

156 Special forms of complementary interpolatory functions of second kind

$\theta, \epsilon?$

$$P = \text{row} \left[\pi(\alpha[\xi] \| m, i; \kappa(\omega)) \right]^{p=[i]}$$

$$Q = \left[\pi(\alpha[\xi] \| m, h; \kappa(\omega)) \pi(\alpha[\xi] \| m, j+1, i-j | \omega_{\xi}(m+\kappa(\omega))) \right]$$

$$\frac{\left\{ \pi(\alpha[\xi] \| m, i+1, j-i) \left[\frac{f_{\xi}(m+\kappa(\omega)) - \Delta(\xi; \alpha, f \| m, i+1, j-i-1 | \omega_{\xi}(m+\kappa(\omega)))}{\pi(\alpha[\xi] \| m, i+1, j-i | \omega_{\xi}(m+\kappa(\omega)))} \right] \right\}}{\pi(\alpha[\xi] \| m, i+1, j-i | \omega_{\xi}(m+\kappa(\omega)))}$$

$$+ \Delta(\xi; \alpha; f \| m, i+1, j-i-1) \left. \right\}^{p=[i]}$$

$$\tilde{p} = P, \tilde{q} = Q \langle K \rangle$$

i) Over $\mathbb{N}B(\rho, \epsilon)$ $R(\tilde{p}, \tilde{q}, \tilde{c})$ is quot. two polynoms. denom of degree $\leq i$ num of deg. $\leq j$

ii) $\{P, Q\} \in \mathcal{C}'\{\kappa, \xi; \omega, f \| m, i, j\}$

iii) Results of "Th.?" with interp. props of $R(\tilde{p}, \tilde{q}, \tilde{c})$ hold

iv) Hom const. system \mathcal{C} of "Th.?" assumes form $\tilde{\mathcal{C}}$:

$$\theta = \xi[m, m, i] \kappa \quad \epsilon = \xi[m, i, j, m, i, j] \rho$$

$$\Delta = \text{diag} [f_{\chi}] \quad (\chi = \epsilon)$$

$$\tilde{\Delta}' = \text{diag} [\pi(\alpha[\xi] \| m, j+1, j-i | \omega_{\chi})] \quad (\chi = \epsilon)$$

$$\tilde{\Delta}'' = \text{diag} [\Delta(\xi; \alpha; f \| m, j+1, j-i-1 | \omega_{\chi})] \quad (\chi = \epsilon)$$

$$\tilde{D} = \text{diag} [\pi(\alpha[\xi] \| m, j+1, i-j | \omega_{\chi})] \quad (\chi = \theta)$$

$$\tilde{D}' = \text{diag} \left[\frac{f_{\chi} - \Delta(\xi; \alpha; f \| m, j+1, j-i-1 | \omega_{\chi})}{\pi(\alpha[\xi] \| m, j+1, j-i | \omega_{\chi})} \right] \quad (\chi = \theta)$$

$$\tilde{C} = \left[\prod_{z=0}^{p-k} (\alpha[\xi] \| m, i; \nu | \alpha_z) \right]$$

$$\tilde{C}'' = \left[\prod_{z=0}^{p-[h]} (\alpha[\xi] \| m, h; \nu | \alpha_z) \right] \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} \begin{matrix} [i-j-1] \\ \vdots \\ [i-1] \end{matrix} \Big]^{p-k}$$

Then $\tilde{C} = \Delta \tilde{C}' - \{ \tilde{\Delta}' \tilde{C}'' \tilde{D}' + \tilde{\Delta}'' \tilde{C}'' \tilde{D} \}$.

a) $j \leq i$ (i.e. $h=j$): $\tilde{\Delta}' = \tilde{I}(i-1)$, $\tilde{\Delta}'' = 0 \text{ diag}(k | i-1)$,

$$\tilde{D}' = \text{diag} [f_x] \quad (x=0)$$

$$\tilde{C} = \Delta \tilde{C}' - \left[\prod_{z=0}^{p-[j]} (\alpha[\xi] \| m, j; \nu | \alpha_z) \prod_{z=0}^{p-[j]} (\alpha[\xi] \| m, j+1, i-j | \alpha_{\frac{z}{2}(m+\nu)}) f_{\frac{z}{2}(m+\nu)} \right] \begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix} \begin{matrix} [i-j-1] \\ \vdots \\ [i-1] \end{matrix} \Big]^{p-[j]}$$

b) $i < j$ (i.e. $h=i$): $\tilde{D} = \tilde{I}(i)$

$$\tilde{C} = \Delta \tilde{C}' - \left[\prod_{z=0}^{p-k} (\alpha[\xi] \| m, i; \nu | \alpha_z) \right.$$

$$\left. \left\{ \frac{\prod_{z=0}^{p-k} (\alpha[\xi] \| m, j+1, j-i | \alpha_z) \left\{ f_{\frac{z}{2}(m+\nu)} - \Delta(\frac{z}{2}; \alpha, f \| m, j+1, j-i-1 | \alpha_{\frac{z}{2}(m+\nu)}) \right\}}{\prod_{z=0}^{p-k} (\alpha[\xi] \| m, j+1, j-i | \alpha_{\frac{z}{2}(m+\nu)})} + \Delta(\frac{z}{2}; \alpha, f \| m, j+1, j-i-1 | \alpha_z) \right\} \right]^{p-k}$$

\tilde{C} : $\tilde{C} \tilde{C} = 0_{[i-1]}$: $\exists k \in \frac{1}{2}(m, m+i)$ $\nu(\alpha_k) \neq 0$

$$R(\tilde{p}, \tilde{q}; \tilde{c} | \alpha_k) = f_k$$

v) $B = \left[\prod_{z=0}^{p-\omega} (\alpha[\xi] \| m, j+z+1, j+z+1) \right]^{p-\omega}$

a) $\tilde{A} = B \tilde{C} \left[\frac{\prod_{z=0}^{p-\omega} (\alpha[\xi] \| m, i, i-j-z-1 | \alpha_{m+\nu})}{\prod_{z=0}^{p-\omega} (\alpha[\xi] \| m, i+1, i+z-1 | \alpha_{m+\nu})} \right]$

$$\left\{ \Delta(\frac{z}{2}; \alpha; f \| m, j+z+1 | \alpha_{m+\nu}) - \Delta(\frac{z}{2}; \alpha; f \| m, i+1, i+z-i | \alpha_{m+\nu}) \right\} \Big]^{p-\omega}$$

$$b) \text{ col } \{K|i\}. \quad C\tilde{c} = O_{[i-1]} \Leftrightarrow A\tilde{c} = U_{[i-1]}$$

$$c) \tilde{c} \uparrow: \tilde{c} = D\tilde{c}$$

$$R(\tilde{p}, \tilde{q}, \tilde{c}) = \frac{|\tilde{q} H \tilde{A}|}{|\tilde{p} H \tilde{A}|} = \frac{|q| |A|}{|p| |A|} = R(p, q, c) \quad \langle NS | \tilde{p} H \tilde{A} \rangle = NS | p H A \rangle$$

(vi) $\mathcal{B} \in \text{seq}(\bar{N})$: $\mathcal{B} [m, m_{i+j}]$ obtained from $\mathcal{B} [m, m_{i+j}]$ by

rearrangement of first $j+1$ members: $\mathcal{B} [m, m_{i+j}] = \mathcal{B} [m, m_{i+j}]$

$\mathcal{B} [m_{i+j}, m_{i+j}] = \mathcal{B} [m_{i+j}, m_{i+j}]$. Write \tilde{C}, D above as $\tilde{C}(\mathcal{B}), D(\mathcal{B})$

define $\tilde{C}(\mathcal{B}), D(\mathcal{B})$ by $\mathcal{B} := \mathcal{B}$. $H(\mathcal{B}, \mathcal{B}) \in M \{K | i \times i\}$:

$$H(\mathcal{B}, \mathcal{B}) = \left[\left[U[\mathcal{B}(\mathcal{B}; \alpha; \pi(\alpha [\mathcal{B}] \| m, \nu | \langle \alpha \rangle \| m, \nu))] (z, \nu = U_{[i]} \right) \mid O_{[i-j-1]}^{[i-j-1]} \right] \Bigg| \Bigg| \\ : \left[O_{[i-j-1]}^{[i-j-1]} \mid I(i-j-1) \right] \Bigg| \Bigg|_K$$

a) \tilde{p}, \tilde{q} above as $\tilde{p}(\mathcal{B}), \tilde{q}(\mathcal{B})$; $\tilde{p}(\mathcal{B}), \tilde{q}(\mathcal{B})$ same by $\mathcal{B} := \mathcal{B}$

$$\tilde{p}(\mathcal{B}) = \tilde{p}(\mathcal{B}) D(\mathcal{B})^{-1} H(\mathcal{B}, \mathcal{B}) D(\mathcal{B})$$

$\tilde{q}(\mathcal{B}), \tilde{q}(\mathcal{B})$ related similarly.

b) Select $\tilde{c}(\mathcal{B}) \in \text{col } \{K|i\}$; set

$$\tilde{c}(\mathcal{B}) = D(\mathcal{B})^{-1} H(\mathcal{B}, \mathcal{B}) D(\mathcal{B}) \tilde{c}(\mathcal{B})$$

$$R\{\tilde{p}(\mathcal{B}), \tilde{q}(\mathcal{B}), \tilde{c}(\mathcal{B})\} = R\{\tilde{p}(\mathcal{B}), \tilde{q}(\mathcal{B}), \tilde{c}(\mathcal{B})\}$$

$$\langle NS \{ \tilde{p}(\mathcal{B}) \tilde{c}(\mathcal{B}) \} \rangle = NS \{ \tilde{p}(\mathcal{B}) \tilde{c}(\mathcal{B}) \}$$

$$c) \tilde{C}(\mathcal{B}) = \tilde{C}(\mathcal{B}) D(\mathcal{B})^{-1} H(\mathcal{B}, \mathcal{B}) D(\mathcal{B})$$

$$d) \tilde{C}(\mathcal{B}) \tilde{c}(\mathcal{B}) = O_{[i-1]} \Leftrightarrow \tilde{C}(\mathcal{B}) \tilde{c}(\mathcal{B}) = O_{[i-1]}$$

$$e) \tilde{C}(\mathcal{B}) \tilde{c}(\mathcal{B}) = O_{[i-1]} :$$

$$R(\tilde{p}(s), \tilde{q}(s); \tilde{c}(s)) = \frac{|\tilde{q}(s)| |\tilde{c}(s)|}{|\tilde{p}(s)| |\tilde{c}(s)|} = \frac{|\tilde{q}(s)|}{|\tilde{p}(s)|} =$$

$$R\{\tilde{p}(s), \tilde{q}(s); \tilde{c}(s)\} \in NS\{|\tilde{p}(s)| |\tilde{c}(s)|\} = NS\{|\tilde{p}(s)|\}$$

f) B, \tilde{A} in $(v) := B(s), \tilde{A}(s)$. $B(s), \tilde{A}(s) : s := s$

$$\tilde{B}(s) = B(s), \quad \tilde{A}(s) = \tilde{A}(s) D(s)^{-1} H(s, s) D(s)$$

(vii) p, q, c function systems & can construct systems dealt with in Th?. Set

$$D = \left[\begin{array}{c} \pi(\alpha[z]) \parallel_{m+z+1, i-r} | \alpha_{\nu} \\ \hline z = \nu \end{array} \right] \begin{array}{l} D = \theta \\ z = \nu \end{array}$$

a) $\tilde{p} = pD, \tilde{q} = qD \in K$

b) Select $\tilde{c} \in \text{col}\{K|i\}$; let $c = D\tilde{c} : R(p, q; c) = R(\tilde{p}, \tilde{q}; \tilde{c}) \in NS\{pc\}$
 $= NS\{\tilde{p}\tilde{c}\}$; each $k \in \mathbb{Z}[m, m+i]$, $R(p, q; c|_{\nu_k}) = f_k \Leftrightarrow$

$$R(\tilde{p}, \tilde{q}; \tilde{c}|_{\nu_k}) = f_k$$

c) $\tilde{c} = cD$

d) $\tilde{c} \in \text{col}\{K|i\}$; $\tilde{c} = 0_{[i-1]} \Leftrightarrow c = D\tilde{c} : c = 0_{[i-1]}$

e) $c = D\tilde{c}$. Either $c = 0_{[i-1]}$, $\tilde{c} = 0_{[i-1]}$

$$R(p, q; c) = \frac{|q||c|}{|p||c|} = \frac{|\tilde{q}||\tilde{c}|}{|\tilde{p}||\tilde{c}|} = R(\tilde{p}, \tilde{q}; \tilde{c}) \in NS\{|p||c|\} = NS\{|\tilde{p}||\tilde{c}|\}$$

21 domains with cancellation and mapping systems with complete cancellation

Sedom $\mathcal{B}(\mathcal{S})$: complete system of $\mathcal{B} \subseteq \mathcal{S} \times \mathcal{S} \subseteq K$ for wh.

$$\{\mathcal{B}(\alpha, \beta) \cap \mathcal{B}(\beta, \delta)\} \subseteq \mathcal{B}(\alpha, \delta)$$

all $\alpha, \beta, \delta \in \mathcal{S}$.

