

The classical calculus of finite differences deals with the behaviour of a function  $f$  of a complex variable  $z$  with respect to a system of distinct points  $\gamma_0, \gamma_1, \dots$  in the finite part of the  $z$ -plane. The algebraic part of the theory deals with the construction of interpolatory polynomials and rational functions of  $z$  whose values agree with those of  $f$  when  $z = \gamma_0, \gamma_1, \dots$  for a finite number of members of  $\Gamma$ . bounds are obtained for the error term in the analytic part ~~examining the behaviour of~~ expressing the difference between the value of  $f$  and that of an interpolatory function as  $z$  varies over a set and the behaviour of the error term as the number of members of  $\Gamma$  taken into account increases indefinitely also is examined.

Both the interpolatory functions and their associated error terms are expressible by formulae (truncated Newton series involving divided differences, Lagrange polynomials, truncated Thiele's continued fractions, and so on) and their associated error terms are expressible by formulae in which the members of  $\Gamma$  only occur in the form differences  $\gamma_i - \gamma_j$  the argument  $z$  only occurs as the argument of  $f$  and otherwise in the differences of the form  $z - \gamma_k$ .

The classical theory may be extended in a very simple way. Functions  $\phi_0(z), \phi_1, \dots$  having the properties that  $\phi_i(z) + \phi_j(z)$  for all  $z \in M$  and unequal  $i, j$  and that  $\phi_k(z_k) = 0$  for all  $k$  are introduced. Differences  $\gamma_i - \gamma_j$  and  $z - \gamma_k$  in the classical theory are replaced by  $\phi_j(z) - \phi_i(z)$  and  $\phi_k(z)$  respectively.

The classical theory is recovered from its extension by taking  $\phi_k(z)$  to be  $z - \gamma_k$  for all  $k$ . The direct algebraic interpolatory properties of the interpolatory functions are preserved in the extension of the theory. Thus an extended interpolatory polynomial possesses the interpolatory properties of its original form; the same is true of interpolatory rational functions which in their original form cannot be reduced by the cancellation of common numerator and denominator factors which ~~are zero~~ vanish at one or more of the points  $\gamma_0, \gamma_1, \dots$

concerned. Naturally the associated error term<sup>62</sup> changes form under extension and <sup>their</sup> behaviour must be made the subject of special investigation.

Many methods of computational mathematics derive from the classical calculus of finite differences and suffer in consequence of the restrictions imposed in this theory. For example, many methods for the numerical solution of differential equations are based upon the local or approximation of the solution by means of polynomials. When the exact solution has decreasing exponential form over the nonnegative real axis, problems of numerical instability are encountered; they derive from the discrepant behaviour of the approximating forms and the solution. By extension of the calculus of finite differences, as suggested above, such difficulties may be alleviated.

It should perhaps also be remarked that

Interpolatory polynomials lose their simple form under extension, becoming quotients of functions; rational functions are similarly affected.

Consider basing infinite difference theory on  $\Phi$ , not  $P$ . Also  $\gamma(0), \gamma(1), \dots$  as sequence  
 $\phi_j(\gamma(0, j)) = 0 (j=0, 1, \dots)$

The base of a subsequence mapping

Let  $\Xi, \Phi \in \text{seq } K[B]$  such that  $\Xi \leq \Phi$  (composition), and set  $\bar{\Phi} := \Phi[\Xi]$ .  $\|\bar{\Phi}\|_{\text{seq } K[B]}$  is  $\Xi$ .

In particular  $\|\Gamma[i, j]\|_{\text{seq } K[B]}$  is  $[i, j]$  and  $\|\bar{\Gamma}[h]\|_{\text{seq } K[B]}$  is  $[\bar{h}, \|\Gamma\|]$ .  
 $\|\Gamma[z]\| = [\Gamma[z]]$

The base size of a sequence mapping

Let  $\Xi \in \text{seq } K[B]$ , let  $n \in \text{fn}\{\bar{\Gamma}[B]\}$  and  $\Xi \in \text{seq } K[B \setminus \{n\}]$   
 $|\Xi| \in \bar{\Gamma}[B]$  is  $n$   $|\Xi| \in \text{fn}\{\bar{\Gamma}[B]\}$  is  $n$ .

$[\Xi]$  is written as  $[\Xi]$

In particular  $[\Gamma[z]]$  is  $z$ , and in this case  $\text{fn}\{[\Gamma[z]]\} = [\Gamma[z]]$

The conjugation of integer mappings in the presence of a sequence mapping

$h \in \text{fn}\{\bar{\Gamma}[B]\}$  and  $\Gamma \in \text{seq } K[B]$  being presented,  $\bar{h} \in \text{fn}\{\tilde{\Gamma}[B]\}$  is defined by setting  $\bar{h} := |\Gamma| - h$

(R. With  $i, j \in \text{fn}\{\bar{\Gamma}[B]\}$ ,  $[i, j]$  is  $\bar{\Phi} := \phi : B \rightarrow \text{seq } \bar{\Gamma}$  where for each  $z \in B$ ,  $\phi(z)$  is  $\langle i(z), \dots, j(z) \rangle$ .)

Addition of sequences

Let  $\Xi, \Phi \in \text{seq } K[B]$   $\Xi : B \rightarrow \text{seq } K$   $\Xi, \Phi \in \text{seq } K[B]$  with  $\Xi \leq \Phi$ ,  $\Xi := \text{fn}(\Xi)$ ,  $\phi := \text{fn}(\Phi)$ .  $\Xi + \Phi$  is  $\bar{\Phi} \in \text{seq } K[B]$  where, with  $\phi := \text{fn}(\Phi)$ ,  $\phi(z)$  is obtain for each  $z \in B$ ,  $\phi(z)$  is obtained by setting  $\phi(z)$  after  $\Xi(z)$ .

Type extensions of special sequence mappings

Let  $n \in \text{fn}\{\bar{N}[B]\}$  and  $\equiv \in \text{seq}\bar{N}[B, n]$

i) Extensions to sequence mappings

Let  $h, k \in \text{fn}\{\bar{N}[B]\}$

- a) Let  $n \in \text{fn}\{\bar{N}[B]\}$  and  $\equiv \in \text{seq}\bar{N}[B, n]$ .  $[h, k] \langle \equiv \rangle$  is the sequence mapping  $\phi: B \rightarrow \text{seq}\bar{N}$  for which, for each  $z \in B$ ,  $\phi(z)$  is the sequence  $[h(z) + \frac{1}{2}(z|z), k(z) + \frac{1}{2}(z|z)] \langle z := [n(z)] \rangle$ .
- b)  $[h, k-i] \langle \equiv \rangle$  is written as  $[h, k] \langle \equiv \rangle$  and the abbreviations  $(h, k) \langle \equiv \rangle$ ,  $(h, k) \langle \equiv \rangle$  are to be interpreted similarly.  $[0, k] \langle \equiv \rangle$  is written as  $[k] \langle \equiv \rangle$ , and the notations  $[k] \langle \equiv \rangle$ ,  $(k) \langle \equiv \rangle$  have analogous meanings.

c) In the special case in which  $\equiv$  is  $[i, j]$ , where  $i, j \in \text{fn}\{\bar{N}[B]\}$ ,  $[h, k] \langle \equiv \rangle$  is written as  $[h, k] \langle i, j \rangle$ .  $[h, k] \langle i, j-i \rangle$  is written as  $[h, k] \langle i, j \rangle$ ,  $[h, k] \langle 0, j \rangle$  as  $[h, k] \langle j \rangle$  and so on; these abbreviations are combined with those described above.

ii) Extensions to double sequence mappings

Let  $i, j \in \text{fn}\{\bar{N}[B]\}$

- a)  $\langle \equiv, j \rangle$  is the double sequence mapping  $\Theta: B \rightarrow \text{seq}^2 \bar{N}[B]$  for which, for each  $z \in B$ ,  $\Theta(z)$  is the double sequence for which  $\Theta(z|z)$  is  $[\frac{1}{2}(z|z), j(z)] \langle z := [n(z)] \rangle$ .  $\langle j, \equiv \rangle$  is  $\Theta': B \rightarrow \text{seq}^2 \bar{N}[B]$  where now  $\Theta'(z|z)$  is  $[\frac{1}{2}(z|z), \frac{1}{2}(z|z)] \langle z := [n(z)] \rangle$ .
- b)  $\langle \equiv, j-i \rangle$  and  $\langle j+i, \equiv \rangle$  are written as  $\langle \equiv, j \rangle$  and  $\langle j, \equiv \rangle$  respectively.  $\langle 0, \equiv \rangle$  is written as  $\langle \equiv \rangle$  and  $\langle 1, \equiv \rangle$  as  $\langle \equiv \rangle$ .

iii) Type extension of functions of special sequence mappings  $\{B \rightarrow T\}$

a) A mapping of the form  $\psi: \overline{\mathbb{N}}[B] \times \overline{\mathbb{N}}[B] \times \dots \rightarrow \overline{T}$ ,  
 whose function values are expressed as  $\psi(\dots [i, j] \dots | z), [i(z), j(z)]$ ,  
 $T$  being a suitable target set, is extended to the form

$\psi: \dots \times \overline{\mathbb{N}}[B] \times \overline{\mathbb{N}}[B] \times \text{seq } \overline{\mathbb{N}}[B] \times \dots \rightarrow \{B \rightarrow \text{seq } T\}$ . Denoting  
 the function in the former mapping by  $\psi'$ ,  $\psi(\dots [i, j] \dots | z)$   
 is, for each  $z \in B$ , the sequence  $\psi'(\dots [h(z) + \frac{1}{3}(z|z), k(z) + \frac{2}{3}(z|z)] \dots | z)$   
 in  $\langle z := [n(z)] \rangle$   
 whose function values are expressed  
 which produces a function expressed as  $\psi(\dots [i, j] \dots) : B \rightarrow T$   
 which has values  $\psi(\dots [i(z), j(z)] \dots | z)$  for each  $z \in B$ ,  $T$  being  
 a suitable target set

b) Let  $\bar{\Psi}$  be a mapping of the form  $\psi: \dots \times \overline{\mathbb{N}}[B] \times \overline{\mathbb{N}}[B] \times \dots \rightarrow \{B \rightarrow T\}$ ,  
 $T$  being a suitable target set, whose function values are  
 appropriately expressed in the form  $\psi(\dots [i, j] \dots)$   
 in the sense that this member of  $\{B \rightarrow T\}$  depends upon  
 the function is specified in terms of the special sequence  
 mapping  $[i, j]$ , and depends upon  $i$  and  $j$  in no other way.

a)  $\bar{\Psi}$  is extended to the form  $\psi: \dots \times \overline{\mathbb{N}}[B] \times \overline{\mathbb{N}}[B] \times \text{seq } \overline{\mathbb{N}}[B] \times \dots \rightarrow \{B \rightarrow \text{seq } T\}$  as follows: denote the function in the  
 former mapping by  $\psi'$ ; for each  $z \in B$ ,  
 the value of  $\psi(\dots [i, j] \dots | z)$  of  $\psi(\dots [k, l] \dots | z)$   
 is, for such  $z \in \psi(\dots [i, j] \dots | z)$  of  $\psi(\dots [k, l] \dots | z)$   
 is the sequence  $\psi'(\dots [h(z) + \frac{1}{3}(z|z), k(z) + \frac{2}{3}(z|z)] \dots | z)$   
 $\langle z := [n(z)] \rangle$

b)  $\bar{\psi}$  is also extended, this time to the form  $\psi: \dots \times \text{seq}\bar{R}[B] \times \bar{R}[B] \times \dots \rightarrow \{B \rightarrow \text{seq}T\}$ , by stipulating that, the value

$\psi(\dots, \underline{\Delta^{\equiv}}, j, \dots | z)$  &  $\psi(\dots, \underline{\Delta^{\equiv}}, j, \dots)$  should be the sequence  $\psi(\dots [z(z|r), j(z)] \dots | z) \langle z := [n(z)] \rangle$  where  $\psi'$  is as before.  $\bar{\psi}$  is further extended to the form  $\psi: \dots \times \bar{R}[B] \times \text{seq}\bar{R}[B] \times \dots \rightarrow \{B \rightarrow \text{seq}T\}$  in the same way, now with  $\psi(\dots [i, \underline{\Delta^{\equiv}}])$  <sup>in terms of</sup> ~~by making use of~~ values of  $\psi(\dots \langle i, \underline{\Delta^{\equiv}} \dots)$ .

c)  $\psi(\dots [i, j-1] \langle \equiv \rangle \dots)$  in (a) is written as  $\psi(\dots [i, j] \langle \equiv \rangle \dots)$ ,  $\psi(\dots \underline{\Delta^{\equiv}}, j \dots)$  in (b) as  $\psi(\dots, \underline{\Delta^{\equiv}}, j, \dots)$ . Expressions such as  $\psi(\dots \langle i, j \rangle \langle \equiv \rangle \dots)$ ,  $\psi(\dots (i, \underline{\Delta^{\equiv}}) \dots)$ , ... are to be interpreted in the same way.

d) Mappings  $\bar{\psi}$  of the form  $\psi: \dots \times \bar{R}[B] \times \dots \rightarrow \{B \rightarrow T\}$  whose function values are appropriately expressed in the form  $\psi(\dots [i] \dots)$  in the sense that this member of  $\{B \rightarrow T\}$  is specified in terms of the sequence special sequence mapping  $[i, j]$  are extended as in (a) to the form  $\psi: \dots \times \bar{R}[B] \times \text{seq}\bar{R}[B] \times \dots \rightarrow \{B \rightarrow \text{seq}T\}$  in terms of values of  $\psi(\dots [j] \langle \equiv \rangle \dots)$ . Abbreviations such as  $\psi(\dots [j] \langle \equiv \rangle \dots)$  and  $\psi(\dots \langle j \rangle \langle \equiv \rangle \dots)$  are to be analogous to those described above.

<sup>Further</sup> cases in which mappings of the form above form have been extended by a convention which permits <sup>special</sup> sequence mappings to be specified by reference to an end value a single integer mapping function are treated similarly. Thus, if  $\psi(\dots, \bar{P}[j], \dots)$  is to be interpreted as  $\psi(\dots, P[|P|-j], \dots)$

$\psi(\dots \bar{P}[j] \Leftrightarrow \dots)$  is to be interpreted as  $\psi(\dots \Gamma[|\Gamma|-j, |\Gamma|] \Leftrightarrow \dots)$ , abbreviations such as  $\psi(\dots \bar{P}(j] \Leftrightarrow \dots)$  being construed in a similar sense.

Mappings  $\bar{\Psi}$  of the above form are also extended as in (b) to the form  $\psi: \dots \times \text{seq} \bar{N}[B] \times \dots \rightarrow \{B \rightarrow T\}$  in terms of values of  $\psi(\dots \Leftrightarrow \dots)$ : for each  $z \in B$ ,  $\psi(\dots \Leftrightarrow \dots | z)$  is the sequence  $\psi'(\dots [\frac{1}{3}(z|z)] \dots) \langle z := [n(z)] \rangle$ ,  $\psi'$  having replaced  $\psi$  in the specification of  $\bar{\Psi}$ . The abbreviation  $\psi(\dots \Leftarrow \dots)$  is to be interpreted in terms of values of  $\psi'(\dots [\frac{1}{3}(z|z)] \dots) \langle z := [n(z)] \rangle$ . Further cases in which mappings  $\bar{\Psi}$  have been extended by convention are treated in the same way. Thus, referring to the example used in the preceding paragraph  $\psi(\dots \bar{P} \Leftrightarrow \dots)$  is to be interpreted in terms of the sequence  $\psi'(\dots, \Gamma[|\Gamma| - \frac{1}{3}(z|z), |\Gamma|] \dots | z) \langle z := [n(z)] \rangle$ ,  $\psi(\dots \bar{P} (\Leftarrow \dots))$  in terms of the sequence  $\psi'(\dots, \Gamma(1|\Gamma| - \frac{1}{3}(z|z), |\Gamma|] \# \dots | z) \langle z := [n(z)] \rangle$  and  $\psi(\dots \bar{P} \Leftarrow \dots)$  by use of the expression  $\Gamma[1|\Gamma| - \frac{1}{3}(z|z), |\Gamma|]$ .

iv) Type extension of sequence mappings to double sequence mappings

v) Type extension of mappings  $\bar{\Psi}$  of the form  $\psi: \dots \times \bar{N}[B] \times \bar{N}[B] \times \dots \rightarrow \{B \rightarrow \text{seq} T\}$  whose function values are now in  $\{B \rightarrow \text{seq} T\}$  but otherwise as described at the beginning of the

preceding clause is effected analogously. With regard to the extension considered in subclause (iii) the analogue is as follows: for each  $z \in B$ , the value  $\psi(\dots [h(z), k(z)] \leqslant \dots | z)$  of  $\psi(\dots [h, k] \leqslant \dots)$  is now the sequence  $\psi'(\dots [h(z) + \tilde{g}(z|z)], k(z) + \tilde{g}(z|z)] \dots | z) \langle z := [n(z)] \rangle$  of sequences  $\psi'(\dots [h(z) + \tilde{g}(z|z), k(z) + \tilde{g}(z|z)] \dots | z | \nu) \langle \nu := [m(z)] \rangle$  where ~~with~~  $m := |\Psi|$ . The relevant conventions and extensions are as before.

v) Type extension of composition sequence mappings to double sequence mappings

The extension of composition sequence mappings is treated as a special case of sequence mapping extension. Let  $\Phi$  be the mapping  $\phi: B \rightarrow \text{seq } K$ ,  $i \in \text{fn}\{\bar{N}[B]\}$  ~~such that~~  $i \neq \text{fn}\{\bar{N}[B]\}$  and  $\Xi$  be ~~such that~~ the sequence mapping  $\tilde{g}: B \rightarrow \text{seq } \bar{N}$  with  $j + \Xi \leq \Phi$  (composition); set  $n := |\Xi|$ .

a) ~~Let  $j + \Xi \leq \Phi$  (composition). Let  $j \in \text{fn}\{\bar{N}[B]\}$  and  $j + \Xi \leq \Phi$  (composition).~~

$\Phi[i, j] \leq \Phi$  (composition)  $\Rightarrow \Phi[i, j] \leq \Phi$  (composition) is the double sequence mapping  $\phi': B \rightarrow \text{seq}^2 K$  for which, for each  $z \in B$ ,  $\phi'(z)$  is the sequence  $\phi[i(z) + \tilde{g}(z|z), j(z) + \tilde{g}(z|z)] \langle z := [n(z)] \rangle$  of sequences  $\phi(z | i(z) + \tilde{g}(z|z) + \nu)$ ,  $\langle \nu := [j(z) - i(z)] \rangle$ .

b)  $\Phi[i, \Xi] \leq \Phi$  (composition).

$\Phi[i, \Xi] \leq \Phi$  (composition) is the double sequence mapping  $\phi'': B \rightarrow \text{seq}^2 K$  for which, for each  $z \in B$ ,  $\phi''(z)$  is the sequence  $\phi[i(z), \tilde{g}(z|z)]$

$\langle z := [n(z)] \rangle$  of sequences  $\phi(z | i(z) \mapsto v) \langle v := [i(z), \xi(z|z)] \rangle$ .

c) Subject to the condition  $i \leq |\Phi| \langle B \rangle$ , (but with that ~~upper~~  
~~lower~~  $\Phi[=, i]$  is defined in a similar way.

