( ) Let $a d \sim b c \bmod I$, $a \equiv c \leq b$ and $b, d \in \Delta'\{Ra(W, I), I\}$ where $I \in B_i \cap B_{x;qf}(W)$. Then $MS\{W, I | a/b\} = MS\{W, I | c/d\}$, both sets being nonvoid.

From ( ), $MS\{W, I | a/b\}$ is nonvoid and $MS\{W, I | a/b\} \subseteq MS\{W, I | c/d\}$ when $ad \sim bc \bmod I$, $c \leq b$ and $b \in \Delta'\{Ra(W, I), I\}$, where $I \in B_i \cap B_{x;qf}(W)$. $x \equiv a \bmod I$ for each $x \in MS\{W, I | a/b\}$, so that, since $a \equiv c \bmod I$, $x \equiv c$ for each such x: $MS\{W, I | a/b\} \subseteq MS\{W, I | c/d\}$. Since $d | c \bmod I$, $c \leq d \bmod I$, from ( ). ~~It now follows that $MS\{W, I | c/d\}$~~ With this addition, the stated conditions become symmetric in the pairs a, b and c, d: $MS\{W, I | c/d\} \subseteq MS\{W, I | a/b\}$.

—o—

$bx = a + s$ $\quad dy = c + t$ $\quad a \leq b$ $\quad c \leq d$ $\quad c \leq a$ $\quad a, b, d \in \Delta'\{Ra(W, I), I\}$

$x \equiv a$ $\quad y \equiv c$ $\quad \exists z \in W$ $\quad xz = y + w$ $\quad w \in I$ $\quad I \in B_i(W)$.

$bdxz = bdy + bdw$ $\quad dz(a+s) = b(c+t) + bdw$

$adz = bc + bt + bdw - dzs$ $\quad z \in S\{W, I | bc/ad\}$

$z \equiv y \equiv c$ $\quad c \leq a \leq b \to c \equiv bc$ if $I \in B_i(W) \cap B_{x;qf}$

$\to z \in MS\{W, I | bc/ad\}$

~~bd~~ $adz = bc + r$ $\quad adz' = bc + r'$ $\quad z' \equiv bc$ $\quad z = z' + q$ $\quad bdq \in I$

xz' $\quad adbxz' = abc + ar' + bcs + r's$ $\quad z' \equiv bc \to z' \leq ab$

$c \leq ab$

$dxz' = c + h$ $\quad h \in I$

$adbx^*(z'+q) = abc + ar' + bcs + r's \quad bdq \in I$

$\to dx$

$x \in MS\{W, I \mid a/b\} \quad z \in MS\{W, I \mid bc/ad\} \quad a, b, d \in \Delta'\{Ra(W, I), I\}$

$a \le b \quad bc \le ad \quad c \le b \mid \to c \le a \quad c \le d \qquad I \in B_z(W) \cap B_{x;q}f$

$bx = a + s \quad \cancel{ab}\ adz = bc + r \quad adbxz = abc + ar + bcs + rs$

$x \equiv a \quad z \equiv bc \quad x \le ab \quad z \le c \quad z \le b \to z \le a \quad z \le ab$

$ab\{dxz - c\} \in I \to xz \in S\{W, I \mid c/d\}$

$xz \equiv abc \quad c \le a \le b \to xz \equiv c \quad xz \in MS\{W, I \mid c/d\}$

(2) $x MS\{W, I \mid bc/ad\} \subseteq MS\{W, I \mid c/d\}$. x fixed in $MS\{W, I \mid a/b\}$

$MS\{W, I \mid a/b\}\, MS\{W, I \mid bc/ad\} \subseteq MS\{W, I \mid c/d\}$

( ) $c \le a \le b \quad \cancel{bc \le ad} \quad c \le d \quad a, b, d \in \Delta'\{Ra(W, I), I\} \quad I \in B_z(W)$

x fixed in $MS\{W, I \mid a/b\}$. Each $y \in MS\{W, I \mid c/d\}$

$\exists\ z(x,y) \in S\{W, I \mid bc/ad\} \quad w(x,y,z) \in I$ such that

$x z(x,y) = y + w(x,y,z) \quad \forall\ MS\{W, I \mid a/b\} \subseteq \Delta'\{Ra(W, I), I\}$

$bx = a + s \quad dy = c + t \quad s, t \in I \quad x \equiv a \quad y \equiv c \quad x \in \Delta'\{Ra(W, I), I\}$

$y \le x$ $\quad W$ contains z: $xz = y + w \quad bdxz = bdy + bdw$

$dz(a+s) = b(c+t) + bdw \quad \cancel{adz = bc + bt + bdw - dzs}$

$adz = bc + r \quad r = bt + bdw - dzs \in I \to z \in S\{W, I \mid bc/ad\}$

$a \le b \quad ad \mid bc \quad c \le a \quad c \le a \le b$

1′ (4) $a \leq b \quad bc \leq ad \quad c \leq b \quad a, b, d \in \Delta' \{Ra(W, I \mid I\} \quad I \in B_i(W) \cap B_{x;qf}$

(1) x fixed in $MS\{W, I \mid a/b\}$ Each $y \in MS\{W, I \mid c/d\}$

$\exists \; z(x,y) \in MS\{W, I \mid bc/ad\}$, $w(x,y,z) \in I$ such that

$x z(x,y) = y + w(x,y,z)$

$c \leq b \quad bc \leq ad \rightarrow c \leq a, \; c \leq d$

As in ( ), $\exists \; z \in S\{W, I \mid bc/ad\} \quad xz = y + w \quad w \in I$

$x \equiv a \quad y \equiv c \quad z \equiv y \equiv c \quad c \leq b \rightarrow c \equiv bc \quad z \equiv bc \rightarrow z \in MS(bc/ad)$.

—o—

$bx = a + s \quad by = c + t \quad xz = y + w \rightarrow az = c + r \quad c \leq a \leq b \quad I \in B_i(W)$

$bxz = by + bw \quad (a+s)z = c + t + bw \quad az = c + r \quad r = t + bw - sz$

$x \equiv a \quad y \equiv \cancel{x} \quad z \equiv y \equiv c \qquad a, b \in \Delta'\{Ra(W, I), I\}$

$bx = a + s \quad az = c + r \qquad ab\, xz = ac + ar + cs + rs$

$x \equiv a \quad z \equiv c \quad a \leq c \quad bxz - c \leq a \quad bxz = c + t \quad \cancel{xz} \in S\{W, I \mid c/b\}$

$xz \equiv ac \equiv c \qquad xz \in MS\{W, I \mid c/b\}$

—o—

$c \leq a \leq b \quad a, b \in \Delta'\{Ra(W, I), I\} \quad I \in B_i(W)$

(1) x fixed in $MS\{W, I \mid a/b\}$ Each $y \in MS\{W, I \mid c/b\}$

$\exists \; z(x,y) \in MS\{W, I \mid c/a\} \quad w(x,y,z) \in I$ such that

$x z(x,y) = y + w(x,y,z)$

$MS\{W, I \mid a/b\} \in \Delta'\{Ra(W, I), I\}$

$bx = a + s \quad az = c + r$

$bxz = c + r + zs$

$bx = a + s \quad by = c + t \quad s, t \in I \quad x \equiv a \quad y \equiv c \quad x \in \Delta'\{Ra(W, I), I\}$

$y \leq x$ W contains z: $xz = y + w \quad bxz = by + bw$

$az = c + r \quad r = t + bw - sz \quad z \in S\{W, I \mid c/a\}$

$x \in \Delta'\{Ra(W, I), I\} \quad y \leq x \rightarrow MS\{W, I \mid y/x\}$ nonvoid $z \equiv y \equiv c$

$\rightarrow z \in MS\{W, I \mid c/a\}$.

(2) In addition $I \in B_{x;qf}(W)$ $MS\{W, I | a/b\} MS\{W, I | c/a\} \subseteq MS\{W, I | c/b\}$:

$bx = a + s$ $az = c + r$ $abxz = ac + av + cs + rs$

$x \equiv a$ $z \equiv c$ $\cancel{a \le c}$ $c \le a$ $\rightarrow$ $bxz - \cancel{c} \le a$ $\rightarrow$ $bxz = c + t$

$xz \in S\{W, I | c/b\}$ $xz \equiv ac \equiv c$ $\rightarrow$ $xz \in MS\{W, I | c/b\}$.

—o—

( ) Let $c \le a \le b$, $c \le d$ mod I and $a, b, d \in \Delta'\{Ra(W, I), I\}$, where $I \in B_z(W)$. "(1) $MS\{W, I | a/b\}$ ~~is monoid~~ and $MS\{W, I | c/d\}$ are monoid. ~~(2) Let $x \in MS\{W, I | a/b\}$ be fixed. To each $y \in MS\{W, I | c/d\}$ correspond $z(x, y) \in S\{W, I | bc/ad\}$ and~~ (2) $MS\{W, I | a/b\} | MS\{W, I | c/d\}$ $[S\{W, I | bc/ad\}, I]$." ~~$w(x, y, z) \in I$ for which $xz(x, y) = y + w(x, y, z)$.~~

From ( ), $MS\{W, I | a/b\}$ is monoid if $b \in \Delta'\{Ra(W, I), I\}$ ~~and~~ and $a \le b$ mod I, where $I \in B_z(W)$. ~~Furthermore~~ If, in addition, $a \in \Delta'\{Ra(W, I), I\}$, $MS\{W, I | a/b\} \subseteq \Delta'\{Ra(W, I), I\}$ from ( ). Similarly $MS\{W, I | c/d\}$ is monoid.

Select $x \in MS\{W, I | a/b\}$ and $y \in MS\{W, I | c/d\}$. Let ~~x~~ $bx = a + s$ where $s \in I$ and $x \equiv a$ mod I, and $dy = c + t$ where $t \in I$ and $y \equiv c$ mod I. Since $a \le c$ mod I, $y \le x$ mod I. Since $x \in \Delta'\{Ra(W, I), I\}$ and ~~$y \le x$ mod I~~ ~~a $c \le a$ mod~~ $y \le x$ mod I, $MS\{W, I | y/x\}$ contains z such that $xz = y + w$ where $w \in I$. Then $bdxz = bdy + bdw$ and ~~a~~ $adz = bc + r$ where $r = bt + bdw - dzs$ $\in I$ since $I \in B_z(W)$: $z \in S\{W, I | bc/ad\}$

1( ) Let $a \leq b$, $bc \leq ad$, $c \leq b$ mod I and $a, b, d \in \Delta'\{Ra(W,I), I\}$ where $I \in B_i \cap B_{x;qf}(W)$.

