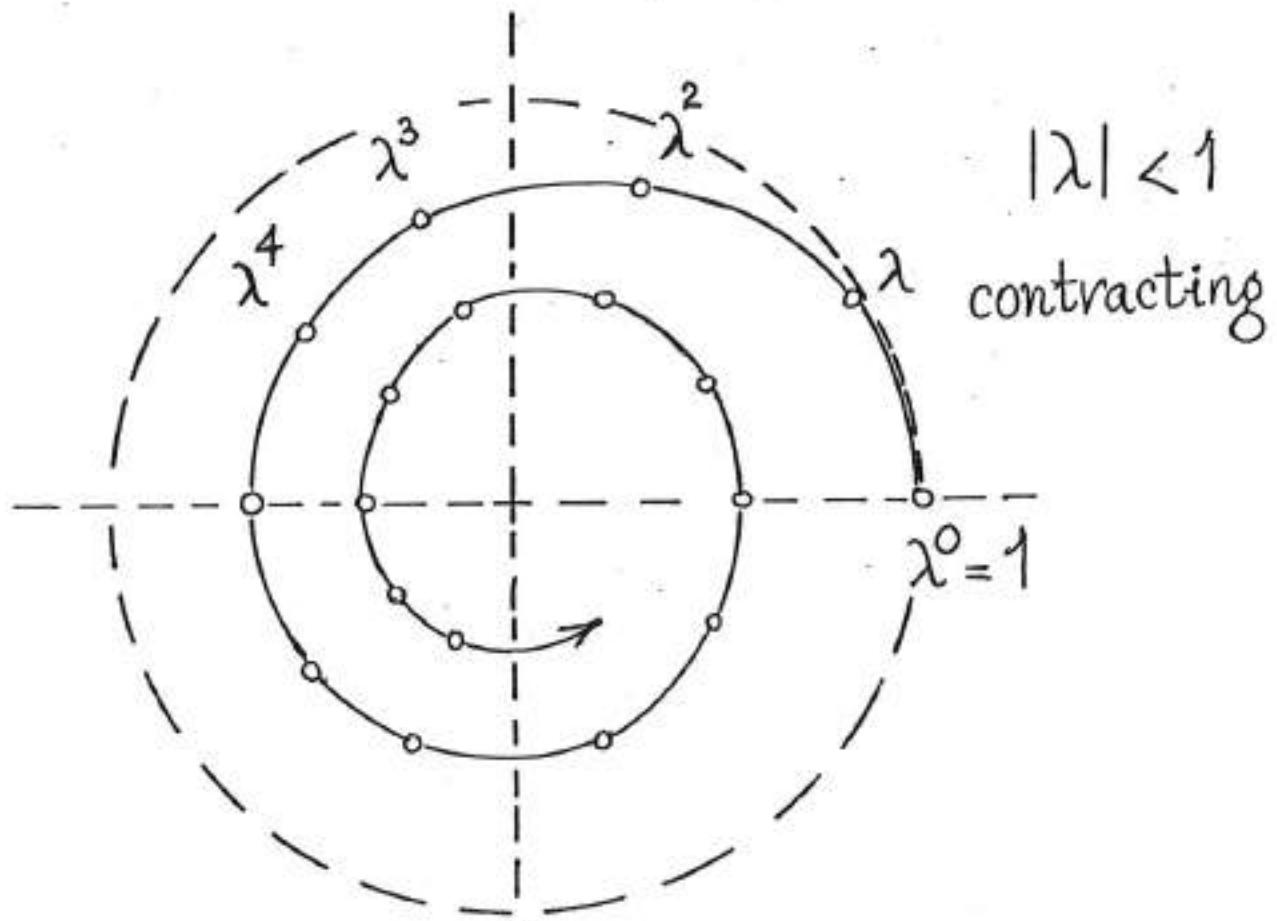
The connection between the theory of optimal
price determination and that of the continued fraction
integral was pointed out in passing in

W — A numerical method for estimating parameters
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Fixed complex number $\lambda \neq 0, 1$

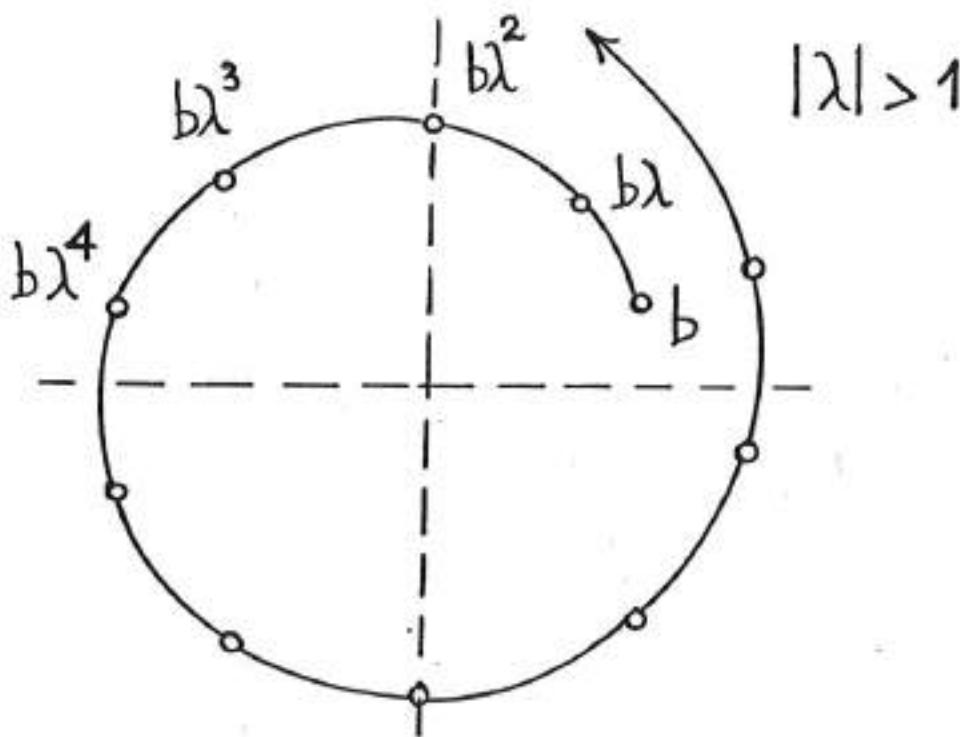
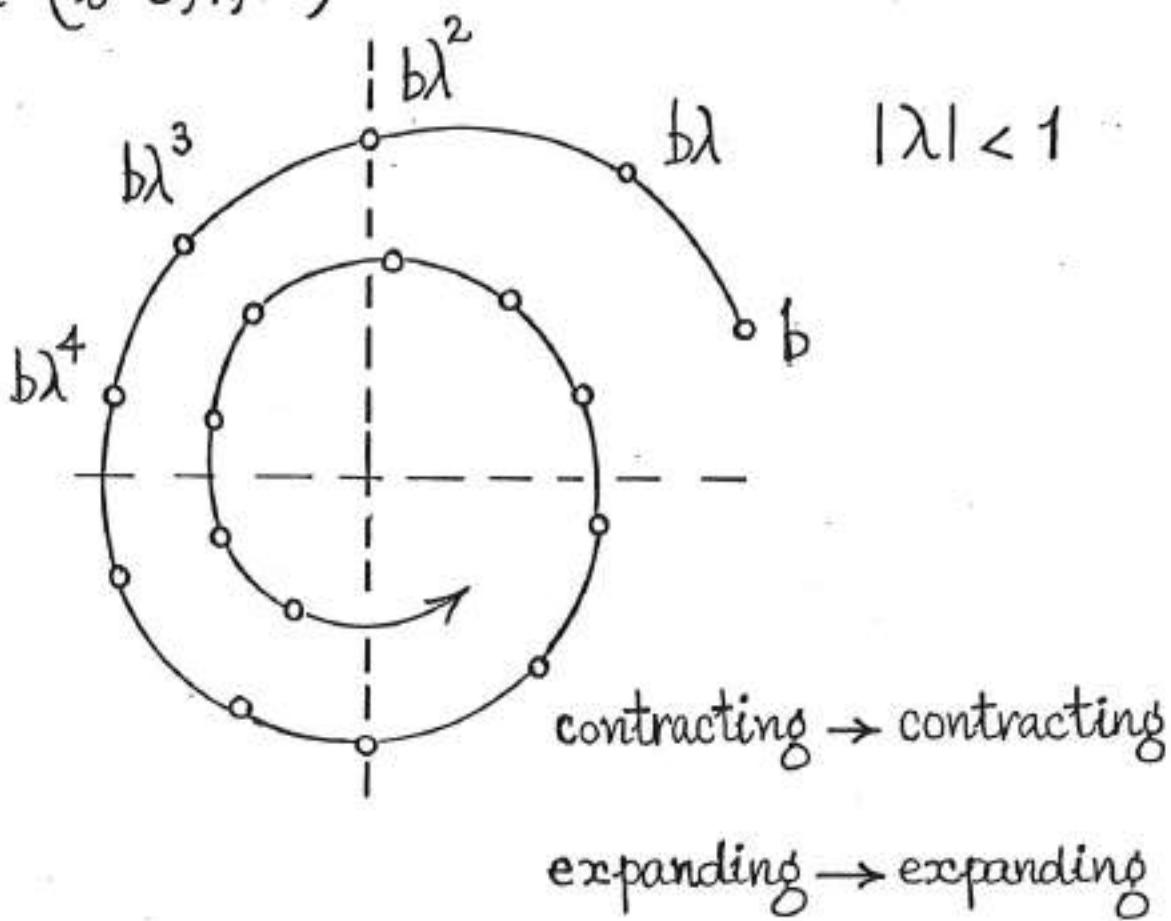