The allocation of matrix mappings.  $\text{def } \{ij\}^k$

Let  $i, j, h, k \in \mathbb{N} \setminus \{0\}$  and  $A \in K[B]$  with  $a \in \{ij\}^k$ .

a) With  $B$

$\{a \in \{K[B]\}_{[i]}^{[j]}\}$

a) With  $\text{befn}\{K[B]\}_{[i]}^{[j]}$ ,  $a \in \text{befn}$   
specification

The allocation of matrix mappings of fixed type

Let  $h, i, j, k \in \mathbb{N}$  and  $h \in [k], i \in [j]$ .

a) The mappings  $a(z, v) : B \rightarrow K \langle z := [h, k], v := [i, j] \rangle$  being prescribed, the allocation

$A := [a(z, v) \langle z, v := M\{[h, k], [i, j]\} \rangle]$

indicates that defines the mapping  $A : B \rightarrow K_{[k-h]}^{[j-i]}$  for which, for each  $z \in B$ , the element in the  $(z+i)^{\text{th}}$  row and  $(j+i)^{\text{th}}$  column of  $A(z)$  is  $a(h+z, i+v | z) \langle z := [h, k], v := [i, j] \rangle$

b) The mappings  $a'(z, v) : B \rightarrow K \langle z := [h, k], v := [i, \min(z-h, j)] \rangle$  being prescribed, the allocation

$A' := [a'(z, v) \langle z, v := L\{[h, k], [i, j]\} \rangle]$

defines the mapping  $A' : B \rightarrow K_{[k-h]}^{[j-i]}$  for which, for each  $z \in B$ , the element in the  $(z+i)^{\text{th}}$  row and  $(j+i)^{\text{th}}$  column of  $A'(z)$  is  $a'(h+z, i+v | z) \langle z := [h, k], v := [\min(z-h, j-i), j-i] \rangle$  and zero for  $z := [h, k], v := [\min(z-h, j-i), j-i]$ .

c) The mappings  $a''(z, v) : B \rightarrow K \langle v := [i, j], z := [h, \min(j-i, k)] \rangle$  being prescribed, the allocation

$A'' := [a''(z, v) \langle z, v := U\{[h, k], [i, j]\} \rangle]$

defines the mapping  $A'': B \rightarrow K^{[j-i]}_{[k-h]}$  for which, for each  $z \in B$ , the element in the  $(z+i)^{th}$  row and  $(j-h)^{th}$  column of  $A''(z)$  is  
 $a''(h+z, i \mapsto | z) \quad (j := [j-i], z := [\min(j, k-h)] \rightarrow \text{and zero for } j := (j-i), z := (\min(j, k-h), k-h]).$

### Compound matrix mappings

Let  $h, i, j, k \in \text{fn}\{\bar{N}[B]\}$  and  $a \in \text{fn}\{K[B]^{[j]}_{[i]}\}$ .

a) With  $a+b \in \text{fn}\{K[B]^{[k]}_{[i]}\}$ ,  $a+b$  denotes the function  $e$  occurring in the mapping  $e: B \rightarrow K[B]^{[j+h+i]}_{[i]}$  for which, for each  $z \in B$ ,  $e(z)$  is the matrix  $\alpha(z)$  whose first  $j(z)+1$  columns are those of  $a(z)$  and whose next  $k(z)+1$  columns are those of  $b(z)$ , ordering being preserved in both cases.

b) With  $c \in \text{fn}\{K[B]^{[j]}_{[h]}\}$ ,  $a+c$  denotes the function  $f$  occurring in the mapping  $f: B \rightarrow K[B]^{[i]}_{[i+h+i]}$  defined by columnwise matrix adjunction analogous to the above.

c) With  $d \in \text{fn}\{K[B]^{[k]}_{[h]}\}$ ,  $a+d$  denotes the function  $g$  occurring in the mapping  $g: B \rightarrow K^{[j+h+i]}_{[i+h+i]}$  for which, for each  $z \in B$ ,  $g(z)$  is the matrix whose first  $i(z)+1$  rows are formed by columnwise adjunction of  $a(z)$  and  $O^{[k(z)]}_{[i(z)]}$  as in (a) and whose next  $h(z)+1$  columns are formed by columnwise adjunction of  $O^{[j(z)]}_{[i(z)]}$   $O^{[j(z)]}_{[h(z)]}$  and  $d(z)$  also as in (a).

d) In (a), with  $k \in \mathbb{N}$  and  $h \in \mathbb{N}$ ,  $M\{[h, k-i], [i, j]\}$  is written as  $M\{[h, k], [i, j]\}$ . The further units are treated similarly.

$M\{[0,k], [i,j]\}$  is written as  $M\{[k], [i,j]\}$ ; ~~for the case~~  
in which  $i=0$  is treated similarly.

When  $h=i, k=j$ ,  $M\{[h,k], [i,j]\}$  is written simply  
as  $M[h,k]$ .

The above conventions are applied conjointly.

The symbols  $\mathcal{L}\{[h,k], [i,j]\}$  and  $\mathcal{R}\{[h,k], [i,j]\}$  are  
treated in the same way.

The extension of sequence mappings to diagonal matrix mappings  
and column and row <sup>vector</sup> matrix mappings

$\exists \mapsto \text{Let } n \in \text{fn}\{\mathcal{R}[\emptyset]\} \text{ and } \Xi \text{ be the mapping } \xi : \mathbb{B} \rightarrow \text{seq } K[n].$

a)  $\Xi$  is tacitly extended to the matrix mapping form

$\xi : \mathbb{B} \rightarrow [K, n]$  by taking, for each  $z \in \mathbb{B}$ ,  $\xi(z)$  to be the  
diagonal matrix whose successive diagonal elements are  
 $\xi(x|z) \langle x := [n(z)] \rangle$  in order and whose remaining  
elements are zero.

b) ~~col~~  $[\xi]$  is the mapping  $c\xi : \mathbb{B} \rightarrow$  vector

b)  $\Xi$  is extended to column matrix mapping form by taking  
by taking  $\text{col}[\Xi]$  to be the mapping  $c\xi : \mathbb{B} \rightarrow K[n]$  for  
which, for each  $z \in \mathbb{B}$ , the successive elements of  $c\xi(z)$  are  
 $\xi(x|z) \langle x := [n(z)] \rangle$ .

c)  $\Xi$  is extended to row vector matrix mapping form by taking  
 $\text{row}[\Xi]$  to be the mapping  $r\xi : \mathbb{B} \rightarrow K^{[n]}$  defined in a similar way.

Bidiagonal matrix mappings derived from sequence mappings.

Let  $\text{nefn}\{\mathbb{R}[B]\}$  and  $\equiv$  be the mapping  $\S: B \rightarrow \text{seq } K[n]$ .

i) Unit lower and upper diagonal matrix mappings.

is the function  $\phi$  occurring in

a)  $\text{ULD}[\equiv] \in \{\mathbb{R}[B], n+1\}$  is the matrix mapping

$\phi: B \rightarrow [K, n+1]$  defined by taking, for each  $z \in B$ ,  $\phi(z)$

to be the matrix for which  $\phi(z)_0^0 = \phi(z)_{n+1}^{n+1} = 1$   $\langle z := n(z) \rangle$  and

$\phi(z)_{x+1}^x = \frac{1}{2}(z|x) \quad \langle x := [n(z)] \rangle$ , the remaining elements being

zero

function  $\psi$  occurring in the

b)  $\text{ULD}[\equiv]$  is the matrix mapping  $\psi: B \rightarrow [K, n+1]$  defined

by taking, for each  $z \in B$ ,  $\psi(z)$  to be the matrix for

which  $\psi(z)_0^{n+1} \quad \psi(z)_{n+1}^{n+1} = \psi(z)_{x+1}^x = 1 \quad \langle x := [n(z)] \rangle$  and

$\psi(z)_x^0 = \frac{1}{2}(z|x) \quad \langle x := [n(z)] \rangle$ , the remaining elements being zero.

ii) Annihilative matrix mappings

a)  $\text{ann}[\equiv]$  is the function  $\Theta$  occurring in the matrix mapping

$\Theta: B \rightarrow [K, n+1]$  defined by taking, for each  $z \in B$ ,  $\Theta(z)$  to

be the matrix for which  $\Theta(z)_0^0 = 1$ ,  $\Theta(z)_{x+1}^x = -\Theta(z)_x^x =$

$\frac{1}{2}(z|x) \quad \langle x := [n(z)] \rangle$ , the remaining elements being zero.

b)  $\overline{\text{ann}}[\equiv]$  is the function  $\lambda$  occurring in the matrix mapping

$\lambda: B \rightarrow [K, n+1]$  defined by taking, for each  $z \in B$ ,  $\lambda(z)$  to

be the matrix for which  $\lambda(z)_{n(z)+1}^{n(z)+1} = 1$ ,  $\lambda(z)_x^x = -\lambda(z)_x^{x+1} = \frac{1}{2}(z|x)$

$\langle x := [n(z)] \rangle$ , the remaining elements being zero

Matrix operator mappings occurring in the theory calculus of finite differences

Triangular matrix operator mappings

~~M[P]~~ Mappings constructed from divided difference multipliers

Def.

$$M[R] : [u(R[z]) \langle z, \omega := L[z] \rangle] \quad L$$

$$\bar{M}[R] \quad u(\bar{R}[\bar{z}]z) \langle z, \omega := U[R] \rangle \quad u$$

$$M^{-1}[R] \quad [u^{-1}(R[z]) \langle z, \omega := L[R] \rangle] \quad L$$

$$\bar{M}^{-1}[R] \quad [u^{-1}(\bar{R}[\bar{z}]z) \langle z, \omega := U[R] \rangle] \quad u$$

$$MM^*[R] \quad [S(R[z], z), u[R]] \langle z, \omega := L[R] \rangle \quad L$$

$$\bar{M}\bar{M}^{-1}[R] \quad [S(\bar{R}[\bar{z}], (\bar{z})) \langle z, \omega := M[R] \rangle]$$

$$M\bar{M}^{-1}[R] \quad [S(R[z], \pi(\bar{R}(\bar{z}))) \langle z, \omega := M[R] \rangle]$$

$$N[R] \quad [u(R[z], z)] \langle z, \omega := L[R] \rangle \quad uL[K]$$

$$N^{-1}[R] \quad [u(R[z], z)] \langle z, \omega := L[R] \rangle \quad uL[K]$$

$$\bar{M} \quad [z, \omega = z]$$

i) Structural properties

a)  $M, \bar{M}, M^{-1}, \bar{M}^{-1}, N, N^{-1}$  totally nonzero and therefore nonsingular

b)  $\bar{M}M^{-1}, M\bar{M}^{-1}$  have triangular form: <sup>unit</sup> elements &

$\bar{M}M^{-1}[R]$  being zero when  $|z| < |R|$ , those of  $M\bar{M}^{-1}$  being zero when  $|R| < |z|$ . The <sup>the</sup> principal backward diagonal elements (i.e. those for which  $|z| = |R|$ ) being unity

$MM^*$  is nonsingular.

i) Algebraic properties

$$M \text{ann}^{\prime}[K] \quad \bar{M} \in \overline{\text{ann}}[K]$$

ii) Relationships

b)  $\{M\}^{-1} = M^{-1} \quad \{\bar{M}\}^{-1} = \bar{M}^{-1} \quad \bar{M}M^{-1} = \bar{M} \cdot M^{-1} \quad M\bar{M}^{-1} = M \cdot \bar{M}^{-1}$

a)  $\bar{M}, \bar{M}^{-1}$  conjugates  $\Rightarrow M, M^{-1}$

c)  $M^*M = u \quad MM^* = [S(\rho, z), \mu[\rho]] \cdot \langle z, z \rangle = L[\rho] \rangle$   
 $= [M|u]$

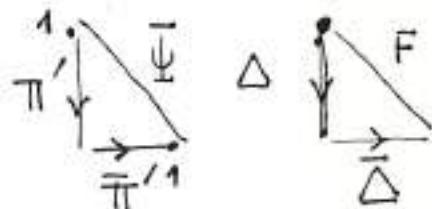
$$\{MM^*\}^{-1} = [S(\rho[\rho, z], \frac{1}{\mu[\rho]}) \langle z, z \rangle = L[\rho]]$$

$$= [M | \frac{1}{\mu}]$$

$$dM = \text{diag}[M]?$$

d)  $N.dM = M \quad \bar{N}.d\bar{M} = \bar{M}$

$$\{N\}^{-1} = \bar{N}^T \quad (\text{defining } \bar{N})$$



iv) Transformation properties

$$M[\rho] \text{col}[F] = \text{col}[S(\rho[\langle \rho \rangle], F)]$$

$$\bar{M}[\rho] \text{col}[F] = \text{col}[S(\bar{\rho}[\langle \bar{\rho} \rangle], F)] \quad \langle \bar{\rho} \rangle \text{ runs means } [\rho] \text{ backwards}$$

$$\bar{M}M^{-1}[\rho] \text{col}[S(\rho[\langle \rho \rangle], F)] = \text{col}[S(\bar{\rho}[\langle \bar{\rho} \rangle], F)]$$

$$M\bar{M}^{-1}[\rho] \text{col}[S(\bar{\rho}[\langle \bar{\rho} \rangle], F)] = \text{col}[S(\rho[\langle \rho \rangle], F)]$$

$$M \text{col}[\Psi] = \text{col}[\bar{\pi}'] \quad \bar{M} \text{col}[\bar{\Psi}] = \text{col}[\pi']$$

$$\bar{M}M^{-1} \text{col}[\bar{\pi}'] = \text{col}[\pi'] \quad M^{-1}\bar{M} \text{col}[\pi'] = \text{col}[\bar{\pi}']$$

$$\bar{M} \frac{1}{\mu} \otimes \text{col}[\Theta] = \text{col}[\pi'] \quad M \frac{1}{\mu} \text{col}[\Theta] = \text{col}[\bar{\pi}']$$

Heading of previous section: Rotation operators?  $M := R$  etc?

with  $\bar{\pi}, \bar{\pi}', \bar{\pi}''$  as forward sequences of factorial polynomials

$\bar{\pi}, \bar{\pi}', \bar{\pi}''$  - backward  $\sim$   $\sim$  " "

$\bar{\psi}, \bar{\phi}$  as central sequences of polynomials

and rotation by  $R \text{col}[\bar{\psi}] = \text{col}[\bar{\pi}'] \quad \bar{R} \text{col}[\bar{\psi}] = \text{col}[\bar{\pi}'] ?$

=

$$\left\{ \begin{array}{l} \text{Jacobi} \\ \text{Tridiagonal} \end{array} \right. \text{col} \left[ \frac{F}{\Phi U} \right] = \Lambda [U, F]$$

Polynomial operator mappings

Def.

$$\bar{\pi} M[r] := [\pi(r[z]), \mu(r[z], v) \langle z, v \rangle := \mathcal{L}[r]]$$

$$\bar{\pi}' \bar{M}[r] := [\pi(\bar{r}(\bar{z})) \mu(\bar{r}(\bar{z}), \bar{v}) \langle z, v \rangle := \mathcal{U}[r]]$$

$$M^{-1} \pi'^{-1}[r] := \left[ \frac{\mu^{-1}(r[v], z)}{\pi(r[v])} \langle z, v \rangle := \mathcal{L}[r] \right]$$

$$\bar{M}^{-1} \bar{\pi}'^{-1}[r] := \left[ \frac{\mu^{-1}(\bar{r}(\bar{z}), v)}{\pi(\bar{r}(\bar{z}))} \langle z, v \rangle := \mathcal{U}[r] \right]$$

$$\Theta[r] := [\theta(r[z], v) \langle z, v \rangle := \mathcal{L}[r]]$$

$$\bar{\Theta}[r] := [\theta(\bar{r}(\bar{z}), \bar{v}) \langle z, v \rangle := \mathcal{U}[r]]$$

$$\Theta' [r] := \left[ \frac{\mu^{-1}(r[v], z)}{\pi(r[v])} - \frac{\mu^{-1}(r[v], \bar{z})}{\pi(r[v])} \langle z, v \rangle := \mathcal{L}[r] \right]$$

$$\bar{\Theta}' [r] := \left[ \frac{\mu^{-1}(\bar{r}(\bar{z}), v)}{\pi(\bar{r}(\bar{z}))} - \frac{\mu^{-1}(\bar{r}(\bar{z}), \bar{v})}{\pi(\bar{r}(\bar{z}))} \langle z, v \rangle := \mathcal{U}[r] \right]$$

i) Structural properties Let  $x \in \langle P \rangle$

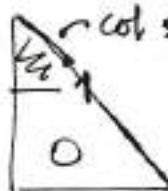
a)  $\bar{\pi}'\bar{m}[P]$  totally nonzero and therefore nonsingular over  $K \setminus [P]$ .

We  $x \in \langle P \rangle$ ,  $z = \gamma(x)$ : all elements in lower triangular part  $\triangleright$

$\bar{\pi}'\bar{m}[P|z]_{[x]}$  are nonzero,  $\bar{\pi}'\bar{m}[P|z]_{-[x]} = \circ^{[P]}_{[x]}$

column sums in  $\bar{\pi}'\bar{m}[P|z]^{-x}$  are zero and  $\bar{\pi}'\bar{m}[P|z]_x^x = 1$ :

$\bar{\pi}'\bar{m}[P|z] \in \text{det}[\mathbb{K}]$ .  $\begin{cases} \text{col sums zero} \\ \text{ann } L[K] \end{cases}$  ?



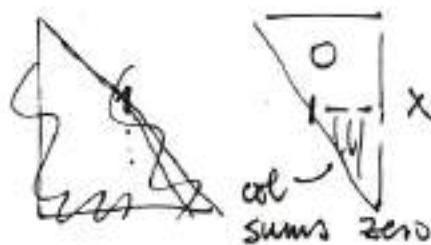
$\bar{\pi}'\bar{m}[P]$  totally nonzero and therefore nonsingular over  $K \setminus [P]$

$x \in \langle P \rangle$ ,  $z = \gamma(x)$ : all elements in upper triangular part  $\triangleright$

$\bar{\pi}'\bar{m}[P|z]_{-[x]}^{[P-x]}$  are nonzero,  $\bar{\pi}'\bar{m}[P|z]_{-[x]} = \circ^{[P-x]}_{[x]}$

column sums in  $\bar{\pi}'\bar{m}[P|z]^{-x}$  are zero and  $\bar{\pi}'\bar{m}[P|z]_x^x = 1$

$\bar{\pi}'\bar{m}[P|z]_{-[x]}^{-x} \in \text{det}[\mathbb{K}]$   $\overline{\text{ann }} U[K] ?$



$\bar{m}^{-1}\bar{\pi}'^{-1}[P]$  is totally nonzero and therefore nonsingular over  $K \setminus [P]$

$\bar{m}^{-1}\bar{\pi}'^{-1}[P] \dots K \setminus [P]$

$\bullet [P]$  is totally nonzero over  $K \setminus P$  and nonsingular over  $K \setminus [P]$ .