(1) $MS\{W,I|a/b\}$, ~~and~~ $MS\{W,I|c/d\}$ and $MS\{W,I|bc/ad\}$ are nonvoid.

~~(2) Let $x \in MS\{W,I|a/b\}$ be fixed. To each $y \in MS\{W,I|c/d\}$ correspond $z(x,y) \in MS\{W,I|bc/ad\}$ and $w(x,y,z) \in I$ for which $xz(x,y) = y + w(x,y,z)$~~

(2) $MS\{W,I|a/b\} | MS\{W,I|c/d\}$ $[MS\{W,I|bc/ad\}, I]$

(3) $MS\{W,I|a/b\}MS\{W,I|bc/ad\} \subseteq MS\{W,I|c/d\}$

(4) $MS\{W,I|a/b\}MS\{W,I|bc/ad\} \subseteq MS\{W,I|c/d\}$

The conditions $c \leq b$, $bc \leq ad$ mod I and $I \in B_{x;si} \cap B_{x;qf}(W)$ ~~too~~ in conjunction imply that $c \leq a$, $c \leq d$ mod I. Since $a, d \in \Delta'\{Ra(W,I), I\}$, $ad \in \Delta'\{Ra(W,I), I\}$ from ( ). The conditions $a \leq b$, $c \leq d$, $bc \leq ad$ mod I and $b, d, ad \in \Delta\{Ra(W,I); I\}$ imply the result of (1), from ( ).

In the proof of ( ) it was shown that with $x \in MS\{W,I|a/b\}$ and $y \in MS\{W,I|c/d\}$ ~~From ( ), it follows that with x and y as described in (2), $S\{W,I|bc/ad\}$ contains z for which $xz = y + w$~~

( ), $MS\{W,I|y/x\}$ contains z for which $xz = y + w$ where $w \in I$ and that this z belongs to $S\{W,I|bc/ad\}$. Thus $z \equiv y \equiv c$ mod I. Since $c \leq b$ mod I and $I \in B_{x;si} \cap B_{x;qf}(W)$, $c \equiv bc$ mod I from ( ): $z \equiv bc$ mod I and $z \in MS\{W,I|bc/ad\}$.

~~To prove the result of (3), select $x \in MS\{W,I|$~~

The result of (3) follows from ( ).

To prove the result of (4), select $x \in MS\{W,I|a/b\}$ and $z \in MS\{W,I|bc/ad\}$ so that, from ( ), $xz \in MS\{W,I|c/d\}$. Since

$x \equiv a^{a_a}, z \equiv bc \bmod I$, ~~it~~ follows from ( ) that $xz \equiv abc \bmod I$.

The conditions $c \leq a \leq b$ imply from ( ) that $xz \equiv c \bmod I$:

$xz \in MS\{W, I \mid c/d\}$.

( ) Let $c \leq a \leq b \bmod I$, ~~$ad \mid bc \bmod I$~~ and $b \in \Delta'\{Ra(W,I), I\}^r$ where $I \in B_{\dot{z}} \cap B_{x;qf}(W)$.

(1) $MS\{W, I \mid a/b\}$ is nonvoid.

(2) $MS\{W, I \mid a/b\} S\{W, I \mid bc/ad\} \subseteq S\{W, I \mid c/d\}$

Since $a \leq b \bmod I$, ~~and~~ $b \in \Delta'\{Ra(W,I), I\}$ and $I \in B_{\dot{z}}(W)$,

$MS\{W, I \mid a/b\}$ is nonvoid, from ( ).

If $S\{W, I \mid bc/ad\}$ is void, the result of (2) is correct.

Assuming that $ad \mid bc \bmod I$, select $x \in MS\{W, I \mid a/b\}$ and $z \in S\{W, I \mid bc/ad\}$, so that $bx = a + s$ where $s \in I$ and $x \equiv a \bmod I$, and $adz = bc + r$ where $r \in I$. Then $ab(dxz - c) = t$ where $t = ar + bcs + rs \in I$ since $I \in B_{x;si}(W) \cap B_{+;c}$. Since $a \leq b \bmod I$ and $I \in B_{x;si} \cap B_{x;qf}(W)$, $ab \equiv a \bmod I$ from ( ), ~~and~~ $x \equiv ab \bmod I$ and $dxz \leq ab \bmod I$. Also $c \leq a \equiv ab \bmod I$:

$dxz - c \leq ab \bmod I$ from ( ). From ( ), $dxz - c \in I$:

$xz \in S\{W, I \mid c/d\}$.

( ) Let $I \in B_{x;si}(W) \cap B_{+c}$. (1) $S\{W, I \mid a/b\} S\{W, I \mid c/a\} \subseteq S\{W, I \mid c/b\}$ for all $a, b, c \in W$.

(2) If in addition $I \in B_{x;qf}(W)$ and $c \leq a \bmod I$,

$MS\{W, I \mid a/b\} MS\{W, I \mid c/a\} \subseteq MS\{W, I \mid c/b\}$.

If either $S\{W, I|a/b\}$ or $S\{W, I|c/a\}$ is ~~nonvoid~~ void, ... result of (1) is correct. Assuming both sets to be nonvoid, let $bx = a+s$ and $az = c+t$ where $s, t \in I$. Then $bxz = c+w$ where $w = r+zs \in I$, since $I \in B_{x;si}(W) \cap B_{+;c}$: $xz \in S\{W, I|c/b\}$.

For the proof of part (2), ~~so that~~ let x, z above be such that $x \equiv a$, $z \equiv c$ mod I. Then $xz \equiv ac$ mod I and, since ~~Let~~ $c \leq a$ mod I and $I \in B_{x;qf}(W)$, $ac \equiv c$ mod I: $xz \in MS\{W, I|c/b\}$.

) Let $c \leq a \leq b$ mod I and $a, b \in \Delta'\{Ra(W, I), I\}$ where $I \in B_2(W)$.

1) $MS\{W, I|a/b\}$, $MS\{W, I|c/a\}$ and MS~~$\{c/b\}$~~$\{W, I|c/b\}$ are non...

~~Let $x \in MS\{W, I|a/b\}$ be fixed. To each $y \in MS\{W, I|c/b\}$~~
$MS\{W, I|a/b\} | MS\{$~~c/b~~ $W, I|c/b\}$ $[MS\{W, I|c/a\}, I]$
~~correspond $z(x,y) \in MS\{W, I|c/a\}$ and $w(x,y,z) \in I$ for which $xz(x,y) = y + w(x,y,z)$.~~

~~(3) $S\{W, I|a/b\}$ $S\{W$~~ ~~3) $MS\{W, I|a/b\} | MS\{W, I|c/b\}$ mod I~~

~~(4)~~ (3) If ~~$I \in B_{x;qf}(W$~~ in addition ~~$B_{x;qf}(W)$, $MS\{W, I|a/b\} MS\{W, I|c/a\} \subseteq MS\{W, I|c/b\}$.~~

The conditions $a \leq b$, $c \leq a$, $c \leq b$ ~~mod I~~ and $b, c \in \Delta'\{Ra(W, I), I\}$ imply, from ( ), the results of (1). ¶ Select $x \in MS\{W, I|a/b\}$ and $y \in MS\{W, I|c/b\}$.
~~Let $bx = a+s$ where $s \in I$ and $x \equiv a$ mod I.~~
From ( ), $MS\{W, I|a/b\} \in \Delta\{Ra(W, I), I\}$, since $a \in \Delta'\{Ra(W, I), I\}$ and $I \in B_2(W)$, ~~Let $bx = a+s$ where $s \in I$ and $x \equiv a$ mod I~~ so ~~that $x \in \Delta'\{$~~~~$Ra(W, I), I\}$.~~ Let $by = c+t$ where $t \in I$ and $y \equiv c$ mod I. Since $c \leq a$ mod I, $y \leq x$ mod I. Also $x \in \Delta'\{Ra(W, I), I\}$ so that from ( ), $MS\{W, I|y/x\}$ contains z for which $xz = y + w$

where $w \in I$ and $z \equiv y \equiv c \bmod I$. Then $bxz = by + bw$, so that $az = c + r$ where $r = t + bw - sz \in I$ since $I \in B_i(W)$: $z \in S\{W, I|c/a\}$. and, since $z \equiv c \bmod I$, $z \in MS\{W, I|c/a\}$.

~~Since z $c \leq a \bmod I$, the result of (3) follows from ( ).~~

When $a \in Ri(W, I)$ $S\{W, I|a/b\} \equiv MS\{W, I|a/b\}$, $S\{W, I|c/a\} \equiv MS\{W, I|c/a\}$ and $S\{W, I|c/b\} \equiv MS\{W, I|c/b\}$ so that the result of (1) may be reformulated as

(*) $MS\{W, I|a/b\}\, MS\{W, I|c/a\} \leq MS\{W, I|c/b\}$

Let ~~If~~ $I \in B_{x;si} \cap B_{x;qf}(W) \cap B_{+c}$ and $c \leq a \bmod I$. (*) holds for all $b \in W$.

