

Delayed versions of interpolatory formulae

96.

In the following a delay parameter r is introduced into the standard formulae used in interpolation theory. Much of the standard theory can be extended directly to delayed form.

Definition. In the following $\bar{N}' := N'' := N''' := \bar{N}$

i) The delayed divided difference multiplier mapping

$\mu: \bar{N} \times \text{seq}'(K | \geq \bar{N} + N' + N'') \times N' \times N'' \times \bar{N} \rightarrow K$ is defined by setting

$$\mu(r; \alpha || m; i, \omega) := \frac{\delta(\phi(\alpha || m, \omega); \alpha || m || m+1, i-1)}{\delta(\phi(\alpha || m, \omega); \alpha || m || m+1, i-2)}$$

$$\mu(r; \alpha || m; i, \omega) := \frac{\delta(\phi(\alpha || m || m+1, i-2), \alpha || m, \omega)}{\delta(\phi(\alpha || m || m+1, i-2-1), \alpha || m, \omega-1)}$$

ii) The delayed divided difference mapping $\delta: \bar{N} \times \text{seq}(K | \geq \bar{N} + N' + N'') \times \text{seq}'(K | \geq \bar{N} + N' + N'') \times N' \times N'' \rightarrow K$ is defined by setting

$$\delta(r; f; \alpha || m, i) := \sum \mu(r; \alpha || m; i, \omega) f_{m+\omega} \quad \langle \omega = [i] \rangle$$

iii) The delayed factorial polynomial mapping $\pi: \bar{N} \times \text{seq}'(K | \geq \bar{N} + N' + N'') \times N' \times N'' \rightarrow \overline{\{K \rightarrow K\}}$ is defined by setting

$$\pi(r; \alpha || m; i, \omega) := \pi(\alpha || m, \omega) \pi(\alpha || m || m+1, i-1)$$

$\boxed{\begin{array}{l} \text{def } \pi(\alpha || m, \omega) \\ \text{with } \omega \in \bar{N} ?? \end{array}}$

iv) The delayed Lagrange multiplier mapping $\lambda: \bar{N} \times \text{seq}(K | \geq \bar{N} + N' + N'') \times N' \times N'' \times \bar{N} \rightarrow \{K \rightarrow K\}$ is defined by setting

$$\lambda(r; \alpha || m, i, \omega) := \mu(r; \alpha || m; i, \omega) \pi(r; \alpha || m; i, \omega)$$

v) The delayed Lagrange interpolation polynomial mapping $\Lambda: \overline{\mathbb{N}}^{\times} \times \text{seq}(\mathbb{K} | \Delta \mathbb{N}' + \mathbb{N}''') \times \text{seq}'(\mathbb{K} | \Delta \overline{\mathbb{N}} + \mathbb{N}' + \mathbb{N}'''') \times \mathbb{N}' \times \mathbb{N}''' \rightarrow \{\mathbb{K} \rightarrow \mathbb{K}\}$ is defined by setting

$$\Lambda(r; f; \alpha || m, i) := \sum \lambda(r; \omega || m; i, \omega) f_{m\omega} \quad \langle \omega := [i] \rangle$$

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Two sequences formed from values of factorial polynomials are defined.

i) ~~Definition~~, ii) The mapping ~~Θ~~ : Set $\mathbb{N}' := \overline{\mathbb{N}}$ $\forall \bar{n} \in \mathbb{N}'$

ii) The mapping $\Theta: \text{seq}'(\mathbb{K} | \Delta \overline{\mathbb{N}} + \mathbb{N}') \times \overline{\mathbb{N}} \times \mathbb{N}' \rightarrow \text{seq}(\mathbb{K})$ is defined by setting ??

$$\Theta(\alpha || n, j | \omega) = \pi(\alpha || n, j | \omega) \quad \pi(\alpha || n, j | \omega) \langle \omega := [\omega] \rangle$$

ii) The mapping $\phi: \text{seq}' \text{seq}(\mathbb{K} | \Delta \overline{\mathbb{N}} + \mathbb{N}') \times \overline{\mathbb{N}} \times \mathbb{N}' \rightarrow \text{seq}(\mathbb{K})$ is defined by setting

$$\phi(\alpha || n, j | \omega) = \frac{1}{\pi(\alpha || n, j | \omega)} \quad \langle \omega := \overline{\mathbb{N}} \setminus [n, j] \rangle$$

and taking the members of the ^{nb} sequence $\phi(\alpha || n, j | [n, j])$ to be arbitrarily assigned members of \mathbb{K} . (In subsequent use of the sequence mapping ϕ , members of the subsequence $\phi(\alpha || n, j | [n, j])$ are not involved)

() Let $r, m, i \in \bar{\mathbb{N}}$, ~~$\omega \in \bar{\mathbb{N}}$~~ and $\alpha \in \text{seq}'(K \models r m n i)$

i) $\mu(r: \alpha || m; i, \omega) = \frac{1}{2} \delta(\phi(\alpha || m, \omega); \alpha || m n i, i - \omega) - \delta(\phi(\alpha || m, \omega + 1); \alpha || m n i + 1, i - \omega - 1)$ q255

ii) ($r=0$):

$$\mu(0: \alpha || m; i, \omega) = \mu(\alpha || m; i, \omega)$$

b) If $r > 0$

$$\begin{aligned} \mu(r: \alpha || m; i, \omega) &= (\alpha_{m,i} - \alpha_{m,n,i}) \delta(\phi(\alpha || m n i, i - \omega); \alpha || m, \omega) \\ &= (\alpha_{m,n,i} - \alpha_{m,i}) \delta(\phi(\alpha || m, \omega + 1); \alpha || m n i, i - \omega) \end{aligned}$$

c) ~~$\mu(r: \alpha || m; i, \omega)$~~ :

$$\mu(r: \alpha || m; i, \omega) = \frac{1}{\pi(\alpha || m n i + 1, i | \alpha_m)}$$

~~$\omega = i$~~ and

$$\mu(r: \alpha || m; i, i) = \frac{1}{\pi(\alpha || m, i | \alpha_{m,i})}$$

iii) ~~$\mu(r: \alpha || m; i, \omega)$~~ With $i > 0$

$$\left\{ \sum_i \mu(r: \alpha || m; i, \omega) \langle \omega := [i] \rangle \right\} = 0$$

In the theorems of this section, integers are specified in the form $r, m, i \in \bar{\mathbb{N}}$. Specifications of the form $r, m, i \in \bar{\mathbb{N}}$ are given occur and array specifications of the form $\alpha \in \text{seq}'(K \models r m n i)$ follow. Special results concerning general values of r, m, n and i are stated, but it may occur that special results concerning the case in which $r=0$, for example, are given. In the latter

case, the array specification of α is to be interpreted in the form α_0 ,
 $\alpha_1 \text{ seq}'(K | \geq m_i)$ and to draw attention to this fact the special
constant is preceded by the parenthetic comment ($r=0$). Special
values of other integers and further array specifications are
treated in a similar way.

$r, m, i \in \mathbb{N}$ and $\alpha \in \text{seq}'(\mathbb{K} | \geq r+m+i)$ follows

iii) Let $f \in \text{seq}(\mathbb{K} | \geq r+m+i)$

a) The divided differences possess the following telescoping property: with

$j \in [i]$ and $f \in \text{seq}(\mathbb{K} | \geq r+m+i)$,

$$\delta(r : \delta(r-1 : f ; \alpha || \langle [\alpha] - r-i \rangle, i-j), \alpha || m, j) = \delta(r : f ; \alpha || m, i)$$

b) In particular

$[r+m+i]$

$\delta(r : \delta(r-k : f ; \alpha || \langle [\alpha] - r-i \rangle, k), r) ; \alpha || m, i) = \delta(f ; \alpha || m, i)$

i) Let $f \in \text{seq}(\mathbb{K} | \geq m+i)$

$$\delta(0 : f ; \alpha || m, i) = \delta(f ; \alpha || m, i)$$

Let $m_i > 0$ and $n \in [m_i]$ and $g_x \in \mathbb{K} \langle x := [n] \rangle$. Define the n^{th} degree power polynomial mapping $g : \mathbb{K} \rightarrow \mathbb{K}$ by setting

$$g(x) = \sum g_{\omega} x^{\omega} \quad \langle \omega := [n] \rangle$$

and $f \in \text{seq}(\mathbb{K} | m+i)$ by setting $f = g(\alpha[m+i])$.

$$f_n = g(\alpha[n])$$

$$\delta(r : \delta(f ; \alpha || \langle [m+i] \rangle, r) ; \alpha || m, i) = 0$$

such that $f[m, m+i]$ is

b) let $m_i > 0$ and $f' \in \text{seq}(\mathbb{K} | m+i)$ be a constant sequence, so that $f'_\omega = F \in \mathbb{K} \langle \omega := [m, m+i] \rangle$.

$$\delta(r : f' ; \alpha || m, i) = 0$$

Let $r, m, i \in \mathbb{N}$ and $\alpha \in \text{seq}'(\mathbb{K} | \geq r+m+i)$

$$i) \pi(r : \alpha || m, i, z | \alpha_\omega) = 0 \quad \langle \omega := [m, m+2] + (m+i), m+i \rangle$$

and

$$\pi(r : \alpha || m, i, z | z) = 0 \quad \langle z := \mathbb{K} \setminus \alpha [m, m+2] + (m+i, m+i] \rangle$$

ii) Let $i > 0$ and ~~for all~~ define $\gamma: [i] \rightarrow \{K \rightarrow K\}$ by setting (28)

$$\gamma(j) := \pi(\alpha || m, j-1) \pi(\alpha || m_{j+1}, i-j)$$

Then for $j := [i]$

$$\text{and } \pi(r: \alpha || m; i, j) = \pi(\alpha || m_{j-1}, 1) \gamma(j) \quad \langle K \rangle_{j := [i]} \rightarrow$$

$$\pi(r: \alpha || m; i, j-1) = \pi(\alpha || m_{j+1}) \gamma(j) \quad \langle K \rangle_j \langle K \rangle$$

both and for $j := (i)$

$$\pi(\alpha || m_{j-1}) \gamma(j) = \pi(\alpha || m_{j+1}) \gamma(j+1) \quad \langle K \rangle$$

iii) Thm p. 95.

~~with $i > 0$ at~~

~~With $f \in \text{seq}(K | r + m_i)$~~

The divided differences satisfy the following recursion:

$$\delta(r: f; \alpha || m, i) = \frac{\delta(r: f; \alpha || m_{i+1}, i-1) - \delta(r: f; \alpha || m, i-1)}{x_{m_{i+1}} - \alpha_m}$$

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iv) Th. m p. 95

~~with $i > 0$~~

$$v) \left\{ \sum_w \mu(r: \alpha || m; i, w) \quad \langle w := [i] \rangle \right\} = 0$$

Let $r, m, i \in \bar{\mathbb{N}}$ ~~def~~ $\sim \mathbb{N}$ and $\alpha \in \text{seq}'(K | r + m_i)$

p 280. \Rightarrow

~~ii) $\lambda(0; m; i, \omega) = \lambda(m; i, \omega) \quad \langle K \rangle$~~

~~iii) $\lambda(0: \alpha || m; i, \omega) = \lambda(\alpha || m; i, \omega)$~~

~~iv) b) $\lambda(r: \alpha || m; i, 0) = \frac{\pi(\alpha || m_{i+1}; i)}{\pi(\alpha || m_{i+1}; i | \alpha_m)} \quad \langle K \rangle$~~

~~v) $\lambda(r: \alpha || m; i, i) = \frac{\pi(\alpha || m, i)}{\pi(\alpha || m, i | \alpha_m)} \quad \langle K \rangle$~~

~~vi) $\lambda(r: \alpha || m; i, \omega | \alpha_\omega) = 0 \quad \langle \omega := [m, m\omega] + (m\omega_2, m\omega_1] \rangle$~~

(Remark p 131)

Let $i, m, r \in \mathbb{N}$, $\alpha \in \text{seq}'(K \models i+m+r)$ and $f \in \text{seq}(K \models i+m+r)$.

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1] Define $X[r:\alpha \parallel m, [i]] \in \mathcal{L}[K|i]$ by setting

$$X[r:\alpha \parallel m, [i]] := \mathcal{L}[\mu(r:\alpha \parallel m; i, \omega)] \quad \begin{cases} \omega := [\varepsilon] \\ z := [i] \end{cases}$$

$$\text{i)} \quad X[r:\alpha \parallel m, [i]] \text{ col } [\delta(f; \alpha \parallel m+z, r)]_{z := [i]} = \text{col } [\delta(f; \alpha \parallel m; r+z)]_{z := [i]}$$

$$\text{ii)} \quad X[r:\alpha \parallel m, [i]] = \mathcal{L}[\delta(\phi(m+r+\omega+1, z-\omega); \alpha \parallel m, \omega)]_{\begin{cases} \omega := [\varepsilon] \\ z := [i] \end{cases}} \quad ? \quad \mathcal{L}\mathcal{I}[i]^{-1}$$

$$= \mathcal{L}[\delta(\phi(m, \omega); \alpha \parallel m+r, z-\omega)]_{z := [i]} \quad \mathcal{L}\mathcal{I}[i]^{-1}$$

$$= \mathcal{L}[\delta(\Theta(m, z); \alpha \parallel m+r, z-\omega)]$$

$$\text{iii)} \quad X[r:\alpha \parallel m, [i]]^{-1} = \mathcal{L}[\delta(\Theta(m, r+\omega); \alpha \parallel m+r, \omega)]$$

$$= \mathcal{L}[\delta(\Theta(m, r+\omega); \alpha \parallel m+z, r)]$$

$$= \mathcal{L}[\mu(\alpha \parallel m+z; r, r-z+\omega)] \mathcal{L}[\pi(\alpha \parallel m, r+\omega) | \alpha_{m+r}]$$

$$\text{iv)} \quad \mathcal{L}[\pi(\alpha \parallel m, r+\omega) | \alpha_{m+r}]^{-1} = \mathcal{L}[\mu(\alpha \parallel m; r+z, r+\omega)]$$

$$\text{v)} \quad \mathcal{L}[\mu(\alpha \parallel m+z; r, r-z+\omega)]^{-1} = \mathcal{L}[\pi(\alpha \parallel m+r, r) | \alpha_{m+r}] \mathcal{L}\mathcal{I}[i]^{-1}$$

6) \check{X} for

2] Define $\check{X}[r:\alpha \parallel m, [i]] \in \mathcal{U}[K|i]$ by setting

$$X[r:\alpha \parallel m, [i]] := [\mu(r:\alpha \parallel m+r; i-\omega, \omega-\omega)]$$

$$\cancel{\mathcal{U}[\delta(\phi(m+r, \omega-\omega); \alpha \parallel m+r, i-\omega) - \delta(\phi(m+r, \omega-\omega+1); \alpha \parallel m+r+1, i-\omega-1)]}$$

$$\text{i)} \quad \check{X}[r:\alpha \parallel m, [i]] \text{ col } [\delta(f; \alpha \parallel m+z, r)] = \text{col } [\delta(f; \alpha \parallel m+z, r+i-\omega)]$$

$$\text{ii)} \quad \check{X}[r:\alpha \parallel m, [i]] = \cancel{\mathcal{U}[\delta(\phi(m+r, \omega-\omega+1, i-\omega); \alpha \parallel m+z, \omega-\omega)] - \delta(\phi(m+r, i-\omega+1); \alpha \parallel m+z, \omega-\omega-1)]}$$

$$\text{iii)} \quad \check{X}[r:\alpha \parallel m, [i]] = \mathcal{U}[\mu(\alpha \parallel m+z, i-\omega, \omega-\omega)]$$

c) If $r > 0$

$$\check{X}[r : \alpha || m, [i]] =$$

$$\mathcal{U}[(\alpha_{mz} - \alpha_{mz}) \delta(\phi(m+z, d-z+1); \alpha || m || z, i-\omega)] =$$

$$= \mathcal{U}[$$

$$\text{iii)} \quad \check{X}[r : \alpha || m, [i]] =$$

$$\mathcal{U}[\delta(\phi(m+z, d-z; \alpha || m || z, i-\omega))] \mathcal{U}[i]^{-1} \Leftarrow$$

$$= \mathcal{U}[\delta(\phi(m+z+1, i-\omega); \alpha || m+z, d-z) - \delta(\phi(m+z, i-\omega+1); \alpha || m+z, d-z)] \Leftarrow$$

$$\Leftarrow \mathcal{U}[\delta(\phi(m+z+1, i-\omega); \alpha || m+z, d-z)] \mathcal{U}[i]^{-1}$$

$$\text{b)} \quad \check{X}[0 : \alpha || m, [i]] = \mathcal{U}[\pi(\alpha || m+z, i-\omega, d-z)]$$

b) If $r > 0$

$$\check{X}[r : \alpha || m, [i]] =$$

$$\mathcal{U}[(\alpha_{mz} - \alpha_{mz}) \delta(\phi(m+z, d-z+1); \alpha || m || z, i-\omega)] \Leftarrow$$

$$= \mathcal{U}[(\alpha_{mz} - \alpha_{mz}) \delta(\phi(m+z, i-\omega+1); \alpha || m+z, d-\omega)]$$

$$\text{iii)} \quad \check{X}[r : \alpha || m, [i]]^{-1} =$$

$$\mathcal{U}[\delta(\Theta(m+z+1, i-\omega); \alpha || m+z, d-\omega)]$$

$$= \mathcal{U}[\delta(\Theta(m+z+1, r+i-\omega); \alpha || m+z, r)]$$

$$= \mathcal{U}[\mu(\alpha || m+z; r, d-\omega)] \mathcal{U}[\pi(\alpha || m+z+1, r+i-\omega | \alpha_{m+z})]$$

$$\text{b)} \quad \mathcal{U}[\pi(\alpha || m+z+1, r+i-\omega | \alpha_{m+z})]^{-1} = \mathcal{U}[\mu(\alpha || m+z, r+i-\omega, d-\omega)]$$

$$\text{c)} \quad \mathcal{U}[\mu(\alpha || m+z; r, d-\omega)]^{-1} = \mathcal{U}[\pi(\alpha || m+z+1, r | \alpha_{m+z})]$$

3] Define $\tilde{X}[r: \alpha || m, [i]] \in [\mathbb{K}|i]$ by setting

$$\tilde{X}[r: \alpha || m, [i]] := [\delta(\Theta(m, r+z); \alpha || m+z, r+i-z)]$$

$$i) \tilde{X}[r: \alpha || m, [i]] := [\delta(\Theta(m, r); \alpha || m+r, i-r)]$$

$$ii) \tilde{X}[r: \alpha || m, [i]] \text{ col } [\delta(f; \alpha || m, r+z)] = \text{col } [\delta(f; \alpha || m+z, r+i-z)]$$

$$iii) \tilde{X}[r: \alpha || m, [i]] = [\delta(\Theta(m, r+z); \alpha || m+z, r+i-z)]$$

$$= \tilde{X}[\alpha || m; r, [i]]^M \times [\alpha || m; r, [i]]^{-1}$$

$$iv) \{\tilde{X}[r: \alpha || m; [i]]\}^{-1} = [\delta(\Theta(m+r+1, r+i-2); \alpha || m, r+z)]$$

$$= [\delta(\Theta(m+r+1, i-z); \alpha || m, z)]$$

$$= \tilde{X}[\alpha || m; r, [i]]^M \times [\alpha || m; r, [i]]^{-1}$$

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From the above, $(\tilde{X}[r: \alpha || m, [i]])$ and $(\tilde{X}[r: \alpha || m, [i]]^{-1})$ have triangular form: the elements with suffices z, r in the former are zero when $r+z < i$; those in the latter are zero when $r+z > i$.

Let $i, m, r \in \mathbb{N}$, $\alpha \in \text{seq}'(\mathbb{K} \cong i+m+r)$ and $f \in \text{seq}(\mathbb{K} \cong i+m+r)$. Set

$$\mathbb{K} := \mathbb{K} \setminus \alpha[m, m+i] \text{ and } \mathbb{K} := \mathbb{K} \setminus \alpha(m+r, m+r+i) \quad \leftarrow \text{p. 156. D. D.}\}$$

1] Define $\tilde{Y}[r: \alpha || m, [i]] \in \{\mathbb{K} \rightarrow \mathcal{L}[\mathbb{K}|i]\}$ by setting

$$Y[r: \alpha || m, [i]] := \mathcal{L}[\lambda(r: \alpha || m; z, z)]$$

$$i) Y[r: \alpha || m, [i]] \text{ col } [\Delta(f; \alpha || m+z, r)] = \text{col } [\Delta(f; \alpha || m; r+z)] \langle \mathbb{K} \rangle$$

$$ii) \{Y[r: \alpha || m, [i]]\}^{-1} = \mathcal{L} \left[\frac{\Delta(\Theta(m, r); \alpha || m+r, z-r) - \Delta(\Theta(m, r); \alpha || m+r+1, z-r-1)}{\pi(\alpha || m, r)} \right] \langle \tilde{\mathbb{K}} \rangle$$

2] Define $\tilde{Y}[r: \alpha || m, [i]] \in \{\mathbb{K} \rightarrow \mathcal{U}[\mathbb{K}|i]\}$ by setting

$$Y[r: \alpha || m, [i]] := \mathcal{U}[\lambda(r: \alpha || m+z, i-r, z-r)]$$

$$i) Y[r: \alpha || m, [i]] \text{ col } [\Delta(f; \alpha || m+z, r)] = \text{col } [\Delta(f; \alpha || m+z, r+i-z)] \langle \mathbb{K} \rangle$$

$$ii) \tilde{Y}[r: \alpha || m, [i]]^{-1} = U \left[\frac{\Lambda(\Theta(m+r+z, i-z); \alpha || m+z, z-z) - \Lambda(\Theta(m+r+z+1, i-z); \alpha || m+z, z-z-1)}{\pi(\alpha || m+r+z+1, i-z)} \right]$$

$\langle \hat{R} \rangle$

3] Define $\tilde{Y}[r: \alpha || m, [i]] \in \{\tilde{K} \rightarrow [K|i]\}$ by setting

$$\tilde{Y}[r: \alpha || m, [i]] := \left[\frac{\Lambda(\Theta(m, z); \alpha || m+r+z, i-z) - \Lambda(\Theta(m, z); \alpha || m+r+z+1, i-z-1)}{\pi(\alpha || m, z)} \right]$$

$$i) \tilde{Y}[r: \alpha || m, [i]] \text{ col}[\Lambda(f; \alpha || m, r+z)] = \text{col}[\Lambda(f; \alpha || m+z, r+i-z)] \langle \tilde{R} \rangle$$

$$ii) \tilde{Y}[r: \alpha || m, [i]] = \tilde{Y}[r: \alpha || m, [i]] \{ \tilde{Y}[r: \alpha || m, [i]] \}^{-1} \langle \tilde{R} \rangle$$

4] Define $\hat{Y}[r: \alpha || m, [i]] \in \{\hat{K} \rightarrow [K|i]\}$ by setting

$$\hat{Y}[r: \alpha || m, [i]] := \left[\frac{\Lambda(\Theta(m+r+z+1, i-z); \alpha || m, z) - \Lambda(\Theta(m+r+z+1, i-z); \alpha || m, z-1)}{\pi(\alpha || m+r+z+1, i-z)} \right]$$

$$i) \hat{Y}[r: \alpha || m, [i]] \text{ col}[\Lambda(f; \alpha || m+z, r+i-z)] = \text{col}[\Lambda(f; \alpha || m, r+z)] \langle \hat{R} \rangle$$

$$ii) \hat{Y}[r: \alpha || m, [i]] = Y[r: \alpha || m, [i]] \{ \hat{Y}[r: \alpha || m, [i]] \}^{-1} \langle \hat{R} \rangle$$

$$iii) \hat{Y}[r: \alpha || m, [i]] = \{ \tilde{Y}[r: \alpha || m, [i]] \}^{-1} \langle \tilde{R} \cap \hat{R} \rangle$$

$$5i) Y[r: \alpha || m, [i]] I_{[i]} = \tilde{Y}[r: \alpha || m, [i]] I_{[i]} = I_{[i]} \langle R \rangle$$

$$ii) \{ Y[r: \alpha || m, [i]] \}^{-1} I_{[i]} = \tilde{Y}[r: \alpha || m, [i]] I_{[i]} = I_{[i]} \langle \tilde{R} \rangle$$

$$iii) \{ \tilde{Y}[r: \alpha || m, [i]] \}^{-1} I_{[i]} = \hat{Y}[r: \alpha || m, [i]] I_{[i]} = I_{[i]} \langle \hat{R} \rangle$$

6] Define $D, D', D'' \in \{K \rightarrow [K|i]\}$ by setting

$$D := \text{diag} [\pi(\alpha || m+r+1)], D' := \text{diag} [\pi(\alpha || m+r, r+1)]$$

and

$$D'' := \text{diag} [\pi(\alpha || m+r, r+i-w+1)]$$

$$\text{Y}[\text{r.allm}, [\epsilon]] = \mathcal{D} \times [\text{r.allm}, [\epsilon]] \{\mathcal{D}'\}^{-1}$$

$$\tilde{\text{Y}}[\text{r.allm}, [\epsilon]] = \mathcal{D}'' \tilde{\text{X}}[\text{r.allm}, [\epsilon]] \{\mathcal{D}'\}^{-1}$$

$$\text{Y}[\text{r.allm}, [\epsilon]] = \mathcal{D}'' \text{X}[\text{r.allm}, [\epsilon]] \{\mathcal{D}\}^{-1}$$

$$\hat{\text{Y}}[\text{r.allm}, [\epsilon]] = \mathcal{D} \{\text{X}[\text{r.allm}, [\epsilon]]\}^{-1} \{\mathcal{D}''\}^{-1}$$

over $\mathbb{K} \setminus \alpha[m, m_{\text{min}}]$ in all cases.

$$Y[r:\alpha \parallel m, [i]]^{-1} = \text{diag}[T(\alpha \parallel m \parallel \alpha, r+i)] \times [r:\alpha \parallel m, [i]]^{-1} \text{diag}[\frac{1}{\pi(\alpha \parallel m, r+i)}]$$

III.

$$\langle K \setminus \alpha[m, m+r+i] \rangle$$

$$Y[r:\alpha \parallel m, [i]]_{[i]} = I_{[i]} \langle K \rangle$$

$$Y[r:\alpha \parallel m, [i]]_{[i]} = I_{[i]} \langle K \setminus \alpha[m, m+r+i] \rangle$$

$$Y[r:\alpha \parallel m, [i]] = \text{diag}[\pi(\alpha \parallel m, r+i)] \times [r:\alpha \parallel m, [i]] \text{diag}[\pi(\alpha \parallel m+r+i)]^{-1}$$

$$\langle K \setminus \alpha[m, m+r+i] \rangle$$

$$= \cancel{\frac{Y[r:\alpha \parallel m, [i]]^{-1}}{I_{[i]}}} \quad \text{How possible that } Y^{-1} I_{[i]} = I_{[i]}$$

$$= \frac{\Delta(\Theta(m, r); \alpha \parallel m \parallel \alpha, r-i) - \Delta(\Theta(m, r); \alpha \parallel m \parallel \alpha + 1, r-i-1)}{\pi(\alpha \parallel m, r)} \quad | \cancel{\langle K \setminus \alpha[m, i] \rangle}$$

$$= L \left[\frac{\delta(\Theta(m, r); \alpha \parallel m+r, r) \pi(\alpha \parallel m+r, r+1)}{\pi(\alpha \parallel m, r+1)} \right] \quad | \cancel{\langle K \setminus \alpha[m, i] \rangle}$$

$$Y^{-1} I_{[i]} \Rightarrow \frac{\Delta(\Theta(m, r); \alpha \parallel m+r, r)}{\pi(\alpha \parallel m, r)} \quad | \cancel{\pi(\alpha \parallel m, r)} \quad \frac{\alpha_{mn} - \alpha_m}{\alpha_{m+n} - \alpha_m}$$

$$= 1 \quad | \cancel{\pi(\alpha \parallel m, r)} \quad \frac{\alpha_{mn} - \alpha_m + \frac{\alpha_{m+n} - \alpha_{mn}}{\alpha_{m+n} - \alpha_{mn}}(r - m)}{\alpha_{m+n} - \alpha_{mn}}$$

$$\mu(r: \alpha \parallel m; \tilde{\tau}, \tilde{\nu}) = \delta(\phi(m+r), \tilde{\tau}-\tilde{\nu}); \alpha \parallel m, \tilde{\nu}) - \delta(\phi(m+r), \tilde{\tau}-\tilde{\nu}+1); \alpha \parallel m, \tilde{\nu}-1)$$

$$= \delta(\phi(m, \tilde{\nu}); \alpha \parallel m+r, \tilde{\tau}-\tilde{\nu}) - \delta(\phi(m, \tilde{\nu}+1); \alpha \parallel m+r+1, \tilde{\tau}-\tilde{\nu}-1)$$

$$\check{X}: \delta(\phi(m+\tau, \tilde{\nu}-\tilde{\tau}); \alpha \parallel m+r, \tilde{\nu}-\tilde{\tau}) - \delta(\phi(m+\tau, \tilde{\nu}-\tilde{\tau}+1); \alpha \parallel m+r+1, \tilde{\nu}-\tilde{\tau}-1)$$

$$= \delta(\phi(m+r+1, \tilde{\nu}-\tilde{\tau}); \alpha \parallel m+\tau, \tilde{\nu}-\tilde{\tau}) - \delta(\phi(m+r), \tilde{\nu}-\tilde{\tau}+1); \alpha \parallel m+\tau, \tilde{\nu}-\tilde{\tau}-1)$$

$$m' = m+\tau \quad \nu' = \nu-\tau \quad m'+r+\tau = m+r+\tau \Rightarrow r=r' \quad \tau'-\nu' = \tilde{\nu}-\tilde{\tau} \quad \tau' = \tilde{\nu}-\tilde{\tau}+\tau-\tau$$

$$\cancel{\check{X}}: [\mu(r: \alpha \parallel m+\tau; \tilde{\nu}-\tilde{\tau}, \tilde{\nu}-\tilde{\tau})]$$

$$m'+r+\tau = m+r+\tau \quad \tau'-\nu' = \tilde{\nu}-\tilde{\tau} \quad m' = m+\tau, \nu' = \tilde{\nu}-\tilde{\tau}$$

$$k' = r \quad j' = r+k, \quad r+k-j' = r-n$$

$$\frac{\pi(\alpha \parallel m \parallel m+1, z-r)}{\pi(\alpha \parallel m+z+1, r-z)}$$

$$m' = m+n+1 \quad m+r+j' = m \quad r+k-j' = k \quad z' = k \quad r'-r' = j \quad r' = r+k-j$$

$$\pi(\alpha \parallel m, z-r) \quad m+j+1 = m-r-k+z-r'+j$$

$$\pi(\alpha \parallel m+z-r, r+k-z) \quad \pi(\alpha \parallel m, z)$$

$$r+k-z \leq z \Rightarrow \pi(\alpha \parallel m+z-r, r+k-z) \quad r+k-z, m+z-r$$

$$z+r \leq z \Rightarrow r \leq 0$$

$$z' = k \quad r+k' = j \quad r+k-j = r+k-j \quad m+r+k = m+r+k-j+j = m+z-r+k$$

$$\pi(\alpha \parallel m+n+1, z-r-r) \quad \pi(\alpha \parallel m+z+r \parallel 1)$$

$$\pi(\alpha \parallel m+z, r+k-z+1) \quad \pi(\alpha \parallel m, z) \Leftarrow \pi(\alpha \parallel m, r+k+1) \text{ when } r+k-z+1 \geq 0 \\ r+k-z+1 \leq 1 \\ r+k+1 \leq z$$

