Extension of determinantal relationships, algorithmic recursions, etc. to noncommutative, possibly nonassociative elements, by use of linear algebraic equations

$$\begin{matrix} a_1 & c_1 \\ a_2 & b_2 \end{matrix}$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & \alpha \end{vmatrix} \begin{vmatrix} a_1 & c_1 \\ a_3 & \delta \end{vmatrix} - \begin{vmatrix} a_1 & c_1 \\ a_2 & \beta \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_3 & \gamma \end{vmatrix} = a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & \alpha & \beta \\ a_3 & \gamma & \delta \end{vmatrix}$$

$$(a_1\alpha - a_2 b_1)(a_1\delta - a_3 c_1) - (a_1\beta - a_2 c_1)(a_1\gamma - a_3 b_1)$$

$$a_1\{a_1\alpha\delta - a_3 c_1\alpha - a_2 b_1\delta - a_1\beta\gamma + a_3 b_1\beta + a_2 c_1\gamma\}$$

$$a_1(\alpha\delta - \beta\gamma) - b_1(a_2\delta - a_3\beta) + c_1(a_2\gamma - a_3\alpha)$$

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & \alpha & \beta \\ a_3 & \gamma & \delta \end{matrix}$$

$$\frac{\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & \alpha & 0 \\ a_3 & \gamma & 1 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & \alpha & \beta \\ a_3 & \gamma & \delta \end{vmatrix}} \cdot \frac{\begin{vmatrix} a_1 & c_1 \\ a_3 & \delta \end{vmatrix}}{\begin{vmatrix} a_1 & 0 \\ a_3 & 1 \end{vmatrix}} + \frac{\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & \beta \\ a_3 & 1 & \delta \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & \alpha & \beta \\ a_3 & \gamma & \delta \end{vmatrix}} \frac{\begin{vmatrix} a_1 & b_1 \\ a_3 & \gamma \end{vmatrix}}{\begin{vmatrix} a_1 & 0 \\ a_3 & 1 \end{vmatrix}} = 1$$

$$\begin{aligned} a_1 x_1 + b_1 x_2 + c_1 x_3 &= 0 \\ a_2 x_1 + \alpha x_2 + \beta x_3 &= 0 \\ a_3 x_1 + \gamma x_2 + \delta x_3 &= 1 \end{aligned} \qquad \begin{aligned} a_1 y_1 &= c_1 \\ a_3 y_1 + y_2 &= \delta \\ a_1 z_1 &= b_1 \\ a_3 z_1 + z_2 &= \gamma \end{aligned}$$

$$\begin{matrix} a_1 & 0 & c_1 \\ a_2 & 1 & \beta \\ a_3 & 0 & \delta \end{matrix} \qquad \begin{matrix} a_1 & b_1 & 0 \\ a_2 & \alpha & 1 \\ a_3 & \gamma & 0 \end{matrix} \qquad x_3 y_2 + x_2 z_2 = 1$$

$$a_1 \quad -$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ a_2 & \alpha & \beta \\ a_3 & \gamma & \delta \end{vmatrix}$$

$$x_3 y_2 = \cancel{x_3 a_1} \; x_3 \delta - x_3 a_3 y_1$$

$$x_2 z_2 = x_2 \gamma - x_2 a_3 z_1$$

$$\text{sum} = 1 - a_3 x_1 - x_3 a_3 y_1 - x_2 a_3 z_1$$

$$y_2 x_3 + z_2 x_2 = 1 - \cancel{a_3 y_1 x_3} \; a_3 x_1 - a_3 y_1 x_3 - a_3 z_1 x_2$$

$$= 1 - a_3 (x_1 + y_1 x_3 + z_1 x_2)$$

$$= 1 - a_3 a_1^{-1} \{ a_1 x_1 + a_1 y_1 x_3 + a_1 z_1 x_2 \}$$

$$= 1 - a_3 a_1^{-1} \{ a_1 x_1 + c_1 x_3 + b_1 x_2 \} = 1$$

$$\begin{vmatrix} a_1 & \cdots & a_{r-1} & a_r \\ x_1 & & x_{r-1} & x_r \\ y_1 & & y_{r-1} & \alpha \end{vmatrix} \begin{vmatrix} a_1 & \cdots & a_{r-1} & a_{r+1} \\ x_1 & & x_{r-1} & x_{r+1} \\ z_1 & & z_{r-1} & \delta \end{vmatrix} - \begin{vmatrix} a_1 & a_{r-1} & a_{r+1} \\ x_1 & x_{r-1} & x_{r+1} \\ y_1 & y_{r-1} & \beta \end{vmatrix} \begin{vmatrix} a_1 & a_{r-1} & a_r \\ x_1 & x_{r-1} & x_r \\ z_1 & z_{r-1} & \gamma \end{vmatrix},$$