$c|b,\ b|a \bmod I$ $\quad S\{W, I|b/c\}|a \bmod I$? $\quad I \in B_{x;si}(W) \cap B_{+;c}$

$cy = b + t \quad bx = a + s \quad (cy - t)x = a + s \quad$ or $y(cx) = a + s + tx$

$a|d \quad az = d + w \ \| \ d|a \quad dz = a + w \quad y(cx) = dz + s + tx - w$

$c|b \quad b|a \quad c|a \quad c \in Ri \quad S\{W, I|b/c\} | S\{W, I|a/c\}$

$cy = b + t \quad bx = a + s \quad cz = a + w \quad ycx = cz + s + tx - w$

$c \in Ri \rightarrow yx = z + r \quad r \in I \quad I \in B_i(W)$

each $y \in S\{W, I|b/c\}$ fixed, to each $z \in S\{W, I|a/c\}$ corresponds

$b := a \quad c := b \quad a := c \quad b|a \quad a|c \quad b|c \quad b \in Ri(W, I)$

$x \in S\{W, I|a/b\}$ fixed. To each $y \in S\{W, I|c/b\}$ correspond $z(x, y) \in S\{W, I|c/a\}$ and $w(x, y, z) \in I$ for which $xz(x, y) = y + w(x, y, z)$

$bx = a + s \quad by = c + t \quad az = c + r \quad bxz = c + r + zs$

$b(xz - y) = r + zs - t \quad b \in Ri \rightarrow xz = y + w$

$b|a,\ a|c$, $b|c \bmod I$, $b \in Ri(W, I)$ $\quad I \in B_i(W)$.

1) With $x' \in S\{W, I|a/b\}$, $y' \in S\{W, I|c/b\}$ fixed $S\{W, I|a/b\} \equiv x' + I$ and $S\{W, I|c/b\} \equiv y' + I$

2) $MS\{W, I|a/b\} \equiv S\{W, I|a/b\}$ and $MS\{W, I|c/b\} \doteq S\{W, I|c/b\}$

3) Corresponding to each triplet $x \in S\{W, I | a/b\}$, $y \in S\{W, I | c/b\}$ and $z \in S\{W, I | \overset{c/a}{\cancel{a/c}}\}$, I contains $w(x,y,z)$ for which $xz = y + w(x,y,z)$

4) $S\{W, I | a/b\} | S\{W, I | c/b\}$ and I and $S\{W, I | \overset{c/a}{\cancel{a/c}}\} | S\{W, I | c/b\}$ mod I.

5) $S\{W, I | a/b\} S\{W, I | \overset{c/a}{\cancel{a/c}}\} \subseteq S\{W, I | c/b\}$ remove

$b \in R_i$ $I \in B_i$ 1, 2 from (,).

$bx = a + s$ $by = c + t$ $az = c + r$ $r, s, t \in I$ $bxz = az + zs = c + r + zs$

$b(xz - y) = r + zs - t \in I$ $b \in R_i$ $xz = y + w$ $w \in I$ 3)

$x \in S(a/b)$ $y \in S(c/b)$ W, I contain z, w $xz = y + w$ first res. 4

~~$x \in S(a$~~ $z \in S(c/a)$ $y \in S(c/b)$ " x, w " second

$bxz = c + r + zs$ 5) from ( )

—o—

$b|a$ $a|c$ $b|c$ $I \in B_{x;si}(W) \cap B_{+;c}$ from (1) with div. conditions

$S\{W, I | a/b\} S\{W, I | c/a\} \subseteq S\{W, I | c/b\}$

$bx = a + s$ $by = c + t$ $az = c + r$ $r, s, t \in I \Rightarrow bxz = az + zs = c + \cancel{r + zs}\, w$

$w = r + zs \in I$

—o—

$b | a$, $a | c$ $S\{W, I | a/b\} | c$ mod I $I \in B_{x;si} \cap B_{+;c}$

$bx = a + s$ $az = c + r$ $bxz = az + zs = c + t$ $t = r + zs$

—o—

(2) triplet? $bx = a + s$ $by = c + t$ $az = c + r \Rightarrow b(xz - y) \in I$

$x \equiv a$ $y \equiv c$ $z \equiv c$

3) with $I \in B_i \cap B_{x;qf}(W)$ triplet as 3) above + 4 + 5) modified

—o—

( ) $c \le a \le b$ mod I $a, b \in \Delta'\{Ra(W, I), I\}$ $I \in B_i \cap B_{x;qf}(W)$

(1) $MS\{W, I | a/b\}$ $MS\{W, I | c/a\}$ $MS\{W, I | c/b\}$ monoid

(2) With $x' \in MS\{W, I | a/b\}$ fixed, $MS\{W, I | a/b\} \equiv x' + I$; the composition of the sets $MS\{W, I | c/a\}$ and $MS\{W, I | c/b\}$ is similar.

(3) Corresponding to each triplet $x \in MS\{W, I \mid a/b\}$, $y \in MS\{W, I \mid c/b\}$ and 10
$z \in MS\{W, I \mid c/a\}$, I contains $w(x,y,z)$ for which $xz = y + w(x,y,z)$

4) $MS\{W, I \mid a/b\} \mid MS\{W, I \mid c/b\}$ mod I and $MS\{W, I \mid c/a\} \mid MS\{W, I \mid c/b\}$ mod I

5) $MS\{W, I \mid a/b\} MS\{W, I \mid c/a\} \subseteq MS\{W, I \mid c/b\}$

~~$a \leq b, c \leq a, c \leq b, b, a \in \Delta'\{$~~ (1) from (1)

2) $a \leq b$ $c \leq a$ $c \leq b$ $b, a \in \Delta'\{Ra(W, I), I\}$ $I \in B_z \cap B_{x;qf}(W)$ from ( )

3) $bx = a + s$ $by = c + t$ $az = c + r$ $r, s, t \in I$ $bxz = az + zs$

$b(xz - y) = p$ $p = r + zs - t \in I$ $I \in B_z(W)$ $x \leq a$ $y \leq c$

$xz \leq a \leq b$ $y \leq b$ $xz - y \leq b$ $xz - y = w$ $w \in I$ $I \in B_{x;qf}$

$x \in MS(a/b)$ $y \in MS(c/b)$ W, I contain z, w : $xz = y + w$ first part 4

$z \in MS(c/a)$ $y \in MS(c/b)$ ..... x, w " second

$x \in MS(a/b)$ $z \in MS(c/a)$ y any member of $MS(c/b)$

$xz = y + w$ $w \in I$ ~~[illegible]~~ $y + w \in MS(c/b)$ $xz \in MS(c/b)$

—o—

$C + I \subseteq C$

$A \mid C$ $B \mid C$ mod I $AB \subseteq C \Leftrightarrow$ corresponding to each triplet $a \in A$ $b \in B$

$c \in C$ $\exists\, w(a,b,c) \in I$ $ab = c + w$ $\Leftarrow v$

$ax = c + r$ $by = c + s$

~~Let $C+$~~ With $A, B, C, I \subseteq W$, let $C + I \subseteq C$. ~~For e~~ Corresponding to each triplet ~~a, b, c~~ $a \in A$ $b \in B$ $c \in C$ let I contain $w(a,b,c)$ ~~$\in I$~~ for which $ab = c + w(a,b,c)$. Then $A \mid C$, $B \mid C$ mod I and $AB \subseteq C$

$a \in A$ $c \in C$ W, I contain b, w such that $ab = c + w$: $A \mid C$ mod I

similarly $B \mid C$ mod I. $a \in A$ $b \in B$ select $c \in C$ arb. $ab = c + w$ $w \in I$

$C + I \subseteq C \to c + w \in C$: $ab \in C$; $AB \subseteq C$.

$A.d[B, I].C$, $B.d[A, I].C$ and $AB \subseteq C$

1 $b|a$ $d|c$ $ad|bc$ $\cancel{b,d\in Ri}$ $I\in B_i(W)$ $a\in Ri$ $d\in Ri$

$bx=a+s$ $dy=c+t$ $adz=bc+r$ $\cancel{bdxy=ac}$

$adbxz=abc+ar+bcs+rs$ $adby=abc+abt$

$\cancel{a}\{adb\{xz-y\}=ar+bcs+rs-abt$ $a\in Ri\to b\in Ri$

$xz-y=W\in I$

$ad|ac$ $bc|ac$ $b|a$ $ad|\cancel{b}c$ $bx=a+s$ $adz=\cancel{b}c+r$

$bdzx=adz+dzs=\cancel{b}c+r+dzs$

$b|a$ $ac|d$ $\to S\{W,I|a/b\}|d \bmod I$ $ac|d\to a|d$

$b|a$ $a|d$ $\to S\{W,I|a/b\}|d.$

$b|a$ $d|c$ $ad|bc \bmod I$ $a,d\in Ri(W,I)$ $I\in B_i(W)$

1) With $x'\in S\{W,I|a/b\}$ fixed, $S\{W,I|a/b\}\equiv x'+I$; the composition of the sets $S\{W,I|c/d\}$ and $S\{W,I|bc/ad\}$ is similar

2) $MS\{W,I|a/b\}\equiv S\{W,I|a/b\}$; similar results hold for $S(c/d)$ & $S(bc/ad)$

3) Corresponding to each triplet $x\in S(a/b)$ $y\in S(c/b)$ $z\in S(bc/ad)$ I contains $W(x,y,z)$ for which $xz=y+W(x,y,z)$

4) $S(a/b)|S(c/b)$ and $S(bc/ad)|S(c|b) \bmod I$

5) $S(a/b)S(bc/ad)\subseteq S(c/b)$?

( ): $b|a \bmod I$ $a\in Ri\to b\in Ri$ $a,d\in Ri\to ad\in Ri$ from ( ).

$b|a$ $d|c$ $ad|bc$ $b,d,ad\in Ri\to$ results of (1)

$\cancel{b\in Ri}$, $b,d,ad\in Ri$ $I\in B_i(W)$? $\to$ results of (2)

$bx=a+s$ $dy=c+t$ $adz=bc+r$ $adbxz=abc+ar+bcs+rs$

$adby=abc+abt$ $adb\{xz-y\}=p$ $p=ar+bcs+rs-abt\in I$ $I\in B_i(W)$

$adb\in Ri$ $\cancel{xz-y\in I}$ hence $xz-y=W\in I\to(3)$

(4,5) from ( ).