λ^i ($i = 0, 1, \dots$): Spiral, center at origin



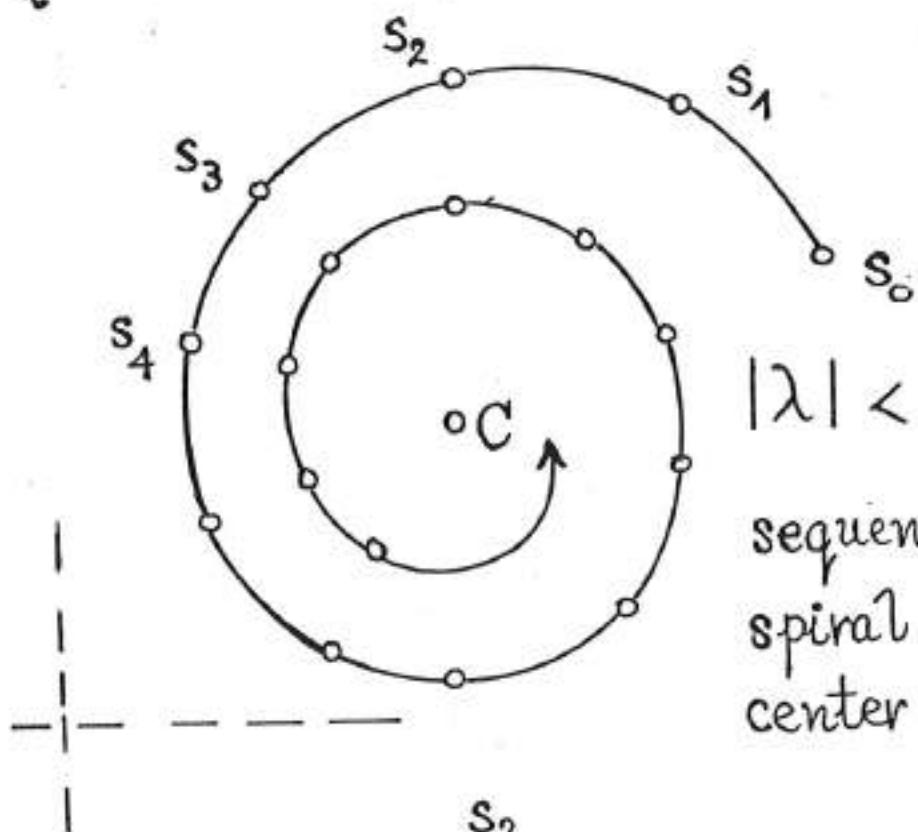
multiplication by constant coefficient $b \neq 0$

$$b\lambda^i \quad (i=0,1,\dots)$$



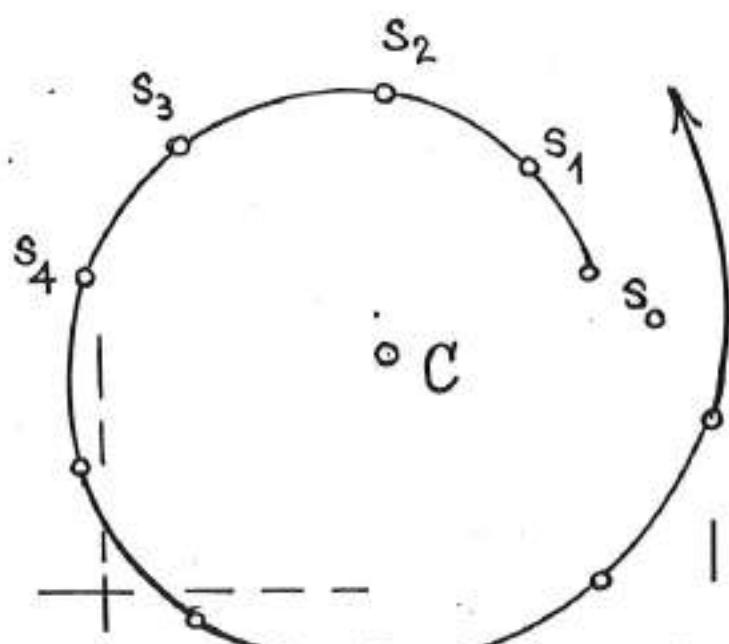
displacement by constant C

$$s_i = C + b\lambda^i \quad (i=0,1,\dots) \text{ First order spiral sequence}$$



$$|\lambda| < 1:$$

sequence converges to C
spiral contracts towards
center C



$$|\lambda| > 1:$$

sequence diverges
spiral expands away
from C

If $\lim_{i \rightarrow \infty} s_i$ is finite

$$\lim_{i \rightarrow \infty} s_i = C$$

Problem: Given three successive members

S_k, S_{k+1}, S_{k+2} of a sequence

It is known that they have the forms

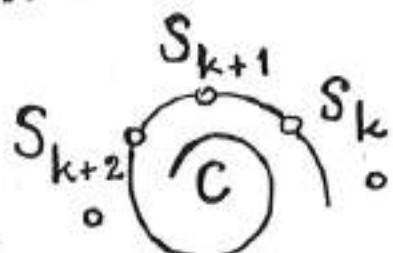
$$S_k = C + b\lambda^k, \quad S_{k+1} = C + b\lambda^{k+1}, \quad S_{k+2} = C + b\lambda^{k+2}$$

but the values of C, b, λ and k are unknown

Find C alone

$$\begin{array}{c} \varepsilon_0^{(k)} \\ \varepsilon_0^{(k+1)} \\ \varepsilon_0^{(k+2)} \\ \hline \end{array} \begin{array}{c} \varepsilon_1^{(k)} \\ \varepsilon_1^{(k+1)} \\ \varepsilon_1^{(k+2)} \\ \hline \end{array} \begin{array}{c} \varepsilon_2^{(k)} \\ \hline \end{array}$$

$$\left| \begin{array}{l} \varepsilon_0^{(k)} = S_k \\ \varepsilon_0^{(k+1)} = S_{k+1} \\ \varepsilon_0^{(k+2)} = S_{k+2} \end{array} \right.$$



$$\varepsilon_1^{(k)} = \frac{1}{\varepsilon_0^{(k+1)} - \varepsilon_0^{(k)}}, \quad \varepsilon_1^{(k+1)} = \frac{1}{\varepsilon_0^{(k+2)} - \varepsilon_0^{(k+1)}}$$

$$\varepsilon_2^{(k)} = \varepsilon_0^{(k+1)} + \frac{1}{\varepsilon_1^{(k+1)} - \varepsilon_1^{(k)}}$$

