

Triangular Systems

Dramatis personae

In this section matrices and matrix functions ^{mainly} of triangular form are defined; their structures are commented upon.

Definition In the following $\mathbb{D} := \text{seq}'(K)$ and, where relevant, $\chi \in [P]$ where $P \in \mathbb{D}$.

~~i) Define $M, \hat{M}, N, \hat{N}, M^{-1}, \hat{M}^{-1} : \mathbb{D} \rightarrow \mathcal{L}[K | \mathbb{D}]$~~

~~by setting~~

~~$$M[P] := \mathcal{L}$$~~

$P \in \mathbb{D}$ is $\chi : \{ |P| \} \rightarrow K$. ~~the~~ ϵ and ν are used as row and column indexes respectively in the ~~sub-diagonal~~ ^{lower and upper triangular} matrix and matrix function allocations given below. In the case of ~~lower triangular matrix and matrix function allocation~~ ^{of lower triangular type}

ϵ traverses the range $\epsilon := [P]$ and ν the range $\nu := [Z]$. In the case of ~~upper triangular~~ ^{upper triangular} allocations the ranges are $\nu := [P]$ and $\epsilon := [Z]$ respectively. χ is used as the index in diagonal matrix and matrix functions ~~resp~~ allocations and traverses the range

χ

$\bar{\nu}$ etc
?

~~$$\chi := [P]$$~~

~~i) Define $M, \hat{M}, N, \hat{N}, M^{-1}, \hat{M}^{-1} : \mathbb{D} \rightarrow \mathcal{L}[K | \mathbb{D}]$ by setting~~

~~$$M[P] := \mathcal{L}[\mu(\chi[\epsilon], \nu)] \quad , \quad \hat{M}[P] := \mathcal{L}[\mu(\bar{\nu}, \epsilon)]$$~~

~~$$N[P] := \mathcal{L}[\mu(\bar{\nu}, \epsilon), \nu] \quad , \quad \hat{N}[P] := \mathcal{L}[\mu(\chi[\nu], \epsilon)]$$~~

$$M[P] := \mathcal{L}[\delta(\mu(P); \gamma[\rho, \epsilon])]$$

$$M^{-1}[P] := \mathcal{L}[\mu^{-1}(\gamma[\rho], \epsilon)] \quad \hat{M}^{-1}[P] := \mathcal{L}[\mu^{-1}(\bar{\gamma}[\epsilon], \rho)]$$

All of the above matrices are totally nonsingular, except for $M[P]$, which is nonsingular. ~~Mean'(K)~~ Mean'(K) and $\hat{M}^{annT}(K)$.

ii) Define $N, \hat{N}: \mathbb{D} \rightarrow \mathcal{U}\mathcal{L}[K || \mathbb{D}]$ by setting
 the inverse of $M[P]$ is expressible as
 $N[P] := \{M[P]\}^{-1} = \mathcal{L}[\delta(\frac{1}{\mu(P)}; \gamma[\rho, \epsilon])]$

ii) Define $\hat{M}^T M^{-1}, M \hat{M}^{-1T}: [K || \mathbb{D}^T]$ by setting

$$\hat{M}^T M^{-1}[P] := \mathcal{L}[\delta(\pi(\gamma[z], \bar{\gamma}[\rho]))]$$

$$M \hat{M}^{-1T}[P] := \mathcal{L}[\delta(\pi(\bar{\gamma}[\rho], \gamma[z]))]$$

~~iii) The elements of $\hat{M}^T M^{-1}[P]$ are zero when $\tau \leq |P|$, those~~

The above matrices are of triangular form, the elements of $\hat{M}^T M^{-1}[P]$ being zero when $\tau \leq |P|$, those of $M \hat{M}^{-1T}[P]$ being zero when $|P| < \tau$. The backward diagonal elements (i.e. those for which $\tau = |P|$) are unity. ~~Those parts of the above matrices that are not identically zero, as described above, are totally~~

iii) Define $N, \hat{N}: \mathbb{D} \rightarrow \mathcal{U}\mathcal{L}[K || \mathbb{D}]$ by setting

$$N[P] := \mathcal{L}[\mu(\gamma[\rho, \epsilon], \rho)] \quad \hat{N}[P] := \mathcal{L}[\mu(\gamma[\rho, \epsilon], \epsilon)]$$

$N[P]$ and $\hat{N}[P]$ are totally nonsingular.

iv) Define $\Pi M, L\Pi M, \acute{M}\acute{\Pi}, \acute{M}\acute{\Pi}L, L\Pi M\acute{M}\acute{\Pi}, \Pi M\acute{M}\acute{\Pi}L: \mathbb{D} \rightarrow \{K \rightarrow K\}$ by setting

$$\Pi M[\rho] := \mathcal{L}[\pi(\delta[\epsilon])\mu(\delta[\epsilon], \nu)], L\Pi M[\rho] := \mathcal{L}[\pi\mu(\delta[\epsilon], \nu)]$$

$$\acute{M}\acute{\Pi}[\rho] := \mathcal{L}[\pi(\bar{\delta}[\rho])\mu(\bar{\delta}[\rho], \epsilon)], \acute{M}\acute{\Pi}L[\rho] := \mathcal{L}[\pi\mu(\bar{\delta}[\rho], \epsilon)]$$

$$L\Pi M\acute{M}\acute{\Pi}[\rho] := \mathcal{L}[\delta(\bar{\pi}\mu(\rho); \bar{\delta}[\rho], \epsilon)\pi(\bar{\delta}[\rho], \epsilon)]$$

$$\Pi M\acute{M}\acute{\Pi}L[\rho] := \mathcal{L}[\Lambda(\pi\mu(\rho); \bar{\delta}[\rho], \epsilon)]$$

and $\Pi M \in \text{ann}(K)$ with $\delta \in [\rho]$

$\Pi M[\rho]$ is totally nonsingular over $K \setminus [\rho]^*$. When $z = \delta(x)$, all elements in the lower triangular part of $\Pi M[\rho|z]_{[x]}$ are nonzero, $\Pi M[\rho|z]_{-[x]} = O_{[\rho]-[x]}^{[\rho]}$, and the column sums in $\Pi M[\rho|z]^{-x}$ are zero, and $\Pi M[\rho|z]_x = 1: \Pi M[\rho|z] \in \text{ann}^T(x)$.

for: $L\Pi M[\rho]$ is totally nonsingular over $K \setminus [\rho]^* \setminus \text{When } z = \delta(x)$ with $x \in [\rho]^*$, all elements in the lower triangular part of $L\Pi M[\rho|z]_{[x]}^{[x]}$ are nonzero, ~~all elements in $L\Pi M[\rho|z]_{-[x]}^{-x}$ are zero~~, $L\Pi M[\rho|z]_{-[x]}^{-x} = O_{[|\rho|-x]}^{[\rho]}$, the remaining elements in the lower triangular part of $L\Pi M[\rho|z]$ are unity: $L\Pi M[\rho|z]_{-[x]}^x = I_{[|\rho|-x]}$