$bx=a+s$ $ay=b+t$

$axy=\cancel{b}a+s+tx$

$A.d[B,I]$. corresponding to $\forall$ each pair $a \in A$, ~~$b \in B$~~ $c \in C$ $\exists$ $b(a,c)$ in B and $w(a,b,c) \in I$ such that $a\,b(a,c) = c + w(a,b,c)$

—o—

$b.d[W,I].a \quad bx = a + w$

$B.s[W,I].A \quad O(W,I|b) \subseteq O(W,I|a)$ all $a \in A, b \in B$

$b.d[W,I].\ \{b:a:b\}.d[W,I].\{a:c:c\}$

$ms\{W,I|a/b\}.d[ms\{W,I|c/a\},I].\ ms\{W,I|c/b\}$

$B.ms[W,I].A \quad O(W,I|b) \equiv O(W,I|a)$

$b.s[W,I].a.s[W,I].c \quad b.s[W,I].a.s.c \quad \gamma.s[W,I].\beta.s.\delta$

$\gamma(s[W,I])\beta(s)\delta$

$A|C[B,I] \quad A.d.C[B,I] \quad b.s.a,\ \gamma.s.\beta\ [W,I]$

—o—

( ) ~~The notation $B|C$~~ With $A, B, C, I \subseteq W$, the notation $B|C[A,I]$ indicates that corresponding to each pair $b \in B$ and $c \in C$, ~~A contains $a(b,c)$~~ A and I contain $a(b,c)$ and $w(a,b,c)$ respectively for which $a(b,c)\,b = c + w(a,b,c)$. The conjoint statement $D|E[A,I], B|C[A,I]$ is written as $D|E, B|C[A,I]$ and similar further abbreviations are adopted.

( ) With $A, B, C, I \subseteq W$, let $C + I \subseteq C$. Corresponding to each triplet $a \in A$, $b \in B$ and $c \in C$ let I contain $w(a,b,c)$ for which $ab = c + w(a,b,c)$. Then $A|C[B,I]$, $B|C[A,I]$ and $AB \subseteq C$.

Select $a \in A$, $c \in C$. B and I contain b and w respectively for which $ab = c + w$: $A|C[B,I]$. Similarly $B|C[A,I]$.

With $a \in A$, $b \in B$ fixed, select $c \in C$ arbitrarily. Then $ab = c + w$ 13
where $w \in I$. Since $C + I \subseteq C$, $c + w \in C$: $ab \in C$ and $AB \subseteq C$.

( ) Let $b|a$, $a|c$ $[W, I]$ where $I \in B_{x;si}(W) \cap B_{+;c}$. Then

$S\{W, I|a/b\} | c$ ~~$[bW, I]$~~ $[bS\{W, I|c/a\}, I]$

Select x in $S\{W, I|a/b\}$ which is nonvoid, since $b|a$ $[W, I]$; $S\{W, I|c/a\}$ and I
~~Since $a|c$ $[W, I]$,~~ $bx = a + s$ where $s \in I$. Since $a|c$ $[W, I]$, Y
~~W and I~~ contain z and r for which $az = c + r$. Then $(bz)x = az + zs$
$= c + t$ where $t = r + zs \in I$ since $I \in B_{x;si}(W) \cap B_{+;c}$: c is divisible
mod I by x, the cofactor bz of x lying in $bS\{W, I|c/a\}$.

—o—

( ) Let $b|a$, $a|c$, $b|c$ $[W, I]$ and $b \in Ri(W, I)$, where $I \in B_z(W)$.

(1) With $x' \in S\{W, I|a/b\}$ fixed, $S\{W, I|a/b\} = x' + I$; the composition of $S\{W, I|c/b\}$ is similar

(2) $MS\{W, I|a/b\} = S\{W, I|a/b\}$ and $MS\{W, I|c/b\} = S\{W, I|c/b\}$

(3) Corresponding to each triplet $x \in MS\{W, I|a/b\}$, $y \in MS\{W, I|c/b\}$ and $z \in S\{W, I|c/a\}$, I contains $w(x, y, z)$ for which $xz = y + w(x, y, z)$

(4) $MS\{W, I|a/b\} | MS\{W, I|c/b\}$ $[S\{W, I|c/a\}, I]$ and
$S\{W, I|c/a\} | MS\{W, I|c/b\}$ $[MS\{W, I|a/b\} | I]$

(5) $MS\{W, I|a/b\} S\{W, I|c/a\} \subseteq MS\{W, I|c/b\}$ all three sets being nonvoid.

~~$b \in Ri(W, I)$ where $I \in B_z(W)$,~~ The results of (1, 2) follow from (,) since $b \in Ri(W, I)$ and $I \in B_z(W)$. To prove the result of (3), let $bx = a + s$, $by = c + t$ and $az = c + r$ where $r, s, t \in I$. Then

$bxz = az + zs = c + r + zs$ and $b(xz - y) = r + zs - t \in I$ since $I \in B_i(W)$. Hence $xz - y = w \in I$ since $b \in R_i(W, I)$. The results of (4,5) follow from ( ).

( ) Let $c \leq a \leq b \bmod I$ and $a, b \in \Delta'\{Ra(W, I), I\}$ where $I \in B_i \cap B_{x;qf}(W)$.

(1) $MS\{W, I \mid a/b\}$, $MS\{W, I \mid c/a\}$ and $MS\{W, I \mid c/b\}$ are nonvoid

(2) With $x' \in MS\{W, I \mid a/b\}$ fixed, $MS\{W, I \mid a/b\} = x' + I$; the sets $MS\{W, I \mid c/a\}$ and $MS\{W, I \mid c/b\}$ are composed in a similar way.

(3) Corresponding to each triplet $x \in MS\{W, I \mid a/b\}$, $y \in MS\{W, I \mid c/b\}$ and $z \in MS\{W, I \mid c/a\}$, I contains $w(x, y, z)$ for which $xz = y + w(x, y, z)$.

(4) $MS\{W, I \mid a/b\} \mid MS\{W, I \mid c/b\}\ [MS\{W, I \mid c/a\}, I]$ and $MS\{W, I \mid c/a\} \mid MS\{W, I \mid c/b\}\ [MS\{W, I \mid a/b\}, I]$.

(5) $MS\{W, I \mid a/b\}\, MS\{W, I \mid c/a\} \subset MS\{W, I \mid c/b\}$.

The results of (1) follow directly from (1), since the assumed conditions contain those of ( ). Since $a \leq b, c \leq a, c \leq b \bmod I$, and $b, a \in \Delta'\{Ra(W, I), I\}$ where $I \in B_i \cap B_{x;qf}(W)$, the results of (2) follow from ( ). To prove part (3), let $bx = a + s$, $by = c + t$ and $az = c + r$ where $r, s, t \in I$. Then $bxz = az + zs$ and, in particular, $x \equiv a$, $y \equiv c \bmod I$, $b(xz - y) = p$ where $p = r + zs - t \in I$ since $I \in B_i(W)$. $xz \leq a \leq b \bmod I$ since $x \equiv a \leq b \bmod I$; $y \leq b \bmod I$ since $y \equiv c \leq b \bmod I$. Hence $xz - y \leq b \bmod I$ and $xz - y = w \in I$ since $I \in B_{x;qf}(W)$. The results of (4,5) follow from ( )

( ) Let $b|a$, $d|c$, $ad|bc$ $[W, I]$ and $a, d \in Ri(W, I)$, where $I \in$

(1) With $x' \in S\{W, I \,|\, a/b\}$ fixed, $S\{W, I \,|\, a/b\} = x' + I$; the compo[sition] of $S\{W, I \,|\, c/d\}$ and $S\{W, I \,|\, bc/ad\}$ is similar.

2) $MS\{W, I \,|\, a/b\} = S\{W, I \,|\, a/b\}$; similar results hold for $S\{W, I \,|\, c/d\}$ and $S\{W, I \,|\, bc/ad\}$

3) Corresponding to each triplet $x \in MS\{W, I \,|\, a/b\}$, $y \in MS\{W, I \,|\, c/d\}$ and $z \in MS\{W, I \,|\, bc/ad\}$, I contains $w(x, y, z)$ for which $xz = y + w$

4) $MS\{W, I \,|\, a/b\} \,|\, MS\{W, I \,|\, c/d\}$ $[MS\{W, I \,|\, bc/ad\}, I]$ and $MS\{W, I \,|\, bc/ad\} \,|\, MS\{W, I \,|\, c/a\}$ $[MS\{W, I \,|\, a/b\}, I]$

5) $MS\{W, I \,|\, a/b\}\, MS\{W, I \,|\, bc/ad\} \subseteq MS\{W, I \,|\, c/b\}$.

Since $b|a$ ~~mod I~~ $[W, I]$, $a \in Ri(W, I)$ and $I \in B_{x;si}(W) \cap B_{+;lc}$, $b \in Ri(W, I)$ from ( ). From ( ), $ad \in Ri(W, I)$ since $a, d \in Ri(W, I)$. The results of (1) now follow from ( ), since $b|a$, $d|c$, $ad|bc$ $[W, I]$ and $b, d, ad \in Ri(W, I)$. The results of (2) follow from ( ), since $b, d, ad \in Ri(W, I)$ and $I \in B_i(W)$.

To prove the result of (3), let $bx = a + s$, $dy = c + t$ and $adz = bc + r$ where $r, s, t \in I$. Then $adbxz = abc + ar + bcs + rs$ and $adby = abc + abt$, so that $adb(xz - y) = p$ where $p = ar + bcs + rs - abt \in I$ since $I \in B_i(W)$. Since $b, ad \in Ri(W, I)$, $adb \in Ri(W, I)$ also. Thus $xz - y = w \in I$. The results of (4, 5) follow from ( ).

$ub+v \in Ra(W,I) \quad bv \in I \quad (ub+v)x = ub+s \quad (ub+v)y = v+t$

$(ub+v)(x+y) = ub+v+s+t$ | $(ub+v)x = ua+s$

$f \in Ra(W,I) \quad fe = f+r \quad r \in I \quad c \in W \quad ec$

$f(ec-c) = cr \quad f \in Ri \rightarrow ec = c+w \quad w \in I$

$Ra(W,I) + I \leq Ra(W,I)$?