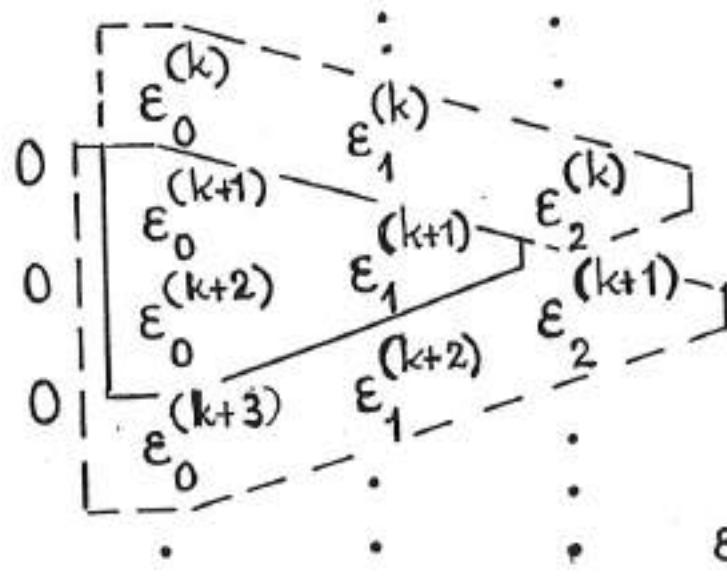
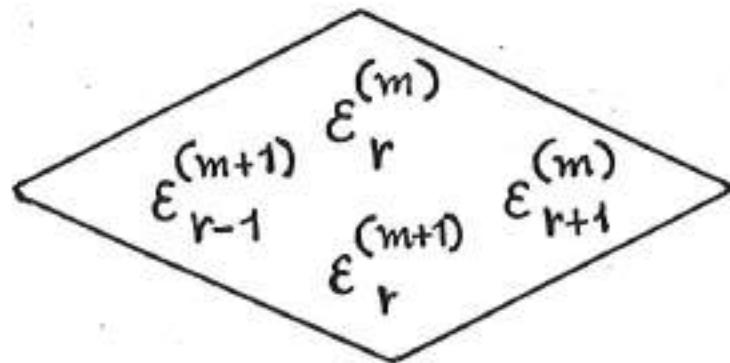
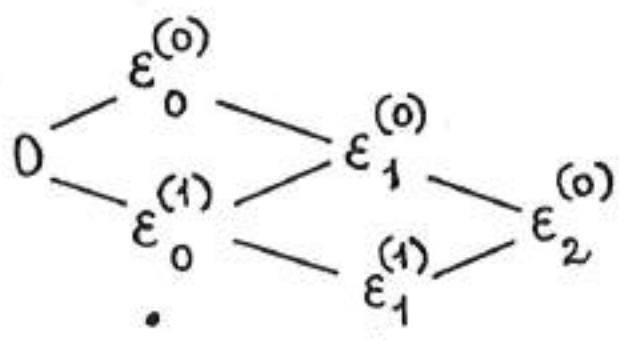
$$\varepsilon_1^{(k)} = \frac{1}{C + b\lambda^{k+1} - C - b\lambda^k} = \frac{1}{b\lambda^k(\lambda - 1)}, \quad \varepsilon_1^{(k+1)} = \frac{1}{b\lambda^{k+1}(\lambda - 1)}$$

$$\varepsilon_2^{(k)} = C + b\lambda^{k+1} + \frac{b\lambda^{k+1}(\lambda - 1)}{1 - \lambda} = C$$

Second problem

$$S_{k+1} = C' + b' \lambda'^{k+1}, S_{k+2} = C' + b' \lambda'^{k+2}, S_{k+3} = C' + b' \lambda'^{k+3}$$

Find C'



$$\left. \begin{aligned} \varepsilon_0^{(k+1)} &= S_{k+1} \\ \varepsilon_0^{(k+2)} &= S_{k+2} \\ \varepsilon_0^{(k+3)} &= S_{k+3} \end{aligned} \right\} \begin{aligned} \varepsilon_2 &= C \\ \varepsilon_0^{(k+1)} &= S_{k+1} \\ \varepsilon_0^{(k+2)} &= S_{k+2} \end{aligned}$$

$$\varepsilon_0^{(i)} = S_i = C + b \lambda^i$$

$$\varepsilon_2^{(i)} = C \quad (i=0,1,\dots)$$

$$\varepsilon_{-1}^{(m)} = 0 \quad (m=1,2,\dots)$$

$$\varepsilon_1^{(m)} = \frac{1}{\varepsilon_0^{(m+1)} - \varepsilon_0^{(m)}} \Rightarrow \varepsilon_1^{(m)} = \varepsilon_{-1}^{(m+1)} + \frac{1}{\varepsilon_0^{(m+1)} - \varepsilon_0^{(m)}}$$

$$\varepsilon_2^{(m)} = \varepsilon_0^{(m)} + \frac{1}{\varepsilon_1^{(m+1)} - \varepsilon_1^{(m)}}$$

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + \frac{1}{\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)}} \quad (r=0,1; m=0,1,\dots)$$

Second order spiral sequence : For $i = 0, 1, \dots$

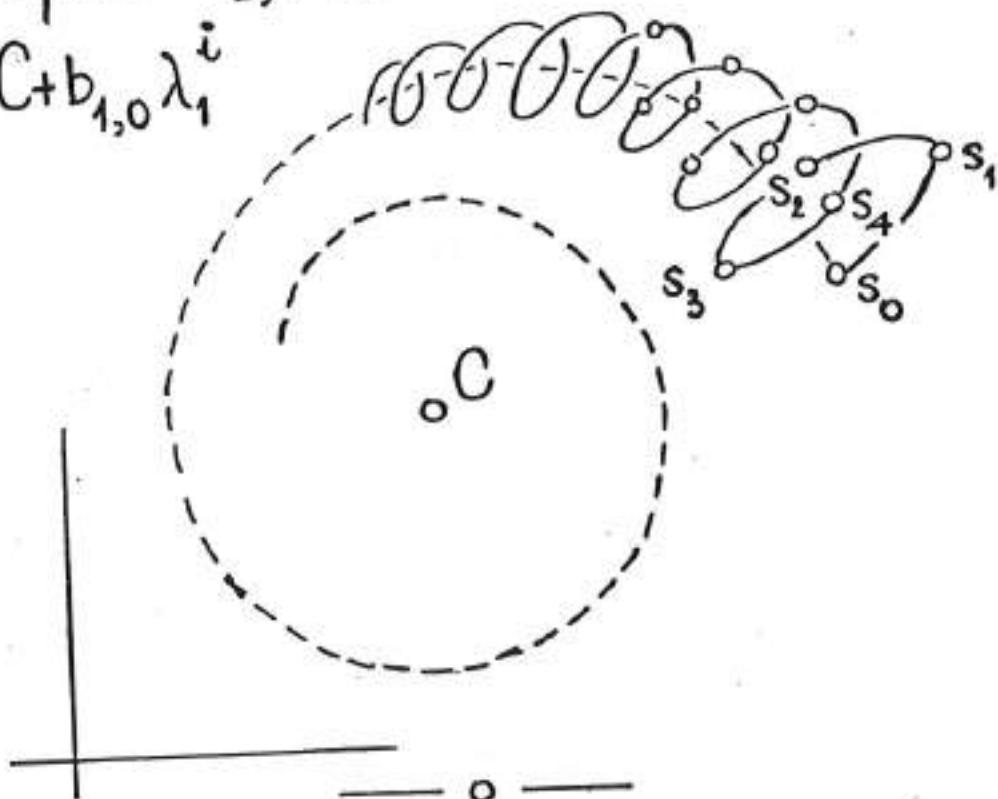
$$s_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{\nu=0}^{\tau(j)} b_{j,\nu} i^\nu \right\}; \quad \sum_{j=1}^h \{\tau(j)+1\} = 2$$

a) $h=1; \tau(1)=1: s_i = C + \lambda_1^i \{b_{1,0} + b_{1,1} i\}$

b) $h=2; \tau(1)=\tau(2)=0: s_i = C + b_{1,0} \lambda_1^i + b_{2,0} \lambda_2^i$

Spiral $b_{2,0} \lambda_2^i$ with center moving on spiral

$$C + b_{1,0} \lambda_1^i$$



n^{th} order spiral sequence

$$s_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{\nu=0}^{\tau(j)} b_{j,\nu} i^\nu \right\}; \quad \sum_{j=1}^h \{\tau(j)+1\} = n$$

($i = 0, 1, \dots$). If $\lim_{i \rightarrow \infty} s_i$ is finite, $\lim_{i \rightarrow \infty} s_i = C$