$\acute{M}\acute{\Pi}[\rho]$ is totally nonsingular over $K \setminus ([\rho])$, When $z = \delta(x)$ and $\acute{M}\acute{\Pi}[\rho] \in \text{ann}^T(K)$ with $x \in [\rho]$ all elements in the lower triangular part of $\acute{M}\acute{\Pi}[\rho|z]_{[x]}^{-[x]}$ are nonzero, $\acute{M}\acute{\Pi}[\rho|z]_{[x]}^{[x]} = O_{[x]}^{[x]}$, the row sums in $\acute{M}\acute{\Pi}[\rho|z]$

are zero, and $\hat{M}\hat{\Pi}[P|z]_{\chi}^{\lambda} = 1$. $\hat{M}\hat{\Pi}[P|z] \in \text{ann}(\chi)$,
and $\hat{M}\hat{\Pi}[P] \in \text{perm}(K)$

kt? $\hat{M}\hat{\Pi}[P]$ is ~~totally~~ nonsingular over $K \setminus [P]$. When $z = \delta(\chi)$ with $\chi \in [P]$, all elements in the lower triangular part of $\hat{M}\hat{\Pi}[P|z]^{-\chi}$ are nonzero, all elements in $\hat{M}\hat{\Pi}[P|z]_{-x}^{[x]}$ are zero: $\hat{M}\hat{\Pi}[P|z]_{-x}^{[x]} = 0_{[P]}^{[x]}$; the remaining elements in the lower triangular part of $\hat{M}\hat{\Pi}[P|z]$ are unity: $\hat{M}\hat{\Pi}[P|z] = I^{[x]}$

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or

\Rightarrow The row sum of $\hat{M}\hat{\Pi}[P]_z$ is $\Delta(\pi(\mu(P)); \delta[z]) \langle z := [P] \rangle$

v) Define $M^{-1}\Pi^{-1}, M^{-1}\Pi^{-1}L^{-1} : \mathbb{D} \rightarrow \{K \setminus \{0\}\} \rightarrow \mathcal{L}[K || \mathbb{D}]$
by setting

$$M^{-1}\Pi^{-1}[P] := \mathcal{L} \left[\frac{\mu^{-1}(\delta[\nu], z)}{\pi(\delta[\nu])} \right]$$

$$M^{-1}\Pi^{-1}L^{-1}[P] := \mathcal{L} \left[\frac{\mu^{-1}(\delta[\nu], z)}{\pi(\delta[\nu])} - \frac{\mu^{-1}(\delta[\nu], z)}{\pi(\delta[\nu])} \right]$$

Both ~~of the above~~ $M^{-1}\Pi^{-1}[P]$ and $M^{-1}\Pi^{-1}L^{-1}[P]$ are totally nonsingular over $K \setminus [P]$, and $M^{-1}\Pi^{-1}L^{-1}[P] \in \text{perm}(K \setminus [P])$.

vi) Define $\hat{\Pi}^{-1}\hat{M}^{-1}, \hat{L}^{-1}\hat{\Pi}^{-1}\hat{M}^{-1} : \mathbb{D} \rightarrow \{K \setminus \{0\}\} \rightarrow \mathcal{L}[K || \mathbb{D}]$

by setting

$$\hat{\Pi}^{-1}\hat{M}^{-1}[P] := \mathcal{L} \left[\frac{\mu^{-1}(\bar{\delta}[z], \nu)}{\pi(\bar{\delta}[z])} \right]$$

$$\hat{L}^{-1}\hat{\Pi}^{-1}\hat{M}^{-1}[P] := \mathcal{L} \left[\frac{\mu^{-1}(\bar{\delta}[z], \nu)}{\pi(\bar{\delta}[z])} - \frac{\mu^{-1}(\bar{\delta}[z], \nu)}{\pi(\bar{\delta}[z])} \right]$$

Both $\hat{T}^{-1}\hat{M}^{-1}[\rho]$ and $L^{-1}\hat{\Pi}^{-1}\hat{M}^{-1}[\rho]$ are totally nonsingular over $K \setminus \langle \rho \rangle$ and $L^{-1}\hat{\Pi}^{-1}\hat{M}^{-1}[\rho] \in \text{perm}^T(K \setminus \langle \rho \rangle)$.

vii) Define ~~$T: \mathbb{D} \rightarrow \mathcal{L}[K \parallel \mathbb{D}]$~~ $T, T^{-1}: \mathbb{D} \rightarrow \mathcal{L}[K \parallel \mathbb{D}]$ and ~~$\hat{T}: \mathbb{D} \rightarrow \mathcal{U}[K \parallel \mathbb{D}]$~~ $\hat{T}, \hat{T}^{-1}: \mathbb{D} \rightarrow \mathcal{U}[K \parallel \mathbb{D}]$ by setting

$$T[\rho] := \mathcal{L}[\mu^{-1}(\gamma(\rho, z), 0)], T^{-1}[\rho] := \mathcal{U}\mathcal{L}\mathbb{D}[\langle \rho \rangle - \gamma(0)]$$

$$\hat{T}[\rho] := \mathcal{U}[\mu^{-1}(\gamma(z, \rho), |\rho|)], \hat{T}^{-1}[\rho] := \mathcal{U}\mathcal{U}\mathbb{D}[\langle \rho \rangle - \gamma(|\rho|)]$$

viii) Define $S, S^{-1}: \mathbb{D} \rightarrow \mathcal{U}\mathcal{U}[K \parallel \mathbb{D}]$ by setting

$$S[\rho] := \mathcal{U}\mathcal{U}[\sigma(\gamma(\rho), z)], S^{-1}[\rho] := \mathcal{U}\mathcal{U}[\sigma(\langle \rho \rangle; \gamma(z))]$$

and $M^{-1}S^{-1}, SM^{-1}, \hat{M}S^{-1}: \mathbb{D} \rightarrow [K \parallel \mathbb{D}]$ by setting

$$M^{-1}S^{-1}[\rho] := V[\text{pow}[\rho], \rho]$$

$$SM^{-1}[\rho] := \begin{bmatrix} \sigma(\gamma[\langle \rho \rangle - \rho], z) \end{bmatrix}$$

$\text{seq}'(K \parallel \mathbb{D}) \times \mathbb{D}$

$$\hat{M}S^{-1}[\rho] := \begin{bmatrix} \mathcal{U}(\rho, z)\gamma(z)^{\rho} \end{bmatrix}$$

ix) Define ~~$W: \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{U}\mathcal{U}[K \parallel \mathbb{D}]$~~ $W, \hat{W}: \mathbb{D} \times \mathbb{D} \rightarrow \mathcal{U}\mathcal{U}[K \parallel \mathbb{D}]$ by setting

$$W[B/\rho] := \mathcal{U}[\sigma(\pi(\gamma(\rho)), \beta(z))]$$

$$\hat{W}[B/\rho] := \mathcal{U}[\sigma(\pi(\beta(z)), \bar{\gamma}(\rho))]$$

$W[B/\rho]$ is formed from the elements of $[B]$ and $\langle \rho \rangle$, $\hat{W}[B/\rho]$ from those of $[B]$ and $\langle \rho \rangle$.

x) Define $dM, dN, dN': \mathbb{D} \rightarrow \text{diag}[K | |D|]$ by setting

$$dM[P] := \text{diag}[\mu(P)]$$

$$dN[P] := \text{diag}[\mu(\delta[\omega], \omega)], dN'[P] := \text{diag}[\mu(\bar{\delta}[\omega], \omega)]$$

xi) Define $d\pi, d\pi', d\pi'', d\pi''', d\pi''', d\pi'''' : \mathbb{D} \rightarrow \{K \rightarrow \text{diag}[K | |D|]$
by setting

$$d\pi[P] = \text{diag}[\pi(\delta[\omega])] \quad d\pi'[P] := \text{diag}[\pi(\bar{\delta}[\omega])]$$

$$d\pi''[P] := \text{diag}[\pi(\frac{1}{2}\delta[\omega, \omega])] \quad d\pi'''[P] := \text{diag}[\pi(\bar{\delta}[\omega])]$$

$$d\pi''''[P] := \text{diag}[\pi(\bar{\delta}[\omega])] \quad d\pi''''[P] = \text{diag}[\pi\mu(P)]$$

$d\pi[P]$ is formed from the elements of $[P]_{\neq}$ and $d\pi''[P]$ from

those of $[P]$. $d\pi[P]$ is nonsingular over $K \setminus [P]$, $d\pi''[P]$ over $K \setminus [P]$,
and the remaining functions over $K \setminus P$.