$x \in \langle P \rangle$ ,  $z = \gamma(x)$ : all elements in lower triangular part of

$\Theta[r|z]_{[x]}^{(x)}$  are nonzero; all elements of  $\Theta[r|z]_{-[x]}^{-x}$  are zero:

$\Theta[r|z]_{-[x]}^{-x} = O_{[x]}$ ; the remaining elements in the lower triangular part of  $\Theta[r|z]$  are unity:  $\Theta[r|z]_{-[x]}^x = I_{[x]}$

$$\begin{matrix} x & \uparrow \\ \square & | \\ 0 & ; \\ \downarrow & \square \end{matrix}$$

$\bar{\Theta}[r]$  is totally nonzero over  $K \setminus P$  and nonsingular over  $K \setminus (P)$

$x \in \langle r \rangle, z = \gamma(x)$ : all elements in upper triangular part  $\bar{\Theta}[r|z]_{[x]}^{-x}$

$\bar{\Theta}[r|z]_{-[x]}^{-x}$  are nonzero; all elements of  $\bar{\Theta}[r|z]_{[x]}^{upper}$  are zero

$\bar{\Theta}[r|z]_{[x]}^{-x} = O_{[x]}$ ; the remaining elements in the lower triangular part of  $\bar{\Theta}[r|z]$  are unity:  $\Theta[r|z]_{[x]}^x = I_{[x]}$

$$\begin{matrix} f \rightarrow & \uparrow & \square & 1 & \square \\ & \nwarrow & | & | & \searrow \\ & & 0 & 1 & 0 \\ & & ; & & \square \end{matrix}$$

$\Theta^{-1}[r]$  is nonsingular over  $K \setminus [r]$  and  $\bar{\Theta}^{-1}[r]$  nonsingular over  $K \setminus (P)$

ii) Algebraic properties

$$\pi' M \in \text{ann}'[K] \quad \bar{\pi}' \bar{M} \in \overline{\text{ann}}[K]$$

$$\Theta, \bar{\Theta} \in \text{perm}[K] \quad \Theta^{-1} \in \text{perm}[K \setminus [r]]$$

$$\bar{\Theta}^{-1} \in \text{perm}[K \setminus (P)]$$

$$\begin{array}{ccc}
 & z = \gamma_0 & z = \gamma_1 \\
 1 & & 1 \\
 \begin{matrix} z - \gamma_1 & z - \gamma_0 \\ \hline \gamma_0 - \gamma_1 & \gamma_1 - \gamma_0 \end{matrix} & \begin{matrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{matrix} & \\
 \begin{matrix} \cancel{(z - \gamma_1)(z - \gamma_2)} & \cancel{(z - \gamma_0)(z - \gamma_2)} & \cancel{(z - \gamma_0)(z - \gamma_1)} \\ (\gamma_0 - \gamma_1)(\gamma_1 - \gamma_2) & (\gamma_1 - \gamma_0)(\gamma_1 - \gamma_2) & (\gamma_1 - \gamma_0)(\gamma_0 - \gamma_1) \end{matrix} & z = \gamma_2 & \begin{matrix} 1 \\ \gamma_2 - \gamma_1 \\ \hline \gamma_0 - \gamma_1 & \gamma_1 - \gamma_0 \end{matrix} \\
 \begin{matrix} (z - \gamma_1)(z - \gamma_2) & (z - \gamma_0)(z - \gamma_2) & (z - \gamma_0)(z - \gamma_1) \\ (\gamma_0 - \gamma_1)(\gamma_0 - \gamma_2) & (\gamma_1 - \gamma_0)(\gamma_1 - \gamma_2) & (\gamma_2 - \gamma_0)(\gamma_2 - \gamma_1) \end{matrix} & \begin{matrix} 0 & 0 & 1 \\ z = \gamma_0 & & z = \gamma_1 \end{matrix} & \\
 \begin{matrix} z - \gamma_2 & z - \gamma_1 \\ \hline \gamma_1 - \gamma_2 & \gamma_2 - \gamma_1 \end{matrix} & \begin{matrix} 1 & 0 & 0 & 0 & 1 \\ \gamma_0 - \gamma_2 & \gamma_0 - \gamma_1 & & 1 & 0 \\ \hline \gamma_1 - \gamma_2 & \gamma_2 - \gamma_1 & & & 1 \end{matrix} & \\
 \begin{matrix} \cancel{\gamma} & 1 & z = \gamma_2 & 0 & 0 & 1 \\ & & & & & 0 & 1 \end{matrix} & & \\
 \hline
 \end{array}$$

### iii) Relationships

a) Multiplicative relationships

$$\pi' M = \pi' \cdot M \quad \bar{\pi}' \bar{M} = \bar{\pi}' \cdot \bar{M}$$

$$\Theta = L[\Gamma] \cdot \pi' M \quad \bar{\Theta} = U[\Gamma] \bar{\pi}' \bar{M} \quad (M\pi')^* (M\pi') = \Theta$$

### b) Conjugation

conjugate of  $\pi' M$  is  $\bar{\pi}' \bar{M}$ , of  $M^{-1} \pi'$  is  $\bar{M}^{-1} \bar{\pi}'^{-1}$ , so  $\Theta$

$\Theta$  is  $\bar{\Theta}$ , or  $\bar{\Theta}^{-1}$  is  $\Theta^{-1}$

### c) Inversion

$$\{\pi' M\}^{-1} = M^{-1} \pi' \quad \{\bar{\pi}' \bar{M}\} = \bar{M}^{-1} \bar{\pi}'^{-1} \quad \{\Theta\}^{-1} = \Theta^{-1} \quad \{\bar{\Theta}\}^{-1} = \bar{\Theta}^{-1}$$

### d) Fixed vectors [invariance]

$\text{col}[\underline{\Phi}]$  is fixed with respect to  $\pi' M$  in the sense that

$$\pi' M \cdot \text{col}[\underline{\Phi}] \neq \text{col}[\underline{\Phi}], \quad \bar{\pi}' \bar{M} \cdot \text{col}[\underline{\Phi}] = \text{col}[\underline{\Phi}]$$

e) Similarity product relationships

683 ex seq. ??

$$\pi' m = \{\bar{\pi}\}^{-1} [m, \bar{\pi}']$$

$$(\pi' m)^* = \bar{\Phi} [\pi' | \bar{\pi}' m^*]$$

$$\pi' m (\pi' m)^* = [L[I] | \pi' m (\pi' m)^* \{ \bar{\Phi} \}^{-1}]$$

~~$$(\pi' m)^* (m \pi') = \Theta$$~~

$$\Theta(\pi' m)^* = \bar{\Phi} [\pi' | [m | \Theta]]$$

$$= \bar{\Phi} [\pi' | \{ (m \pi') (m \pi')^* \}] ?$$

~~$$= \bar{\Phi} [\pi' | \{ \pi' m | \bar{\pi}' m^* \}]$$~~

$$= [L[I] | \bar{\pi}' m \Theta^*]$$

$$\pi' m \Theta^* = [\pi' | [\bar{\Phi} [m | \Theta]]]$$

$$= [\pi' | [\bar{\Phi}, m \bar{\pi}' \bar{\pi}' m^*]]$$

$$= [\bar{\Phi} | \pi' m] (\pi' m)^* \{ \bar{\Phi} \}^{-1}$$

$$= [L[I]^{-1} | \Theta(\pi' m)^*]$$

wf) Transformation properties

$$\pi' m[r] \text{col}[F] = \text{col}[\delta(r[\langle r \rangle], F)]$$

$$\text{col}[\delta(r[z], F) \pi(r[z]) \langle z := \langle r \rangle \rangle]$$

$$\bar{\pi}' \bar{m}[r] \text{col}[F] = \text{col}[\delta(\bar{r}[\bar{z}], F) \pi(\bar{r}(\bar{z})) \langle z := \langle r \rangle \rangle]$$

$$\Theta[r] \text{col}[F] = \text{col}[\lambda(r[\langle r \rangle], F)]$$

$$\bar{\Theta}[r] \text{col}[F] = \text{col}[\lambda(\bar{r}[\langle \bar{r} \rangle], F)]$$

$$\Theta \text{col}[\bar{\Psi}] = \bar{\pi}' \text{col}[2\bar{\pi}] = \omega[\bar{\Psi}] + \bar{\pi} \text{col}[2\bar{\pi}']$$

$$\bar{\Theta} \text{col}[\bar{\Psi}] = \pi' \text{col}[2\pi] = \omega[\bar{\Psi}] + \pi \text{col}[2\pi']$$

$$2\pi[r] := \frac{d}{dz}\pi(r[\omega]|_z) \quad \langle \omega \rangle = \langle r \rangle$$

$$2\bar{\pi}[r|z] = \frac{d}{dz}\pi(\bar{r}[\bar{\omega}]|z) \quad \langle \omega \rangle = \langle r \rangle$$

=

Elements of  $\Theta^{-1}[r]$

$$\frac{\mu^{-1}(r[\nu])z \phi(r|z)}{\pi(r[\nu])} \quad z \in \mathbb{C}$$

$$\frac{\mu^{-1}(r[z])z}{\pi(r[z])} \quad z \in \mathbb{C}$$

$\notin \bar{\Theta}^{-1}[r]$

$$\frac{\mu^{-1}(\bar{r}(\bar{\nu})z) \phi(r|z)}{\pi(\bar{r}(\bar{\nu}))} \quad z \in \mathbb{C}$$

$$\frac{\mu^{-1}(\bar{r}(\bar{\nu})z)}{\pi(\bar{r}(\bar{\nu}))} \quad z \in \mathbb{C}$$

=

The exponential operator

$$E[r] := [\sigma(r[z], \omega) \langle z, \omega \rangle = \langle r \rangle] \quad (E \in \mathcal{UZ})$$

$$\bar{E}[r] := [\sigma(\bar{r}(\bar{\nu}), \bar{\omega}) \langle z, \omega \rangle = \bar{u} \langle r \rangle] \quad (\bar{E} \in \mathcal{UU})$$

$$E^{-1}[r] := [\delta(r[\nu], r^\tau) \langle z, \omega \rangle = \mathcal{L} \langle r \rangle] \quad \mathcal{UZ}$$

$$\bar{E}^{-1}[r] := [\delta(\bar{r}(\bar{\nu}), \bar{r}^{\bar{\tau}}) \langle z, \omega \rangle = \bar{u} \langle r \rangle] \quad \mathcal{UU}$$

i) Properties Relationships

ii) Inversion

$$\{E\}^{-1} = E^{-1} \quad \{\bar{E}\}^{-1} = \bar{E}^{-1}$$

b) Conjugation The conjugates of  $E$  and  $E^{-1}$  are  $\bar{E}$  and  $\bar{E}^{-1}$  respectively.

iii) Displacement

$$k \in \mathbb{R} \quad I[k] + E[r] = E[O[k] + r]$$

$$\text{also } E[r] + I[k] = E[r + O(k)]$$

??

iv) Exponential properties

$$a) k \in \mathbb{R} \quad E[O[k] + r] = E[B[k] + x(r)] \quad \begin{matrix} \cancel{E[B]} \\ E[O < B > + r] \end{matrix} \quad E[B + x(r)] \\ = E[B[k] + (r+x)] \quad = E[B + (r+x)) \text{ etc.}$$

$$|r|=0 \quad E[r] = \begin{pmatrix} 1 & \cdot \\ -x(0) & 1 \end{pmatrix} \quad \begin{matrix} \text{reverse} \\ \cancel{x \text{ commutes}} \end{matrix} \quad E(x) = E[x(0)+x]$$

$$b) k=0 \quad E[r] E[x(r)] = E[r+x] \quad (\text{noncommutative})$$

$$c) x=0 \quad E[O[k] + r] E[B[k] + O(r)] = E[B[k] + r]$$

$$d) E[O + (r)] E[y + O(r)] = E[y + (r)]$$

$$e) E[x(r)] E[y(r)] = E[y(r)] E[x(r)] = E(x(r)y(r)) \\ E^{-1}[x(r)] = E[-x(r)] \quad \begin{matrix} \cancel{\text{true that }} E(x^p) E(y^p) \\ = E(xy)^p? \end{matrix}$$

iv) Transformation properties

$$E[r] \cancel{\text{ col }} [p \text{ row}] = \text{ col } [\pi \text{ row}]$$

$$E \text{ col } [p \text{ row}] = \text{ col } [\pi']$$

$$\bar{E}[r] \cancel{\text{ col }} [\bar{p} \text{ row}] = \text{ col } [\bar{\pi}'(r)]$$

$$\bar{E} \text{ col } [\bar{p} \text{ row}] = \text{ col } [\bar{\pi}']$$

$$\text{leading to } \bar{M}^{-1} E \text{ col } [p \text{ row}] = \text{ col } [\bar{\Psi}] \quad E^{-1} \bar{M} \text{ col } [\bar{\Psi}] = \text{ col } [\bar{p} \text{ row}]$$

- $\neg\theta$

$\neg\theta(\theta)$

$\neg\theta(\theta)$

1

$\neg\theta(\theta)$

$x^2\theta(0)\theta(1) - x\theta(0) - x\theta(1)$

$\neg\theta(\theta)$

$\neg(x\theta y)\theta(\theta) \Rightarrow 1$

$\neg(x\theta y)(\theta(\theta)) \Rightarrow 1$

$x^2\theta(0)\theta(1) + xy(\theta(0)\theta(1))\theta(0) + y^2\theta(0)\theta(0)$

$(x+y)^2\theta(0)\theta(1) - xy\theta(0)\theta(1) + xy\theta(0)$

$M \text{ col } \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{col } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E \text{ col } [p_{\theta\theta}] = "$

$M^{-1} E \text{ col } [p_{\theta\theta}] = \theta\theta$

$E^{-1} M \text{ col } \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \theta\theta$

$\theta(\Gamma[\tau]\nu)$

$\sigma(\Gamma[\tau], |\Gamma|, |\Gamma| - \nu)$

$= \theta(\bar{\tau}(\bar{\tau}], \bar{\nu}) / M^{-1} E$

$\text{E}[r] = [p_{\theta\theta}[\Gamma|x] | \in [\Gamma]]$

=

$\overline{E}[\Gamma]$

1  $\neg\theta(\theta) \quad \theta(\theta)$

1  $-2x \quad x^2$

1  $-x \quad 1$

1  $x^2 + 2x\theta(0) - \theta(0)\theta(0)$

0  $\theta(0)\theta(1) - \theta(0) - \theta(1)$

0  $\theta(0)\theta(1) - \theta(0)\theta(0) \quad 1$

0  $\theta(0)\theta(0) - \theta(0)\theta(0) \quad 1$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E[\Gamma] & 0 \\ 0 & 0 \end{bmatrix}$

defining  $\sigma$  as stated

$[\sigma(\Gamma \langle \langle \Gamma \rangle \rangle - \bar{\tau})]$

$\bar{\tau}$

$E^{-1} M : V' [p_{\theta\theta}]$

$E[\Gamma] M[\Gamma] = V [p_{\theta\theta}[\Gamma] | \Gamma]^{-1} \quad V \left[ \begin{array}{c} p_{\theta\theta} \\ \equiv \end{array} \right]$

$\left[ \begin{array}{c} \theta(\theta) \theta(\tau) | \nu \\ \langle \tau, \nu := \mu[\langle \Gamma \rangle, \langle \equiv \rangle] \end{array} \right]$

$V[\theta = |\Gamma|]$

$\exists(\theta) | \nu$

$\langle \tau, \nu := \mu[\langle \equiv \rangle, \langle \Gamma \rangle]$

$M[\Gamma] V [p_{\theta\theta}[\Gamma] | \Gamma] = \theta(\Gamma[\tau], \Gamma^\nu)$

$E[\Gamma] \text{ col } [\theta(\tau)]^{\Gamma \rightarrow} \quad \nu := \mu[\Gamma]$

$\theta(\theta) \quad \theta(\Gamma) \quad E V^T = \{M^{-1}\}_1^1$

$\theta(\theta) \quad \theta(\Gamma) \quad E V^T = M^{-1}$

$\theta(\theta)^2 \quad T \quad V' = M^{-1}$

?  $E[\Gamma] V [p_{\theta\theta}[\Gamma] | \Gamma] = [\pi(\Gamma[\tau] | \theta(\nu))]^{(\theta)} = \{M^{-1}[\Gamma]\}_{V'}^{-1} = E^T$

$M^T[\Gamma] E[\Gamma] = \{V [p_{\theta\theta}[\Gamma] | \Gamma]\}^{-1}$

1 1 1

0  $\theta(\nu) - \theta(\theta) \quad \theta(\theta) - \theta(\theta)$

0 0  $(\theta(\theta) - \theta(\theta))(\theta(\theta) - \theta(\theta))$

$\mu^{-1}(\Gamma[\tau], \nu) \quad \pi(\Gamma[\nu], \theta(\tau))$

$\mu^{-1}(\bar{\Gamma}[\nu], \tau) \quad \pi(\bar{\Gamma}[\nu] | \theta(\tau))$

$V' \cdot \exists(\theta) | \nu$

ii) With  $\nabla[\Xi|r] := [\frac{\partial}{\partial}(f(z)|v) \langle z, v \rangle = M[\langle r \rangle, \langle \Xi \rangle] \rangle]$

$$M^{-1}[r]\{E^T[r]\}^{-1} = \nabla[\text{pow}[r]|r] \quad \text{Lede } \Rightarrow \text{Vandermonde} \\ M^{-1} \in \mathbb{M} \quad E^T \in \mathbb{M}$$

$$E^T[r] M[r] = \{\nabla[\text{pow}[r]|r]\}^{-1}$$

$$\bar{M}^{-1}[r] E[r] = [\alpha(r \langle \langle r \rangle - z), v \langle z, v \rangle = M \langle r \rangle \rangle]$$

$$E^{-1}[r] \bar{M}[r] = \nabla[\text{pow}[r]|r]^T \mu[r] \quad N?$$

relationship involving  $E[r+O(k)]$  can be picked up from continued product formula, setting last  $k$  members of sequence zero

iii) Displacement Exponential properties  $\bar{E}(r) + I[k] = \bar{E}(r+O(k))$  |  
 with a)  $\bar{E}(r+O(k)) \bar{E}(x \langle r \rangle + B[k])$  | ii) Special values  
 $= \bar{E}((r+x) + B[k])$  |  $E[x+O(r)] = \{ \text{pow}[r|x] \}$   
 $\bar{E}(O[r]+x) =$

b)  $\bar{E}(r) \bar{E}(x \langle r \rangle) = \bar{E}(r+x)$  |  $\{ \text{ul}[\text{pow}[r|x]] \}$

c)  $\bar{E}(r+O(k)) \bar{E}(O \langle r \rangle + B[k]) = \bar{E}(r+B[k])$

d)  $\bar{E}(r) + O \bar{E}(O \langle r \rangle + y) = \bar{E}(r+y)$

e)  $\bar{E}(x \langle r \rangle) \bar{E}(y \langle r \rangle) = \bar{E}(y \langle r \rangle) \bar{E}(x \langle r \rangle) = \bar{E}((xy) \langle r \rangle)$

$$\bar{E}^{-1}(x \langle r \rangle) = \bar{E}(-x \langle r \rangle)$$

also add to (ii)  $E[xr] = [\text{pow}[r|x] | E[r]]$

$$\bar{E}(xr) = [\bar{\text{pow}}[r|x] | \bar{E}(r)]$$

also  $\bar{M}^{-1}[r] \{E^T[r]\}^{-1} = \nabla[\bar{\text{pow}}[r]|r]$

$$E^T[r] \bar{M}[r] = \{\bar{\text{pow}}[r]|r\}^{-1}$$

also  $M^{-1}[r] \bar{E}[r] = [\alpha(r \langle \langle r \rangle - z), v \langle z, v \rangle = M \langle r \rangle \rangle]$

$$E^{-1}[r] M[r] = \nabla[\bar{\text{pow}}[r]|r]^T \mu[r]$$

$$\begin{array}{ccccc}
 & & \left[ \begin{matrix} \gamma_0 - \gamma_1 & \gamma_0 - \gamma_2 \\ \gamma_1 - \gamma_2 & \gamma_1 - \gamma_2 \end{matrix} \right] & \xrightarrow{\gamma_1 - \gamma_2} & \\
 (\gamma_0 - \gamma_1)(\gamma_0 - \gamma_2) & (\gamma_0 - \gamma_2) & 1 & \gamma_1 - \gamma_2 & \gamma_0 - \gamma_2 - \gamma_1 - \gamma_2 \\
 & & 1 & \gamma_0 \gamma_2 - \gamma_0 \gamma_1 & 1 \\
 \overline{M}^{-1} E & \gamma_1 - \gamma_2 & -\gamma_0 & = & \gamma_0 \gamma_2 & -\gamma_0 - \gamma_2 \\
 & & 1 & & \gamma_0 \gamma_1 & -\gamma_0 - \gamma_1 \\
 & & \gamma_0 \gamma_1 - \gamma_0 \gamma_1 & & & 1 \\
 & & 1 & & & 1 \\
 & & 1 & -\gamma_0 - \gamma_2 & \gamma_1 \gamma_2 & -\gamma_1 - \gamma_2 & \gamma_1 + \gamma_2 \\
 & & 1 & -\gamma_2 & 1 & -\gamma_0 - \gamma_2 & \gamma_0 + \gamma_2 \\
 & & 1 & \gamma_2 - \gamma_0 (\gamma_2 - \gamma_0)(\gamma_0 - \gamma_1) & 1 & -\gamma_0 - \gamma_1 & \gamma_0 \gamma_1 \\
 & & & & 1 & & 
 \end{array}$$