Let $m, i \in \mathbb{N}$, $\alpha \in \text{seq}'(K \models m+i)$ and $f \in \text{seq}(K \models m+i)$.

$$i) \quad \pi(\alpha \parallel m, i) \{ \Delta(f; \alpha \parallel m, i) - \Delta(f; \alpha \parallel m+1, i-1) \} = \\ \pi(\alpha \parallel m+1, i) \{ \Delta(f; \alpha \parallel m, i) - \Delta(f; \alpha \parallel m, i-1) \} \quad \langle K \rangle$$

$$ii) \quad \{ \pi(\alpha \parallel m, i) - \pi(\alpha \parallel m+1, i) \} \Delta(f; \alpha \parallel m, i) = \\ \pi(\alpha \parallel m, i) \Delta(f; \alpha \parallel m+1, i) - \pi(\alpha \parallel m+1, i) \Delta(f; \alpha \parallel m, i-1)$$

$$iii) \quad (\alpha_{m+i} - \alpha_m) \Delta(f; \alpha \parallel m, i) = \\ (z - \alpha_m) \Delta(f; \alpha \parallel m+1, i-1) - (z - \alpha_{m+i}) \Delta(f; \alpha \parallel m, i)$$

$$\frac{\Delta(1||j|z)}{\pi(\alpha||m+z, k)} = \frac{\Delta(1||k|z)}{\pi(\alpha||m+z, j)}$$

$$\pi(\alpha||m+z, j) \Delta(1||j|z) = \pi(\alpha||m+z, k) \Delta(1||k|z)$$

$$\frac{\Delta(1||j|0)}{\Delta(1||j|1)} = \frac{\pi(\alpha||m, k)}{\pi(\alpha||m+1, k)} = \frac{\pi(\alpha||m, l)}{\pi(\alpha||m+k, l)}$$

$$\frac{\Delta(1||k|0)}{\Delta(1||k|1)} = \frac{\pi(\alpha||m, l)}{\pi(\alpha||m+j, l)}$$

$$\pi(\alpha||m+j, l) \pi(\alpha||m, j) \Delta(1||j|0) = \pi(\alpha||m, l) \pi(\alpha||m, k) \Delta(1||k|0)$$

$$\pi(\alpha||m+1, j) \Delta(1||j|0) = \pi(\alpha||m, k) \Delta(1||k|1)$$

$$\pi(\alpha||m, l) \pi(\alpha||m, j) \Delta(1||j|1) = \pi(\alpha||m+k, l) \pi(\alpha||m, k) \Delta(1||k|0)$$

$$\pi(\alpha||m, j) \Delta(1||j|1) = \pi(\alpha||m+1, k) \Delta(1||k|0)$$

$$\pi(\alpha||m+z, j) \Delta(1||j|z) = \pi(\alpha||m+z, k) \Delta(1||k|z)$$

$$\tilde{\Pi}(\chi|\omega) = \pi(\alpha||m+z, \chi)$$

$$\tilde{\Pi}(j|\omega) \Delta(j|\omega) = \tilde{\Pi}(k|\omega) \Delta(j|\omega)$$

Remark p 212 Let $j, k, m, r \in \mathbb{N}$

1] Let $k \in \mathbb{N}$ and $\alpha \in \text{seq}'(|K| \geq m + \max(j-k, k+r))$. So for $\chi := j, k$ set

and $\omega := [r]$

$$\Pi(\chi|\omega) := \pi(\alpha||m+\chi+r, r+k-\chi)$$

for $\omega := [\omega]$ and

$$\begin{aligned} \Delta(\chi|\omega) := & \Lambda(\Theta(\alpha||m, \chi); \alpha||m+j+k-\chi, r+\chi-j) \\ & - \Lambda(\Theta(\alpha||m, \chi); \alpha||m+j+k-\chi+\omega, r+\chi-j-1) \end{aligned}$$

For $z, \omega := [1]$

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$$\bar{\pi}(j|z) \Delta(j|\omega) = \bar{\pi}(k|z) \Delta(k|\omega) \quad \langle K \rangle$$

2] Let $k \in [j+n]$ and $\alpha \in \text{seq}'(|K| \geq m+j+n)$. For $\chi := j, k$ and $\omega := [1]$ set

$$\bar{\pi}(\chi|\omega) := \bar{\pi}(\alpha || m+\omega, \chi)$$

and

$$\begin{aligned} \bar{\Delta}(\chi|\omega) := & \Delta(\theta(\alpha || m+\chi+1, j+n-\chi); \alpha || m, j+k-\chi) - \\ & \Delta(\theta(\alpha || m+\chi+1, j+n-\chi); \alpha || m+\omega, j+k-\chi-1) \end{aligned}$$

For $z, \omega := [i]$

$$\bar{\pi}(j|z) \bar{\Delta}(j|\omega) = \bar{\pi}(k|\omega) \bar{\Delta}(k|z) \quad \langle K \rangle$$

=

In the following theorem all matrices concerned are in $[K|i]$ and all column vectors in $\text{col}[K|i]$. Matrix elements are indicated by use of the ^{row}_{columns} ^{index}_{suffix z} and ^{column index}_{suffix ω} . Index range specifications are omitted from matrix declarations. Thus the complete declaration

$$X[r: \alpha || m, [i]] := L[\mu(r: \alpha || m; z, \omega)] \underset{\langle z := [i] \rangle}{\underset{\langle \omega := [z] \rangle}{\langle \omega := [z] \rangle}}$$

is abbreviated to

$$X[r: \alpha || m, [i]] := L[\mu(r: \alpha || m; z, \omega)]$$

All column vectors concerned are in $\text{col}[K|i]$, their elements are indicated by use of the suffix z , and range specifications are omitted from column declarations.

In the following theorem matrix-mapping matrix specifications are abbreviated elements are indicated by use of row ^{index} suffix z and column index v and, as in the previous theorem, suffix index range specifications are omitted from mapping matrix specifications. A similar convention concerning ~~map~~ ~~map~~ column mappings is adopted.

$$? \text{ Set } \mu^{(-1)}(r; \alpha || m; z, v) := \delta(\Theta(\alpha || m; z); \alpha || m + v, z - v) \\ = \delta(\Theta(\alpha || m, v); \alpha || m + z, r)$$

$$\text{on p. 106 } \tilde{X}[r; \alpha || m, [i]] := \mathcal{U}[\mu(r; \alpha || m + z; i - v, v - e)]$$

$$m' = m + v + 1, \quad v' + v' = r + i - v \quad m' + z' = m + z, \quad r' = r$$

$$z' = m + z - m - v - 1 = z - v - 1 \quad v' = i - v$$

$$\tilde{X}[r; \alpha || m, [i]]^{-1} = \mathcal{U}[\mu^{(-1)}(r; \alpha || m + v + 1; z - v - 1, i - v)]$$

$$m' = m \quad z' = r + v \quad v' + v' = z \quad z' - v' = r + i - v \quad r' + z' = r + i \quad r' = i - v$$

$$v' = z + v - i$$

$$\tilde{X}[r; \alpha || m, [i]] = [\mu^{(-1)}(i - v; \alpha || m; r + v, z + v - i)]$$

$$m' = m \quad z' = r + v \quad m' + v' = m + z, \quad z' - v' = r \quad v' = v \quad r' = z - v$$

$$\mu^{(-1)}(r; \alpha || m; z, v) = \mu^{(-1)}(z - v; \alpha || m; r + v, v) \quad (v \leq z)$$

$$z' - v' = i - v \quad m' = m \quad v' + v' = r + v, \quad v' = z + v - i \quad z' = z \\ r' = r + v - z - v + i = r + i - z$$

$$\tilde{X}[r; \alpha || m, [i]] = [\mu^{(-1)}(r + i - z; \alpha || m; z, z + v - i)]$$

$$m' = m + v + 1 \quad z' = r + i - v - 1 \quad m' + v' = m \quad z' - v' = r + v$$

$$v' = r + i - v - r - z = i - z - v \quad r' = m - m - v - 1 - i + z + v = z - i - 1$$

$$\Sigma_1 - \Sigma = 1$$

$$\alpha_{mrrrii} - \alpha_m \quad \alpha_{mm}$$

$$\Sigma - \Sigma_1 = \sum \delta(-F \parallel m, \omega) \delta(\dots G \parallel m + \omega, i - \omega)$$

$$+ \sum \delta(\dots F \parallel m, \omega) \{ (\alpha_{mrrrii} - \alpha_{mm}) \delta(\dots G \parallel m + \omega, i - \omega) \\ - \delta(\dots G \parallel m + \omega, i - \omega) \}$$

$$\delta(r; G; \alpha \parallel m + \omega, i - \omega) = -\delta(r; G; \alpha \parallel m, i - \omega)$$

~~$$+ (\alpha_{mrrrii} - \alpha_{mm}) \delta(r; G; \alpha \parallel m + \omega, i - \omega)$$~~

$$\Sigma - \Sigma_1 = \sum (\alpha_{m+r+\omega+i} - \alpha_m) \delta(\dots F \parallel m, \omega) \delta(\dots G \parallel m + \omega, i - \omega)$$

$$+ \sum (\alpha_{mrrrii} - \alpha_{mm}) \delta(\dots F \parallel m, \omega) \delta(\dots G \parallel m + \omega, i - \omega)$$

$$= (\alpha_{mrrrii} - \alpha_m) \delta(\dots F \parallel m, \omega) \delta(G \parallel m, i + \omega)$$

$$+ (\alpha_{mrrrii} - \alpha_m) \delta(\dots F \parallel m, i + \omega) \delta(G \parallel m + \omega, 0)$$

$$+ \sum (\alpha_{mrrrii} - \alpha_m + \alpha_{mrrrii} - \alpha_{mm}) \delta(\dots F \parallel m, \omega) \delta(\dots G \parallel m + \omega, i - \omega)$$

cancellation only takes place when $r=0$ $\langle \omega := (i) \rangle$

$$\alpha_{mrrrii} - \alpha_{mm} = \alpha_{mrrrii} - \alpha_m + \alpha_m - \alpha_{mm}$$

$$= \alpha_{mrrrii} - \alpha_m + \alpha_m - \alpha_{m+r+i} + (\alpha_{mrrrii} - \alpha_{mm})$$

$$m, \dots, mri \quad mri - \omega, \omega \quad m, \dots i - \omega$$

$$\delta \Sigma (f \otimes e \otimes f; \alpha \parallel m, i) = \sum_i \delta(e; \alpha \parallel m, \omega) \delta(f; \alpha \parallel m + \omega, i - \omega) \quad \langle \omega := [i] \rangle$$

3 other forms obtained by ^{interchanging} $e \otimes f$ and using summation index

$\omega' = i - \omega$. $e[m, mri]$ or $[e, f][m, mri]$ reverse order form obtained by carrying out e, f interchange & index transposition normal

$$\delta(f \times \tau(\alpha \| m+z+1, i-\omega; \langle \alpha \rangle); \alpha \| m+\omega, i-\omega)$$

$$= \sum_i \delta(f; \alpha \| m+\omega, \omega) \delta(\tau(\alpha \| m+z+1, i-\omega; \langle \alpha \rangle); \alpha \| m+\omega, i-\omega)$$

$$= \delta(f; \alpha \| m+\omega, z-\omega)$$

$$\omega = z - \omega$$

$$\omega > z - \omega \quad m + \omega > m + z$$

$$\text{II} \quad M[\alpha \| m, [i]]^{-1} \text{diag}[f_{m+\omega}] M[\alpha \| m, [i]] = [\delta(f; \alpha \| m+\omega, z-\omega)]$$

$$= M[\alpha \| m, [i]] \text{diag}[f_{m+\omega}] M[\alpha \| m, [i]]$$

— —

$$\sum_m \delta(\theta(m, z); \alpha \| m+r+\omega, z-\omega) \delta(f; \alpha \| m; r+\omega) \quad \langle \omega := [z] \rangle$$

$$m+\omega, z+r-\omega \quad \omega \quad \langle \omega := [r, r+z] \rangle$$

$$\omega < r \Rightarrow z+r-\omega > z \Rightarrow \delta(\theta \dots) = 0 \quad = \langle \omega := [r+z] \rangle$$

$$M = \delta(f \times \theta(m, z); \alpha \| m, z+r)$$

$$= \sum_i \delta(\theta(m, z); \alpha \| m, \omega) \delta(f; \alpha \| m+\omega, z+r-\omega)$$

$$M = \delta(f; \alpha \| m+z, r) \quad (\text{proving } X[r; \alpha \| m, [i]]^{-1} \text{col}[\delta(f; \alpha \| m, r+z)] = 0)$$

$$\text{prob: } X[r; \alpha \| m, [i]]^{-1} \text{col}[\delta(f; \alpha \| m; r+z)] = \text{col}[\delta(f; \alpha \| m+z, r)]$$

$$X[r; \alpha \| m, [i]] := \mathcal{L}[\alpha(r; \alpha \| m; z, \omega)]$$

$$:= \left[\begin{array}{c} \delta(\phi(\alpha \| m, \omega); \alpha \| m+r+\omega, z-\omega) \\ -\delta(\phi(\alpha \| m, \omega); \alpha \| m+r+\omega, z-\omega) \end{array} \right]$$

$$\sum_m \delta(\theta(m, z); \alpha \| m+r+\omega, z-\omega) \delta(\phi(\alpha \| m, \omega); \alpha \| m+r+\omega, \omega-\omega)$$

$$\langle \omega := [\omega, \tau] \rangle$$

$$m+r+\omega, z-\omega$$

$$\omega := [z-\omega]$$

$$= \delta(\theta(m, z) \times \phi(m, \omega); \alpha \| m+r+\omega, z-\omega) = 0 \text{ when}$$

$$= 1 \text{ when } z \geq \omega$$

$$\sum \delta(\theta(m, z); \alpha \| m + \omega, z - \omega) \delta(\phi(m, \omega+1); \alpha \| m + \omega + 1, \omega - \omega - 1)$$

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$$\omega := [z, \bar{z}]$$

$$= \delta(\theta(m, z), \phi(m, \omega+1); \alpha \| m + \omega, z - \omega) \quad z > v$$

$$= 1 \text{ when } z \geq \omega + 1 = 1 \text{ when } z = v$$

$$z = v : 1 \text{ from prev p.}$$

$$\| \times [r: \alpha \| m, [i]] = \mathcal{L} [\delta(\phi(m, \omega); \alpha \| m + \omega, z - \omega)] \mathcal{L} I [i]^{-1}$$

=

$$\delta(r: \frac{1}{\pi(\alpha \| \langle m \rangle, r+1 | z)}; \alpha \| m, 1) = \frac{1}{x_{m+r+1} - \alpha_m} \left\{ \frac{1}{\prod_{\omega=1}^{r+1} (z - \alpha_{m+\omega})} - \frac{1}{\prod_{\omega=0}^r (z - \alpha_{m+\omega})} \right\}$$

$$z - \alpha_m = z + \alpha_{m+r+1}$$

$$\| \delta(r: \frac{1}{\pi(\alpha \| \langle m \rangle, r+1 | z)}; \alpha \| m, i) = \frac{1}{\pi(\alpha \| \langle m \rangle, r+i+1 | z)}$$

$$\delta(r: \pi(\alpha \| \langle m \rangle, r+1 | z)) = \frac{1}{\alpha_{m+r+1} - \alpha_m} \left\{ \prod_{\omega=1}^{r+1} (z - \alpha_{m+\omega}) - \prod_{\omega=0}^r (z - \alpha_{m+\omega}) \right\}$$

$$= -\pi(\alpha \| m+1, r | z)$$

$$\sum \delta(\theta(m, z); \alpha \| m + \omega, z - \omega) \delta(\phi(m + \omega + 1, \omega - \omega); \alpha \| m, \omega)$$

$$\mu(\alpha \| m; \rho, \chi) \frac{1}{\prod_{k=0}^{\rho-1} (\alpha_{m+k} - x_{m+\omega+k+1})}$$

$$\phi(n, j | \bar{N}) = \frac{1}{\pi(\alpha \| n, j | \alpha_k)}$$

$$= \delta(r: \frac{1}{\pi(\alpha \| \langle n \rangle, r+1 | \alpha_k)}; \alpha \| n, j-r-1)$$

$$\phi(n, j | \bar{N} \setminus [n, j]) = \delta(r: \frac{1}{\pi(\alpha \| \langle n \rangle, r+1 | \alpha[\bar{N} \setminus [n, j]])}; \alpha \| n, j-r-1)$$

$$\phi(\alpha \parallel \langle n, j \rangle) = \delta(r; \phi(\alpha \parallel \langle n \rangle, r+1); \alpha \parallel n, j-r-1) \quad r \in [j] \quad \text{HO.}$$

$$= \delta(\phi(\alpha \parallel \langle n \rangle, 1); \alpha \parallel n, j-1)$$

$$\phi(\alpha \parallel n, j \mid k) = \delta(r; \phi(\alpha \parallel \langle \frac{n}{k+j} \rangle, r+1 \mid k); \alpha \parallel n, j-r-1)$$

$$= \delta(\phi(\alpha \parallel \langle \frac{n}{k+j} \rangle, 1 \mid k); \alpha \parallel n, j-1)$$

④ $[n \mid j-r-1]$ or $[\alpha]_{-r-1} \neq \dots$ i.e. $[\alpha]_{-1} \parallel k := [\alpha] \setminus [n, j]$

$$\delta'(f; \alpha \parallel m, 1) = \delta(f; \alpha \parallel m, 1)$$

$$\delta'(f; \alpha \parallel m, 2) =$$

$$\delta(\pi(\alpha \parallel [\alpha] - n), r+1)$$

$$\frac{\pi(\alpha \parallel m, r+2) - \pi(\alpha \parallel m+1, r+1)}{\alpha_{m+r+1} - \alpha_m} = \frac{\pi(\alpha \parallel m+1, r+1)}{\alpha_{m+r+1} - \alpha_m}$$

$$\frac{\pi(\alpha \parallel m+2, r+1) - \pi(\alpha \parallel m+1, r+1)}{\alpha_{m+r+2} - \alpha_m} = -\pi(\alpha \parallel m+2, r+2)$$

$$\begin{array}{ll} a & \frac{b-a}{\alpha_{m+r+1} - \alpha_m} \\ b & \frac{1}{\alpha_{m+r+1} - \alpha_{m+1}} \\ c & \frac{c-b}{\alpha_{m+r+2} - \alpha_{m+1}} - \frac{b-a}{\alpha_{m+r+1} - \alpha_m} \end{array}$$

$$\frac{a}{\pi(\alpha \parallel m, 2 \mid \alpha_{m+r+1})} = \frac{-b \cdot (\alpha_{m+r+1} + \alpha_{m+r+2} - \alpha_m - \alpha_{m+1})}{(\alpha_{m+r+1} - \alpha_m)(\alpha_{m+r+2} - \alpha_m)}$$

$$\frac{c}{\pi(\alpha \parallel m+1, 2 \mid \alpha_{m+1})} = \begin{cases} a: \mu(r-1; \alpha \parallel m, 2, 2) \\ c: \mu(r-1; \alpha \parallel m+1, 2, 0) \end{cases}$$

$$-b: \frac{1}{(\alpha_{m+r+1} - \alpha_m)(\alpha_{m+r+2} - \alpha_m)} + \frac{1}{(\alpha_{m+r+1} - \alpha_m)(\alpha_{m+r+2} - \alpha_m)} = 0$$

Simplest proof of [1] p. 166

Show $\times[r: \alpha||m, [i]]^{-1} \text{col } [\delta(f: \alpha||m; r+z)] = \text{col } [\delta(f: \alpha||m+z, r)]$ as on p 137.

Then show $\times[r: \alpha||m, [i]]^{-1} \times[r: \alpha||m, [i]] = [i]$ as on pp. 137, 138 and hence (i) of [1] on p. 166

$$\frac{\pi(\alpha||m+1, h|z) - \pi(\alpha||m, h|z)}{\alpha_{m+r+1} - \alpha_m} = \frac{\alpha_m - \alpha_{m+h}}{\alpha_{m+r+1} - \alpha_m} \pi(m+1, h-1|z)$$

$$\frac{\alpha_{m+r+1} - \alpha_{m+h+1}}{\alpha_{m+r+2} - \alpha_{m+1}} \pi(m+2, h-1|z) - \frac{\alpha_m - \alpha_{m+h}}{\alpha_{m+r+1} - \alpha_m} \pi(m+1, h-1|z)$$

$$= \left\{ \frac{(z - \alpha_{m+h})(\alpha_{m+r+1} - \alpha_{m+h+1})}{\alpha_{m+r+2} - \alpha_m} - \frac{(z - \alpha_{m+r+1})(\alpha_m - \alpha_{m+h})}{\alpha_{m+r+1} - \alpha_m} \right\} \pi(m+2, h-2|z)$$

$$z: \alpha_{m+1} \alpha_{m+r+1} - \alpha_{m+r+1} \alpha_m - \alpha_{m+h+1} \alpha_{m+r+1} + \alpha_m \alpha_{m+h+1}$$

$$- \alpha_m \alpha_{m+r+2} + \alpha_m \alpha_{m+1} + \alpha_{m+h} \alpha_{m+r+2} - \alpha_{m+1} \alpha_{m+h}$$

second difference produces polynomial of degree $h-2$ in z only if $h=r+1$. second diff is then

$$-\frac{\{\pi(m+2, h-1|z) - \pi(m+1, h-1)\}}{\alpha_{m+h+1} - \alpha_m}$$

$$\sum_{r=0}^{\infty} \left[r : \alpha \| m, [i] \right]^{-1} \text{col} \left[S(f; \alpha \| m+r, r+i-r) \right] = \text{col} \left[S(f; m+r, r) \right]$$

$$\sum_{\omega} \delta(\theta(m+r+1, i-r); \omega || m+r, m-r) \delta(f; \omega || m+r, m-r - \omega) \langle \omega := [z, \bar{z}] \rangle$$

$$\omega := (i-\pi, r+i-\pi] \quad \omega_- = (i-\pi, r+i-\pi] \quad \delta(\Theta_-) = 0$$

$$\Rightarrow \delta(f \times \theta(mn+r+i-1); \alpha) \models m+r, r+i-1$$

$$= \sum_i S(f; \alpha || m+z, \omega) S(\Theta(m+r+z+1, i-\epsilon); \alpha || m+z+\omega, r+i-z-\omega)$$

$$\omega < r : \delta(\Theta_{\dots}) = 0 \quad \Rightarrow \quad \Theta(m_{H+2} + 1, i - z | m + z + \omega) = 0$$

$$= S(f; \alpha \|m+z, r) \quad \omega = [r, rr^{-1}z]$$

$$\hat{X}_{[r:\infty][m,[\bar{e}]]}^{-1}\hat{X}_{[r:\infty][m,[\bar{e}]]}$$

$$\langle \omega := [\tau, \nu] \rangle$$

$$\sum_{\omega} \delta(\theta(m_{\text{trt}} + z + 1, i - \tau); \alpha \| m + z, \omega - \varepsilon) \delta(\phi(m_{\text{trt}} + 1, i - \tau); \alpha \| m + \bar{\omega}, \bar{\tau} - \omega) \quad \langle \omega := [2-2] \rangle$$

$$= S(\theta(m+r+z+1, i-\epsilon) \times \phi(m+r+2, i-2); \alpha || m, i - r) = 1$$

$$\bar{x}[\alpha || m; r, [i]] \times [\alpha || m; r, [i]]^{-1}$$

$$\times [\alpha \| m, r, [i]] \text{ col} [\delta(\theta(m, r_{12})) ; \alpha \| m + z, r)] = \text{ col} [\delta(\theta(m, r_{12})) ; \alpha \| m, r + z)]$$

$$\overline{x}_{[\alpha||m;r,[i]]} \dot{x}_{[\alpha||m;r,[i]]}^{-1}$$

$$X[\alpha || m; r, [i]] \text{ col } [\delta(\Theta(m\hat{\alpha}+1, rr_i-2); \alpha || m+z, r)] = \text{col}_k$$

$$= \text{col}[\theta(m+1, r+1); \alpha(m, r)]$$

$$S\left(\frac{e}{z-\langle\alpha\rangle}, \alpha||m, \omega\right) \Delta(z||m, \omega H|z) = \Delta(e; \alpha||m, \omega|z)$$

$$\Lambda(\exp; \alpha || m, i) = \sum \Lambda(e; \alpha || m, \omega) \cdot \frac{\pi(\alpha || m, i+1 | z)}{\pi(\alpha || m, \omega H | z)} g(f; \alpha || m \# \omega, i - \omega)$$

$$\pi(\alpha \parallel m; i, i-\omega | z) S(f; \alpha \parallel m+\omega, i-\omega)$$

$$= \Delta(f; \alpha \parallel m+\omega, i-\omega | z) - \Delta(f; \alpha \parallel m+\omega+1, i-\omega | z)$$

$$\Lambda(e \times f; \alpha \parallel m, i) = \sum_i \Lambda(e; \alpha \parallel m, \omega) \left\{ \begin{array}{l} \Lambda(f; \alpha \parallel m+\omega, i-\omega | z) \\ - \Lambda(f; \alpha \parallel m+\omega+1, i-\omega | z) \end{array} \right. \quad \langle \omega := [i] \rangle$$

$$S(e; \alpha \parallel m, \omega) \frac{\pi(\alpha \parallel m; i+1 | z)}{\pi(\alpha \parallel m+\omega, i-\omega+1)}$$

$$= \frac{\pi(\alpha \parallel m, \omega | z)}{\pi(\alpha \parallel m+\omega, i-\omega+1)} \quad \langle \omega := [i] \rangle$$

$$\Lambda(e \times f; \alpha \parallel m, i) = \sum_i \Lambda(f; \alpha \parallel m+\omega, i-\omega) \left\{ \Lambda(e; \alpha \parallel m, \omega) - \Lambda(e; \alpha \parallel m, \omega-1) \right\} \quad \text{interchanging}$$

6 other forms obtained by reversing e, f and using summation index
 $\omega' = i-\omega$ $\alpha, e, f [m, m+1]$ reverse order forms of above are obtained by
carrying out e, f interchanging & index reversal

$\kappa \in [i]$

$$\lambda(\alpha \parallel m; i, \kappa) = \lambda(\alpha \parallel m; \kappa, \kappa) \lambda(\alpha \parallel m+\kappa; i-\kappa, 0)$$

$$\sum \lambda(\alpha \parallel m; \omega, \kappa) \left\{ \lambda(\alpha \parallel m+\omega; i-\omega, \kappa-\omega) - \lambda(\alpha \parallel m+\omega+1; i-\omega-1, \kappa-\omega-1) \right\} = 0 \quad \langle \omega := [\kappa, x] \rangle \quad (x > \kappa)$$

$$\sum \lambda(\alpha \parallel m; \omega, \kappa) \left\{ \lambda(\alpha \parallel m+\omega; i-\omega, \kappa-\omega) - \lambda(\alpha \parallel m+\omega+1; i-\omega-1, \kappa-\omega-1) \right\} = 0 \quad (\kappa > \omega) \quad \langle \omega := [\kappa, \omega] \rangle$$

$$= \lambda(\alpha \parallel m; i, \kappa) \quad (\kappa = \omega)$$

$$\Lambda[\alpha \parallel m; [i]] = \mathcal{L}[\lambda(\alpha \parallel m; \tau, \nu)]$$

$$\Lambda[\alpha \parallel m, [i]] = \mathcal{L}[\lambda(\alpha \parallel m+\omega; i-\omega, \tau-\omega) - \lambda(\alpha \parallel m+\omega+1; i-\omega-1, \tau-\omega-1)]$$

$$\Lambda[\alpha \parallel m, [i]] \Lambda[\alpha \parallel m, [i]] = \text{diag}[\lambda(\alpha \parallel m; i, \omega)]$$

$$c = v + (\alpha_{m+3} - \alpha_{m+2}) b$$

$$= 1 + \alpha_{m+2} - \alpha_m + (\alpha_{m+2} - \alpha_m)(\alpha_{m+2} - \alpha_{m+1})$$

$$+ \alpha_{m+2} - \alpha_{m+2} + (\alpha_{m+3} - \alpha_{m+2})(\alpha_{m+2} - \alpha_m + \alpha_{m+3} - \alpha_{m+1}) + \pi(\alpha \parallel m, 3 \mid \alpha_{m+3})$$

$$\cancel{\frac{\pi(\alpha \parallel m, 3 \mid \alpha_{m+3}) - \pi(\alpha \parallel m, 3 \mid \alpha_{m+2})}{\alpha_{m+3} - \alpha_{m+2}}}$$

$$= 1 + (\alpha_{m+2} - \alpha_m)(1 + \alpha_{m+2} - \alpha_{m+1})$$

$$+ (\alpha_{m+3} - \alpha_{m+2})(1 + \alpha_{m+2} + \alpha_{m+3} - \alpha_m - \alpha_{m+1})$$

$$= 1 + (\alpha_{m+2} - \alpha_m)(1 -$$

$$= 1 + (\alpha_{m+3} - \alpha_m + (\alpha_{m+2} - \alpha_m)(\alpha_{m+2} - \alpha_{m+1})$$

$$+ (\alpha_{m+3} - \alpha_{m+2})(\alpha_{m+2} + \alpha_{m+3} - \alpha_m - \alpha_{m+1})$$

$$= 1 + \alpha_{m+3} - \alpha_m + (\alpha_{m+2} - \alpha_m)(\alpha_{m+3} - \alpha_{m+1}) + (\alpha_{m+3} - \alpha_{m+2})(\alpha_{m+3} - \alpha_{m+1})$$

$$= 1 + \alpha_{m+3} - \alpha_m + (\alpha_{m+3} - \alpha_{m+1})(\alpha_{m+2} - \alpha_m + \alpha_{m+3} - \alpha_{m+2})$$

$$c = \sum_i \pi(\alpha \parallel m, \omega \mid \alpha_{m+3}) \langle \omega := [3] \rangle = S(f; \alpha \parallel m+3, \omega)$$

$$S(f; \alpha \parallel m+z, \omega) = \sum_{\tau \in \mathbb{Z}} S(\pi(\alpha \parallel m, \omega \mid \langle \alpha \rangle); \alpha \parallel m+z, \omega) \langle \omega := [z, \tau] \rangle$$

$$[S(f; \alpha \parallel m+z, \omega)] = \mathcal{L}[$$

$$b = S(f; \alpha \parallel m+2, 1)$$

Define $g(i): \mathbb{R} \rightarrow \mathbb{R}$ by setting $g(i) := \sum_i \pi(\alpha \parallel m, \omega) \langle \omega := [i] \rangle$

$$\text{then } S(g(i); \alpha \parallel m+z, \omega) = \sum_i S(\pi(\alpha \parallel m, \omega \mid \langle \alpha \rangle); \alpha \parallel m+z, \omega) \langle \omega := [i, z] \rangle$$

$\Leftrightarrow z := [i], \omega := [i-z]$ and $S(g(i); \alpha \parallel m, \omega) = 1 \quad \omega := [i]$

$$g(i) = 1 + \pi(\alpha \parallel m, 1) g(i-1)$$

$\mathbb{Y}[\dots]^{-1} \mathbb{Y}$

150.

