On rational approximations to the exponential function

by

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Abstract

Precise locations are given for the roots with large modulus of the equation (*) $P_{\mu,\nu}(z) = e^z$, where $P_{\mu,\nu}$ is the approximating fraction (Padé quotient) of order (μ,ν) derived from the exponential series. With $n>m$, set $\beta_{m,n} = \frac{1}{2}\{n-m-4[(n-m)/4]\}$ ([...] denotes the integer part) and $\gamma_m = \frac{1}{2}\{1-(-1)^m\}$. Let $C_{m,n;k}$ $(k=1,2,...)$ be the points of intersection in the $z=x+iy$ plane of the equispaced lines parallel to the real axis $y = (2k+\beta_{m,n}+\gamma_m)\pi$ $(k=1,2,...)$ and the exponential curve $y = \gamma_m\pi+(n!/m!)^{1/(n-m)}\exp\{x/(n-m)\}$; let $C_{m,n;-k}$ be the complex conjugate of $C_{m,n;k}$ $(k=1,2,...)$. Let $\varepsilon>0$ be a fixed arbitrarily small number. If $\nu>\mu$, there exists a positive integer K, depending only upon μ, ν and ε, such that one and only one root of equation (*) lies within the circle with centre $C_{\mu,\nu;k}$ and radius $\varepsilon, \pm k=K,K+1,...$ and these are the only roots of equation (*) to lie outside the circle $|z| = |C_{\mu,\nu;k}-\varepsilon|$; when $\nu<\mu$, the preceding results with $C_{\mu,\nu;k}$ replaced by $-C_{\nu,\mu;k}$ hold; when $\mu=\nu$, the stated results are to be modified by taking the centres $C_{\mu,\mu;k}$ to be on the imaginary axis at the points $2ki$ if μ is even and $(2k+1)i$ if μ is odd, and replacing the small circles of radius ε by segments of the imaginary axis of length 2ε.

The approximating fraction (Näherungsbruch) of order (μ,ν) $(\mu,\nu\geq 0)$ derived from the series

$$(1) \qquad \sum_{k=0}^{\infty} c_k z^k$$

with $c_0 \neq 0$ is that irreducible quotient $P_{\mu,\nu}$ of two polynomials, the denominator of degree $\leq \mu$, the numerator of degree $\leq \nu$, whose series expansion in ascending powers of z agrees with (1) for the largest possible number of initial terms. The rational functions $P_{\mu,\nu}$ were studied by Jacobi and Frobenius, and by Padé who investigated the properties of the complete table, obtained by setting $\mu,\nu=0,1,\ldots,$ of such quotients (for an account of the theory, see Ch. 5 of [3]).

The approximating fractions for the exponential series may be expressed (see [3]§42) in simple closed form :

$$(2) \qquad P_{\mu,\nu}(z) = \frac{\sum_{j=0}^{\nu} (\mu+\nu-j)!\binom{\nu}{j} z^j}{\sum_{j=0}^{\mu} (\mu+\nu-j)!\binom{\mu}{j} (-z)^j}$$

for $\mu,\nu=0,1,\ldots$. The series expansion of the function (2) in ascending powers of z agrees with the exponential series as far as the term $\{(\mu+\nu)!\}^{-1} z^{\mu+\nu}$. Basing his work on general theory due to Pontrjagin [4], the author derived the following result [7]: let $P_{\mu,\nu}$ be the approximating fraction of order (μ,ν) derived from the exponential series; if $\nu>\mu$ $\langle\nu<\mu\rangle$, there is an unbounded number of values of z with arbitrarily large positive $\langle$negative$\rangle$

real part for which

(3) $$e^z = P_{\mu,\nu}(z) \ .$$

It is of some interest to know both that equation (3) is satisfied at all for nonzero z and indeed that there are unboundedly many such values of z located in a prescribed manner. Nevertheless the above result is unhelpful in the sense that a half-plane is a large place in which to look for the roots of any equation. The result is also to a certain extent empty : the general theory upon which its derivation is based leads to the conclusion that an equation of the form (3) holds for any irreducible quotient $P_{\mu,\nu}$ of two polynomials, the numerator and denominator of degrees μ and ν respectively; special knowledge concerning the coefficients of the approximating fractions derived from the exponential series is not used.