$\gamma_1 \gamma_2$

i. The gradient operator

$$G(r) = [\pi(r(\omega, \tau)) \langle z, \omega \rangle := \cancel{u} \langle r \rangle] \quad ux$$

$$\bar{G}(r) = [\pi(r(z, \omega)) \langle z, \omega \rangle := u \langle r \rangle] \quad uu$$

$$G^{-1}(r) = u \cancel{u} \cancel{u} [-\bar{\Phi}(r)]$$

$$\bar{G}^{-1}(r) = u \cancel{u} u [-\bar{\Phi}(r)]$$

App. m

ii) Relationships

- a)  $\bar{G}, \bar{G}^{-1}$  conjugates  $\mathcal{G}, \mathcal{G}^{-1}$ .  $G, \bar{G}$  inverses of  $\mathcal{G}, \bar{\mathcal{G}}$

$$b) G = [\pi | L[I]] \quad \bar{G} = [\bar{\pi} | U[I]] \quad G = [\bar{\Phi} E | L[I]] \quad \bar{G} = [\bar{\Phi} \bar{E} | U[I]]$$

iii) Transformation properties

$$G(r|x) \text{ col } [\pi' [r|y]] = \frac{\text{col } [\pi[r|x] - \pi[r|y]]}{x-y}$$

$$\bar{G}(r|x) \text{ col } [\bar{\pi}' [r|y]] = \frac{\text{col } [\bar{\pi}[r|x] - \bar{\pi}[r|y]]}{x-y}$$

$$G(r|\gamma(\omega)) \text{ col } [\pi'(r)] = \text{ col } [\pi''(r)]$$

$$\bar{G}(r|\gamma(\mu)) \text{ col } [\bar{\pi}'(r)] = \text{ col } [\bar{\pi}''(r)]$$

or define as  
displacement operator?

$$\begin{matrix} 1 & 1 & 1 \\ \gamma(\omega)-x & 1 & \beta(\omega)-y & 1 & \gamma(\omega)+\beta(\omega)-x-y & 1 \\ \gamma(\mu)-x & 1 & \beta(\mu)-y & 1 & (\gamma(\mu)-x)(\beta(\omega)-y) & \\ & & & & \beta=\beta & (\gamma(\mu)-x)(\gamma(\omega)-y) \end{matrix}$$

$$E[x+O(r)] = L[\rho_{\text{sw}}[r|x]]^{-1} \quad (*)$$

$$E(O[r]+x) = U[\overline{\rho}_{\text{sw}}[r|x]]^{-1} \quad | \quad L, P \text{ commute}$$

$$\Rightarrow [\Phi E P | L] = [\Phi E | L]$$

$$A(r|x/y) := I[r] + (x-y)$$

$$(x-y)[O^L + [U(\pi(r), x)|x] \langle z, z \rangle - K(r)]. + O[r]]$$

$$A(r|x/y) = \{G(r|y)\}^{-1} G(r|x) \quad (+)$$

$$A(r|x/y) = I[r] + (x-y)[O^L + [G(r|y)|x]. + O[r]]$$

$$= \cancel{O^L + E}$$

$$= [E[O^L + \cancel{G(r|y)}] \| L[\rho_{\text{sw}}[r|x]] \{ L[\rho_{\text{sw}}[r|y]]\}^{-1}] \downarrow \text{same reason}$$

$$= E[y + (r)] L[\rho_{\text{sw}}[r|x]] E^{-1} [O^L + (r)]$$

$$= E[y + (r)] E^{-1} [x + (r)] \quad \text{from (iii) + (*)}$$

$$= u \mathcal{Z}[-\Phi(r|x)]^{-1} u \mathcal{Z}[-\Phi(r|y)]$$

$$= u \mathcal{Z}[-\Phi(r|y)] \{ u \mathcal{Z}[-\Phi(r|x)] \}^{-1} \Rightarrow (+)$$

$$\frac{A(r|x/y) - I[r]}{x-y} = \{G(r|y)\}^{-1} \frac{G(r|x) - G(r|y)}{x-y}$$

$$\text{col}[0\langle\omega\rangle + r(\omega,\omega)] \frac{\pi(r(\omega,\omega)|x)}{\langle\omega:=\omega, r\rangle} \langle\omega:=\omega, r\rangle$$

$$= \{G(r|y)\}^{-1} \text{col}[0\langle\omega\rangle + \frac{\pi(r(\omega,\omega)|x) - \pi(r(\omega,\omega)|y)}{x-y} \langle\omega:=\omega, r\rangle]$$

$$G(\bar{r}(\bar{\omega})|y)\}^{-1} \text{col}\left[\frac{\pi'(\bar{r}(\bar{\omega})|x) - \pi'(\bar{r}(\bar{\omega})|y)}{x-y}\right]$$

$$\frac{1}{x-\gamma_1}, \frac{1}{y-\gamma_0} = \frac{x-\gamma_0}{(x-\gamma_0)(x-\gamma_1)}, \frac{y-\gamma_0}{(y-\gamma_0)(y-\gamma_1)} = \text{col}[\pi'(\bar{r}(\bar{\omega})|x)]$$

$$\frac{x-y}{x+y-\gamma_0-\gamma_1}, \frac{x-y}{x+y-\gamma_0-\gamma_1} \quad (\rightarrow \text{Always same def of } A)$$

$$G(r|y) \left\{ I[r] + (x-y)[0^r + [G(r|x) + O_{[r]}]] \right\}$$

$$= G(r|\cancel{x}) \Leftarrow G(r|y)^{<r>$$

$$G(r|y) I[r]^{<r>} + (x-y) G(r|y) [0^r + G(r|x)]$$

$$= G(r|x)^{<r>} + (x-y) G(r|y)^{<r>} G(r|x)$$

$$G(r|y)^{<r>} G(r|x) = \frac{G(r|x)^{<r>} - G(r|y)^{<r>}}{x-y}$$

$$G((r|y) G(r|x)) = \frac{G(r|x)^{<r>} - G(r|y)^{<r>}}{x-y}$$

$$[G(r|x) - G(r|y)]_{-0}$$

$$\Gamma = \gamma_0 \gamma_1 \gamma_2$$

$$\frac{1}{y-\gamma_2} \cdot \frac{1}{x-\gamma_1} = \frac{x-\gamma_1}{(x-\gamma_1)(x-\gamma_2)} \frac{1}{x-\gamma_2} - \frac{y-\gamma_1}{(y-\gamma_1)(y-\gamma_2)} \frac{1}{y-\gamma_2}$$

$$\frac{1}{x-\gamma_1} \cdot \frac{1}{x-\gamma_2} = \{G(r|y)\}^{-1} G(r|x)$$

$$\frac{1}{(x-\gamma_1)(x-\gamma_2)} \frac{1}{x-\gamma_2} = \bar{G}^T[r|x] \bar{G}^{T^{-1}}[r|y]$$

$$G(r|x) G^*(r|y)^* = G(r|y) G(r|x)^* \quad \text{independent of } x$$

$$u2\mathcal{L}[-\bar{\Phi}(r|x)] u2\mathcal{L}[-\bar{\Phi}(r|y)] = E^{-1}[x+r] u2\mathcal{L}[-\bar{\Phi}(r|y)]$$

$$= u2\mathcal{L}[-\bar{\Phi}(r|y)] u2\mathcal{L}[-\bar{\Phi}(r|\cancel{y})] \quad \text{i.e. } G(r|x) \in E[x+r]$$

$$z \rightarrow z-1 \quad z-2 \quad z-1 \quad z=1 \quad \text{independent of } x$$

$$(x-\gamma_{z-1})(y-\gamma_{z-1}) = (y-\gamma_{z-1})(x-\gamma_{z-1}) \quad (z, z-1)$$

$$x-\gamma_{z-1} + y - \gamma_z = y - \gamma_z + x - \gamma_{z-1} \quad (z, z-1)$$

$$u2\mathcal{L}[\equiv] u2\mathcal{L}[(\sqcup)] = u2\mathcal{L}[\sqcup] u2\mathcal{L}[\equiv] \quad \text{as}$$

$$\xi_{z-1} \omega_{z-1} = \omega_{z-1} \xi_{z-1} \quad \text{but } u2\mathcal{L}[\equiv] u2\mathcal{L}[\equiv+\infty]$$

$$\xi_{z-1} + \omega_z = \omega_z + \xi_{z-1} \quad \text{only if } \xi_z - \xi_{z-1} = \omega_z - \omega_{z-1}$$

$$\begin{matrix} 1 & 1 & ? & 1 & 1 & 1 \\ \xi_0 & 1 & \omega_1 & 1 & \xi_1 & 1 \\ & & & & \xi_0 + \omega_1 & \xi_1 + \omega_0 \\ & & & & \xi_1 & 1 \\ & & & & \omega_1 \xi_1 & \xi_1 + \omega_2 & 1 \\ & & & & \omega_1 \xi_1 & \xi_1 + \omega_2 & 1 \\ & & & & \omega_1 \xi_1 & \xi_2 + \omega_1 & 1 \end{matrix}$$

$$E^{-1}[r] u2\mathcal{L}[-\bar{\Phi}(r|\gamma_{(0)})] \quad \text{independent of } \gamma_{(0)}$$

$$\delta(r[\nu], r^{\tau}) + (\gamma_{(0)} - \gamma_{(0)}) \delta(r[\nu], r^{\tau}) = \delta(r[\nu], r^{\tau})$$

i) Relationships

a)  $\bar{G}, \bar{G}^{-1}$  conjugates  $\mathcal{G}, \mathcal{G}^{-1}$ .  $\mathcal{G}^{-1}, \bar{\mathcal{G}}^{-1}$  inverses  $\mathcal{G}, \bar{\mathcal{G}}$ .

$$b) \mathcal{G} = [\pi \parallel L[I]] = [\Phi E \parallel L[I]] = [\Phi EP \parallel L]$$

$$\bar{\mathcal{G}} = [\bar{\pi} \parallel U[I]] = [\bar{\Phi} \bar{E} \parallel U[I]] = [\bar{\Phi} \bar{E} \bar{P} \parallel U]$$

$$\mathcal{G}([r]_y) \mathcal{G}([r])_x = \frac{[G(r|x) - G(r|y)]}{x-y}$$

$$\bar{\mathcal{G}}([r]_y) \bar{\mathcal{G}}([r])_x = \frac{[\bar{G}(r|x) - \bar{G}(r|y)]}{x-y}$$

ii) Transformation properties

a) The Gradient properties

$$G(r|x) \text{col} [\pi'([r|y])] = \frac{\text{col} [\pi(r|x) - \pi(r|y)]}{x-y}$$

$$\bar{G}(r|x) \text{col} [\bar{\pi}'(r|y)] = \frac{\text{col} [\bar{\pi}(r|x) - \bar{\pi}(r|y)]}{x-y}$$

b) Logarithmic derivative properties

The operators  $G$  and  $\bar{G}$  have logarithmic derivative properties with respect to the exponential operators  $E$  and  $\bar{E}$ .

$G(r|x) E[x + (r)]$  and  $\bar{G}(r|x) \bar{E}([r] + x)$  are independent of  $x \in K$

c) Index displacement properties

$$G(r|\gamma(\omega)) \text{col} [\pi'([r])] = \text{col} [\pi''([r])] \quad \text{any } y$$

$$\bar{G}(r|\gamma(\omega)) \text{col} [\bar{\pi}'(r)] = \text{col} [\bar{\pi}''(r)] \quad \begin{aligned} G(r|\gamma(\omega)) E[r] \\ = E^r(r) \end{aligned}$$

## The argument interchange operator

$$A(r|x/y) := I[r] + (x-y)$$

$$(x-y) [O^{[r]}_+ + L[\pi(r), z]|x\rangle \langle z, z] = L[r] + O_{[r]}]$$

i) Properties

a) Conjugation

$$A \text{ is } *-\text{symmetric} \quad \bar{A} = A^\top$$

b) Inversion

$$\{A(r|x/y)\}^{-1} = A(r|y/x)$$

c) Multiplication

The matrices  $A(r|x/y)$  multiply commutatively, are not in general closed with respect to multiplication, but transform according to the special law

$$T(r|x/y) T(r|y/z) = T(r|x/z)$$

ii) Connections with the power sequence, the exponential operator and the gradient operator

$$\begin{aligned} A(r|x/y) &= [E[0+(r)] \parallel L[\text{pow}[r|x]] \{ L[\text{pow}[r|y]]\}^{-1}] \\ &= E[y+(r)] L[\text{pow}[r|x]] E^{-1}[0+(r)] \\ &= I[r] + (x-y) [O^{[r]}_+ + G(r)|x\rangle \langle x, r] + O_{[r]}] \\ &= \{G(r|y)\}^{-1} G(r|x) \\ &= E[y+(r)] \cancel{\{E^{-1}[x+(r)]\}} \end{aligned}$$

iii) Further relationships

$$A(r|x/y) = \{ u \mathcal{L} [-\Phi(r|x)] \}^{-1} u \mathcal{L} [-\Phi(r|y)] \\ = \{ u \mathcal{L} [-\Phi(r|y)] \} \{ u \mathcal{L} [-\Phi(r|x)] \}^{-1}$$

mit?

iv) Special values

a)  $A(r|x/x) = I[r]$

b)  $A(x < r > | y/x) = L[\rho_{\text{av}}[r|y-x]]$

v) Transformation properties

$$A(r|x/y) \text{col} [\pi'[r|y]] = \text{col} [\pi'[r|x]]$$

$$\bar{A}(r|x/y) \text{col} [\bar{\pi}'(r|y)] = \text{col} [\bar{\pi}'(r|x)]$$

$$\text{row} [\bar{\pi}'(r|y)] A(r|x/y) = \text{row} [\bar{\pi}'(r|x)]$$

$$\text{row} [\pi'(r|y)] \bar{A}(r|x/y) = \text{row} [\pi'(r|x)]$$

$$= u \mathcal{L} [(y+(r))-x] A(r|x/y) \quad \begin{matrix} \text{clear up mixed expressions} \\ (y+(r))-x : \text{outer brackets} \\ \text{do not denote open interv.} \end{matrix}$$

$$= 1 + G^{-1}(r|y)$$

\* i.e.  $A(r|x/y) = \{ u \mathcal{L} [(y+(r))-x] [1 + G^{-1}(r|y)] \}$

=

$$\lambda[r, r]: \lambda(r[\omega], F) \quad \langle \omega := [r] \rangle$$

$$\hat{\lambda}[r, F]: \lambda(r[\omega_H], F) \quad \langle \omega := [r] \rangle$$

$$\{ \pi[r|x] \}^{-1} \{ \hat{\lambda}[r, F|x] G(r|x) \pi'[r|y] \\ - G(r|\cancel{x}) \lambda[r, F|y] \pi'[r|y] \} \\ \xrightarrow{x-y} \frac{\hat{\lambda}[r, F|x] - \hat{\lambda}[r, F|y]}{x-y}$$

$$S[B/P] = [S(\Gamma[\omega], \pi(B[\omega])) \langle z, \omega := \mathcal{L}[\Gamma] \rangle] \in \mathcal{U}$$

$$\bar{S}[B/\Gamma] = [S(\bar{\Gamma}[\bar{\omega}], \pi(\bar{B}[\bar{\omega}]) \langle z, \omega := \omega \langle \Gamma \rangle \rangle] \in \mathcal{U}$$

$$S[B/P] = V[\pi'[B] \downarrow \Gamma]^T M[\Gamma]^T$$

$$= E[B] V[\rho_m[\Gamma] \downarrow \Gamma]^T M[\Gamma]^T$$

$$= E[B] E[\Gamma] M^{-1}[P] M[P]^T$$

$$= E[B] E^{-1}[\Gamma]$$

$$\bar{S}[B/\Gamma] = V[\bar{\pi}'(B) \downarrow \Gamma]^T \bar{M}[\Gamma]^T$$

$$= \bar{E}[B] V[\bar{\rho}_m[\Gamma] \downarrow \Gamma]^T \bar{M}[\Gamma]^T$$

$$= \bar{E}[B] \bar{E}^{-1}[\Gamma] \bar{M}^{-1}[\Gamma] \bar{M}[\Gamma]^T$$

$$= \bar{E}[B] \bar{E}^{-1}[\Gamma]$$

$$S[B/P] \Leftarrow \text{col}[\pi'[\Gamma]] = \text{col}[\pi'[B]]$$

$$\bar{S}[B/\Gamma] \text{col}[\bar{\pi}'(\Gamma)] = \text{col}[\bar{\pi}'(B)]$$

$$S[y^+(\Gamma) / x^+(\Gamma)] = A(\Gamma | x/y)$$

$$\bar{S}[(\Gamma)^+ y / (\Gamma)^+ x] = \bar{A}(\Gamma | x/y) = A(\Gamma | x/y)^T$$

$$= \begin{array}{ccccc} & & & & x^2 - (\gamma_1 + \gamma_2)x + \gamma_1 \gamma_2 - x^2 + \gamma_2 x + \gamma_1 x \\ & & & & \gamma_1 \gamma_2 - \gamma_1 - \gamma_2 & \gamma_1 \gamma_2 - \gamma_1 - \gamma_2 \\ \gamma_1 & & 1 & & -x & 1 \\ x - \gamma_1 & & & & & \\ \end{array} \quad \text{G displaces to operator}$$

$$(x-\gamma_1)(x-\gamma_2) x - \gamma_2 \not\rightarrow 1 \quad \gamma_1 x \quad -x - \gamma_1 \not\rightarrow 1 \quad 1$$

$$(x-\gamma_1)(x-\gamma_2)(x-\gamma_3) (x-\gamma_2)(x-\gamma_3) x - \gamma_3 \not\rightarrow \gamma_1 \gamma_2 x - \gamma_1 \gamma_2 + x(\gamma_1 + \gamma_2) - x - \gamma_1 - \gamma_2 \quad 1$$

$$\begin{array}{ccccc} 1 & & & & \\ -\gamma_1 & 1 & & & \\ \gamma_1 \gamma_2 - \gamma_1 - \gamma_2 & & 1 & & \end{array}$$

## Sequence interchange operators

$$S[B/r] := [S(r[p], \pi(B[z])) \langle z, p \rangle = L \langle r \rangle] \quad \leftarrow \text{def}$$

$$\bar{S}[B/r] := [S(\bar{r}[\bar{p}], \pi(\bar{B}[\bar{z}])) \langle z, p \rangle = U \langle r \rangle] \quad \leftarrow \text{def}$$

i) Properties

a) Conjugation and inversion

Conjugate of  $S$  is  $\bar{S}$  (wrt wrt with  $B, r$ ).