$\omega := [\omega, z]$

$$\sum \delta(\Theta(m, z); \alpha \| m \omega, z - \omega) \prod_{\omega=1}^r (\alpha \| m + \omega + 1, z - \omega) \mu(r; \alpha \| m, \omega, z)$$

$T(\alpha \| m, z)$

$\prod (\alpha \| m, \omega) \prod (\alpha \| m + r + 1, \omega - \omega)$

$\pi: \prod (\alpha \| m + r + 1, z - \omega) \prod (\alpha \| m, z)$

$\prod (\alpha \| m + r, z - \omega) \quad \prod (\alpha \| m, z)$

$\sum \delta(\Theta(m, z); \alpha \| m \omega, z - \omega) \mu(r; \alpha \| m, \omega, z) \quad \langle \omega := [\omega, z] \rangle$

$$= \begin{cases} 0 & \omega \neq z \\ 1 & \omega = z \end{cases} \text{ from } \mathbb{X}[\dots]^{-1} \mathbb{X}[\dots] \text{ prof.}$$

$\pi \text{ factor} = 1 \text{ when } z = \omega \quad \text{i.e. } \mathbb{Y}[\dots]^{-1} \mathbb{Y} = \mathbb{I}[i]$

 \equiv

$\lambda(r; \alpha \| m, 0, 0) = 1$

$\lambda(r; \alpha \| m, 1, 0) = \mu(r; \alpha \| m; 1, 0) \pi(\alpha \| m + r + 1, 1) = \frac{\pi(\alpha \| m + r + 1, 1)}{\pi(\alpha \| m + r + 1, 1 | \alpha_m)}$

$\lambda(r; \alpha \| m, 0, 1) = \mu(r; \alpha \| m; 1, 1) \pi(\alpha \| m, 1) = \frac{\pi(\alpha \| m, 1)}{\pi(\alpha \| m, 1 | \alpha_{m+1})}$

$\lambda(r; \alpha \| m, 0, 0) = \Delta(f; \alpha \| m; r)$

$\lambda(r; \alpha \| m; 1, 0) = \Delta(f; \alpha \| m; r+1)$

$$\frac{z - \alpha_{m+r+1}}{\alpha_m - \alpha_{m+r+1}} \cdot \frac{\prod_{\omega=1}^r (z - \alpha_{m+\omega})}{\prod_{\omega=1}^r (\alpha_m - \alpha_{m+\omega})} = \lambda(\alpha \| m; r+1, 0)$$

$$\frac{\prod_{\omega=1}^r (z - \alpha_{m+\omega})}{\prod_{\omega=1}^r (\alpha_m - \alpha_{m+\omega})} = \lambda(\alpha \| m; r, 0)$$

$$\lambda(r: \alpha || m; z, o) = \frac{\pi(\alpha || m || r + 1, z)}{\pi(\alpha || m || r + 1, z | \alpha_m)}$$

$$\lambda(r: \alpha || m; z, o) \underbrace{\prod_{\omega=1}^r (z - \alpha_{m+\omega})}_{\prod_{\omega=1}^r (\alpha_m - \alpha_{m+\omega})} = \lambda(\alpha || m; r + z, o)$$

$$\stackrel{(6)}{f}_\omega = \lambda(\alpha || m; r + i, o | \alpha_\omega) \quad \omega = [m, m+i]$$

$$\lambda(r: \alpha || m; z, o) \cdot \lambda(\alpha || m; r + m, o) = \Delta(f^{(\omega)}; \alpha || m, r + z)$$

$$\mu(r: \alpha || m; z, o) = \pi(m | z) \pi(\alpha || m || r + 1, z - o) \quad \pi(m | z, r + 1)$$

$$\psi \lambda(r: \alpha || m; z, o) = \pi(m | z) \pi(\alpha || m || r + 1, z - o) \psi \mu(r: \alpha || m; z, o) \\ = \Delta(f^{(\omega)}; \alpha || m, r + z)$$

$$S(f^{(\omega)}; \alpha || m, r + z) \pi(\alpha || m, r + z)$$

$$= \pi(m | z) \psi \left\{ \pi(\alpha || m || r + 1, z - o) \mu(r: \alpha || m; z, o) \right. \\ \left. - \pi(\alpha || m || r + 1, z - o - 1) \mu(r: \alpha || m; z - 1, o) \right\}$$

$$= \pi(m | z) \psi \pi(\alpha || m || r + 1, z - o - 1)$$

$$\left\{ (z - \alpha_{m+r+z}) \mu(r: \alpha || m; z, o) - \mu(r: \alpha || m; z - 1, o) \right\}$$

$$f^{(\omega)} \in \text{seq}(\mathbb{N} | i) \quad \omega := [i] \quad B: [\Delta(f^{(\omega)}; \alpha || m, r)]$$

$$Y[\dots]B = [\Delta(f^{(\omega)}; \alpha || m, r)]$$

$$Y[\dots]^{-1} Y B = [\Delta(f^{(\omega)}; \alpha || m + z, r)] = B \rightarrow YY^{-1} = I[i]$$

Choose B ~~such~~ nonsingular e.g. $f^{(2)} = \Theta(\alpha || m, o)$, $\Delta(f^{(2)}; \alpha || m + z, r) = 0$ ~~unless~~
 $\Delta(f^{(\omega)}, \alpha || m + z, r) = \pi(\alpha || m, o | \alpha_{m+\omega}) \lambda(\alpha || m, o | \alpha_{m+\omega}, m, r, o)$

$$f^{(\omega)} = \Theta(\alpha || m, r+2) \quad \Delta(f^{(\omega)}, \alpha || m+z, r) = 0 \quad z < 2 \quad 152.$$

$$\begin{aligned} \Delta(f^{(\omega)}, \alpha || m+z, r) &= \pi(\alpha || m, r+2 | \alpha_{m+z}) \lambda(\alpha || m+z; r, r) \\ &= \frac{\pi(\alpha || m, r+2 | \alpha_{m+z})}{\pi(\alpha || m+z, r | \alpha_{m+z})} \pi(\alpha || m+z, r) \end{aligned}$$

This proves $Y^{-1}Y = I \langle K \setminus \alpha[m, m+i] \rangle$ (since $Y^{-1}Y = I$ as in p 150)

— o —

$$* Y[r: \alpha || m, [i]]^{-1} \text{col} [\Delta(f; \alpha || m+z, r+i-\omega)] \quad \omega := \text{left}[z, i]$$

$$\sum \underbrace{\delta(\Theta(m+r+z+1, i-\omega); \alpha || m+z, \omega-z) \pi(\alpha || m+z, \omega-z)}_{\pi(\alpha || m+r+z+1, i-\omega)} \Delta(f; \alpha || m+z, r+i-\omega) \quad \omega := [i-\omega] \\ m+z+\omega \quad r+i-\omega \\ \omega := [r+i-\omega]$$

$$= \Delta(f \times \Theta(m+r+z+1, i-\omega); \alpha || m+z, r+i-\omega) \quad \omega := [r+i-\omega]$$

$$= \sum \underbrace{\delta(\ell \pi \Delta(f; \alpha || m+z, \omega))}_{\pi(\alpha || m+r+z+1, i-\omega)} \pi(\alpha || m+z+\omega+1, r+i-z-\omega) \delta \quad \omega := [r+i-\omega] \\ \delta(\Theta(\alpha || m+r+z+1, i-\omega); \alpha || m+z+\omega, r+i-\omega-\omega)$$

$$\omega := [r] : \delta(\dots) = 0 \quad \omega := (r, r+i-\omega) : \Theta(\rightarrow) = 0$$

$$\Theta(\dots | \omega_{m+z+\omega}) = 0 \Rightarrow \delta(\dots) = 0$$

$$- \Delta(f; \alpha || m+z, r) \underbrace{\pi(\alpha || m+z+r+1, i-\omega)}_{\pi(\alpha || m+z+r+1, i-\omega)} = \Delta(f; \alpha || m+z, r)$$

=

$$\sum_{m+z+\omega} \Delta(f; \alpha || m+\omega, r) \lambda(r: \alpha || m+z, i-\omega, \omega-\omega) \quad \omega := \text{left}[z, i] \\ \omega \quad \omega := [i-\omega]$$

$$= \Delta(f; \alpha || m+z, r+i-\omega)$$

=

$$\tilde{Y}[\dots] \text{ wt } [\Delta(f; \alpha \| m, r+i)]$$

\tilde{Y} : zero when $i > z$

$$\sum \frac{\delta(\Theta(m, r); \alpha \| m+r+\omega, i-\omega) \pi(\alpha \| m+r+\omega, i-\omega)}{\pi(\alpha \| m, r)} \Delta(f; \alpha \| m, r+i)$$

$$\langle \omega := [i-r, i] \rangle$$

$$= \langle \omega := [i] \rangle$$

$$(\omega < i - z : i - \omega > z \\ \Rightarrow \delta(\Theta..) = 0)$$

$$= \Delta(f \times \Theta(m, r), \alpha \| m+r, i)$$

$$\pi(\alpha \| m, r)$$

$$= \sum$$

$$m+\omega, i+r-\omega$$

$$m+2\omega+1, 2r-\omega$$

$$m, \omega$$

$$\omega := [r, r+i]$$

$$= \omega \hat{\pi}_{\alpha}:=[r+i]$$

$$\omega < r : 2r-\omega > i > z$$

$$\frac{\Delta(f \times \Theta(m, r); \alpha \| m, r+i)}{\pi(\alpha \| m, r)}$$

$$= \sum \frac{\Delta(f; \alpha \| m+\omega, r+i-\omega) \pi(\alpha \| m, \omega)}{\pi(\alpha \| m, r)} \delta(\Theta(m, r); \alpha \| m, \omega)$$

$$\triangle \Delta$$

$$\tilde{Y}[\dots] Y[\dots]$$

$$\langle \omega := [\max(i-z, 0), i] \rangle$$

$$\sum \frac{\delta(\Theta(m, r); \alpha \| m+r+\omega, i-\omega) \pi(\alpha \| m+r+\omega+1, i-\omega)}{\pi(\alpha \| m, r)}$$

$$\omega < \max(i-z, 0)$$

$$\Rightarrow \omega < i-z \Rightarrow i-\omega > z$$

$$\Rightarrow \delta(\Theta..) = 0$$

$$\left\{ \delta(\phi(\alpha \| m, \omega); \alpha \| m+r, \omega-\omega) - \delta(\phi(\alpha \| m, \omega+\omega); \alpha \| m+r+1, \omega-\omega-1) \right\}$$

$$\langle \omega := [i, i] \rangle$$

$$\pi(\alpha \| m, \omega) \pi(\alpha \| m+r+\omega+1, \omega-\omega)$$

$$\pi(m, r) \frac{\pi(\alpha \| m+r+\omega+1, i-\omega)}{\pi(m, r)} \sum \delta(\phi(\alpha \| m, \omega); \alpha \| m+r, \omega-\omega)$$

$$\delta(\Theta(m, r); \alpha \| m+r+\omega, i-\omega)$$

-

"

$$v := v+1$$

$$\langle \omega := [i-\omega] \rangle$$

$$\omega \quad m+r+\omega, i-\omega-\omega$$

$$\frac{\pi(\alpha/m)}{\pi(\alpha/m, z)} \overline{\pi}(\alpha // m + r + 1, i - r) \cdot \left\{ \begin{array}{l} \delta(\Theta(m, z) \times \phi(m, r); \alpha // m + r + 1, i - r) \\ - \delta(\Theta(m, z) \times \phi(m, r + 1); \alpha // m + r + 1, i - r - 1) \end{array} \right\}$$

$$\begin{array}{ll} z > r & i - r \geq r - r \Rightarrow \delta(\Theta \times \phi) = 0 \\ z < r & \Theta(m, z) \times \phi(m, r) = \phi(m + z, r - z) \\ z \geq r & i - r - 1 \geq r - z - r \end{array} \quad \begin{array}{ll} z = r & i - r - 1 = r - z - 1 \\ i - r = r - z & \\ \text{diff} = 0 & \end{array}$$

$$= \frac{\pi(m)}{\pi(\alpha/m, z)} \overline{\pi}(\alpha // m + r + 1, i - r) \cdot \left\{ \begin{array}{l} \delta(\phi(m + z, r - z); \alpha // m + r + 1, i - r) \\ - \delta(\phi(m + z, r - z + 1); \alpha // m + r + 1, i - r - 1) \end{array} \right\}$$

$$\begin{array}{ll} m' = m + z & r' = r - z \\ d' = r - z & m'rr'r' = mrrr \\ i' - d' = r - z & \\ r' = \check{m} + \check{r} + \check{r} - \check{m} - \check{z} - \check{d} + \check{z} & r' = r \\ i' = i - r + r - z = i - z & \end{array}$$

$$\frac{\pi(\alpha/m)}{\pi(\alpha/m, z)} \overline{\pi}(\alpha // m + r + 1, i - r) \mu(r; \alpha // m + z; i - z, r - z) = \lambda(r; \alpha // m + z; i - z, r - z)$$

$$\overline{\pi}(r; \alpha // m + z; i - z, r - z) = \overline{\pi}(\alpha // m + z, r - z) \pi(\alpha // m + r + 1, i - r)$$

$$\tilde{Y}[..] Y[..] = \tilde{Y}[..]$$

$$\tilde{Y}[..] \text{ col } [\Lambda(f; \alpha // m + z, r + i - z)] \quad \langle \omega := [i] \rangle$$

$$\sum \frac{\delta(\Theta(\alpha // m + r + 1, i - r); \alpha // m, \omega) \pi(\alpha // m, \omega) \Lambda(f; \alpha // m + \omega, r + i - \omega)}{\pi(\alpha // m + r + 1, i - r)} \quad \omega := [r + i] \quad \langle \delta(\Theta \dots m, \omega) = 0 \rangle_{r \geq i - r}$$

$$= \delta(f \times \Theta(\alpha // m + r + 1, i - r); \alpha // m, r + i) \quad \langle \omega := [r + i] \rangle$$

$$= \sum \frac{\Lambda(f; \alpha // m, \omega) \pi(\alpha // m + \omega + 1, r + i - \omega) \delta(f; \Theta(\alpha // m + r + 1, i - r); \alpha // m + \omega, r + i - \omega)}{\pi(\alpha // m + r + 1, i - r)} \quad \begin{array}{l} \delta = 0 \quad r - \omega - i \leq r + i \\ \omega \geq r + 2 \Rightarrow \Theta = 0 \end{array}$$

$$= \Lambda(f; \alpha // m, r + i)$$

$\hat{Y}[-] \hat{Y}[..]$

$$\pi(r: \alpha || m+z; i-\tau, \tau-\sigma) = \pi(\alpha || m+z, \tau-\sigma) \pi(\alpha || m+r+\tau+1, i-\tau)$$

$$\mu(r: \alpha || m+z; i-\tau, \tau-\sigma) = \delta(\phi(\alpha || m+r+1, i-\tau); \alpha || m+z, \tau-\sigma) - \delta(\phi(\alpha || m+r+1, i-\tau+1); \alpha || m+z, \tau-\sigma-1)$$

$$\sum \underbrace{\delta(\Theta(m+r+1, i-\tau); \alpha || m, \omega) \pi(\alpha || m, \omega)}_{\pi(\alpha || m, \omega)} \cancel{\pi(\alpha || m+r+1, i-\tau)} \pi(\alpha || m+r+1, i-\tau)$$

$$\left\{ \delta \{ \phi(\alpha || m+r+1, i-\tau); \alpha || m+\omega, \tau-\omega) - \delta(\phi(\alpha || m+r+1, i-\tau+1); \alpha || m+\omega, \tau-\omega-1) \right\}$$

$\langle \omega := \{i\} \rangle$

$$\begin{aligned} & \frac{\pi(\alpha || m, \omega) \pi(\alpha || m+r+1, i-\tau)}{\pi(\alpha || m+r+z+1, i-\tau)} \\ &= \frac{\delta(\Theta(m+r+z+1, i-\tau) \times \phi(m+r+1, i-\tau); \alpha || m, \omega)}{\pi(\alpha || m+r+z+1, i-\tau)} \quad \tau \leq \omega \Leftrightarrow 0 \\ & \quad - \delta(\Theta(m+r+z+1, i-\tau) \times \phi(m+r+1, i-\tau+1); \alpha || m, \omega-1) \quad \tau \leq \omega-1 \Leftrightarrow 0 \end{aligned}$$

$$\{...\} = \{ \cancel{\delta(\Theta(m+r+z+1, i-\tau); \alpha || m, \omega)} \}$$

$\tau = \omega$

$$\{...\} = -\delta(\phi(m+r+1, 1); \alpha || m, \omega-1)$$

$\tau > \omega$

$$= \mu(r: \alpha || m, z, \omega)$$

$$\{...\} = \delta(\phi(m+r+1, \tau-1); \alpha || m, \omega) - \delta(\phi(m+r+1, \tau-\omega+1); \alpha || m, \omega-1)$$

$$\tau \geq \omega \quad \text{faktur} = \pi(\alpha || m, \omega) \pi(\alpha || m+r+1, \tau-\omega) = \pi(r: \alpha || m, z, \omega)$$

 $\hat{Y}[..] \hat{Y}[..] = Y[..]$

$$\frac{1}{\pi(\alpha || m+r+1, \tau-\omega)} X[..] I[i] = I[i]^0 X[..]^{-1} I[i]^0 = I[i]^0$$

$$X[..] I[i] = I[i]^i \quad X[..]^{-1} I[i]^i = I[i]$$

$$X[..] I[i]^0 = I[i]^i \quad X[..]^{-1} I[i]^i = I[i]^0$$

from p. 108

$$1ii) \{Y[\dots]\}^{-1} = \dots = \text{diag}_{\hat{D}^{-1}} \left[\frac{\Lambda(\Theta(\alpha||m, z); \alpha||m+1, z-1)}{\pi(\alpha||m, z)} \right] L \bar{I}[i]^{-1}$$

$$= \text{diag}_{\hat{D}^{-1}} \left[\frac{\pi(\alpha||m, z)}{\Lambda(\Theta(\alpha||m, z); \alpha||m+1, z-1)} \right]$$

$$2ii) \hat{Y}[\dots]^{-1} = \dots = \text{diag}_{\hat{D}^{-1}}$$

$$= \text{diag}_{\hat{D}^{-1}} \left[\frac{\Lambda(\Theta(\alpha||m+1, z-1); \alpha||m+2, z-2)}{\pi(\alpha||m+1, z-1)} \right] L \bar{I}[i]^{-1}$$

$$= \text{diag}_{\hat{D}^{-1}} \left[\frac{\pi(\alpha||m+1, z-1)}{\Lambda(\Theta(\alpha||m+1, z-1); \alpha||m+2, z-2)} \right]$$

$$3ii) \tilde{Y}[\dots] = \text{diag}_{\hat{D}^{-1}} \left[\frac{\Lambda(\Theta(\alpha||m, z); \alpha||m+1, z-1)}{\pi(\alpha||m, z)} \right] L \bar{I}[i]^{-1}$$

$$= \text{diag}_{\hat{D}^{-1}} \left[\frac{\pi(\alpha||m, z)}{\Lambda(\Theta(\alpha||m, z); \alpha||m+1, z-1)} \right] \text{diag}_{\hat{D}^{-1}}$$

$$= \hat{Y}[\dots] \{Y[\dots]\}^{-1}$$

$$4ii) \hat{\tilde{Y}}[\dots] = \text{diag}_{\hat{D}^{-1}} \left[\frac{\Lambda(\Theta(\alpha||m+1, z-1); \alpha||m, z)}{\pi(\alpha||m+1, z-1)} \right] L \bar{I}[i]^{-1}$$

$$= \text{diag}_{\hat{D}^{-1}} \left[\frac{\pi(\alpha||m+1, z-1)}{\Lambda(\Theta(\alpha||m+1, z-1); \alpha||m, z)} \right] \text{diag}_{\hat{D}^{-1}}$$

$$= Y[\dots] \{\hat{Y}[\dots]\}^{-1}$$

Define $\tilde{D}, \hat{D} \in \mathbb{K} \rightarrow [K|i]$ by setting

$$\tilde{D} := \text{diag} [\pi(\alpha||m, z)] \quad \hat{D} := \text{diag} [\pi(\alpha||m+1, z-1)]$$

$$4] \quad X[r:\alpha || m, [i]] I_{[i]} = \tilde{X}[r:\alpha || m, [i]]^{-1} I[i]^i - I[i]^o$$

$$\tilde{X}[r:\alpha || m, [i]]^{-1} I[i]^o = \tilde{X}[r:\alpha || m, [i]]^{-1} I[i]^i = I[i]^i$$

$$\tilde{X}[r:\alpha || m, [i]] I_{[i]} = \tilde{X}[r:\alpha || m, [i]] I[i]^o = I[i]^i$$

$$\theta(m, i) \theta(m+i, j) = \theta(m, i+j) \quad \parallel \prod_{k=1}^n \oplus \left(\sum_{\omega} \nu(\omega) \langle \omega : [x] \rangle, \nu(x) \right)$$

$$\theta(a, b) \theta(a+b, c) = \theta(a, b+c) = \theta(\nu(\alpha), \{\sum \nu(\omega) \langle \omega : (m) \rangle\})$$

$$\theta(a, b+c) \theta(a+b+c, d) = \theta(a, b+c+d)$$

$$\theta(a, b) \theta(a+b, c) \theta(a+b+c, d) = \theta(a, b+c+d)$$

$$\theta(\dots \times \theta(a+b+c+\dots + x, y) \theta(a+b+c+\dots + x+y, z) = \theta(a, b+c+d+\dots + x+y+z)$$

$$a, b, \dots, y, z \in \overline{\mathbb{N}} \quad \alpha \in \text{seq}'(|K| \geq a+b+\dots+y+z-1)$$

ϕ similarly

$$m \leq n \quad n+j \leq m+i \quad \theta(m, i) \times \phi(n, j) = \theta(m, n-m) \theta(n+j, m+i-n-j)$$

$$\phi(m, i) \times \theta(n, j) = \phi \quad \phi$$

$$m \leq n \quad m+i \leq n+j \quad \theta(m, i) \times \phi(n, j) = \theta(m, n-m) \phi(m+i, n+j-m-i)$$

$$\phi \quad \theta \quad \phi \quad \theta$$

// i) Let

i) Let $a, b, \dots, y, z \in \overline{\mathbb{N}}$ and $\alpha \in \text{seq}^{\#}(|K| \geq a+b+\dots+y+z-1)$.

$$\begin{aligned} & \theta(\alpha_1, \alpha_2 \parallel a, b) \times \theta(\alpha_3, \alpha_4 \parallel a+b, c) \times \theta(\alpha_5, \alpha_6 \parallel a+b+c, d) \times \\ & \dots \times \theta(\alpha_{2k}, \alpha_{2k+1} \parallel a+b+c+\dots+x, y) \times \theta(\alpha_{2k+2}, \alpha_{2k+3} \parallel a+b+c+\dots+x+y, z) \\ & = \theta(\alpha, \beta \parallel a, b+c+\dots+x+y+z) \end{aligned}$$

ii) With $\alpha \in \text{seq}'(|K| \geq a+b+\dots+y+z-1)$ a relationship similar to the above with θ replaced by ϕ holds for members of subsequences containing $a+b+\dots+y+z$.
(the constituent sequences) with index $\#$ in the range from $[\alpha] - [\alpha, \vee]$

2] Let $n \in \mathbb{N}$, $\alpha \in \text{seq}(\bar{\mathbb{N}} | \geq n)$ and $\omega \in \text{seq}(\bar{\mathbb{N}} | \geq n)$. Let $\alpha \in \text{seq}(K | \geq \sum_{\omega} \omega) < \omega = [n]$.

$$\prod_{x=1}^n \Theta(\alpha || \sum_{\omega} \omega < \omega = [x], \omega(x)) = \Theta(\alpha || \omega), \{ \sum_{\omega} \omega < \omega = [n] \}$$

ii) With $\alpha \in \text{seq}'(K | \geq \{ \sum_{\omega} \omega < \omega = [n] \} - 1)$ a relationship similar to the above with Θ replaced by ϕ holds for subsequences containing members with index in the range $[\alpha] - [\omega(0)]$, $\{ \sum_{\omega} \omega < \omega = [n] \}$ of the constituent sequences.

$$\underline{\omega \in \text{seq}(\bar{\mathbb{N}} | \geq \alpha)} \quad \text{non decr. } \omega(0) = 0 \quad \alpha \in$$

$$\prod_{x=0}^{i-1} \Theta(\alpha, \beta || m + \omega(x), \omega(x+1) - \omega(x)) = \Theta(\alpha || m, \omega(i))$$

$$\underline{\text{all } \beta \in \{ \beta \mid \beta \notin \{ \alpha[m, \omega(i)] \} : = \}}$$

$$\prod_{x=0}^{i-1} \phi(\alpha, \beta || m + \omega(x), \omega(x+1) - \omega(x) | \equiv) = \phi(\alpha, \beta || m, \omega(i) | \equiv)$$

$$\equiv : \text{all } \beta \in \{ \beta \mid \beta \notin \{ \alpha[m, \omega(i)] \} \} = ' m, m \omega(i)$$

$$\begin{aligned} \beta_j = \alpha_j & \quad j \in [n, i] \Rightarrow \phi(n, j | \beta_j) \in K \quad \Theta(m, i | \beta_j) = \beta_j - \alpha_j = 0 \\ & \rightarrow \Theta(m, i | \beta_j) \phi(n, j | \beta_j) = 0 \quad \text{but} \quad \Theta(m, n - m | \beta_j) \neq 0 \\ & \quad \Theta(n, j, m, j | \beta_j) \neq 0 \end{aligned}$$

$$\begin{aligned} \text{Thus } \Theta(\alpha, \beta || m, i | \equiv) \phi(\alpha, \beta || n, j | \equiv) &= \Theta(\alpha, \beta || m, i | \equiv) \phi(\alpha, \beta || n, j | \equiv) \\ &= \Theta(\alpha, \beta || m, i | \equiv) \end{aligned}$$

$$\therefore \beta_j = \alpha_j \quad j \in [n, m \omega(i)] \quad \phi(n, j | \beta_j) \in K \quad \Theta(m, i | \beta_j) = 0$$

$$\alpha_{mn} \alpha_{m+n} \alpha_{m+1} - \alpha_{mn} \alpha_{m+n} \alpha_m + \alpha_m \alpha_{mn} \alpha_{m+2} - \alpha_m \alpha_{mn}^2 = \\ = \sum_{\omega=0}^K \Theta(\alpha \beta || n(\omega), j(\omega)) ; \beta || m, i) = \begin{cases} 1 & (i=z) \\ 0 & (i>z) \end{cases}$$

$$\sum_{\omega=0}^K j(\omega) = z$$

$$\beta \in \text{seq}'(K | \geq m+r_i) \quad \alpha \in \text{seq}(K | \geq \max \{n(\omega) + j(\omega) - 1\} \langle \omega := [x] \rangle)$$

Two sequences formed from values of factorial polynomials are defined

Definition Set $N' := N$

i) The mapping $\Theta: \text{seq}''(K | \geq \bar{N} + N') \times \text{seq}(K) \times \bar{N} \times N' \rightarrow \text{seq}(K)$ is defined by setting

$$\Theta(\alpha, \beta || n, j | \omega) := \pi(\alpha || n, j | \beta_\omega) \quad \langle \omega := [\beta] \rangle$$

ii) $\Theta(\alpha, \alpha || n, j)$ is written simply as $\Theta(\alpha || n, j)$; in this way a mapping $\Theta: \text{seq}'(K | \geq \bar{N} + N') \times \text{seq}(K) \times \bar{N} \times N' \rightarrow \text{seq}(K)$ is defined

ii) The mapping $\phi: \text{seq}(K | \geq \bar{N} + N') \times \text{seq}(K) \times \bar{N} \times N' \rightarrow \text{seq}(K)$ is defined by setting

$$\phi(\alpha, \beta || n, j | \omega) := \frac{1}{\pi(\alpha || n, j | \beta_\omega)}$$

for all $\omega \in [\delta]$ for which $\beta_\omega \notin \alpha[n, j]$ and taking the remaining members of the sequence $\phi(\alpha, \beta || n, j) \langle \omega \in [\delta] \rangle := \phi(\alpha, \beta || n, j | [\beta])$ to be arbitrarily assigned members of K . (In later use of the sequence mapping ϕ , members of the sequence $\phi(\alpha, \beta || n, j)$ not defined by use of the above allocation are not involved.)

iii) $\phi(\alpha, \alpha || n, j)$ is written simply as $\phi(\alpha || n, j)$; a further mapping is defined as in (ii) above, in the way.