$$= \begin{vmatrix} a_1 & a_{r-1} \\ x_1 & x_{r-1} \end{vmatrix} \begin{vmatrix} a_1 & a_{r-1} & a_r & a_{r+1} \\ x_1 & x_{r-1} & x_r & x_{r+1} \\ y_1 & y_{r-1} & \alpha & \beta \\ z_1 & z_{r-1} & \gamma & \delta \end{vmatrix}$$

$$\frac{\begin{vmatrix} a_1 & a_{r-1} & a_{r+1} \\ & & \\ x_1 & x_{r-1} & x_{r+1} \\ z_1 & z_{r-1} & \delta \end{vmatrix} \begin{vmatrix} a_1 & a_{r-1} & a_r & 0 \\ & & & \\ x_1 & x_{r-1} & x_r & 0 \\ y_1 & y_{r-1} & \alpha & 0 \\ z_1 & z_{r-1} & \gamma & 1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_{r-1} & 0 \\ & & \\ x_1 & x_{r-1} & 0 \\ z_1 & z_{r-1} & 1 \end{vmatrix} \begin{vmatrix} a_1 & a_{r-1} & a_r & a_{r+1} \\ & & & \\ & & \alpha & \beta \\ z_1 & z_{r-1} & \gamma & \delta \end{vmatrix}} + \frac{\begin{vmatrix} a_1 & a_{r-1} & a_r \\ x_1 & x_{r-1} & x_r \\ z_1 & z_{r-1} & \gamma \end{vmatrix} \begin{vmatrix} a_1 & a_{r-1} & 0 & a_{r+1} \\ x_1 & x_{r-1} & 0 & x_{r+1} \\ y_1 & y_{r-1} & 0 & \beta \\ z_1 & z_{r-1} & 1 & \delta \end{vmatrix}}{\begin{vmatrix} a_1 & a_{r-1} & 0 \\ & & \\ x_1 & x_{r-1} & 0 \\ z_1 & z_{r-1} & 1 \end{vmatrix} \quad ''} = 1$$

$$a_1\theta_1 + a_2\theta_2 + \ldots + a_{r-1}\theta_{r-1} + a_r\theta_r + a_{r+1}\theta_{r+1} = 0$$

$$\begin{aligned}
x_1\theta_1 + x_2\theta_2 \quad & + x_{r-1}\theta_{r-1} + x_r\theta_r + x_{r+1}\theta_{r+1} = 0 \\
y_1\theta_1 + y_2\theta_2 \quad & + y_{r-1}\theta_{r-1} + \alpha\theta_r + \beta\theta_{r+1} = 0 \\
z_1\theta_1 + z_2\theta_2 \quad & + z_{r-1}\theta_{r-1} + \gamma\theta_r + \delta\theta_{r+1} = 1
\end{aligned}$$

$$a_1\phi_1 + \cdots + a_{r-1}\phi_{r-1} = a_{r+1}$$

$$-a_1\phi_1\theta_{r+1} \quad -a_{r-1}\phi_{r-1}\theta_{r+1} + a_{r+1}\theta_{r+1} = 0$$

$$x_1\phi_1 \quad + x_{r-1}\phi_{r-1} = x_{r+1}$$

$$-x_1\phi_1\theta_{r+1} \cdots -x_{r-1}\phi_{r-1}\theta_{r+1} + x_{r+1}\theta_{r+1} = 0$$

$$z_1\phi_1 \quad + z_{r-1}\phi_{r-1} + \phi_r = \delta \quad -z_1\phi_1\theta_{r+1} \quad - z_{r-1}\phi_{r-1}\theta_{r+1} + \delta_1\theta_{r+1} - \phi_r\theta_{r+1} = 0$$

$$a_1\psi_1 \quad + a_{r-1}\psi_{r-1} = a_r$$

$$x_1\psi_1 \quad + x_{r-1}\psi_{r-1} = x_r$$

$$z_1\psi_1 \quad + z_{r-1}\psi_{r-1} + \psi_r = \gamma \quad - \langle \underset{\sim}{z}, \underset{\sim}{\theta} + \underset{\sim}{\phi}\theta_{r+1} + \underset{\sim}{\psi}\theta_r \rangle$$