Inverses of  $S[B/r]$  &  $\bar{S}[B/r]$  are  $S[r/B] \leftarrow \bar{S}[r/B]$   
respectively  $\langle K \rangle$

b) Multiplication

$$S[A/B] S[B/r] = S[A/r] \langle K \rangle$$

Similarly for  $\bar{S}$

ii) Connections with factorial and power sequences and with ~~the~~ exponential operators

$$\begin{aligned} S[B/r] &= V[\pi'[B] | r]^T M[r]^T \\ &= E[B] V[\text{pow}[r] | r]^T M[r]^T \\ &= E[B] E^{-1}[r] \end{aligned}$$

$$\begin{aligned} \bar{S}[B/r] &= V[\bar{\pi}'[B] | r]^T \bar{M}[r]^T \\ &= \bar{E}[B] V[\overline{\text{pow}}[r] | r]^T \bar{M}[r]^T \\ &= \bar{E}[B] \bar{E}^{-1}[r] \end{aligned}$$

iii) Connections with argument interchange operators

$$A(r|x/y) = S[y+(r)/x+(r)]$$

$$\begin{aligned} \bar{A}(r|x/y) &= A(r|x/y)^T \\ &= \bar{S}([r]+y/[r]+x)] \end{aligned}$$

iv) Transformation properties

$$S[B/P] \text{col}[\pi'[r]] = \text{col}[\pi'[B]]$$

$$\bar{S}[B/P] \text{col}[\bar{\pi}'[r]] = \text{col}[\bar{\pi}'[B]]$$

$$\sum_{\omega} \mu(\bar{r}(\bar{o}]\omega) A(r|\gamma(\omega)/y) \left\langle \omega := (r) \right\rangle = 0 \quad (*)$$

$$\text{now } z+1 \leq \omega \quad z+1 \neq \omega \quad z \leq |r|-1 \quad \omega \leq z \quad z < |r|$$

$$\sum_{\omega} \mu(\bar{r}(\bar{o}]\omega) \pi(r(\omega, z] | \gamma(\omega)) \{ \gamma(\omega) - y \} \left\langle \omega := (r) \right\rangle$$

$$= 0 \text{ when } z-\omega+1 < |r|-1 \quad \text{by terms} = 0 \text{ when } \omega < \omega \leq z$$

$\omega = 0$  terms zero for  $\omega \leq z$

$$\text{sum becomes } \sum_{\omega} \mu(\bar{r}(\bar{o}]\omega) \pi(r(z] | \gamma(\omega)) \{ \gamma(\omega) - y \} \left\langle \omega := (z, |r|) \right\rangle$$

$$\sum_{\omega} \frac{\pi(r(z] | \gamma(\omega)) \{ \gamma(\omega) - y \}}{\pi(r(\omega) | \gamma(\omega)) \pi(r(\omega, |r|] | \gamma(\omega))}$$

$$\sum_{\omega} \frac{\{ \gamma(\omega) - y \}}{\pi(r(\omega, \omega) | \gamma(\omega)) \pi(r(\omega, |r|] | \gamma(\omega))} \left\langle \omega := (z, |r|) \right\rangle$$

$$= \delta(r(z], r-y)$$

$$z = |r|-1 \quad \omega = 0 : \text{one term } \mu(\bar{r}(\bar{o}], |r|) \pi(r(|r| | r|) | \gamma(|r|)) \{ \gamma(|r|) - y \}$$

$$= \{ \gamma(|r|) - y \} \quad \text{unless} \quad 1, \dots, |r| \quad 2, \dots, |r|-1 + 1$$

$$z = |r|-1 \quad \omega = 1 \quad \mu(\bar{r}(\bar{o}], 1) \pi(r(1, |r|-1] | \gamma(1)) \{ \gamma(1) - y \} \\ + \mu(\bar{r}(\bar{o}], |r|) \pi(r(|r| | r|) | \gamma(|r|)) \{ \gamma(|r|) - y \}$$

$|r| = 3 : 1, 2, 3 \quad 2, \dots, |r|-1 \text{ also numbers}$

$$= \frac{\gamma(1) - y}{\gamma(1) - \gamma(|r|)} + \frac{\gamma(|r|) - y}{\gamma(|r|) - \gamma(1)} = 1 \Rightarrow \text{relationship } (*) \text{ is false}$$

$$\begin{array}{ll}
 \text{Ns} [\Gamma, F] [\delta(\Gamma[\nu, \epsilon], F) \langle z \rangle := \mathcal{L}\{\Gamma\}] \triangleq D & \Delta = D^o \\
 \text{Ns} \dots [z, \nu] \xrightarrow{u} \text{Ns} \dots & \Delta = D^o \\
 \text{Os} [\delta(\Gamma[\nu, \epsilon], F) \pi(D[\nu, \epsilon])] \subset \text{Os} T \text{ for Newton form} \\
 \bar{\Theta}_s [z, \nu] [z, \nu] u & \text{Ns} D T \Lambda t = \bar{T}^o \\
 \Lambda[\Gamma, F] \triangleq [\lambda(\Gamma[D, \nu], F)] \dots \bar{\Lambda} \dots \Gamma[\nu, \nu] & \Delta = D^o \bar{\Delta} = \bar{D}^o \\
 \text{Conjugation} \quad D \text{Ns} \Lambda *-\text{symmetric} & N = \Delta^o \\
 & N := M \alpha? \\
 & N := M \alpha?
 \end{array}$$

$$\begin{aligned}
 \Theta[\Gamma] [\theta(\Gamma[\nu, \nu])] \quad \Theta^{-1}[\Gamma] &= \frac{\Phi \left[ \phi(\Gamma[\nu]) \mu^{-1}(\Gamma[\nu]) \nu \right]}{\pi(\Gamma[\nu])} \quad N := M d M^{-1} \\
 &= \bar{\Phi} M^{-1} \bar{\pi}^{-1} \quad \Theta^{-1}[\Gamma] = \bar{\Phi} \bar{M}^{-1} \bar{\pi}^{-1}
 \end{aligned}$$

$$\Lambda = L T \rightarrow \bar{\Lambda} = U \bar{T}$$

$$\begin{array}{l}
 \text{Inversion} \\
 \{D[\Gamma, F]\}^{-1} = D[\Gamma, \frac{1}{F}] \quad \text{also } \bar{T} \quad \Lambda^{-1} = L^{-1} \Lambda \left[ \Gamma, \frac{1}{F} \right] L^{-1} \\
 \text{Invariance of sequences under transformation} \\
 (\bar{z} - \bar{\gamma}_0)(\bar{z} - \bar{\gamma}_1) - (\bar{z} - \bar{\gamma}_1)(\bar{z} - \bar{\gamma}_0) = 0 \quad \Lambda[\bar{\Phi}] = \bar{\Phi} \\
 \frac{(z - \gamma_0)(z - \gamma_1)}{\gamma_1 - \gamma_0} = 1 \quad \left. \begin{array}{l} D[\Gamma, F] = F \quad F \text{ is const} \\ \bar{T}[\Gamma, F] = F \quad " \end{array} \right\} \\
 \Lambda[k\bar{\Phi}] = k\bar{\Phi} \quad k \in \mathbb{K}
 \end{array}$$

Similarity product expression

$$D[\Gamma, F] = [M[\Gamma] \parallel F] \quad T[\Gamma, F] = [\Theta[\Gamma] \parallel F] \quad \text{As} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \overline{D} \dots$$

Invariance with respect to the similarity product

$$D = [\bar{\pi}' L \pi' \parallel D] = \bar{\pi}' \left[ \frac{1}{\pi'} \cdot L^{-1} \bar{\pi}^{-1} \parallel D \right]$$

$$\bar{T} = [L \cdot \frac{1}{\Phi} \parallel \bar{T}] = [\bar{\Phi} L^{-1} \parallel \bar{T}]$$

$$\begin{aligned}
 T[\Gamma, F] &= [\bar{M} \bar{\pi}' \bar{\Phi}^{-1} \parallel F] \\
 &= [\bar{M} \bar{\pi}' L \bar{\pi}^{-1} \parallel F] \\
 \text{but what is } \bar{M} \bar{\pi}' & \\
 \text{Have missed out duplicating} \\
 \text{det. of } \bar{\pi}' \text{ are suitable} \\
 \text{for sim. prod. without removal} \\
 \text{of zero factors}
 \end{aligned}$$

$$\tilde{\pi}[r]: \tilde{\pi}[r] := \tilde{\pi}''[r] \langle B \setminus \{r\} \rangle$$

$$z = \gamma(x) \quad x \in \langle r \rangle$$

$$\tilde{\pi}[r|z] = \frac{1}{\pi[r[x]]|z]}$$

Trick from pp. 570, 571

involve algebraic  
removal of vanishing  
factors does not work.

Since reciprocal of zero factor  
is involved in similarity product

$$\cancel{\text{Ansatz}} \langle \omega := [x] \rangle + \frac{1}{\pi(r[\omega]|z)} \langle \omega := (x, r) | z \rangle$$

$$\Lambda = [? \| D] L \quad ? = x, y$$

$$\tau[r, f] = [\tilde{\pi}[r] \| D[r, f]]$$

$$\begin{aligned} & (\gamma_1 - \gamma_0)x - y \\ & (\gamma_1 - \gamma_0)x = y \end{aligned}$$

$$\begin{array}{ccccc} 1 & & [M, F] & 1 & F_0 \\ \frac{(z - \gamma_0)}{\gamma_0 - \gamma_{01}} & \frac{1}{F_0} & \parallel & \frac{1}{\gamma_0 - \gamma_1} & \frac{1}{\gamma_1 - \gamma_0} \\ & F_0 & & F_0 & F_1 \\ & \frac{F_1 - F_0}{\gamma_1 - \gamma_0} \cdot \frac{y}{x} & F_1 & & \frac{F_1 - F_0}{\gamma_1 - \gamma_0} \\ \frac{1}{F_0} & & & & F_1 \\ \frac{(z - \gamma_0)(F_1 - F_0)}{\gamma_0 - \gamma_1} & F_1 & & & \end{array}$$

$$T = \tilde{\pi}[r] D \tilde{\pi}'[r]^{-1}$$

$$\tilde{\pi}[r] \quad 1, z - \gamma_0$$

$$T \text{ should be } [S[r[\omega], z] \pi(r[\omega], z)]$$

$$\tilde{\pi}[r] \quad z - \gamma_0, (z - \gamma_0)(z - \gamma_1)$$

$$\chi=0 \quad \tilde{\pi}[r|\gamma(\omega)] = \frac{1}{\gamma(0) - \gamma(1)}, 1 \quad \frac{F_0}{\gamma(0) - \gamma(1)}$$

$$z - \gamma_1 - (z - \gamma_0)x =$$

$$\chi=1 \quad \frac{\gamma_1 - \gamma_0}{\gamma_1 - \gamma_0}, 1$$

$$(1, -\gamma_1) \frac{F_1 - F_0}{\gamma_1 - \gamma_0} F_1$$

$$\frac{(z - \gamma_1)F_0 - (z - \gamma_0)F_1}{\gamma_0 - \gamma_1}$$

$$\mathcal{U} \quad T = [\tilde{\pi}''[r] | D[r, f]] \quad \langle K \setminus \{r\} \rangle$$

$$\Theta[r]: \frac{1}{z - \gamma_1}, \frac{z - \gamma_0}{\gamma_1 - \gamma_0} F_0, \quad \frac{1}{z - \gamma_1}, \frac{\gamma_1 - \gamma_0}{z - \gamma_0} F_1 \quad \checkmark \quad F_0 \quad F_1$$

$$T = [\Theta[r] \| F]$$

Divided difference, Newton term and Lagrange form arrays derived from special sequences

$D[r, \underline{\Phi}[r]]$  written simply as  $D[\underline{\Phi}]$  etc.

$$D[r] = r + I[r] - L^{-1}[I[r]] = uL\Delta[r]$$

$uL\Delta[r], uL\Delta[-\underline{\Phi}]$

$[\pi^{-1} | L] \underline{\Phi}^{-1}$

$$D[\underline{\Phi}] = -uL\Delta[-\underline{\Phi}]$$

$\pi' L \pi'$  commute

$$= \underline{\Phi}[r] - I[r] + L^{-1}[I[r]]$$

with each other and all  $D$

$$D[\frac{1}{\underline{\Phi}}] = \underline{\Phi}^{-1} [\pi'^{-1} | L] = [\pi^{-1} | L] \underline{\Phi}^{-1} \quad L \underline{\Phi}^{-1} L \underline{\Phi} \text{ commute}$$

with each other and all  $I$

$$D[\bar{\Psi}] = \bar{\pi}' L \pi'$$

$$\tau[r] = u\Delta L[\underline{\Phi}[r]] - I[r] + r = -r \cancel{L^{-1}} \quad \begin{matrix} \text{coefficients} \\ \text{are fun. of } z \end{matrix}$$

$$\tau[\bar{\Phi}] = \bar{\Phi} - u\Delta L[\underline{\Phi}[r]] + I[r] = \cancel{\bar{\Phi} L^{-1}} + \bar{\Phi} + r$$

$$\Rightarrow \bar{\tau}\left[\frac{1}{r}\right] = -L\left[\frac{1}{r}\right]$$

$$= -\bar{\Phi} L^{-1} + z < r >$$

$$\tau\left[\frac{1}{\underline{\Phi}}\right] = L \underline{\Phi}^{-1}$$

$$\tau[z < r >] = z < r >$$

$$\tau[\bar{\Psi}] = L \bar{\Psi}$$

$$\Lambda[r] = \{L[I[r]] - I[r]\} \underline{\Phi}[r] + L[r]r = L[r]z - \underline{\Phi}[r]$$

$$= \underline{\Phi}[r] \{L[r] - I[r]\} + r L[r] = z L[r] - \bar{\Phi}[r]$$

$$\Lambda[\underline{\Phi}] = \bar{\Phi}$$

$$\Lambda[z < r >] = z L[r]$$

$$\Lambda\left[\frac{1}{\underline{\Phi}}\right] = L \bar{\Phi}^{-1} L$$

$$\Lambda[r] = z L[r] - \bar{\Phi} \cancel{F_0} = \cancel{F_0} \quad \begin{matrix} \text{no: again} \\ \text{coeffs in } \Lambda \end{matrix}$$

$$\Lambda[\bar{\Psi}] = L \bar{\Psi} L$$

$$\frac{(z - \gamma_0) F_0 - (z - \gamma_1) F_1}{\gamma_1 - \gamma_0} \quad z \approx \gamma_1$$

$$D\Phi, \quad D\Phi[r] := \mathcal{U}[r(\rho(\omega, \tau))]$$

simpler  $D$ ?

$$D'\Phi^{-1}[r] = \mathcal{U}\mathcal{Z}[-\Phi[r]]$$

for displacement operator  
or  $G$  for gradient operator

$$\{D\Phi\}^{-1} = D'^{-1}\Phi^{-1}$$

$$\bar{\Phi}[r|y] + (x-y) D\bar{\Phi}[r|x] = \bar{\Phi}[r|x] E(r|x/y) \quad \textcircled{c}$$

$$\begin{aligned} & \bar{\Phi}[r|x] E(r|x/y) D'^{-1}\bar{\Phi}^{-1}[r|x] - \bar{\Phi}[r|y] D'^{-1}\bar{\Phi}^{-1}[r|x] \\ &= \bar{\Phi}[r|x] \mathcal{U}\mathcal{Z}[-\bar{\Phi}[r|y]] - \bar{\Phi}[r|y] \mathcal{U}\mathcal{Z}[-\bar{\Phi}[r|x]] \end{aligned}$$

diag terms:  $x-\gamma_\omega - y + \gamma_\omega = x-y$

line diag terms  $-(x-\gamma(\omega))(y-\gamma(\omega)) - (y-\gamma(\omega))(x-\gamma(\omega)) = 0$

$$= (x-y) I[r]$$

~~$\bar{\Phi}[r|y] D\bar{\Phi}(r|x)$~~

$$\bar{\Phi}[r|y] \cancel{\Phi^{-1}}[r|x] + (x-y) I[r] = \bar{\Phi}[r|x] \cancel{\Phi^{-1}}[r|y] \quad \textcircled{d}$$

diag terms:  $y-\gamma(\omega) + x-y = x-\gamma(\omega)$

off diag terms:  $(y-\gamma(\omega))(x-\gamma(\omega)) = (x-\gamma(\omega))(y-\gamma(\omega))$

$$T \cancel{\Phi}(r|x/y) = D'^{-1}\bar{\Phi}^{-1}[r|y] D\bar{\Phi}[r|x] \quad \textcircled{e} \quad \text{and } \textcircled{d} \Rightarrow \textcircled{e}$$

$$T(r|x/y) = D\bar{\Phi}[r|x] \cancel{D'^{-1}\bar{\Phi}^{-1}[r|y]}$$

$$D\bar{\Phi} = [\bar{\Phi} \mathbb{I}, L] = [\bar{\Phi} E, L]$$

$$[\bar{\Phi} EP, L] = [\bar{\Phi} E, L] \text{ since } E \text{ and } P \text{ commute}$$

$$P[\bar{P}]: P[r|x] := u \mathcal{L}[x^{\tau \rightarrow}] \langle z := [r], \omega := [z] \rangle$$

$$P^{-1}[\bar{P}], P^{-1}[r|x] := u \mathcal{L}[-x I(r)]$$

$\{P\}^{-1} = P^{-1}$   $P[r]$  not closed with respect to multiplication over  $K$ . Nevertheless commute  $\| L \bar{P} L_{P \otimes S[r \mapsto z]} \| \cdot L \bar{P}_{S \otimes}$

$$E[\bar{P}], E[r] := u \mathcal{L}[s(r[\omega], \omega)] \langle z := [r], \omega := [z] \rangle$$

$$E^{-1}[r] := u \mathcal{L}[s(r[\omega], \{r\}^\omega)]$$

$$k \in \mathbb{N} \quad \text{drop}^k E[r] = E[\langle \circ \langle k \rangle + r \rangle]$$

$$E[\langle \circ \langle k \rangle + r \rangle] E[\langle \beta \langle k \rangle + x I \langle r \rangle \rangle] =$$

$$E[\langle \beta \langle k \rangle + [r+x] \rangle] \quad |r|=0 \quad E[r] = \binom{1}{-\gamma(\omega)} \circ$$

$$k=0: E[r] E[x I \langle r \rangle] = E[r+x] \quad E[\gamma(\omega)] E[x] = E[\gamma(\omega) + x] \quad x \text{ commutative}$$

$$x=0: E[\langle \circ \langle k \rangle + r \rangle] E[\langle \beta \langle k \rangle + \circ \langle r \rangle \rangle] = E[\langle \beta \langle k \rangle + r \rangle]$$

$$E[\langle \circ \langle r \rangle \rangle] E[y + \circ \langle r \rangle] = E[\langle y + \langle r \rangle \rangle]$$

$$E[x I \langle r \rangle] E[y I \langle r \rangle] = E[(xy) I \langle r \rangle] \quad (\text{commutative})$$

$$E^{-1}[x I \langle r \rangle] = E[-x I \langle r \rangle]$$

$$E[r] P[r \cancel{\otimes}]^\circ = c \bar{P}[r \cancel{\otimes}] \langle K \rangle$$

$$\text{drop } S[(r)^{-1} S[r]] = u \mathcal{L}[-\gamma(\omega) I(r)] = P[r | \gamma(\omega)]^{-1}$$

$$\tau: \tau(r|x/y) := I[r] + (x-y) \left[ O^{\{r\}} / \left[ L \left[ \tau(r|y, \bar{x}) | x \right] \langle \bar{x}, \bar{y} = [r] \rangle \times O_{[r]} \right] \right]$$