Sedom, $\mathcal{B}: \mathcal{S} \times \mathcal{S} \subseteq K$

i) each $\alpha \in \mathcal{S}$: $\mathcal{B}'_{\alpha}: \mathcal{S} \subseteq K$: $\mathcal{B}'_{\alpha}(\beta) = \mathcal{B}(\alpha, \beta)$ for $\beta \in \mathcal{S}$

$\mathcal{B}'_{\alpha} \in \text{const}(\mathcal{S})$ ($\alpha \in \mathcal{S}$). Then $\mathcal{B} \in \mathcal{B}(\mathcal{S})$

ii) each $\beta \in \mathcal{S}$: $\mathcal{B}''_{\beta}: \mathcal{S} \subseteq K$: $\mathcal{B}''_{\beta}(\alpha) = \mathcal{B}(\alpha, \beta)$ $\alpha \in \mathcal{S}$.

$\mathcal{B}''_{\beta} \in \text{const}(\mathcal{S})$ ($\beta \in \mathcal{S}$). Then $\mathcal{B} \in \mathcal{B}(\mathcal{S})$

23, 20:

$i \in \bar{\mathbb{N}}$, $\mathcal{S}_0 = \mathcal{S}_1 = \mathcal{S}$ sedom, $\mathcal{B} \in \mathcal{B}(\mathcal{S})$. $\mathcal{G}(\mathcal{S}_0, \mathcal{B}|_i)$: complete

system of $\mathcal{G}: \mathcal{S}_0 \times \mathcal{S}_1 \rightarrow \{\mathcal{B}(\mathcal{S}_0, \mathcal{S}_1) \rightarrow \{K \mid i, i\}\}$ which

a) possess the cancellation property

$$\mathcal{G}(\alpha/\beta) \mathcal{G}(\beta/\delta) = \mathcal{G}(\alpha/\delta) \langle \mathcal{B}(\alpha/\beta) \cap \mathcal{B}(\beta/\delta) \rangle$$

all $\alpha, \beta, \delta \in \mathcal{S}$ and

b) possess internal reduction property $\mathcal{G}(\alpha/\alpha) = I(i) \langle \mathcal{B}(\alpha, \alpha) \rangle$

all $\alpha \in \mathcal{S}$.

Mapping \mathcal{G} possessing above properties: said mapping system with complete cancellation

20 union of mapping systems with complete cancellation

$i \in \bar{N}$, $S \in \text{dom}$, $B \in \mathcal{B}(S)$, $G \in \mathcal{G}(S, B | i)$

For each ordered pair $\alpha, \beta \in S$

$$a) \{B(\alpha, \beta) \cap B(\beta, \alpha)\} \subseteq NS\{G(\alpha/\beta)\}$$

$$b) G(\alpha/\beta)^{-1} = G(\beta/\alpha) \langle B(\alpha, \beta) \cap B(\beta, \alpha) \rangle$$

21
Existence and extension of classes of mapping systems with complete cancellation

$i \in \bar{N}$, $S \in \text{dom}$, $B \in \mathcal{B}(S)$, $G: S \times S \rightarrow \{B(S, S) \rightarrow \{K | i, i\}\}$:

$G(\alpha/\beta) = I(i)$ ($\alpha, \beta \in S$). Then $G \in \mathcal{G}\{S, B\}$

$i \in \bar{N}$, $S' = S'' \in \text{dom}$, $B': S \subseteq K$, $D: S \rightarrow \{B'(S) \rightarrow \{K | i, i\}\}$

i) $B \in \mathcal{B}(S)$, $G \in \mathcal{G}(S, B | i)$, $\hat{B}: S \times S \subseteq K$:

$$\hat{B}(\alpha, \beta) = B'(\alpha) \cap B(\alpha, \beta) \cap NS\{D(\beta)\}$$

$\hat{G}: S' \times S'' \rightarrow \{\hat{B}(S', S'') \rightarrow \{K | i, i\}\}$:

$$\hat{G}(\alpha/\beta) = D(\alpha)G(\alpha/\beta)D(\beta)^{-1} \quad \langle \hat{B}(\alpha, \beta) \rangle$$

Then $\hat{G} \in \mathcal{G}\{S, \hat{B}\}$

ii) $\tilde{B}: S \times S \subseteq K$: $\tilde{B}(\alpha, \beta) = B'(\alpha) \cap NS\{D(\beta)\}$

$\tilde{G}: S' \times S'' \rightarrow \{\tilde{B}(S', S'') \rightarrow \{K | i, i\}\}$: $\tilde{G}(\alpha/\beta) = D(\alpha)D(\beta)^{-1}$

Then $\tilde{G} \in \mathcal{G}\{S, \tilde{B}\}$

23. examples of mapping systems defined from divided differences of factorial polynomials. Translation properties of such systems

$$i \in \mathbb{N}, \mathcal{S} = \text{seq}'(\mathbb{K} | \geq i + \mathbb{N}) \times \mathbb{N}.$$

$$i) \mathcal{G}: \mathcal{S} \times \mathcal{S} \rightarrow [\mathbb{K} | i \times i]:$$

$$\mathcal{G}(\alpha, m / \beta, n) = L \left[\mathcal{S}(\alpha; \pi(\beta \| m + \nu + 1, i - \nu) | \langle \alpha \rangle) \| m + \nu, i - \nu \right] (z, \nu = [i])$$

a) each pair $\{\alpha, m\}, \{\beta, n\} \in \mathcal{S}$

$$\mathcal{G}(\alpha, m / \beta, n) =$$

$$U \left[\mu(\alpha \| m + \nu; i - \nu, \nu - \nu) \right] (z, \nu = [i]) \left[\pi(\beta \| m + \nu + 1, i - \nu | \alpha_{m + \nu}) \right] (z, \nu = [i])$$

$$b) \mathcal{G} \in \mathcal{bb}(\mathcal{S}, \mathbb{K})$$

$$ii) \hat{\mathcal{G}}: \mathcal{S} \times \mathcal{S} \rightarrow [\mathbb{K} | i \times i]:$$

$$\hat{\mathcal{G}}(\alpha, m / \beta, n) = U \left[\mathcal{S}(\alpha; \pi(\beta \| n, \nu) | \langle \alpha \rangle) \| m, \nu \right] (z, \nu = [i])$$

a) $\hat{\mathcal{G}}(\alpha, m)$ each pair $\{\alpha, m\}, \{\beta, n\} \in \mathcal{S}$

$$\hat{\mathcal{G}}(\alpha, m / \beta, n) =$$

$$L \left[\mu(\alpha \| m, \nu; \nu) \right] (z, \nu = [i]) \left[\pi(\beta \| n, \nu | \alpha_{m + \nu}) \right] (z, \nu = [i])$$

$$b) \hat{\mathcal{G}} \in \mathcal{bb}(\mathcal{S}, \mathbb{K})$$

$i \in \mathbb{N}$, $\alpha \in \text{seq}'(\mathbb{K} | \geq i)$, $m \in \mathbb{N}$; $\mathbb{S}(m)$ complete system of $\frac{1}{\mathbb{S}} \in \text{seq}'(\mathbb{N})$
for which $\frac{1}{\mathbb{S}}[m, m+i] \subseteq [\alpha]$, $\mathbb{S} = \mathbb{S}(\mathbb{N}) \times \mathbb{N}$, $[j, k] \subseteq [\alpha]$.

$f \in \text{seq}(\mathbb{K} | \geq k)$

1a) $H: \mathbb{S} \times \mathbb{S} \rightarrow [\mathbb{K} | i \times i]$:

$$H(\frac{1}{\mathbb{S}}, m / \frac{1}{\mathbb{B}}, n) = L[\mathcal{D}(\frac{1}{\mathbb{S}}: \alpha; \pi(\alpha[\mathbb{B}]) \| m, \tau+1, i-\tau) | \langle \alpha \rangle) \| m, \tau, i-\tau] \\ (\tau, \tau = [i])$$

a) each pair $\{\frac{1}{\mathbb{S}}, m\}, \{\frac{1}{\mathbb{B}}, n\} \in \mathbb{S}$

$$H(\frac{1}{\mathbb{S}}, m / \frac{1}{\mathbb{B}}, n) =$$

$$U[\mu(\alpha[\frac{1}{\mathbb{S}}] \| m, \tau; i-\tau, \tau-\tau)] (\tau, \tau = [i]) [\pi(\alpha[\mathbb{B}] \| n, \tau+1, i-\tau) | \alpha_{\frac{1}{\mathbb{S}}(m, \tau)}] \\ (\tau, \tau = [i])$$

b) $H \in \mathcal{B}\mathcal{B}(\mathbb{S}, \mathbb{K})$

ii) \mathbb{S}' : complete system of pairs $\{\frac{1}{\mathbb{S}}, m\} \in \mathbb{S}$ for which $m \geq k-i$,
 $\frac{1}{\mathbb{S}}[m+i-k, m+i] \subseteq [\min(|\alpha|, |f|)]$, $\mathbb{S}'' : \mathbb{S}' \xrightarrow{?} \mathbb{S}' : \{\frac{1}{\mathbb{S}}, m\} \in \mathbb{S}'$

$\mathbb{S}''(\frac{1}{\mathbb{S}}, m)$: complete system of pairs $\{\frac{1}{\mathbb{B}}, n\} \in \mathbb{S}'$ for which
 $\frac{1}{\mathbb{S}}[m+i-j, m+i] \equiv \frac{1}{\mathbb{B}}[n+i-j, n+i]$, $\frac{1}{\mathbb{S}}[m+i-k, m+i-j] =$
 $\frac{1}{\mathbb{B}}[n+i-k, n+i-j]$, $V: \mathbb{S}' \rightarrow [\mathbb{K} | (k-j) \times i]$:

$$V / \frac{1}{\mathbb{S}}, m) = [\mathcal{D}(\frac{1}{\mathbb{S}}: \alpha; f \| m, \tau-i+\tau) | \langle \alpha \rangle] \\ \tau = [i] \\ \tau = [j, k]$$

For each $\{\frac{1}{\mathbb{S}}, m\} \in \mathbb{S}'$, $\{\frac{1}{\mathbb{B}}, n\} \in \mathbb{S}''(\frac{1}{\mathbb{S}}, m)$

$$V / \frac{1}{\mathbb{S}}, m) = H(\frac{1}{\mathbb{S}}, m / \frac{1}{\mathbb{B}}, n) V / \frac{1}{\mathbb{B}}, n)$$

2] $\hat{H}: \mathbb{S} \times \mathbb{S} \rightarrow [\mathbb{K} | i \times i]$:

$$\hat{H}(\frac{1}{\mathbb{S}}, m / \frac{1}{\mathbb{B}}, n) = U[\mathcal{D}(\frac{1}{\mathbb{S}}: \alpha; \pi(\alpha[\mathbb{B}]) \| m, \tau) | \langle \alpha \rangle) | m, \tau] (\tau, \tau = [i])$$

ia) each pair $\{\frac{1}{\mathbb{S}}, m\}, \{\frac{1}{\mathbb{B}}, n\} \in \mathbb{S}$

$$\hat{H}(\frac{1}{\mathbb{S}}, m / \frac{1}{\mathbb{B}}, n) =$$

$$L[\mu(\alpha[\frac{1}{\mathbb{S}}] \| m, \tau) | \langle \alpha \rangle] (\tau, \tau = [i]) [\pi(\alpha[\mathbb{B}]) \| n, \tau) | \alpha_{\frac{1}{\mathbb{S}}(m, \tau)}] (\tau, \tau = [i])$$

b) $\text{ob}(\mathbb{S}, K)$

ii) \mathbb{S}' : complete system of pairs $\{\xi, m\} \in \mathbb{S}$ for which
 $\xi[m, m+k] \in [\min(|a|, |f|)]$. \mathbb{S}'' : $\mathbb{S}' \times \mathbb{S}' \rightarrow \mathbb{S}'$: $\{\xi, m\} \in \mathbb{S}'$,
 $\xi''(\xi, m)$: complete system of pairs $\{\eta, n\} \in \mathbb{S}'$ for which
 $\xi[m, m+j] \equiv \eta[n, n+j]$, $\xi(m+j, m+k) = \eta(n+j, n+k)$. $\check{V}: \mathbb{S}' \rightarrow [K | (a_j) \times \delta_j]$
 $\check{V}/\xi, m = [\delta(\xi, a; f \| m \mapsto, z - \nu)]$ $\begin{matrix} p = [i] \\ z = [j - k] \end{matrix}$

For each $\{\xi, m\} \in \mathbb{S}'$, $\{\eta, n\} \in \mathbb{S}''(\xi, m)$

$$\check{V}/\xi, m = \hat{H}(\xi, m / \eta, n) \check{V}/\eta, n$$

2.