$xc \quad yc \qquad (ub+v)xc = ubc+sc \quad (ub+v)yc = vc+tc$

$xcyc \in I \qquad xW + xW \leq xW \quad (xW)^2 \leq xW$

$(xW)(yW) \leq I$

$u'b+v' \in Ra(W,I)$ etc $\qquad x'c = xc$ all $c \in W$

$(ub+v)xc = ubc+sc \quad (u'b+v')x'c = u'bc+s'c$

$(ub+v)(u'b+v')(xc-x'c) = c\{ub(u'b+v') - u'b(ub+v)\}$
$+ (u'b+v')s - (ub+v)s'$

$x = x' \bmod I \rightarrow xc = x'c \bmod I$ all $c \in W$

$(ub+v)e = ub+v+r \quad (u'b+v')e' = u'b+v'+r' \quad e' = e+p \quad p \in I$
$(u'b+v')e = ub+v+r-(u'b+v')p$

$ub^2e = ub^2+bv+br \qquad u,u' \leq b \quad I \in I_{\ldots} \Rightarrow u = u' \bmod I$
$u'b^2e = ub^2+bv+br-(u'b+v')bp$

$x: \quad x_1(b;u,v;r) \quad (ub+v)x_1 = ub+r$
$x_2(b;u,v;s) \quad (ub+v)x_2 = v+s$

$(ub+v)x_1x_2 \in I \rightarrow x_1x_2 \in I \quad x_1v \in I \quad x_2b \in I$

$x_{\substack{1\\2}}(b;u,v;r \mid c) = \substack{x_1c\\x_2c} \qquad x_1(c+d) = x_1(c) + x_1(d)$
$x_1(cd) = x_1(c)x_2(d) \bmod I$

$X_1(b : u, v ; r \mid W) : x_1 W$
$\quad 2 \qquad\qquad\qquad x_2 W$

$X_1(W) + X_1(W) \subseteq X_1(W) \quad 0 \in W \to 0 \in X_1(W)$

$X_1(W) X_1(W) \subseteq X_1(W)$ $+ \underline{I}$ $\qquad b.\mathrm{sd}[W, \underline{I}].a$

$(ub+v)x_1^2 = (ub+v)x_1 + r - vx_1$

$(ub+v)(x_1^2 - x_1) \in \underline{I} \quad x_1^2 = x_1 \bmod \underline{I} \quad x_1^2 = x_1 + w$

$(x_1 c)(x_1 b) = x_1^2 bc = x_1 bc + bcw$ bcw ~~does~~ may not have

form $x_1 e$: $X_1(W) X_1(W) \not\subseteq X_1(W)$ $\quad \|\quad x_1, x_2 \in \Delta'\{Ra(W, I), \underline{I}\}$

—o—

$x_1 + x_2 \in Ra(W, I) \quad x_1 x_2 \in \underline{I}$

$X_1(W) + \underline{I}$: commutative ring

—o—

$I \in B_2(W)$

$x_1 y = x_1 c \qquad x_1 \in \Delta'\{Ra(W, I), I\} \quad x_1 c \leq x_1 \quad MS\{x_1 c / x_1\}$ monoid

$x_2 z = x_2 c \qquad y + z = c \bmod \underline{I}$?

$y \leq x_1 c \quad x_1 x_2 c \in \underline{I} \to x_2 y \in \underline{I} \quad x_1 z \in \underline{I}$

$(x_1 + x_2) y = x_1 c \bmod \underline{I} \qquad (x_1 + x_2)(y + z - c) \in \underline{I}$

$(x_1 + x_2) z = x_2 c \bmod \underline{I} \qquad y + z = c \bmod \underline{I}$

—o—

$\{X_1(W) + \underline{I}\} + \{X_1(W) + \underline{I}\} = X_1(W) + \underline{I}$

$\{X_1(W) + I\}\{X_1(W) + I\} \subseteq X_1(W) + \underline{I}$

| all members of $MS\{x_1 c / x_1\}$ = mod $\underline{I}$

Let $b \in \Delta'\{Ra(W, I), I\}$ ~~so that~~ and W contain u, v for which $ub + v \in Ra(W, I)$, $bv \in \underline{I}$. ~~W then contains x~~

(1) ~~W contains x_1~~ W and I contain x_1, x_2 and s, t respectively for which $(ub+v)x_1 = ub + r \quad (ub+v)x_2 = v + s$.

~~(2) Set $X_1(b; u, v, r \mid W) = x_1 W$ and define $X_2(W)$ similarly.~~
~~$X_1(W) =$~~

(2) a) $x_1 v, x_2 b, x_1 x_2 \in \underline{I}$; b) $x_1^2 = x_1$, $x_2^2 = x_2$ mod $\underline{I}$ c) $x_1 + x_2 \in Ra(W, I)$ and $(x_1+x_2)c = c$ mod I for all $c \in W$

(3) Set $X_1(W) = X_1(b; u, v, r \mid W) = x_1 W$ and define $X_2(W)$ similarly

a) $X_1(W) X_2(W) \le \underline{I}$

e) $x_1 c = c$ mod $\underline{I}$ for all $c \in X_1(W)$

b) $X_1(W) + X_1(W) = X_1(W)$ ~~with a similar result for $X_2(W)$~~ d) $X_1(W)$ is a commutative ring

c) $X_1(W) X_1(W) \le X_1(W) + \underline{I}$ is ——— $X_2(W)$

f) Results corresponding to b) c) d) hold for $X_2(W)$

4) Set $X_1(W, \underline{I}) = X_1(W) + \underline{I}$ and define $X_2(W)$ similarly

? ~~MS~~ $b \| a$ mod I $\Leftrightarrow$ MS$\{W, I \mid a/b\}$ nonvoid

a) $X_1(W, I) X_2(W, I) \le \underline{I}$

b) $X_1(W, I) + X_1(W, I) = X_1(W, I)$

c) $X_1(W, I) X_1(W, I) \le X_1(W, I)$

$c \| b$ $b \| a \to c \| a$?

$cy = b + t$ $bx = a + s$ $x = a$ $y = b$

$cxy = a + xt + s$ $xy \equiv a$?

d) $X_1(W, I)$ is a commutative ring

e) $x_1 c = c$ mod I for all $c \in X_1(W, I)$

$b \| a \to a \le b$ $I \in B_{x;qf}$

f) Results corresponding to b - c) hold for $X_2(W, I)$

$b \in \Delta'$ $b \| a$ iff $a \le b$??

MS$\{x_1 c / x_1\} \le X_1(W)$?

5) a) $x_1, x_2 \in \Delta'\{Ra(W, I), I\}$

b) For all $c \in W$, MS$\{x_1 c / x_1\}$ and MS$\{x_2 c / c_2\}$ are nonvoid

???

c) With $y_1(c) \in$ MS$\{x_1 c / x_1\}$, and $y_2(c) \in$ MS$\{x_2 c / c_2\}$, $y_1(c) + y_2(c) = c$ mod I for all $c \in W$

(many $y_1(c)$ not all = mod $\underline{I}$)

6) Let W also contain $u'v'$ for which $u'b + v' \in Ra(W, I)$, $bv \in \underline{I}$ so that W and I contain $x_1' x_2'$ and r', s' for which $(u'b + v') x_1' = u'b + r'$, $(u'b + v') x_2' = v' + s'$. Then $x_1 = x_1'$, $x_2 = x_2'$ mod $\underline{I}$.

— o —

$d_1 x = a_1$ $x \le a_1 \le d_1$ $d_1 + d_2 \in R_i$ $d_1 d_2 \in \underline{I}$ $d_1 d_2 \in I \to x d_2 \in \underline{I}$

$d_2 y = a_2$ $y \le a_2 \le d_2$ $x \le d_1$

$(d_1 + d_2) x = a_1$ mod $\underline{I}$ $(d_1 + d_2) x' = a_1$

~~(d_1+d_2)~~ $bx = a$ $b + v \in R_i$ $bv \in \underline{I}$ $x \le b$ $xv \in I$ $(b + v) x = a$ mod $\underline{I}$

$bx = a \mod I \quad \exists x \quad x \equiv a \leq b$

$ubx = ua \mod I \quad (ub+v)x = ua + vx \mod I \quad bx \in I \rightarrow vx \in I$

$ub+v \in R_I \quad bx' = a \quad x' \leq b \quad x = x' \mod I$

MS nonvoid | $x \in$ MS MS $= x + I$ | MS $= S \cap N(b)$ ‖ $u\ v$ ..

( ) Let $b \in \Delta'\{W, Ra(W,I), I\}$ and $a \leq b \mod I$, where $I \in B_I(W)$

(1) MS$\{W, I|a/b\}$ is nonvoid

(2) All members of MS$\{W, I/a/b\}$ are equal mod I. x being any member of MS$\{W, I|a/b\}$, $x + \underset{EN}{I \equiv}$ MS$\{W, I|a/b\}$

(3) MS$\{W, I|a/b\} \equiv S\{W, I|a/b\} \cap N(W, I|b)$

(4) Let u, v be an $Ra(W,I), I$ lifting factor and displaced orthogonal complement pair of b. Since $ub+v \in Ra(W,I)$, W contains a number x for which $(ub+v)x = ~~u~~ ua \mod I$. This x is ~~a~~ a representative member of MS$\{W, I|a/b\}$ and MS$\{W, I|a/b\} \equiv x + I$. The determination of MS$\{W, I/a/b\}$ in this way is independent of the lifting factor – displaced orthogonal complement pair used (and of the direct orthogonal complement possibly used in the same way).

~~The first result of stated result of Part (4) is proved first.~~

Since $b \in \Delta'\{W, Ra(W,I), I\}$, W contains u, v for which $ub+v \in Ra(W,I)$, $bv \in I$ or a simpler condition whose treatment is correspondingly simpler holds. ~~W also contains x~~ Since $ub+v \in Ra(W,I)$, W contains x for which $(ub+v)x = ua + v$ where

$w \in \underline{I}$. ~~uba~~ $(ub+v)bx =$ ~~$uba + bw$ and~~ $ub\,a$ ~~$+ av$~~ $+ bw$ ~~$- av$~~ and hence $(ub+v)(bx-a) = bw - av$. Since $bv \in I$ and ~~av~~ $a \leq b \bmod \underline{I}$, $av \in I$: $bw - av \in \underline{I}$. Also since $I \in B_{x;si}(W) \cap B_{+;c}$: $(ub+v) \in Ri(W, \underline{I})$, so that $bx = a+s$ where $s \in I$: $x \in S\{W, I|a/b\}$. If $ay \in I$, $uay \in I$ since $I \in B_{x;si}(W)$ and $(ub+v)xy = uay + wy \in \underline{I}$ since $I \in B_{x;si}(W) \cap B_{+;c}$. Again since $(ub+v) \in Ri(W, I)$, $xy \in I$: $x \leq a \bmod I$. From ( ), $a \leq x \bmod I$ for all $x \in S\{W, \underline{I}|a/b\}$: $x \equiv a \bmod \underline{I}$ and $x \in MS\{W, I|a/b\}$. The result of (1) has been obtained.

