The expression of moments as continued fraction integrals

$$\phi(t) = e^{-zt}\int_a^b e^{ut}\,dw(u) \qquad (\mathrm{CF})\int_t^\infty \phi(\xi)\,d\xi = e^{-zt}\int_a^b \frac{e^{ut}\,dw(u)}{z-u}$$

$$(\mathrm{CF})\int_t^\infty \xi\,\phi(\xi)\,d\xi = -\frac{\partial}{\partial z}\left\{e^{-zt}\int_a^b \frac{e^{ut}\,dw(u)}{z-u}\right\}$$

$$= te^{-zt}\int_a^b \frac{e^{ut}\,dw(u)}{z-u} + e^{-zt}\int_a^b \frac{e^{ut}\,dw(u)}{(z-u)^2}$$

$t + \frac{1}{z-u}$

$$= e^{-zt}\int_a^b \frac{\{zt - tu + 1\}e^{ut}\,dw(u)}{(z-u)^2}$$

$zt - tu + 1$

$$\cancel{\left(-\frac{\partial}{\partial z}\right)^r e^{-zt}f(z)}$$

$$\left(-\frac{\partial}{\partial z}\right)^r e^{-zt}\cdot\frac{1}{z-u} = (-1)^r\sum_{s=0}^r \binom{r}{s}(-t)^{r-s}\frac{s!}{(z-u)^{s+1}}\cdot(-1)^s\; e^{-zt}$$

$$= \sum_{s=0}^r \binom{r}{s} t^{r-s}\frac{s!}{(z-u)^{s+1}}$$

$$\sum_{s=0}^r \binom{r}{s} s!\,\rho^s = \frac{1}{\rho}\sum_{s=0}^r \frac{\binom{r}{s}s!}{\left(\frac{1}{\rho}\right)^{s+1}}$$

$\boxed{\sigma>0}$ $$\frac{s!}{\sigma^{s+1}} = \int_0^\infty e^{-\sigma u}u^s\,ds$$

$$= \frac{1}{\rho}\int_0^\infty e^{-\frac{1}{\rho}u}(1+u)^r\,du$$

$$\sum_{s=0}^r \binom{r}{s}\frac{t^{r-s}s!}{(z-u)^{s+1}} = \frac{t^r}{(z-u)}\cdot\sum_{s=0}^r \binom{r}{s}s!\left(\frac{1}{t(z-u)}\right)^s$$

$$= \frac{t^r}{\cancel{z-u}}\cdot t\,\cancel{(z-u)}\int_0^\infty e^{-t(z-u)\tau}(1+\tau)^r\,d\tau$$

$$\left(-\frac{\partial}{\partial z}\right)^r(\mathrm{CF})\int_t^\infty \phi(\xi)\,d\xi = e^{-zt}\int_a^b \frac{e^{ut}\,t^{r+1}\int_0^\infty e^{-t(z-u)\tau}(1+\tau)^r\,d\tau\,dw(u)}{z-u}$$

$$(\,t(z-u)>0\,)$$

$$\sum_{s=0}^{\infty} \frac{\left(\frac{-\partial}{\partial z}\right)^{s} (CF) \int_t^\infty \phi(\xi)\, d\xi}{\lambda^{s+1}}$$

$$= e^{-zt} \int_a^b \frac{t e^{ut} \int_0^\infty e^{-t(z-u)\tau} d\tau \, dw(u)}{(z-u)\{\lambda - t(1+\tau)\}} \qquad t(z-u) > 0$$

$$\sigma < 0 \qquad \frac{s!}{\sigma^{s+1}} = (-1)^{s+1} \int_0^\infty e^{\sigma u} u^s \, ds$$

$$\sum_{s=0}^{r} \binom{r}{s} s!\, \rho^s = \frac{1}{\rho} \sum_{s=0}^{r} \frac{\binom{r}{s} s!}{\left(\frac{1}{\rho}\right)^{s+1}} = \frac{1}{\rho} \int_0^\infty e^{\frac{1}{\rho}\tau} \sum_{s=0}^{r} (-1)^{s+1} \tau^s \binom{r}{s} d\tau$$

$$= -\frac{1}{\rho} \int_0^\infty e^{\frac{1}{\rho}\tau} (1-\tau)^r d\tau$$

$$\frac{t^r}{z-u} \sum_{s=0}^{r} \binom{r}{s} s! \left(\frac{1}{t(z-u)}\right)^s = -\frac{t^r}{z-u}\, t(z-u) \int_0^\infty e^{t(z-u)\tau} (1-\tau)^r d\tau$$

$t(z-u) < 0$:

$$\left(\frac{-\partial}{\partial z}\right)^r (CF) \int_t^\infty \phi(\xi)\, d\xi = -\frac{e^{-zt} \int_a^b e^{ut} t^{r+1} \int_0^\infty e^{t(z-u)\tau} (1-\tau)^r d\tau \, dw(u)}{z-u}$$

$$\sum_{s=0}^{\infty} \frac{\left(-\frac{\partial}{\partial z}\right)^s (CF) \int_t^\infty \phi(\xi)\, d\xi}{\lambda^{s+1}} = -e^{-zt} \int_a^b \frac{e^{ut} t \int_0^\infty e^{t(z-u)\tau} d\tau \, dw(u)}{(z-u)\{\lambda - t(1-\tau)\}}$$

$\lim t \to 0$

$$\left(\frac{-\partial}{\partial z}\right)^r (CF) \int_t^\infty \phi(\xi)\, d\xi = \cancel{e^{z}} \int_a^b \frac{dw(u)}{(z-u)^{r+1}}$$

$$\sum_{s=0}^{\infty} \frac{\left(\frac{-\partial}{\partial z}\right)^s CF \int_t^\infty \phi(\xi)\, d\xi}{\lambda^{r+1}} = \int_a^b \frac{dw(u)}{\lambda(z-u) - 1}$$