Results concerning the precise asymptotic location of the roots of equations involving transcendental functions and rational functions of a prescribed system can be based upon two supporting structures. The first is a theory of the roots of equations involving the transcendental functions themselves together with auxiliary functions of known asymptotic behaviour. The second is a further theory concerning the asymptotic behaviour of all rational functions of the prescribed system. The required results may then be obtained by replacing the auxiliary functions of the first theory by the rational functions of the second.

The classical theory of transcendental functions abounds in asymptotic estimates for the zeros of such functions, any one of which may be taken as a supporting result of the first type. Upon this occasion, however, use will be made of an early result due to Hardy [1,2] : let $r = 4r' + R$ $(0 \le R \le 3)$ be a nonnegative integer (with r', R also integers) and let $P(z) \sim \rho z^r$ $(-\infty<\rho<\infty,\ \rho \neq 0)$ as $|z|$ tends to infinity, and let $\varepsilon>0$ be a fixed arbitrarily small real number; there exists a positive integer K, which depends upon ε and P, such that one and only one root of the equation

$$(4) \qquad e^z = P(z)$$

lies within the circle with centre

$$C_k = \ln|\rho| + r\ln\{(2|k|+\tfrac{1}{2}R)\pi\} + i\{2k+\tfrac{1}{2}(R+1-\mathrm{sign}(\rho))\}\pi$$

and radius $\varepsilon, \pm k=K,K+1,\ldots,$ and these are the only roots of equation (4) to lie outside the circle $|z| = |C_K-\varepsilon|$. Results involving combinations of exponential and polynomial functions, of this nature, have many applications; a survey of more recent work in this direction is given in [5].

The approximating fractions derived from a series of the form (1) constitute a well-defined system. It has been shown [6] by the author that the functions $P_{\mu,\nu}$ with $\nu \ge \mu$ derived from the series (1) with coefficients

$$(5) \qquad c_k = \prod_{\tau=0}^{k-1} \left(\frac{A - q^{\alpha+\tau}}{C - q^{\gamma+\tau}} \right)$$

may be expressed in simple closed form. An exact expression may, in particular, be derived for $\rho_{\mu,\nu} = \eta_{\mu,\nu;\nu} / \delta_{\mu,\nu;\mu}$, where $\eta_{\mu,\nu;\nu}$ and $\delta_{\mu,\nu;\mu}$ are the coefficients of z^{ν} and z^{μ} in the numerator and denominator polynomials of $P_{\mu,\nu}$. For large $|z|$

$$(6) \qquad P_{\mu,\nu}(z) \sim \rho_{\mu,\nu}\, z^{\nu-\mu} .$$

Letting A or C tend to infinity and introducing a suitable change of variable z, or taking A=1 or C=1 and letting q tend to unity, a number of special forms of the above general approximating fraction $P_{\mu,\nu}$ may be obtained. Each special form is accompanied by a corresponding form of the constant $\rho_{\mu,\nu}$ and, in consequence, a corresponding estimate (6). These estimates furnish examples of supporting results of the second type mentioned above. They may be used in conjunction with Hardy's result to furnish precise estimates of the location of the roots of equations (3) with $\nu \geq \mu$, where now $P_{\mu,\nu}$ represents an approximating fraction derived from the series with coefficients given by formula (5), or one of the special fractions obtainable from this general form.