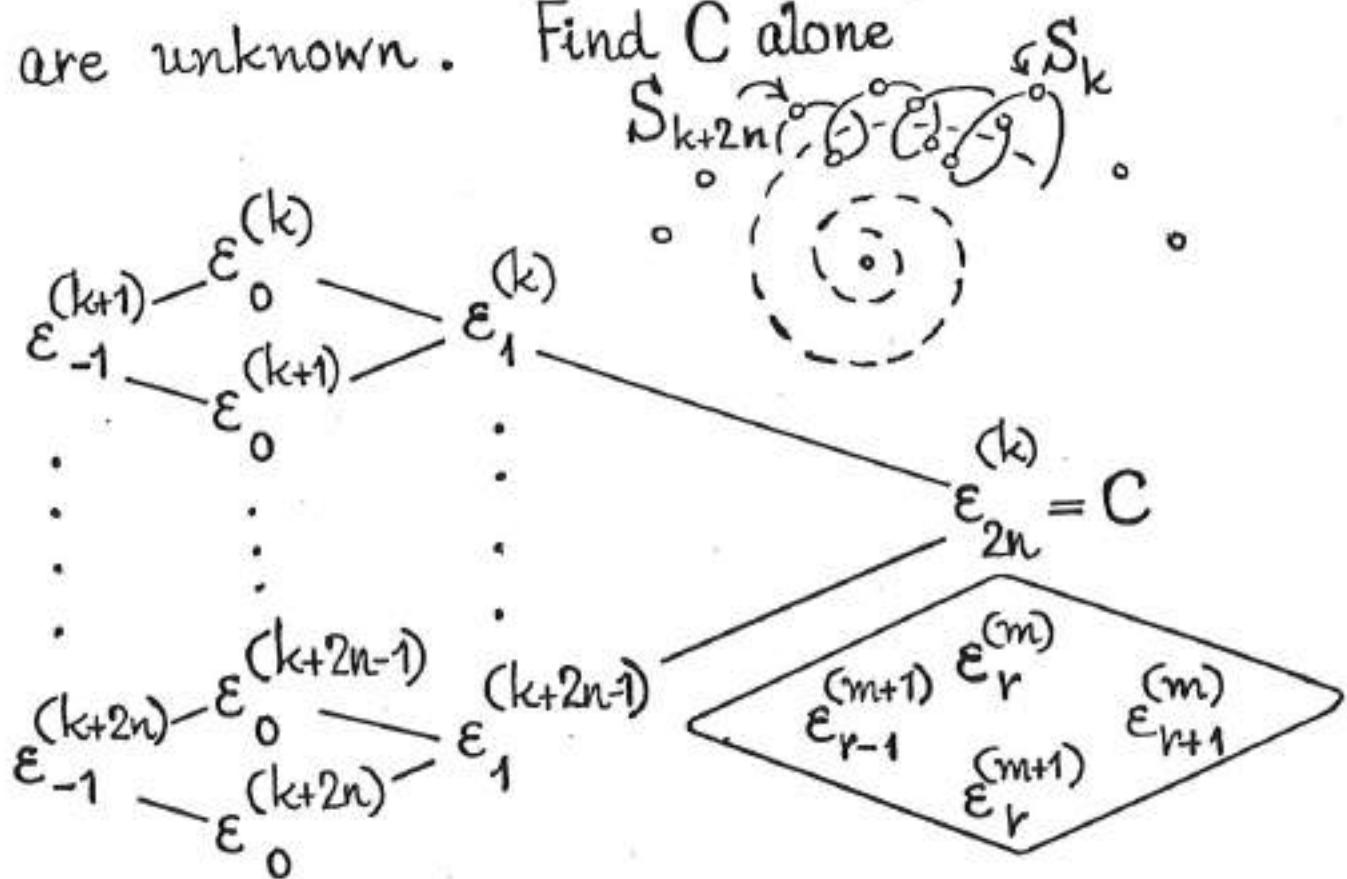
Problem: Given $2n+1$ successive members S_i

$(i=k, \dots, k+2n)$ of a sequence

It is known that they have the form

$$S_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{v=0}^{\tau(j)} b_{j,v} i^v \right\}; \sum_{j=1}^h \{\tau(j)+1\} = n$$

but the values of $C, h, \lambda_j, \tau(j), b_{j,v}$ and k are unknown. Find C alone



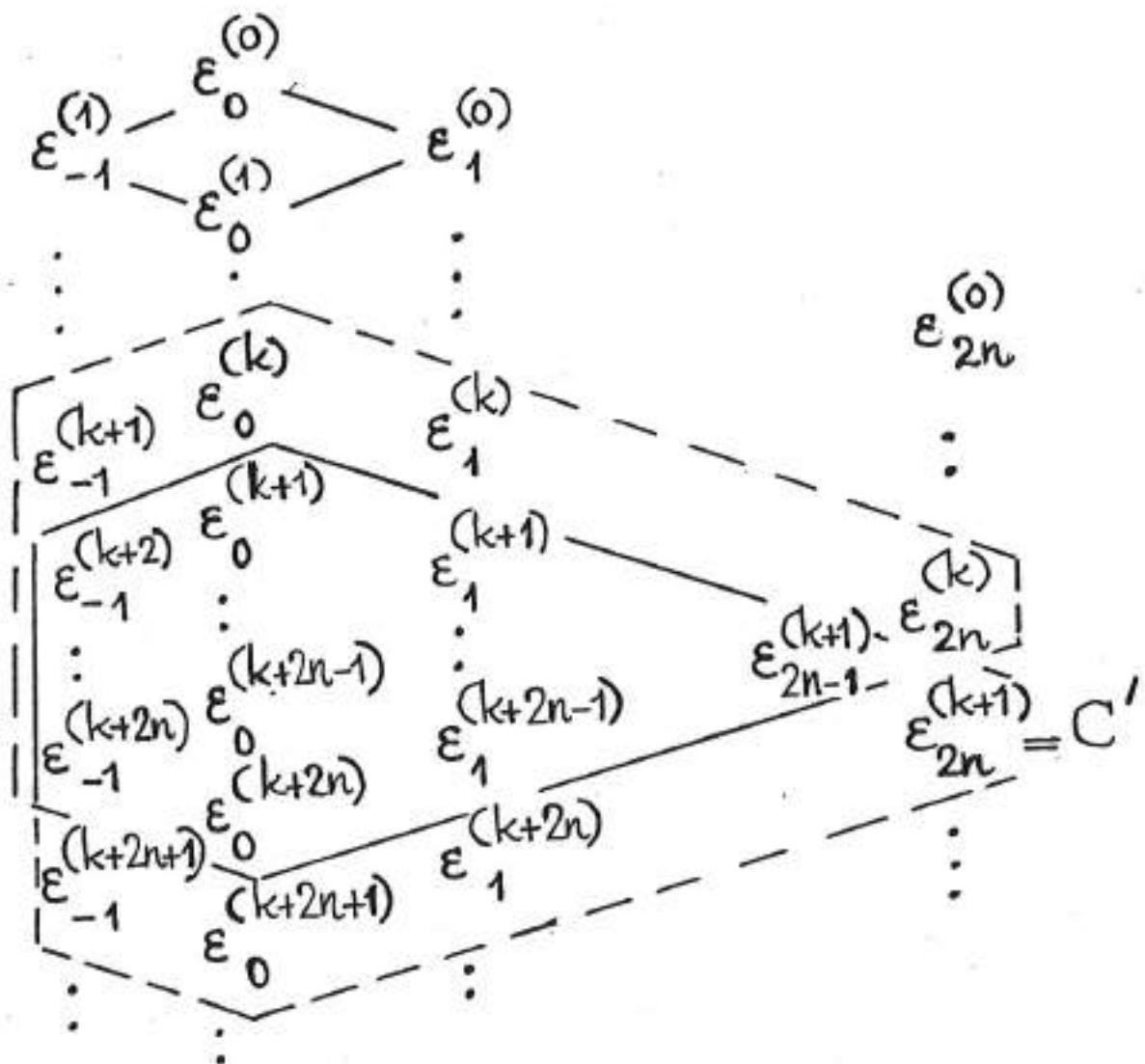
$$\epsilon_{-1}^{(i)} = 0 \quad (i=k+1, \dots, k+2n), \quad \epsilon_0^{(i)} = S_i \quad (i=k, \dots, k+2n)$$

$$\epsilon_{r+1}^{(m)} = \epsilon_{r-1}^{(m+1)} + \frac{1}{\epsilon_r^{(m+1)} - \epsilon_r^{(m)}}$$

Second problem: For $i = k+1, k+2, \dots, k+2n+1$

$$S_i = C' + \sum_{j=1}^{h'} \lambda_j' i \left\{ \sum_{\nu=0}^{\tau'(j)} b_{j,\nu}' i^\nu \right\}; \quad \sum_{j=1}^{h'} \{\tau'(j)+1\} = n$$

Find C'