$i \in \mathbb{N}$, $\alpha \in \text{seq}'(K | \geq i)$. $m \in \bar{\mathbb{N}}$: $\mathfrak{S}(m)$: complete system of $\mathfrak{S} \in \text{seq}'(N)$ for which $\mathfrak{S}[m, m+i] \subseteq [i]$. $\mathfrak{S} := \mathfrak{S}(i) \times N$; $\mathfrak{S}' = \mathfrak{S}'' = \mathfrak{S}$.

$[j, k] \subseteq [i]$, $f \in \text{seq}(K | \geq k)$

1] $\mathfrak{B}: \mathfrak{S} \subseteq K$: $\mathfrak{B}(\mathfrak{S}, n) := K \setminus \alpha[\mathfrak{B}(n, n+i)]$.

$K: \mathfrak{S}' \times \mathfrak{S}'' \rightarrow \{\mathfrak{B}(\mathfrak{S}'') \rightarrow [K | i \times i]\}$:

$K(\mathfrak{S}, m / \mathfrak{B}, n) =$

$$\left[\frac{\pi(\alpha[\mathfrak{S}] \| m+i, i-\tau) \delta(\mathfrak{S}: \alpha; \pi(\alpha[\mathfrak{B}] \| n+i, i-\rho) \langle \alpha \rangle \| m+i, i-\tau)}{\pi(\alpha[\mathfrak{B}] \| n+i, i-\rho)} \right]$$

$(\tau, \rho = [i]) \uparrow$

ia) each pair $\{\mathfrak{S}, m\}, \{\mathfrak{B}, n\} \in \mathfrak{S}$

$\langle \mathfrak{B}(\mathfrak{B}, n) \rangle$

$K(\mathfrak{S}, m / \mathfrak{B}, n) =$

$$\cup \left[\pi(\alpha[\mathfrak{S}] \| m+i, i-\tau) \mu(\alpha[\mathfrak{S}] \| m+i, i-\tau, \rho-\tau) \right] (\tau, \rho = [i])$$

$$\left[\frac{\pi(\alpha[\mathfrak{B}] \| n+i, i-\rho) \alpha_{\mathfrak{S}}(m+i)}{\pi(\alpha[\mathfrak{B}] \| n+i, i-\rho)} \right] (\tau, \rho = [i]) \langle \mathfrak{B}(\mathfrak{B}, n) \rangle?$$

b) $K \in \text{bb}(\mathfrak{S}, \mathfrak{B})?$

ii) \mathfrak{S}' : complete system of pairs $\{\mathfrak{S}, m\} \in \mathfrak{S}$ for which $m \geq k-i$,

$\mathfrak{S}[m-i-k, m+i] \subseteq [\min(|\alpha|, |f|)]$. $\mathfrak{S}'': \mathfrak{S}' \xrightarrow{?} \mathfrak{S}' : \{\mathfrak{S}, m\} \in \mathfrak{S}'$,

$\mathfrak{S}''(\mathfrak{S}, m)$: complete system of pairs $\{\mathfrak{B}, n\} \in \mathfrak{S}'$ for which $\mathfrak{S}[m-i-j, m+i] =$

$\mathfrak{B}[n-i-j, n+i]$, $\mathfrak{S}[m-i-k, m-i-j] = \mathfrak{B}[n-i-k, n-i-j]$. [as in (ii) prev. th.]

$W: \mathfrak{S}' \rightarrow \{K \rightarrow [K | (k-i) \times i]\}$:

$$W / \mathfrak{S}, m := \left[\Delta(\mathfrak{S}: \alpha; f \| m+i-\tau, \rho-i+\tau) \right]_{\substack{\rho = [i] \\ \tau = [j, k]}}$$

Each $\{\xi, m\} \in \mathcal{S}'$, $\{\eta, n\} \in \mathcal{S}''(\xi, m)$

$$W/\xi, m = K(\xi, m / \eta, n) W / \eta, n \quad \langle B(\eta, n) \rangle$$

2] $\mathcal{B}: \mathcal{S} \subseteq K: \mathcal{B}(\xi, n) = K \setminus \alpha[\xi, n, n, i]$.

$\hat{K}: \mathcal{S}' \times \mathcal{S}'' \rightarrow \{B(\mathcal{S}'') \rightarrow [K | i, x_i]\}$:

$$\hat{K}(\xi, m / \eta, n) :=$$

$$\cup \left[\frac{\pi(\alpha[\xi] \| m, \tau) \delta(\xi: \alpha; \pi(\alpha[\eta] \| n, \nu) \langle \alpha \rangle \| m, \tau)}{\pi(\alpha[\eta] \| n, \nu)} \right]_{(z, \nu = [i])}$$

$\langle B(\eta, n) \rangle?$

ia) each pair $\{\xi, m\}, \{\eta, n\} \in \mathcal{S}$

$$\hat{K}(\xi, m / \eta, n) =$$

$$L[\pi(\alpha[\xi] \| m, \tau), \mu(\alpha[\eta] \| n, \nu)]_{(z, \nu = [i])}$$

$$\left[\frac{\pi(\alpha[\eta] \| n, \nu | \alpha_{\xi}(m, \tau))}{\pi(\alpha[\eta] \| n, \nu)} \right]_{(z, \nu = [i])} \quad \langle B(\eta, n) \rangle?$$

b) $\hat{K} \in \text{bb}(\mathcal{S}, \mathcal{B})$

ii) \mathcal{S}' : complete system of pairs $\{\xi, m\} \in \mathcal{S}$ for which

$\xi[m, m+k] \in [\min(|\alpha|, |f|)]$, $\mathcal{S}'': \mathcal{S}' \times \mathcal{S}' \rightarrow \mathcal{S}'$: $\{\xi, m\} \in \mathcal{S}'$,

$\mathcal{S}''(\xi, m)$: complete system of pairs $\{\eta, n\} \in \mathcal{S}'$ for which

$\xi[m, m+j] = \eta[n, n+j]$, $\xi[m+j, m+k] = \eta[n+j, n+k]$ [as in

2ii) prev. th.]. $\hat{W}: \mathcal{S}' \rightarrow \{K \rightarrow [K | (k-j) \times i]\}$:

$$\hat{W}/\xi, m := \left[\Delta(\xi: \alpha; f \| m, \tau, z, \nu) \right]_{z=[j, k]}^{\nu=[i]} \quad \langle B(\eta, n) \rangle^{K?}$$

Each $\{\xi, m\} \in \mathcal{S}'$ and $\{\eta, n\} \in \mathcal{S}''(\xi, m)$

$$\hat{W}/\xi, m = \hat{K}(\xi, m / \eta, n) \hat{W}(\eta, n) \quad \langle \hat{B}(\eta, n) \rangle$$

approximation by the use of rational functions

Definition With $i \in \bar{N}$, $p, q: \mathbb{K} \rightarrow \text{ms}[K|i]$ and $c \in \mathcal{R}[K|i]$,

$R(p, q || c): \text{NS}(pc) \rightarrow \mathbb{K}$ is defined by setting

$$R(p, q || c) = \frac{qc}{pc}$$

$\alpha \in \text{seq}'(\mathbb{K})$ and $f \in \text{seq}(\mathbb{K})$ being suitably prescribed, use of the rational function $R(p, q || c)$ for the purposes of interpolation based upon conditions of the form

$$() \quad R(p, q || c | \alpha_k) = f_k$$

will be ~~considered~~ studied.

In the first use of $R(p, q || c)$ considered, p and q are not subject to further prescription ~~and~~ ^{but} c is selected from the coefficient space generated by a homogeneous constraint system derived by imposing condition () for the i values $\{k := \exists [n, n+i] \text{ where } n \in \bar{N} \text{ and } \exists \in \text{seq}'(\bar{N}) \text{ are suitably defined.}$

In the second use, it is supposed that certain of the components of q possess interpolatory properties of the form $q(\omega | \alpha_k) = p(\omega | \alpha_k) f_k$ and that, for the argument value α_k , the remaining components of q together with their counterparts in p assume the value zero. Subject to the requisite loose

concerning c , the denominator of $R(p, q | c | \omega_k)$ appears as a nonzero factor of the numerator: cancellation induces relationship (). Interpolation of this form over the range of values $k := \frac{1}{s}[m, m_{ij}]$ where $m, j \in \bar{N}$, is treated. In this direct method c is, subject to mild constraints, disposable.

Setting $n = m_{ij} + 1$ in the above, the two methods may be used in conjunction to obtain interpolation properties of the form () over the range $k := \frac{1}{s}[m, m_{ij}]$. c is determined by the derived interpolation conditions over the range $k := \frac{1}{s}(m_{ij}, m_{ij})$; the properties ascribed to p and q induce direct interpolation over the range $k := \frac{1}{s}[m, m_{ij}]$.

Two kinds of pairs p, q inducing the direct interpolation properties described in the penultimate paragraph are dealt with. In the first, the components of p are akin to Lagrange multipliers; in the second they are akin to factorial polynomials. Assuming that the components in question are actually Lagrange multipliers and factorial polynomials respectively, the formulae involved may be refined, relationships between the two kinds of mapping systems p may be established (as may also be done for the corresponding systems q) and special results concerning rearrangement of $\frac{1}{s}$ may be obtained.

1 d interpolation

Definition Let $i \in \bar{N}$, $n \in \bar{N}$, $p, q \in K \rightarrow \text{row}[K|i]$, $\xi \in \text{seq}'(\bar{N}|\geq n+i)$,

$\bar{r} := \max \xi(\omega) \langle \omega := [n, n+i] \rangle$, $\alpha \in \text{seq}'(K|\geq \bar{r})$, $f \in \text{seq}(K|\geq \bar{r})$

and $\rho \equiv [i]$.

$C := C(p, q \| \xi : \alpha; f \| \rho, n, i) \in [K|i-1, i]$ is the homogeneous constraint system defined by setting

$$C := \left[p(\omega | \alpha_{\xi(n+\omega)}) f_{\xi(n+\omega)} - q(\omega | \alpha_{\xi(n+\omega)}) \right]_{\omega := [i]}$$

Let $i \in \bar{N}$, $n \in \bar{N}$, $p, q: K \rightarrow \text{row}[K|i]$, $\xi \in \text{seq}(\bar{N}|\geq n+i)$, $\bar{r} := \max \xi(\omega) \langle \omega := [n, n+i] \rangle$, $\alpha \in \text{seq}'(K|\geq \bar{r})$, $f \in \text{seq}(K|\geq \bar{r})$ and $\rho \equiv [i]$. Set

$C := C(p, q \| \xi : \alpha; f \| \rho, n, i)$ and let $c \in \text{col}[K|i]$ satisfy the relationship $Cc = 0_{[i]}$.

i) $R(p, q \| c)$ possesses the desired interpolatory property that for each $k \in \xi[n, n+i]$ for which $\alpha_k \in \text{NS}\{p\}$

$$R(p, q \| c | \alpha_k) = f_k$$

$$ii) \quad R(p, q \| c) = \frac{|q| |c|}{|p| |c|} \langle \text{NS}\{p \| c\} \rangle$$

2] Let $r \in \bar{N}$ and $\epsilon \in \text{seq}'(\bar{N}|\geq r+i)$ be such that $\epsilon[r, r+i] \equiv \xi[n, n+i]$

so that $\gamma \equiv [i]$ exists for which $\epsilon(r+\gamma) = \xi(n+\gamma(\gamma))$ ($\gamma \equiv [i]$). Denote

C above by $C(\xi, n)$ and set $C(\epsilon, r) := C(p, q \| \epsilon : \alpha; f \| \rho, r, i)$

i) $C(\epsilon, r)$ is a row rearranged form of $C(\xi, n)$:

$$C(s, r) = C(\xi, n)_{(inv p) \times p} = I(i-1)_{(inv p) \times p} C(\xi, n)$$

- ii) The coefficient spaces generated by the two homogeneous constraint systems $C(s, r)$ and $C(\xi, n)$ are identical: $c' \in \text{col}[K_i]$ satisfies the relationship $C(s, r)c' = 0_{1 \times i}$ if and only if $C(\xi, n)c' = 0_i$.
- iii) The result of clause (ii) with C replaced by $C(s, r)$ holds and $NS\{1-p \parallel C(s, r)\} = NS\{1-p \parallel C(\xi, n)\}$.

Direct interpolation by the use of complementary interpolatory functions.