$E(r)$  in  $E([r])$  etc

$$\tau(r|x/y) = \left[ \text{diag } E[(r), P[r, x]P[r, y]]^{-1} \right]$$

$$= E[y + (r)] P[r, x] E^{-1}[\langle o + (r) \rangle]$$

$$= E[y + (r)] E^{-1}[\langle x + (r) \rangle] \quad (\text{non commutative product})$$

$$= u_2 L[-\Phi(\#|x)]^{-1} u_2 L[-\Phi(r|y)]$$

$$= u_2 L[-\Phi(r|y)] u_2 L[-\Phi(r|x)]^{-1}$$

Matrices  $\tau(r|..)$  multiply commutatively, not closed with respect to multiplication, but transform according to the rule

$$\tau(r|x/y) \tau(r|y/z) = \tau(r|x/z)$$

$$\tau(o[r]|x/o) = P[r|x]$$

$$\tau(x[r]|x/y) = P[r|y-x]^{-1} \left\{ \tau(x[r]|y/x) = P[r|y-x] \right. \left. \Rightarrow \text{replace} \right\}$$

$$\{\tau(r|x/y)\}^{-1} = \tau(r|y/x) \quad || \quad \tau(r|x/x) = I[r]$$

$\tau(1|x/y)$  is \*-symmetric

$$\tau(r|x/y) \circ \overline{\Pi}[r|y] = E[y + (r)] P[r, x] E^{-1}[\langle o + (r) \rangle] - E[r] P[r, y]$$

$$= \text{diag } E[(r)] P[r, x] P[r, y]^{-1} E^{-1}[\langle o + (r) \rangle] E[r] P[r, y]^*$$

$$= \text{diag } E[(r)] P[r|\gamma(\circ)]^{-1} P[r, x] I[r]^*$$

$$= E[r] P[r, x]^* = \overline{\Pi}[r|x] \quad || \quad r \overline{\Pi}[r|y] \tau(r|x/y) = \overline{r \Pi}[r|x]$$

$$u2\mathcal{L}[\langle y + (r) \rangle - x] \top (r|x/y) = \text{displ}$$

$$\text{displ} u2\mathcal{L}[-\bar{\Phi} \cancel{(r)} [r|y]]$$

$$= \text{displ } D' \cancel{\Phi}^{-1}(r)y)$$

$$D\bar{\Phi}(r|x)\bar{\pi}[r|y] = \frac{c\bar{\Phi}\pi[r|x] - c\bar{\Phi}\pi[r|y]}{x-y}$$

$$\pi[r] = \pi(r|\omega)$$

$$\pi[r] = \pi(r|\omega)$$

$$\pi(r) = \pi(r(\omega))$$

$$\bar{D}\bar{\Phi}'(r) := u2u[-\bar{\Phi}(r)]$$

$$\bar{D}\bar{\Phi}(r|x)\bar{\pi}[r|y] = \frac{c\bar{\Phi}\bar{\pi}[r|x] - c\bar{\Phi}\bar{\pi}[r|y]}{x-y}$$

Displacement

$$D\bar{\Phi}(r|\gamma(\tau))\bar{\pi}[r] = \text{col}[\pi(r(\tau))] \langle z := [r] \rangle$$

$$(-\pi(r)?)$$

$$D\bar{\Phi}(r|\gamma(\tau))\bar{\pi}(r) = \text{col}[\pi(r[z, l])] \langle z := [r] \rangle$$

$$(-\pi(r)?)$$

$$\lambda[r, F] : \lambda(r[\omega], F) \langle \omega := [r] \rangle; \lambda'[r, F] : \lambda(r[\omega_H], F) \langle \omega := [r] \rangle$$

$$\bar{\Phi}\pi[r|x]^{-1} \{ \lambda'[r, F|x] D\bar{\Phi}(r|x)\pi[r|y] - D\bar{\Phi}(r|x)\lambda[r, F|y]\pi[r|y]$$

$$= \lambda'[r, F|x] - \lambda'[r, r|y]$$

$$N[P] := \mathbb{Z}[\mu(P[\omega, \infty])] \quad c\Delta[P, F] = \text{wt } [\delta(P[\omega], F)]$$

$$M[P]c[F] = c\Delta[P, F] \quad (\because c\Delta := Mc) ? \quad [M[P]F]^o ?$$

$$b[P] := \text{diag}[\mu(P[\omega])\omega] \quad N[P]c[F] = c\Delta[P, F']$$

$$F' = \underline{F}$$

$$b[P]^{-1}c\Delta[P, F] = b[P]^{-1}M[P]c[F] = [b^{-1}, N]c[F]$$

$$(\because \text{def. } b[P] \text{ as } \text{diag} \left[ \frac{1}{\mu(P[\omega])\omega} \right] = [\pi(P[\omega])\delta(\omega)])$$

$$= dM^{-1}[P] ?$$

$$\tau(r, 0 | x, y) := \text{val}[\langle x + (r) \rangle - y]$$

$$\tau(r, \omega | x, y) := \overline{\text{disp}}^{|\Gamma|-\omega-1} \begin{pmatrix} 1 & 0 \\ x - \delta(\omega) & 1 \end{pmatrix} \quad \langle \omega := (r) \rangle$$

$$\tau(k, r | x, y) = \overline{\mathcal{L}}[\pi(r(k+z), k+z) | x] \quad \langle z, \omega := |\Gamma| - k \rangle$$

$$k := (r)$$

$$\tau(0, r | x, y) = \tau(r | x, y)$$

$$\text{disp} \tau(h+1, r | x, y) \tau(r, h | x, y) = \tau(h, r | x, y)$$

$$\text{disp}^{k-h} \tau(k, r | x, y) \left\{ \prod \text{disp}^{\omega-h} \tau(r, \omega | x, y) \langle \omega := [h, k] \rangle \right\}$$

$$= \tau(h, r | x, y)$$

$\tau$  \*-symmetric  $\bar{\tau}(k, r | x, y) = \tau(k, r | x, y)^T$ . Corresponding relationships directly available.

$\tau(r | x, y)^{-1} = \tau(r | y, \underline{x})$  leads directly to decomposition & inverse.

$$D\bar{\Phi}(k, r] = \mathcal{L}[\pi(r[k+z, k+z])] \quad \langle z, \omega := [r| - k] \rangle$$

$$D'\bar{\Phi}'(k, r] = u2\mathcal{L}[-\bar{\Phi}(r[k])]$$

$$d\phi(r, \omega) = \overline{disp}^{|r|-\omega-1} \begin{pmatrix} 1 & 0 \\ x-\gamma(\omega) & 1 \end{pmatrix} \quad \frac{1}{\phi(r|\omega)} ?$$

$$disp D\bar{\Phi}(h+1, r] x(r, h) = D\bar{\Phi}(h, r]$$

$$D\bar{\Phi}(h, r] = disp^{k-h} D\bar{\Phi}(k, r] \Pi disp^{\omega-h} d\phi(r, \omega) \quad \langle \omega := [h, k] \rangle$$

$$\bar{D}\bar{\Phi}(k, r] = u2\mathcal{L}[\pi(r[k+z, k+z])]$$

$$D'\bar{\Phi}'(k, r] = u2u[-\bar{\Phi}(r[k])]$$

$$\bar{d}\phi(r, \omega) = disp^{|r|-\omega-1} \begin{pmatrix} 1 & \infty - \gamma(|r|-\omega-1) \\ 0 & 1 \end{pmatrix} \quad \phi(r| |r|-\omega)$$

$$\bar{D}\bar{\Phi}(h, r) = \overline{disp} \bar{D}\bar{\Phi}(h+1, r) \bar{d}\phi(r, h)$$

=

T satisfies

$$\left\{ \sum \mu(r(|r|), \omega) T(r|\gamma(\omega)/y) \quad \langle \omega := (r) \rangle \right\} = 0$$

$$\left\{ \sum \mu(r(|r|), \omega) T(r|\gamma(\omega)/y) \quad \langle \omega := (r) \rangle \right\} = I[r] - \mathcal{L}I[r]^{-1}$$

=

$$B \in \text{dom} \quad a, b: B \rightarrow T \quad T: B \times B \rightarrow T' \quad W: B \rightarrow T$$

$$Wa = b \quad \langle B \rangle + T(x, Y) a(Y) = a(x)$$

$$\Rightarrow \int_W(x) T(x, Y) W(Y)^{-1} b(Y) = b(x)$$

=

$$W[B/r] = E[B] E[r]^{-1}$$

addition to Bidirectional matrix mappings derived from sequence mappings

## ii) Permanent matrix mappings

a)  $\text{perm}[\equiv]$  is the function  $\Theta$  occurring in the matrix mapping

$\Theta: B \rightarrow [K, n_H]$  defined by taking for each  $z \in B$ ,  $\Theta(z)$  to be

the matrix for which  $\Theta(z)_x^y = 1$ ,  $\Theta(z)_{xH}^y = \frac{1}{2}(z|x)$ ,  $\Theta(z)_{x+1}^y = 1 - \frac{1}{2}(z|x)$   $\langle x := [n(z)] \rangle$ , (here differs from previous usage  $\Theta[i, \omega] = \dots$ )  
the remaining element being zero.

b)  $\overline{\text{perm}}[\equiv]$  is the function  $\lambda$  occurring in the matrix mapping  $\lambda: B \rightarrow [K, n_H]$

defined by taking for each  $z \in B$ ,  $\lambda(z)$  to be the matrix for which  $\lambda(z)_{n(z)H}^{n(z)H} = 1$ ,

$\lambda(z)_x^y = 1 - \frac{1}{2}(z|x)$ ,  $\lambda(z)_x^{yH} = \frac{1}{2}(z|x)$   $\langle x := [n(z)] \rangle$ , the

remaining elements being zero.

=

Square Triangular matrix mappings derived from sequence mappings

a)  $L[\equiv]$  is the function  $\Theta$  occurring in the matrix mapping  $\Theta: B \rightarrow [K, n_H]$

defined by taking, for each  $z \in B$ ,  $\Theta(z)$  to be the matrix

for which  $\Theta(z)_z^y = \frac{1}{2}(z|z-y)$   $\langle z := [n(z)], y := [z] \rangle$ , the

remaining elements being zero.

b)  $U[\equiv]$  is the function  $\lambda$  occurring in the matrix mapping  $\lambda: B \rightarrow [K, n_H]$

defined by taking, for each  $z \in B$ ,  $\lambda(z)$  to be the matrix for which

$\lambda(z)_z^y = \frac{1}{2}(z|y-z)$   $\langle y := [n(z)], z := [z] \rangle$ , the remaining elements

being zero.

$I[\equiv]$  becomes  $L[I[\equiv]]$        $L[I[\equiv]]^{-1}$  becomes  $\text{ann}[I[\equiv]]$

$U[I[\equiv]]$        $U[I[\equiv]]^{-1}$        $\text{ann}[I[\equiv]]$

$\text{perm}[\Theta[\equiv]]$  is  $I[\equiv]$ , also  $\overline{\text{perm}}[\Theta[\equiv]]$ ;  $\overline{U[\equiv]} = \overline{L[\equiv]}$

$U[I[\equiv]] = L[I[I[\equiv]]] = L[I[\equiv]]^T$

1. 1. \* - symmetric

## Sequences

$$\pi[r] = \pi(r[\tau]) \quad \langle z := [r] \rangle \quad \pi(r[<[r]>])? \quad \text{simply } \pi(r[[r]])$$

$$= \pi(r[\langle r \rangle]) \quad \text{this is } \pi(r[i, \langle r \rangle]) \text{ with } \begin{array}{l} \text{by direct} \\ i=0 \end{array} \text{extension}$$

*(not described in notes)*

$$\pi'[r] = \pi(r[\tau]) \quad \langle z := [r] \rangle$$

$$= \pi(r[\langle r \rangle])$$

*redraft notes in terms of either*  
 $[i, \Xi] \leftrightarrow [\Xi, j]$

$$\pi''[r] = \pi(r[\tau]) \quad \langle z := [r] \rangle$$

$$= \pi(r[\langle r \rangle])$$

$[i, \Xi] \nrightarrow$  for  $i=0$   
 or  $[i, \Xi]$  alone if this  
 fits in with transformation  
 properties of  $M[[h, k]]$

$$\bar{\pi}[r] = \pi(\bar{r}[|r|-z, |r|])$$

$$= \pi(\bar{r}[\tau]) \quad \langle z := [r] \rangle$$

$$= \pi(\bar{r}[\langle r \rangle]) \quad \text{as described in notes}$$

*simply  $\pi(\bar{r}[[r]])$*   
*by direct extension*

$$\bar{\pi}'[r] = \pi(r(|r|-z, |r|))$$

$$= \pi(\bar{r}(\tau)) \quad \langle z := [r] \rangle$$

$$= \pi(\bar{r}[\langle r \rangle]) \quad \text{as described in notes}$$

$\langle [r] \rangle$   
 written as  $\langle r \rangle$  when  
 $r \in \text{seq}[K] ??$

$$\bar{\pi}''[r] = \pi(r(|r|-z, |r|))$$

$$= \pi(\bar{r}[\langle r \rangle])$$

*consider writing  $[r]$  as*

$\langle r \rangle$   
*leading to  $\pi(r[\langle r \rangle])$*   
*& for  $\Pi$  etc*

$\bar{\Phi}[r] : \pi(r[\omega, \omega]) \quad \langle \omega := [r] \rangle$

$$= \pi(r[0, 0] \langle [r] \rangle) \quad \text{as defined in notes}$$

$$\bar{\Psi}[r] = \pi(r[\omega]) \pi(\bar{r}(\bar{\omega})) \quad \langle \omega := [r] \rangle$$

$$\mu[\Gamma] \quad \mu(\Gamma, \omega) \langle \omega := [\Gamma] \rangle \\ \mu(\Gamma, \langle [\Gamma] \rangle)$$

$$\theta[\Gamma] \quad \theta(\Gamma, \omega) \langle \omega := [\Gamma] \rangle \\ \theta(\Gamma, \langle [\Gamma] \rangle)$$

$$dM[\Gamma] \quad \frac{1}{\pi(\Gamma(\omega) | \gamma(\omega))} \langle \omega := [\Gamma] \rangle$$

$$dM^{-1}[\Gamma] \quad \pi(\Gamma(\omega) | \gamma(\omega)) \langle \omega := [\Gamma] \rangle$$

$$d\bar{M}[\Gamma] \quad \frac{1}{\pi(\Gamma(\omega, |\Gamma|) | \gamma(\omega))} = \frac{1}{\pi(\bar{\Gamma}(\bar{\omega}) | \gamma(\omega))} \langle \omega := [\Gamma] \rangle$$

$$d\bar{M}^{-1}[\Gamma] = \pi(\bar{\Gamma}(\bar{\omega}) | \gamma(\omega)) \langle \omega := [\Gamma] \rangle$$

$$p\omega[\Gamma] \quad p\omega[\Gamma | z] = z^\omega \langle \omega := [\Gamma] \rangle \\ = z^{[\Gamma]} \quad \overline{p\omega}[\Gamma] = \overline{z}^{|\Gamma|-\omega} \langle \omega := [\Gamma] \rangle \\ = \overline{z}^{|\Gamma|-<\Gamma>}$$

$$\Phi\pi' = \phi(0)\pi'' = \pi \quad \bar{\Phi}\bar{\pi}' = \phi(|\Gamma|)\bar{\pi}'' = \bar{\pi}$$

$$\bar{\Psi} = \pi' \bar{\pi}' \quad \Theta = \mu \bar{\Psi} \quad \mu = dM \cdot d\bar{M} \quad dM^{-1} = \frac{1}{dM} \quad d\bar{M}^{-1} = \frac{1}{d\bar{M}}$$

$$p\omega[\#|1] = \perp$$

conjugates  $\bar{\pi} \pi' \pi''$   $dM$  are  $\bar{\pi}, \bar{\pi}', \bar{\pi}''$ ,  $d\bar{M}$ ,  $\overline{p\omega}$

$\bar{\Phi}, \bar{\Psi}, \mu, \Theta$  self conjugate

## The binomial function

Let  $h, i, j, k \in \text{fin}\{\bar{\aleph}[B]\}$ , and  $P \in \text{seq}[K]$ .