~~x) Define $d\pi'' : \mathbb{D} \times \{ \text{diag}[D] \} \rightarrow$~~

xii) With $P \in \mathbb{D}$ and $x \in [P]_{\neq}$ define $d\pi''''[P, x] := K \setminus \delta[x] \rightarrow \text{diag}[K | P]$

by setting

$$d\pi''''[P, x] := \text{diag} \left[\left\langle \frac{1}{\pi(\delta[\omega])} \langle \omega := [x] \rangle \right\rangle + \left\langle \pi(\bar{\delta}[\omega]) \langle \omega := [P] - [x] \rangle \right\rangle \right]$$

$$d\pi''''[P, 0] = d\pi''[P] \langle K \rangle \text{ and } d\pi''''[P, |P|_H] = \{d\pi''[P]\}^{-1}$$

$\langle K \setminus P \rangle$. $d\pi''''[P, x]$ is nonsingular over $\{K \setminus P\} \cup \delta(x)$

xiii) Let $\mathbb{B} \subseteq K$.

a) Define $d : \{B \rightarrow \text{seq}(K)\} \rightarrow \text{diag}[K]$ by setting

$$d[F] := \text{diag}[F]$$

~~b) Define $\Delta : \{B \rightarrow \text{seq}(K | |D|)\} \times \mathbb{D} \rightarrow \{B \rightarrow \mathcal{L}[K | |D|]\}$~~

For each $z \in B$, $d[F|z]$ reduces to a multiple of $I[F]$ if and only if all components of $F(z)$ are equal. $NS\{d[F]\} \equiv NS\{F\}$.

b) Define ~~$\Delta: B \rightarrow$~~ $\Delta, \Delta L^{-1}, \Lambda: \{B \rightarrow \text{seq}(K|D|) \times D \rightarrow \{B \rightarrow \mathcal{L}[K|D|]\}$ by setting

$$\Delta[F, \rho] := \mathcal{L}[\delta(F; \gamma[\rho, e])]$$

$$\Delta L^{-1}[F, \rho] := \mathcal{L}[\delta(F; \gamma[\rho, e]) \pi(\gamma[\rho, e])]$$

$$\Lambda[F, \rho] := \mathcal{L}[\Lambda(F; \gamma[\rho, e])]$$

For each $z \in B$, $\Delta[F, \rho|z]$ reduces to diagonal form if and only if all components of $F(z)$ are equal and then becomes $d[F|z]$, a multiple of $I[\rho]$. The same holds true for $\Delta L^{-1}[F, \rho]$.

For each $z \in B$, ~~$\Lambda[F, \rho|z]$ also reduces to a constant multiple~~ ~~of~~ all components of $\Delta[F, \rho|z]$ are equal if and only if all components of $F(z)$ are equal; $\Lambda[F, \rho|z]$ then reduces to $\mathcal{L}[I[\rho]d[F|z]]$.

$$NS\{\Delta[F, \rho]\} \equiv NS\{\Delta L^{-1}[F, \rho]\} \equiv NS\{\Lambda[F, \rho]\} = NS\{F\}.$$

$$\{\Delta[F, \rho]\}^{-1} = \Delta\left[\frac{1}{F}, \rho\right] \quad \langle NS\{F\} \rangle$$

$$\{\Delta L^{-1}[F, \rho]\}^{-1} = \Delta L^{-1}\left[\frac{1}{F}, \rho\right] \quad \langle NS\{F\} \rangle$$

$$\{\Lambda[F, \rho]\}^{-1} = \mathcal{L}[I[\rho]^{-1} \Lambda\left[\frac{1}{F}, \rho\right] \mathcal{L}[I[\rho]]^{-1}] \quad \langle NS\{F\} \rangle$$

Let F be constant over $\tilde{B} \subseteq B$, $\Delta[F, \Delta]$ is constant over \tilde{B}

a) ~~$\Delta[F, \Delta]$ is constant over \tilde{B} and $\Delta^{-1}[F, \Delta]$ and $\Delta[F, \Delta]$~~

b) ~~Δ^{-1} are constant over \tilde{B} if and only if all components~~

c) ~~F are ^{equal} constant over \tilde{B} , these ~~two~~ ~~form~~ $\Delta^{-1}[F, \Delta|z]$ and~~

~~$\Delta[F, \Delta|z]$ with $z \in \tilde{B}$ then take the special forms described above.~~

~~$z \in \tilde{B}$~~ When $x \in (P)$ and $z = \gamma(x)$ is in B , $\Delta^{-1}[F, P|z]$ splits up into two triangular ^{sub}matrices separated by a

rectangular zero matrix: $\Delta^{-1}[F, P|z]_{-x}^{x} = \begin{matrix} & x \\ & \begin{bmatrix} & \\ & \end{bmatrix} \\ -x & \end{matrix}$

$\Delta[F, \Delta|z]$ contains a triangular submatrix whose ^{successive} columns

~~consist of copies of an one copies of corresponding~~ are vectors with equal components, namely, in each case, copies of the corresponding components of $F(z)$: $\Delta[F, P|z]_{-x}^{-x} =$

$$\mathcal{L}[I[|P|-\lambda]d[F|z]_{-x}^{-x}]$$

iv) 3 pp back

~~over $K \setminus (P)$~~
 The inverse of $\mathcal{L}\pi\mu[P]$ is expressible as

$$\{\mathcal{L}\pi\mu[P]\}^{-1} = \mathcal{L}\left[\Delta\left(\frac{1}{\pi\mu(P)}, \gamma(\rho, \epsilon)\right)\right] \langle K \setminus (P) \rangle$$

and that of $\mathcal{L}\pi\mu\pi L[P]$ as

$$\{\mathcal{L}\pi\mu\pi L[P]\}^{-1} = \mathcal{L}[P]^{-1} \mathcal{L}\left[\Delta\left(\frac{1}{\pi\mu(P)}, \gamma(\rho, \epsilon)\right)\right] \mathcal{L}[P]^{-1} \langle K \setminus (P) \rangle$$

and that of $\mathcal{L}\pi\mu\pi L[P]$ as

$$\{\mathcal{L}\pi\mu\pi L[P]\}^{-1} = \mathcal{L}\left[\frac{\delta\left(\frac{1}{\mu(P)}; \gamma(\rho, \epsilon)\right)}{\pi(\gamma(\rho))} - \frac{\delta\left(\frac{1}{\mu(P)}; \gamma(\rho, \epsilon)\right)}{\pi(\gamma(\rho))}\right] \langle K \setminus (P) \rangle$$

Conversing the last case

$$\{ \prod_{\rho \in P} \pi[\rho|z] \}^{-1} = \frac{\delta\left(\frac{z - \langle \rho \rangle}{\mu(\rho)}, \delta[\rho, \epsilon]\right)}{\pi(\delta[z]|z)\pi(\delta[\rho]|z)}$$

when $z \in K \setminus P$.

Relationships between linear triangular systems

connecting the matrices and matrix functions defined above

Various relationships, most of them suggested by the names given to the functions involved, are now established.