Let $m, r \in \bar{\mathbb{N}}$ and $\alpha \in \text{seq}'(\mathbb{K} | \geq m+r)$, $\beta \in \text{seq}(\mathbb{K})$

$$S(r; \Theta(\alpha, \beta) \parallel \langle [\alpha|_r - 1], r+1 | \xi \rangle; \beta \parallel m, 1) = \Theta(\alpha, \beta | m+1, r | \xi)$$

$$\text{for } \xi := [\beta]$$

Let $n, j \in \bar{\mathbb{N}}$, $\alpha \in \text{seq}(\mathbb{K} | \geq n+j-1)$ and $\beta \in \text{seq}(\mathbb{K})$. The function Θ acts as a filter in the sense that for all $\xi \in [\beta]$ for which $\beta_\xi \notin \alpha[n, n+j]$, $\Theta(\alpha, \beta | n, j | \xi) = 0$.

In particular, the subsequence $\Theta(\alpha | n, j | [n, j])$ consists exclusively of zero members.

Let $n, j, k, r \in \bar{\mathbb{N}}$ and $m, \omega \in \text{seq}(\bar{\mathbb{N}} | \geq k)$. Set $\Xi := \sum_{\omega} \zeta(\omega) \zeta_{\omega} \cdot \omega$.

Let $\alpha \in \text{seq}(\mathbb{K} | \geq m + \max\{m(\omega), j(\omega) - 1\} < \omega := [k] >)$ and $\beta \in \text{seq}'(\mathbb{K} | \geq m+r+n+j)$

$$S(r; \prod_{\omega \in \Xi} \Theta(\alpha, \beta | m(\omega), j(\omega)) < \omega := [k] >; \beta | m, j) = \begin{cases} 1 & (j = r) \\ 0 & (j > r) \end{cases}$$

Let $m, i \in \bar{\mathbb{N}}$ and $\omega \in \text{seq}(\bar{\mathbb{N}} | \geq i)$ be nondecreasing with $\omega(0) = 0$.

Let $\alpha \in \text{seq}(\mathbb{K} | \geq m+\omega(i)-1)$ and $\beta \in \text{seq}(\mathbb{K})$.

i) $\left\{ \prod \Theta(\alpha, \beta | m+\omega(x), \omega(x+1)-\omega(x)) < x := [i] > \right\} = \Theta(\alpha, \beta | m, \omega(i))$
 $\text{seq}(\bar{\mathbb{N}})$ be obtained by arranging

ii) Let Ξ be the set of all $\xi \in [\beta]$ for which $\beta_\xi \notin \alpha[m, \omega(i)]$
arranged in some order.

$$\left\{ \prod \phi(\alpha, \beta | m+\omega(x), \omega(x+1)-\omega(x)) < x := [i] > \right\} = \phi(\alpha, \beta | m, \omega(i) | \Xi)$$

Let $i, j, m, n \in \bar{N}$, $\alpha \in \text{seq}(\mathbb{K})$ ($| \geq \max(m_i, n_j) - 1$) and $\beta \in \text{seq}(\mathbb{K})$. Let $\Xi \in \text{seq}(\bar{N})$ be obtained by arranging all $\beta \in [\beta]$ for which $\beta_{\beta} \notin \alpha[m, n_j]$ in some order, and let $\Xi' \in \text{seq}(\bar{N})$ be obtained determined by the condition $\beta_{\beta} \notin \alpha[m, m_i]$ in a similar way.

i) If $m \leq n$ and $n_j \leq m_i$

$$\Theta(\alpha, \beta \parallel m, i \mid \Xi) \times \phi(\alpha, \beta \parallel n, j \mid \Xi) \neq \Theta(\alpha, \beta \parallel m, n-m \mid \Xi) \times \Theta(\alpha, \beta \parallel n_j, m_i-n_j \mid \Xi)$$

and

$$\phi(\alpha, \beta \parallel m, i \mid \Xi') \times \Theta(\alpha, \beta \parallel n, j \mid \Xi') \neq \phi(\alpha, \beta \parallel m, n-m \mid \Xi') \times \phi(\alpha, \beta \parallel n_j, m_i-n_j \mid \Xi')$$

ii) If $m \leq n$ and $m_i \leq n_j$

$$\Theta(\alpha, \beta \parallel m, i \mid \Xi) \times \phi(\alpha, \beta \parallel n, j \mid \Xi) \neq \Theta(\alpha, \beta \parallel m, n-m \mid \Xi) \times \phi(\alpha, \beta \parallel m_i, n_j-m_i \mid \Xi)$$

and

$$\phi(\alpha, \beta \parallel m, i \mid \Xi') \times \Theta(\alpha, \beta \parallel n, j \mid \Xi') \neq \phi(\alpha, \beta \parallel m, n-m \mid \Xi') \times \Theta(\alpha, \beta \parallel m_i, n_j-m_i \mid \Xi')$$

Let $m, r, i \in \bar{N}$ and $\alpha \in \text{seq}'(\mathbb{K})$ ($| \geq m+r+i$), $\beta \in \text{seq}(\mathbb{K})$

$$S(r; \phi(\alpha, \beta \parallel \langle |\alpha| - r - i \rangle, r+1 \mid \beta); \alpha \parallel m, i) = \phi(\alpha, \beta \parallel m, r+r+1 \mid \beta)$$

for $\beta := [\beta]$

$$\sum \left\{ \sum_i \mu(\alpha \parallel m; \omega, i) f_{m+i} \langle z := [\omega] \rangle \right\} \left\{ \sum_i \mu(\alpha \parallel m + \omega; i - \omega, \bar{\omega}) \frac{f_{m+i}}{f_{m+\omega+i}} \right\}_{\substack{D-\omega \\ \langle \omega := [i-\bar{\omega}] \rangle \\ \langle \omega := [i] \rangle}}$$

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$$\sum \sum \sum_i \mu(\alpha \parallel m; \omega, i) \mu(\alpha \parallel m + \omega; i - \omega, \bar{\omega}) \langle \omega := [z, \bar{\omega}] \rangle \langle z := [\omega] \rangle \langle \omega := [i] \rangle$$

$$= \sum_i \mu(\alpha \parallel m; z, \omega) \mu(\alpha \parallel m + z; i - z, 0) f_{m+z} g_{m+z}$$

$$\sum \sum \sum_i \langle z := [\omega] \rangle \langle \omega :=$$

Let $m, i \in \bar{\mathbb{N}}$ and $\alpha \in \text{seq}'(|K| \geq m i)$

i) For $z := [i]$

$$\mu(\alpha \parallel m; i, z) = \mu(\alpha \parallel m; z, \omega) \mu(\alpha \parallel m + z; i - z, 0)$$

ii) For all $z, \bar{\omega} \in [i]$ such that $z \neq \bar{\omega}$

$$\left\{ \sum_i \mu(\alpha \parallel m; \omega, \bar{\omega}) \mu(\alpha \parallel m + \omega; i - \omega, \bar{\omega} - \omega) \langle \omega := [z, \bar{\omega}] \rangle \right\} = 0$$

Let $m, i \in \bar{\mathbb{N}}$ and $\alpha \in \text{seq}'(|K| \geq m i)$, and $e, f \in \text{seq}(|K| \geq m i)$

$$\delta(e \times f; \alpha \parallel m, i) = \sum_i \delta(e; \alpha \parallel m, \omega) \delta(f; \alpha \parallel m + \omega, i - \omega) \langle \omega := [i] \rangle$$

Proof:

$$\text{a)} \quad \sum \sum \sum_i \mu(\alpha \parallel m; \omega, i) \mu(\alpha \parallel m + \omega; i - \omega, \bar{\omega} - \omega) \frac{f_{m+z}}{f_{m+\omega+z}} \frac{g_{m+\omega}}{g_{m+z}}$$

$$\langle \omega := [z, \bar{\omega}] \rangle \langle z := [\omega], \bar{\omega} := [\bar{\omega}] \rangle \langle \omega := [i] \rangle$$

$$= \sum_i \mu(\alpha \parallel m; z, \omega) \mu(\alpha \parallel m + z; i - z, 0) f_{m+z} g_{m+z}$$

The first sum is rearranged as

$$\sum_i \left\{ \sum_j \langle z := [\omega], \bar{\omega} := [\omega, i] \rangle \right\} \langle \omega := [z] \rangle$$

Then

$$\sum \left\{ \sum \mu(\alpha || m; \omega, \nu) f_{m+\nu} \langle \omega := [\omega] \rangle \right\} \left\{ \sum \mu(\alpha || m+\omega; i-\omega, i-\omega) f_{m+\omega} \langle \omega := [\omega, i] \rangle \right\} \\ \langle \omega := [i] \rangle$$

$$= \sum \mu(\alpha || m; i, i) e_{m+i} f_{m+i}$$

$$\text{b) } S(exf; \alpha || m, i) = \frac{e_m f_{m+1} - e_m f_m}{\alpha_{m+1} - \alpha_m} = \frac{(e_{m+1} - e_m) f_{m+1}}{\alpha_{m+1} - \alpha_m} \\ = \frac{e_m (f_{m+1} - f_m) + (e_{m+1} - e_m) f_m}{\alpha_{m+1} - \alpha_m}$$

The result is true for $i=1$. Assume true with i replaced by $i+1$.

$$S(exf; \alpha || m, i+1) = \sum_i S(e; \alpha$$

as stated so that for suitable α, e and f ,

$$S(exf; \alpha || m+1, i) = \sum_i S(e; \alpha || m+1, \omega) S(f; \alpha || m+\omega+1, i-\omega) \langle \omega := [i] \rangle$$

Then

$$S(exf; \alpha || m, i+1) = \frac{S(exf; \alpha || m+1, i) - S(exf; \alpha || m, i)}{\alpha_{m+1} - \alpha_m}$$

$$= (\alpha_{m+1} - \alpha_m)^{-1} \left\{ \sum (\alpha_{m+\omega+1} - \alpha_m) S(e; \alpha || m, \omega+1) S(f; \alpha || m+\omega+1, i-\omega) \right. \\ \left. + \sum (\alpha_{m+\omega} - \alpha_m) S(e; \alpha || m, \omega) S(f; \alpha || m+\omega, i-\omega+1) \right\}$$

$$= (\alpha_{m+1} - \alpha_m)^{-1} \left[(\alpha_{m+1} - \alpha_m) \{ S(e; \alpha || m, 0) S(f; \alpha || m, i+1) \right. \\ \left. + S(e; \alpha || m, i+1) S(f; \alpha || m+1, 0) \} \right]$$

$$+ \sum (\alpha_{m+\omega} - \alpha_m + \alpha_{m+i+1} - \alpha_{m+\omega}) S(e; \alpha || m, \omega) S(f; \alpha || m+\omega, i-\omega+1) \langle \omega := (i) \rangle$$

$$= \sum S(e; \alpha || m, \omega) S(f; \alpha || m+\omega, i-\omega+1) \langle \omega := [i+1] \rangle.$$

There are 3 other forms of the result stated which may be obtained by interchanging e and f and using the summation index $\omega := i-\omega$. The $w, e, f [m, m+i]$ reverse order form of the stated result is obtained by carrying out e, f interchange and index reversal.

Let $m, i \in \bar{N}$, $\alpha \in \text{seg}'(K \geq m_i)$ and $f \in \text{seg}(K \geq m_i)$. For

$$\cancel{\Delta} z := K \setminus \alpha [m, m_i]$$

$$\underline{\Lambda}(f; \alpha // m, i/z) = \delta\left(f \times \frac{1}{z - \langle \alpha \rangle}\right)$$

$$\pi(\alpha // m, i+1/z) \delta\left(f \times \frac{1}{z - \langle \alpha \rangle}; \alpha // m, i\right)$$

Let $m, i \in \bar{N}$, $\alpha \in \text{seg}'(K \geq m_i)$ and $f \in \text{seg}(K \geq m_i)$. With summation being effected for $\omega := [e]$ in each case

$$\begin{aligned} \underline{\Lambda}(ef; \alpha // m, i) &= \sum_i \delta(e; \alpha // m, \omega) \pi(\alpha // m, \omega) \underline{\Lambda}(f; \alpha // m_{\omega}, i-\omega) \\ &= \sum_i \delta(e; \alpha // m_{\omega}, i-\omega) \pi(\alpha // m_{\omega+1}, i-\omega) \underline{\Lambda}(f; \alpha // m, \omega) \\ &= \sum_i \left\{ \underline{\Lambda}(e; \alpha // m, \omega) - \underline{\Lambda}(e; \alpha // m, \omega-1) \right\} \underline{\Lambda}(f; \alpha // m_{\omega}, i-\omega) \\ &= \sum_i \left\{ \underline{\Lambda}(e; \alpha // m_{\omega}, i-\omega) - \underline{\Lambda}(e; \alpha // m_{\omega+1}, \omega-1) \right\} \underline{\Lambda}(f; \alpha // m, \omega) \langle K \rangle \end{aligned}$$

(Further forms of the above results may be obtained by interchanging e and f on the right hand side and replacing the index of summation by $\omega' = i - \omega$.)

- $\boxed{\begin{array}{l} m' = m + \omega \quad i' = i - \omega \quad m' + j' = m \omega \\ \delta(\phi(m+\omega+1, i-\omega)) \times e; \alpha // m_{\omega+1}, i-\omega \quad j' = \omega - \omega \end{array}}$
- $\sum_i \delta(e // m, \omega) \pi(\alpha // m, \omega) \underline{\Lambda}(f // m_{\omega}, i-\omega)$
 - $\sum_i \underline{\Lambda}(e // m, \omega) \delta(f // m_{\omega}, i-\omega) \pi(\alpha // m_{\omega+1}, i-\omega)$
 - $\sum_i \delta(e // m_{\omega}, i-\omega) \pi(\alpha // m_{\omega+1}, i-\omega) \underline{\Lambda}(f // m, \omega)$
 - $= \sum_i \underline{\Lambda}(e // m, \omega) \delta(f // m_{\omega}, i-\omega) \pi(\alpha // m_{\omega+1}, i-\omega) \langle \omega = [k] \rangle$
 - $= \sum_i \underline{\Lambda}(e // m, \omega) \delta(f // m_{\omega}, i-\omega) \pi(\alpha // m_{\omega+k}, i-k-\omega) \langle \omega = [k] \rangle$

Let $m, i \in \overline{N}$, $\alpha \in \text{seq}'(K \setminus m_i)$ and, $f \in \text{seq}(K \setminus m_i)$ and $j \in [i]$

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i) If $f_{m\omega} = \emptyset \langle \omega := [j] \rangle$ then

$$\delta(f; \alpha // m, i) = \delta(\phi(m, j) \times f; \alpha // m \setminus j, i - j)$$

ii) If $f_{m\omega} = \emptyset \langle \omega := (j, i] \rangle$ then

$$\delta(f; \alpha // m, i) = \delta(\phi(m \setminus j + 1, i - j) \times f // m, j)$$

Let $m, i \in \overline{N}$, $\alpha \in \text{seq}'(K \setminus m_i)$ and, $f \in \text{seq}(K \setminus m_i)$ and $j, k \in [i]$.

1] Let $e \in \text{seq}(K \setminus m_i)$

i) With ~~$j \in [i]$~~ , let $e_{m\omega} = \emptyset \langle \omega := [j] \rangle$. For $\omega := [j]$

$$\begin{aligned}\delta(exf; \alpha // m, i) &= \sum \delta(e; \alpha // m, \omega) \delta(f; \alpha // m \setminus \omega, i - \omega) \langle \omega := [j], i \rangle \\ &= \sum \delta(\phi(\alpha // m, j) \times e; \alpha // m \setminus j, \omega - j) \delta(f; \alpha // m \setminus \omega, i - \omega)\end{aligned}$$

ii) With ~~$j \in [i]$~~ , let $e_{m\omega} = \emptyset \langle \omega := (j, i] \rangle$. For $\omega := [j, i]$

$$\begin{aligned}\delta(exf; \alpha // m, i) &= \sum \delta(e; \alpha // m \setminus \omega, i - \omega) \delta(f; \alpha // m, \omega) \langle \omega := [j] \rangle \\ &= \sum \delta(\phi(\alpha // m \setminus j + 1, i - j) \times e; \alpha // m \setminus j, \omega - j) \delta(f; \alpha // m, \omega)\end{aligned}$$

iii) With ~~$k \in [i]$~~ let $g_x \in K \langle x := [k] \rangle$ exist such that, with $g: K \rightarrow K$ defined by setting

$$g(z) := \sum g_x z^x \langle x := [k] \rangle$$

The numbers of the subsequence $e[m, m_i]$ have the proper polynomial representation

$$e_{m\omega} = g(\alpha_{m\omega}) \langle \omega := [i] \rangle$$

For $\omega := [k, i]$

$$\delta(\exp; \alpha // m, i) = \sum \delta(e; \alpha // m, \omega) \delta(f; \alpha // m_{\omega}, i - \omega) \quad \langle \omega := [r] \rangle$$

$$= \sum \delta(e; \alpha // m_{\omega}, i - \omega) \delta(f; \alpha // m, \omega) \quad \langle \omega := [i - e, \frac{i}{r}] \rangle$$

w) With j, k , ℓ fixed, let the conditions upon e of (i, iii) hold in conjunction.

$$\text{For } \omega := [j], z := [k, i]$$

$$\delta(\exp; \alpha // m, i) = \sum \delta(e; \alpha // m, \omega) \delta(f; \alpha // m_{\omega}, i - \omega) \quad \langle \omega := [\ell, z] \rangle$$

$$= \sum \delta(\phi(\alpha // m, \omega) \times e // m_{\omega}, \omega - \omega) \delta(f; \alpha // m_{\omega}, i - \omega) \quad \langle \omega := [\ell, z] \rangle$$

v) With j, k, ℓ fixed, let the conditions upon e of (ii, iii) hold in conjunction.

$$\text{For } \omega := [j, i], z := [k, i]$$

$$\delta(\exp; \alpha // m, i) = \sum \delta(e; \alpha // m_{\omega}, i - \omega) \delta(f; \alpha // m, \omega) \quad \langle \omega := [i - z, z] \rangle$$

$$= \sum \delta(\phi(\alpha // m + 1, i - \omega) \times e; \alpha // m_{\omega}, \omega - \omega) \delta(f; \alpha // m, \omega) \quad \langle \omega := [i - z, z] \rangle$$

2i) With j, k fixed,

$$\delta(\Theta(\alpha // m, j) \times f; \alpha // m, i) = \delta(f; \alpha // m_j, i - j)$$

$$= \delta(\Delta(f; \alpha // m, i / \langle \alpha \rangle); \alpha // m_j, i - j)$$

$$= \sum \delta(\Theta(\alpha // m, j); \alpha // m_{\omega}, i - \omega) \delta(f; \alpha // m, \omega) \quad \langle \omega :=$$

Also, for $r := [j, i]$,

$$\delta(\Theta(\alpha // m, j) \times f; \alpha // m, i) //$$

$$= \sum \delta(\Theta(\alpha // m, j); \alpha // m_{\omega}, i - \omega) \delta(f; \alpha // m, \omega) \quad \langle \omega := [i - r, i] \rangle$$

$$= \sum \delta(\Theta(m // m_j + 1, i - j))$$

$$= \sum \delta(\Theta(m // m_j); \alpha // m, \omega); m_j, i - j) \delta(f; \alpha // m, \omega) \quad \langle \omega := [i - r, i] \rangle$$

i) ~~with j < i~~

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$$\begin{aligned}\delta(\Theta(\alpha // m_{i,j+1}, i-j) \times f; \alpha // m, i) &= \delta(f; \alpha // m, j) \\ &= \delta(\Lambda(f; \alpha // m, i / \langle \alpha \rangle); \alpha // m, j)\end{aligned}$$

Also, for $\tau := [i-j, i]$,

$$\begin{aligned}\delta(\Theta(\alpha // m_{i,j+1}, i-j) \times f; \alpha // m, i) &\leftarrow \\ &= \sum_i \delta(\Theta(\alpha // m_{i,j+1}, i-j); \alpha // m, \omega) \delta(f; \alpha // m_{i,j}, i-\omega) \quad \langle \omega := L[\tau] \rangle \\ &= \sum_i \delta(\Theta(\alpha // m_{i,j+1}, i-\omega); \alpha // m, j) \delta(f; \alpha // m_{i,j}, i-\omega) \quad \langle \omega := L[\tau] \rangle\end{aligned}$$

Let $m, i \in \mathbb{N}$, $\alpha \in \text{seq}'(K // \geq m_i)$ and $f \in \text{seq}(K // \geq m_i)$. Over K

$$\begin{aligned}\Lambda(f; \alpha // m, i) &= \sum_i \delta(f; \alpha // m, \omega) \pi_i(\alpha // m, \omega) \quad \langle \omega := L[i] \rangle \\ &= \sum_i \delta(f; \alpha // m_{i,j}, i-\omega) \pi_i(\alpha // m_{i,j}, i-\omega) \quad \langle \omega := L[i] \rangle \\ &= \sum_i \delta(f; \alpha // m_{i-\omega}, \omega) \pi_i(\alpha // m_{i-\omega+1}, \omega) \quad \langle \omega := L[\tau] \rangle\end{aligned}$$

With $j \in [i]$

$$\begin{aligned}\Lambda(f; \alpha // m, i) - \Lambda(f; \alpha // m, j) &= \sum_{\omega=0}^{i-j} \delta(f; \alpha // m, j+\omega+1) \pi_i(\alpha // m, j+\omega+1) \quad \langle \omega := L[i-j] \rangle \\ \text{and } \Lambda(f; \alpha // m, i) - \Lambda(f; \alpha // m_{i-j}, j) &= \sum_{\omega=0}^{i-j} \delta(f; \alpha // m_{i-j}, i-\omega) \pi_i(\alpha // m_{i-j}+1, i-\omega) \\ &\quad \langle \omega := L[i-j] \rangle\end{aligned}$$

again over K in both cases

$$\begin{aligned}f_{m_{i,j}} = 0 \quad \omega := L[j] \quad \Lambda(f; \alpha // m, i) &= \sum_{\omega=0}^{i-j} \frac{\prod_{\chi=0}^{j-1} (z - \alpha_{m+\chi})}{\prod_{\chi=0}^{j-1} (\alpha_{m+j+\omega} - \alpha_{m+\chi})} \frac{\prod_{\chi=0}^{i-j} \langle \omega \rangle}{\prod_{\chi=0}^{i-j} (\alpha_{m+j+\omega} - \alpha_{m+j+\chi})} \\ &= \pi_i(\alpha // m, j) \Lambda(f; \alpha // m_{i,j}, i-j) \\ f_{m_{i,j}} = 0 \quad \omega := (j, i] \quad \Lambda(f; \alpha // m, i) &= \pi_i(\alpha // m_{i,j+1}, i-j) \Lambda(f; \alpha // m_{i,j+1}, i-j) \times f // m, j\end{aligned}$$

Let $m, i \in \overline{N}$, $\alpha \in \text{seq}'(K \geq m_i)$, $f \in \text{seq}(K \geq m_i)$ and $j \in [i]$

i) If $f_{m+i} = 0 \langle \omega := [j] \rangle$ then

$$\Lambda(f; \alpha // m, i) = \bar{\pi}(\alpha // m, j) \Lambda(\phi(m, j) \times f // m_{i-j}, i-j) \quad \langle K \rangle$$

ii) If $f_{m+i} = 0 \langle \omega := (j, i] \rangle$ then

$$\Lambda(f; \alpha // m, i) = \bar{\pi}(\alpha // m_{i-j+1}, i-j) \Lambda(\phi(\alpha // m_{i-j+1}, i-j) \times f // m_j) \quad \langle K \rangle$$

Let $m, i \in \overline{N}$, $\alpha \in \text{seq}'(K \geq m_i)$, $f \in \text{seq}(K \geq m_i)$ and $j, k \in [i]$

1] Let $e \in \text{seq}(K \geq m_i)$

i) Let $e_{m+i} = 0 \langle \omega := [j] \rangle$. For $\nu := [j]$

$$\begin{aligned} \Lambda(e \times f; \alpha // m, i) &= \sum \delta(e; \alpha // m, \omega) \pi(\alpha // m, \omega) \Lambda(f; \alpha // m_{i+\omega}, i-\omega) \\ &= \sum \delta(\phi(\alpha // m, \nu) \times e; \alpha // m_{i+\nu}, \nu-\omega) \bar{\pi}(\alpha // m, \omega) \Lambda(f; \alpha // m_{i+\omega}, i-\omega) \\ &= \sum \Lambda(e; \alpha // m, \omega) \pi(\alpha // m_{i+\omega+1}, i-\omega) \delta(f; \alpha // m_{i+\omega}, i-\omega) \\ &= \sum \Lambda(\phi(\alpha // m, \nu) \times e; \alpha // m_{i+\nu}, \nu-\omega) \pi(\nu-\omega; \alpha // m; i+\nu-\omega, \omega) \\ &\quad \delta(f; \alpha // m_{i+\omega}, i-\omega) \end{aligned}$$

summation being over the range $\omega := [\nu, i]$ in all cases.

ii) Let $e_{m+i} = 0 \langle \omega := (j, i] \rangle$. For $\nu := [j, i]$

$$\begin{aligned} \Lambda(e \times f; \alpha // m, i) &= \sum \delta(e; \alpha // m_{i+\nu}, i-\omega) \pi(\alpha // m_{i+\nu+1}, i-\omega) \Lambda(f; \alpha // m, \omega) \\ &= \sum \delta(\phi(\alpha // m_{i+\nu+1}, i-\omega) \times e; \alpha // m_{i+\nu}, \nu-\omega) \bar{\pi}(\alpha // m_{i+\nu+1}, i-\omega) \\ &\quad \Lambda(f; \alpha // m, \omega) \\ &= \sum \Lambda(e; \alpha // m_{i+\nu}, i-\omega) \pi(\alpha // m, \omega) \delta(f; \alpha // m, \omega) \\ &= \sum \Lambda(\phi(\alpha // m_{i+\nu+1}, i-\omega) \times e; \alpha // m_{i+\nu}, \nu-\omega) \\ &\quad \pi(\nu-\omega; \alpha // m; i-\nu+\omega, \omega) \delta(f; \alpha // m, \omega) \end{aligned}$$

summation being over the range $\omega := [\nu]$ in each case.

iii) Let $g_{x \in K} \langle x := [k] \rangle$ exist such that, with $g: K \rightarrow K$ defined by setting

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$$g(z) := \sum_i g_x z^x \langle x := [k] \rangle$$

the members of the ^{sub}sequence $e[m, m+i]$ have the power polynomial representation

$$e_{m+i} = g(e_{m+i}) \quad \langle \omega := [i, i] \rangle$$

For $z := [k, i]$

$$\begin{aligned} \Delta(e \times f; \alpha // m, i) &= \sum \delta(e; \alpha // m, \omega) \pi(\alpha // m, \omega) \Delta(f; \alpha // m+i, i-\omega) \langle \omega := [i, i] \rangle, \\ &= \left\{ \sum \Delta(e; \alpha // m, \omega) \delta(f; \alpha // m+i, i-\omega) \pi(\alpha // m+i, i-\omega) \langle \omega := [i, i] \rangle \right\} \\ &\quad + g \Delta(f; \alpha // m+i, i-\omega) \\ &= \sum \delta(e; \alpha // m+i, i-\omega) \delta(f; \alpha // m, \omega) \pi(\alpha // m, \omega) \langle \omega := [i-\omega, i] \rangle \\ &= \sum \delta(e; \alpha // m+i, i-\omega) \pi(\alpha // m+i, i-\omega) \Delta(f; \alpha // m, \omega) \langle \omega := [i-\omega, i] \rangle \\ &= \left\{ \sum \Delta(e; \alpha // m+i, i-\omega) \delta(f; \alpha // m, \omega) \pi(\alpha // m, \omega) \langle \omega := [i-\omega, i] \rangle \right\} \\ &\quad + g \Delta(f; \alpha // m, i-\omega) \end{aligned}$$

iv) Let the conditions upon e of (i, iii) hold in conjunction. For $\nu := [j, j]$, $z := [k, i]$

$$\begin{aligned} \Delta(e \times f; \alpha // m, i) &= \sum \delta(e; \alpha // m, \omega) \pi(\alpha // m, \omega) \Delta(f; \alpha // m+i, i-\omega) \cancel{\langle \omega := [\nu, \nu] \rangle} \\ &= \sum \delta(\phi(\alpha // m, \omega) \times e; \alpha // m+\nu, \omega-\nu) \Delta(f; \alpha // m+i, i-\omega) \cancel{\langle \omega := [\nu, \nu] \rangle} \\ &= \left\{ \sum \Delta(e; \alpha // m, \omega) \delta(f; \alpha // m+i, i-\omega) \pi(\alpha // m+i, i-\omega) \cancel{\langle \omega := [\nu, \nu] \rangle} \right\} \\ &\quad + g \Delta(f; \alpha // m+i, i-\omega) \\ &= \left\{ \sum \Delta(\phi(\alpha // m, \omega) \times e; \alpha // m+\nu, \omega-\nu) \pi(\omega-\nu; \alpha // m, i+\nu-\omega, \nu) \right. \\ &\quad \left. \delta(f; \alpha // m+i, i-\omega) \right\} + g \Delta(f; \alpha // m+i, i-\omega) \end{aligned}$$

summation being over the range $\omega := [\nu, \nu]$ in each case.