The interpolatory property expressible in the form $R(p, q | c | \omega_k) = f_k$ is induced by the assumptions firstly that $\omega_k \in NS\{p, c\}$ and secondly that p and q are complementary interpolatory mapping systems in the sense that, as ν ranges through $[i]$, where the relationship $q(\omega | \omega_k) = p(\omega | \omega_k) f_k$ does not hold $p(\omega | \omega_k)$ and $q(\omega | \omega_k)$ are both assumed to be zero. Interpolation over a range of values $k := \mathfrak{z}[m, m+j]$ is considered.

The above description of p and q reveals the mechanism of direct interpolation. However, the identification of complementary interpolatory mapping systems is perhaps most easily carried out by fixing the index ν in the range $[i]$ and examining the properties of the components $p(\omega | \omega_k)$ and $q(\omega | \omega_k)$ as k ranges through $\mathfrak{z}[m, m+j]$. The following definitions are presented in such terms.

$i, j, m \in \overline{\mathbb{N}}$ with $h := \min(i, j)$ and $\kappa := [i]$. With $i' \in \overline{\mathbb{N}}$ as specified below, let $\xi \in \text{seq}'(\mathbb{N} | \geq m + \max(i', j))$, $\bar{e}(i') := \max \xi(\nu) \langle \nu := [m, m + \max(i', j)] \rangle$, $\tau := \max \xi(\nu) \langle \nu := [m, m + j] \rangle$, $\alpha \in \text{seq}'(\mathbb{K} | \geq \bar{e}(i'))$ and $f \in \text{seq}(\mathbb{K} | \geq \tau)$

i) Let $i' = i$.

$\mathcal{C}(\xi; \alpha; f | \kappa; m, i, j)$ is the complete system of ordered pairs of complementary interpolatory mapping systems $p, q: \mathbb{K} \rightarrow \text{row } [|\mathbb{K}|; i]$ for which

$\alpha)$ for $\nu := [i]$, $p(\nu | \alpha_{\xi(m + \kappa(\nu))}) \neq 0$ and $p(\nu | \alpha_k) = 0 \langle k := \xi [[m, m + \kappa(\nu)] - (m + \kappa(\nu))] \rangle$ and

$\beta)$ for $\nu := \text{inv } \kappa [h]$, $q(\nu | \alpha_k) = p(\nu | \alpha_k) f_k \langle k := \xi [(m + \kappa(\nu)) + (m + i, m + j)] \rangle$ and $q(\nu | \alpha_k) = 0 \langle k := \xi [[m, m + h] - (m + \kappa(\nu))] \rangle$ and, for $\nu := [i] - \text{inv } \kappa [h]$, $q(\nu) = 0 \langle \mathbb{K} \rangle$

ii) Let $i' = j$.

$\mathcal{C}(\xi; \alpha; f | \kappa; m, i, j)$ is the complete system of ordered pairs of complementary interpolatory mapping systems $p, q: \mathbb{K} \rightarrow \text{row } [|\mathbb{K}|; i]$ for which

$\alpha)$ for $\nu := [i]$, $p(\nu | \alpha_k) = q(\nu | \alpha_k) = 0 \langle k := \xi [[m, m + h] - (m + \kappa(\nu))] \rangle$ and

$\beta)$ for $\nu := \text{inv } \kappa [h]$, $p(\nu | \alpha_{\xi(m + \kappa(\nu))}) \neq 0$ and $q(\nu | \alpha_k) = p(\nu | \alpha_k) f_k \langle k := \xi [(m + \kappa(\nu)) + (m + i, m + j)] \rangle$.

2) let $i \geq 0$.

i) let $i' = i - 1$.

$\bar{C}'(\xi; \alpha; f \| \kappa; m, i, j)$ is the complete system of ordered pairs of complementary interpolatory mapping systems $p, q: \mathbb{K} \rightarrow \text{row}[\mathbb{K} | i]$ for which

$\alpha)$ for $\nu := [i] - \text{inv} \kappa(0)$, $p(\omega | \alpha_k) = 0 \langle k := \xi[m, m + \kappa(\omega)] \rangle$

$\beta)$ for $\nu := \text{inv} \kappa[h]$, $q(\omega | \alpha_k) = p(\omega | \alpha_k) f_k \langle k := \xi[m + \kappa(\omega), m + j] \rangle$

$\beta')$ for $\nu := \text{inv} \kappa[h]$, $q(\omega | \alpha_k) = 0 \langle k := \xi[m, m + \kappa(\omega)] \rangle$

$\beta'')$ for $\nu := \text{inv} \kappa(j, i)$, $q(\omega) = 0 \langle k \rangle$

ii) let $i' = j$.

$\bar{C}'(\xi; \alpha; f \| \kappa; m, i, j)$ is the complete system of ordered pairs of complementary interpolatory mapping systems $p, q: \mathbb{K} \rightarrow \text{row}[\mathbb{K} | i]$ for which

$\alpha)$ for $\nu := \text{inv} \kappa(i)$, $p(\omega | \alpha_k) = q(\omega | \alpha_k) = 0 \langle \nu := \xi[m, m + \min(j+1, \kappa(\omega))] \rangle$

$\beta)$ for $\nu := \text{inv} \kappa[h]$, $q(\omega | \alpha_k) = p(\omega | \alpha_k) f_k \langle k := \xi[m + \kappa(\omega), m + j] \rangle$

The conditions imposed upon $\{p, q\}$ in (1ii) are minimal: they suffice to ensure that with $\alpha_k \in \xi[m, m + j]$, the rational function $R(p, q \| c | z)$, whose denominator is nonzero when $z = \alpha_k$, has the interpolative property $R(p, q \| c | \alpha_k) = f_k$.

The conditions imposed upon p in (1i) involve α_k ($k := \frac{1}{2} [m, m + \max(i, s)]$); they are independent of j and are, when $j < i$, more stringent than those imposed in (1ii). One consequence of the greater stringency is that the same mapping system p may feature in all systems $b(\frac{1}{2} : \alpha; f \parallel k; m, i, j)$ for $j = \overline{N}$. Similar remarks may be made concerning part [2].

The condition of the form $q(w) = 0$ ($k < \nu := \text{inv } k(j, i)$) imposed in (1i) takes effect when $j < i$. A similar remark concerning the q of clause (2i) may be made. These conditions take effect in subsequent theory in which a connection between two special mapping systems q of the kinds considered is established.

In [14] of the above definition, it is supposed that for each $w \in \text{inv } k[h]$ $q(w)$ has interpolatory properties over two ranges, $k := \frac{1}{2} (m + k(w))$ and $k := \frac{1}{2} (m_i, m_j)$. Functions of this sort, possessing relatively complicated interpolatory properties, may be synthesised from two functions possessing simpler properties. The first part of the following theorem deals with such a synthesis and the second, which is used later, with a special case concerning Lagrange interpolation polynomials.

The synthesis of functions possessing interpolatory properties over two disjoint ranges

1] Let $g, r \in \overline{\mathbb{R}}$, $\beta \in \text{seq}'(\mathbb{K} | \geq g)$, $\gamma \in \text{seq}'(\mathbb{K} | \geq r)$, $s \in \text{seq}(\mathbb{K} | \geq g)$ and $t \in \text{seq}(\mathbb{K} | \geq r)$. Let

a) $\mathcal{L}: \text{seq}(\mathbb{K} | g) \times \mathbb{K} \rightarrow \mathbb{K}$ be such that $\mathcal{L}(e, \beta) = e$ for all $e \in \text{seq}(\mathbb{K} | g)$.

b) $\mathcal{L}': \mathbb{K} \rightarrow \mathbb{K}$ be such that $\mathcal{L}'(\gamma[r]) = t[r]$

c) $\Pi: \mathbb{K} \rightarrow \mathbb{K}$ be such that $\Pi(\beta_k) \neq 0 \langle k = [g] \rangle$ and $\Pi(\gamma_k) = 0 \langle k = [r] \rangle$

d) \mathcal{L} and Π be such that $\mathcal{L}\{\Pi(\beta[g])|\gamma_k\} \neq 0 \langle k = [r] \rangle$

Define $Q: \text{NS}\{\mathcal{L}\{\Pi(\beta[g])\}\} \rightarrow \mathbb{K}$ by setting

$$Q(z) = \frac{\Pi(z) [\mathcal{L}(s[g]|z) - \mathcal{L}\{\mathcal{L}'(\beta[g])|z\}] + \mathcal{L}'(z)}{\mathcal{L}\{\Pi(\beta[g])|z\}}$$

Then

$$Q\{\beta[g] + \gamma[r]\} = s[g] + t[r]$$

2] Let $n, r, \omega \in \overline{\mathbb{R}}$ with $\omega \notin [n, n+r]$; set $r' = \max(\omega, n+r)$. Let $\alpha \in \text{seq}'(\mathbb{K} | \geq r')$, $f \in \text{seq}(\mathbb{K} | \geq r')$. Define the power polynomial mapping $\Delta' := \Delta'(\alpha; f || n, r, \omega): \mathbb{K} \rightarrow \mathbb{K}$ by setting

$$\Delta'(\omega) = \frac{\Pi(\alpha || n, r+1) \{f\} - \Delta(\alpha; f || n, r | \omega, \omega)}{\Pi(\alpha || n, r+1 | \omega, \omega)} + \Delta(\alpha; f || n, r)$$

Then, with $\omega := \omega + [n, n+r]$

$$\Delta'(\alpha[\omega]) = f[\omega]$$

Interpolatory functions may, of course, be synthesised in other ways. For example, let g, r, β, γ, s and t be as in part [1] of the above theorem, and $\Pi, \Pi', \mathcal{L}, \mathcal{L}': \mathbb{K} \rightarrow \mathbb{K}$ be such that $\Pi(\beta[g]) = 0[g]$, $\Pi(\gamma_k) \neq 0 \langle k := [r] \rangle$, $\Pi'(\beta_k) \neq 0 \langle k := [g] \rangle$, $\Pi'(\gamma[r]) = 0[r]$, $\mathcal{L}(\beta[g]) = s[g]$ and $\mathcal{L}'(\gamma[r]) = t[r]$. Define $Q: \mathcal{NS}\{\Pi + \Pi'\} \rightarrow \mathbb{K}$

by setting

$$Q := \frac{\Pi \mathcal{L}' + \Pi' \mathcal{L}}{\Pi + \Pi'}$$

Then

$$Q(\beta[g] + \gamma[r]) = s[g] + t[r]$$

When Π, \dots, \mathcal{L}' are power polynomials, Q is an interpolatory power rational function. As part [2] of the above theorem indicates, the slightly more contrived synthesis considered here permits Q to be an interpolatory power polynomial.

Let $i, j, m \in \mathbb{N}$ and $\kappa := [i]$. With $i' \in \mathbb{N}$ as specified below, let

$\xi \in \text{seq}'(\mathbb{N} | \geq m + \max(i', j))$, $\bar{\xi}(i') := \max \xi(\rho) \langle \rho := [m, m + \max(i', j)] \rangle$

$\alpha := \max \xi(\rho) \langle \rho := [m, m+j] \rangle$, $\alpha \in \text{seq}'(\mathbb{K} | \geq \bar{\xi}(i'))$, and $f \in \text{seq}(\mathbb{K} | \geq \alpha)$,
and $\beta \in \text{seq}(\mathbb{N})$.