Let x' be any member of $S\{W, I|a/b\}$ satisfying the condition $x' \leq b \bmod \underline{I}$. (Since $x \equiv a \leq b \bmod I$, such x' exist). Then $bx' = a+s'$ where $s' \in \underline{I}$ and $x'v \in \underline{I}$, since $bv \in I$ and $x' \leq b \bmod I$. $(ub+v)x' = ua + r$ where $r = us' + vx' \in \underline{I}$ since $\underline{I} \in B_{x;si}(W) \cap B_{+;c}$. Hence $(ub+v)(x' - x) = r - w \in \underline{I}$ since $I \in B_{+;ls}$. $ub+v \in Ri(W, \underline{I})$, so that $x' = x + p$ where $p \in I$. This conclusion holds in particular for all x' such that $x' \equiv a \bmod \underline{I}$: $MS\{W, I|a/b\} \subseteq x + \underline{I}$. From ( ), $x + \underline{I} \subseteq MS\{W, I|a/b\}$ when $x \in MS\{W, \underline{I}|a/b\}$ and $I \in B_i(W)$. The result of (2) has been derived.

From the above $S\{W, I|a/b\} \cap FN(W, I|b) \subseteq MS\{W, I|ab\}$. For all $x' \in MS\{W, I|a/b\}$, $x' \in S\{W, I|a/b\}$ and $x' \equiv a \leq b \bmod \underline{I}$: $MS\{W, I|a/b\} \subseteq S\{W, I|a/b\}$. The result of (3) follows.

That $MS\{W, I|a/b\}$ may be constructed as described in

1 part (4) has been shown above. The definition of $MS\{W, I|a/b\}$ is independent of any lifting factor-displaced orthogonal complement pairs which may or may not be associated with b. The determination of $MS\{W, I|a/b\}$ as described is independent of the pair used.

$a \equiv b \bmod I \quad a = b + s \quad s \in I$

1): $b \in \Delta'\{W, Ra(W,I), I\}$

$x_1^2 = x_1 + p'$

2) $(ub+v)x_1 v = t, \; t = ubv + vr \in I \quad I \in B_{x;si}(W) \cap B_{+;c}$

$(ub+v) \in Ri \to x_1 v \in I \; \| \; (ub+v)x_1 x_2 = x_1 v + x_1 s \in I$ ''

$x_1 x_2 \in I \; \| \; (ub+v)x_1^2 = (ub+v)x_1 + r - vx_1 \quad (ub+v)(x_1^2 - x_1) = p$

$p \in I \quad I \in B_{x;si}(W) \cap B_{+;ls} \quad x_1^2 - x_1 \in I \quad x_2^2 = x_2$ simly.

$(ub+v)(x_1+x_2) = ub+v+r+s$ ~~ub+~~ $ub+v \in Ra$, Ra factored ? $I \in i$

$\to x_1 + x_2 \in Ra$ & $r+s \in I$ if $I \in B_{+;c}$

$(ub+v)\{(x_1+x_2)c - c\} = cr + cs \quad (x_1+x_2)c \equiv c \bmod I \quad ub+v \in Ri$

$I \in B_{x;si}(W) \cap B_{+;c}$

3) $c \in X_1 \; d \in X_2 \quad c = c'x_1 \in X_1 \quad d = d'x_2 \in X_2 \quad cd = c'd'x_1x_2 \in I$ (no cond. I

$I \in B_{x;si}(W) \cap B_{+;c} \; \| \;$ ~~c,d~~ $c = c'x_1 \quad d = d'x_1 \in X_1 \quad c+d = (c'+d')x_1 \in X_1$

$cd = (c'd'x_1)x_1 \in X_1 \; \| \; x_1 c \equiv c \bmod I$

no condition on I

$c = c'x_1 \quad x_1 c = c'x_1^2 \quad I \in B_{x;si}(W) \cap B_{+;ls} \quad c'x_1^2 = c'x_1 + c'p' \quad p' \in I$

4)

$I \in B_{x;si}(W) \cap B_{+;c} \to X_1 X_2 \in I$ also $(X_1+I)(X_2+I) \in I \quad c \in X_1 \; d \in X_2$

~~$cd \in I$~~ $(c+s)(d+t) \in I \; \| \; c+d+s+t = (c'+d')x_1 + s + t \quad I \in B_{+;c}$

~~$0 \in X_1(W, I$~~ $0 \in X_1 \quad 0 \in I \to 0 \in X_1(W,I) \quad \to X_1(W,I) \subseteq X_1(W,I) + X_1(W,I)$

$0 \in I$ & $I \in B_{+;c} \; \| \; X_1 X_1 \in X_1$ no condition on $I \quad (X_1+I)(X_1+I) \subseteq X_1 + I$

$I \in B_{x;si}(W) \cap B_{+;c}$

3e) X_1 commutative ring without conditions on $\underline{I}$ closed wrt add & mult (b, c)

$c = c'x_1$ $d = d'x_1$ $c - d = (c' - d')x_1 \in X_1$ add assoc in W ∴ in X_1

$0 = 0x_1$ in X_1 $c = c'x_1 \in X_1$ $-c = -c'x_1 \in X_1$ also add comm in W ∴ in X_1

mult dist wrt add in W ∴ in X_1

$\overline{d} \in Ri(W, I)$ $d = d'x_1 \in Ri(X_1, I_1)$ ~~xI~~ $I_1 = x_1 \underline{I}$?

~~a~~ $g = g'x_1$ $dg \in \underline{I}_1$ $d'g'x_1^2 = x_1 s$ $s \in \underline{I}$?

$g'x_1^2 \in \underline{I}$ $\underline{I} \in B_{x;qf} \to g'x_1 \in \underline{I} \to g \in \underline{I}_1$ no.

$d \in Ri(X_1, \underline{I}_1)$ $g \in X_1$ $dg \in I_1 \to g \in I_1$

$d'g'x_1^2 \in \underline{I}_1$ $d'g'x_1^2 = x_1 a'$

$D_1 : x_1 Ri(W, I)$ $d \in D_1$ $g \in X_1$ $dg \in \underline{I}_1 \to g \in I_1$?

$d'g'x_1^2 = x_1 s$ $s \in \underline{I}$ $\to d'g'x_1 \in \underline{I}$ if $I \in B_{x;qf} \to g'x_1 \in I$ $(d' \in Ri(W, I))$

does not imply $g' \in \underline{I}$: $g \in \overline{O}(W, I | x_1) \to g = g'x_1 \in \underline{I}$

not $g = g'x_1$ $g' \in \underline{I}$

define ~~I~~ $\underline{I}_1 = X_1 \cap \underline{I}$ $dg \in \underline{I}_1 \to d'g'x_1^2 \in \underline{I} \to g'x_1^2 \in \underline{I}$

$\underline{I} \in B_{x;si}(W) \cap B_{+;ls}$ $x_1^2 = x_1 + p$ $p \in I$ $g'(x_1 + p) \in \underline{I} \to g'x_1 \in \underline{I}$

$\to g \in I_1$ // note $\underline{I}_1' =$ ~~...~~ $x_1 \underline{I}$ $\underline{I}_1' \subseteq \underline{I}_1$ possible that $ax_1 \in I_1$ $ax_1 \notin I_1'$

eg $a \in \overline{O}(W, I | x_1)$ $a \notin \underline{I}$ // $I_1 = x_1 \overline{O}(W, I | x_1)$.

$d'g'x_1^2 \in \underline{I}$ only when $g'x_1 \in \underline{I} \Rightarrow d' \in Ri(W, \underline{I})$?

suppose ~~$d'g'x_1^2 \in \underline{I}$ with~~ ~~$d'g'x_1$~~ $d'g'x_1 \in \underline{I}$ with $g'x_1 \notin \underline{I}$

$d' \notin Ri$ $\exists\, a' \notin \underline{I}$ for which $d'a' \in \underline{I} \to d'a'x_1^2 \in \underline{I} \to d'a'x_1 \in \underline{I}$

~~a's~~ $x_1 \leq d'$ $d'g'x_1^2 \in \underline{I} \to d'g'x_1 \in \underline{I} \to g'x_1 \in \underline{I}$ if $I \in B_{x;qf}$

$x_1 \leq d'$ $d'g'x_1^2 \in I \to g'x_1^3 \in \underline{I} \to g'x_1 \in \underline{I}$ // $x_1^n = x_1$ mod $\underline{I}$ $\to g'x_1 \in \underline{I}_1$

D: all d' for which $x_1 \leq d'$ $x_1 D \subseteq Ri(X_1, I_1)$

$d'g'x_1^2$ only when $g'x_1 \in I$ $\quad d'g' \in I \to d'g'x_1^2 \in I \to g'x_1 \in I$
$\to x_1 \leq d'$ $\quad Ri(X_1, I_1) \leq D$.