$$\phi_r\theta_{r+1} + \psi_r\theta_r = 1 \qquad - \langle \underset{\sim}{z}, A^{-1}\{ \underset{\sim}{A}\underset{\sim}{\theta} + \underset{\sim}{A}\underset{\sim}{\phi}\theta_{r+1} + \underset{\sim}{A}\underset{\sim}{\psi}\theta_r$$

$$\delta\theta_{r+1} - (z_1\phi_1 + \cdots + z_{r-1}\phi_{r-1})\theta_{r+1} \qquad \underset{\sim}{A}\underset{\sim}{\theta} + \begin{pmatrix} a_{r+1} \\ x_{r+1} \end{pmatrix}\theta_{r+1} + \begin{pmatrix} a_r \\ x_r \end{pmatrix}\theta_r$$

$$+ \gamma\theta_r - (z_1\psi_1 + \cdots z_{r-1}\psi_{r-1})\theta_r$$

$$= 1 - (z_1\theta_1 + \cdots z_{r-1}\theta_{r-1}) - (z_1\phi_1 + \cdots z_{r-1}\phi_{r-1})\theta_{r+1}$$

$$- (z_1\psi_1 \cdots z_{r-1}\psi_{r-1})\theta_r \qquad z_1 a^{-1}\{a_1\theta_1 + a_1\phi_1\theta_{r+1} + a_1\psi_1\theta_r\}$$

$$= 1 - z_1\{\theta_1 + \phi_1\theta_{r+1} + \psi_1\theta_r\}$$

$$\cdots$$

$$- z_{r-1}\{\theta_{r-1} + \phi_{r-1}\theta_{r+1} + \psi_{r-1}\theta_r\}$$

$$z_1 A_1^{-1}\{A_1\theta_1 + A_1\phi_1$$

$$\frac{\begin{vmatrix} a_1 & \cdots & a_{r-1} & a_r \\ x_1 & & x_{r-1} & x_r \\ y_1 & & y_{r-1} & y_r \end{vmatrix}}{\begin{vmatrix} a_1 & & & a_{r+1} \\ & & x_{r-1} & x_{r+1} \\ y_1 & & y_{r-1} & y_{r+1} \end{vmatrix}} - \frac{\begin{vmatrix} a_1 & a_{r-1} & a_r \\ x_1 & x_{r-1} & x_r \\ z_1 & z_{r-1} & z_r \end{vmatrix}}{\begin{vmatrix} a_1 & & a_{r+1} \\ & x_{r-1} & x_{r+1} \\ z_1 & z_{r-1} & z_{r+1} \end{vmatrix}} = \frac{\begin{vmatrix} a_1 & a_{r-1} \\ a_{r-1} & x_{r-1} \end{vmatrix} \begin{vmatrix} a_{r+1} & & a_{r+1} \\ z_1 & & z_{r+1} \end{vmatrix}}{\begin{vmatrix} a_1 & & a_{r+1} \\ & x_{r-1} & x_{r+1} \\ y_1 & y_{r-1} & y_{r+1} \end{vmatrix} \begin{vmatrix} a_1 & a_{r-1} & a_{r+1} \\ a_{r-1} & x_{r-1} & x_{r+1} \\ z_1 & z_{r-1} & z_{r+1} \end{vmatrix}}$$

$x = r-1 \quad y = r \quad z = r+1$

$$\left[\begin{array}{cc|c} \underset{\sim}{A} & \underset{\sim}{A}_{1|r-1,r+1} & \underset{\sim}{A}_{1|r-1,r} \\ \underset{\sim}{A}_{r,1|r-1} & A_{r,r+1} & A_{r,r} \end{array}\right]_r \qquad \left[\begin{array}{cc|c} A_{1|r,1|r-1} & A_{1|r,r+1} & A_{1|r,r} \end{array}\right]_r$$

$$- \left[\begin{array}{cc|c} \underset{\sim}{A} & \underset{\sim}{A}_{1|r-1,r+1} & A_{1|r-1,r} \\ \underset{\sim}{A}_{r+1,1|r-1} & A_{r+1,r+1} & A_{r+1,r} \end{array}\right]_r$$

$-e_r =$

$0 \; 0 \; a_1$

$$= \left[\begin{array}{cc|c} \underset{\sim}{A} & \underset{\sim}{A}_{1,r-1,r+1} & \underset{\sim}{0} \\ A_{r,1|r-1} & A_{r,r+1} & 1 \end{array}\right]_r \qquad \left[\begin{array}{cc|c} A_{1|r+1,1|r-1} e_r & A_{1|r+1,r+1} & A_{1|r+1,r} \end{array}\right]_r$$