1a) Let $i' \leq i$. Set $b(j) := b(\xi; \alpha; f || \kappa; m, i, j)$.

a) $b(j)$ is nonvoid and is determined from the subsequences

$\alpha [\xi [m, m + \max(i, j)]]$ and $f [m, m+j]$ of α and f respectively.

b) If $j' \in [j)$, $b(j) \subset b(j')$

c) Let $|s| \geq r + \max(i, j)$ and $s [r, r + \max(i, j)] = \xi [m, m + \max(i, j)]$

$b(s; \alpha; f || \kappa; r, i, j) = b(j)$

ii) Let $i' = j$. Set $\bar{b}(j) := \bar{b}(\xi; \alpha; f || \kappa; m, i, j)$

a) $\bar{b}(j)$ is nonvoid and is determined from the subsequences

$\alpha [\xi [m, m+j]]$ and $f [\xi [m, m+j]]$ of α and f respectively.

b) The result of ib) with b replaced by \bar{b} holds

c) Let $|s| \geq r+j$ and $s [r, r+j] = \xi [m, m+j]$. (These conditions upon

s are implied by those imposed in (ic) above), $\bar{b}(s; \alpha; f || \kappa; r, i, j)$

$= \bar{b}(j)$

iii) Let $i' = i$ and define $b(j)$ and $\bar{b}(j)$ as above.

a) If $j < i$, $b(j) \subset \bar{b}(j)$

b) If $i \leq j$, $b(j) = \bar{b}(j)$

2] Let $i > 0$

i) Let $i' = i-1$. Set $b'(j) := b'(\xi; \alpha; f || \kappa; m, i, j)$.

a) $b'(j)$ is nonvoid and is determined from the subsequences

$\alpha [\xi [m, m + \max(i-1, j)]]$ and $f [m, m+j]$ of α and f respectively.

b) the result of subclause (1ib) with \bar{b} replaced by \bar{b}' holds

c) Let $|S| \geq r + \max(i-1, j)$ and $S \in [r, r + \max(i-1, j)] = \frac{1}{2} [m, m + \max(i-1, j)]$

(These conditions are implied by those imposed upon S in (1ic) above;

they imply those implied imposed upon S in (1iic). $\bar{b}'(S; \alpha; f \| \kappa; r; i, j) = \bar{b}'(j)$.)

ii) Let $i = j$. Set $\bar{b}'(j) := \bar{b}(S; \alpha; f \| \kappa; m; i, j)$. The results of clause (1ii) with \bar{b} replaced by \bar{b}' hold

Let $\{p, q\} \in \mathcal{B}(\mathbb{Z}; \alpha; f \| \kappa; m, i, j)$ and fixed $k \in \mathbb{Z}[m, m+i, j]$ certain
 function values $\varphi(\omega | \alpha_k)$ are assumed to be zero: the denominator
 of $R(p, q \| c | \alpha_k)$ reduces to a sum formed over values of ν
 belonging to a subsequence $\Omega(k)$ of $[i, j]$. Similar considerations
 apply to the mapping system q , and indeed all mapping
 systems concerned in the above definition. The following
 theorem provides an alternative definition of $\mathcal{B}(\mathbb{Z}; \alpha; f \| \kappa; m, i, j)$
 in terms of sequences of the form $\Omega(k)$; it deals with
 relationships between such sequences and invariance properties.
 Since derived interpolatory properties of $R(p, q \| c)$ over the
 range of α_k with $k := \mathbb{Z}(m+i, m+i+j)$ are treated, properties
 of sequences of the form $\Omega(k)$ over the complete range
 $k := \mathbb{Z}[m, m+i, j]$ are investigated.

Let $i, j, m, r \in \mathbb{N}$ with $h := \min(i, j)$ and $\iota := [i, j]$. Let $\mathbb{Z} \in \text{seq}'(\mathbb{N} | \geq m+i, j)$, $\tau := \max \mathbb{Z}(\omega) \langle \nu := [m, m+i, j] \rangle$, $\tau' := \max \mathbb{Z}(\omega) \langle \nu := [m, m+i, j] \rangle$, $\alpha \in \text{seq}'(\mathbb{K} | \geq \tau')$ and $f \in \text{seq}(\mathbb{K} | \geq \tau)$. Define $\chi: \mathbb{Z}[m, m+i, j] \rightarrow [i, j]$ by setting $\chi(k) := \text{inv } \mathbb{Z}(k) - m$. Let $\alpha \in \text{seq}'(\mathbb{N} | \geq r+i, j)$.

1] Define $\Omega, \Xi, \bar{\Omega}: \mathbb{Z}[m, m+i, j] \rightarrow \text{seq}'([i, j])$ by setting
 $\Omega(k) := \text{inv } \kappa(\chi(k)) \langle k := [m, m+i, j] \rangle$ and $\Omega(k) := [i, j] \langle k := \mathbb{Z}(m+i, m+i, j) \rangle$, $\Xi(k) := \Omega(k) \langle k := \mathbb{Z}[m, m+i, j] \rangle$ when $i \leq j$ and
 $\Xi(k) := \Omega(k) \langle k := \mathbb{Z}[m, m+i, j] \rangle$, $\Xi(k) := \text{asc}(\text{inv } \kappa[j]) \langle k := \mathbb{Z}[m+i, m+i, j] \rangle$ when $j < i$, and finally $\bar{\Omega}(k) := \Omega(k) \langle k :=$

$\xi[m, m+j] \rangle, \Omega(k) := [i] \langle k := \xi(m_{i,j}, m_{i+j}) \rangle$

a) $\{p, q\} \in \mathcal{L}(\xi; \alpha; f \parallel \kappa; m, i, j)$ if and only if

$\alpha)$ for $k := \xi[m, m+i], p(\Omega(k) | \alpha_k) \neq 0$ and $p(\omega | \alpha_k) = 0 \langle \nu := [i] - \Omega(k) \rangle$

$\beta)$ for $k := \xi[m, m+i], q(\omega | \alpha_k) = p(\omega | \alpha_k) f_k \langle \nu := \equiv(k) \rangle$

$\beta')$ for $k := \xi[m, m+i], q(\omega | \alpha_k) = 0 \langle \nu = \text{asc}(\text{inv } \kappa[h]) - \equiv(k) \rangle$ and

$\beta'')$ if $j < i, q(\omega) = 0 \langle \nu := \text{asc}(\text{inv } \kappa(j, i)) \rangle$

b) If $\{p, q\} \in \mathcal{L}(\xi; \alpha; f \parallel \kappa; m, i, j)$

$\alpha)$ $p(\omega | \alpha_k) = 0 \langle \nu := [i] - \Omega(k) \rangle$ and

$\beta)$ $q(\omega | \alpha_k) = 0 \langle \nu := [i] - \equiv(k) \rangle$

for $k := \xi[m, m+i]$.

c) $|\Omega(k)| = 0 \langle k := \xi[m, m+i] \rangle$ and $|\Omega(k)| = i \langle k := \xi(m_{i,j}, m_{i+j}) \rangle$

d) If $i \leq j$ the results of c) hold with Ω replaced by \equiv hold ($\Omega(k) = \equiv(k) \langle k := \xi[m, m+i] \rangle$).

e) If $j < i, |\equiv(k)| = 0 \langle k := \xi[m, m+i] \rangle$ and $|\equiv(k)| = j < i$

$\langle k := \xi(m_{i,j}, m_{i+j}) \rangle, \Omega(k) = \equiv(k) \langle k := \xi[m, m+i] \rangle,$

$|\Omega(k)| \leq |\equiv(k)| \langle k := \xi(m_{i,j}, m_{i+j}) \rangle$ and $\equiv(k) \leq \Omega(k) \langle k :=$

$\xi(m_{i,j}, m_{i+j}) \rangle$

f) In the following $r, \bar{r} \in \mathbb{N}$. Let $\omega \in \text{seq}'(\mathbb{N} \upharpoonright_{\leq r+i})$ be such that

$\omega[r, r+i]$ may be obtained from $\xi[m, m+i]$ by preserving

$\xi[m, m+i]$ and rearranging $\xi(m_{i,j}, m_{i+j})$ so that $\omega[r, r+i] =$

$\xi[m, m+i]$ and $\omega(r_{i,j}, r_{i+j}) = \xi(m_{i,j}, m_{i+j})$. Denote Ω defined above

by $\Omega(\xi, m)$ and let $\Omega(\omega, r): \omega[r, r+i] \rightarrow \text{seq}'([i])$ be defined in

the same way in terms of ω and $\bar{\omega}$.

For each $k \in \omega[\bar{r}, \bar{r}+i+j] \equiv \xi[m, m+i+j]$, $\Omega(\omega, \bar{r} | k) = \Omega(\xi, m | k)$.

Let $\bar{\omega} \in \text{seq}'(\mathbb{N} | \geq \bar{r}, \bar{r}+i+j)$ be such that $\bar{\omega}[\bar{r}, \bar{r}+h] \equiv \xi[m, m+h]$ and $\bar{\omega}[\bar{r}+h, \bar{r}+i+j] \equiv \xi[m+h, m+i+j]$ (with $r=\bar{r}$, the conditions imposed upon ω imply those imposed upon $\bar{\omega}$). Denote \equiv defined above by $\equiv(\xi, m)$ and define $\equiv(\bar{\omega}, \bar{r}) : \bar{\omega}[\bar{r}, \bar{r}+i+j] \rightarrow \text{seq}'([i])$ analogously.

For each $k \in \bar{\omega}[\bar{r}, \bar{r}+i+j] \equiv \xi[m, m+i+j]$, $\equiv(\bar{\omega}, \bar{r} | k) = \equiv(\xi, m | k)$.

ii) $\{p, q\} \in \bar{\mathcal{L}}(\xi; \alpha; f \| \kappa; m, i, j)$ if and only if

a) for $k := \xi[m, m+h]$, $p(\Omega(k) | \alpha_k) \neq 0$ and $p(\omega | \alpha_k) = q(\omega | \alpha_k) = 0$

$\langle \nu := [i] - \bar{\Omega}(k) \rangle$ and

$\beta) q(\omega | \alpha_k) = p(\omega | \alpha_k) f_k \langle \nu := \bar{\Omega}(k) \rangle$

b) If $\{p, q\} \in \bar{\mathcal{L}}(\xi; \alpha; f \| \kappa; m, i, j)$, $p(\omega | \alpha_k) = q(\omega | \alpha_k) = 0 \langle \nu := [i] - \bar{\Omega}(k) \rangle$ for $k := \xi[m, m+i+j]$

c) $|\bar{\Omega}(k)| = 0 \langle k := \xi[m, m+h] \rangle$ and $|\bar{\Omega}(k)| = i \langle k := \xi[m+h, m+i+j] \rangle$

d) Let $\bar{\omega}, \bar{r}$ be as in subclass (1if), denote $\bar{\Omega}$ defined above by $\bar{\Omega}(\xi, m)$ and let $\bar{\Omega}(\bar{\omega}, \bar{r}) : \bar{\omega}[\bar{r}, \bar{r}+i+j] \rightarrow \text{seq}'([i])$ be defined in the same way in terms of $\bar{\omega}$ and \bar{r} .

For each $k \in \bar{\omega}[\bar{r}, \bar{r}+i+j] \equiv \xi[m, m+i+j]$, $\bar{\Omega}(\bar{\omega}, \bar{r} | k) = \bar{\Omega}(\xi, m | k)$

ii) $\Omega(k) = \bar{\Omega}(k) \langle k := \xi[[m, m+h] + (m+i, m+i+j)] \rangle$ and

$\Omega(k) \subseteq \bar{\Omega}(k) \langle k := \xi[m+i, m+i+j] \rangle$.

2. Let $i > 0$. Define $\Omega', \equiv', \bar{\Omega}': \mathbb{Z}[m, m+i+j] \rightarrow \text{seq}'([i])$ by setting $\Omega'(k) = \text{asc}(\text{inv } \kappa[\min(i, \chi(k))])$ and $\equiv'(k) := \text{asc}(\text{inv } \kappa[\min(i, j, \chi(k))])$ both for $k := \mathbb{Z}[m, m+i+j]$ and, finally, $\bar{\Omega}'(k) = \text{asc}(\text{inv } \kappa[\chi(k)])$, $\langle k := \mathbb{Z}[m, m+i] \rangle$, $\bar{\Omega}'(k) = [i]$, $\langle k := \mathbb{Z}(m+i, m+i+j) \rangle$

ia) $\{p, q\} \in \mathcal{B}'(\mathbb{Z}; \alpha; f \| \kappa; m, i, j)$ if and only if

$\alpha)$ for $k := \mathbb{Z}[m, m+i]$, $p(\omega | \omega_k) = 0 \langle \nu := [i] - \Omega'(k) \rangle$

$\beta)$ for $k := \mathbb{Z}[m, m+i+j]$, $q(\omega | \omega_k) = p(\omega | \omega_k) \upharpoonright_k \langle \nu := \equiv'(k) \rangle$

$\beta')$ if $i \leq j$, for $k := \mathbb{Z}[m, m+i+j]$, $q(\omega | \omega_k) = 0 \langle \nu := [i] - \equiv'(k) \rangle$

$\beta'')$ if $j < i$, for $k := \mathbb{Z}[m, m+i+j]$, $q(\omega | \omega_k) = 0 \langle \nu := \text{inv } \kappa(\chi(k), j) \rangle$

and $q(\omega) = 0 \langle \kappa \rangle \langle \nu := \text{inv } \kappa([j]) \rangle$

b) If $\{p, q\} \in \mathcal{B}'(\mathbb{Z}; \alpha; f \| \kappa; m, i, j)$

$\alpha)$ $p(\omega | \omega_k) = 0 \langle \nu := [i] - \Omega'(k) \rangle$ and

$\beta)$ $q(\omega | \omega_k) = 0 \langle \nu := [i] - \equiv'(k) \rangle$

for $k := \mathbb{Z}[m, m+i+j]$

c) $|\Omega'(k)| = \chi(k) < i \langle k := \mathbb{Z}[m, m+i] \rangle$ and $|\bar{\Omega}'(k)| = i \langle k := \mathbb{Z}[m+i, m+i+j] \rangle$

d) If $i \leq j$ the results of (c) with Ω' replaced by \equiv' hold and $\Omega'(k) = \equiv'(k) \langle k := \mathbb{Z}[m, m+i+j] \rangle$

e) If $j < i$, $|\equiv'(k)| = \chi(k) \leq j < i \langle k := \mathbb{Z}[m, m+i+j] \rangle$ and $|\equiv'(k)| = j < i \langle k := \mathbb{Z}(m+i, m+i+j) \rangle$. $\Omega'(k) = \equiv'(k) \langle k := \mathbb{Z}[m, m+i+j] \rangle$ and $\equiv'(k) \subset \Omega'(k) \langle k := \mathbb{Z}(m+i, m+i+j) \rangle$

f) In the following $\bar{r}, \bar{r}' \in \mathbb{N}$.