v) Let the conditions upon e of (ii, iii) hold in conjunction. For $\omega := [j, i]$ and $z := [k, i]$

$$\begin{aligned}
 \Lambda(e \times f; \alpha // m, i) &= \sum_i \delta(e; \alpha // m + \omega, i - \omega) \pi(\alpha // m + \omega, i - \omega) \Lambda(f; \alpha // m, \omega) \quad \langle \omega := [i - z, z] \rangle \\
 &= \sum_i \delta(\phi(\alpha // m + \omega, i - \omega) \times e; \alpha // m + \omega, i - \omega) \pi(\alpha // m + \omega, i - \omega) \Lambda(f; \alpha // m, \omega) \\
 &\quad \langle \omega := [i - z, z] \rangle \\
 &= \sum_i \Lambda(e; \alpha // m + \omega, i - \omega) \delta(f; \alpha // m, \omega) \pi(\alpha // m, \omega) \\
 &\quad + g \Lambda(f; \alpha // m, i - z) \\
 &= \sum_i \Lambda(\phi(\alpha // m + \omega, i - \omega) \times e; \alpha // m + \omega, i - \omega) \pi(i - \omega; \alpha // m; i - z + \omega, \omega) \\
 &\quad \delta(f; \alpha // m, \omega) + g \Lambda(f; \alpha // m, i - z)
 \end{aligned}$$

summation being over the range $\omega := [i - z, z]$ in each case

$$\begin{aligned}
 2c) \quad \Lambda(\Theta(\alpha // m, j) \times f; \alpha // m, i) &= \pi(\alpha // m, j) \Lambda(f; \alpha // m + j, i - j) \\
 &= \pi(\alpha // m, j) \Lambda(\Lambda(f; \alpha // m, i / \langle \alpha \rangle); \alpha // m + j, i - j)
 \end{aligned}$$

Also, for $z := [j, i]$,

$$\begin{aligned}
 \Lambda(\Theta(\alpha // m, j) \times f; \alpha // m, i) &= \\
 &\sum_i \delta(\Theta(\alpha // m, j); \alpha // m + \omega, i - \omega) \pi(\alpha // m + \omega, i - \omega) \Lambda(f; \alpha // m, \omega) \quad \langle \omega := [i - z, z] \rangle \\
 &= \sum_i \Lambda(\Theta(\alpha // m, j); \alpha // m + \omega, i - \omega) \delta(f; \alpha // m, \omega) \pi(\alpha // m, \omega) \quad \langle \omega := [i - z, z] \rangle \\
 &\quad + \pi(\alpha // m, j) \Lambda(f; \alpha // m, i - z) \\
 &= \sum_i \delta(\Theta(\alpha // m, \omega); \alpha // m + j, i - j) \pi(\alpha // m + \omega, i - \omega) \Lambda(f; \alpha // m, \omega) \quad \langle \omega := [i - z, z] \rangle \\
 &= \pi(\alpha // m, j) \left\{ \sum_i \Lambda(\Theta(\alpha // m, \omega); \alpha // m + j, i - j) \delta(f; \alpha // m, \omega) \quad \langle \omega := [i - z, z] \rangle \right\} \\
 &\quad + \Lambda(f; \alpha // m, i - z)
 \end{aligned}$$

$$ii) \quad \Delta(\Theta(\alpha // m_{ij+1}, i-j) \times f; \alpha // m, i) = \overline{\pi}(\alpha // m_{ij+1}, i-j) \Delta(f; \alpha // m, j) \quad 17^e$$

$$= \pi(\alpha // m_{ij+1}, i-j) \Delta(\Delta(f; \alpha // m, i // \alpha); \alpha // m, j)$$

Also, for $\tau := [i-j, i]$,

$$\Delta(\Theta(\alpha // m_{ij+1}, i-j) \times f; \alpha // m, i) \leftarrow$$

$$= \sum \delta(\Theta(\alpha // m_{ij+1}, i-j); \alpha // m, \omega) \delta(f; \alpha // m_{\omega}, i - \omega) \pi(\alpha // m_{\omega+1}, i - \omega) \langle \omega := [\tau] \rangle$$

$$= \sum \Delta(\Theta(\alpha // m_{ij+1}, i-j), \alpha // m, \omega) \leftarrow$$

$$= \sum \delta(\Theta(\alpha // m_{ij+1}, i-j); \alpha // m, \omega) \pi(\alpha // m, \omega) \Delta(f; \alpha // m_{\omega}, i - \omega) \langle \omega := [\tau] \rangle$$

$$= \sum \Delta(\Theta(\alpha // m_{ij+1}, i-j); \alpha // m, \omega) \delta(f; \alpha // m_{\omega}, i - \omega) \pi(\alpha // m_{\omega+1}, i - \omega) \langle \omega := [\tau] \rangle$$

$$+ \pi(\alpha // m_{ij+1}, i-j) \Delta(f; \alpha // m + z, i - \tau)$$

$$\therefore = \sum \delta(\Theta(\alpha // m_{ij+1}, i-j); \alpha // m_{\omega}, i - \omega) \pi(\alpha // m_{\omega+1}, i - \omega) \Delta(f; \alpha // m, \omega) \langle \omega := [i - \tau, i] \rangle$$

$$\therefore = \sum \Delta(\Theta(\alpha // m_{ij+1}, i-j), \alpha // m_{\omega}, i - \omega) \delta(f; \alpha // m, \omega) \pi(\alpha // m, \omega) \langle \omega := [i - \tau, i] \rangle$$

$$+ \pi(m_{ij+1}, i-j) \Delta(f; \alpha // m, i - \tau) \leftarrow$$

$$= \sum \delta(\Theta(\alpha // m_{\omega+1}, i - \omega); \alpha // m, j) \pi(\alpha // m, \omega) \Delta(f; \alpha // m_{\omega}, i - \omega) \langle \omega := [\tau] \rangle$$

$$= \pi(\alpha // m_{ij+1}, i-j) \{ \{ \sum \Delta(\Theta(\alpha // m_{\omega+1}, i - \omega); \alpha // m, j) \delta(f; \alpha // m_{\omega}, i - \omega) \langle \omega := [\tau] \rangle$$

$$+ \Delta(f; \alpha // m + z, i - \tau) \} \}$$

$$\Pi \pi(\alpha || m+j, k-j) \wedge (\Theta(\alpha || m, j); \alpha || m+k, r) = \Pi(\alpha || m+k, j-k) \wedge (\Theta(\alpha || m, k); \alpha || m+j, r+k-j)$$

check $r \geq j$

$$\Pi(\alpha || m+j, k-j) \pi(\alpha || m, j) = \Pi(\alpha || m+k, j-k) \pi(\alpha || m, k)$$

$$\Pi(\alpha || m, k) =$$

$$k \geq j \quad \Pi(\alpha || m, k) = \Pi(\alpha || m, k) \quad j \geq k \quad \Pi(\alpha || m, j) = \Pi(\alpha || m, j)$$

$$\sum_{\substack{k \\ k=j+r \\ r \geq j}} \frac{\prod_{x=j+1}^k (\alpha_{m+n} - \alpha_{m+x})}{\prod_{x=0}^k (\alpha_{m+n} - \alpha_{m+x})} \quad \langle n := [k] \rangle \quad \omega \in (j, r+j] \quad \text{term} = 0$$

$$k > j \quad \langle n := [j] \rangle \quad \text{demon} := \prod_{x=0}^j (\dots) \prod_{x=j+1}^k (\alpha_{m+n} - \alpha_{m+x})$$

$$\text{num} \Rightarrow \prod_{x=k+1}^{r+j} (\alpha_{m+n} - \alpha_{m+x}) \rightarrow S(\Theta(\alpha || m+k+1, r+j-k); \alpha || m, j)$$

$$k \leq j \quad \prod_{x=k+1}^j (\alpha_{m+n} - \alpha_{m+x}) \quad \text{above + below : zero for } \omega \in [k, j]$$

$$= \prod_{x=j+1}^{r+j} (\alpha_{m+n} - \alpha_{m+x}) \prod_{x=0}^k (\omega - \alpha_{m+x})$$

$$k > j \quad \langle n := [j] \rangle \quad \text{num num} \Rightarrow \prod_{x=k+1}^{r+j} (\alpha_{m+n} - \alpha_{m+x}) \prod_{x=0}^j (\omega - \alpha_{m+x}) \prod_{x=j+1}^k (\omega - \alpha_{m+x})$$

$$\underline{\Lambda}(\Theta(m+j+1, r); \alpha || m, k) = \Pi(\alpha || m+j+1, k-j) \underline{\Lambda}(\Theta(m+k+1, j+r-k); \alpha || m, j)$$

$$k \leq j \quad \prod_{x=k+1}^j (\alpha_{m+n} - \alpha_{m+x}) \prod_{x=k+1}^j (\omega - \alpha_{m+x}) \quad \text{above + below}$$

$$\Pi(\alpha || m+k+1, j+r-k) \underline{\Lambda}(\Theta(m+j+1, r); \alpha || m, k) =$$

$$\Pi(\alpha || m+j+1, k-j) \underline{\Lambda}(\Theta(\alpha || m+k+1, j+r-k); \alpha || m, j)$$

$$\int_{[m, m+i]} f(z) dz = \int_m^m f(z) dz + \int_{m+i}^{m+1} f(z) dz$$

$$\pi(\beta \| n, i | z) \delta(f; \alpha \| m, i) = \int_z^m f(z) - \Delta(\beta \| n, i-1 | z)$$

$$\lambda(\alpha \| m; i, \omega | z') = \underline{\pi(\beta \| n, i | z')}$$

$$\begin{aligned} \lambda(\alpha \| m; i, \omega | z') &= \frac{z' - z}{\beta_{n+m-1} - z} \cdot \prod_{x=0}^{i-1} \frac{(z' - \beta_{n+x})}{(\beta_{n+m-1} - \beta_{n+x})} \\ &= \frac{z' - z}{\beta_{n+m-1} - z} \cdot \lambda(\beta \| n; i-1, \omega-1 | z') \end{aligned}$$

$$\Delta(\alpha \| m; i | z') = \frac{\int_z^m \pi(\beta \| n, i | z') dz}{\pi(\beta \| n, i | z)} - \sum_{w=0}^{i-1} (z' - z) \sum_{n=0}^{i-1} \frac{\lambda(\beta \| n; i-1, \omega | z')}{z - \beta_{n+m}} f_{n+m}$$

$$\frac{\Delta(\alpha \| m; i | z')}{\pi(\beta \| n, i | z')} - \frac{\int_z^m dz}{\pi(\beta \| n, i | z)} = - \frac{\Delta(\frac{f}{z - \langle \beta \rangle}; \beta \| n; i-1 | z')}{\pi(\beta \| n, i | z')} \frac{z' - z}{z + z}$$

$z, z' \notin \beta[n, m]$

Let $m, n \in \mathbb{N}$, $i \in \mathbb{N}$, $\beta \in \text{seq}'(|K| \geq n+i-1)$, $z \in K \setminus \beta[n, m]$ and $\alpha \in \text{seq}'(|K| \geq m+i-1)$ be such that $\alpha[m, m+i] = z + \beta[n, m+i]$

$$\begin{aligned} \text{i)} \quad \mu(\alpha \| m; i, \omega) &= \frac{1}{\pi(\beta \| n, i | z)} \quad \langle \omega = 0 \rangle \\ &\quad \frac{\mu(\beta \| n; i-1, \omega-1)}{\beta_{n+m-1} - z} \quad \langle \omega := (i) \rangle \end{aligned}$$

$$\text{ii) With } z' \in K \quad \frac{\pi(\beta \| n, i | z')}{\pi(\beta \| n, i | z)} \quad \langle \omega = 0 \rangle$$

$$\lambda(\alpha \| m; i, \omega | z') = \frac{(z' - z) \lambda(\beta \| n; i-1, \omega-1 | z')}{\beta_{n+m-1} - z} \quad \langle \omega := (i) \rangle$$

2] Let $f(z) \in K$, $f \in \text{seq}(K | \Delta_{n,i-1})$ and $e \in \text{seq}(K | \Delta_{n,i})$ be such that $\frac{\partial}{\partial z} e[m, m_i] = f(z) + \beta[n, n_i]$

$$\frac{\partial}{\partial z} e[m, m_i] = f(z) + \beta[n, n_i]$$

i) $\int(z) - \Lambda(f; \beta \| n, i-1 | z) = S(\frac{\partial}{\partial z} e; \alpha \| m, i) \pi(\beta \| n, i | z)$

ii) With $z' \in K$ such that $z' \in K \setminus \{z + \beta[n, n_i]\}$,

$$\frac{\frac{\Lambda(\frac{\partial}{\partial z} e; \alpha \| m, i | z')}{\pi(\beta \| n, i | z')}}{z' - z} - \frac{f(z)}{\pi(\beta \| n, i | z)} = - \frac{\Lambda(\frac{f}{z-z'}; \beta \| n, i-1 | z')}{\pi(\beta \| n, i | z')}$$

(*)] Let $j, k, m, r \in \overline{\mathbb{N}}$ and $\alpha \in \text{seq}'(K | \Delta_{m+\max(j-1, k), r})$

i) $S(\Theta(\alpha \| m, j); \alpha \| m+k, r) = S(\Theta(m, k); \alpha \| m+j, r+k-j)$

ii) $\pi(\alpha \| m+j, k-j) \Delta (\Theta(\alpha \| m, j); \alpha \| m+k, r) =$
 $\pi(\alpha \| m+k, j-k) \Delta (\Theta(\alpha \| m, k); \alpha \| m+j, r+k-j) \quad \langle K \rangle$

2] Let $j, m, r \in \overline{\mathbb{N}}$, $k \in [j+r]$ and $\alpha \in \text{seq}'(K | \Delta_{m+j+r})$

i) $S(\Theta(\alpha \| m+j+1, r); \alpha \| m, k) = S(\Theta(\alpha \| m+k+1, j+r-k); \alpha \| m, j)$

ii) $\pi(\alpha \| m+k+1, j-k) \Delta (\Theta(\alpha \| m+j+1, r); \alpha \| m, k) =$
 $\pi(\alpha \| m+j+1, k-j) \Delta (\Theta(\alpha \| m+k+1, j+r-k); \alpha \| m, j) \quad \langle K \rangle$

(If $\epsilon(n, r)$ is defined for $r < 0$ to be a zero sequence, the condition $k \in [j+r]$ in [2] above is unnecessary)

() Let $K', K'' \subseteq K$, $g: K' \times K'' \rightarrow K$, $j, k, m, n \in \bar{\mathbb{N}}$, $\alpha \in \text{seq}'(K | \geq n+k)$ 18.

and $\beta \in \text{seq}'(K | \geq n+k)$

$$i) \quad \delta(\delta(g(\langle\langle\alpha\rangle\rangle, \langle\beta\rangle); \alpha || m, j); \beta || n, k) =$$

$$\delta(\delta(g(\langle\alpha\rangle, \langle\langle\beta\rangle\rangle); \beta || n, k); \alpha || m, j)$$

$$ii) \quad \Delta(\delta(g(\langle\langle\alpha\rangle\rangle, \langle\beta\rangle); \alpha || m, j); \beta || n, k) =$$

$$\delta(\Delta(g(\langle\alpha\rangle, \langle\langle\beta\rangle\rangle); \beta || n, k); \alpha || m, j)$$

$$iii) \quad \Delta(\Delta(g(\langle\langle\alpha\rangle\rangle, \langle\beta\rangle); \alpha || m, j); \beta || n, k) =$$

$$\Delta(\Delta(g(\langle\alpha\rangle, \langle\langle\beta\rangle\rangle); \beta || n, k); \alpha || m, j)$$

() Let $j, k, m, n \in \bar{\mathbb{N}}$ with $m+j < n$ and $\alpha \in \text{seq}'(K | \geq n+k)$

$$i) \quad \delta(\phi(\alpha || m, j+1); \alpha || n, k) = -\delta(\phi(\alpha || n, k+1); \alpha || m, j)$$

$$ii) \quad \pi(\alpha || m, j+1) \Delta(\phi(\alpha || m, j+1); \alpha || n, k)$$

$$+ \pi(\alpha || n, k+1) \Delta(\phi(\alpha || n, k+1); \alpha || m, j) = 1 \quad \langle K \rangle$$

— Let $m, i \in \bar{\mathbb{N}}$, $\alpha \in \text{seq}'(K | \geq m+i)$ and $z \in K \setminus \alpha[m, m+i]$.

$$i) \quad \frac{\pi(\alpha || m, i+1 | z') - \pi(\alpha || m, i+1 | z)}{z' - z} = \pi(\alpha || m, i+1 | z) \Delta\left(\frac{1}{z' - z}; \alpha || m, i | z\right)$$

ii) Let $f \in \text{seq}(K | \geq m+i)$.

$$\frac{\pi(\alpha || m, i+1 | z') \Delta(f; \alpha || m, i | z) - \pi(\alpha || m, i+1 | z) \Delta(f; \alpha || m, i | z')}{z' - z}$$

$$= \pi(\alpha || m, i+1 | z') \Delta\left(\frac{1}{z' - z} \times f; \alpha || m, i | z\right)$$

() Let $i, m, n \in K$ s.t. $\alpha \in \text{seq}'(K | \geq m_i)$, $\beta \in \text{seq}'(K | \geq$

Let $\beta \in \text{seq}'(K | \geq m_i)$ be such that $\{\beta[n, n_i]\} \subset K \setminus \{\alpha[n, n_i]\}$ and set $z := \beta[n] \cdot \beta_n$. Let $f \in \text{seq}(K | \geq m_i)$.

$$\delta \left(\frac{\Lambda(f; \alpha || m, i | \langle \beta \rangle)}{\pi(\alpha || m, i+1 | \langle \beta \rangle)} ; \beta || n, k \right) =$$

$$\frac{1}{\pi(\alpha || m, i+1 | z)} \Lambda(\phi(\beta, \langle \alpha \rangle || m+1, k) \times f; \alpha || m, i | z)$$

$$k' = \omega, r' = i - \omega \quad \omega \geq i - z \quad z > j \quad \omega \geq i - j \quad i - \omega \leq j$$

$$\frac{\pi(\alpha || m, \omega, j - \omega)}{\pi(\alpha || m, j, \omega - j)} \Lambda(\theta(\alpha || m, \omega); \alpha || m, j, i - j) \delta(f; \alpha || m, \omega) \pi(\alpha || m, \omega)$$

$$\omega \leq j : \pi(\alpha || m, j) \quad \omega \geq j : \pi(\alpha || m, j)$$

$$\pi(\alpha || m, j) \left\{ \sum \Lambda(\theta(\alpha || m, \omega); \alpha || m, j, i - j) \delta(f; \alpha || m, \omega) \langle \omega : = [i - z, z] \rangle \right. \\ \left. + \Lambda(f; \alpha || m, i - z) \right\} \quad \text{Pfz 2i}$$

$$r' = i - j \quad k' = \omega \quad \omega \in [i] \quad \delta(\theta(m, j+1, i - j); \alpha || m, \omega) = \delta(\theta(m, \omega+1, i - \omega); \alpha || m, j)$$

$$\sum \delta(\theta(\alpha || m, \omega+1, i - \omega); \alpha || m, j) \pi(\alpha || m, \omega) \Lambda(f; \alpha || m, \omega, i - \omega) \langle \omega : = [z] \rangle$$

$$\frac{\pi(\alpha || m, j+1, \omega - j)}{\pi(\alpha || m, \omega+1, j - \omega)} \Lambda(\theta(\alpha || m, \omega+1, i - \omega); \alpha || m, j) \pi(\alpha || m, \omega+1, i - \omega) \delta(f; \alpha || m, \omega, i - \omega)$$

$$\omega \leq j : \pi(\alpha || m, j+1, i - j) \quad \omega \geq j$$

$$\pi(\alpha || m, j+1, i - j) \left\{ \sum \Lambda(\theta(\alpha || m, \omega+1, i - \omega); \alpha || m, j) \delta(f; \alpha || m, \omega, i - \omega) \langle \omega : = [z] \rangle \right. \\ \left. + \Lambda(f; \alpha || m, z, i - z) \right\}$$

$$\hat{\Lambda}^{-1} \Lambda : [\Lambda\left(\frac{1}{\lambda(\alpha||m; i)}, \omega||m+2, z-2\right)] \text{II}(i, \zeta^{-1} \langle K \setminus \alpha[m, i] \rangle) \quad 198.$$

$$\begin{aligned}\lambda(\alpha||m; i, \omega) &= \sum \mu(\alpha||m; \omega, \omega) \pi(\alpha||m, \omega) \quad \langle \omega := [z, \zeta] \rangle \\ &= \sum \mu(\alpha||m+\omega; i-\omega, 2-\omega) \pi(\alpha||m+1, z-\omega) \quad \langle \omega := [z] \rangle\end{aligned}$$

$$\begin{aligned}\mu(\alpha||m; \omega, \omega) &= \delta(\lambda(\alpha||m; i, \omega) | \langle \alpha \rangle); \alpha||m, z) \\ z < 2 \quad \lambda(\alpha||m; i, \omega) | \alpha_{m+\omega} &= 0 \quad \langle \omega := [z] \rangle\end{aligned}$$

$$\mu(\alpha||m+2; i-2, z-2) = \delta(\lambda(\alpha||m; i, z) | \langle \alpha \rangle); \alpha||m+2, z-2)$$

Def. Set $\bar{N}' := \overline{N}_{M, M}$ Remark 199.

i) The mappings $\text{seq}'(K \cong \bar{N} + \bar{N}') \times \bar{N} \times \bar{N} \rightarrow \{K|\bar{N}\}$ are defined by setting

$$M[\alpha||m, [i]] := L[\mu(\alpha||m; z, \omega)] \quad \langle \omega := [z] \rangle$$

and

$$M[\alpha||m, [i]] := L[\mu(\alpha||m+2; i-2, z-2)] \quad \langle \omega := [z] \rangle$$

ii) The mappings $\Lambda, \tilde{\Lambda}, \text{seq}, \text{seq}' : \text{seq}'(K \cong \bar{N} + \bar{N}') \times \bar{N} \times \bar{N} \rightarrow \{K \rightarrow L[K|\bar{N}]\}$ are defined by setting

$$\Lambda[\alpha||m, [i]] := L[\lambda(\alpha||m; z, \omega)]$$

$$\tilde{\Lambda}[\alpha||m, [i]] := L[\lambda(\alpha||m+2; i-2, z-2) - \lambda(\alpha||m+1; i-2-1, z-2-1)]$$

$$\cancel{\Lambda}[\alpha||m, [i]] := \cancel{L}[\Lambda(f; \alpha||m+2, z-2) - \Lambda(\alpha||$$

iii) The mapping $A : \text{seq}(K \cong \bar{N} + \bar{N}') \times \text{seq}'(K \cong \bar{N} + \bar{N}') \times \bar{N} \times \bar{N} \rightarrow L[K|\bar{N}]$ is defined by setting

$$A[f; \alpha||m, [i]] := L[\delta(f; \alpha||m+2, z-2)]$$

iv) The mappings $\tilde{\Lambda}, \hat{\Lambda} : \text{seq}(|K| \geq \bar{N} + N') \times \text{seq}'(|K| \geq N + N') \times N' \times N \rightarrow ^B$

$\{K \rightarrow L[K | \bar{N}]\}$ are defined by setting

$$\tilde{\Lambda}[f; \alpha || m, [i]] := L[\Lambda(f; \alpha || m_{\leq i}, z - i) - \Lambda(f; \alpha || m_{\leq i+1}, z - i - 1)]$$

and

$$\hat{\Lambda}[f; \alpha || m, [i]] := L[\Lambda(f; \alpha || m_{\leq i}, z - i)]$$

v) $\mu(\alpha || m; i)$ and $\pi(\alpha || m; i)$ are the sequences $\mu(\alpha || m; i, w) \langle w := [i] \rangle$ and $\pi(\alpha || m; i, w) \langle w := [i] \rangle$

definition

with i as prescribed,

In the following all matrices and function/matrix functions are, in $[K | i]$ and $\{K \rightarrow [K | i]\}$. The row and column indexes are consistently z and ω respectively; ~~and the~~ index ranges are omitted from allocations. Thus the first allocation of (i) below is to be understood as

$$M[\alpha || m, [i]] := L[\mu(\alpha || m; z_{\leq i})] \langle z := [i], \omega := [z] \rangle$$

and so on.

() Let $m, i \in \bar{N}$ and $\alpha \in \text{seq}'(|K| \geq m + i)$.

$|[m, m+i]$?

$$i) M[\alpha || m, [i]] M[\alpha || m, [i]] = d[\mu(\alpha || m; i)]$$

$$M[\alpha || m, [i]] M[\alpha || m, [i]] = [S(\mu(\alpha || m; i); \alpha || m_{\leq i}, z - i)]$$

$$M[\alpha || m, [i]]^{-1} = L[\pi(\alpha || m, z | \alpha_{m+i})]$$

$$= L[\Theta(\alpha || m, z | m + i)]$$

$$M[\alpha || m, [i]]^{-1} = L[\pi(\alpha || m + i, i - z | \alpha_{m+i})]$$

$$= L[\Theta(\alpha || m + i, i - z | m + i)]$$

$$\{M[\alpha||m, [i]]\}^{-1} \{M[\alpha||m, [i]]\}^{-1} = d\left[\frac{1}{\mu_{\alpha||m; i}}\right]$$

$$\{\tilde{M}[\alpha||m, [i]]\}^{-1} \{M[\alpha||m, [i]]\}^{-1} = S\left(\frac{1}{\mu_{\alpha||m; i}}; \alpha||m \bar{\alpha}, z-\nu\right)$$

Now K in each case

$$\Lambda[\alpha||m, [i]]^{-1} = L\left[\frac{\pi(\alpha||m+2, i)}{\pi(\alpha||m, i)}\right] \quad \text{when } z=\nu$$

$$\begin{aligned} ii) \quad \Lambda[\alpha||m, [i]]^{-1} &= L\left[\frac{\pi(\alpha||m, 2|\alpha_{m+2}) - \pi(\alpha||m, 2+1|\alpha_{m+2})}{\pi(\alpha||m, 2)}\right] \\ &= L\left[\frac{\pi(\alpha||m, 2|\alpha_{m+2})}{\pi(\alpha||m, 2)}\right] L[i]^{-1} \langle K \setminus \alpha[m, i] \rangle \end{aligned}$$

$$\tilde{\Lambda}[\alpha||m, [i]]^{-1} = L\left[\frac{\pi(\alpha||m+2+1, i-\nu|\alpha_{m+2})}{\pi(\alpha||m+2, i-\nu)}\right] \langle K \setminus \alpha[m, i] \rangle$$

$$\tilde{\Lambda}[\alpha||m, [i]] \Lambda[\alpha||m, [i]] = d[\lambda(\alpha||m; i|[m, m+i])] \langle K \rangle$$

$$\Lambda[\alpha||m, [i]] \tilde{\Lambda}[\alpha||m, [i]] = [\lambda(\lambda(\alpha||m; i); \alpha||m+2, z-\nu)] L[i]^{-1} \langle K \rangle$$

$$\{\Lambda[\alpha||m, [i]]\}^{-1} \{\tilde{\Lambda}[\alpha||m, [i]]\}^{-1} = d\left[\frac{1}{\lambda(\alpha||m; i|[m, m+i])}\right] \langle K \setminus \alpha[m, m+i] \rangle$$

$$\{\tilde{\Lambda}[\alpha||m, [i]]\}^{-1} \{\Lambda[\alpha||m, [i]]\}^{-1} =$$

$$\left[\lambda\left(\frac{1}{\lambda(\alpha||m; i)}; \alpha||m+2, z-\nu\right) \right] L[i]^{-1} \langle K \setminus \alpha[m, m+i] \rangle$$

def $\lambda(f.. \text{ in terms of } f)$
 $f \in \{K \rightarrow \text{seq}(K)\}$

iii) ~~Def~~ Let

() Let $m, i \in \mathbb{N}$, $\alpha \in \text{seq}'(K \setminus m, i)$ and $f \in \text{seq}(K \setminus m, i)$

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i) $A[f; \alpha \parallel m, [i]]$ is expressible directly in terms of $f[m, m_i]$ by means of the formulae

$$A[f; \alpha \parallel m, [i]] = M[\alpha \parallel m, [i]] d[f[m, m_i]] \{M[\alpha \parallel m, [i]]\}^{-1} \\ = \{\bar{M}[\alpha \parallel m, [i]]\}^{-1} d[f[m, m_i]] \bar{M}[\alpha \parallel m, [i]]$$

~~ii)~~ $\hat{A}[f; \alpha \parallel m, [i]] = L[\Lambda(f; \alpha \parallel m_{i+2}, z-2)] L[i]^{-1} \langle K \rangle$

b) $\hat{A}[f; \alpha \parallel m, [i]]$ is expressible directly in terms of $f[m, m_i]$ by means of the formulae

$$\hat{A}[f; \alpha \parallel m, [i]] = \Lambda[\alpha \parallel m, [i]] d[f[m, m_i]] \{\Lambda[\alpha \parallel m, [i]]\}^{-1} \langle K \setminus \alpha[m, i] \rangle \\ = \{\Lambda[\alpha \parallel m, [i]]\}^{-1} d[f[m, m_i]] \Lambda[\alpha \parallel m, [i]] \langle K \setminus \alpha[m, m_i] \rangle \\ = L[\delta(f; \alpha \parallel m_{i+2}, z-2) \pi(\alpha \parallel m_{i+2}, z-2)]$$