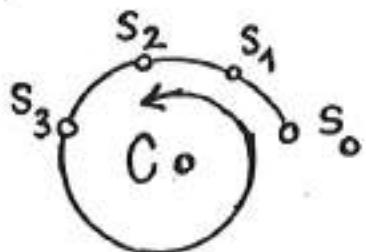
$$\varepsilon_{-1}^{(i)} = 0 \quad (i = k+2, \dots, k+2n+1), \quad \varepsilon_0^{(i)} = S_i \quad (i = k+1, \dots, k+2n+1)$$

$$\varepsilon_0^{(i)} = S_i = C + \sum_{j=1}^{h'} \lambda_j^i \left\{ \sum_{\nu=0}^{\tau(j)} b_{j,\nu} i^\nu \right\}; \quad \sum_{j=1}^{h'} \{\tau(j)+1\} = n$$

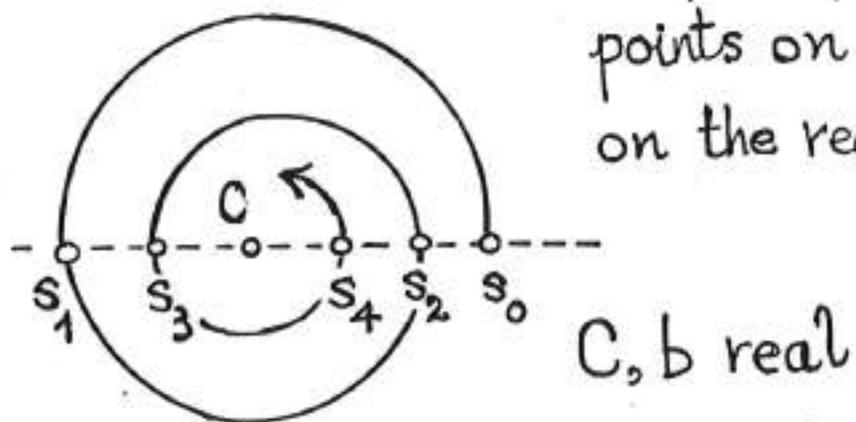
$$(i = 0, 1, \dots) \Rightarrow \varepsilon_{2n}^{(i)} = C \quad (i = 0, 1, \dots)$$

Complex number theory

$$s_i = C + b\lambda^i \quad (i=0,1,\dots)$$

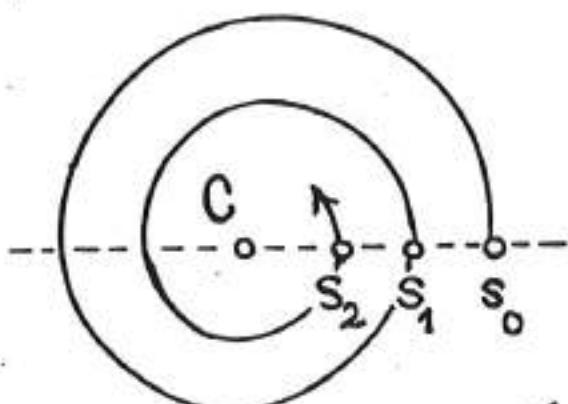


Real number theory: the spiral is in the complex plane, but the points on the spiral are on the real axis



C, b real

$\arg(\lambda) = \pi : \lambda$ real and negative



C, b real, $\arg(\lambda) = 2\pi$
 λ real and positive

The numbers $\varepsilon_r^{(m)}$ derived from a real initial sequence $\varepsilon_0^{(m)} = s_m$ ($m=0,1,\dots$) are real

Estimation of the limit of a sequence by spiral fitting

Problem: Given a sequence of complex or real numbers S_0, S_1, S_2, \dots

Estimate its limit in terms of the values of its initial members



Solution

Fit a first order spiral sequence to S_0, S_1 and S_2 ; find its center. This center is the estimate of

the limit derived from S_0, S_1 and S_2 : $\epsilon_2^{(0)}$

Fit a first order spiral sequence to S_1, S_2 and S_3 ;

... limit derived from S_1, S_2 and S_3 : $\epsilon_2^{(1)}$

Fit a second order spiral sequence to S_0, S_1, \dots, S_4 ;

... limit derived from S_0, S_1, \dots, S_4 : $\epsilon_4^{(0)}$

Systematic solution: ϵ -algorithm

Determine numbers $\epsilon_r^{(m)}$ from $\epsilon_0^{(m)} = S_m$ ($m=0, 1, \dots$)

$\epsilon_{2r}^{(m)}$ is the center of the r^{th} order spiral sequence fitted

to S_m, \dots, S_{m+2r} and is an estimate of the limit

Examples:

Partial sums of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln(2) \approx 0.693\dots$$

$$\varepsilon_0^{(m)} = S_m = \sum_{i=0}^{m-1} \frac{(-1)^i}{i+1}$$

Centers of first order
spiral sequences

$$\begin{array}{ccccccc} 0 & \varepsilon_0^{(0)} = 0.0 & & \varepsilon_1^{(0)} = 1 & & \varepsilon_2^{(0)} = 0.667 & \downarrow \\ 0 & \varepsilon_0^{(1)} = 1.0 & & & & & * \\ 0 & \varepsilon_0^{(2)} = 0.5 & & \varepsilon_1^{(1)} = -2 & & \varepsilon_2^{(1)} = 0.7 & \varepsilon_4^{(0)} = 0.692 \\ 0 & \varepsilon_0^{(3)} = 0.833 & & \varepsilon_1^{(2)} = 3 & & \varepsilon_2^{(2)} = 0.690 & \\ 0 & \varepsilon_0^{(4)} = 0.583 & & \varepsilon_1^{(3)} = -4 & & & \uparrow \end{array}$$

center of second
order spiral sequence
fitted to S_0, S_1, \dots, S_4

$$\varepsilon_1^{(0)} = 0 + \frac{1}{1-0} = 1$$

$$\varepsilon_1^{(1)} = 0 + \frac{1}{0.5-1} = \frac{1}{(-\frac{1}{2})} = -2$$

$$\varepsilon_2^{(0)} = 1 + \frac{1}{-2-1} = \frac{2}{3} = 0.667$$

$$S_m = \sum_{i=0}^{m-1} \frac{(-1)^i}{i+1} \quad (m=0,1,\dots)$$

$$S_m \rightarrow \ln(2) = 0.69314\ 71805\ 6$$

$$\varepsilon_0^{(0)} = 0$$

$$\downarrow \varepsilon_0^{(1)} = 1 \rightarrow \varepsilon_2^{(0)}$$

$$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \quad \downarrow \rightarrow$$

$$\varepsilon_0^{(8)} = 0.63\dots$$

Center of fourth order spiral
sequence fitted to S_0, \dots, S_8

$$\downarrow \rightarrow \varepsilon_8^{(0)} = 0.69314\ 6\dots$$

$$\varepsilon_0^{(0)} = 0$$

$$\varepsilon_0^{(1)} = 1$$

$$\vdots$$

$$\varepsilon_0^{(8)} = 0.69314\ 6\dots$$

$$\varepsilon_8^{(0)} = 0.69314\ 71805\ 3$$

$|\ln(2) - S_m| \div \frac{1}{2^m}$: Calculation of the value

of the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \dots$ correct to

ten decimals requires summation of ten billion

terms. Repeated use of the ε -algorithm yields

the same accuracy from only eight terms.

$$S_m = \sum_{i=0}^{m-1} \binom{-\frac{1}{2}}{i} \frac{1}{i+1} \quad (m=0,1,\dots)$$

$$S_m \rightarrow 2(\sqrt{2}-1) \doteq 0.82842\ 71247\ 43\dots$$

$$\varepsilon_0^{(0)} = 0$$

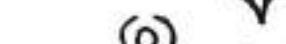


$$\varepsilon_0^{(1)} = 1 \rightarrow \varepsilon_2^{(0)}$$



$$\varepsilon_0^{(6)} = 0.81\dots$$

Center of third order spiral
sequence fitted to S_0, \dots, S_6



$$\rightarrow \varepsilon_6^{(0)} = 0.82840\dots$$

— o —

$$\varepsilon_0^{(0)} = 0$$

$$\varepsilon_0^{(1)} = 1$$



$$\varepsilon_0^{(6)} = 0.82840\dots$$

$$\varepsilon_6^{(0)} = 0.82842\ 71247\ 49$$

$|2(\sqrt{2}-1) - S_m| \div \frac{1}{2^m}$: Calculation of the value

of the sum of the series correct to eleven
decimals requires summation of one hundred
billion terms. Repeated use of the ε -algorithm
yields the same accuracy from only six terms