Let $\omega \in \text{seq}'(\mathbb{N} | \geq r+i+j)$ be such that $\omega[r', r+i] = \mathbb{Z}[m, m+i]$ and $\omega[m+i, m+i+j] = \mathbb{Z}[m+i, m+i+j]$ (with $r = r'$ the conditions

imposed upon ω in subclause (1if) imply those imposed upon ω' here which, with $r' = \bar{r}$, in turn imply those imposed upon $\bar{\omega}$ in that subclause. Denote $\bar{\Omega}'$ defined above by $\bar{\Omega}'(\xi, m)$ and define $\bar{\Omega}'(\omega', r')$: $\omega' [r', r'+i+j] \rightarrow \text{seq}'([i])$ in terms of ω' and r' analogously.

For each $k \in \omega' [r', r'+i+j] \equiv \xi [m, m+i+j]$, $\bar{\Omega}'(\omega', r' | k) \equiv \bar{\Omega}'(\xi, m | k)$

Let $\bar{\omega}' \in \text{seq}'(N | \geq r'+i+j)$ be such that $\bar{\omega}' [r', r'+\min(i-1, j)] = \xi [m, m+\min(i-1, j)]$ and $\bar{\omega}' (r'+\min(i-1, j), r'+i+j) \equiv \xi (m+\min(i-1, j), m+i+j)$ (either of the conditions imposed, with $\bar{r}' = r$, upon ω in subclause (1if) or the conditions imposed, with $\bar{r}' = \bar{r}$ upon $\bar{\omega}$, or the conditions imposed, with $\bar{r}' = r'$, upon ω' above imply those imposed upon $\bar{\omega}'$ here). Denote \equiv' defined above by $\equiv'(\xi, m)$ and define $\equiv'(\bar{\omega}', \bar{r}')$ in terms of $\bar{\omega}'$ and \bar{r}' analogously.

For each $k \in \bar{\omega}' [r', r'+i+j] \equiv \xi [m, m+i+j]$, $\equiv'(\bar{\omega}', \bar{r}' | k) \equiv \equiv'(\xi, m | k)$.

ii) $\{p, q\} \in \bar{\mathcal{O}}'(\xi: \alpha; f \| \kappa; m, i, j)$ if and only if

a) for $k := \xi [m, m+\min(i, j+1)]$, $p(\omega | \alpha_k) = q(\omega | \alpha_k) = 0 \langle \omega := [i] - \bar{\Omega}'(k) \rangle$

and

$\beta)$ for $k := \xi [m, m+i]$, $q(\omega | \alpha_k) = p(\omega | \alpha_k) f_k \langle \omega := \bar{\Omega}'(k) \rangle$

b) If $\{p, q\} \in \bar{\mathcal{O}}'(\xi: \alpha; f \| \kappa; m, i, i)$, $p(\omega | \alpha_k) = q(\omega | \alpha_k) = 0$

$\langle \omega := [i] - \bar{\Omega}'(k) \rangle$ for $k := \xi [m, m+i]$

c) Let $\bar{\omega}', \bar{r}'$ be as in subclause (2if), denote $\bar{\Omega}'$ defined above

by $\bar{\Omega}'(\xi, m)$ and let $\bar{\Omega}'(\bar{\omega}', \bar{r}') : \bar{\omega}'[\bar{r}', \bar{r}' + i + j] \rightarrow \text{seq}'([i])$ be defined in the same way in terms of $\bar{\omega}'$ and \bar{r}' .

For each $k \in \bar{\omega}'[\bar{r}', \bar{r}' + i + j] = \xi[m, m + i + j]$, $\bar{\Omega}'(\bar{\omega}', \bar{r}' | k) = \bar{\Omega}'(\xi, m | k)$.

iii) $\bar{\Omega}'(k) = \bar{\Omega}'(k) \langle k := \xi[[m, m+h] + (m+i, m+i+j)] \rangle$ and

$\bar{\Omega}'(k) \leq \bar{\Omega}'(k) \langle k := \xi(m+h, m+i) \rangle$

Direct interpolation

Let $i, j, m \in \bar{N}$ and $\nu := [i]$. With $i' \in N$ as specified below, let $\xi \in \text{seq}'(N | \geq m + \max(i', j))$, $\nu := [m, m + \max(i', j)]$,

$\tau := \max \xi(\omega) \langle \nu := [m, m + \max(i', j)] \rangle$, $\alpha \in \text{seq}'(K | \geq \tau(i'))$ and $f \in \text{seq}(K | \geq \tau)$

1] Define $\Omega: \xi[m, m + \min(i, j)] \rightarrow \text{seq}'([i])$ as in Th (1i)

i) Let $i' = i$ and $\{p, q\} \in \mathcal{B}(\xi; \alpha; f \| \kappa; m, i, j)$

For each $k \in \xi[m, m + \min(i, j)]$ for which $c\{\Omega(k)\} \neq 0$,

$R(p, q \| c | \alpha_k)$ reduces to

$$R(p, q \| c | \alpha_k) = \frac{c\{\Omega(k)\} p\{\Omega(k) | \alpha_k\} f_k}{c\{\Omega(k)\} p\{\Omega(k) | \alpha_k\}} = f_k$$

For each $k \in \xi[m + i, m + j]$ for which $p(\alpha_k) c \neq 0$, $R(p, q \| c | \alpha_k) = f_k$

ii) Let $i' = j$ and $\{p, q\} \in \mathcal{B}(\xi; \alpha; f \| \kappa; m, i, j)$. The results of (i)

hold as stated for the newly defined pair $\{p, q\}$

2] ^{Let $i > 0$.} Define $\Omega': \xi[m, m + j] \rightarrow \text{seq}'([i])$ as in Th. (2i)

i) Let $i' = i - 1$ and $\{p, q\} \in \mathcal{B}'(\xi; \alpha; f \| \kappa; m, i, j)$

For each $k \in \xi[m, m + j]$ set

$$s'(k) = \sum_{\omega} c(\omega) p(\omega | \alpha_k) \langle \nu := \Omega'(k) \rangle$$

and, if $s'(k) \neq 0$,

$$t'(k) = \frac{\sum_{\omega} c(\omega) q(\omega | \alpha_k) \langle \nu := \Omega'(k) \rangle}{s'(k)}$$

For all $k \in \xi[m, m + j]$ for which $s'(k) \neq 0$

$$R(p, q \| c | \alpha_k) = t'(k) = f_k$$

ii) Let $i' = j$ and $\{p, q\} \in \mathcal{B}'(\xi; \alpha; f \| \kappa; m, i, j)$

The result of (2i) above holds for the newly defined pair $\{p, q\}$.

direct and derived interpolation.

Let $i \in \mathbb{N}$, $j, m, r \in \overline{\mathbb{N}}$, $\xi \in \text{seq}'(\overline{\mathbb{N}} \mid \geq m+i_j)$, $\xi' := \text{max}_{\xi} \xi(\omega) \langle \omega := [m, m+i_j] \rangle$
 $\alpha \in \text{seq}'(k \mid \geq r')$, $f \in \text{seq}(k \mid \geq r')$, $\kappa := [i]$ and $\rho := [i]$. With

$p, q: k \rightarrow m[k \mid i]$ specified in the various cases considered below, set
 $C := C(p, q \parallel \xi: \alpha; f \parallel \rho: m+i_j^{+1}i)$ and, in each case, let $c \in \text{col}[k \mid i]$
 satisfy the relationship $Cc = O_{[i]}$.

1] Let $\{p, q\} \in \mathcal{C}(\xi: \alpha; f \parallel \kappa: m, i, j)$

ia) The direct interpolation results of the form

$$R(p, q \parallel c \mid \omega_k) = f_k$$

given in Th. (1i) hold over the range $k := \xi[m, m+i_j]$.

b) Define $\Omega, \Xi: \xi[m, m+i_j] \rightarrow \text{seq}'([i])$ as in Th. (1i). With

$k \in \xi(m+i_j, m+i_j]$, set

$$s(k) := \sum_1 c(\omega) p(\omega \mid \omega_k) \langle \omega := \Omega(k) \rangle$$

and, if $s(k) \neq 0$,

$$t(k) := \frac{\sum_1 c(\omega) q(\omega \mid \omega_k) \langle \omega := \Xi(k) \rangle}{s(k)}$$

For all $k \in \xi(m+i_j, m+i_j]$ for which $s(k) \neq 0$

$$R(p, q \parallel c \mid \omega_k) = t(k) = f_k$$

i) $R(p, q \parallel c) = \frac{|q \parallel c|}{|p \parallel c|} \langle N^S \{p \parallel c\} \rangle$

ii) Let $\alpha \in \text{seq}'(\overline{\mathbb{N}} \mid \geq r+i_j)$ be such that $\alpha[r, r+i_j]$ may be obtained
 from $\xi[m, m+i_j]$ by preserving $\xi[m, m+\text{max}(i, j)]$ and rearranging

$\xi(m+\max(i,j), m+ij]$, so that $\alpha[r, r+\max(i,j)] = \xi[m, m+\max(i,j)]$ and $\alpha(r+\max(i,j), r+ij] \equiv \xi(m+\max(i,j), m+ij]$ and $\delta \equiv [i]$ for which $\alpha(r+j+z+1) = \xi(m+j+\delta(z)+1)$ ($z := [i]$) exists. (If $j < i$, $\delta(z) = z$ ($z := [i-j]$)). Denote C above by $C(\xi)$.

a) $C(\alpha) := C(p, q \parallel \alpha; f \parallel p: r+j+1, i)$ is a row rearranged form of $C(\xi)$ and relationships () of Th (2i) with r, n replaced by $r+j+1, m+j+1$ hold. The coefficient spaces generated by $C(\xi)$ and $C(\alpha)$ are identical: c' 's col $[k|i]$ satisfies the relationship $C(\xi)c' = 0_{[i]}$ if and only if $C(\alpha)c' = 0_{[i]}$. In particular, $C(\alpha)c = 0_{[i]}$.

b) $\{p, q\} \in \mathcal{B}(\alpha; f \parallel \kappa: r, i, j)$

c) Denote $\Omega, \Xi: \xi[m, m+ij] \rightarrow \text{seq}'([i])$ defined in Th. [1] in terms of ξ and m by $\Omega(\xi, m), \Xi(\xi, m)$ and let $\Omega(\alpha, r), \Xi(\alpha, r): \alpha[r, r+ij] \rightarrow \text{seq}'([i])$ be defined in the same way in terms of α and r . For each $k \in \alpha[r, r+ij] \equiv \xi[m, m+ij]$, $\Omega(\xi, m|k) = \Omega(\alpha, r|k)$ and $\Xi(\xi, m|k) = \Xi(\alpha, r|k)$.

The direct interpolation result of subclass (ia) holds over the range $k := \alpha[r, r+ij] = \xi[m, m+ij]$ and in the supporting result of Theorem (1c) $\Omega(k)$ may be taken to be $\Omega(\xi, m|k)$ or $\Omega(\alpha, r|k)$, since these two single member sequences are equal.

d) The result of clause (ii) with C replaced by $C(\alpha)$ holds over $NS\{ |p \parallel C(\alpha) | \} = NS\{ |p \parallel C(\xi) | \}$.

2.] Let $\{p, q\} \in \bar{L}(\xi: \alpha; f \parallel \kappa: m, i, j)$.

The direct interpolation results of the form () given in Th. (1ii) hold over the range $k := \xi[m, m+j]$. With $\bar{Q}: \xi[m, m+j] \rightarrow \text{seq}'([i])$ defined as in Th. (1ii) the derived interpolation results of subclause (1ib) above with Ω, \equiv both replaced by \bar{Q} hold. The result of clause (1ii) above holds for the newly defined p and q .

With $\epsilon \in \text{seq}'(N \mid \geq r+i_j)$ such that $\epsilon[r, r+i_j] = \xi[m, m+j]$ and $\epsilon(r+i_j, r+i_j) = \xi(m+j, m+i_j)$, so that $\delta = [i]$ for which $\epsilon(r+i_j+z+1) = \xi(m+j+\delta(z)+1)$ ($z_1 = [i]$) exists (the subsequent restriction upon δ as given in clause (1iii) above now being unnecessary) the result of subclause (1iia) holds and now $\{p, q\} \in \bar{L}(\epsilon: \alpha; f \parallel \kappa: r, i, j)$.