() Let $P \in \text{seq}'(K)$

i) $M[P]M[P] = dM[P] \quad M[P]M[P] = mM[P]$

$$\{M[P]\}^{-1} = M^{-1}[P] \quad \{M[P]\}^{-1} = M^{-1}[P]$$

~~$$M^T M^{-1}[P] =$$~~

$$M[P]^T \{M[P]\}^{-1} = M^T M^{-1}[P], \quad M[P] \{M[P]^T\}^{-1} = M M^{-T}[P]$$

~~$$M[P] = N[P] dN[P] = M[P], \quad dN[P] N^{-1}[P] = M[P]$$~~

$$\{N[P]\}^{-1} = N^{-1}[P] \quad dN[P] dN[P] = dM[P]$$

ii) $\pi M[P] = d\pi[P]M[P] \langle K \rangle, \quad M\pi[P] = M[P]d\pi[P] \langle K \rangle$

~~$$\{\pi M[P]\}^{-1} = M^{-1}\pi^{-1}[P] \langle K \setminus P \rangle, \quad \{M\pi[P]\}^{-1} = \pi^{-1}M^{-1}[P]$$~~

~~$$L[P]\pi M[P] = L\pi M[P] = d\pi[P]M[P] \{d\pi[P]\} \langle K \rangle$$~~

~~$$= d\pi[P]M[P] \{d\pi[P]\}^{-1} \langle K \setminus P \rangle$$~~

$$M\pi[P]L[P] = M\pi[L[P]] \langle K \rangle$$

$$= \{d\pi[P]\}^{-1} M[P] d\pi[P] \langle K \setminus P \rangle$$

~~$$\{\pi M[P]\}^{-1} = M^{-1}\pi^{-1}[P] \langle K \setminus P \rangle, \quad \{M\pi[P]\}^{-1} = \pi^{-1}M^{-1}[P] \langle K \setminus P \rangle$$~~

~~$$L\pi M[P]M\pi[P] = L\pi M\pi[P] \langle K \rangle, \quad \pi M[P]M\pi[L[P]] = \pi M\pi[L[P]] \langle K \rangle$$~~

$$\{\Pi M [P]\}^{-1} = M^{-1} \Pi^{-1} [P] \langle K \setminus [P] \rangle, \{\acute{M} \acute{\Pi} [P]\}^{-1} = \acute{\Pi}^{-1} \acute{M}^{-1} [P] \langle K \setminus [P] \rangle$$

$$\{L \Pi M [P]\}^{-1} = M^{-1} \Pi^{-1} L^{-1} [P] \langle K \setminus [P] \rangle, \{\acute{M} \acute{\Pi} L [P]\}^{-1} = L^{-1} \acute{\Pi}^{-1} \acute{M}^{-1} [P] \langle K \setminus [P] \rangle$$

$$ii) \{T [P]\}^{-1} = T^{-1} [P] \quad \{\acute{T} [P]\}^{-1} = \acute{T}^{-1} [P]$$

(The first of these results follows from the more general relationship)

$$\{Z [T(\delta \omega, \epsilon) | z]\}^{-1} = u Z Z [P] - z$$

and its reverse order form)

$$iii) \{S [P]\}^{-1} = S^{-1} [P], \{M [P] S [P]\}^{-1} = M^{-1} S^{-1} [P]$$

$$\{S [P] M [P]\}^{-1} = S M^{-1} [P], \{M [P] S [P]\}^{-1} = M S^{-1} [P]$$

$$S = \nabla, \\ M = \Delta$$

(In particular $S [P] M [P] = \{V [p_{\text{own}} [P], P]\}^{-1}$)

v) Let $B \in \text{seq}'(K | | P |)$.

$$W [B/P] = M [B] V [\Pi [P], B], \acute{W} [B/P] = V [\acute{\Pi} [B], P] \acute{M} [P]$$

$$\{W [B/P]\}^{-1} = W [P/B]; \{\acute{W} [B/P]\}^{-1} = \acute{W} [B/P]$$

$$W [P/P] = \acute{W} [P/P] = I [P]$$

The functions W, \acute{W} transform according to the following rule:
with $A \in \text{seq}'(K | | P |)$

$$W [A/B] W [B/P] = W [A/P], \acute{W} [A/B] \acute{W} [B/P] = \acute{W} [A/P].$$

vi) Let $F \in B \subseteq K$ and $F: B \rightarrow \text{seq}(K \setminus \{0\})$

$$a) \Delta L^{-1}[F, \Gamma] = d\pi[\Gamma] \Delta[F, \Gamma] \{d\pi[\Gamma]\}^{-1} \langle B \setminus \Gamma \rangle$$

When $x \in [\Gamma]$ and $z = \gamma(x) \in B$

$$\Delta L^{-1}[F, \Gamma|z] = \{d\pi[\Gamma, x]\}^{-1} \Delta[F, \Gamma|z] d\pi[\Gamma, x]$$

$$b) \Delta[F, \Gamma] = \Delta L^{-1}[F, \Gamma] \mathcal{L}[\Gamma] \langle B \rangle$$

$$= d\pi[\Gamma] \Delta[F, \Gamma] \mathcal{L}\left[\frac{1}{\pi(\gamma[z])}\right] \langle B \setminus \Gamma \rangle$$

When $x \in [\Gamma]$ and $z = \gamma(x) \in B$

$$\Delta[F, \Gamma|z] = \{d\pi[\Gamma, x]\}^{-1} \Delta[F, \Gamma|z] \mathcal{L}[\Gamma] \mathcal{L}[\Gamma]$$

where
 $\mathcal{L}[\Gamma, z]$

vii) The func and $\Delta L^{-1}[F, \Gamma]$

c) The functions $\Delta[F, \Gamma]$ may be expressed in terms of similarity transformations of the diagonal matrix function $d[F]$

$$\Delta[F, \Gamma] = M[\Gamma] d[F] M^{-1}[\Gamma]$$

$$= \acute{M}^{-1}[\Gamma] d[F] \acute{M}[\Gamma]$$

$$\Delta L^{-1}[F, \Gamma] = \pi M[\Gamma] d[F] M^{-1} \pi^{-1}[\Gamma] \langle B \setminus \Gamma \rangle$$

$$= \acute{\pi}^{-1} \acute{M}^{-1}[\Gamma] d[F] \acute{M} \acute{\pi}[\Gamma] \langle B \setminus \Gamma \rangle$$

When $x \in [\Gamma]$ and $z = \gamma(x) \in B$

$$\Delta L^{-1}[F, \Gamma|z] = \{d\pi[\Gamma, x]\}^{-1} M[\Gamma] d[F] M^{-1}[\Gamma] d\pi[\Gamma, x]$$

$$= \{d\pi[\Gamma, x]\}^{-1} \acute{M}^{-1}[\Gamma] d[F] \acute{M}[\Gamma] d\pi[\Gamma, x]$$

d) The function $\Delta[F, \rho]$ may also be expressed simply in terms of $d[F]$.

$$\begin{aligned}\Delta[F, \rho] &= \pi M[\rho] d[F] M^{-1} \pi^{-1} L[\rho] \quad \langle B \setminus \rho \rangle \\ &= \pi^{-1} M^{-1}[\rho] d[F] M \pi[\rho] I[\rho] \langle B \setminus \rho \rangle\end{aligned}$$

When $x \in \rho$ and $z = \gamma(x) \in B$

$$\begin{aligned}\Delta[F, \rho | z] &= \{ \det \pi[\rho, x] \}^{-1} M[\rho] d[F|z] M^{-1}[\rho] \det \pi[\rho, x] I[\rho] \\ &= \{ \det \pi[\rho, x] \}^{-1} M[\rho] d[F|z] M[\rho] \det \pi[\rho, x] I[\rho]\end{aligned}$$

Ring isomorphisms

The similarity transformations stated in clause (vi) of the above theorem suggest isomorphisms between systems of sequence functions and corresponding matrix functions

() Let $\rho \in \text{seq}'(K)$ and $B \subseteq K$.

i) The complete system of sequence functions in $\{B \rightarrow \text{seq}(K | |\rho|)\}$ form a commutative ring which may be denoted by \mathbb{F} . \mathbb{F} possesses a unit element — the constant function $I(|\rho|)$. A member F of \mathbb{F} is invertible if and only if $NS\{F\} = B$.

ii) The matrix functions $\Delta[F, \rho]$ in $\{B \rightarrow [K | |\rho|]\}$ derived from the members F of \mathbb{F} form a ring \mathbb{F}' which is isomorphic to \mathbb{F} .