~~iii)~~ $\hat{A}[f; \alpha \parallel m, [i]]$ is expressible directly in terms of $f[m, m_i]$ by means of the formulae

$$\hat{A}[f; \alpha \parallel m, [i]] = \Lambda[\alpha \parallel m, [i]] d[f[m, m_i]] L\left[\frac{\pi(\alpha \parallel m, z \mid \alpha_{m+2})}{\pi(\alpha \parallel m, z)}\right] \langle K \setminus \alpha[m, i] \rangle \\ = \{\Lambda[\alpha \parallel m, [i]]\}^{-1} d[f[m, m_i]] L[\lambda(\alpha \parallel m_{i+2}; i-2, z-2)]$$

~~iv)~~ Let

$$\pi(\alpha \parallel m, z) \pi(\alpha \parallel m_{i+2}, z-1, z-2) \underbrace{\pi(\alpha \parallel m, z+1)}_{\pi(\alpha \parallel m+2, z+1)} \left| \begin{array}{c} \text{diag}[\pi(\alpha \parallel m+2, i-2, z)] \\ M[\dots] d[\overbrace{1}^z \overbrace{i(\alpha \parallel m+2, 1)}^{z-1}] L[i]^{-1} \end{array} \right.$$

$$\Lambda[\dots] = \text{diag}[\pi(\alpha \parallel m, z+1)] M[\dots] \text{diag}[\overbrace{1}^z \overbrace{\pi(\alpha \parallel m+2, z+1)}^{z-1}]^{-1}$$

$$\pi(\alpha \parallel m_{i+2}; i-2, z-2) = \pi(\alpha \parallel m_{i+2}, i-2+1) \\ z - \alpha_{m+2}$$

$$\Lambda_{\alpha}[\dots] = \text{diag} [\pi(\alpha || m, \omega_H)] M[\dots] \text{diag} [\pi(\alpha || m \omega, 1)]^{-1}$$

$$\tilde{\Lambda}_{\alpha}[\dots] = \text{diag} [\pi(\alpha || m \omega, 1)]^{-1} \tilde{M}[\dots] \text{diag} [\pi(\alpha || m \omega, \omega - \omega_H)] L[i]^{-1}$$

 $\tau-1, z$ τ, z

$$-\pi(\alpha || m + z, \tau - z + 1 | z) - \pi(\alpha || m + z, \tau - z + 1 | z)$$

$$-\pi(\alpha || m + z, \tau - z + 1 | z) \pi(\alpha || m \omega, \tau | z) - \pi(\alpha || m + z, \tau - z + 1 | z) \pi(\alpha || m, \tau + 1 | z)$$

$$\lambda(\alpha || m \omega; \tau - \omega, \tau - \omega) - \lambda(\alpha || m \omega + 1; \tau - \omega - 1, \tau - \omega - 1)$$

$$\frac{\prod_{\chi=0}^i (z - \alpha_{m+\chi})}{\prod_{\chi=0}^i (\alpha_{m+z} - \alpha_{m+\chi})} - \frac{\prod_{\chi=0+1}^i (z - \alpha_{m+\chi})}{\prod_{\chi=0+1}^i (\alpha_{m+z} - \alpha_{m+\chi})} \Rightarrow 1 - \frac{\alpha_{m+z} - \alpha_{m+\omega}}{z - \alpha_{m+\omega}}$$

$$= \frac{z - \alpha_{m+z}}{z - \alpha_{m+\omega}}$$

$$\tilde{\Lambda}_{\alpha || m, [i]} = \text{diag} [\pi(\alpha || m + \omega | 1)] L[\lambda(\alpha || m \omega, \tau - \omega, \tau - \omega)] \text{diag} [\pi(\alpha || m \omega | 1)]$$

$$= \tilde{M}[\dots] \text{diag} [\pi(\alpha || m + \omega + 1, \tau - \omega)]$$

$$\tilde{\Lambda}_{\alpha}[\dots] \Lambda_{\alpha}[\dots] = \text{diag} [\pi(\alpha || m, i_H)] \text{diag} [\mu(\alpha || m; i)] \text{diag} [\pi(\alpha || m \omega, 1)]^{-1}$$

$$= d[\lambda(\alpha || m, i)]$$

() Let $i, m \in \mathbb{N}$ and $\alpha \in \text{seq}'(K | \geq m + i)$

$$1] \quad \Lambda_{\alpha || m, [i]} = \text{diag} [\pi(\alpha || m, \omega_H)] M_{\alpha || m, [i]} \text{diag} [\pi(\alpha || m \omega, 1)]^{-1} \quad \langle K \setminus \alpha |_{m, m+i} \rangle$$

and

$$\tilde{\Lambda}_{\alpha || m, [i]} = \tilde{M}_{\alpha || m, [i]} \text{diag} [\pi(\alpha || m \omega + 1, \tau - \omega)] \quad \langle K \rangle$$

2] Let $f \in \text{seq}(K | \geq m + i)$

$$i) \quad \tilde{\Lambda}_{\alpha || m, i} = \text{diag} [\pi(\alpha || m \omega + 1)] A[f; \alpha || m, i] \text{diag} [\pi(\alpha || m, \omega_H)]^{-1} \quad \langle K \setminus \alpha |_{m, m+i} \rangle$$

$$ii) \quad \tilde{A}[f; \alpha || m, [i]] = \tilde{A}[f; \alpha || m, [i]] L[i]$$

$$= \text{diag}[\pi(\alpha || m, \omega_i)] \# A[f; \alpha || m, [i]] L\left[\frac{1}{\pi(\alpha || m, \omega_i)}\right]$$

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() Let $m, i \in \mathbb{N}$ and $\alpha \in \text{seq}'(K | \geq m_i)$.

1] The complete system \mathcal{B} of all subsequences of the form $f[m, m_i]$ extracted from the sequences f in $\text{seq}(K | \geq m_i)$ form a commutative ring \mathbb{F} , which may be denoted by F . If

2i] The matrices $A[f; \alpha || m, [i]] \in [K | i]$ derived from the possesses a unit element $1, 1, \dots, 1$. A subsequence $f[m, m_i]$ is an invertible member of \mathbb{F} if and only if it is devoid of zero components

2ii] The matrices $A[f; \alpha || m, [i]] \in [K | i]$ derived from the subsequences $f[m, m_i]$ of \mathbb{F} form a ring \mathbb{F}' which is isomorphic to \mathbb{F}

ii) In particular, with $e, f \in \text{seq}(K | \geq m_i)$

$$A[e + f, \alpha || m, [i]] = A[e; \alpha || m, [i]] + A[f; \alpha || m, [i]]$$

$$A[ef, \alpha || m, [i]] = A[e; \alpha || m, [i]] A[f; \alpha || m, [i]]$$

iii) $I[\bar{e}]$ is in \mathbb{F}' and $A[f; \alpha || m, [i]]$ is nonsingular in \mathbb{F}' if and only if the subsequence $f[m, m_i]$ from which it is derived is invertible in \mathbb{F} .

3a] The matrix functions $\hat{\Lambda}[f; \alpha || m, [i]]$ in $\{K \rightarrow [K[i]]\}$ form a ring \hat{F} isomorphic to F

b) $\hat{\Lambda}[f; \alpha || m, [i]]$ is constant over K if and only if all members components of the subsequence $f[m, mi]$ from which it is derived are equal

c) $\hat{\Lambda}[f; \alpha || m, [i]]|_z$ is either singular for all $z \in K$ or nonsingular for all $z \in K$; the latter case holds if and only if the corresponding subsequence $f[m, mi]$ is invertible in F .

d) \hat{F} contains the constant nonsingular unit element $I[\bar{i}]$.
ii) Relationships similar to those given in clause (1i) hold for the members of \hat{F} over K

4i] The matrix functions $\hat{\Lambda}[f; \alpha || m, [i]]$ in $\{K \rightarrow [K[i]]\}$ form a ring \hat{F} in which the product of $A, B \in \hat{F}$ is taken to be

$$A \hat{\times} B := A \hat{\Lambda}[i] B$$

($I[\bar{i}]^{-1}$ is the unit element in \hat{F} . A is nonsingular with respect to multiplication in $\{K \rightarrow [K[i]]\}$ if and only if is nonsingular with respect to the product $\hat{\times}$ in \hat{F} ; if this is the case the inverse of A in \hat{F} is $(I[i]^{-1} A^{-1} I[\bar{i}])$. \hat{F} is isomorphic to F . The remarks of (3ib-d) above hold, mutatis mutandis, with respect to \hat{F} .

ii) Relationships similar to those given in clause (1i) (the product in the second now being taken to be $\hat{\times}$ as defined above) hold for the members of \hat{F} over K .

$$\textcircled{4} \quad (\text{Ad}[f]B)^0 = Af \quad \text{all } f \in \text{col}[[\mathbb{K}|i]] \text{ if. } B^0 = J_{[i]}$$

$$\textcircled{b} \quad (\text{Ad}[f]A^{-1})^{\circ} = Af \quad . \quad \text{if } \text{Ad}(\{A\}^{-1})^{\circ} = I_{[i]}$$

④ ⑤ \nexists $AI_{[i]} = I[i]^o$

d) iff $A = \begin{pmatrix} I \\ A' \end{pmatrix}$ where $A' I_{[e]} = J_{[e]}$

$$\text{4: } A\mathbb{I}_{[i]} = A(\mathbb{I}[i])^\circ = A'(\mathbb{I}[i]^{-1}\mathbb{I}\mathbb{I}[i])^\circ = A'\mathbb{I}[i]^\circ$$

A possesses property (b) $A_0 \hat{I}_{[i]} = 1$, $A_z \hat{J}_{[i]} = 0$ $\forall z > 0 \Rightarrow A \hat{J}_{[i]} = \hat{I}_{[i]}$

$$A = \text{diag}(c) + A' \quad \text{where } A' \text{ is symmetric} \quad \text{and } A'_{\text{perm}} \circ \text{diag}(A) = A'$$

$$A^{-1} A I_{[i]} = A^{-1} I^{[i]} \circ = (A^{-1})^{\circ} \Rightarrow \textcircled{b}$$

$$\textcircled{d} \Rightarrow A[I_{[i]}] = A'[I[i]]^{-1} \cdot I[i] = A'[I[i]]$$

$$\textcircled{2} := A = \{I[i]^{-1}A' \mid A'I_{[i]} = I_{[i]}\}$$

$$(d) A_{[i,j]} = \{I[i]^{-1} A' I_{[i,j]} = \{I[i]^{-1}\}_{[i,j]} = \{I[i]\}^{-1} (\{I[i]\})^{\circ} = \{I[i]\}^{\circ}$$

$$A[I_{ij}] = J[i] \xrightarrow{\text{def}} A'[I_{\substack{i \\ j}}] = J[i] I[i] \xrightarrow{\text{def}} (J[I[i]] I[i])^o = \text{Hoc } J_{ij}$$

$$A' \begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix} A' \{I[i]^{-1} = \begin{pmatrix} a \\ 2b-1 \end{pmatrix} \begin{pmatrix} 1-b \\ 1-b \end{pmatrix} A[I_{[i]} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$LI[i]^{-1}A' = \begin{matrix} a & 1-a \\ b-a & a-b \end{matrix} = A \quad A]_{[i]} = \begin{matrix} 1 \\ 0 \end{matrix} ? \quad \text{when } a=1$$

$$\begin{array}{llll} a & 1-a & c & 1-c \\ b & 1-b & d & 1-d \end{array}$$

$$A]_{[i]} =]_{[i]} \quad B]_{[i]} =]_{[i]} \Rightarrow AB]_{[i]} =]_{[i]} \quad \text{Reann' if}$$

$$A = I^{-1} A' \quad A'_{[i]} = i_{[i]} \quad A'B' = IAI^{-1}B \quad (\{A\}^{-1})^0 = I_{[i]}$$

$$B = JI^{-1}B' \quad B' = -J^{-1}JI^{-1}A'B' = A'IB \quad \checkmark$$

mult class with $A \times B = A \sqcup B$

$$A^{-1} = \frac{1}{\det A} \text{adj } A$$

$$x,y \in \mathbb{K} \quad x+y=1 \quad (xA+yB)_{[i,j]} = I_{[i,j]} \quad [I[i]] A d[f] A^{-1} [I[i]]^{-1} [I[i]] = \\ (B d[f] B^{-1} [I[i]])^o = Bf$$

$$CI_{[i]} = I_{[i]}^c \quad C = U I_{[i]}^{-1} A' \quad A' \text{ permanent}$$

Let $i \in \bar{\mathbb{N}}$ and $K' \subseteq K$

extend to $i, j \in \mathbb{N}$??

i) A mapping $A: K' \rightarrow [K'|i]$ for which

$$AI_{[i]} = I_{[i]}^o \quad \langle K' \rangle$$

is said to be annihilatory over K' , $\text{ann}(K'|i)$ is the complete system of such mappings and $\text{ann}'(K'|i)$ the complete subsystem of all such mappings that are nonsingular over K' .

ii) A mapping $B: K' \rightarrow [K|i]$ for which

$$BI_{[i]} = I_{[i]} \quad \langle K' \rangle$$

is said to be permanent over K' ; $\text{perm}(K'|i)$ and $\text{perm}'(K'|i)$ are the complete system and of such mappings and the complete subsystem of such mappings that are nonsingular over K' respectively.

() Let $i \in \bar{\mathbb{N}}$ and $K' \subseteq K$.

i) The mappings of $\text{ann}(K'|i)$ are preserved during the formation

of an arithmetic mean; with $A, A' \in \text{ann}(K'|i)$ and $a, a': K' \rightarrow K$ such that $a \circ a' = 1 \langle K' \rangle$, $aA + a'A \in \text{ann}(K'|i)$. The same result holds for $\text{perm}(K'|i)$

ii) The mappings of $\text{ann}(K'|i)$ is closed with respect to the multiplicative operator \times defined by setting $A \times A' := A \otimes I_{[i]} A'$ over K' ; if $A, A' \in \text{ann}(K'|i)$ then $A \times A' \in \text{ann}(K'|i)$ also. The same result holds for $\text{ann}'(K'|i)$

b) $\text{perm}(K'|i)$ is closed with respect to multiplication in $[K|i]$.
and $\text{perm}'(K'|i)$ are

\rightarrow folgt.

2.i] $A \in \text{ann}(\mathbb{K}'[i])$ if and only if $B \in \text{perm}(\mathbb{K}'[i])$, where 207.

$B = \sum_{k=1}^n L_i[k] A^{\langle k' \rangle}$. The same result holds with regard to the systems $\text{ann}'(\mathbb{K}'[i])$ and $\text{perm}'(\mathbb{K}'[i])$

1. \rightarrow $\text{ann}'(\mathbb{K}'[i])$
- $A \in \text{ann}'(\mathbb{K}'[i])$ if and only if $(\{A\}^{-1})^{\circ} = I_{[i]}$
 - $B \in \text{perm}'(\mathbb{K}'[i])$ if and only if $B^{-1} \in \text{perm}'(\mathbb{K}[i])$

3.i] The mappings systems $A, C: \mathbb{K}' \rightarrow [\mathbb{K}[i]]$ satisfy the relationship

$$(Ad[f]B)^{\circ} = Af \quad \langle \mathbb{K}' \rangle$$

for all $f: \mathbb{K}' \rightarrow \text{col}[\mathbb{K}[i]]$ if and only if $B^{\circ} = I_{[i]} \langle \mathbb{K}' \rangle$.

ii) The mapping $A: \mathbb{K}' \rightarrow [\mathbb{K}[i]]$ satisfies the relationship

$$(Ad[f]A^{-1})^{\circ} = Af \quad \langle \mathbb{K}' \rangle$$

if and only for all $f: \mathbb{K}' \rightarrow [\mathbb{K}[i]]$ if and only if $A \in \text{ann}'(\mathbb{K}[i])$

iii) The mapping $B: \mathbb{K}' \rightarrow [\mathbb{K}[i]]$ satisfies the relationship

$$(Bd[f]B^{-1}L_i[i])^{\circ} = Bf \quad \langle \mathbb{K}' \rangle$$

for all $f: \mathbb{K}' \rightarrow [\mathbb{K}[i]]$ if and only if $B \in \text{perm}'(\mathbb{K}[i])$

$$A_{[i,j]} = I_{[i,j]}^{\circ} \text{ prot. ann } A \times A' = A \cup [i] A' \quad [i][i] I_{[i]}^{\circ} = I_{[i]}$$

$$(\{A\}^{-1})^{\circ} = I_{[i]} \quad B^{\circ} = I_{[i]} \quad \text{②} + \text{④} \Rightarrow \text{⑤} \quad \text{④} \leftarrow \text{③} \Rightarrow \text{②}$$

$$AB_{[i]} = I_{[i]}^{\circ} \quad \text{A} \in \text{ann} \text{ ⑥} \quad B \in \text{perm} \Rightarrow AB \in \text{ann} \text{ ⑦} \quad A_{[i,j]} = J_{[i,j]}^{\circ} \quad A' \text{ moves } \mathbb{K}' \rightarrow$$

$$A_{[i,j]} = AB_{[i]} I_{[j]} \quad B_{[i]} = x \quad Ax = J_{[i]}^{\circ} \quad A(J_{[i]} - J_{[i,j]}) = 0$$

$A \in \text{ann}$ $B \in \text{perm}$ $L_i[i]AB \in \text{perm}$

2nd of the three conditions

$$(1) A \in \text{ann}(K'|_i) \quad (2) B \in \text{perm}(K'|_i) \quad (3) AB \in \text{ann}(K'|_i)$$

~~(1,2)~~ (1,2) imply (3), and (2,3) imply (1). If A^* is nonsingular over K' , conditions (1,3) imply (2).

b) The above ^{observations relate} results apply in equal measure to the three conditions (1), 2)

& and

$$(3') I[i]AB \in \text{perm}(K'|_i).$$

It is possible to define a post-annihilatory mapping $A: K' \rightarrow L(K|_i)$ in terms of the relationship

$$AI_{[i]} = I[i]^i$$

and develop its theory as above. ~~is~~ The operator \times in (1ii) must be defined by $A \times B := \alpha I[i]A'$. The relationship stated in (1iii) is now $(\{A^{-1}\})^i = I[i]$. That in (2i) is $B := \alpha I[i]A$, and in (2ii b) is $\alpha I[i]AB \in \text{perm}(K'|_i)$. The result of (3i) is presented in terms of the relationships $(\text{Ad}[f]B)^i = Af \langle K' \rangle$ and $B^i = I[i]$, and the relationships of (3ii, iii) become $(\text{Ad}[f]A^{-1})^i = Af \langle K' \rangle$ and $(B \text{d}[f]B^{-1}\alpha I[i])^i = Bf \langle K' \rangle$.

The above treatment and its original counterpart may be subsumed under a general theory of annihilatory mappings based upon the relationship $AI_{[i]} = I[i]^i$ with $\alpha I[i]$.)

Let $K' \subseteq K$ and $F, F', \Lambda_0 : K' \rightarrow [K|i]$ be such that

$$\Lambda_0 F = F' \Lambda_0 \quad \langle K' \rangle$$

$$F' = \Lambda_0 F \{ \Lambda_0 \}^{-1} \quad \langle \text{NS}(\Lambda_0) \rangle$$

$$\begin{aligned} \text{Pf } \Lambda, \Lambda_0 F &= \\ \Lambda_1 F &= \Lambda_0^{-1} \Lambda_0 \Lambda_1 F \end{aligned}$$

- a) If $\Lambda_1 : \text{NS}(\Lambda_0) \rightarrow [K|i]$ exists such that, with $D : \text{NS}(\Lambda_0) \rightarrow [K|i]$ defined by setting

$$D := \Lambda_1 \Delta_0$$

$$FD = DF$$

$$\begin{aligned} \Delta_0^{-1} F' \Lambda_0 \Lambda_1 \Delta_0 &= \\ &= \Lambda_1 \Delta_0 \{ \Delta_0 \}^{-1} F' \Delta_0 \\ F' \Delta_0 \Delta_1 &= \Delta_0 \Delta_1 F' \\ &= \end{aligned}$$

Λ_1 commutes with F also

$$\Lambda_1 F \Lambda_1^{-1} = F = \Lambda_1 \Delta_0 F (\Lambda_1 \Delta_0)^{-1}$$

$$\Lambda_1 F' = \Lambda_1 \Delta_0^{-1} F \Delta_0$$

$$F' \Lambda_1 = \Delta_0^{-1} F \Delta_0 \Delta_1$$

and

$$F' = \{ \Delta_1 \}^{-1} F \Delta_1 \quad \langle \text{NS}(\Lambda_0, \Lambda_1) \rangle$$

$$\Lambda_1 F' = F \Lambda_1 \quad \langle \text{NS}(\Lambda_0) \rangle$$

In the following theorem it is shown how one similarity transformation may be induced another.

In special cases it may be shown that Λ_1 defined over a source domain K' satisfies the permutable relationship over K' . In this case the final result obtains over $\text{NS}(\Lambda_1)$

$$A = BC \langle K \rangle A \begin{bmatrix} h, h_{ij} \\ k, k_{hi} \end{bmatrix} = B \begin{bmatrix} h, h_{ki} \\ k, k_{hi} \end{bmatrix} C \begin{bmatrix} h, h_{ij} \\ h, h_{ki} \end{bmatrix} \quad \langle K \rangle \quad h \in [k|i]$$

$$A, B, C \in K \rightarrow [K] \geq \max(h_{ij}, k_{hi})$$

$$\text{and etc } [a_{k+r, h+r}]_{[i]}^{[j]} = [b_{k+r, h+r}]_{[i]}^{[k_{ri}-h]} [c_{h+r, h+r}]_{[k_{ri}-h]}^{[j]}$$

$$A[k, h, [i, j]] = B[k, h, [i, k_{ri}-h]] C[h, h, [k_{ri}-h, j]]$$

$$A[f; \alpha || m; k, h, [i, j]] \quad \text{also } \tilde{\Lambda}$$

$$A[e; \alpha || m; k, h, [i, i]] = A[e; \alpha || m; k, h, [i, k_{ri}-h]]$$

$\tilde{f}I[n]_{[h, k+i]}^{[h, k+i]} = fI[k+i-h] \text{ for } \hat{\Lambda}$

Remark p. 212 \rightarrow Let $i, j, k \in \mathbb{N}$, $h \in [k+i]$ and, $n \in \text{seq}'(K | \geq k+i)$,
 $m+k+i$

and $f \in \text{seq}(K | \geq k+i, m+k+i)$. Set $n = \max(k+i, h+j)$

i) $A[f; \alpha || m; k, h, [i, j]]_{\epsilon [k+i, j]}$ is the submatrix function of

$A[f; \alpha || m, [i, j]]$ defined by setting

$$A[f; \alpha || m; k, h, [i, j]] := A[f; \alpha || m, [i, j]]_{[k, k+i]}^{[h, h+j]} \\ := [S(f; \alpha || m + h + j, k - h + i - 2)]_{[i]}^{[j]} : K \rightarrow [k+i, j]$$

ii) $\tilde{\Lambda}[f; \alpha || m; k, h, [i, j]]$ and $\hat{\Lambda}[f; \alpha || m; k, h, [i, j]]$ are the similarly defined submatrix functions derived from $\tilde{\Lambda}[f; \alpha || m, [i, j]]$ and $\hat{\Lambda}[f; \alpha || m, [i, j]]$ respectively.

() \checkmark Let $i, j, k \in \mathbb{N}$, $h \in [k+i]$ and, $n \geq \max(h+j, k+i)$, ~~let the~~ and $K' \subseteq K$. Let the ~~map~~ matrix mappings $X, Y, Z : K' \rightarrow L[K | n]$ satisfy the relationship

$$\cancel{X \cdot Y \cdot Z \leftarrow K'}$$

Define the submatrix mapping $\mathcal{X}[k, h, [i, j]]$ of X by with $K' \subseteq K$. Let $x(z, z) : K' \rightarrow K \langle z := [n], z := [z] \rangle$ and define the matrix mapping $X, \cancel{Y, Z} : K' \rightarrow L[K | n]$ by setting

$$X := L[x(z, z)] \langle z := [n], z := [z] \rangle .$$

and Define the matrix mappings $Y, Z : K' \rightarrow L[K | n]$ similarly. Define the submatrix Let

$$\cancel{X = YZ \leftarrow K'}$$

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Define the submatrix mapping function $\hat{X}[k, h, [i, j]]$ of X by
 setting $\hat{X}[k, h, [i, j]] : \mathbb{K}' \rightarrow [\mathbb{K}|i, j]$

$$\begin{aligned} X[k, h, [i, j]] &:= X_{[k, k+i]}^{[h, h+j]} \\ &:= \left[\hat{x}_{(k+z, h+z)} \right]_{z=[i]} \end{aligned}$$

and define the submatrix mappings $\hat{Y}[k, h, [i, k+i-h]] : \mathbb{K}' \rightarrow [\mathbb{K}|i, k+i-h]$ and $\hat{Z}[k, h, [i, k+i-h, j]] : \mathbb{K}' \rightarrow [\mathbb{K}|k+i-h, j]$ similarly.

If

$$X = YZ \quad \langle \mathbb{K}' \rangle$$

then

$$X[k, h, [i, j]] = Y[k, h, [i, k+i-h]] Z[h, h, [k+i-h, j]] \quad \langle \mathbb{K}' \rangle$$

2] Let $m, i, j, k \in \mathbb{N}$, $h \in [\mathbb{K}|i]$, $\alpha \in \text{seq}'(\mathbb{K}| \cong m+k)$ and $e, f \in \text{seq}(\mathbb{K}| \cong m+k)$.

$$\begin{aligned} A[\hat{e}; \alpha || m; k, h, [i, j]] &= A[e; \alpha || m; k, h, [i, k+i-h]] \\ &\quad A[f; \alpha || m; h, h, [k+i-h, j]] \end{aligned}$$

$$\hat{\Lambda}[\hat{e} \circ \hat{f}; \alpha || m; k, h, [i, j]] = \hat{\Lambda}[e; \alpha || m; k, h, [i, k+i-h]] \\ A[f; \alpha || m; h, h, [k+i-h, j]] \quad \langle \mathbb{K} \rangle$$

and

$$\hat{\Lambda}[\hat{e} \circ \hat{f}; \alpha || m; k, h, [i, j]] = \hat{\Lambda}[e; \alpha || m; k, h, [i, k+i-h]] \hat{\Lambda}[[k+i-h]] \\ A[f; \alpha || m; h, h, [k+i-h, j]] \quad \langle \mathbb{K} \rangle.$$

$\xrightarrow{110}$ The formation of a submatrix $X_{[k, k+i]}^{[h, h+j]}$ of the matrix 212.
 product $X = YZ$ with $Y, Z \in [K|n]$ requires, in general,
 the multiplication of $Y_{[k, k+i]}$ in $[K|i, n]$ and $Z_{[h, h+j]}^{[k, k+i]}$
 in $[K|n, j]$. If however, Y and Z are both of triangular
 form, multiplication of two submatrices of reduced dimension
The same considerations hold true for matrix mappings.
 suffices. The minimal dimension requirements are specified
 in the following theorem, which also deals, in particular, with
 the multiplication of triangular matrices and matrix functions
 of the form A, \tilde{A} and \hat{A} .