Denoting $\bar{Q}: \xi[m, m+j] \rightarrow \text{seq}'([i])$ defined in Th. (1) in terms of ξ and m by $\bar{Q}(\xi, m)$ and, letting $\bar{Q}(\epsilon, r): \epsilon[r, r+i_j] \rightarrow \text{seq}'([i])$ be defined in the same way in terms of ϵ and r ,

$\bar{Q}(\epsilon, r \mid k) = \bar{Q}(\xi, m \mid k)$ for $k := \epsilon[r, r+i_j] = \xi[m, m+j]$. The direct interpolation result of the new version of subclause (1ia) holds over the range $k := \epsilon[r, r+i_j] = \xi[m, m+j]$ and, in the supporting result of Th. (1ii), $\bar{Q}(k)$ may be taken to be $\bar{Q}(\xi, m \mid k)$ or $\bar{Q}(\epsilon, r \mid k)$.

The derived interpolation result of the new version of subclause (1ib) holds for each $k \in \epsilon(r+i_j, r+i_j) = \xi(m+i_j, m+i_j)$ and $\bar{Q}(k)$ in that version may be taken to be either $\bar{Q}(\xi, m \mid k)$ or $\bar{Q}(\epsilon, r \mid k)$. The modified

result of clause (ii) above with C replaced by C' holds as in subclause (iii) above for the new pair p, q .

3] A theory based upon a pair $\{p, q\} \in \bar{b}'(\frac{1}{2}: \alpha; f \| \kappa : m, i, j)$ may also be adapted from part [1]. The direct interpolation results of the counterparts to subclauses (i, a, b) are based upon Theorem (2i) and $\bar{\Omega}' \equiv'$ as defined in Th. [2]. Now $\sigma \in \text{seq}'(N \geq r+i, j)$ is such that $\sigma \upharpoonright [r, r+\max(i-1, j)] = \frac{1}{2} \upharpoonright [m, m+\max(i-1, j)]$ and $\sigma \upharpoonright (r+\max(i-1, j), r+i, j] \cong \frac{1}{2} \upharpoonright (m+\max(i-1, j), m+i, j]$ so that $\delta \geq Li$ for which $\sigma(r+j+z+1) = \frac{1}{2}(m+j+\delta(z)+1)$ ($z := Li$) exists (if $j < i-1, \delta(z) = z < z := Li-j-1$).

4] Finally, in the theory adapted from part [1] for the pair $\{p, q\} \in \bar{b}'(\frac{1}{2}: \alpha; f \| \kappa : m, i, j)$, Theorem (2ii) is used and $\bar{\Omega}'$ is as defined in Theorem [2]. σ is as described in part [2] of the present theorem.

Examples

i) ξ is taken to be $[m, i, j]$ and, where relevant, κ, ρ and ω are taken to be the natural orderings $\kappa := [i], \rho := \omega := [i]$

Now

$$\Pi = [\pi(\alpha \| m, \nu)]^{\nu=[i]}, \quad \mathcal{L} = [\Delta(\alpha; f \| m, \nu; j-\nu)]^{\nu=[i]}$$

The components of these complementary interpolatory mapping systems possess the following properties: for $\nu=[i]$

$$\alpha') \pi(\alpha \| m, \nu | \alpha_k) = 0 \quad \langle k = [m, m, \nu] \rangle$$

$$\beta') \text{ if } \nu \leq j, \Delta(\alpha; f \| m, \nu; j-\nu | \alpha_k) = f_k \quad \langle k = [m, \nu, m, j] \rangle$$

$$\text{if } j < \nu, \Delta(\alpha; f \| m, \nu; j-\nu) = 0 \quad \langle k \rangle$$

$$\text{In this case } \Omega(\xi; \alpha \| \kappa; m, i, j; k) = [\min(i, k)]$$

The theory of this example may be obtained from parts [1, 2] of th. by replacing α [3] by α and, where appropriate, discarding the symbols ξ, κ, ρ and ω and the attached parentheses. In particular

$$C = C(\rho, \omega \| \xi; \alpha; f; \rho; m, j, i)_{\nu} \quad \nu=[i]$$

$$= [\pi(\alpha \| m, \nu | \alpha_{m+j+z+1}) \{ f_{m+j+z+1} - \Delta(\alpha; f \| m, \nu; j-\nu | \alpha_{m+j+z+1}) \}]_{z=[i]}$$

$$A = A(\xi; \omega, \kappa \| m, i, j)$$

$$= [\delta(\alpha; f \| m, \nu; j+z-\nu+1)]_{z=[i]}^{\nu=[i]}$$

$$B = B(\xi; \rho, \omega \| m, i, j)$$

$$= \mathcal{L}[\mu(\alpha \| m; j+z+1, j+\nu+1)] \langle z, \nu = [i] \rangle$$

ii) ξ_k is taken to be the reverse order sequence $m_{i+j}, m_{i+j-1}, \dots, m$ and, where relevant, κ, ρ and ω are taken to be the reverse orderings $\kappa = i, i-1, \dots, 0$ and $\rho = \omega = i-1, i-2, \dots, 0$.

Now

$$\Pi = [\pi(\alpha \parallel m_{i+j} + 1, i-j)]^{D=[i]} \quad \mathcal{L} = [\Delta(\alpha; f \parallel m_{i+j-i}, j-i)]^{D=[i]}$$

The components of these complementary interpolatory mapping systems possess the following properties: for $D=[i]$

$$\alpha') \pi(\alpha \parallel m_{i+j} + 1, i-j | \alpha_k) = 0 \quad \langle k = (m_{i+j}, m_{i+j}) \rangle$$

$$\beta') \text{ if } i-j \leq D, \Delta(\alpha; f \parallel m_{i+j-i}, j-i | \alpha_k) = f_k \quad \langle k = [m_{i+j-i}, m_{i+j-i}] \rangle$$

$$\text{if } D < i-j, \Delta(\alpha; f \parallel m_{i+j-i}, j-i) = 0 \quad \langle k \rangle$$

$$\text{In this case } \Omega(\xi: \alpha \parallel \kappa: m, i, j; k) = [\max(0, k-j), i]$$

$$\hat{C} = C(\rho, q \parallel \xi: \alpha; f: \rho; m_{i+j}, i)$$

$$= [\pi(\alpha \parallel m_{i+j} + 1, i-j | \alpha_{m+z}) \{ f_{m+z} - \Delta(\alpha \parallel m_{i+j-i}, j-i | \alpha_{m+z}) \}]_{z=i}^{D=[i]}$$

$$\hat{A} = A(\xi: \omega; \kappa \parallel m, i, j)$$

$$= [\delta(\alpha; f \parallel m_{i+j-i}, j-i | \alpha_{m+z})]_{z=i}^{D=[i]}$$

$$\hat{B} = B(\xi; \rho, \omega \parallel m, i, j)$$

$$= \bigcup_{z=i}^{D=[i]} [\mu(\alpha \parallel m_{i+j-i}, j-i | \alpha_{m+z})] \quad \langle z, D=[i] \rangle$$

$$(\xi: \text{confirmatory proof that } \hat{A}\hat{B} = \hat{A} = \hat{B}\hat{C})$$

iii) To illustrate part [B] of Th. $\mathcal{B} [m, m; i, j]$, \mathcal{B} , κ , ρ and ω are taken to be as in example (i) and \mathcal{B} is $\mathcal{B} [n, m; i, j]$ is taken to be $m; j, m; j-1, \dots, m, m; j+1, \dots, m; i, j$. (recip $\mathcal{B} [n, m; i, j] = \mathcal{B} [n, m; i, j]$)

With $i' = \min(i, j)$

$$\Pi = \left[\Pi(\alpha \| m; j-d+1, d) \right]^{d=i'} \left[\Pi(\alpha \| m; j+d+1) \right]^{d=i-i'}$$

$$\Lambda = \left[\Lambda(\alpha \| m; j-d) \right]^{d=i'} \left| 0^{(i-j)} \right.$$

The components of these complementary interpolatory mapping systems possess the following properties. For $d =$

$\alpha')$ For $d' \in [i', i']$, $\Pi(\alpha \| m; j-d+1, d | \alpha_k) = 0 \langle k = (m; j-d, m; j) \rangle$

For $d' \in (j, i]$, $\Pi(\alpha \| m, d | \alpha_k) = 0 \langle k = [m, m+d] \rangle$

$\beta')$ For $d' \in [i']$, $\Lambda(\alpha \| m; j-d | \alpha_k) = f_k \langle k = [m, m; j-d] \rangle$

For $k \in [0, j]$, $\Omega(\mathcal{B}; \alpha \| \kappa; n, i, j; k) = [\min(i, j-k)]$ and

for $k \in (j, i+j]$, $\Omega(\mathcal{B}; \alpha \| \kappa; n, i, j; k) = [\min(i, k)]$

$$H(\mathcal{B} / \mathcal{B}) = H(\mathcal{B}, \mathcal{B}; \kappa \| m, i, j)$$

$$= \left[U \left[\mathcal{D}(\alpha; \Pi(\alpha \| m; j-d+1, d | \langle \alpha \rangle) \| m, d) \right]^{d=i'} \left| 0^{(i-j)} \right. \right] \left| 0^{(i-j)} \right.$$

$$\left[0^{(i')} \right. \left. \left| I^{(i-j-1)} \right. \right]$$

The mapping systems $C(\mathcal{B})$, $B(\mathcal{B})$ and $A(\mathcal{B})$ satisfy the relationships given in part [B] of Th.

v) its second illustration of part [3] of Theorem is obtained by

taking $\mathcal{B}[m, m; i, j]$, k, p and w as in example (ii) and

$\mathcal{B}[n, m; i, j]$ to be $m; i, m; i+1, \dots, m; i+j, m; i-1, \dots, m$. (recip $\mathcal{B}[n, m; i, j] = \mathcal{B}[n, m; i, j]$)

With $i' = \min(i, j)$,

$$\Pi = [\Pi(\alpha \| m; i+j+1, i-p)]^{p=L(i-j)} | [\Pi(\alpha \| m; i, i'-p)]^{p=L(i')}$$

$$\Sigma = O^{L(i-j)} | [\Delta(\alpha; f \| m; i'-p, j-i'+1)]^{p=L(i')}$$

The components of these complementary interpolatory mapping systems possess the following properties

α' For $p \in [0, i-j]$, $\Pi(\alpha \| m; i+j+1, i-p | \alpha_k) = 0 \langle k = [m; i+j+1, m; i, j] \rangle$

For $p \in [\max(0, i-j), i]$: $\Pi(\alpha \| m; i, i-p | \alpha_k) = 0 \langle k = [m; i, m; 2i-p] \rangle$

β' For $p \in [\max(0, i-j), i]$: $\Delta(\alpha; f \| m; 2i-p, j-i'+1 | \alpha_k) = f_k$

$\langle k = [m; 2i-p, m; i, j] \rangle$

In this case

$$\mathcal{R}(\mathcal{B}; \alpha \| \kappa; m, i, j; k) \neq [\max(0, k-j), i] \quad \langle k \in [i] \rangle$$

$$= [\max(0, 2i-k+1), i] \quad \langle k \in (i, i+j] \rangle$$

$$H(\mathcal{B}/\mathcal{B}) = H(\mathcal{B}, \mathcal{B}; \kappa \| m, i, j) =$$

$$[I(i-j-1) | O_{Li-j}^{[j]}] \| [O_{[j]}^{[i-j]} | L[\mathcal{B}(\alpha; \Pi(\alpha \| m; i, i-p) \langle \alpha \rangle)]$$

$$m; i, j-i'+1, i'-1] \langle (i, 2-i') \rangle$$

Again the mapping systems $\mathcal{C}(\mathcal{B})$, $\mathcal{B}(\mathcal{B})$ and $\mathcal{A}(\mathcal{B})$ satisfy the relationships given in part [3] of Th.

With $\{p, q\} \in \mathcal{B}(\frac{1}{2}; \alpha; f \| \kappa; m, i, j)$ and fixed $k \in \mathbb{N}[m, m+i+j]$ certain function values $p(\omega | \omega_k)$ are assumed to be zero: the denominator values of ω belonging to $R(p, q | \mathcal{C} | \omega_k)$ reduces to a sum formed over a subsequence $\Omega(k)$ of values $\{i\}$. Similar considerations apply to the mapping system of and indeed to all mapping systems concerned in the above definition. The following theorem provides an alternative definition of $\mathcal{B}(\frac{1}{2}; \alpha; f \| \kappa; m, i, j)$ is provided in terms of sequences of the form $\Omega(k)$

in the process, it deals with relationships between such sequences and invariance properties. Since interdomain interpretation properties of $R(p, q | \mathcal{C})$ over the range of ω_k with $k = \mathbb{N}[m, m+i+j]$, properties of these atom sequences of the form $\Omega(k)$ over the complete ranges $k \in \mathbb{N}[m, m+i+j]$ are investigated.