$A_{(r,r-1)} A_{(r)}$

$\cancel{\left[A_{1|r,1|r+1} \,|\, e_r \right]}$

$$\begin{matrix} a_1 & a_{r-1} & 0 & a_{r+1} \\ x_1 & x_{r-1} & 0 & x_{r+1} \\ y_1 & y_{r-1} & 1 & y_{r+1} \\ z_1 & z_{r-1} & 0 & z_{r+1} \end{matrix}$$

$\left[A_{1|r,1|r-1} \; A_{1|r,r+1} \,|\, e_r \right]$

. last column of $\left[A_{1|r,1|r-1} \; A_{1,r,r+1} \right]^{-1}$

$$a_{1,1}\theta_1 \quad + a_{1,r-1}\theta_{r-1} + a_{1,r+1}\theta_r = a_{1,r} \qquad (\theta_1 \ldots \theta_{r-1})^T = A^{-1}\{A_{1|r-1,r} - A_{1|r-1,r+1}\theta_r\}$$

$$a_{r-1,1}\theta_1 \qquad a_{r-1,r-1}\theta_{r-1} + a_{r-1,r+1}\theta_r = a_{r-1,r}$$

$$a_{r,1}\theta_1 \qquad a_{r,r-1}\theta_{r-1} + a_{r,r+1}\theta_r = a_{r,r} \qquad \langle A_{r,1|r-1}, A^{-1}\{A_{1|r-1,r} - A_{1|r-1,r}\theta_r\}\rangle + a_{r,r+1}\theta_r = a_{r,r}$$

$$a_{1,1}\phi_1 \qquad a_{1,r-1}\phi_{r-1} + a_{1,r+1}\phi_r = a_{1,r} \qquad \{a_{r,r+1} - \langle A_{r,1|r-1}, A^{-1}A_{1|r-1,r}\rangle\}\theta_r = a_{r,r} - \langle A_{r,1|r-1}, A^{-1}A_{1|r-1,r}\rangle$$

$$a_{r-1,1}\phi_1 \qquad a_{r-1,r-1}\phi_{r-1} + a_{r-1,r+1}\phi_r = a_{r-1,r}$$

$$a_{r+1,1}\phi_1 \qquad a_{r+1,r-1}\phi_{r-1} + a_{r+1,r+1}\phi_r = a_{r+1,r} \qquad \{a_{r+1,r+1} - \langle A_{r+1,1|r-1}, A^{-1}A_{1|r-1,r}\rangle\}\phi_r = a_{r+1,r} - \langle A_{r+1,1|r-1}, A^{-1}A_{1|r-1,r}\rangle$$

$$a_{1,1}\psi_1 \qquad a_{1,r-1}\psi_{r-1} + a_{1,r+1}\psi_r = 0 \qquad (\psi_1 \ldots \psi_{r-1})^T = -A^{-1}A_{1|r-1,r+1}\psi_r$$

$$a_{r-1,1}\psi_1 \qquad a_{r-1,r-1}\psi_{r-1} + a_{r-1,r+1}\psi_r = 0 \qquad \{a_{r,r+1} - \langle A_{r,1|r-1}, A^{-1}A_{1|r-1,r+1}\rangle\}\psi_r$$

$$a_{r,1}\psi_1 \qquad a_{r,r-1}\psi_{r-1} + a_{r,r+1}\psi_r = 1$$

$$a_{1,1}\tau_1 \qquad + a_{1,r-1}\tau_{r-1} \qquad + a_{1,r+1}\tau_{r+1} = a_{1,r}$$

$$a_{r-1,1}\tau_1 \qquad a_{r-1,r-1}\tau_{r-1} \qquad + a_{r-1,r+1}\tau_{r+1} = a_{r-1,r}$$

$$a_{r,1}\tau_1 \qquad a_{r,r-1}\tau_{r-1} + \tau_r + a_{r,r+1}\tau_{r+1} = a_{r,r}$$

$$a_{r+1,1}\tau_1 \qquad a_{r+1,r-1}\tau_{r-1} \qquad + a_{r+1,r+1}\tau_{r+1} = a_{r+1,r}$$

$$\theta_r - \phi_r = \psi_r\tau_r \quad \Big| \quad (\tau_1 \ldots \tau_{r-1})^T = A^{-1}\{A_{1|r-1,r} - A_{1|r-1,r+1}\tau_{r+1}\}$$