$\xrightarrow{110}$ Corresponding to the results of part [i] of Th. ~~110~~ concern
 divided differences derived from sequences of the form $\Theta(\alpha||m, i)$.
 A number of
 Corresponding results concerning differences of Lagrange interpolatory
 functions may be stated. The following theorem presents these results
 in succinct form.

$$\delta(\phi(\alpha||m, j+1); \alpha||n, k) = -\delta(\phi(n, k+i); \alpha||m, j)$$

$$j' \Rightarrow m' = m \quad n' = m+i-1, k' = i-1 \quad || \text{ two forms for } \mu(r, \dots) \text{ consistent}$$

$$j'+1 \Rightarrow m' = m \quad n' = m+i \quad k' = i-2$$

$$j' \Rightarrow m' = m \quad n' = m+i \quad k' = i-2 \quad k+1 = i-2+1 \quad \text{two forms of } \mu(r, b) \text{ consistent}$$

$$\phi(\alpha||m, 2) | x_{m+i}) = (\alpha_{m+i} - \alpha_{m+2}) \phi(\alpha||m, 2+1 | \alpha_{m+i})$$

(110)
 a second result of $\mu(r, b)$

$$\begin{aligned} \delta(\phi(\alpha||m, 2); \alpha||m+i, i-2) &= (\alpha_{m+i} - \alpha_{m+2}) \delta(\phi(\alpha||m, 2+1); \alpha||m+i, i-2) \\ &\quad + \delta(\phi(\alpha||m, 2+1); \alpha||m+i+1, i-2-1) \end{aligned}$$

(6) $\alpha_{m+2} - \alpha_{m+1}$

$$\prod_{\chi=0}^{j-1} \pi(\alpha || m, i | \beta_{n+\chi}) = (-1)^{\sum_{\chi=0}^{j-1}} \prod_{\chi=0}^{j-1} \pi(\beta || n, j | \alpha_{m+\chi})$$

$$\begin{aligned} \prod_{\chi=0}^{j-1} \pi(\alpha || m, i+1 | \beta_{n+\chi}) &= \prod_{\chi=0}^{j-1} (\beta_{n+\chi} - \alpha_{m+\chi}) \prod_{\chi=0}^{j-1} \pi(\alpha || m, i | \beta_{n+\chi}) \\ &= (-1)^j \pi(\beta || n, j | \alpha_{m+\chi}) + (-1)^{i+j} \prod_{\chi=0}^{j-1} \pi(\beta || n, j | \alpha_{m+\chi}) \\ &= (-1)^{(i+1)j} \prod_{\chi=0}^j \pi(\beta || n, j | \alpha_{m+\chi}) \end{aligned}$$

$$(\alpha_{m+2} - \alpha_{m+1}) \{ (\alpha_{m+1} - \alpha_{m+2}) \{ (\alpha_{m+1} - \alpha_m) - (\alpha_{m+1} + \alpha_{m+2}) \}$$

$$\mu(r-1; \alpha || m; i+1, 2H)$$

$$\begin{aligned} &= (\alpha_{m+2} - \alpha_{m+1}) \{ (\alpha_{m+1} - \alpha_{m+2}) \mu(r-1; \alpha || m+1; i, 2) \\ &\quad - (\alpha_{m+2} - \alpha_{m+1}) \mu(r; \alpha || m; i, 2H) \} \end{aligned}$$

$m+2, m+1$
min const

$$(6) (\alpha_{m+2H} - \alpha_{m+1H}) (\alpha_{m+1} - \alpha_m) \mu(r-1; \alpha || m; i+1, 2H)$$

$$= (\alpha_{m+1} - \alpha_{m+2H}) \mu(r-1; \alpha || m+1; i, 2) - (\alpha_{m+2H} - \alpha_{m+1}) \mu(r; \alpha || m; i, 2H)$$

$$(6') (\alpha_{m+1H} - \alpha_m) \mu'(r-1; \alpha || m; i+1, 2H)$$

1, 7, 8, 9 from group

$$= \mu'(r-1; \alpha || m+1; i, 2) - \mu'(r; \alpha || m; i, 2H)$$

$$(7) : i := i+1$$

$$(\alpha_{m+1} - \alpha_{m+2H-1}) \mu(r; \alpha || m; i, 2) + (\alpha_{m+1H-1} - \alpha_{m+2-1}) \mu(r-1; \alpha || m; i+1, 2)$$

$$= (\alpha_{m+2} - \alpha_{m+1}) \mu(r; \alpha || m; i, 2)$$

$$(\alpha_{m+2} - \alpha_{m+1})$$

$$(2) : (\alpha_{m+2} - \alpha_{m+2H-1}) \mu(r; \alpha || m; i, 2) + (\alpha_{m+1H-1} - \alpha_{m+2H-1}) \mu(r-1; \alpha || m; i+1, 2)$$

$$= (\alpha_{m+2} - \alpha_{m+1}) \mu(r-1; \alpha || m; i, 2)$$

$$(\alpha_{m+2} - \alpha_{m+1})$$

$$(\alpha_{m+1H-1} - \alpha_{m+2H-1}) (\alpha_{m+2} - \alpha_{m+1})$$

20 i viii 21 cdf using

$$(\alpha_{m_1} - \alpha_m) \mu''(m, a+1, b, c) - (\alpha_{m_1} - \alpha_b) \mu''(m, a, b, c+1) = \\ \mu''(m_1, a+1, b, c) - \mu''(m, a, b+1, c+1)$$

$$\mu''(m, a, b+1, c+1) + (\alpha_{m_1} - \alpha_m) \mu''(m, a+1, b, c) = \\ \mu''(m_1, a+1, b, c) + (\alpha_{m_1} - \alpha_b) \mu''(m, a, b, c+1)$$

(c) Let $r, m, i, \omega \in \mathbb{N}$ and $\alpha \in \text{seq}'(K / \Delta_{m+i})$

i) $\mu(r; \alpha // m; i, \omega) = \delta(\phi(\alpha // m_{r+i}, \omega-1); \alpha // m, \omega) - \delta(\phi(\alpha // m_{r+i}, \omega-2+1); \alpha // m, \omega-1)$

by

ii) ($r=0$)

$$\mu(0; \alpha // m, i, \omega) = \mu(\alpha // m, i, \omega)$$

b) If $r > 0$

$$\begin{aligned} \mu(r; \alpha // m; i, \omega) &= (\alpha_{m_{r+i}} - \alpha_{m+i}) \delta(\phi(\alpha // m, \omega+1); \alpha // m_{r+i}, \omega-1) \\ &= (\alpha_{m_{r+i}} - \alpha_{m_{r+i}}) \delta(\phi(\alpha // m_{r+i}, \omega-1+1); \alpha // m, \omega) \\ &= (\alpha_{m_{r+i}} - \alpha_{m_{r+i}}) \sum_j \mu(0; \alpha // \\ &\quad \xrightarrow{\text{if}} \\ &= (\alpha_{m_{r+i}} - \alpha_{m_{r+i}}) \sum_j \mu(0; \alpha // m; i+r, \omega) \Pi(\alpha // m_{r+i}, r-1 / \alpha_{m_{r+i}}) \\ &\quad \langle \omega := [k(\frac{r}{r+i}), k+i] \rangle \\ &= (\alpha_{m_{r+i}} - \alpha_{m_{r+i}}) \sum_j \mu(0; \alpha // m; i+r, \omega) \Pi(\alpha // m_{r+i}, r-1 / \alpha_{m_{r+i}}) \\ &\quad \langle \omega := [k(\frac{r}{r+i}), k+i] \rangle \end{aligned}$$

where, in the last two formulae, $k(\frac{r}{r+i}) \in [r, r+i]$.

$$c) \mu(r; \alpha || m; i, b) = \frac{1}{\pi(\alpha || m || r, i | \alpha_m)}$$

$$\mu(r; \alpha || m; i, i) = \frac{1}{\pi(\alpha || m, i | \alpha_{mri})}$$

iii) With $i=0$

$$\left\{ \sum_i \mu(r; \alpha || m; i, \omega) \langle \omega : = [i] \rangle \right\} = 0$$

iv) With $r > 0$ but $\alpha > \alpha_m$, set $a = mri$, $b = mrrb$, $c = mrrr$ and

$$\mu'(m, a, b, c) = \underline{\mu(r; \alpha || m; i, i)}$$

(In all but relationships (3, 6, 8, 10) it is assumed that $|\alpha| > \alpha_m$) are satisfied

a) The figure showing homogeneous three-term relationships holds

- 1) $\mu'(m, a, b+r, c+r) = \mu'(m, a, b, c) + (\alpha_{c+r} - \alpha_b) \mu'(m, a, b, c+r)$
- 2) $\mu'(m, a+r, b, c+r+1) = \mu'(m, a+r, b, c) + (\alpha_{c+r+1} - \alpha_{a+r}) \mu'(m, a+r, b, c+r)$
- 3) $\mu'(m, a+r, b+r, c) = \mu'(m, a, b, c) + (\alpha_{a+r} - \alpha_b) \mu'(m, a+r, b, c)$
- 4) $(\alpha_{c+r} - \alpha_{a+r}) \mu'(m, a+r, b+r, c+r) =$
 $(\alpha_b - \alpha_{a+r}) \mu'(m, a+r, b, c) + (\alpha_{c+r} - \alpha_b) \mu'(m, a, b, c+r)$
- 5) $\mu'(m+r, a, b, c+r) = \mu'(m, a, b, c) + (\alpha_{c+r} - \alpha_m) \mu'(m, a, b, c+r)$
- 6) ~~$\mu'(m+r, a, b, c) = \mu'(m, a, b+r, c) + (\alpha_b - \alpha_m) \mu'(m, a, b, c+r)$~~
- 7) $(\alpha_{c+r} - \alpha_m) \mu'(m, a, b+r, c+r) =$
 $(\alpha_b - \alpha_m) \mu'(m, a, b, c) + (\alpha_{c+r} - \alpha_b) \mu'(m+r, a, b, c+r)$

$$8) \mu'(m+1, a+1, b, c) = \mu'(m, a, b, c) + (\alpha_{an} - \alpha_m) \mu'(m, a+1, b, c)$$

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$$9) (\alpha_{c+1} - \alpha_{a+1}) \mu'(m+1, a+1, b, c+1) = (\cancel{\alpha_{c+1}} - \cancel{\alpha_{a+1}})$$

$$(\alpha_{c+1} - \alpha_m) \mu'(m, a, b, c+1) + (\alpha_m - \alpha_{an}) \mu'(m, a+1, b, c)$$

$$10) (\alpha_{a+1} - \alpha_m) \mu'(m, a+1, b+1, c) =$$

$$11) (\cancel{\alpha_{c+1}} - \cancel{\alpha_{a+1}}) \mu'(m, a+1, b+1, c+1) \Rightarrow$$

$$(\alpha_b - \alpha_m) \mu'(m, a, b, c) + (\alpha_{an} - \alpha_b) \mu'(m+1, a+1, b, c)$$

$$(\cancel{\alpha_b} - \cancel{\alpha_{an}}) \mu'(m, a+1, b, c) + (\cancel{\alpha_{c+1}} - \cancel{\alpha_b}) \mu'(m, a, b, c+1)$$

b) The above relationships may be arranged in the scheme

	a	b	c	i
m	1	2	3	4
a		5	6	7
b			8	9
c			10	11

in the function values $\mu'(\overset{m, a, b, c}{\text{abc...}})$

which indicates the variables held constant in the above relationships.

(Thus, in relationships 2, 5, 8 and 9, each of which belongs to either to the column with label b or to the row with that label, b is constant. In relationship 7, a and ~~r is already declared~~ i = ~~r~~ $\alpha_{an} - m - b$ are both constant.)

c) In addition to the variables held constant as indicated in the above scheme, further variables are constant in the function values $\mu'(m, a, b, c)$ occurring on the right hand sides of the above relationships. Thus, on the right hand side of (1, 2), m, a, b (i.e. r and d) are constant; in (3, 8) m, b, c (i.e. r_{11} , r_{12} and i_{12}); in (4, 9) m, b, i (i.e. r_{12}); in (5) m, a, b (i.e. r_{11} and d); in (6) m, a, c (i.e. d and r_{11}); in (7) a, b, i (i.e. r).

Similar remarks may be made concerning any pair of function values selected from each of the above relationships.

d) Delayed divided difference multipliers satisfy a number of homogeneous four-term relationships, for example

$$\mu'(m, a, b+1, c+1) + (\alpha_{bm} - \alpha_m) \mu'(m, a+1, b, c) = \\ \mu'(m+1, a+1, b, c) + (\alpha_{cm} - \alpha_b) \mu'(m, a, b, c+1)$$

$\downarrow^{(v)}$

The result of (iii) follows from the definition of μ :

$$\left\{ \sum \mu(r; \alpha || m; i, \omega) \langle \omega: -1, 1 \rangle \right\} = \delta(\phi(\alpha || m, \omega); \alpha || m+r, i) = 0$$

Relationship (1) may be obtained from the first of the formulae given in (ib), using the difference relationship for $\delta(\phi(\alpha || m, 2+1); \alpha || m+r, i-2)$; (2,3) may be (2,3,5,6) may be obtained by expressing $\phi(\alpha || m, \omega)$ as $\phi(\alpha || m, 2+1) \Theta(m+r | 1)$ or $\phi(\alpha || m+r, i-2) \Theta(\alpha || m+r, i-2)$ as $\phi(\alpha || m+r, i-2+1) \Theta(\alpha || m+r | 1)$; (4) follows by combining (1,2) and (7) by combining (5,6); (8) is derived as for (1), using the second of the formulae given in (ib); (9) is obtained by combining (5,8) and (10) by combining (3,8). Relationship (5) also follows directly from either the third or fourth of formulae (ib).

In any of the five groups of four relationships, in each of which one of the variables m, a, b, c, i is constant, as indicated in the scheme displayed in (ivb), any relationship may be obtained from directly from any remaining pair. Thus δ follows may be determined directly from (2,5), from (2,9) or from (5,9).

v) Relationships corresponding to (1-10) above hold for the delayed divided difference multipliers $\mu(r; \alpha || m; i, \omega)$. They are, except in the case of relationship (5), more complicated than those given. In the exceptional case, the relationship in question may be presented as

$$\mu(r; \alpha || m; i+r, \omega) = \frac{\mu(r; \alpha || m+1; i, \omega) - \mu(r; \alpha || m; i, \omega)}{\alpha_{m+r+1} - \alpha_m}$$

$$\delta(q, r, m) \leq \delta(r, \delta(0, r, m)) ; \alpha || m, 1) \quad \delta(0, r, m) = \delta(0, f; \alpha || m, r)$$

$$\delta(0,r,m+1) \quad \delta(r,\delta(0,r,\langle \rangle); \alpha || m, k) \quad \delta(r,\delta(0,r,\langle \rangle); \alpha || m, k)$$

$$\delta(r+k) \delta(0; f; \alpha ||\beta||^M, r)$$

$$S(r+k, S(0, r, \langle \rangle); \alpha || m, k); \alpha || m, 1)$$

SFRK

$$\delta(r; \delta(0; f; \alpha || \langle \rangle, r); \alpha || m, k) = \delta(r; f; \alpha || m, i + h)$$

$\delta(r+k; \delta(r; \delta(0; f; \alpha || \ll \gg, r); \alpha || \langle \rangle, k); \alpha || m, o)$

$$S(r:f;\alpha \parallel m,k) = S(r+j;S(r:f;\alpha \parallel \langle \rangle,j) \parallel m,k)$$

$$f \in \text{seq}(\kappa | \geq m_{j+k})$$

$\left[\frac{1}{2}t - \frac{1}{2} \right]$

$$S(r; f, \alpha || \omega, j) \quad \omega := [m, m+k]$$

$$j = i - k \approx j$$

$$\delta(r; f; \alpha \| m, j+k) = \delta(r+j, \delta$$

$r := r + \delta$ $j := i + j$ $f \in \text{seq}(K | \exists m \in j \cdot k)$

$$\delta(r; f; \alpha || m, i+j+k) = \delta(r+i+j; \delta(r; f; \alpha || \langle \rangle, \#) || m, k)$$

$$= \delta(r+i+j; \delta(\cancel{r+i}; \delta(r+\cancel{j}; \alpha \langle \cancel{k}, j \rangle) \langle i, j \rangle) \langle m, k \rangle)$$

~~if~~ \rightarrow

[H-~~o~~-j]
[metk]

$$r+i_j = r(n) \quad r+i = r(n-1) \quad \xrightarrow{i \leftarrow i-1} \quad r = r(n-2) \quad \text{for } k := -\omega$$

$$\delta(r(n-2); f: \alpha \amalg m, r(n) - r(n-2) + \omega) =$$

$\delta(r(n)) : \delta(r(n-1)) : \delta(r(n-2)) : f_{\text{max}} \ll [f_1 - r(n-1) + r(n-2)] \gg, r(n-1) - r(n-2)$

$$\| \langle \xi [f] - r(n) + r(n-2) \rangle \|_{m, \omega} \quad \omega :=$$

$$\begin{aligned} \delta(r:f;\alpha || \omega, 0) & \langle \omega := [m, m+i+j+k] \rangle \\ \delta(r:f;\alpha || \omega, i) & \langle \omega := [m, m+i+j+k] \rangle \\ \delta(r:f;\alpha || \omega, i+j) & \langle \omega := [m, m+i+j+k] \rangle \\ \delta(r:f;\alpha || \omega, i+j+k) & \langle \omega := [m, m+i+j+k] \rangle \end{aligned}$$

$$\delta(r:f;\alpha || \omega, i+j+k) = \delta(r+i; \delta(r:f;\alpha || \langle \rangle, i); \alpha || \omega, j) \quad \omega := [m, m+i+j+k]$$

$$\delta(r:f;\alpha || m, i+j+k)$$

$$\delta(r:f;\alpha || \omega, 0) \quad \langle \omega := [m, m+i+j+k] \rangle \quad F_\omega$$

$$\delta(r:f;\alpha || \omega, i) = \delta(r+i : \delta(r:f;\alpha || \langle \rangle, i); \alpha || \omega, 0) \quad \langle \omega := [m, m+i+j+k] \rangle$$

$$\delta(r+i : \delta(r:f;\alpha || \langle \rangle, i); \alpha || \omega, j)$$

$$= \delta(r+i+j, \alpha ||$$

$$= \delta(r:f;\alpha || \omega, i+j)$$

$$= \delta(r+i+j : \delta(r+i : \delta(r:f;\alpha || \langle \rangle, i); \alpha || \langle \rangle, j); \alpha || \omega, 0)$$

$$\begin{matrix} \uparrow \\ [m+i+j+k] \end{matrix} \quad \begin{matrix} \uparrow \\ [m+j+k] \end{matrix}$$

$$\langle \omega := [m, m+k] \rangle$$

$$f \in \text{seq}(K | \overset{r+}{\geq} m+i+j+k) \quad f \in \text{seq}(K | \geq m+i+j+k)$$

a) Let $h, e, j, k \in \mathbb{R}$.

Delayed divided differences possess the following telescoping property.

With $\alpha \in \text{seq}'(K | \geq m+i+j+k)$ and $f \in \text{seq}(K | \overset{r+}{\geq} m+i+j+k)$ it is supposed that the subsequence of $f \in \text{seq}(K | \geq m+i+j+k)$ comes from the initial values

that the subsequence $\{f_{[m, m+i+j+k]}\}$ of $f \in \text{seq}(K | \geq m+i+j+k)$ is composed of the successive k^{th} order delayed divided differences of f , so that are constructed, so that

$$f_\omega := \delta(h:f; \alpha || \omega, 0) \quad \langle \omega := [m, m+i+j+k] \rangle$$

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Successively the delayed divided differences $\delta(h; f; \alpha || \omega, i) <_{\omega := [m, m+j+k]} = \delta(h; f; \alpha || \omega, i)$ are constructed and used as the initial values sequence $F[m, m+j+k]$ for the construction of $(hri)^{th}$ order delayed divided differences, so that

$$\delta(r+i; \delta(r; F; \alpha || \omega, i); \alpha || \omega, 0) =$$

$$\underline{\delta(r; F; \alpha || \omega, i) <_{\omega := [m, m+j+k]}} \rightarrow$$

with the subsequence $F[m, m+j+k]$ of $F \in \text{seq}$

$$F_\omega = \delta(r; f; \alpha || \omega, i) <_{\omega := [m, m+j+k]}$$

and

$$\underline{\delta(r+i; F; \alpha || \omega, 0) = F_\omega <_{\omega := [m, m+j+k]}}$$

$$\delta(hri; F; \alpha || \omega, 0) = F_\omega = \delta(h; f; \alpha || \omega, i) <_{\omega := [m, m+j+k]}$$

The delayed divided differences $\delta(hri; F; \alpha || \omega, j) <_{\omega := [m, m+k]}$ are constructed and used as the initial values sequence for the construction of $(hri;j)^{th}$ order delayed divided differences, and in particular for the construction of $\delta(rri;j)$

$$\underline{\delta(rri;j; \delta(hri; F; \alpha || \omega, k), j); \alpha || m, k)}$$

In the above process, in particular,

$$\delta(hri;j; \delta(hri; F; \alpha || \omega, k), j); \alpha || m, k) \Leftrightarrow$$

$$= \delta(hri; F; \alpha || m, j+k)$$

$$= \delta(hri; f; \alpha || m, i+j+k)$$

b) As a special case of the above result, it follows that with $f \in \text{seq}(|K| \geq m+n)$

$$\delta(r; \delta(f; \alpha || \omega, i), r); \alpha || m, i) = \delta(f; \alpha || m, r+i)$$

The delayed divided differences dealt with in subclause (ii) may be displayed in the following scheme ²⁶:

$$\begin{aligned}
 f_\omega &= \delta(h; f; \alpha \| \omega, 0) \xrightarrow{i+1} \\
 F_\omega &= \delta(h; f; \alpha \| \omega, i) \xrightarrow{j+1} \\
 \delta(h_{ri}; \delta(F_\omega; \alpha \| \omega, j)) &\xrightarrow{k+1} \\
 &\vdots \\
 \delta(h_{ri+j}; \delta(h_{ri}; F_\omega; \alpha \| \langle m, k \rangle), j); \alpha \| m, k
 \end{aligned}$$

The subclause Two components, indicated by trapezoidal regions in the above diagram, of a telescoping process are considered. It is evident that the results given may be extended to an arbitrary number of components.

- at least arbitrary
- $\text{pp}(K|n)$ is the complete system of mapping $g: K \rightarrow K$ (i.e. each g has a representation of the form

$$g(x) = \sum g_\omega x^{\omega} \langle \omega: [n] \rangle$$

- where the $g_\omega \in K \langle \omega: [n] \rangle$ depend upon the g in question). In fact g_ω are independent $\omega \in \text{seq}'(K \geq r_{\min})$
- i) Let $r_{\min} \geq 0$, $n \in [r_{\min}]$ and $g \in \text{pp}(K|n)$ and define $f \in \text{seq}(K \geq r_{\min})$ by setting $f := g(\alpha[m, n])$.

$$\delta(r; \delta(f; \alpha \| \langle [m, n] \rangle, r); \alpha \| m, i) = 0$$

- ii) Let $i \geq 0$ and $f' \in \text{seq}(K \geq r_{\min})$ be such that $f'[m, m+i]$ is a constant subsequence, so that $f'_\omega = F \in K \langle \omega: [m, m+i] \rangle$. With $\alpha \in \text{seq}'(K \geq r_{\min})$,

$$\delta(r; f'; \alpha \| m, i) = 0$$

$$\pi(\alpha \parallel m_{ij-1}, i) \pi(\alpha \parallel m_{j-1}) \pi(\alpha \parallel m_{ij+r+1}, i-j) =$$

$$* \quad \pi(\alpha \parallel m_{ij}) \pi(\alpha \parallel m_{ij+r+1}, i-j) = \pi(r; \alpha \parallel m_{ij}, i-j)$$

$$\pi(\alpha \parallel m_{ij-1}) \pi(\alpha \parallel m_{ij+r+1}, i-j) \pi(\alpha \parallel m_{ij+r}, i) =$$

$$\pi(\alpha \parallel m_{ij-1}) \pi(\alpha \parallel m_{ij+r}, i-j+1) = \pi(r; \alpha \parallel m_{ij}, i, j-1)$$

$$* = \pi(\alpha \parallel m_{ij}) \pi(\alpha \parallel m_{ij+r+1}, i) \pi(\alpha \parallel m_{ij+r+2}, i-j-1)$$

$$\pi(r; \alpha \parallel m_{ij}, i, \otimes) = \pi(\alpha \parallel m_{ij+r+1}, i)$$

$$\pi(r; \alpha \parallel m_{ij}, i, \otimes) = \pi(\alpha \parallel m_{ij}, i)$$

Let $h, e, j, k \in K$

iii) Delayed Lagrange forms possess the following telescoping property. With $\omega \in \text{seq}'(K \rightarrow h \parallel m_{ij+r+k})$ it is supposed that the ~~seq~~ subsequence $g[m, m_{ij+r+k}]$ of $g \in \text{seq}(K \rightarrow K \parallel m_{ij+r+k})$ constitutes the initial value sequence from which h^{th} order delayed Lagrange forms are constructed, so that

$$g_\omega := \Lambda(h; g; \alpha \parallel \omega, \otimes) \stackrel{\langle K \rangle}{\sim} \langle \omega := [m, m_{ij+r+k}] \rangle \quad \text{for}$$

The delayed Lagrange forms $\Lambda(h; g; \alpha \parallel \omega, i) \langle \omega := [m, m_{ij+r+k}] \rangle$ are constructed and used as the initial value sequence $G[m, m_{ij+r+k}]$ of $G \in \text{seq}(K \rightarrow K \parallel m_{ij+r+k})$ in the construction of $(hri)^{\text{th}}$ order Lagrange forms, so that

$$\text{for } \Lambda(hri; G; \alpha \parallel \omega, \otimes) = G_\omega := \Lambda(h; f; \alpha \parallel \omega, i) \langle \omega := [m, m_{ij+r+k}] \rangle$$

The delayed Lagrange forms $\Lambda(hri; G; \alpha \parallel \omega, j) \langle \omega := [m, m_{ij+r+k}] \rangle$ are constructed and used as the initial values for the construction of $(hrij)^{\text{th}}$ order delayed Lagrange forms.

In the above process, in particular,

$$\begin{aligned} & \Lambda(hrij; \Lambda(hri; G; \alpha \parallel \langle [m_{ij+r+k}], j); \alpha \parallel m_{ij+r+k}) \\ &= \Lambda(hri; G; \alpha \parallel m_{ij+r+k}) \quad \langle K \rangle \\ &= \Lambda(h; g; \alpha \parallel m_{ij+r+k}) \quad \langle K \rangle \end{aligned}$$

b) As a special case of the above result, it follows that with $\text{geseq}(K \rightarrow K | \leq mri)$

$$\Delta(r; \Delta(g; \alpha \| \langle [i] \rangle, r); \alpha \| m, i) = \Delta(g; \alpha \| m, ri) \langle K \rangle$$

(The results of clause (ii) are paraphrases of the results of clause () of Theorem. The remarks ~~concern~~ subsequent to that theorem apply, mutatis mutandis, with equal force here)

iv) Let $ri \geq 0, n \in [ri]$, $g \in pp(K|n)$ and define $f \in \text{seq}(K|mri)$ by setting $f := g(\alpha[mri])$.

$$\Delta(r; \Delta(f; \alpha \| \langle [mri] \rangle, r); \alpha \| m, i) = g \langle K \rangle$$

b) Let $\text{geseq}(K \rightarrow K | \leq mri)$ be such that $g'_{[m, mri]}$ consists of copies of the same function, so that g'_{ω} for some mapping $G: K \rightarrow K$, $g'_{\omega} = G \langle K \rangle \langle \omega := [m, mri] \rangle$.

$$\Delta(r, g'; \alpha \| m, i) = G \langle K \rangle$$

the subsequence

If the successive components of $f'_{[m, mri]}$ are of $f \in \text{seq}(K | \leq mri)$ are successive values of an n^{th} degree power polynomial, in the sense that for some $g \in pp(K|n)$, $f'_{[m, mri]} = g(\alpha[m, mri])$, then for some $\alpha \in \text{seq}'(K | \leq mri)$, then, with $n \in [i]$,

$$S(f; \alpha \| m, i) = 0$$

It is not however ^{generally} true that with $r \geq 0$ and the length of f and α suitably extended,

$$S(r; f; \alpha \| m, i) = 0.$$

Nevertheless this result is true when $n=0$. It also holds when the components of $f[m, m_i]$ are divided differences of the successive values of an n^{th} degree fully poised polynomial. 265.

$$\pi(r; \alpha || m; i, s) = \overline{\pi}(\alpha || m, s) \overline{\pi}(\alpha || m + 2H, i - 2)$$

$$\pi'(m; a, b, c) = \overline{\pi}(\alpha || m, a - m) \overline{\pi}(\alpha || b + 1, c - b)$$

$$\lambda' = \mu' \pi'$$

$$\frac{\lambda'(m, a, b+1, c+1)}{\pi(\alpha || m, a - m) \overline{\pi}(\alpha || b+2, c - b)} = \frac{\lambda'(m, a, b, c)}{\pi(\alpha || m, a - m) \overline{\pi}(b+1, c - b)} \\ + (\alpha_{c+1} - \alpha_b) \frac{\lambda'(m, a, b, c+1)}{\overline{\pi}(\alpha || m, a - m) \overline{\pi}(b+1, c - b+1)}$$

$$1. (z - \alpha_b) \lambda'(m, a, b+1, c+1) = (z - \alpha_{c+1}) \lambda'(m, a, b, c) \\ + (\alpha_{c+1} - \alpha_b) \lambda'(m, a, b, c+1)$$

$$\frac{\lambda'(m, a, b, c+1)}{(m, a - m)(b+1, c - b+1)} = \frac{\lambda'(m, a+1, b, c)}{(m, a - m+1)(b+1, c - b)} + \frac{(\alpha_m - \alpha_{m+1}) \lambda'(m, a+1, b, c+1)}{(m, a - m+1)(b+1, c - b+1)}$$

$$2. (z - \alpha_a) \lambda'(m, a, b, c+1) = (z - \alpha_{c+1}) \lambda'(m, a+1, b, c) + (\alpha_{c+1} - \alpha_{m+1}) \lambda'(m, a+1, b, c+1)$$

$$\frac{\lambda'(m, a+1, b+1, c)}{(m, a - m+1)(b+2, c - b-1)} = \frac{\lambda'(m, a, b, c)}{(m, a - m)(b+1, c - b)} + \frac{(\alpha_{m+1} - \alpha_b) \lambda'(m, a+1, b, c)}{(m, a - m+1)(b+1, c - b)}$$

$$3. (z - \alpha_{b+1}) \lambda'(m, a+1, b+1, c) = (z - \alpha_a) \lambda'(m, a, b, c) + (\alpha_a - \alpha_b) \lambda'(m, a+1, b, c)$$

$$\frac{(\alpha_{c+1} - \alpha_{a+1}) \lambda'(m, a+1, b+1, c+1)}{(m, a - m+1)(b+2, c - b)} = \frac{(\alpha_b - \alpha_{m+1}) \lambda'(m, a+1, b, c)}{(m, a - m+1)(b+1, c - b)} \\ + \frac{(\alpha_{m+1} - \alpha_b) \lambda'(m, a, b, c+1)}{(m, a - m)(b+1, c - b+1)}$$

$$4. (z - \alpha_{b+1})(\alpha_{ch} - \alpha_m) \lambda'(m, a+1, b+1, ch) =$$

$$(z - \alpha_{c+1})(\alpha_b - \alpha_m) \lambda'(m, a+1, b, c) + (z - \alpha_a)(\alpha_{ch} - \alpha_b) \lambda'(m, a, b, ch)$$

$$\frac{\lambda'(m, a, b, ch)}{(m+1, a-m-1)(b+1, c-b)} = \frac{\lambda'(m, a, b, c)}{(m, a-m)(b+1, c-b)} + \frac{(\alpha_{ch} - \alpha_m) \lambda'(m, a, b, ch)}{(m, a-m)(b+1, c-b)}$$

$$5. (z - \alpha_m) \lambda'(m+1, a, b, ch) = (z - \alpha_{c+1}) \lambda'(m, a, b, c) + (\alpha_{ch} - \alpha_m) \lambda'(m, a, b, ch)$$

$$\frac{\lambda'(m, a, b, c)}{(m+1, a-m-1)(b+1, c-b)} = \frac{\lambda'(m, a, b+1, c)}{(m, a-m)(b+2, c-b-1)} + \frac{(\alpha_b - \alpha_m) \lambda'(m, a, b, c)}{(m, a-m)(b+1, c-b)}$$

$$6. (z - \alpha_m) \lambda'(m, a, b, c) = (z - \alpha_{b+1}) \lambda'(m, a, b+1, c) + (\alpha_b - \alpha_m) \lambda'(m, a, b, c)$$

$$\frac{(\alpha_{ch} - \alpha_m) \lambda'(m, a, b+1, ch)}{(m, a-m)(b+2, c-b)} = \frac{(\alpha_b - \alpha_m) \lambda'(m, a, b, c)}{(m, a-m)(b+1, c-b)} + \frac{(\alpha_{c+1} - \alpha_b) \lambda'(m+1, a, b, ch)}{(m+1, a-m-1)(b+1, c-b+1)}$$

$$7. (z - \alpha_{b+1})(\alpha_{ch} - \alpha_m) \lambda'(m, a, b+1, ch) = (z - \alpha_{c+1})(\alpha_b - \alpha_m) \lambda'(m, a, b, c)$$

$$+ (z - \alpha_m)(\alpha_{ch} - \alpha_b) \lambda'(m+1, a, b, ch)$$

$$\frac{\lambda'(m+1, a+1, b, c)}{(m+1, a-m)(b+1, c-b)} = \frac{\lambda'(m, a, b, c)}{(m, a-m)(b+1, c-b)} + \frac{(\alpha_{a+1} - \alpha_m) \lambda'(m, a+1, b, c)}{(m, a-m+1)(b+1, c-b)}$$

$$8. (z - \alpha_m) \lambda'(m+1, a+1, b, c) = (z - \alpha_a) \lambda'(m, a, b, c) + (\alpha_{ch} - \alpha_m) \lambda'(m, a+1, b, c)$$

$$\frac{(\alpha_{c+1} - \alpha_{ch}) \lambda'(m+1, a+1, b, ch)}{(m+1, a-m)(b+1, c-b+1)} = \frac{(\alpha_{ch} - \alpha_m) \lambda'(m, a, b, ch)}{(m, a-m)(b+1, c-b+1)} + \frac{(\alpha_m - \alpha_{a+1}) \lambda'(m, a+1, b, c)}{(m, a-m+1)(b+1, c-b)}$$

$$9. (z - \alpha_m)(\alpha_{c+1} - \alpha_{ch}) \lambda'(m+1, a+1, b, ch) =$$

$$(z - \alpha_a)(\alpha_{c+1} - \alpha_m) \lambda'(m, a, b, ch) + (z - \alpha_{c+1})(\alpha_m - \alpha_{ch}) \lambda'(m, a+1, b, c)$$

$$\frac{(\alpha_{ch} - \alpha_m) \lambda'(m, a+1, b+1, c)}{(m, a-m+1)(b+2, c-b-1)} = \frac{(\alpha_b - \alpha_m) \lambda'(m, a, b, c)}{(m, a-m)(b+1, c-b)} + \frac{(\alpha_{ch} - \alpha_b) \lambda'(m, a+1, b, c)}{(m+1, a-m)(b+1, c-b)}$$

$$10. (z - \alpha_{b+1})(\alpha_{ch} - \alpha_m) \lambda'(m, a, b+1, c) = (z - \alpha_a)(\alpha_b - \alpha_m) \lambda'(m, a, b, c)$$

$$+ (z - \alpha_m)(\alpha_{a+1} - \alpha_b) \lambda'(m+1, a+1, b, c)$$

w) With $r > 0$, set $a = m+r$, $b = m+r+1$, $c = m+r+2$ and, with $z \in K$,

$$\lambda'(r \not\in \mathbb{Q}, m, a, b, c) = \frac{\lambda(r: \alpha(m; i, j) | z)}{\alpha_{m+r} - \alpha_{m+r}}$$

(In all but relationships (3, 6, 8, 10) to follow it is assumed that $|z| > r+m+i$)

g) The following homogeneous three term relationships are satisfied.