— · —

A all a st $ay = x_1 + p$ $\quad y \in W$ $p \in I$ $\quad a | x_1 \bmod I$

Ax_1 $\quad cx_1 \in X_1$ $\quad \exists y \in W$ $p \in I$ $\quad cay = cx_1 + cp$

$c\underbrace{ax_1y}$ $\quad (ax_1)(yx_1) = cx_1 + x_1q$ $\quad q \in \mathcal{O}(x_1)$

$(ax_1)(cyx_1) = cx_1^3 + pcx_1^2 = cx_1 + q$, $\quad q \in I$ may not be in I_1

$x_1^2 = x_1 + t$ $\quad x_1^3 = x_1^2 + x_1t = x_1 + t + x_1t$

~~or $X_1 \ni x_1$~~ x_1 may not be contained in X_1 ‖ $x_1^2 \in X_1$ but x_1^2 does not function as unit element with respect to $I_1 = x_1\mathcal{O}(W, I/x_1)$ in X_1 ‖ $x_1^2 = x_1 + p$ $\quad \cancel{x_1^2c = x_1c + pc}$ $\quad x_1^2(x_1c) = x_1(x_1c) + x_1cp$

$= x_1c + pc + x_1cp$ $\quad pc \notin I_1$ possible

$(ax_1)(yx_1) = cx_1 + x_1p$ $\quad p \in \mathcal{O}(x_1)$ $\quad ay$

‖ $x_1 \in X_1$
$(ax_1)yx_1 = x_1 + r$
$a(x_1yx_1) = x_1 + r \to a | x_1$

$ay = x_1 + p$ $\quad a\, cy = cx_1 + cp$ $\quad (ax_1)(cyx_1) = cx_1^2 + cpx_1$
$= cx_1 + cpx_1 + cq$

A: all a such that $a | x_1 \bmod I$ $\quad Ra(W, I) \leq A$

$Ax_1 \overset{+I_1}{\leq} Ra(X_1(W, I), I)$ ‖ $yx_1^2 = x_1(yx_1)$ ‖ $c = x_1$ $\quad x_1(ayx_1 - x_1) \in I$

$(ax_1)(yx_1) = cx_1 + r$ $\quad ay - x_1 \in I$ $\quad c = x_1^2$ $\quad x_1^2(ay - x_1) \in I$

$c \in Ra(W, I)$ $\quad \cancel{yx_1^2 or}$ $cd = yx_1^2 + t$ ‖ Take $cx_1 \in Ra(X_1(W, I), I)$ or $cx_1 \leq x_1$

$a(cd - t) = cx_1 + r$ $\quad acd = cx_1 + r + at$ $\quad c(ad - x_1) = r + at$

$c \in Ri(W, I) \to ad - x_1 \in I$ $\quad ad \equiv x_1 \bmod I$ $\quad a | x_1 \bmod I$

$Ra(W, I)$ nonradical?! $Ra(X_1(W, I), I) \leq Ax_1 + I$

All members of $X_1(W,I)$ have the form ax_1+s with $a\in W$, $s\in \underline{I}$. $b=a x_1+s$

This is true in particular of $Ra(X_1(W,I),I)$. Select $\cancel{ax_1+s}\in Ra(X_1(W,I),I)$

To any $c\in X_1(W,\underline{I})$ correspond $\underline{z\in X_1(W,I)}$ $z(b,c)\in X_1(W,I)$ and $t\in I$ such that

$\cancel{az = c+t}$ ~~let~~ $bz(b,c)=c+t$ Let $c=x_1$. and

$z(b,x_1)$ has the form yx_1+w where $y\in W$, $w\in I$:

$(a^{\xi}x_1+s)(yx_1+w)=x_1+t$ $a^{\xi}(x_1^2y)=x_1+p$ where

$p=t-\cancel{sw}\ \cancel{a^{\xi}x_1w - yx_1s}\ x_1(a^{\xi}w+ys)-sw\in\underline{I}$ $I\in B_{x;si}(W)\cap B_{t;is}$

$a\,|\,x_1 \bmod \underline{I}$: $b\in Ax_1+\underline{I}$: $Ra\{X_1(W,I),I\}\subseteq Ax_1+\underline{I}$

—∘—

Select a such that $a|x_1 \bmod \underline{I}$ and $s\in\underline{I}$. Set $b=ax_1+s$

Let $ay=x_1+t$ Select $c=dx_1+w\in X_1(W,I)$

$dx_1+w=d(ay-t)+w=day+w-dt$ $x_1^2=x_1+p$ $p\in\underline{I}$ $ayx_1=x_1^2+tx_1$

$\cancel{dx}$ $c=d(x_1^2-p)+w=dx_1^2+w-dp=d(ayx_1-tx_1)+w-dp$

$=dyax_1+w-dp-dtx_1=dyb+w-dp-dtx_1-dys$

$x_1b=ax_1^2+sx_1=ax_1+ap+sx_1=b+ap+sx_1-s$

$b=x_1b+s-ap-sx_1$

$c=dyx_1b+\cancel{dys}-dyap-dysx_1+w-dp-dtx_1-\cancel{dys}$

$bz=c+dy(ap+sx_1)+d(p+tx_1)-w$ $z=dyx_1$

$z=dyx_1\in X_1(W,I)$ $I\in B_2(W)$

—∘—

$b=ax_1+s$. $ay=x_1+t$ $c=dx_1+w$ show $z\in X_1(W,I)$ $r\in\underline{I}$ $bz=c+r$

$x_1^2=x_1+p$ ‖ $c=dx_1^2+w-dp$ $x_1^2=ayx_1-tx_1$ $x_1=ayx_1-tx_1-p$

$c=dx_1(ayx_1-tx_1-p)\overset{+w-dp}{=}(dyx_1)(ax_1)-dx_1(tx_1+p)+w-dp$

$=dyx_1(ax_1+s)-dx_1(tx_1+p+ys)^{+w-dp}$ $bz=c+r$ $z=dyx_1$

$I\in B_2(W)$

$r=dx_1(tx_1+p+ys)+dp-w$ $\cancel{I\in B(W)\cap B}$

A: all a such that $a|x_1 \bmod I$. $x_1^2 = x_1 + p$ $p \in I$

1) $I \in B_{x;si}(W) \cap B_{+;c}$ $Ra\{X_1(W,I), I\} \leq Ax_1 + I$

2) $I \in B_i(W)$ $\cancel{x_1^2 = x_1 + p \ \ p \in I}$ $Ra\{X_1(W,I), I\} \equiv Ax_1 + I$ (reverse sign of q) ~~in preceding~~ proof

$X_1(W,I) \equiv x_1 W + I \rightarrow x_1 = x_1^2 + (-p) \in X_1(W,I)$

$I \in B_{+;ls}$ $x_1 = x_1^2 + (-p)$ $-p \in I$.

$xW + I \equiv N(W,I|x)$

$I \in B_{x;si}(W) \cap B_{+;c}$ $x_1^2 = x_1 + p$ $p \in I \rightarrow x_1^r = x_1 + q$ $q \in I$ $r = 2,3,\ldots$

~~true~~ true when $r=2$. assume true $x_1^{r+1} = x_1^2 + x_1 q = x_1 + x_1 q + p$

D: all d such that $x_1 \leq d \bmod I$ $\cancel{I \in B_{x;si}(W) \cap B_{+;c}}$

$X_1 = x_1 W$ $I_1 = x_1 O(W, I|x_1)$

2) $x_1 D \equiv Ri(X_1, I_1)$ $I \in B_i(W)$ $x_1^2 = x_1 + p$ $p \in I$

$d' \in x_1 D$, ~~select~~ $\underset{0}{g}' = \underset{0}{g} x_1 \in X_1$ $d' \underset{0}{g}' = d \underset{0}{g} x_1^2 \in I_1 \rightarrow d\underset{0}{g} x_1^2 \in I$ $I \in B_{x;si}(W) \cap B_{+;c}$

$\rightarrow \underset{0}{g} x_1^3 \in I$ $\underset{0}{g} x_1^3 = w \in I$ $\underset{0}{g}(x_1 + q) = w \vee \underset{0}{g} x_1 = w - \underset{0}{g} q \in I$ $I \in B_{+;ls}$

$\rightarrow \underset{0}{g} \in O(W, I|x_1) \rightarrow \underset{0}{g} x_1 \in I_1 \rightarrow \underset{0}{g}' \in I_1 \rightarrow d' \in Ri(X_1, I_1) : x_1 D \leq Ri(X_1, I_1)$

1) $Ri(X_1, I_1) \leq x_1 D$ ~~no conditions on I~~ $I \in B_{x;si}(W)$ no conditions on x_1

$d\underset{0}{g} x_1^2 \in I_1$ only when $\underset{0}{g} x_1 \in I$. $d\underset{0}{g} \in I \rightarrow d\underset{0}{g} x_1^2 \in I \rightarrow \underset{0}{g} x_1 \in I$

$\rightarrow x_1 \leq d \bmod I$

$N(W,I|x) = xW + I$ All members of $N(W,I|x)$ have the form $dx + s$ $s \in I$ $d \in W$

1) $Ri\{N(W,I|x), I\} \leq xD + I$ select $d' \in Ri\{N(W,I|x), I\}$ Thus $d' = dx + s$ for some $d \in W$, $s \in I$

~~all d'.~~

$\cancel{(d\underset{0}{g} + I)}$ $(dx + s)(\underset{0}{g}x + t)$ ~~only when~~ $\in I$ only when $\underset{0}{g}x + t \in I$

$d\underset{0}{g} \in I \rightarrow d\underset{0}{g}x^2 \in I \rightarrow d\underset{0}{g}x^2 + \ldots + \underset{0}{g}xs + \ldots \in I \xrightarrow{I \in B_{x;si}(W) \cap B_{+;c}} (dx + s)\underset{0}{g}x$

$\rightarrow$ ~~$\underset{0}{g}x \ldots \in I$~~ $\rightarrow \underset{0}{g}x \in I$ ~~$(I \in B_{+;ls})$~~ $: x \leq d \bmod I$ $I \in B_{x;si}(W) \cap B_{+;c}$

1 2) $xD+I \equiv Ri\{N(W,I|x), I\}$

$Ri\{N(W,I|x), I\} \leq xD+I$ from (1)

select $d' = dx+s$ $(s\in I) \in xD+I$ If for any $g' = gx+t \in N(W, I/x)$.