Let  $P \in \text{seq}[K]$  and  $h, i, j, k \in [P]$ .  $B(P[i, j] / [h, k])$  is the function  $\phi$  occurring in the mapping  $\phi \rightarrow$  Set  $M := [h, k]$  when  $j < h$  or  $k < i$ ,  $M := (j, k)$  when  $h \leq j$

$B(P[i, j] / [h, k])$  is the function  $\phi$  occurring in the mapping  $\phi: B \rightarrow K \setminus \{[h, k] \setminus \{[i, j]\}\}$  defined as follows. When either  $j < h$  or  $k < i$ ,  $\phi$  is

$$\phi \mapsto \frac{\pi(P[i, j])}{\pi(P[h, k])} \quad (j < h \text{ or } k < i)$$

and if  $\frac{\pi(P[i, h])}{\pi((j, k))} \quad (h < j \text{ and } h \leq j \leq k)$

$$\phi := \frac{\pi(P(k, j))}{\pi(P[h, i])} \quad (i \leq k \leq j)$$

$$\frac{\pi(P[i, h]) \pi(P(k, j))}{\pi(P[h, i]) \pi(P(j, k))} \quad (\text{otherwise})$$

$B(P[i, j-i] / [h, k])$  is written as  $B(P[i, j] / [h, k])$  and  $B(P[0, j] / [h, k])$  as  $B(P[j] / [h, k])$ ; analogous contractions are also employed.

conjugate of  $\text{wla}[\equiv]$  is  $\text{wla}[\equiv]$

$$\bar{z} := |z| - z$$

$$\bar{\Gamma}[z] = \gamma[(|z|-z, |z|)]$$

$$\bar{\Gamma}(z) = \gamma(|z|-z, |z|)$$

$$\bar{\Gamma}(|z|) = \gamma(|z|) = (r)$$

$$\Gamma[z] = \gamma[z] \quad \Gamma[z] = \gamma[z]$$

$$\Gamma(z) = \gamma(z)$$

conjugate of  $\bar{\Gamma}(z)$  is  $\gamma(z) = \Gamma(z) \neq \Gamma$  if  $\Gamma$  is  $\bar{\Gamma}$  of  $(r)$  in  $(r)$

$$\text{if } \bar{\Gamma}(z) \text{ is } \Gamma(z)$$

$$\omega := [z, |z|] = \omega := [|\bar{z}|, \bar{\Gamma}(\bar{z})] \text{ i.e. } \omega := [\bar{\Gamma}(\bar{z})] \quad \text{then} \\ \omega := (z, |z|) = \omega := \omega[\bar{\Gamma}(\bar{z})] \quad \text{etc.} \quad \omega := [\bar{\Gamma}(\bar{z})] \\ \equiv \omega := [\bar{\Gamma}(\bar{z})]$$

$$\pi[r] \langle \pi(\bar{\Gamma}(z)) \rangle z := [r] \parallel \bar{\pi}(r) \pi(r(z, |z|)) \parallel \bar{\pi}[r] \\ \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad \text{is} \\ \text{in above iteration}$$

$$\bar{\pi}(r) \text{ conjugate of } \pi(r) \\ \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$\text{col}[\Gamma(r)] \text{ written as } \bar{\pi}(r) \text{ etc}$$

=

In exposition  $\sim - *$  have considered  $\sim$  as rearrangement and then reversal to obtain

In above  $\sim$  is rearrangement of  $r$  but reversal occurs in sequences of different lengths

In above iteration  $\bar{\Gamma}(|z|) = (r)$  in alternative  $\bar{\Gamma}(0) = (r)$   
conjugate of  $\Gamma(z)$  is  $\bar{\Gamma}(\bar{z})$  .. and is  $\bar{\Gamma}(\bar{z})$

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$|z| - z, |z|$

Mexico

$0, |z| - z$

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Consider  $\bar{\Gamma}[z] = \gamma[z, |z|]$  etc  
when conjugate  $\Gamma(z)$  is  $\bar{\Gamma}(\bar{z})$

$\Gamma$  is  $\bar{\Gamma}$  of  $(r)$  in  $(r)$

$$\omega := [z, |z|] = \omega := [|\bar{z}|, \bar{\Gamma}(\bar{z})] \text{ i.e. } \omega := [\bar{\Gamma}(\bar{z})] \quad \text{then} \\ \omega := (z, |z|) = \omega := \omega[\bar{\Gamma}(\bar{z})] \quad \text{etc.} \quad \omega := [\bar{\Gamma}(\bar{z})]$$

$$\pi[r] \langle \pi(\bar{\Gamma}(z)) \rangle z := [r] \parallel \bar{\pi}(r) \pi(r(z, |z|)) \parallel \bar{\pi}[r] \\ \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad \text{is} \\ \text{in above iteration}$$

$$\bar{\phi}(r) \pi(\gamma(z, z)) \text{ and} \\ \pi[r] = \bar{\phi}(r) \pi(r) \quad \langle \pi(\bar{\Gamma}(z)) \rangle \\ = \bar{\phi}(r) \pi(r) \quad \text{in alternative iteration}$$

$$\bar{\pi}(r) = \bar{\pi}(r) \bar{\phi}(r)$$

etc

use  $\bar{\pi}(r) \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$   
etc dispenses with  $\bar{\pi}(r) \bar{\pi}(r)$   
etc

Use  $\bar{\Gamma}(z)$  etc dispenses with  $\Gamma(z)$  etc.

$$\gamma(x) - \gamma(x+|\Gamma|-\omega\tau) \quad \chi := [\omega] \quad \omega - 1 + |\Gamma| - \omega\tau = |\Gamma|$$

$$\Gamma[\omega] - \bar{\Gamma}(\omega)$$

$$\textcircled{34} = \mathbb{D} := \text{nsseq}(K) \quad \mathbb{D}'(\mathbb{D}) = \{[\mathbb{D}]\} \quad \mathbb{D}''(\mathbb{D}) = \{[\mathbb{D}]\} \quad \begin{matrix} \mathbb{N}'(\mathbb{D}) \\ \mathbb{N}''(\mathbb{D}) \end{matrix} ;$$

$\Gamma \in \mathbb{D}$  is  $\gamma: \{[\Gamma]\} \rightarrow K$  ( $\{|\Gamma|\} \rightarrow K?$ )

$$m, m^{-1}: \mathbb{D} \times \mathbb{D}'(\mathbb{D}) \rightarrow \mathcal{L}[K | |\mathbb{D}| - \mathbb{D}'(\mathbb{D})]$$

$$m[\Gamma, \omega] := \text{ann } \mathcal{L}\left[\{\gamma(x+\omega+1) - \gamma(x)\}^{-1}\right] \langle x := [\bar{\omega}] \rangle$$

$$m^{-1}[\Gamma, \omega] := \text{diag } [\gamma(x+\omega+1) - \gamma(x)] \langle x := [\bar{\omega}-1] \rangle$$

$$\mu(k; \Gamma[\tau], \nu) = \delta\left(\Gamma[\nu], \frac{1}{\pi(\Gamma[k+\tau], k+\tau)}\right) - \delta\left(\Gamma[\nu], \frac{1}{\pi(\Gamma[k+\tau], k+\tau)}\right)$$

$$\mu^{-1}(k; \Gamma[\tau], \nu) = \delta\left(\Gamma[k+\tau], \pi(\Gamma[\tau])\right)$$

$$M, M^{-1}: \mathbb{D}''(\mathbb{D}) \times \mathbb{D} \rightarrow \mathcal{L}[K, |\mathbb{D}| - |\mathbb{D}''(\mathbb{D})|]$$

$$M[k, \Gamma] = \mathcal{L}\left[\mu(k; \Gamma[\tau], \nu)\right] \langle \tau := [\bar{k}], \nu := [\tau] \rangle$$

$$M^{-1}[k, \Gamma] = \mathcal{L}\left[\mu^{-1}(k; \Gamma[\tau], \nu)\right]$$

$$\{m[\Gamma, \omega]\}^{-1} = m^{-1}[\Gamma, \omega] \langle \omega := [\nu] \rangle$$

$$\mu(k; \Gamma[\tau], \nu) = \delta\left(\Gamma[k+\tau], \frac{1}{\pi(\Gamma[\nu])}\right) - \delta\left(\Gamma[k+\tau], \frac{1}{\pi(\Gamma[\nu])}\right)$$

$$(k > 0) \quad = \{\gamma(\omega) - \gamma(k+\omega)\} \delta\left(\Gamma[\nu], \frac{1}{\pi(\Gamma[k+\tau], k+\tau)}\right) \\ = \{\gamma(k+\omega) - \gamma(\omega)\} \delta\left(\Gamma[k+\tau], \frac{1}{\pi(\Gamma[\nu])}\right)$$

$$\mu^{-1}(k; \Gamma[\tau], \nu) = \delta\left(\Gamma[\tau, k+\tau], \pi(\Gamma[k+\tau])\right)$$

$$M[k, \Gamma] = \left\{ \mathcal{L}\left[-\delta\left(\Gamma[\nu], \frac{1}{\pi(\Gamma[k+\tau], k+\tau)}\right)\right] \langle \tau, \nu := [\bar{k}] \rangle \right\}^{-1} \\ = \left\{ \mathcal{L}\left[\delta\left(\Gamma[k+\tau], \frac{1}{\pi(\Gamma[\nu])}\right)\right] \right\}^{-1}$$

$$M'[\Gamma[h, |\Gamma|]]^{-1} = [\mu(\Gamma[h, h+z]_2) \quad z := \bar{h}, \bar{z} := [z]]$$

$$\uparrow_{h \in ?}$$

$$M'[\Gamma[h, k_H]] = m'[\Gamma, k] M'$$

$$[I[k-h] + m'[\Gamma, k]] M'[\Gamma[h, k]]$$

$$\{M'[\Gamma[h, k]]\}^{-1} = \{m'[\Gamma[h, k_H]]\}^{-1} [I[k-h] + m'[\Gamma, k]]$$

$$m'[\Gamma, |\Gamma|-1] = \frac{1}{1 - \gamma_{|\Gamma|} - \gamma_{|\Gamma|-1}}$$

last 3 terms of  $M'[\Gamma[h, |\Gamma|-1]]$

$$\mu(\Gamma[h, |\Gamma|-1], |\Gamma|-1) = \frac{1}{\pi(\Gamma[h, |\Gamma|-2]) \gamma_{|\Gamma|-1}}$$

$$\mu(\Gamma[h, |\Gamma|], |\Gamma|-1) + \mu(\Gamma[h, |\Gamma|], |\Gamma|) \quad \mu(\Gamma[h, |\Gamma|], |\Gamma|) (\gamma_{|\Gamma|} - \gamma_{|\Gamma|})$$

$$\frac{\frac{1}{\pi(\Gamma[h, |\Gamma|-2]) \gamma_{|\Gamma|-1}}}{\pi(\Gamma[h, |\Gamma|-2]) \gamma_{|\Gamma|-1}} \quad \frac{1}{\pi(\Gamma[h, |\Gamma|-1]) \gamma_{|\Gamma|}}$$

$$\frac{1}{\pi(\Gamma[h, |\Gamma|-2]) \gamma_{|\Gamma|}} \quad \frac{1}{\pi(\Gamma[h, |\Gamma|])} \quad \frac{1}{\pi(\Gamma[h, |\Gamma|])}$$

$$= \delta(\Gamma[|\Gamma|-1, |\Gamma|], \frac{1}{\pi(\Gamma[h, |\Gamma|-1])}) \quad \delta(\Gamma[z, \tau] \frac{1}{\pi(\Gamma[h, z])})$$

$$z=1 \quad \lambda=0$$

except when  $\lambda = z = |\Gamma|$

factor in numerator

$$\mu(\Gamma[h, h+1], h) = \frac{1}{\gamma_h - \gamma_{hH}}$$

$$\pi(\Gamma[|\Gamma|-1, z])$$

$$\{M[k, \Gamma]\}^{-1} = M^{-1}[k, \Gamma] \langle k := [\Gamma] \rangle$$

$$h \in [\Gamma] \quad k \in [h, |\Gamma|] \quad (k \in [\bar{\Gamma}[h]]?)$$

$$M[h, \Gamma] = \text{disp}^{k-h} M[k, \Gamma] \{ \prod \text{disp}^{\omega-h} m[\Gamma, \omega] \langle \omega := [h, k] \rangle \}$$

$$k = h+1$$

$$M[h, \Gamma] = \text{disp} M[h+1, \Gamma] m[\Gamma, h]$$

leads to

$$M[k, \Gamma] = \prod \text{disp}^{\omega-k} m[\Gamma, \omega] \langle \omega := [k, |\Gamma|] \rangle \quad \langle \omega := [\Gamma[\bar{k}]] \rangle$$

$$M[\Gamma] = M[0, \Gamma]$$

$\kappa'$

$$? \quad \bar{m}, \bar{m}^{-1}: \mathbb{D} \times \{\mathbb{D}'(\mathbb{D})\} \rightarrow \mathcal{U}[K | |\mathbb{D}| - \mathbb{N}'(\mathbb{D})]$$

$$\bar{m}[\Gamma, \omega] := \text{ann} \mathcal{U}[\{\gamma(x) - \gamma(x_{|\omega+1})\}^{-1}] \langle x := [\bar{\omega}] \rangle$$

$$\bar{m}^{-1}[\Gamma, \omega] := \mathcal{U}[\bar{\omega}] \overline{\text{disp}} \underset{\text{first iteration}}{\text{diag}} [\gamma(x) - \gamma(x_{|\omega+1})] \langle x := \cdot \rangle$$

$$\mu(k; \bar{\Gamma}[\bar{\tau}], \nu) \approx \delta\left(\bar{\Gamma}[\bar{k}_{\tau}], \frac{1}{\pi(\bar{\Gamma}[\bar{\tau}, \bar{\nu}])}\right) - \delta\left(\bar{\Gamma}(\bar{k}_{\nu}), \frac{1}{\pi(\bar{\Gamma}[\bar{\tau}, \bar{\nu}])}\right)$$

$$\mu^{-1}(k, \Gamma[\bar{\nu}], \bar{\tau}) = \delta(\Gamma[\bar{\tau}, k + \bar{\nu}], \pi(\bar{\Gamma}[\bar{\nu}]))$$

$$\bar{M}, \bar{M}^{-1}: \{\mathbb{D}''(\mathbb{D})\} \times \mathbb{D} \rightarrow \mathcal{U}[K | |\mathbb{D}| - \mathbb{N}''(\mathbb{D})]$$

$$? \quad \bar{M}[k, \Gamma] = \mathcal{U}[\mu(k; \bar{\Gamma}[\bar{\tau}], \nu)] \langle \nu := [\bar{k}], \tau := [\bar{\nu}] \rangle$$

$$\bar{M}^{-1}[k, \Gamma] = \mathcal{U}[\mu^{-1}(k, \Gamma[\bar{\nu}], \bar{\tau})]$$

$$\{\bar{m}[\Gamma, \omega]\}^{-1} = \bar{m}^{-1}[\Gamma, \omega] \quad \langle \omega := [\Gamma] \rangle$$

$$\mu(k; \bar{r}[\bar{\pi}], \nu) = \delta(r[\pi, \nu], \frac{1}{\pi(\bar{r}[k_{\pi\nu}])}) - \delta(r[\pi, \nu], \frac{1}{\pi(\bar{r}[k_{\pi\nu}])})$$

$$(k>0) = \{\gamma(k+\nu) - \gamma(\nu)\} \delta(\bar{r}[k_{\pi\nu}], \frac{1}{\pi(r[\pi, \nu])})$$

$$= \{\gamma(\nu) - \gamma(k_{\pi\nu})\} \delta(r[\pi, \nu], \frac{1}{\pi(\bar{r}[k_{\pi\nu}])})$$

$$? \quad \mu^{-1}(k, \bar{r}[\nu], \pi) = \delta(r[\pi, \nu], \pi(\bar{r}[k+\pi]))$$

$$\bar{m}[k, r] = u[-\delta(\bar{r}[k_{\pi\nu}], \frac{1}{\pi(r[\pi, \nu])})] u I[\bar{k}]^{-1}$$

$$= u[\delta(r[\pi, \nu], \frac{1}{\pi(\bar{r}[k_{\pi\nu}])})]$$

$$\{\bar{m}[k, r]\}^{-1} = \bar{m}^{-1}[k, r] \quad \langle k := [r] \rangle$$

$$h \in [r] \quad k \in [h, |r|] \quad \langle k \in [\bar{r}[h]] \rangle$$

$$\bar{m}[h, r] = \overline{\text{disp}}^{k-h} \bar{m}[k, r] \prod \overline{\text{disp}}^{\omega-h} \bar{m}[r, \omega] \quad \langle \omega := [h, k] \rangle$$

$$k = h+1$$

$$\bar{m}[h, r] = \overline{\text{disp}} \bar{m}[k, r] \bar{m}[r, h]$$

leads to

$$\bar{m}[k, r] = \prod \overline{\text{disp}}^{\omega-k} \bar{m}[r, \omega] \quad \langle \omega := [\bar{r}[\bar{k}]] \rangle$$

$$\bar{m}[r] = \bar{m}[0, r]$$

$$\tilde{\mu}(k; \Gamma[z]_v) = \delta(\bar{\Gamma}[\bar{k}z], \pi(\Gamma(z)))$$

$$\tilde{\mu}^{-1}(k; \Gamma[z]_v) = \delta(\Gamma[k+z], \pi(\bar{\Gamma}(\bar{z})))$$

$$\bar{M}M^{-1}, M\bar{M}^{-1}: \mathbb{N}''(D) \times D \rightarrow [K_{\mathbb{R}}^2 / |D| - N''(D)]$$

$$\bar{M}M^{-1}[k, \Gamma] := [\tilde{\mu}(k; \Gamma[z]_v)] \quad \langle z, v := [\bar{k}] \rangle$$

$$M\bar{M}^{-1}[k, \Gamma] := [\mu^{-1}(-)]$$

$$\tilde{\mu}(k; \Gamma[z]_v) = 0 \text{ when } z \mapsto < \bar{k}$$

$$\quad \quad \quad = \delta(\bar{\Gamma}[\bar{z}], \pi(\bar{\Gamma}(kz)))$$

$$\tilde{\mu}^{-1}(k; \Gamma[z]_v) = 0 \text{ when } \bar{k} < z \mapsto$$

$$\quad \quad \quad = \delta(\Gamma[z], \pi(\bar{\Gamma}(k+z)))$$

$$\{\bar{M}M^{-1}[k, \Gamma]\}^{-1} = M\bar{M}^{-1}[k, \Gamma] \quad \langle k: -[v] \rangle$$

$$\bar{M}M^{-1}[k, \Gamma] = \bar{M}[k, \Gamma] M^{-1}[k, \Gamma] \quad \dots$$

=

$$\gamma(x + \omega h) \quad \langle x := [\bar{\omega}] \rangle = \bar{\Gamma}(\bar{\omega}) \quad \gamma(x) .. \equiv \Gamma(\bar{\omega})$$

$$\therefore m[\Gamma, \omega] = \text{ann} L \left[ \frac{1}{\bar{\Gamma}(\bar{\omega}) - \Gamma(\bar{\omega})} \right]$$

$$\frac{1}{\delta_0 \vec{x}_1} \frac{1}{\delta_1 \vec{x}_0}$$

$$m^{-1}[\Gamma, \omega] = L[\bar{\omega}] \text{ disp diag } [\bar{\Gamma}(\bar{\omega}) - \Gamma(\bar{\omega})]$$

$$\frac{1}{\gamma_1 \vec{x}_2} \frac{1}{\delta_2 \vec{x}_1}$$

$$\bar{m}[\Gamma, \omega] = \text{ann} U \left[ \frac{1}{\Gamma(\bar{\omega}) - \bar{\Gamma}(\bar{\omega})} \right]$$

$$\frac{1}{\gamma_0 \vec{x}_1} \frac{1}{\gamma_1 \vec{x}_0}$$

$$\bar{m}^{-1}[\Gamma, \omega] = U[\bar{\omega}] \text{ disp diag } [\Gamma(\bar{\omega}) - \bar{\Gamma}(\bar{\omega})]$$

$$\frac{1}{\gamma_1 \vec{x}_2} \frac{1}{\delta_2 \vec{x}_1}$$

ann for ann L,  $\overline{\text{ann}}$  for ann U?