Special form of complementary interpretation functions

Let $i, j, m \in \mathbb{N}$ and $\kappa \equiv [i]$. With $i' \in \mathbb{N}$ as specified below let $\xi \in \text{seq}'(\mathbb{N}) \cong m \text{ max}(i, j)$, $\bar{z}(i') = \text{max } \frac{1}{2}(\omega) \langle \nu = [m, m + \text{max}(i, j)] \rangle$, $\tau = \text{max } \frac{1}{2}(\omega) \langle \nu = [m, m+i+j] \rangle$, $\alpha \in \text{seq}'(\mathbb{K} | \geq \bar{z}(i'))$ and $f \in \text{seq}(\mathbb{K} | \geq \tau)$ and $\omega \in \text{col}(\mathbb{K} | i)$

1] Let $i' \in i$ and, set for ω Define p, q and $\kappa \rightarrow \text{max } [K | i]$ by setting

$$p(\omega) := \prod_{\omega \in \Omega} \{ \pi(\alpha[\frac{1}{2}] \| m; i, \omega) \} \kappa(\omega) \quad \langle \Omega = [i] \rangle \langle \mathbb{K} \rangle$$

$$q(\omega) := \prod_{\omega \in \Omega} \{ \pi(\alpha[\frac{1}{2}] \| m, h; \kappa(\omega)) \pi(\alpha[\frac{1}{2}] \| m+i+1, i-j | \omega_{\frac{1}{2}(m+\kappa(\omega))}) \} \\ \{ \pi(\alpha[\frac{1}{2}] \| m+i+1, j-i) \{ f_{\frac{1}{2}(m+\kappa(\omega))} - \Delta(\frac{1}{2}; \alpha; f \| m+i+1, j-i-1 | \omega_{\frac{1}{2}(m+\kappa(\omega))}) \} \} \\ \pi(\alpha[\frac{1}{2}] \| m+i+1, j-i | \omega_{\frac{1}{2}(m+\kappa(\omega))}) \\ + \Delta(\frac{1}{2}; \alpha; f \| m+i+1, j-i-1) \} \quad \langle \Omega = [i] \rangle \langle \mathbb{K} \rangle$$

both for $\nu = [i]$

i) Over $\mathbb{N} \setminus \{0\}$, $R(p, q | \mathcal{C})$ is the quotient of two polynomials, the denominator and numerator being of degrees $\leq i$ and $\leq j$ respectively

ii) $\{p, q\} \in \mathcal{B}(\frac{1}{2}; \alpha; f \| \kappa; m, i, j)$

iii) $R(p, q || c)$ possesses the interpretability properties described in Th (1i).

2] Let $i' = i - 1$. Define $p', q' \in K \rightarrow \text{ms} [K | i]$ by setting

$$p'(\omega) := \pi(\alpha[\xi] || m, \kappa(\omega)) \quad \langle K \rangle$$

$$q'(\omega) := \pi(\alpha[\xi] || m, \kappa(\omega)) \Delta(\xi: \alpha; f || m, \kappa(\omega), j - \kappa(\omega)) \quad \langle K \rangle$$

both for $i := [i]$

With p, q, c and, where relevant, c_i replaced by p', q', c' at c' respectively

With p, q, c replaced by p', q', c' the result of (1i) above holds, as does that of (1ii) with p, q, c replaced by p', q', c' . $R(p', q' || c')$ possesses the interpretability properties described in Th (2i).

3] Let $i' = i$. Define $\Theta: \xi \in \text{seq}(\mathbb{N})$ by setting

$$\Theta := D(\xi || \kappa; m, i)$$

$\Theta = \xi[m, m_i] \kappa$ and $D \in [K | i, i]$ by setting

$$D = \left[\pi(\alpha[\xi] || m+z+1, i-z | \omega) \right]_{z=\kappa}^{z=\Theta} \quad \begin{matrix} D \geq z = 0 \\ m+z+1, m_i \end{matrix}$$

$$D = \left[\left[\pi(\alpha[\xi] || m+z+1, i-z | \omega) \right]_{z=\kappa}^{z=[i]} \right]_{\kappa}^{z=\Theta}$$

i) $p = p'D$ and $q = q'D \quad \langle K \rangle$

ii) Let $c' = Dc$. $R(p, q || c) = R(p, q || c')$

$$R(p, q || c) = R(p', q' || c') \quad \langle \text{NS}\{pc\} = \text{NS}\{p'c'\} \rangle$$

$$pc = p'e', \quad qc = q'e' \quad \langle K \rangle$$

and \rightarrow

Let $i \in \mathbb{N}$, $j, m, r \in \mathbb{N}$, $\xi \in \text{seq}'(\mathbb{N} | \geq m; i; j)$, $\tau' := \max \xi(\omega) < \rho := [m, m; i; j]$

$\alpha \in \text{seq}'(K | \geq \tau')$, $f \in \text{seq}(K | \geq \tau')$, $\kappa := [i)$ and $\rho := [i)$

1] Define $p, q: K \rightarrow [K | i]$ as in Th [1], set $C := C(p, q | \xi: \alpha; f | \rho, \kappa; i)$

and define $\Theta \in \text{seq}'(\xi[m, m; i] | i)$, $\psi \in \text{seq}'(\xi(m; j, m; i; j) | i-1)$ by

setting $\Theta := \xi[m, m; i] \kappa$, $\psi := \xi(m; j, m; i; j) \rho$

B i) Define $\Delta, \tilde{\Delta}', \tilde{\Delta}'' \in \text{diag}[K | i; \xi_0 - 1]$, $\tilde{D}, \tilde{D}' \in \text{diag}[K | i]$ and

$\tilde{C}', \tilde{C}'' \in [K | i-1, i]$ by setting

$$\Delta := \text{diag}[f_\omega] \langle \omega := \epsilon \rangle$$

$$\tilde{\Delta}' := \text{diag}[\pi(\alpha[\xi] | m; j+1, j-i | \alpha_\omega)] \langle \omega := \epsilon \rangle$$

$$\tilde{\Delta}'' := \text{diag}[\Lambda(\xi: \alpha; f | m; j+1, j-i-1 | \alpha_\omega)] \langle \omega := \epsilon \rangle$$

$$\tilde{D} := \text{diag}[\pi(\alpha[\xi] | m; j-1, i-j | \alpha_\omega)] \langle \omega := \Theta \rangle$$

$$\tilde{D}' := \text{diag} \left[\frac{f_\omega - \Lambda(\xi: \alpha; f | m; j+1, j-i-1 | \alpha_\omega)}{\pi(\alpha[\xi] | m; j+1, j-i | \alpha_\omega)} \right] \langle \omega := \Theta \rangle$$

$$\tilde{C}' := \left[\pi(\alpha[\xi] | m; i, 0 | \alpha_\epsilon) \right]_{z:=\epsilon}^{z=\kappa}$$

$$\tilde{C}'' = \left[\left[\pi(\alpha[\xi] | m; h, 0 | \alpha_\epsilon) \right]_{z:=\epsilon}^{z=h} \mid 0 \begin{matrix} [i-j-1] \\ [i-1] \end{matrix} \right]_{z:=\epsilon}^{z=\kappa}$$

a) $C \in \mathbb{R}^{k \times k}$ has the decomposition

$$C = \Delta \tilde{C}' - \{ \tilde{\Delta}' \tilde{C}'' \tilde{D}' + \tilde{\Delta}'' \tilde{C}'' \} \tilde{D}$$

b) If $j \leq i$, $\tilde{\Delta}' = I(i-1)$, and $\tilde{\Delta}'' = 0 \in \text{diag}[K | i-1]$ in the above

and $\tilde{D}' = \text{diag}[f_\omega] \langle \omega := \Theta \rangle$ in the above, and

$$C = \Delta \tilde{C}' -$$

$$\left[\prod (\alpha[\xi] \| m; j, \nu | \alpha_{\xi}) \prod (\alpha[\xi] \| m; j+1, i-j | \alpha_{\xi(m)}) \right]_{z=0}^{j-i} \left[\prod (\alpha[\xi] \| m; j+1, j-i | \alpha_{\xi(m)}) \right]_{z=0}^{i-j} \left[\prod (\alpha[\xi] \| m; j+1, j-i-1 | \alpha_{\xi(m)}) \right]_{z=0}^{i-j}$$

c) If $i < j$, $\tilde{D} = I(i)$ in the above, and

$$C = \Delta \tilde{C}' - \left[\prod (\alpha[\xi] \| m; i, \nu | \alpha_{\xi}) \right]$$

$$\left\{ \frac{\prod (\alpha[\xi] \| m; j+1, j-i | \alpha_{\xi}) \left\{ \prod_{z=0}^{j-i} \Lambda(\xi; \alpha; f \| m; j+1, j-i-1 | \alpha_{\xi(m)}) \right\}}{\prod (\alpha[\xi] \| m; j+1, j-i | \alpha_{\xi(m)})} \right.$$

$$\left. + \Lambda(\xi; \alpha; f \| m; j+1, j-i-1 | \alpha_{\xi}) \right\}_{z=0}^{j-i}$$

ii) Let c satisfy the relationship $Cc = 0_{(i)}$. The results of Th. [1] hold as stated. $\downarrow := A(\xi; \alpha; f \| m; i, j) \in [K | i-1, i]$ and $B := B(\xi; \alpha; f \| m; i, j)$ interpretation and other

iii) Define $A \in [K | i-1, i]$, $B \in [K | i-1, i]$ by setting $C \in [K | i-1, i-1]$

$$B := \left\{ \prod_{z=0}^{j-i} \left[\mu(\alpha[\xi] \| m; j+z+1, j+i) \right] \right\}_{z=0}^{j-i}$$

$$A := B \tilde{C} \left[\frac{\prod (\alpha[\xi] \| m; i, j, i-j-z-1 | \alpha_{m(z)})}{\prod (\alpha[\xi] \| m; i+1, j+z-\frac{1}{2}+1 | \alpha_{m(z)})} \right]$$

$$\left\{ \Lambda(\xi; \alpha; f \| m; j+z+1 | \alpha_{m(z)}) - \Lambda(\xi; \alpha; f \| m; i+1, j+z-i | \alpha_{m(z)}) \right\}_{z=0}^{j-i}$$

- a) The coefficient spaces generated by the homogeneous constraint systems C and A are identical: $c' \in \text{col}[K | i]$ satisfies the relationship $Cc' = 0_{(i)}$ if and only if $Ac' = 0_{(i)}$. In particular the c of clause (ii) satisfies the relationship $Ac = 0_{(i)}$.
- b) Let c satisfy either of the relationships $Cc = 0_{(i)}$ or $Ac = 0_{(i)}$.

$$R(p, q \| c) = \frac{|q \| A|}{|p \| A|} \quad \langle NS\{p \| A\} \rangle = NS\{p \| c\} \rangle$$

2] Define $p', q': K \rightarrow [K | i]$ as in Th. [2], set $C' := C(p', q' \| \xi; \alpha; f \| p, n, i)$

i) Let $c' \in K^i$ satisfy the relationship $C'c' = 0_{[i]}$. The interpretation and other results of Th. [2] holds as stated.

ii) Define $A' := A(\xi; \alpha; f \| \kappa, \| m, i, i)$ by setting

$$A' = \left[\delta(\xi; \alpha; f \| m, i, i, j+z-2+1) \right]_{z=0}^{D=\kappa}$$

a) $A' = BC'$

b) The coefficient spaces generated by the homogeneous constraint systems C' and A' are identical: $c'' \in \text{col } [K | i]$ satisfies the relationship $C'c'' = 0_{[i]}$ if and only if $A'c'' = 0_{[i]}$.

c) Let c' satisfy either of the relationships $C'c' = 0_{[i]}$ or $A'c' = 0_{[i]}$

$$R(p', q' \| c') = \frac{|q' \| A'|}{|p' \| A'|} \quad \langle NS\{p' \| A'\} \rangle = NS\{p' \| c'\} \rangle$$

3] Define $D \in [K | i, i]$ by setting

$$D = \left[\eta(\alpha \| \xi) \| m+z+1, i-\kappa | \alpha \right]_{z=0}^{D=\Theta}$$

i) $C = C'D$ and $A = A'D$

ii) c satisfies the relationship $Cc = 0_{[i]}$ if and only if $c' = Dc$ satisfies the relationship $C'c' = 0_{[i]}$. The same result with C, C' replaced by A, A' holds