One of the latter special fractions is that obtained from the series with coefficients $c_k = (k!)^{-1}$, i.e. the exponential series. Evidently equation (3) assumes special interest when the approximating fraction $P_{\mu,\nu}$ concerned is derived from the exponential series. The corresponding derivatives of the functions upon each side of equation (3), up to and including that of order $\mu+\nu$, are equal in value when z=0; expressed loosely, for the $P_{\mu,\nu}$ in question, equation (3) is satisfied $\mu+\nu+1$ times at the origin. As will be shown, it is satisfied for unboundedly many values of z whose precise location may be given. The approximating fractions derived from the exponential series are of further interest in that they satisfy the relationships $P_{\mu,\nu}(z) = P_{\nu,\mu}(-z)^{-1}$ $(\mu,\nu=0,1,\ldots)$. If equation (3) is satisfied when $z=z'$, it is also satisfied with μ and ν interchanged and $z=-z'$; results concerning the location of the roots of equation (3) may be given for all $\mu,\nu\gtreqless 0$, and are not confined to the case in which $\nu\geq\mu$. For the approximating fraction $P_{\mu,\nu}$ derived from the exponential series, the term $\rho_{\mu,\nu}$ in formula (6) is given by

$$\rho_{\mu,\nu} = (-1)^{\mu}\mu!/\nu! \ .$$

(This result can be derived from the general theory concerning series with coefficients given by formula (5), or may be taken directly from formula (2).)

The following result may be stated : let $P_{\mu,\nu}(z)$ be the approximating fraction of order (μ,ν) derived from the exponential series $(\mu,\nu\geq 0)$; with $\beta_{m,n} = \tfrac{1}{2}\{n-m-4[(n-m)/4]\}$ ([...] denotes the integer part) and $\gamma_m = \tfrac{1}{2}\{1-(-1)^m\}$, set

$$C_{m,n;k} = \ln\left(\frac{m!}{n!}\right) + (n-m)\ln\{(2|k|+\beta_{m,n})\pi\} + i\{2k+\beta_{m,n}+\gamma_m\}\pi$$

for $m=0,1,\ldots$; $n=m,m+1,\ldots$; $k=\ldots,-1,0,1,\ldots$ and let $\varepsilon>0$ be a fixed arbitrarily small real number; when $\nu\geq\mu$, there exists a positive integer K, which depends upon μ, ν and ε, such that one and only one root of equation (3) lies within the circle with centre $C_{\mu,\nu;k}$ and radius $\varepsilon, \pm k=K,K+1,\ldots$ and these are the only roots of equation (3) to lie outside the circle $|z| = |C_{\mu,\nu;K}-\varepsilon|$; when $\nu<\mu$, the preceding result with $C_{\mu,\nu;k}$ replaced by $-C_{\nu,\mu;k}$ holds true.

When $n>m$, the points $C_{m,n;k}$ with $k\geq K$ lie at the intersections in the $z=x+iy$ plane of the equispaced lines parallel to the real axis $y = (2k+\beta_{m,n}+\gamma_m)\pi$ $(k=K,K+1,\ldots)$ and the exponential curve

$$y = \gamma_m\pi + \left(\frac{n!}{m!}\right)^{1/(n-m)} \exp\left(\frac{x}{n-m}\right).$$

The corresponding points obtained by reversing the sign of k are simply the complex conjugates of the above intersection points. When $\nu>\mu$, the centres $C_{\mu,\nu;k}$ of the circles within which the roots of equations (3) lie are at intersection points as described above with m,n replaced by μ,ν (these centres all lie in the right half-plane $\mathrm{Re}(z)<0$). When $\nu<\mu$, the centres of the circles in question are at the reflections, in the imaginary axis, of the above intersection points as described above with m,n replaced by ν,μ (the centres now lie in the left half-plane $\mathrm{Re}(z)<0$).

When $\mu=\nu$, the centres of the circles within which the roots of equation (3) lie are located upon the imaginary axis at the points $2ki$ when μ is even and $(2k+1)i$ when μ is odd $(\pm k=K,K+1,\ldots)$. However, it has been shown [8,9] that in this case the roots of equation (3) are pure imaginary; they are thus contained in small segments of the imaginary axis of length 2ε and with midpoints at the above system of centres.

References

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