$$\text{ann}[\Xi] = \text{disp diag}[\Xi] L[\|\Xi\|+1]$$

$$\text{ann}[\Xi]^{-1} = L[\|\Xi\|+1]^{-1} \text{ disp diag}[\frac{1}{\Xi}]$$

$$\overline{\text{diag}[\Xi]} = \text{diag}[\Xi] \quad \overline{\text{disp}^\omega \text{diag}[\Xi]} = \overline{\text{disp}^\omega} \text{diag}[\Xi]$$

$$\overline{\text{am}[\Xi]} = \overline{\text{disp}} \text{diag}[\Xi] \& I[|\Xi|+1]$$

$$\overline{\overline{\overline{\Gamma(\omega) - \Gamma(\bar{\omega})}}} = \Gamma(\omega) - \bar{\Gamma}(\bar{\omega})$$

$$M[\Gamma(h, k)]_z^2 =$$

$$\mu(\Gamma(h, k)|z, \bar{z}) =$$

$$\delta(\Gamma[h+\nu, h+z], \frac{\pi(\Gamma[z+h-k])}{\pi(\Gamma[\nu])}) - \delta(\Gamma[h+\nu, h+\bar{z}], \frac{\pi(\Gamma[z+h-k])}{\pi(\Gamma[\nu])})$$

$$= \{\gamma(\nu) - \gamma(h+\nu)\} \delta\left(\Gamma[h+\nu, h+z], \frac{\pi(\Gamma[z+h-k])}{\pi(\Gamma[\nu])}\right) \quad (h > 0)$$

$$M[\Gamma(h, k)] = [\mu(\Gamma(h, k)|z, \bar{z})] \langle z := [\bar{h}], \bar{z} := [z] \rangle$$

$$= \left[ \delta\left(\Gamma[h+\nu, h+z], \frac{\pi([z+h-k])}{\pi(\Gamma[\nu])}\right) \right] \cdot S_I[\bar{h}]^{-1}$$

$$M[\Gamma(h, k)] = \overline{\overline{\text{disp}^{\omega-h} m[\Gamma, \omega]}} \quad \langle \omega := [h, k] \rangle$$

$$(m[\Gamma, \omega] = \text{am} \mathcal{Z} \left[ \frac{1}{\bar{\Gamma}(\bar{\omega}) - \Gamma(\bar{\omega})} \right])$$

$$\mu^{-1}(\Gamma(h, k)|z, \bar{z}) =$$

$$\delta\left(\Gamma[h+\nu, h+z], \frac{\pi(\Gamma[z])}{\pi(\Gamma[z+k-h])}\right) \frac{\pi(\Gamma[z-k+h] | \gamma(\omega+k))}{\pi(\Gamma[z-k+h] | \gamma(\omega+h))}$$

$$\{M[\Gamma(h, k)]\}^{-1} = [\mu^{-1}(\Gamma(h, k)|z, \bar{z})] \langle z := [\bar{h}], \bar{z} := [z] \rangle$$

$$k=|\Gamma|$$

term  $\pi(\Gamma[z+h-k])$  may be deleted from  $\mu$  expressions

terms  $\pi(\Gamma[\vartheta+h-k]), \pi(\Gamma[\vartheta-k+h]|\gamma(\vartheta+h)), \pi(\Gamma[\vartheta-k+h]|\gamma(\vartheta+h))$

in  $\mu^{-1}$  expressions vanish except when  $\vartheta=|\Gamma|-h$ , i.e.  $z=|\Gamma|-h$  as well; single term involved. This  $\mu^{-1}$  term becomes

$$\delta(\Gamma[|\Gamma|, |\Gamma|] \frac{\pi(\Gamma[z])}{\pi(\Gamma[0])}) \pi(\Gamma[0]|\gamma(|\Gamma|)) - \pi(\Gamma[|\Gamma|-h]|\gamma(|\Gamma|)).$$

and above terms may be deleted in this case also

$$\mu(\Gamma[h, |\Gamma|] | z, \vartheta) =$$

$$\delta(\Gamma[\vartheta], \frac{1}{\pi(\Gamma[h+\vartheta, h+z])}) - \delta(\Gamma[\vartheta], \frac{1}{\pi(\Gamma[h+\vartheta, h+z])})$$

$$= \text{prev. } \mu(h; \Gamma[z], \vartheta)$$

$$\mu^{-1}(\Gamma[h, |\Gamma|] | z, \vartheta) = \delta(\Gamma[h+\vartheta, h+z], \pi(\Gamma[z])) \\ = \text{prev. } \mu^{-1}(h; \Gamma[z], \vartheta)$$

$$h=0$$

$$\pi \text{disp}_m[\Gamma, \omega] \langle \omega := [k] \rangle = [\pi(\Gamma[z-k]|\gamma(\omega)) \mu(\Gamma[z], \omega)] \\ \langle z := [\vartheta], \vartheta := [z] \rangle \\ = M[\Gamma[k]]$$

$$M[\Gamma[k]]^{-1} = \left[ \pi(\Gamma[\vartheta]|\gamma(z)) \frac{\pi(\Gamma[\vartheta-k]|\gamma(\vartheta))}{\pi(\Gamma[\vartheta-k]|\gamma(\vartheta)) \pi(\Gamma[\vartheta-k], \gamma(\vartheta))} \right] \\ \langle z := [\vartheta], \vartheta := [z] \rangle$$

$$\bar{\mu}(P(i,j] | \tau, \omega) =$$

$$\delta\left(P[\tau, \omega], \frac{\pi(\bar{P}(i-\tau])}{\pi(\bar{P}(j-\omega])}\right) - \delta\left(P[\tau, \omega], \frac{\pi(\bar{P}(i-\omega])}{\pi(\bar{P}(j-\omega])}\right)$$

$$= \{\delta(\bar{j}+\omega) - \delta(\omega)\} \delta\left(P[\tau, \omega], \frac{\pi(\bar{P}(i-\tau])}{\pi(\bar{P}(j-\omega])}\right) \quad (j < |P|)$$

$$\bar{M}[P(i,j)] = [\bar{\mu}(P(i,j] | \tau, \omega)] \quad \langle \omega := [j], \tau := [\omega, j] \rangle$$

$$= \left[ \delta\left(P[\tau, \omega], \frac{\pi(\bar{P}(i-\tau])}{\pi(\bar{P}(j-\omega])}\right) \right] \cdot \bar{W}[j]^{-1}$$

$$\bar{m}[P, \omega] = \bar{\alpha}_{\text{unif}} \left[ \frac{1}{P[\omega] - \bar{P}(\omega)} \right] \quad \langle \omega := (P) \rangle$$

$$\bar{m}[P(i,j)] = \bar{\prod} \text{disp } j-\omega \bar{m}[P, \omega] \quad \langle \omega := (i, j) \rangle$$

conjugate of  $m[P, \omega]$  is  $\bar{m}[P, \bar{\omega}]$   $\langle \omega := (P) \rangle$

conjugate of  $M[P(h,k)]$  is  $\bar{M}[P(\bar{k}, \bar{h})]$   $\langle k := [P], h := [k] \rangle$

$$\bar{\mu}^{-1}(P(i,j] | \tau, \omega) =$$

$$\delta\left(P[\tau, \omega], \frac{\pi(\bar{P}(j-\tau])}{\pi(\bar{P}(i-\omega])}\right) \frac{\pi(\bar{P}(i-\omega]) | \gamma(\omega))}{\pi(\bar{P}(i-\omega]) | \gamma(\omega))}$$

$$\{\bar{M}[P(i,j)]\}^{-1} = [\bar{\mu}^{-1}(P(i,j] | \tau, \omega)] \quad \langle \tau := [j], \omega := [\omega, j] \rangle$$

$i=0$ : term  $\pi(\bar{P}(i-\tau])$  may be deleted from  $\bar{\mu}$  expressions

terms  $\pi(\bar{P}(i-\omega])$ ,  $\pi(\bar{P}(i-\omega]) | \gamma(\omega))$ ,  $\pi(\bar{P}(i-\omega]) | \gamma(\omega))$  may be deleted from  $\bar{\mu}^{-1}$  expressions

$$j = |\Gamma|$$

$$\omega := (i, |\Gamma|)$$

$$\overline{\prod} \overline{\text{diag}}^{\bar{\omega}} \bar{m}[\bar{\Gamma}, \bar{\omega}] \langle \omega := \xi \bar{\Gamma}[\bar{i}] \rangle =$$

$$[\pi(\bar{\Gamma}(i-\epsilon) | \gamma(\omega)) \mu(\bar{\Gamma}[\bar{i}]^{\mu})] \langle \omega := [\Gamma], z := [\omega, |\Gamma|] \rangle$$

$$= M[(i, |\Gamma|)] = M[\bar{\Gamma}[\bar{i}]] \quad (i := \bar{i}?)$$

$$\{M[(i, |\Gamma|)]\}^{-1} = \left[ \frac{\pi(\bar{\Gamma}(\bar{\omega}) | \gamma(z)) \pi(\bar{\Gamma}[i-\omega] | \gamma(\omega))}{\pi(\bar{\Gamma}[i-\omega] | \gamma(z)) \pi(\bar{\Gamma}(i-\omega) | \gamma(\omega))} \right]$$

$$\langle \omega := [\Gamma], z := \bar{\Gamma}[\bar{\omega}] \rangle.$$

Transformations

$$M[\Gamma[h, k]] \text{ col}[\delta(\Gamma[z, h+z], F)] \langle z := [\bar{h}] \rangle$$

$$= \text{col}[\delta(\Gamma[h+z], F) \langle z := [k-h] \rangle + \text{col}[\delta(\Gamma[z, k+z], F)] \langle z := [\bar{k}] \rangle]$$

$k = |\Gamma|$ : second component omitted

$$\bar{M}[\Gamma(i, j)] \text{ col}[\delta(\Gamma[z, \bar{j}+z], F)] \langle z := [\bar{j}] \rangle$$

$$= \text{col}[\delta(\Gamma[z, \bar{i}+z], F) \langle z := [i] \rangle + \delta(\bar{\Gamma}[\bar{i}+\bar{z}], F) \langle z := [j-i] \rangle]$$

$i = 0$ : first component omitted

second term in second right hand side expression may be expressed as

$$\delta(\bar{\Gamma}[-z], F) \langle z := [-\bar{i}, -\bar{j}] \rangle$$

$$M'[r[h,k]] := [S(r[\nu,\tau], B(r|[h,k]/[k,\nu])) \langle \tau, \nu \rangle := \mathcal{L}\{\|\bar{r}[h]\|\}]$$

$$k = \# | r | := [\pi(r[h,\nu]) | \gamma(\nu)] \quad " \quad ]$$

$$m'[\tau, \omega] := u \mathcal{L} \Delta [\bar{r}(\bar{\omega}) - \gamma(\omega)]$$

$$M'[r[h,k]] = \bar{\Pi} \{ I[\omega-h] + m'[\tau, \omega] \} \quad \langle \omega := [h,k] \rangle$$

$$\bar{M}'[r(i,j)] := [S(r[\tau,\nu], B(r|(i,\nu]/(j,i))) \langle \tau, \nu \rangle := u[i] \langle j \rangle]$$

$$i=0 := [\pi(r[\nu,j]) | \gamma(\nu)] \quad " \quad ]$$

$$\bar{m}'[\tau, \omega] := u \mathcal{L} \Delta [r(\omega) - \gamma(\omega)]$$

$$\bar{M}'[r(i,j)] = \bar{\Pi} [\bar{m}'[\tau, \omega] + I(j-\omega)] \quad \langle \omega := (i,j) \rangle$$

$M[r[h,k]], \dots$  may be expressed as

$$M[r[h,k]] := [S(r[\nu,\tau], B(r|[z-k]/[z-h])) \langle \tau, \nu \rangle := \mathcal{L}\{\|\bar{r}[h]\|\}]$$

$$\{M[r[h,k]]\}^{-1} := [S(r[\nu,\tau], B(r|[z-h]/[z-k])) \phi(r|\nu-k| \gamma(\nu)) \langle " \rangle]$$

$$\bar{M}[r(i,j)] := [S(r[\tau,\nu], B(\bar{r}|(i-\tau)/(j-\nu)) \langle \tau, \nu \rangle := u[j] \langle i \rangle)]$$

$$\{\bar{m}[r(i,j)]\}^{-1} := [S(r[\tau,\nu], B(\bar{r}|(j-\tau)/(i-\nu)) \phi(\bar{r}|i-\nu| \gamma(\nu)) \langle "... \rangle)]$$

conjugate of  $m'[\tau, \omega]$  is  $\bar{m}'[\tau, \bar{\omega}]$

conjugate of  $M[r[h,k]]$  is  $\bar{M}'[r(\bar{k}, \bar{h})]$

inverse of  $M'[r[h,k]]$ ,  $\bar{M}'[r(i,j)]$

# Transformations

$$M'[r[h,k]] = \text{col}[\delta(r\langle h, \|\bar{r}[h]\| \rangle, F)] \\ = \text{col}[f[h,k] + \delta(r\langle k, \|\bar{r}[k]\| \rangle, F)]$$

$k=|\bar{r}|$ : second component omitted

$$\bar{M}'[r(i,j)] = \text{col}[\delta(r\langle [j], j \rangle, F)] \\ = \text{col}[\delta(r\langle [i], i \rangle, F) + f[i,j]]$$

$i=0$ : first component omitted

$$= M[r[h,k]] = \text{col}[\delta(r\langle [h], \|\bar{r}[h]\| \rangle, F)] \\ = \text{col}[\delta(r\langle [h,k] \rangle, F) + \delta(r\langle [k] \rangle, F)]$$

$|\bar{r}|=k$  second component missing

$$\bar{M}[r(i,j)] = \text{col}[\delta(r\langle \bar{j}, j \rangle, F)] \\ = \text{col}[\delta(r\langle \bar{i}, i \rangle, F) + \delta(\bar{r}\langle \bar{j}, \bar{i} \rangle, F)]$$

$\bar{[j]}$ :

$$\Theta[k, r] = \mathcal{L}[\Theta(r[k, k+r], k+r)]$$

$$\bar{\Theta}[r, \omega] = \text{perm} \left\{ 1 - \frac{\Phi(r < \omega)}{\phi(r | \omega)} \right\} \quad (\text{perm}[B] \text{ being } \begin{matrix} 1-\beta(0) & \beta(0) \\ 1-\beta(1) & \beta(1) \end{matrix})$$

$$\bar{\Theta}[r, h] \bar{\Theta}[h, r] = \text{diag } \bar{\Theta}[h+h, r] \quad (\text{check corresp. result for } \bar{H} * \bar{m})$$

$$\bar{\Theta}[r, \omega] = \langle 0 + I[r < \omega] \rangle - \langle 1 + \frac{\Phi(r < \omega)}{\phi(r | \omega)} \rangle I[r < \omega]^{-1}$$

$$\bar{\Theta}[k, r] = \mathcal{U}[\Theta(r(k), \bar{z})]$$

$$\bar{\Theta}[r, \omega] = \overline{\text{perm}} \left\{ 1 - \frac{\Phi(r < \omega)}{\phi(\bar{r} | \omega)} \right\}$$

$\frac{\tau_0 - \tau_2}{z - \tau_2}$	$\frac{z - \tau_0}{z - \tau_2}$	.
.	$\frac{\tau_1 - \tau_2}{z - \tau_2}$	$\frac{z - \tau_1}{z - \tau_2}$

$$= \langle I[r < \omega] + 0 \rangle - \langle \frac{\Phi(r < \omega)}{\phi(\bar{r} | \omega)} + 1 \rangle \mathcal{U}[r < \omega]^{-1}$$

p639  $c\bar{\Psi}$  written as  $r\Pi'[r]$

$M, \bar{M}$  have the transformation properties

$$c\Pi = M c\bar{\Psi}, \bar{c}\Pi = \bar{M} c\bar{\Psi}, \bar{c}\Pi = M \bar{M}^{-1} c\Pi, c\Pi = \bar{M} M^{-1} c\bar{\Pi}$$

$$c\Pi = \bar{M} \mu c\Theta \quad \bar{c}\Pi = M \mu c\Theta$$

$c\bar{\Psi} = \bar{\Pi} M c\bar{\Psi}$  :  $c\bar{\Psi}$  is eigenvector of  $\bar{\Pi} M$  corresponding to eigenvalue 1

$$\bar{\Pi} \Delta \bar{\Phi} \Pi = \Theta c\bar{\Psi} \text{ where } \Delta \bar{\Phi} \Pi, \frac{d}{dz} \Pi (r[\omega] | z) \langle \omega := [r] \rangle$$

$$\Delta \bar{\Phi} \Pi = \Pi + \bar{\Phi} \Delta \Pi \rightarrow c\bar{\Psi} + \bar{\Pi} \bar{\Phi} \Delta \Pi = \Theta c\bar{\Psi}$$

$c\bar{\Psi} = \bar{\Pi} \bar{M} c\bar{\Psi}$  :  $c\bar{\Psi}$  is eigenvector of  $\bar{\Pi} \bar{M}$  corresponding to eigenvalue 1

$$\Theta c\bar{\Psi} = \Pi \Delta \bar{\Phi} \Pi$$

$$X(k, P) = \mathcal{L}[\pi(P[k+\tau, k+\tau])]$$

$$X(k, P)^{-1} = U \mathcal{L}[-\Phi(P[k])] \quad \begin{matrix} 1 \\ \gamma_{k_1-\tau-1} \end{matrix}$$

$$x(P, \omega) = \overline{\text{diag}}^{|\mathcal{P}|-\omega-1} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \gamma_{k_{\omega}-\tau-1} & \end{pmatrix}$$

$$\text{diag } X(h_H, P) x(P, h) = X(h, P)$$

$$X(h, P) = \text{diag}^{k-h} X(k, P) \overline{\text{diag}}^{\omega-h} x(P, \omega) \langle \omega := [h, k] \rangle$$

$$\bar{X}(k, P) = \mathcal{U}[\pi(P[k+\tau, k+\tau])]$$

$$\bar{X}(k, P)^{-1} = U \mathcal{L}[-\Phi(P[k])]$$

$$\bar{x}(P, \omega) = \overline{\text{diag}}^{|\mathcal{P}|-\omega-1} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \gamma_{|\mathcal{P}|-\omega-1} & \end{pmatrix}$$

$$\bar{X}(h, P) = \overline{\text{diag}} \bar{X}(h_H, P) \bar{x}(P, h)$$

$$T(P) = X(P | \gamma(\omega)) \text{ etc}$$

$$N^{-1}(P) c \Delta(P, \frac{F}{P}) = c(F)$$

$$\begin{aligned} b(P)^{-1} c \Delta(P, F) &= b(P)^{-1} M(P) c(F) \\ &= [b^{-1}, N] c(F) \end{aligned}$$

consider defining  $b(P) := \text{diag} \left[ \frac{1}{\mu(P[\omega], \omega)} \right] \mu^{-1}(P[\omega], \omega)$

$$\omega \in [r] \quad X(\omega) := [\mu(r[\nu, \epsilon]\omega)] \quad \langle \nu, \epsilon := [r] \rangle$$

$$X(\omega)X(x) = O[r] \quad \langle \omega := [r], x := [r] - \omega \rangle$$

$$\{X(\omega)\}^2 = X(\omega) \quad \langle \omega := [r] \rangle$$

$$[M, F] = \sum_i X(\omega) f(\omega) \quad \langle \omega := [r] \rangle$$

$$Y(\omega) := [\mu(r[\nu, \epsilon]\omega) \pi(r[\nu, \epsilon]))]$$

$$Y(\omega)Y(x) = O[r]$$

$$\{Y(\omega)\}^2 = Y(\omega)$$

$$[\pi M, F] = \sum_i Y(\omega) f(\omega)$$

$$Z(\omega) := [\theta(r[\nu, \epsilon]\omega)]$$

$$Z(\omega)Z(x) = O[r]$$

$$\{Z(\omega)\}^2 = Z(\omega)$$

$$[\oplus, F] = \sum_i Z(\omega) f(\omega)$$

$$\bar{X}(\omega) := [\mu(r[\nu, \epsilon]\omega)] \dots [\bar{M}, F] = \sum_i \bar{X}(\omega) f(\omega)$$

$$\bar{Y}(\omega) := [\mu(r[\nu, \epsilon]\omega) \pi(r[\nu, \epsilon])] \dots [\bar{\pi}\bar{M}, F] = \sum_i \bar{Y}(\omega) f(\omega)$$

$$\bar{Z}(\omega) := [\theta(r[\nu, \epsilon]\omega)] \dots [\bar{\oplus}, F] = \sum_i \bar{Z}(\omega) f(\omega)$$