Commuting Cayley numbers

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Abstract. Necessary and sufficient conditions are given for α and β, belonging to a division algebra of generalized Cayley numbers, to satisfy the relationship $\alpha\beta=\beta\alpha$.

We consider the alternative algebra $\mathbb{C}\{\mathbf{F},\xi\}$ of the generalized Cayley numbers over a field $\mathbf{F}$ which have the representation $\alpha=(a,B)$, $\beta=(b,B),\ldots$ where $a,A,b,B,\ldots$ are quaternions over $\mathbf{F}$, and multiply according to the rule $\alpha\beta=(ab+\xi\tilde{B}A, Ba+A\tilde{b})$, where $\tilde{b},\tilde{B}$ are conjugate to b,B and $\xi\in\mathbf{F}$ is fixed (for the theory of Cayley numbers and associated references, see [1]).

Theorem. Let $\alpha,\beta\in\mathbb{C}\{\mathbf{F},\xi\}$ where $\mathbf{F}$ is not of characteristic 2 and ξ is not the norm of any quaternion over $\mathbf{F}$. Then $\alpha\beta=\beta\alpha$ if and only if either $\alpha\in\mathbf{F}$ or $\beta=\lambda\alpha+\mu$ $(\lambda,\mu\in\mathbf{F})$.

Proof. If α,β satisfy one of the stated conditions, then $\alpha\beta=\beta\alpha$.

If $\alpha=\hat{\alpha}$ (where $\hat{\alpha}=(\tilde{a},-A)$ is the conjugate of α), and $\mathbf{F}$ is not of characteristic 2, then $\alpha\in\mathbf{F}$. We now discount this possibility; thus, in particular, $\alpha\neq 0$. If $\alpha\beta=\hat{\beta}\hat{\alpha}$, then $\alpha\beta=\omega$ say, where $\omega\in\mathbf{F}$. Since ξ is not the norm of any quaternion over $\mathbf{F}$, $\mathbb{C}\{\mathbf{F},\xi\}$ is, by a theorem of Albert [2], a division algebra, and α^{-1} exists. Hence $\beta=\omega\alpha^{-1}=\omega n(\alpha)^{-1}\hat{\alpha}=\lambda\alpha+\mu$, where $\lambda=-\omega n(\alpha)^{-1}$, $\mu=\omega n(\alpha)^{-1}t(\alpha)$ $(n(\alpha)=\alpha\hat{\alpha}\,(\in\mathbf{F})$ and $t(\alpha)=\alpha+\hat{\alpha}\,(\in\mathbf{F})$ being the norm and trace of α). We now discount the possibility that $\alpha\beta=\hat{\beta}\hat{\alpha}$. By a theorem of Artin [3], all products formed from the two fixed numbers α,β are associative and hence, trivially from $\alpha,\beta,\hat{\alpha},\hat{\beta}$. Since $\alpha\beta=\beta\alpha$, these products are also commutative. Multiplying the relationship $(\hat{\alpha}-\alpha)\beta=(\hat{\beta}-\beta)\alpha+\beta\hat{\alpha}-\beta\alpha$ throughout by $\beta\alpha-\hat{\beta}\hat{\alpha}$, we find that $(2Tn-t\tau)\beta=(2tN-T\tau)\alpha+T^2n-t^2N$ (where $t=t(\alpha)$, $n=n(\alpha)$, $T=t(\beta)$, $N=n(\beta)$, $\tau=t(\alpha\beta)$). Since $\beta\alpha-\hat{\beta}\hat{\alpha}\neq 0$, $\alpha-\hat{\alpha}\neq 0$, the inverse of $2Tn-t\tau=(\beta\alpha-\hat{\beta}\hat{\alpha})(\hat{\alpha}-\alpha)$ exists. Hence β again has a representation of the form $\lambda\alpha+\mu$ $(\lambda,\mu\in\mathbf{F})$.

By symmetry, α and β may be interchanged in the above theorem. We also remark that if $\alpha\beta=\beta\alpha$, then $(tT-2\tau)^2=(4n-t^2)(4N-T^2)$.

References.

1. Schafer R.D., Introduction to nonassociative algebras, Academic Press, New York-London (1966).
2. Albert A.A., Quadratic forms permitting composition, Ann. of Math., 43 (1942) 161-
3. Artin E., Geometric algebra, Wiley, London-New York (1957).

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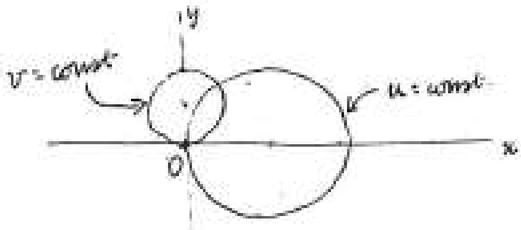
Explanation of physical significance of $\int_a^b \frac{d\sigma(t)}{z-t}$ (see also Rothe, Ollendorff & Pohlhausen, Theory of functions as applied to engineering problems, Dover, p. 106)

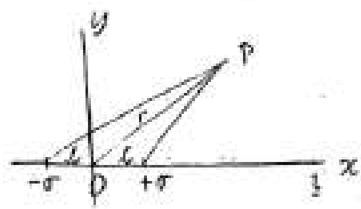
$$W = \frac{1}{z} = \frac{1}{r} e^{-i\theta}, \quad \nabla^2 W = 0.$$

$$W = u + iv$$

$$u = \frac{\cos\theta}{r}, \quad v = -\frac{\sin\theta}{r}.$$

$\nabla u \cdot \nabla v = 0$ (orthogonal surfaces).





$P = P(r, \theta)$

Potential at P $= \varphi_P = -2\sigma \log|z - l\underline{i}| + 2\sigma \log|z + l\underline{i}|$

$$= -2\sigma\left\{\log|z| - l\underline{i} \cdot \frac{1}{|z|}\hat{\underline{z}} + O(l^2)\right\}$$

$$+2\sigma\left\{\log|z| + l\underline{i} \cdot \frac{1}{|z|}\hat{\underline{z}} + O(l^2)\right\}$$

$$\rightarrow \frac{4\mu\cos\theta}{r}, \quad \mu = \lim_{\substack{l \to 0 \\ \sigma \to \infty}} \sigma l.$$

Thus $\mathrm{Re}(W) = \frac{\cos\theta}{r}$ corresponds to a <u>line</u> <u>dipole</u>.