In particular, with $F, G \in \mathbb{F}$,

$$\Delta[F+G, P] = \Delta[F, P] + \Delta[G, P] \quad \langle B \rangle$$

$$\Delta[F \times G, P] = \Delta[F, P] \Delta[G, P] \quad \langle B \rangle$$

F' contains the unit member $I[P]$ corresponding to $I\langle P \rangle$ in F . $\Delta[F, P]$ is nonsingular in F' if and only if F is invertible in F .

iii) The matrix functions ΔL^{-1} in $\{B \rightarrow [K|P]\}$ form a ring \hat{F} isomorphic to F . Relationships analogous to those of clause (ii) hold and the further, *mutatis mutandis*, the further remarks are also valid.

iv) The matrix functions $\Delta[F, P]$ in $\{B \rightarrow [K|P]\}$ form a ring \hat{F} in which the product of $A, B \in \hat{F}$ is taken to be

$$A \bar{\times} B = A \{ \Delta I[P] \}^{-1} B$$

\hat{F} is isomorphic to F .

Relationships analogous to those of clause (ii) (the matrix product in the second row being taken to be $\bar{\times}$ as defined above) hold over B .

$\Delta I[P]$ is the unit element in \hat{F} , corresponding to $I\langle P \rangle$ in F .

$\Delta[F, P]$ is nonsingular with respect to multiplication in $\{B \rightarrow [K|P]\}$ if and only if it is nonsingular with respect to the product $\bar{\times}$ in \hat{F} . ^{and, in turn, if and only if F is invertible in F .} If this is the case the inverse of

$$\Delta[F, P] \text{ in } \hat{F} \text{ is } \Delta\left[\frac{1}{F}, P\right]$$

$$\Delta\left[\frac{1}{F}, P\right] = \Delta I[P] \{ \Delta[F, P] \}^{-1} \Delta I[P].$$

Continued product decompositions

Certain of the triangular matrix and matrix functions considered above may be decomposed in the form of continued products

~~Concerning the matrix~~

The following theorem concerns the matrices $M, \hat{M}, M^{-1}, \hat{M}^{-1}, \hat{M}^T M^{-1}$ and $M \hat{M}^{-1T}$.

() Let $\mathbb{D} := \text{seq}'(K)$. In the following $\Gamma \in \mathbb{D}$ is $\delta: \{\Gamma\} \rightarrow K$. Define
 i) Define $\frac{m, m^{-1}}{m, m^{-1}}: \mathbb{D} \times \{\mathbb{D}\} \rightarrow \mathcal{L}[K | |\mathbb{D}|]$ by setting

$$m[\Gamma, \omega] := \text{ann } \mathcal{L}[\{\delta(\chi \Gamma + \omega) - \delta(\chi)\}^{-1}] \langle \chi := [|\Gamma| - \omega] \rangle$$

$$m^{-1}[\Gamma, \omega] := \mathcal{L}I[|\Gamma| - \omega] \text{diag} [1 + \langle \{\delta(\omega + \chi + 1) - \delta(\chi)\} \rangle \langle \chi := [|\Gamma| - \omega - 1] \rangle]$$

With

$$\mu(k; \delta[z], \nu) = \delta\left(\frac{1}{\pi(\delta[kz], kz)}; \delta[z]\right) - \delta\left(\frac{1}{\pi(\delta[kz], kz)}; \delta[\nu]\right)$$

and

$$\mu^{-1}(k; \delta[z], \nu) = \delta(\pi(\delta[z], \delta[kz, kz]))$$

define $M, M^{-1}: \{\mathbb{D}\} \times \mathbb{D} \rightarrow \mathcal{L}[K | \mathbb{D}]$ by setting

$$M[k, \Gamma] := \mathcal{L}[\mu(k; \delta[z], \nu)] \langle z := [|\Gamma| - k], \nu := [z] \rangle$$

$$M^{-1}[k, \Gamma] := \mathcal{L}[\mu^{-1}(k; \delta[z], \nu)]$$

$$a) \{m[\Gamma, \omega]\}^{-1} = m^{-1}[\Gamma, \omega] \quad \langle \omega := [\Gamma] \rangle$$

$$b) \mu(k; \delta[z], \rho) = \delta\left(\frac{1}{\pi(\delta[\rho])}; \delta[kz], kz\right) - \delta\left(\frac{1}{\pi(\delta[\rho])}; \delta[kz], kz\right)$$

If $k \geq 0$,

$$\mu(k; \delta[z], \rho) =$$

=

$$c) \mu^{-1}(k; \delta[z], \rho) = \delta(\pi(\delta[kz]), \delta[z], kz)$$

$$d) M[k, \Gamma] = \left\{ \mathcal{L}\left[-\delta\left(\frac{1}{\pi(\delta[\rho])}; \delta[\rho]\right) \langle z, \rho := [|\Gamma| - k] \rangle \right\} \mathcal{L}[|\Gamma| - k]$$

$$= \left\{ \mathcal{L}\left[\delta\left(\frac{1}{\pi(\delta[\rho])}; \delta[kz], kz\right) \langle z, \rho := [|\Gamma| - k] \rangle \right\} \mathcal{L}[|\Gamma| - k]$$

$$e) \{M[k, \Gamma]\}^{-1} = M^{-1}[k, \Gamma] \quad \langle k := [\Gamma] \rangle$$

f) With $k \in [\Gamma]$ and $h \in [k]$

$$M[h, \Gamma] = \text{disp}^{k-h} M[k, \Gamma] \left\{ \prod \text{disp}^{\omega-h} m[\Gamma, \omega] \langle \omega := [h, k] \rangle \right\}$$

$$g) M[k, \Gamma] = \prod \text{disp}^{\omega-k} m[\Gamma, \omega] \langle \omega := [k, |\Gamma|] \rangle$$

$$M^{-1}[k, \Gamma] = \overline{\prod} \text{disp}^{\omega-k} m^{-1}[\Gamma, \omega] \langle \omega := [k, |\Gamma|] \rangle$$

$$h) M[\Gamma] = M[0, \Gamma]$$

$$= \prod \text{disp}^{\omega} m[\Gamma, \omega] \langle \omega := [\Gamma] \rangle$$

$$M^{-1}[\Gamma] = M^{-1}[0, \Gamma]$$

$$= \overline{\prod} \text{disp}^{\omega} m^{-1}[\Gamma, \omega] \langle \omega := [\Gamma] \rangle$$

With $k \in \mathbb{R}$

$$\text{disp}^k M[k, \rho] = M[\rho] \left\{ \prod \text{disp}^{\omega} m^{-1}[\rho, \omega] \langle \omega := [\rho] \rangle \right\}$$

$$\text{disp}^k M^{-1}[k, \rho] = \left\{ \prod \text{disp}^{\omega} m[\rho, \omega] \langle \omega := [\rho] \rangle \right\} M^{-1}[\rho]$$

ii) Define $\hat{m}, \hat{m}^{-1}: \mathbb{D} \times \{[\rho]\} \rightarrow \mathcal{U}[K|\mathbb{D}]$ by setting

$$\hat{m}[\rho, \omega] := \text{ann} \mathcal{U} \left[\left\{ \gamma(x) - \gamma(x + \omega) \right\}^{-1} \right] \langle x := [|\rho| - \omega] \rangle$$

$$\hat{m}^{-1}[\rho, \omega] := \mathbb{I}[|\rho| - \omega] \text{diag} \left[\left\{ \gamma(x) - \gamma(x + \omega) \right\} \langle x := [|\rho| - \omega] \rangle + \mathbb{I} \right]$$