Expanding spirals : Divergent series and sequences of partial sums

$$F(r, x) = r! e^x \int_x^\infty \frac{1}{t^{r+1}} e^{-t} dt \quad (x > 0; r=0, 1, \dots)$$

Integration by parts: $I(x) = \int_x^\infty a(t) b(t) dt$

$$B(t) = \int_t^\infty b(t') dt'$$

$$I(x) = - \lim_{t \rightarrow \infty} \{a(t) B(t)\} + a(x) B(x) + \int_x^\infty \frac{da(t)}{dt} B(t) dt$$

$$a(t) = \frac{1}{t^{r+1}}, \quad b(t) = e^{-t}, \quad B(t) = e^{-t}, \quad \frac{da(t)}{dt} = -\frac{r+1}{t^{r+2}}$$

$$\int_x^\infty \frac{1}{t^{r+1}} e^{-t} dt = \frac{e^{-x}}{x^{r+1}} - \int_x^\infty \frac{(r+1)}{t^{r+2}} e^{-t} dt$$

$$\times \text{ by } r! e^{-x}: \quad F(r, x) = \frac{r!}{x^{r+1}} - F(r+1, x)$$

$$+ve = +ve - (+ve)$$

$$F(r, x) < \frac{r!}{x^{r+1}}$$

$$F(r, x) = r! e^x \int_x^\infty \frac{1}{t^{r+1}} e^{-t} dt : F(r, x) < \frac{r!}{x^{r+1}}$$

$$F(r, x) = \frac{r!}{x^{r+1}} - F(r+1, x)$$

Exponential integral : $e^x \int_x^\infty \frac{e^{-t}}{t} dt = F(0, x)$

$$\begin{aligned} F(0, x) &= \frac{0!}{x} - F(1, x) = \frac{0!}{x} - \frac{1!}{x^2} + F(2, x) \\ &= \dots = \sum_{i=0}^{m-1} \frac{(-1)^i i!}{x^{i+1}} + e_m(x) \end{aligned}$$

$$\text{error term } e_m(x) = (-1)^m F(m, x)$$

$$\text{Divergence : } \left| \frac{(-1)^i i!}{x^{i+1}} \right| = \frac{i}{x} \left| \frac{(-1)^{i-1} (i-1)!}{x^i} \right|$$

$i > x$: terms increase in magnitude

$$\text{Semi-convergence : } |e_m(x)| < \left| \frac{(-1)^m m!}{x^{m+1}} \right|$$

Use: calculate $e^{100} \int_{100}^\infty \frac{e^{-t}}{t} dt$ with error less than 0.000 000 1

$$e_3(100) < \frac{3!}{100^4} = 0.000\ 000\ 06 < 0.000\ 000\ 1$$

$$e^{100} \int_{100}^\infty \frac{e^{-t}}{t} dt = \frac{1}{100} - \frac{1}{10\ 000} + \frac{2}{1000\ 000} = 0.009\ 902$$

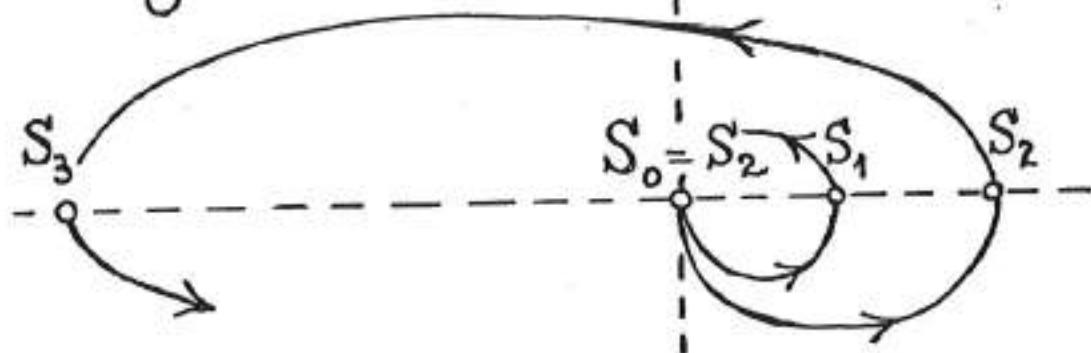
x small: direct use of series unfeasible e.g. $x=1$

$$|e_0(1)| < 1, |e_1(1)| < 1, |e_2(1)| < 2, |e_3(1)| < 6,$$

Series: $0! - 1! + 2! - 3! + 4! - 5! + \dots$
 $1 - 1 + 2 - 6 + 24 - 120 + \dots$