$d'g' \in I$ then $(dx+s)(gx+t) = w \in I$ $dgx^2 = w - dxt - gxs - st \in I$

$(I \in B_c(W)) \to gx^3 \in I$ $(x \leq d)$ $g(x+q) \in I \to gx \in I \to gx+t \in I$

$\to g' \in I$: $d'g' \in I$ only when $g' \in I$: $d' \in Ri\{N(W, I|x), I\}$

—o—

$I \in B_{+;ls}$ $x+a+b+\dots+c = z \in I$ $p+c = z$ $p \in W$ $\cancel{c\in I}$ $c, z \in I$ $\underset{x\in W}{}$

I contains y such that $p = y \in I$ $\|$ $x+y = z$ $\cancel{\&}$ $y, z \in I \to x \in I$

$x+a+b+\dots+c = z \in I \to x \in I$ $\cancel{0+y=z}$ $0+z=z \to 0 \in I$

$x+a+b+\dots+c = z \in I \to x \in I$ then $x+(a+b+\dots+c) = z \to a+b+\dots+c \in I$

$I \in B_{+;ls}(W)$ $\overset{x\in W}{a}, \dots, d \in I$ if $\overset{x+}{a}+\dots+b+\dots+c+\dots+d \in I$

then $\overset{x+}{}$ $b+\dots+c \in I$

$I \in B_{+;c}$ $a, b \in I \to a+b \in I$ ~~if $x+a \in I$ then $x \in I$~~ ?

$\cancel{a+b=c}$ if $x+b = a+b$ $\cancel{\&}$ $a, b \in I \to x \in I$

$I \in B_{+;ls}(W)$ $\cancel{y\in}$ $y \in W$, $z \in I$ $y+z \in I \to y \in I$

if $x \in W$, $a, \dots, d \in I$ and $x+a+\dots+b+\dots+c+\dots+d \in I$

then $x+b+\dots+c \in I$

$I \in B_{+;c}$ $a, b \in I \to a+b \in I$

if $\cancel{a,b\in I}$ $x \in W$, $\cancel{a+b}$ $a, b \in I$ and $x+b = a+b$ then $x \in I$

—o—

$y = \cancel{y}_1 + \cancel{y}_2 \in Ra(W, I)$ $yx_1 = y_1 + s$ $yx_2 = y_2 + t$ $I \in B_{x;si}^{(W)} \cap B_{+;c}$

$x_1c + x_2c = c \overset{+r \quad r\in I}{\cancel{\bmod I}}$ $y(x_1+x_2) = y + s+t$

$y(x_1c + x_2c) = yc + c(s+t)$ $y\{x_1c + x_2c - c\} \in I$. $y \in Ri$

$\to x_1c + x_2c = c \mod I$

$y_1y_2 \in I \to y^2 x_1 x_2 \in I \quad I \in B_{x;si}(W) \cap B_{+;c} \to x_1x_2 \in I \quad (y \in R_2(W,I))$ 27

$(y_1+y_2)x_1 = y_1+s \qquad y_2^2 x_1 \in I \;\|\; x_1c + x_2c = c+p \quad x_1^2cd + x_2^2cd$

$x_1d + x_2d = d+q \qquad = cd + r$

$I \in B_2(W)$

$(x_1c)(x_1d) + (x_2c)(x_2d) = cd \mod I$

not true that $x_1(cd) = (x_1c)(x_1d)$ $\quad\Big|\quad (x_1c + x_1d) + (x_2c + x_2d) = c+d+t$

$x_1(cd) + x_2(cd) = cd \mod I$

$W_1(a) = x_1a \quad W_2(a) = x_2a \qquad W_1(a) + W_2(a) = a \mod I$

$W_1(a+b) = W_1(a) + W_1(b) \qquad W_2(a+b) = W_2(a) + W_2(b)$

$y_1y_2 \in I \qquad W_1(a)W_1(b) + W_2(a)W_2(b) = ab \mod I$

$\cancel{x_1^2 = x_1 + p \quad x_2^2 = x_2 + q \quad p, q \in I \to}$ also

$W_1(ab) = W_1(a)W_1(b) \mod I$ if $\cancel{x_2^2 = x_2 + q}$ $\quad W_2(ab) = W_2(a)W_2(b) \mod I$

$y_1y_2 \in I \quad yy_2x_1 \in I \to y_2x_1 \in I \quad y_1x_2 \in I$

$y_1x_1 = y_1 \mod I \qquad y_1x_1^2 = y_1x_1 = y_1 \mod I$

$yx_1^2 = y_1x_1 = y_1 \mod I \qquad y(x_1^2 - x_1) \in I \to x_1^2 = x_1 \mod I$

$yx_1 = y_1 \mod I$

—o—

$(ub+v)x_1 = ub+s \qquad (ub+v)x_2 = v+t$

$cx_1 \le b$? ✓ $\quad bg \in I \to (ub+v)x_1g \in I \to cx_1g \in I$

$N(W,I|x_1) \le EN(W,I|b) \qquad N(W,I|x_2) \le EN(W,I|v)$

only $b \in \Delta''(R_2(W,I), I) \to x_1 \le b \to cx_1 \le b$

$bg \in I \to cgx_1 \in I \;\|\; b \le u \quad I \in B_{x;qf} \to x_1 \equiv b$

$bg \in I \to dg \in I : \to d = cx_1 \;\|\|\; " \quad x_1g \in I \to dg \in I : \to d = cx_1$

$d = cx_1 + c'x_2 \;\|\; bg \in I \to (dx_1 + dx_2)g \in I$

$d = dx_1 + dx_2 \mod I$ $\quad$ $bg \in I \to dg \in I : \to dx_2 \in I$

$dx_2 \notin I$ $\quad \exists g$ such that $bg \in I$ but $dg \notin I$

$bx_2 \in I$

$b \in \Delta'\{W, Ra(W,I), I\}$ $\quad bv \in I$ $\quad (ub+v)\, bx_2 = bv + bt \in I \to bx_2 \in I$

$b \in \Delta''\{W, Ra(W,I), I\} \to N(W, I | x_1) \subseteq N(W, I | b) \subseteq EN(W, I | b)$

$b \in \Delta'\{W, Ra(W,I), I\} \to N\{W, I | x_1) = EN(W, I | b)$

$b \in \Delta\{W, Ra(W,I), I\}, I \in B_{x;\,qf}$ $\quad N(W, I | x_1) = EN(W, I | x_1)$

$N(W, I | x_1) = x_1 W + I$ $\quad N(W, I | b) = bW + I$ $\quad N(W, I | x_1) \subseteq N(W, I | b)$

cx_1 $\quad (ub+v)\, cx_1 = ubc + cs$ $\quad (ub+v)\, y = uc + w$

$= (ub+v)\, by + cs - bw$

$cx_1 = by + p$ $\quad p \in W$ $\quad cx_1 + q = by + p + q$ $\quad p, q \in I$ $\quad y \in W$

$cx_1 + q \in N(W, I | x_1) \to cx_1 + q \in N(W, I | b) \to N(W, I | x_1) \subseteq N(W, I | b)$

$b \in \Delta''\{W, Ra(W,I), I\}$

weak numerator set $N(W, I | b) \cdot a : b | a \mod I$ ($a \in W$) $\quad b.\, wd[W, I].\, a$

strong " " $\cdot a : b.$ ~~sd~~ $sd[[W, I]].\, a$ $\quad bx = a + s$ $\quad s \in I$, $\quad x \equiv a \mod I$

lond $\cdot a :$ $a \leq b \mod I$ $\quad b.\, s[W, I].\, a$

$a \equiv b \mod I$ $\quad b.\, ms[W, I].\, a$

~~ϕ, θ order~~ $\quad \phi \in DE[A, B]$: for each ~~all~~ pair $a \in A$ and $b \in B$ the relationship $a.\phi.b$ ~~is~~ either ~~true or not true~~ holds or does not ~~for no pair $a \in A$, $b \in B$~~ is known to hold or is known not to hold

$\phi, \theta \in DE[A, B]$ $\quad$ ~~ϕ, t~~ $\quad \phi \leq \theta\, [A|B]$: for all pairs $a \in A$, $b \in B$ for which $a.\phi.b$, $a.\theta.b$ also $\quad \phi.\!\leq\![A|B].\theta$

def $\phi < [A|B]\, \theta$ $\quad \phi = [A|B]\, \theta$

$wd[W, I] . \le \{W|W\} . s[W, I]$

$s[W, I] . \le \{\Delta'(W, Ra(W, I), I) | W\} . sd[W, I]$

strong support, weak support in terms of $L(W, I|b)$

$\phi \in DE[A, B]$. $C: CA \subseteq A$ $CB \subseteq B$ $\phi \in \overset{sa}{AB}\{A, B|C\}$

for all $a \in A$ $b \in B$ for which $a.\phi.b$, $ac.\phi.bc$ ~~also~~ for all $c \in C$ also

$wd[W, I] \in ab\{W, W|W\}$ ra, la sa

$\phi \in$ $C: CB \subseteq B$ $a.\phi.bc$ also for all $c \in C$ also

$wd \in ra, sa\{W, W|W\}$.

$\cancel{bx = a}$ $\cancel{by = a + s}$ $dy = c + t$ $t \in I$

$w_1(d)w_1(y) = w_1(c) \bmod I$ $\cancel{x_1^2}$ $x_1^2 = x_1 + p$ $x_1^2 dy = cx_1 + \cdots$

$\rightarrow \uparrow$ $x_1 y$ $x_2 y$

$\cancel{x_1 d . y_1 = x_1 c + t_1} \rightarrow$ $(x_1 d + W_1) y_1 = x_1 c + t_1$

$(x_2 d + W_2) y_2 = x_2 c + t_2$

$\cancel{uby_1 =}$ $uby_1 = ubc + t'$ $vy_2 =$ $y_1 \equiv x_1 c$ $y_2 \equiv x_2 c$

$(x_1 + x_2) dy_1 = x_1 c$ $(x_1 + x_2) d(y_1 + y_2) = (x_1 + x_2) c$ $d(y_1 + y_2) = c$

$(x_1 + x_2) dy_2 = x_2 c$

$b \in Ra(W, I)$: b $\overline{\text{strongly}}^{\,a}$ divides W $bx = a + s$ $a g \in I \rightarrow bxg \in I \rightarrow sxg \in I$

$x \equiv a \bmod I$ $I \in B_{x; si(W)} \cap B_{1; c}$