With

$$\mu(k; \bar{\gamma}[z], \rho) =$$

$$\delta \left(\frac{1}{\pi(\bar{\gamma}[z, \rho])}, \bar{\gamma}[\rho] \right) - \delta \left(\frac{1}{\pi(\bar{\gamma}[z, \rho])}, \bar{\gamma}[\rho] \right)$$

and

$$\mu^{-1}(k; \bar{\gamma}[z], \bar{z}) = \delta \left(\pi(\bar{\gamma}[\rho]), \bar{\gamma}[z, k+z] \right)$$

define $\hat{M}, \hat{M}^{-1}: \{[\rho]\} \times \mathbb{D} \rightarrow \mathcal{L}[K|\mathbb{D}]$ by setting

$$\hat{M}[k, \rho]^T := \mathcal{U} \left[\mu(k; \bar{\gamma}[z], \rho) \right] \langle z := [|\rho| - k], z := [\rho] \rangle$$

$$\hat{M}^{-1}[k, \rho]^T := \mathcal{U} \left[\mu^{-1}(k; \bar{\gamma}[\rho], z) \right] \langle \rho := [|\rho| - k], z := [\rho] \rangle$$

a) $\left\{ \hat{m}[\rho, \omega] \right\}^{-1} = \hat{m}^{-1}[\rho, \omega] \langle \omega := [\rho] \rangle$

b) $\mu(k; \bar{\gamma}[z], \rho) =$

$$\delta \left(\frac{1}{\pi(\bar{\gamma}[k, \rho])}, \bar{\gamma}[z, \rho] \right) - \delta \left(\frac{1}{\pi(\bar{\gamma}[k, \rho])}, \bar{\gamma}[z, \rho] \right)$$

$\bar{\gamma}$ if $k > 0$

$$\begin{aligned} \mu(k; \bar{\delta}[z], \nu) &= \{\delta(k\nu) - \delta(\nu)\} \delta\left(\frac{1}{\pi(\bar{\delta}[z, \nu])}, \bar{\delta}[k\nu]\right) \\ &= \{\delta(\nu) - \delta(k\nu)\} \delta\left(\frac{1}{\pi(\bar{\delta}[k\nu])}, \bar{\delta}[z, \nu]\right) \end{aligned}$$

c) $\mu^{-1}(k, \bar{\delta}[z], \nu) = \delta(\pi(\bar{\delta}[k\nu]), \bar{\delta}[z, \nu])$

d)
$$\begin{aligned} \acute{M}[k, \rho]^T &= \mathcal{U} \left[-\delta\left(\frac{1}{\pi(\bar{\delta}[z, \nu])}, \bar{\delta}[k\nu]\right) \right] \mathcal{U} \mathbb{I} [|\rho| - k]^{-1} \\ &= \mathcal{U} \left[\delta\left(\frac{1}{\pi(\bar{\delta}[k\nu])}, \bar{\delta}[z, \nu]\right) \right] \mathcal{U} \mathbb{I} [|\rho| - k]^{-1} \end{aligned}$$

e) $\{\acute{M}[k, \rho]\}^{-1} = \acute{M}^{-1}[k, \rho] \quad \langle k := |\rho| \rangle$

f) With $k \in [\rho]$ and $h \in [k]$

$$\acute{M}[h, \rho]^T = \overline{\text{disp}}^{k-h} \acute{M}[k, \rho]^T \{ \prod \overline{\text{disp}}^{\omega-h} \acute{m}[\rho, \omega] \langle \omega := [h, k] \rangle \}$$

g) $\acute{M}[k, \rho]^T = \prod \overline{\text{disp}}^{\omega-k} \acute{m}[\rho, \omega] \langle \omega := [k, |\rho|] \rangle$

$$\acute{M}^{-1}[k, \rho]^T = \prod \overline{\text{disp}}^{\omega-k} \acute{m}^{-1}[\rho, \omega] \langle \omega := [k, |\rho|] \rangle$$

h) $\acute{M}[\rho]^T = \acute{M}[0, \rho]^T$

$$= \prod \overline{\text{disp}}^{\omega} \acute{m}[\rho, \omega] \langle \omega := [\rho] \rangle$$

$$\acute{M}^{-1}[\rho]^T = \prod \overline{\text{disp}}^{\omega} \acute{m}^{-1}[\rho, \omega] \langle \omega := [\rho] \rangle$$

$$= \prod \overline{\text{disp}}^{\omega} \acute{m}^{-1}[\rho, \omega] \langle \omega := [\rho] \rangle$$

With $k \in [\rho]$

$$\overline{\text{disp}}^k \acute{M}[k, \rho]^T = \acute{M}[\rho]^T \{ \prod \overline{\text{disp}}^{\omega} \acute{m}^{-1}[\rho, \omega] \langle \omega := [k] \rangle \}$$

$$\overline{\text{disp}}^k \acute{M}^{-1}[k, \rho]^T = \{ \prod \overline{\text{disp}}^{\omega} \acute{m}[\rho, \omega] \langle \omega := [k] \rangle \} \acute{M}^{-1}[\rho]^T$$

iii) With

$$\tilde{\mu}(k; \gamma[z], \omega) = \delta(\pi(\gamma[z]), \bar{\gamma}[k\omega])$$

and

$$\tilde{\mu}^{-1}(k; \gamma[z], \omega) = \delta(\pi(\bar{\gamma}[\omega]), \gamma[k+z])$$

define $\hat{M}^T M^{-1}, M \hat{M}^{-1T}: \{[D]\} \times \mathcal{D} \rightarrow [K \mid |D| - k]$ by setting

$$\hat{M}^T M^{-1}[k, P] = [\tilde{\mu}(k; \gamma[z], \omega)] \langle z, \omega := [P] - k \rangle$$

$$M \hat{M}^{-1T}[k, P] = [\tilde{\mu}^{-1}(k; \gamma[z], \omega)] \langle z, \omega := [P] - k \rangle$$

a) $\tilde{\mu}(k; \gamma[z], \omega)$ is zero when $z\omega < |P| - k$ and

$$\tilde{\mu}(k; \gamma[z], \omega) = \delta(\pi(\gamma[k\omega]), \bar{\gamma}[z])$$

b) $\tilde{\mu}(k; \gamma[z], \omega)$ is zero when $|P| - k < z\omega$ and

$$\tilde{\mu}^{-1}(k; \gamma[z], \omega) = \delta(\pi(\bar{\gamma}[k+z]), \gamma[\omega])$$

$$c) \quad \{\hat{M}^T M^{-1}[k, P]\}^{-1} = M \hat{M}^{-1T}[k, P] \langle k := [P] \rangle$$

$$d) \quad \hat{M}^T M^{-1}[k, P] = \hat{M}[P]^T M^{-1}[k, P] \langle k := [P] \rangle$$

e) With $k \in [P]$ and $h \in [k]$ ($h, k \in [k]$?)

$$\hat{M}^T M^{-1}[h, P] = \overline{\text{disp}}^{k-h} \hat{M}[k, P]^T \{ \prod \overline{\text{disp}}^{\omega-h} \hat{m}[P, \omega] \langle \omega := [h, k] \rangle \}$$

$$\{ \prod \overline{\text{disp}}^{\omega-h} m^{-1}[P, \omega] \langle \omega := [h, k] \rangle \} \overline{\text{disp}}^{k-h} M^{-1}[k, P]$$

$$f) \quad \hat{M}^T M^{-1}[k, P] = \{ \prod \overline{\text{disp}}^{\omega-k} \hat{m}[P, \omega] \langle \omega := [k, |P|] \rangle \}$$

$$\{ \prod \overline{\text{disp}}^{\omega-k} m^{-1}[P, \omega] \langle \omega := [k, |P|] \rangle \}$$