Partial sums $0, 1, 0, 2, -4, +20, -100, \dots$

Fitting by expanding spiral sequence



$$e^1 \int_1^\infty \frac{e^{-t}}{t} dt \doteq 0.59635$$

center of fifth order spiral
sequence fitted to S_0, \dots, S_{10}

$$\varepsilon_0^{(0)} = 0$$

$$\downarrow^{(1)} \varepsilon_0^{(1)} = 1 \rightarrow$$

$$\vdots \quad \varepsilon_0^{(10)} = -326980$$

$$\varepsilon_0^{(0)} = 0$$

$$\downarrow^{(1)} \varepsilon_0^{(1)} = 1$$

$$\vdots \quad \varepsilon_0^{(10)} = 0.59509$$

$$\varepsilon_{10}^{(0)} \downarrow = 0.59509$$

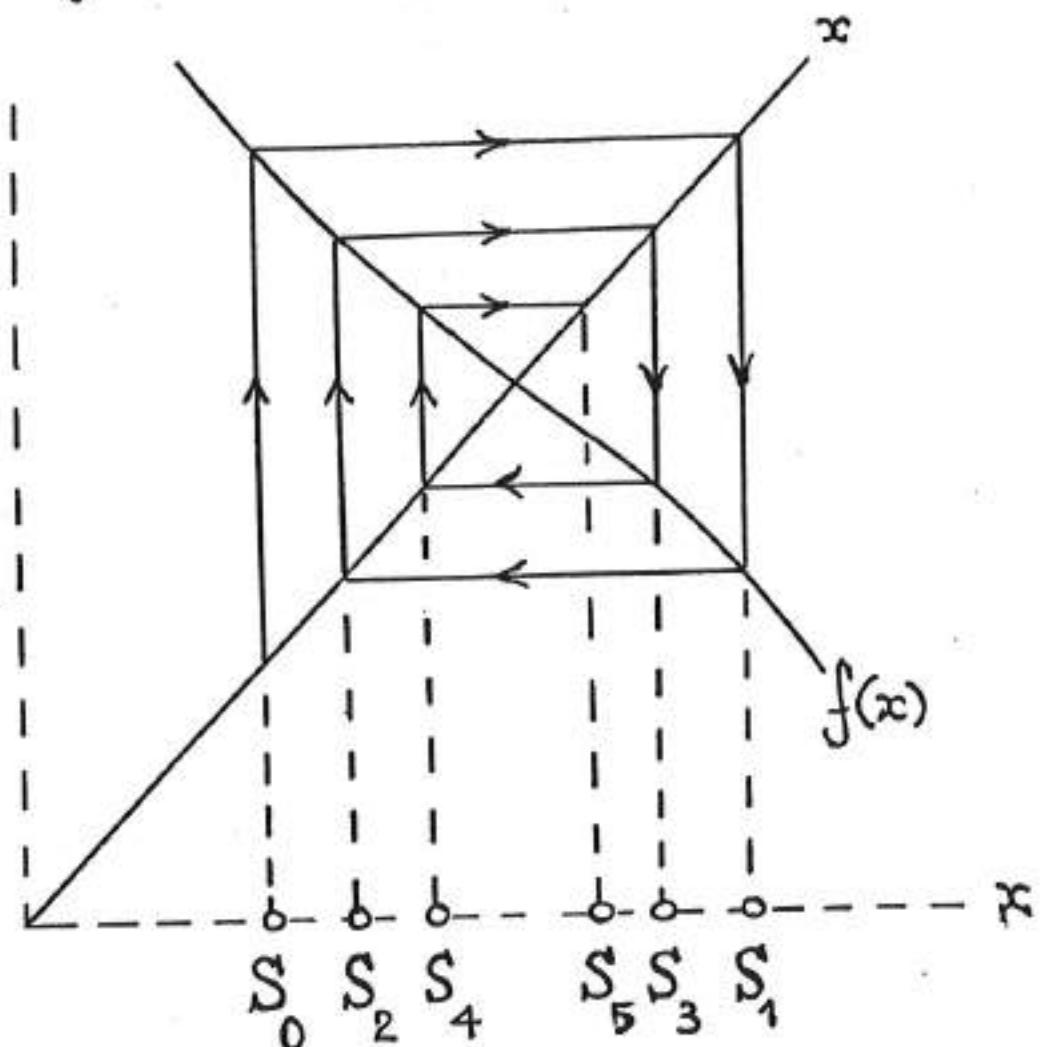
$$\varepsilon_{10}^{(0)} = 0.59639$$

Fixed point of a function f

Value C of x for which $x = f(x)$

Initial approximation S_0 to C

$S_1 = f(S_0)$, $S_2 = f(S_1)$, $S_3 = f(S_2), \dots$



$$C^{(k)} \quad (k=0,1,\dots) \quad C^{(k)} \rightarrow C$$

r^{th} order convergence: $C^{(k+1)} - C = O\{(C^{(0)} - C)^r\}$

$$C^{(1)} - C = O\{(C^{(0)} - C)^r\}$$

$$\begin{aligned} C^{(2)} - C &= O\{(C^{(1)} - C)^r\} = O\{O\{(C^{(0)} - C)^r\}\}^r \\ &= O\{(C^{(0)} - C)^{r^2}\} \end{aligned}$$

$$C^{(k)} - C = O\{(C^{(0)} - C)^{r^k}\} \quad (k=0,1,\dots)$$

$$\text{Special process: } C^{(k+1)}_k - C \doteq (C^{(k)} - C)^r$$

$$C^{(k)} - C \doteq (C^{(0)} - C)^r \quad (k=0,1,\dots)$$

$$\text{Example: } C^{(0)} - C = 0.1 \times \times \dots$$

$$(C^{(0)} - C)^n \doteq 0.0 \underset{\leftarrow(n-1)\rightarrow}{\dots} 0 1 \times \times \dots$$

$$(C^{(0)} - C)^{r^k} \doteq 0.0 \underset{\leftarrow(r^k-1)\rightarrow}{\dots} 0 \times \times \dots$$

Linear convergence: $r = 1$

$$C^{(1)} - C \doteq 0.1 \times \times \dots, C^{(2)} - C \doteq 0.1 \times \times \dots$$

Quadratic convergence: $r = 2$

$$C^{(1)} - C \doteq 0.01 \times \times \dots, C^{(2)} - C \doteq 0.0001 \times \times \dots$$

Cubic convergence: $r = 3$

$$C^{(1)} - C \doteq 0.001 \times \times \dots, C^{(2)} - C \doteq 0.00000001 \times \times \dots$$

High order convergent process for obtaining a fixed point C of a function f without using derivatives of f

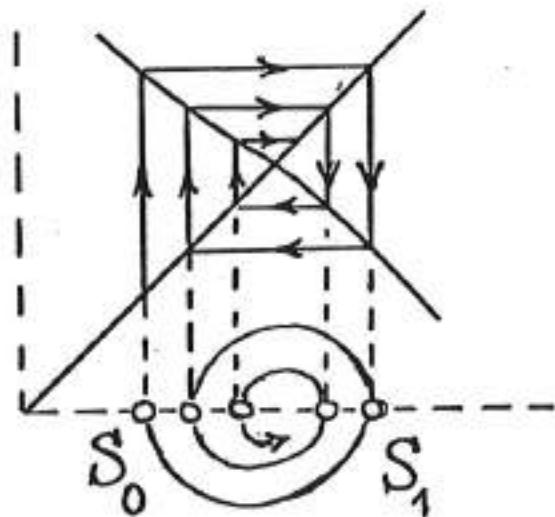
Initial estimate $C^{(0)}$ of C . With $S_0 = C^{(0)}$ iterate $S_{i+1} = f(S_i)$ to obtain S_1, \dots, S_{2r}

Fit r^{th} order spiral sequence to S_0, \dots, S_{2r} and find its center $C^{(1)}$ (compute $C^{(1)} = \varepsilon_{2r}^{(0)}$)

Take $C^{(1)}$ to be a new initial estimate of C

Repeat the process to obtain $C^{(2)}, \dots$

$$C^{(k+1)} - C = O\{(C^{(k)} - C)^r\}$$



Relationship of ε -algorithm applied to members of a field

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + \frac{1}{\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)}}$$

$$\varepsilon_{r-1}^{(m+1)} \quad \varepsilon_r^{(m+1)} \quad \varepsilon_r^{(m)} \quad \varepsilon_{r+1}^{(m)}$$

may be written as

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + (\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)})^{-1}$$

and may then be applied to any mathematical system over which addition, subtraction and the formation of an inverse are defined

Examples

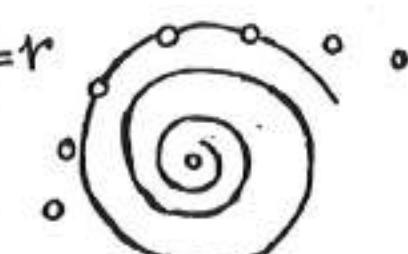
Rings over which an inverse is defined
(e.g. square matrices over a field)

Inner product spaces
(e.g. vectors over a field)

Nonassociative algebras
(e.g. Cayley numbers)

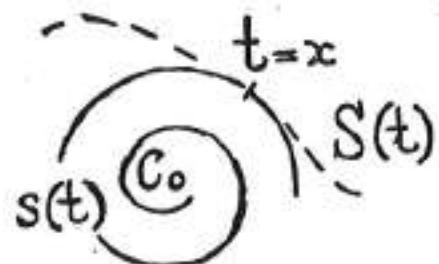
Graphs

S_0, S_1, \dots : Estimation of $\lim_{i \rightarrow \infty} S_i$ by fitting an r^{th} order spiral sequence to a subsequence S_m, \dots, S_{m+2r} by use of relationships, holding for $i = m, \dots, m+2r$

$$S_i = C + \sum_{j=1}^h \lambda_j^i \left\{ \sum_{v=0}^{\tau(j)} b_{j,v} i^v \right\}; \sum_{j=1}^h \{\tau(j)+1\} = r$$


and determining the center $C = \varepsilon_{2r}^{(m)}$

which is an estimate of $\lim_{i \rightarrow \infty} S_i$



Continuous analog

Replacement of variables: i by t , λ_j^i by $e^{-\alpha_j t}$
 n^{th} order spiral function

$$s(t) = C + \sum_{j=1}^h e^{-\alpha_j t} \left\{ \sum_{v=0}^{\tau(j)} b_{j,v} t^v \right\}; \sum_{j=1}^h \{\tau(j)+1\} = n$$

Problem: a) It is known that $s(t)$ has the above form but the values of $C, h, \alpha_j, \tau(j)$ and $b_{j,v}$ are unknown

b) The value $S^{(0)} = \lim_{t \rightarrow \infty} s(t)$ and the derivative values $S^{(i)} = \lim_{t \rightarrow \infty} \frac{d^i s(t)}{dt^i}$ ($i = 1, \dots, 2n$) are given but the value of α itself is not known

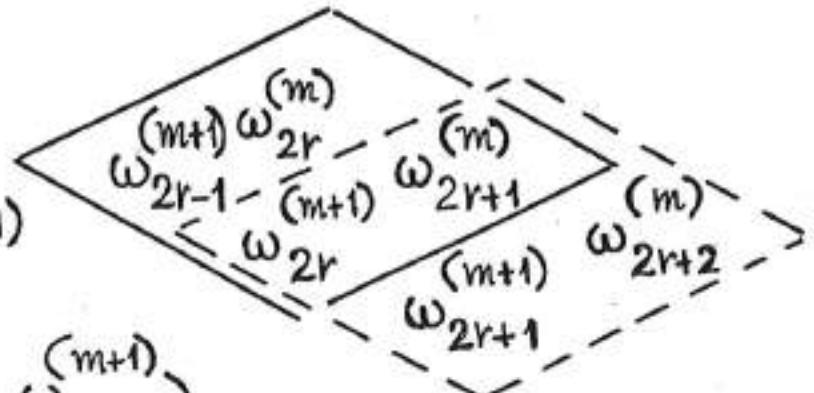
Find C alone

$$\begin{aligned} \omega_{-1}^{(1)} &= 0 - \frac{\omega_0^{(0)}}{S^{(1)}} = S^{(0)} \\ \vdots & \quad \vdots \quad \vdots \\ \vdots & \quad \vdots \quad \vdots \\ \vdots & \quad \vdots \quad \vdots \\ \omega_{-1}^{(2n)} &= 0 - \frac{\omega_0^{(2n-1)}}{S^{(2n-1)}} = S^{(0)} \\ \vdots & \quad \vdots \quad \vdots \end{aligned}$$