$$\langle A_{r+1,1|r-1}, A^{-1}\{A_{1|r-1,r} - A_{1|r-1,r+1}\tau_{r+1}\}\rangle + a_{r+1,r+1}\tau_{r+1} = a_{r+1,r}$$

$$\{a_{r+1,r+1} - \langle A_{r+1,1|r-1}, A^{-1}A_{1|r-1,r+1}\rangle\}\tau_{r+1} = a_{r+1,r} - \langle A_{r+1,1|r-1}, A^{-1}A_{1|r-1,r}\rangle$$

$$\{a_{r+1,r+1} - \langle A_{r+1,1|r-1}, A^{-1}A_{1|r-1,r+1}\rangle\}\tau_{r+1} = \{a_{r+1,r+1} - \langle A_{r+1,1|r-1}, A^{-1}A_{1|r-1,r}\rangle\}\phi_r$$

$$\tau_r = a_{r,r} - a_{r,r+1}\tau_{r+1} - \langle A_{r,1|r-1}, A^{-1}\{A_{1|r-1,r} - A_{1|r-1,r+1}\tau_{r+1}\}\rangle$$

$$= \{a_{r,r+1} - \langle A_{r,1|r-1}, A^{-1}A_{1|r-1,r}\rangle\}\theta_r$$

$$- \{a_{r,r+1} - \langle A_{r,1|r-1}, A^{-1}A_{1|r-1,r+1}\rangle\}\tau_{r+1}$$

$$= \{a_{r,r+1} - \langle A_{r,1|r-1}, A^{-1}A_{1|r-1,r}\rangle\}\theta_r$$

$$- \psi_r^{-1}\tau_{r+1}$$

$$A^{(1)} = \cancel{A_{(r),(r-1)}}\ A$$

$$[A_{(r),(r-1)}, A_{(r),r+1}]$$

$$A^{(1)}\theta = a^{(1)} \quad A^{(2)}\phi = a^{(2)} \quad A^{(1)}\psi = e_r \qquad A^{(2)} = \begin{bmatrix} A_{(r-1),(r-1)} & A_{(r-1),r+1} \\ A_{r+1,(r-1)} & A_{r+1,r+1} \end{bmatrix}$$

$$A^{(1)}\tau' = a^{(1)} - e\tau_r = a^{(1)} - A^{(1)}\psi\tau_r \qquad \tau' = (\tau_1 \dots \tau_{r-1}, \tau_{r+1})$$

$$A^{(1)}\{\tau' - \psi\tau_r\} = a^{(1)} \qquad \theta = \tau' - \psi\tau_r \qquad A^{(3)} = [A_{(r+1),(r-1)}\ e_r' A_{(r+1),\dots}$$

$$A^{(2)}\tau' = a^{(2)} \qquad \phi = \tau' \qquad \theta = \phi - \psi\tau_r$$

$$[A_{(r),r-1}, A_{(r),r+1}]\psi = e$$

$$A^{(1)}\phi = A^{(1)}\theta - e\tau_r = A^{(1)}\theta - \{A^{(1)}\psi\}\tau_r$$

$$\begin{bmatrix} A; \underset{\sim}{u}, \underset{\sim}{v} \\ \hline \underset{\sim}{x} \\ \underset{\sim}{y} \end{bmatrix} \qquad [A\{\underset{\sim}{u}, \underset{\sim}{v} / \underset{\sim}{x}, \underset{\sim}{y}] \qquad [A|\cancel{v}$$

$A_{\cdot,(r)}$ = col. r

$$A_{(r-1),r}$$

$$\left[\begin{array}{cc|c} \underset{\sim}{A}_{(r-1),(r-1)} & \underset{\sim}{A}_{(r-1),r+1} & A_{(r-1),r} \\ \underset{\sim}{A}_{r,(r-1)} & \underset{\sim}{A}_{r,r+1} & A_{r,r} \end{array}\right]$$

$$[A_{(r+1),r-1}, e_r' A_{(r+1),r} \dots = A_{(r+1),r}$$

$$a^{(1)} = A_{(r),r} \qquad a^{(2)} = \begin{pmatrix} A_{(r-1),r} \\ A_{r+1,r} \end{pmatrix}$$

$$[A_{(r),(r-1)}, A_{(r),r+1}]\theta = A_{(r),r} \qquad \begin{bmatrix} A_{(r-1),(r-1)}, A_{(r-1),r+1} \\ A_{r+1,(r-1)}, A_{r+1,r+1} \end{bmatrix}\phi = \begin{pmatrix} A_{(r-1),r} \\ A_{r+1,r} \end{pmatrix}$$