The matrices S and S^{-1} also have continued product decomposition

Let $\mathbb{D} := \text{sq}'(\mathbb{K})$ and $\mathbb{D}'(\mathbb{D}) = \{[\mathbb{D}]\}$. In the following $\Gamma \in \mathbb{D}$ is $\gamma: \{[\Gamma]\} \rightarrow \mathbb{K}$. Define $s, s^{-1}: \mathbb{D} \times \mathbb{D}'(\mathbb{D}) \rightarrow \mathcal{U}[\mathbb{K} | |\mathbb{D}| - \mathbb{D}'(\mathbb{D})]$ by setting

$$s[\Gamma, \omega] := \mathcal{U}[\mathcal{S}[-\gamma(\omega)] \langle [\Gamma] - \omega \rangle]$$

$$s^{-1}[\Gamma, \omega] := \mathcal{U}[\gamma(\omega)^{\mathbb{D}-\omega} \langle \omega := [\Gamma] - \omega, z := \omega \rangle]$$

and $S, S^{-1}: \mathbb{D}'(\mathbb{D}) \times \mathbb{D} \rightarrow \mathcal{U}[\mathbb{K} | |\mathbb{D}| - \mathbb{D}'(\mathbb{D})]$ by setting

$$S[k, \Gamma] := \mathcal{U}[\mathcal{S}(\gamma[k, k+\omega], z) \langle \omega := [\Gamma] - k, z := \omega \rangle]$$

$$S^{-1}[k, \Gamma] := \mathcal{U}[\mathcal{S}(\langle \Gamma \rangle^{\mathbb{D}}; \gamma[k, k+z]) \langle \omega := [\Gamma] - k, z := \omega \rangle]$$

a) $\{s[\Gamma, \omega]\}^{-1} = s^{-1}[\Gamma, \omega] \langle \omega := [\Gamma] \rangle$

b) $\{S[k, \Gamma]\}^{-1} = S^{-1}[k, \Gamma] \langle k := [\Gamma] \rangle$

c) With $k \in [\mathbb{K}]$ and $h \in [k]$

$$S[\mathbb{K}, \mathbb{K}] = \left\{ \prod \text{disp}^{\omega-h} s[\Gamma, \omega] \langle \omega := [h, k] \rangle \right\} \text{disp}^{k-h} S[\mathbb{K}, \mathbb{K}]$$

d) $S[k, \Gamma] = \prod \text{disp}^{\omega-k} \mathcal{S}[\Gamma, \omega] \langle \omega := [k, \Gamma] \rangle$

$$S^{-1}[k, \Gamma] = \prod \text{disp}^{\omega-k} \mathcal{S}^{-1}[\Gamma, \omega] \langle \omega := [k, \Gamma] \rangle$$

e) $S[\Gamma] = S[0, \Gamma]$

$$= \prod \text{disp}^{\omega} s[\Gamma, \omega] \langle \omega := [\Gamma] \rangle$$

$$S^{-1}[\Gamma] = S^{-1}[0, \Gamma]$$

$$= \prod \text{disp}^{\omega} s^{-1}[\Gamma, \omega] \langle \omega := [\Gamma] \rangle$$

With $k \in [r]$

$$\text{disp}^k S[k, r] = S[r] \left\{ \prod \text{disp}^{\omega} s^{-1}[r, \omega] \langle \omega := [k] \rangle \right\}^{\overline{\pi}}$$

$$\text{disp}^k s^{-1}[k, r] = \left\{ \prod \text{disp}^{\omega} s[r, \omega] \right\} \langle \omega := [k] \rangle s^{-1}[r]$$

The continued product decompositions of the matrices W, \hat{W}, W^{-1} and \hat{W}^{-1} are stated in the following theorem.

() Let $\mathbb{D} := \text{seq}'(K)$ and $\mathbb{D}'(\mathbb{D}) := \{[D]\}$. In the following

$B, r \in \mathbb{D}$ are $\beta: \{[B]\} \rightarrow K, \gamma: \{[r]\} \rightarrow K$ respectively. $[\mathbb{D}] - \mathbb{D}'(\mathbb{D})$

i) Define $w, w^{-1}: \{\text{seq}'(K | \mathbb{D})\} \times \mathbb{D} \times \mathbb{D}'(\mathbb{D}) \rightarrow \mathcal{U}[K | \mathbb{D}]$ by setting

$$w[B/r, \omega] := \mathcal{U}[\beta[r | \omega] - \gamma(\omega)]$$

$$w^{-1}[B/r, \omega] := \mathcal{U}[\pi(\beta[z, \omega] | \gamma(\omega))] \langle \omega := [r | \omega], z := [r] \rangle$$

Define $W, W^{-1}: \mathbb{D}'(\mathbb{D}) \times \{\text{seq}'(K | \mathbb{D})\} \times \mathbb{D} \rightarrow \mathcal{U}[K | \mathbb{D} | \mathbb{D}'(\mathbb{D})]$

by setting

$$W[k; B/r] := \mathcal{U}[\delta(\pi(\gamma[k, k\omega]), \beta[z])] \langle \omega := [r | k], z := [r] \rangle$$

$$W^{-1}[k; B/r] := \mathcal{U}[\delta(\pi(\beta[\rho]), \gamma[k, k+z])] \langle \rho := [r | k], z := [r] \rangle$$

a) $\{w[B/r, \omega]\}^{-1} = w^{-1}[B/r, \omega] \langle \omega := [r] \rangle$

b) $\{W[k, B/r]\}^{-1} = W^{-1}[k, B/r] \langle k := [r] \rangle$

c) With $k \in [r]$ and $h \in [k]$

$$W[h, B/r] = \left\{ \prod \text{disp}^{\omega-h} w[B/r, \omega] \langle \omega := [h, k] \rangle \right\} \text{disp}^{k-h} W[k, B/r]$$

d) $W[k, B/r] = \prod \text{disp}^{\omega-k} w[B/r, \omega] \langle \omega := [k, r] \rangle$

$$W^{-1}[k, B/r] = \prod \text{disp}^{\omega-k} w^{-1}[B/r, \omega] \langle \omega := [k, r] \rangle$$

$$e) \quad W[B/r] = W[0, B/r] \\ = \overline{\prod} \text{disp}^\omega W[B/r, \omega] \langle \omega := [r] \rangle$$

$$W^{-1}[B/r] = W^{-1}[0, B/r] \\ = \overline{\prod} \text{disp}^\omega W^{-1}[B/r, \omega] \langle \omega := [r] \rangle$$

$$\text{disp}^k W[k, B/r] = \left\{ \overline{\prod} \text{disp}^\omega W[B/r, \omega] \langle \omega := [k] \rangle \right\} W[B/r]$$

$$\text{disp}^k W^{-1}[k, B/r] = W^{-1}[B/r] \left\{ \overline{\prod} \text{disp}^\omega W^{-1}[B/r, \omega] \langle \omega := [k] \rangle \right\}$$

ii) Define $\acute{W}, \acute{W}^{-1}: \{\text{seq}'(K|D)\} \times D \times D'(D) \rightarrow \mathcal{U}[K|D|D'(D)]$
by setting

$$\acute{W}[B/r, \omega] := \mathcal{U}[\pi(\beta(\omega+z, \omega), \gamma(|r|-\omega))] \langle \omega := [r]-\omega, z := [r] \rangle$$

$$\acute{W}^{-1}[B/r, \omega] := \mathcal{U}[\pi(\beta[\omega], \gamma(|r|-\omega))] \langle \omega := [r]-\omega, z := [r] \rangle$$