Solution

$$\begin{array}{c} \omega_{2n-1}^{(0)} \\ \omega_{2n-1}^{(1)} \end{array} \geq \omega_{2n}^{(0)} = C$$

$$\omega_{2r+1}^{(m)} = \omega_{2r-1}^{(m+1)} + \frac{\omega_{2r}^{(m)}}{\omega_{2r}^{(m+1)}}$$



$$\omega_{2r+2}^{(m)} = \omega_{2r}^{(m+1)} (\omega_{2r+1}^{(m)} - \omega_{2r+1}^{(m+1)})$$

$$\omega_1^{(0)} = 0 + \frac{S^{(0)}}{S^{(1)}} = \frac{S^{(0)}}{S^{(1)}}, \quad \omega_1^{(1)} = \frac{S^{(1)}}{S^{(2)}}$$

$$\omega_2^{(0)} = S^{(1)} \left\{ \frac{S^{(0)}}{S^{(1)}} - \frac{S^{(1)}}{S^{(2)}} \right\} = S^{(0)} - \frac{S^{(1)}{}^2}{S^{(2)}}, \dots$$

Example $n=1$, $s(t) = C + b e^{-\alpha t}$

$$S^{(0)} = C + b e^{-\alpha x}, \quad S^{(1)} = -b \alpha e^{-\alpha x}, \quad S^{(2)} = b \alpha^2 e^{-\alpha x}$$

$$\omega_2^{(0)} = C + b e^{-\alpha x} - \frac{b^2 \alpha^2 e^{-2\alpha x}}{b \alpha^2 e^{-\alpha x}} = C$$

If $\lim_{t \rightarrow \infty} s(t)$ is finite, $\lim_{t \rightarrow \infty} s(t) = C$

ω -algorithm

Estimation of or definition of $\lim_{t \rightarrow \infty} S(t)$ in terms of derivatives at a finite argument value

Construct numbers $\omega_r^{(m)}$ from the initial values $\omega_0^{(m)} = \cancel{S^{(m)}}$ ($m=0,1,\dots$) where $S^{(i)} = \lim_{t \rightarrow \infty} \frac{d^i S(t)}{dt^i}$

$\omega_{2r}^{(0)}$ is the center of the r^{th} order spiral function $s(t)$ fitted to $S(t)$ at $t=\infty$ by use of the relationships

$$\lim_{t \rightarrow \infty} \frac{d^i s(t)}{dt^i} = \lim_{t \rightarrow \infty} \frac{d^i S(t)}{dt^i} \quad (i=0, \dots, 2r)$$

and is an estimate of $\lim_{t \rightarrow \infty} S(t)$

If $S(t)$ has the form

$$C + \sum_{j=1}^h e^{-\alpha_j t} \left\{ \sum_{\nu=0}^{\tau(j)} b_{j,\nu} t^\nu \right\}; \quad \sum_{j=1}^h \{\tau(j)+1\} = n$$

formation of numbers $\omega_r^{(m)}$ terminates with $\omega_{2n}^{(0)} = C$

$$S(x,t) : S^{(i)} = \lim_{t \rightarrow \infty} \frac{\partial^i S(x,t)}{\partial t^i} \quad (i=0,1,\dots)$$

— o —

The continued fraction integral (cf) $\int_x^\infty \psi(t') dt'$

$$S(x,t) = \int_x^t \psi(t') dt' \Rightarrow \lim_{t \rightarrow \infty} S(x,t) = \int_x^\infty \psi(t') dt'$$

$$S^{(0)} = \int_x^\infty \psi(t') dt' = 0 \text{ and for } i=1,2,\dots$$

$$S^{(i)} = \lim_{t \rightarrow \infty} \frac{\partial^i \int_x^t \psi(t') dt'}{\partial t^i} = \frac{d^{i-1} \psi(x)}{dx^{i-1}} = \psi^{(i-1)}(x)$$

Construct numbers $\omega_r^{(m)}$ from the initial values

$$\omega_0^{(0)} = 0, \omega_0^{(m)} = \cancel{\psi^{(m-1)}(x)} \quad (m=1,2,\dots)$$

$\omega_{2r}^{(0)}$ is an estimate of (cf) $\int_x^\infty \psi(t') dt'$

First order continued fraction integral approximation

$$r=1 : \omega_2^{(0)} = S^{(0)} - \frac{S^{(1)^2}}{S^{(2)}} = -\frac{\psi(x)^2}{\psi'(x)}$$

$$\text{If } \psi(t) = \sum_{j=1}^h e^{-\alpha_j t} \left\{ \sum_{\nu=0}^{\tau(j)} b_{j,\nu} t^\nu \right\}; \quad \sum_{j=1}^h \{\tau(j)+1\} = n$$

$$\text{then } \omega_{2n}^{(0)} = (\text{cf}) \int_x^\infty \psi(t') dt'$$

$$n=1 : \omega_2^{(0)} = -\frac{(be^{-\alpha x})^2}{-\alpha be^{-\alpha x}} = \frac{b}{\alpha} e^{-\alpha x} = \int_x^\infty be^{-\alpha t'} dt'$$

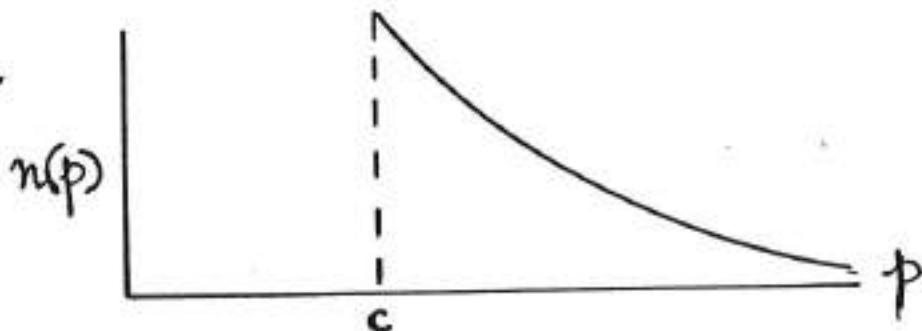
Oil pricing policy

Problem: The owner of a well producing oil with cost price c per barrel wishes to determine

- the price P per barrel which maximizes his profit and
- his profit at the optimal price P

Solution:

- Determine the market-response function (the number of barrels sold weekly at price p) by observation



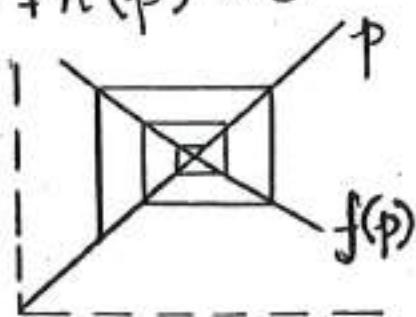
At price p his weekly profit is $n(p)(p-c)$

This attains a maximum when p assumes the

value P for which $\frac{dn(p)}{dp}(p-c) + n(p) = 0$

i.e. when $p=f(p)$ where

$f(p)=c-\frac{n(p)}{-\frac{dn(p)}{dp}}$, i.e. at a fixed point P of f



b) The optimal price P satisfies the condition

$$\frac{dn(P)}{dP}(P-c) + n(P) = 0$$

The profit at this price is

$$n(P)(P-c) = -\frac{n(P)^2}{\frac{dn(P)}{dP}}$$

The right hand side expression is the first order continued fraction integral approximation $\omega_2^{(0)}$ to

$$\int_P^\infty n(p)dp$$

It is equal to this expression if the market response function has the form $n(p)=be^{-\alpha p}$

$$n(P)(P-c) = -\frac{\overline{n(P)}^2}{\frac{dn(P)}{dP}} := n(P)(P-c) = \int_P^\infty n(p)dp$$

Inverse function $p(n)$: $\int_P^\infty n(p)dp = \int_0^n p(n)dn$

Optimality condition

$$P-c = \frac{1}{n(P)} \int_0^{n(P)} p(n)dn = \frac{\int_0^{n(P)} p(n)dn}{\int_0^{n(P)} dn}$$

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