Define $\acute{W}, \acute{W}^{-1}: D'(D) \times \{\text{seq}'(K|D)\} \times D \rightarrow \mathcal{U}[K|D|D'(D)]$ by
setting

$$\acute{W}[k, B/r] := \mathcal{U}[\pi(\beta[k+z], \gamma[r], |r|-k)] \langle \omega := [r]-k, z := [r] \rangle$$

$$? \quad \acute{W}^{-1}[k, B/r] := \mathcal{U}[\pi(\gamma[z, |r|-k], \beta[k])] \langle \omega := [r]-k, z := [r] \rangle$$

$$a) \quad \{\acute{W}[B/r, \omega]\}^{-1} = W^{-1}[B/r, \omega] \langle \omega := [r] \rangle$$

$$b) \quad \{\acute{W}[k, B/r]\}^{-1} = W^{-1}[k, B/r] \langle k := [r] \rangle$$

c) With $k \in [r], h \in [k]$

$$\acute{W}[h, B/r] = \left\{ \overline{\prod} \text{disp}^{\omega-h} \acute{W}[B/r, \omega] \langle \omega := [h, k] \rangle \right\} \text{disp}^{k-h} \acute{W}[k, B/r]$$

$$d) \quad \acute{W}[k, B/r] = \overline{\prod} \text{disp}^{\omega-k} \acute{W}[B/r, \omega] \langle \omega := [k, |r] \rangle$$

$$\acute{W}^{-1}[k, B/r] = \overline{\prod} \text{disp}^{\omega-k} \acute{W}^{-1}[B/r, \omega] \langle \omega := [k, |r] \rangle$$

$$e) \quad \acute{W}[B/P] = \acute{W}[0, B/P]$$

$$= \prod \overline{\text{drop}}^{\omega} \acute{w}[B/P, \omega] \langle \omega := [P] \rangle$$

$$\acute{W}^{-1}[B/P] = \acute{W}^{-1}[0, B/P]$$

$$= \prod \overline{\text{drop}}^{\omega} \acute{w}^{-1}[B/P, \omega]$$

$$\overline{\text{drop}}^k \acute{W}[k, B/P] = \left\{ \prod \overline{\text{drop}}^{\omega} \acute{w}^{-1}[B/P, \omega] \langle \omega := [k] \rangle \right\} \acute{W}[B/P]$$

$$\overline{\text{drop}}^k \acute{W}^{-1}[k, B/P] = \acute{W}^{-1}[B/P] \left\{ \prod \overline{\text{drop}}^{\omega} \acute{w}[B/P, \omega] \langle \omega := [k] \rangle \right\}$$

Transformations

The functions matrices and matrix functions introduced in the preceding sections possess properties of transformation. The vectors and vector functions upon which they operate are now described.

Definition In the following $\mathbb{D} := \text{seq}'(K)$ and $\Gamma \in \mathbb{D}$ is $\gamma: \{[P]\} \rightarrow K$. z and ν are used as indexes in the row and column vector and vector function allocations respectively; they both traverse the range $[P]$

i) Define $c\pi, \hat{c}\pi, c\pi', \hat{c}\pi': \mathbb{D} \rightarrow \{K \rightarrow \text{col}[K | |\mathbb{D}|]\}$ by setting

$$c\pi[P] := \text{col}[\pi(\gamma[z])] \quad \hat{c}\pi[P] := \text{col}[\pi(\gamma[\nu])]$$

$$c\pi'[P] := \text{col}[\pi(\gamma[z])] \quad \hat{c}\pi'[P] := \text{col}[\pi(\gamma[\nu])]$$

ii) Define $r\text{row}, r\pi, r\pi': \mathbb{D} \rightarrow \{K \rightarrow \text{row}[K | |\mathbb{D}|]\}$ by setting

$$r\text{row}[P] := \text{row}[p\text{un}[P]]$$

$$r\pi[P] := \text{row}[\pi(\gamma[\nu])] \quad r\pi'[P] := \text{row}[\pi(\gamma[[P] - \nu])]$$

iii) Let $B \subseteq K$ and set $\mathbb{D}' := \{B \rightarrow \text{seq}(K)\}$

a) Define $c: \mathbb{D}' \rightarrow \{B \rightarrow \text{col}[K | |\mathbb{D}'|]\}$ and

$r: \mathbb{D}' \rightarrow \{B \rightarrow \text{row}[K | |\mathbb{D}'|]\}$ by setting

$$c[F] := \text{col}[F]$$

$$r[F] := \text{row}[F]$$

b) Define $c\Delta, c\pi\Delta, c\Delta: \{B \rightarrow \text{seq}(K | |\mathbb{D}|)\} \times \mathbb{D} \rightarrow \{B \rightarrow \text{col}[K | |\mathbb{D}|]\}$ by setting

$$c\Delta[F, \Gamma] := \text{col}[\delta(F, \gamma[z])], \quad c\pi\Delta[F, \Gamma] := \text{col}[\pi(\gamma[z])\delta(F, \gamma[z])]$$

$$c\Lambda[F, \Gamma] := \text{col}[\Lambda(F, \gamma[z])]$$

$r\Delta$:

c) Define $r\Delta, r\pi\Delta, r\Lambda: \{B \rightarrow \text{seq}(K || D)\} \times D \rightarrow \{B \rightarrow \text{col}(K || D)\}$

by setting

$$r\Delta[F, \Gamma] := \text{row}[\delta(F, \bar{\gamma}[\omega])], \quad r\pi\Delta[F, \Gamma] := \text{row}[\pi(\bar{\gamma}[\omega])\delta(F, \bar{\gamma}[\omega])]$$

$$r\Lambda[F, \Gamma] := \text{row}[\Lambda(F, \bar{\gamma}[\omega])]$$

The transformation properties are so stated in the following

theorem

() Let $D := \text{seq}'(K)$ and $B \subseteq K$. In the following $P \in D$ and $F \in \text{seq}(K || P)$.

$$i) \quad M[\Gamma]c[F] = c\Delta[F, \Gamma] \langle B \rangle, \quad r[F]M[\Gamma] = r\Delta[F, \Gamma] \langle B \rangle$$

$$\pi M[\Gamma]c[F] = c\pi\Delta[F, \Gamma] \langle B \rangle, \quad r[F]M\pi[\Gamma] = r\pi\Delta[F, \Gamma] \langle B \rangle$$

$$L\pi M[\Gamma]c[F] = c\Lambda[F, \Gamma] \langle B \rangle, \quad r[F]M\pi L[\Gamma] = r\Lambda[F, \Gamma] \langle B \rangle$$

$$ii) \quad T[\Gamma]c\pi[\Gamma] = \text{col}[\pi(\bar{\gamma}[\omega])]c\hat{\pi}[\Gamma] \langle K \rangle$$

$$\hat{T}[\Gamma]c\hat{\pi}[\Gamma] = c\hat{\pi}[\Gamma] \langle K \rangle$$

$$iii) \quad r\pi[\Gamma] = r\pi'[\Gamma]M[\Gamma] = r_{\text{pow}}[\Gamma]S[\Gamma] \langle K \rangle$$

$$r\pi'[\Gamma] = r_{\text{pow}}[\Gamma]S\hat{M}^{-1}[\Gamma] \langle K \rangle$$

iv) Let $B \in \text{seq}'(K || P)$

$$r\pi[B]W[B/P] = r\pi[\Gamma] \langle B \rangle, \quad \hat{W}[B/P]c\hat{\pi}[\Gamma] = c\hat{\pi}[B] \langle K \rangle$$