- 1) $(z - \alpha_b) \lambda'(m, a, b+1, c+1 | z) = (z - \alpha_{c+1}) \lambda'(m, a, b, c | z)$
 $+ (\alpha_{c+1} - \alpha_b) \lambda'(m, a, b, c+1 | z)$
- 2) $(z - \alpha_a) \lambda'(m, a, b, c+1 | z) = (z - \alpha_{c+1}) \lambda'(m, a+1, b, c | z)$
 $+ (\alpha_{c+1} - \alpha_{a+1}) \lambda'(m, a+1, b, c+1 | z)$
- 3) $(z - \alpha_{b+1}) \lambda'(m, a+1, b+1, c | z) = (z - \alpha_a) \lambda'(m, a, b, c | z)$
 $+ (\alpha_{a+1} - \alpha_b) \lambda'(m, a+1, b, c | z)$
- 4) $(z - \alpha_{b+1})(\alpha_{c+1} - \alpha_{a+1}) \lambda'(m, a+1, b+1, c+1 | z) =$
 $(z - \alpha_{c+1})(\alpha_b - \alpha_{a+1}) \lambda'(m, a+1, b, c | z) + (z - \alpha_a)(\alpha_{c+1} - \alpha_b) \lambda'(m, a, b, c+1 | z)$
- 5) $(z - \alpha_m) \lambda'(m+1, a, b, c+1 | z) = (z - \alpha_{c+1}) \lambda'(m, a, b, c | z)$
 $+ (\alpha_{c+1} - \alpha_m) \lambda'(m, a, b, c+1 | z)$
- 6) $(z - \alpha_m) \lambda'(m+1, a, b, c | z) = (z - \alpha_{b+1}) \lambda'(m, a, b+1, c | z)$
 $+ (\alpha_b - \alpha_m) \lambda'(m, a, b, c | z)$
- 7) $(z - \alpha_{b+1})(\alpha_{c+1} - \alpha_m) \lambda'(m, a, b+1, c+1 | z) =$
 $(z - \alpha_{c+1})(\alpha_b - \alpha_m) \lambda'(m, a, b, c | z) + (z - \alpha_m)(\alpha_{c+1} - \alpha_b) \lambda'(m+1, a, b, c+1 | z)$
- 8) $(z - \alpha_m) \lambda'(m+1, a+1, b, c | z) = (z - \alpha_a) \lambda'(m, a, b, c | z)$
 $+ (\alpha_{a+1} - \alpha_m) \lambda'(m, a+1, b, c | z)$

$$9) (z - \alpha_m)(\alpha_{c+1} - \alpha_m) \lambda'(m+1, a+1, b, c+1 | z) =$$

$$(z - \alpha_a)(\alpha_{c+1} - \alpha_m) \lambda(m, a, b, c+1 | z) + (z - \alpha_{c+1})(\alpha_m - \alpha_m) \lambda'(m, a+1, b, c | z)$$

$$10) (z - \alpha_{b+1})(\alpha_{a+1} - \alpha_m) \lambda(m, a+1, b+1, c | z) =$$

$$(z - \alpha_a)(\alpha_b - \alpha_m) \lambda(m, a, b, c | z) + (z - \alpha_m)(\alpha_{a+1} - \alpha_b) \lambda'(m+1, a+1, b, c | z)$$

- b) The result of subclauses (ivb) of Th. applies, mutatis mutandis, here and a result analogous to that of subclause (ivd) of that theorem holds.
 c) Relationships corresponding to (1-10) above hold for the functions

λ_{fun}

$$\lambda(r; m, a, b, c) = \lambda($$

values $\lambda(r; \alpha || m; i, v | z)$. They are, except in the case of relationship (5), more complicated than those given. In the exceptional case, the relationship in question may be presented as

$$\lambda(r; \alpha || m; i, v | z) =$$

$$\lambda(r; \alpha || m; i+1, v+1 | z) =$$

$$\frac{(z - \alpha_m) \lambda(r; \alpha || m+1; i, v | z) - (z - \alpha_{m+1}) \lambda(r; \alpha || m; i, v+1 | z)}{\alpha_{m+1} - \alpha_m}$$

\Rightarrow p 280

$r, m, i \in K$, $\alpha \in \text{seq}^*(K | \Delta_{\text{minimality}})$. Def. Z... p274.

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$\epsilon [K|i]$

i) Let $e, f \in \text{seq}(K | \Delta_{\text{minimality}})$ $\delta(r: \dots) \in [K|i]$

ii) $Z[r: \alpha || m, [i]]^0 = I[i]^0$

iii) $\delta(r=0)$, $Z[0: \alpha || m, [i]] = I[i]$

iv) ($i=2$). $Z[r: \alpha || m, [2]]$ has the diagonal form

$[1, \alpha_m | \alpha_{m+1} | \dots] \in [K|2]$ p274

v) ~~Another~~ With $\delta(r: e, f \in \text{seq}(K | \Delta_{\text{minimality}}))$ define $A[r: \dots] \in [K|2]$ by setting p. 274

a) $(A[r: \dots])_2^0 = \delta(r: exf; \alpha || m, 2)$

b) $(A[r: \dots])_2^1 =$

$$\frac{(\alpha_m - \alpha_{m+2})e_m f_{m+2} + (\alpha_m - \alpha_{m+1})e_{m+1} f_m}{(\alpha_{m+2} - \alpha_{m+1})(\alpha_m - \alpha_{m+1})} \dots \text{ p275}$$

This expression reduces to

$$\delta(r: exf; \alpha || m, 3, 1) = \frac{e_m f_{m+2} - e_{m+1} f_m}{\alpha_{m+2} - \alpha_{m+1}}$$

when $r=0$. Otherwise the relationship

$$A[r: exf; \alpha || m, [2]] =$$

$$A[r: f; \alpha || m, [2]] Z[r: f; \alpha || m, [2]] A[r: e; \alpha || m, [2]]$$

is false

Let $r, m, \alpha \in \bar{\mathbb{N}}$ and $\omega \in \text{seq}'(K \models \Delta_{\text{min}})$. Define

$Z[n \omega || m, [i]] \in [K|i]$ by setting

$$Z[r: \alpha || m, [i]] := \left\{ X[r: \alpha || m, [i]]^T \right\}^{-1} d \left[\mu(r: \alpha || m, i) \begin{bmatrix} [i] \\ \vdots \\ [m] \end{bmatrix} \right] \\ \left\{ X[r: \alpha || m, [i]] \right\}^{-1}$$

i) Let $e, f \in \text{seq}(K \models \Delta_{\text{min}})$.

$$S(r: exf; \alpha || m, i) = \text{row}[f[m, mi]] d \left[\mu(r: \alpha || m, i) \begin{bmatrix} [i] \\ \vdots \\ [m] \end{bmatrix} \right] \text{col}[e[m, mi]] \\ = \text{row}[S(r: f; \alpha || m \supset, i \rightarrow)] Z[r: \alpha || m, [i]] \text{col}[S(r: e; \alpha || m, i)]$$

ii) $Z[r: \alpha || m, [i]]^0 = I[i]$

iii) ($r=0$). $Z[0: \alpha || m, [i]] = I[i]$

iv) ($i=2$). $Z[r: \alpha || m, [2]]$ has the diagonal form

$$\left[1, \frac{\alpha_m + \alpha_{mn} - \alpha_{mm} - \alpha_{mnr+2}}{\alpha_m - \alpha_{mnr+2}}, 1 \right]$$

v) With $e, f \in \text{seq}(K \models \Delta_{\text{min}})$ define $A[r: f; \alpha || m, [i]] \in \mathcal{L}[K|i]$ by setting

$$A[r: f; \alpha || m, [i]] := \mathcal{L}[S(r: f; \alpha || m \supset, i \rightarrow)]$$

a) ($i=2$). Define $\hat{Z} \in \mathcal{L}[K|2]$ by setting

~~$$\hat{Z} = A[r: f; \alpha || m, [i]] Z[r: \alpha || m, [i]] A[r: e; \alpha || m, [2]]$$~~

a) $\hat{Z}_2^0 = S(r: exf; \alpha || m, 2)$

b) ~~$\hat{Z}_2^0 = (\alpha_m - \alpha_{mnr+2}) e_{m+2} f_{m+2} + (\alpha_{mn} - \alpha_{mm}) e_{mn} f_{m+2}$~~

$$\hat{Z}_2^1 = \frac{-(\alpha_m + \alpha_{mn} - \alpha_{mm} - \alpha_{mnr+2}) e_{mn} f_{m+1}}{(\alpha_{mnr+2} - \alpha_{mn})(\alpha_m - \alpha_{mnr+2})}$$

This expression reduces to

$$\delta(r: \text{exp}; \alpha || m, i) = \frac{\alpha_{m+2} f_{m+2} - \alpha_m f_m}{\alpha_{m+2} - \alpha_m}$$

when $r=0$. Otherwise, the relationship

$$A[r: \text{exp}; \alpha || m, [2]] =$$

$$A[r: f; \alpha || m, [2]] \neq [r: f; \alpha || m, [2]] A[r: e; \alpha || m, [2]]$$

is false.

Let $r, m, i \in \mathbb{N}$ such that $r \neq 0$

i) ($r=f$). Let $\alpha \in \text{seq}'(K | \geq m)$ and $f \in \text{seq}(K | \geq m)$

$$\delta(0: f; \alpha || m, i) = \delta(f; \alpha || m, i)$$

ii) a) p. 260 b) p. 261

iii) With $\alpha \in \text{seq}'(K | \geq m+1)$ and $f \in \text{seq}(K | \geq m+1)$, divided differences satisfy the following recursion

$$\delta(r, f; \alpha || m, i+1) = \frac{\delta(r, f; \alpha || m+1, i) - \delta(r, f; \alpha || m, i)}{\alpha_{m+1} - \alpha_m}$$

iv) a+b) p. 262

Remark on p. 262 followed by

The result of clause (iii) follows directly from clause (v) of Th.

Let $r, m, i, \alpha \in \bar{\mathbb{N}}$ and $\alpha \in \text{seq}'(\mathbb{K} | \geq r+m+i)$

ii) ($r=0$)

$$\lambda(0 : \alpha || m ; i, \omega) = \lambda(\alpha || m ; i, \omega) \quad \langle \mathbb{K} \rangle$$

b) $\lambda(r : \alpha || m ; i, \omega) = \frac{\pi(\alpha || m || i, i)}{\pi(\alpha || m || i, i | \alpha_m)} \quad \langle \mathbb{K} \rangle$

c) $\lambda(r : \alpha || m ; i, i) = \frac{\pi(\alpha || m, i)}{\pi(\alpha || m, i | \alpha_{m+i})} \quad \langle \mathbb{K} \rangle$

ii) $\lambda(r : \alpha || m ; i, \omega | \alpha_\omega) = 0 \quad \langle \omega := [m, m] + (m+i, m+i] \rangle$

and

$$\lambda(r : \alpha || m ; i, \omega | \alpha_\omega) + 0 \quad \langle z := \mathbb{K} \setminus \alpha [[m, m] + (m+i, m+i]] \rangle$$

iii) $\left\{ \sum \lambda(r : \alpha || m ; i, \omega) \langle \omega := [i] \rangle \right\} = 1 \quad \langle \mathbb{K} \rangle$

and, in particular, with $x \in [r+i]$,

$$\left\{ \sum \lambda(r : \alpha || m ; i, \omega | \alpha_{m+i}) \langle \omega := [i] \cap [x-r, x] \rangle \right\} = 1$$

i) pp. 267, 268

$$\begin{aligned} & (\alpha_{m+i} - \alpha_m) \sum \lambda(r : \alpha || m ; i+1, \omega | z) \langle \omega := [i] \rangle \\ &= \cancel{\sum} \{ (z - \alpha_m) \{ \sum \lambda(r : \alpha || m+1 ; i, \omega | z) - \cancel{\sum} \{ (z - \alpha_{m+i}) \sum \lambda(r : \alpha || m ; i, \omega | z) \langle \omega := [i] \rangle \}} \end{aligned}$$

remark:

The simplest proof of the first result of clause (iii) follows
involves recursive makes use of the concluding result of subclause (ivc) of Th. :

$$\left\{ \sum \lambda(r : \alpha || m ; i+1, \omega | z) \langle \omega := [i] \rangle \right\}$$

$$\left\{ (z - \alpha_m) \left[\sum \lambda(r : \alpha || m+1 ; i, \omega | z) \langle \omega := [i] \rangle \right] - (z - \alpha_{m+i}) \left[\sum \lambda(r : \alpha || m ; i, \omega | z) \langle \omega := [i] \rangle \right] \right\}$$

$\alpha_{m+i} - \alpha_m$

$$= \frac{z - \alpha_m - z + \alpha_{m+1}}{\alpha_{m+1} - \alpha_m} = 1$$

Let $r, m, i \in \mathbb{R}$, and $\alpha \in \text{seq}'(K | \geq mri)$ and $f \in \text{seq}(K | \geq mri)$

\rightarrow Let f

i) Let the mapping system $\Delta : [m, mi] \rightarrow \{K \rightarrow K\}$ possess the interpolation properties

$$\Delta(\omega | \alpha[\omega, r+\omega]) = f[\omega, r+\omega]$$

for $\omega := [m, mi]$. Define the mapping $\tilde{\Delta} : K \rightarrow K$ by setting

$$\tilde{\Delta} := \left\{ \sum_i^{\infty} \lambda(r; \alpha || m; i, \omega) \Delta(\omega) \mid \omega := [i] \right\}$$

$\tilde{\Delta}$ possesses the interpolation property

$$\tilde{\Delta}(\alpha[m, mri]) = f[m, mri].$$

Let $r, m, i \in \mathbb{R}$

i) Let $\alpha \in \text{seq}'(K | \geq mi)$ and $f \in \text{seq}(K | \geq mi)$

$$\Delta(\circ : f; \alpha || m; i) = \Delta(f; \alpha || m; i)$$

ii) Let $\alpha \in \text{seq}'(K | \geq r+mi)$ and $f \in \text{seq}(K | \geq r+mi)$. Delayed Lagrange forms satisfy the following recursion

$$\Delta(r; f; \alpha || m; i+1 | z) =$$

$$\frac{(z - \alpha_m) \Delta(r; f; \alpha || m+1; i | z) - (\lambda - \alpha_{m+1}) \Delta(r; f; \alpha || m; i | z)}{\alpha_{m+1} - \alpha_m}$$

$$\alpha_{m+1} - \alpha_m$$

iii) pp 263, 264

iv " "

remark on p. 264

70.

$$\bar{\pi}(r; \alpha || m; i, \chi) = \bar{\pi}(\alpha || m || r) \sum_i S(\bar{\pi}(\alpha || m, \omega); \omega || m + \chi, r) \pi(\alpha || m || r, i - \omega)$$

(where $\omega := [x+r]$) ??

83.

$$\bar{\pi}(r; \alpha || m; i, \omega) = \sum_i S(\bar{\pi}(r; \alpha || m; i, \omega) \langle \chi \rangle; \alpha || m, \omega) \pi(\alpha || m, \omega)$$

$\langle \omega := \omega, i \rangle$

terms with $\omega < 0$ vanish due to presence of $\bar{\pi}(\alpha || \omega)$ in l.h.s

89.

$$\lambda(\alpha || m; i || r, \omega | z) = \frac{z - \alpha_{m+i}}{\alpha_{m+2} - \alpha_{m+i}} \lambda(\alpha || m; i, \omega | z)$$

96.

Delayed versions of interpolatory formulae

Def's. $\mu(r; \alpha || m; i, \omega)$, $S(r; f; \alpha || m; i)$, $\bar{\pi}(r; \alpha || m; i, \omega)$, $\Delta(r; \theta(\alpha || m, j))$, $\phi(\alpha || m, j)$

255.

Th. Theory of $\mu(r; \alpha || m; i, \omega)$

98.

Preliminary remarks on Th. of the section ($r=0$) etc.

101.

Th. Theory of $\bar{\pi}(r; \alpha || \dots)$

279 280 281 282 remark on p 284

Th. Theory of $S(r; f; \alpha || m; i)$

280 287 288

Th. Theory of $\Delta(r; \alpha || \dots)$

281

Th. Interpolating properties of $\bar{\pi}(r; \alpha || \dots)$

281 283 284

Th. Theory of $\Lambda(r; f; \alpha || \dots)$

282.

131.

Preliminary remark on rotation matrices $X[r; \alpha] \dots$

283

106.

Th. of $X[r; \alpha(m, L)] := \mathcal{L}[g(r; \alpha(m; z, \omega))]$

132 Remark on $Y[\dots]$

108 156

Th. of $Y[r; \alpha(m, L)] := \mathcal{L}[\lambda(r; \alpha(m; z, \omega))]$

122.

Th. $\pi(\alpha(m, i)) \{ \Lambda(f; \alpha(m, i)) - \Lambda(f; \alpha(m, i-1)) \} = \dots$

p.212

Preliminary remark on $\Pi(j|z) \Delta(j|z) = \dots$ (give example to make clearer)

130.

Th. $\Pi(j|z) \Delta(j|z) = \dots$

137

$\delta(f \times \theta(m, z); \alpha(m, z+r)) = \delta(f; \alpha(m+r, r))$ proving $X[r; \alpha(m, L)]^{-1}$

$\text{col}[\delta(f; \alpha(m, z+r))] =$

$\text{col}[\delta(f; \alpha(m, r))]$

Hence

$$\phi(\alpha(n, j)) = \delta(r; \phi(\alpha(\langle [n] \rangle; r), r); \alpha(n, j-r-1)) \quad r \in L_j$$

$$= \delta(\phi(\alpha(\langle [n] \rangle, 1); \alpha(n, j-1))$$

$$\phi(\alpha(n, j|k)) = \delta(r; \phi(\alpha(\langle [\omega] \rangle-1, r+1|k); \alpha(n, j-r-1))$$

$$= \delta(\phi(\alpha(\langle [n+j] \rangle, 1|k); \alpha(n, j-1))$$

141

Remark on proof of [1] p 106

142:

$$g(i) : \mathbb{K} \rightarrow \mathbb{K} : g(i) = \sum_i \pi(\alpha(m, \omega)) \langle \omega : = [i] \rangle$$

$$\delta(g(i|\alpha)); \alpha(m+r, \omega) = \sum \delta(\pi(\alpha(m, \omega | \langle \omega \rangle); \alpha(m+r, \omega)) \langle \omega : = [\omega, z_{\omega}] \rangle$$

$$z := [i] \quad \omega := [i-\epsilon], \quad \delta(g(i); \alpha(m, \omega)) = 1 \quad \langle \omega : = [i] \rangle$$

$$\alpha(z) = 1 + \pi(\alpha(m, 1)) g(i-1)$$

150 152 153 154 155

289

$$\begin{aligned} Y^{-1}Y, \quad & \tilde{Y} \text{ col } [\Lambda(f; \alpha || m+z, m-z)] \quad \tilde{Y} \text{ col } [\Lambda(f; \alpha || m, m)] \\ \tilde{Y}Y, \quad & \tilde{Y} \text{ col } [\Lambda(f; \alpha || m+z, m-z)] \quad \tilde{Y}Y \end{aligned}$$

157

$$\theta(\alpha, \beta || a, b) \wedge \theta(\alpha, \beta || a \wedge b, c) \dots \quad (\text{superseded})$$

158

$$\prod_{\lambda=1}^n \theta(\alpha || \{\sum_i \lambda(\omega) < \omega : = [x] \}, \nu(x)) = \theta(\alpha || \lambda(\omega), \{\sum_i \lambda(\omega) < \omega : = [n]\})$$

160

$$\text{Defn of } \theta(\alpha, \beta || n, j) \quad \phi(\alpha, \beta || n, j)$$

161

Th. θ as a filter

$$\delta(r, \theta || m, 1) =$$

$$\delta(r, \prod (\theta(\alpha, \beta) || \dots$$

$$\prod (\theta(\alpha, \beta) || \dots \quad \prod (\phi(\alpha, \beta) || \dots$$

162 4

$$\theta(\dots) \phi(\dots)$$

$$\delta(r, \phi(\alpha, \beta) || \dots$$

163 164

Th. prod. \Rightarrow orthog groups of μ

Th. $S(\text{exf})$ and proto.

167.

$$\text{Th. } \pi(\alpha || m, z+1 | z) \delta(f \times \frac{1}{z - \langle \alpha \rangle}; \alpha, \dots) = \Lambda$$

Th. $\Lambda(\text{exf})$

172.

Th. $f_{m+n} = 0 \langle \omega := l_j \rangle \Rightarrow \delta(f) = \delta(\phi \times f \dots)$

172 173 174

$e_{mn} = 0 \langle \omega := l_j \rangle \Rightarrow \delta(exf \dots) = \dots$

(use $g \in \text{pump} \dots$; take reverse order from ω ?)

174.

$\Delta(f \dots) = \sum_i \delta \dots \pi_i \dots$

$\Delta(f \dots i) - \Delta(f \dots j) = \dots$

175.

Th. $f_{m+n} = 0 \langle \omega := l_j \rangle \Rightarrow \Delta(f) = \dots$

175 176 177 178

$e_{mn} = 0 \langle \omega := l_j \rangle \Rightarrow \Delta(exf \dots)$

(as for 172 - 174 above)

187.

Th. $\alpha[m, m+i] = z + \beta[n, n+i] : \theta(\alpha || m; i, \omega) = \dots$

$\lambda(\alpha || \dots)$

188.

Th. $\delta(\theta(\alpha || m; i); \alpha || m+k, r) = \dots$

(consider defining $\theta(\alpha || m, r)$ with $r < 0$ as zero sequence)

189

Th. $\delta(\delta(\delta(\alpha \langle \alpha \rangle, \beta \rangle) \dots)$

Th. $\delta(\phi(\alpha || m, i+1); \alpha || m, k) \dots$

Th. $\pi(\dots z') - \frac{\pi(\dots z)}{z' - z}$

190.

Th. def. of $\frac{\Delta(\dots \langle p \rangle)}{\pi(\dots \langle p \rangle)}$

$$\begin{aligned}\lambda(\alpha || m; i, \omega) &= \sum_j \mu(\alpha || m; \omega, j) \pi(\alpha || m, \omega) \langle \omega := [j, i] \rangle \\ &= \sum_j \mu(\alpha || m + \omega; i - \omega, j - \omega) \pi(\alpha || m + \omega, i - \omega) \langle \omega := [j, i] \rangle\end{aligned}$$

$$\Rightarrow \mu(\alpha || m; i, \omega) = \delta(\lambda(\alpha || m; i, \omega | \langle \alpha \rangle); \alpha || m, \omega)$$

$$\text{and } \mu(\alpha || m + \omega; i - \omega, j - \omega) \delta(\lambda(\alpha || m; i, \omega | \langle \alpha \rangle); \alpha || m + \omega, i - \omega)$$

199. Remark on allocation notation

198, 199

Defn. $M[\alpha || m, [i]]$, $\tilde{M}[\dots]$

$$\underline{\Lambda}[a || m, [i]] = L[\lambda(\alpha || m; i, \omega)], \quad \tilde{\Lambda}[\dots] = L[a || m; i - \omega, j - \omega]$$

$$A[f; \alpha || m, [i]] = L[\delta(f; \alpha || m, i, \omega)]$$

$$\tilde{\Lambda}[\dots] = [\underline{\Lambda}(f; \alpha || m, i, \omega) - \underline{\Lambda}(f; \alpha || m + \omega, i - \omega)]$$

$$\hat{\Lambda}[\dots] = L[\dots]$$

$$\mu(\alpha || m; i) \quad \tilde{\Lambda}(\alpha || m; i)$$

199. 200

$$\text{Th. } M[-]^{-1} \quad \tilde{M}[\dots]^{-1} \quad M = d[\alpha \dots] \quad \tilde{M} =$$

$$\hat{\Lambda}[\dots]^{-1} \dots$$

201.

$$\text{Th. } A := M d[\alpha \dots] M^{-1} = \tilde{M}^{-1} d[\alpha \dots] \tilde{M}$$

$$\underline{\Lambda} \quad \hat{\Lambda}$$

202

$$\text{Th. } \underline{\Lambda} = d[\alpha \dots] M d[\alpha \dots]^{-1} \quad \tilde{\Lambda} \quad \hat{\Lambda} = \dots A \dots$$

203

$$\text{Th. } \mathbb{F}: \text{ring of } f[m_1, m_2] \quad A[f \dots] \quad F' \quad \underline{\Lambda}: \mathbb{F} \quad \tilde{\Lambda}: \mathbb{F} \quad \hat{\Lambda}: \mathbb{F} \quad A \times B = A \sqcup [i:j] B$$

204 207-208

$$\text{Defn. } A]_{[i:j]} =]^{[i:j]} \text{ ann. } \quad B]_{[i:j]} =]^{[i:j]} \text{ perm}$$

205 207 208.

Th. Prop of anne perm

209.

Th. Deviation of one similarity transformation from another

212.

Remark on subtraction products derived from $X = YZ$ Y, Z triangular

210

Def $A[f; \alpha || m, k, h, [i, j]] \overset{\sim}{\Lambda} \overset{\wedge}{\Lambda}$

210

Th. Subtraction prod. of triangular mat. $\Rightarrow A, \overset{\wedge}{\Lambda}, \overset{\wedge}{\Delta}$

$$A[e \circ f \dots] = A[e \dots] A[f \dots] \overset{\sim}{\Lambda} \overset{\wedge}{\Delta}$$

221

$$\prod_{x=0}^{j-1} \pi(\alpha || m, i | \beta_{n+x}) = (-1)^{i+j} \prod_{x=0}^{j-1} \pi(\beta || n, j | \alpha_{m+x})$$

262.

Def pp($K|n$) (prop($K|n$)?)

263.

$$\pi(r; \alpha || m; i, o) = \pi(\alpha || m; i) \quad \pi(r; \alpha || m; i, i) = \pi(\alpha || m, i)$$

278.

$$\text{Th. } m \in Z[\dots] = \left\{ \left[\dots \right]^T \right\}^{-1} d[\mu(\dots)] \times [\dots]^{-1}$$

287.

Results from earlier notes

bp 180

Def. Π for $[..]$, L , UL for lower triangular with unit diagonal
 U \rightarrow diag., $UL\Delta$ for unit lower diagonal; $c \uparrow$ for col. row;
 V^χ for χ -fach Verschiebungspunktor

bp 180 (237) $\prod_{\omega} A_\omega \langle \omega := [r] \rangle = A_r A_{r-1} \dots A_0$
 $\xrightarrow{(239)}$

~~ULΔ~~

bp 180 (239) \downarrow ✓ see p. 304
 $\{UL\Delta [z_\chi] \langle \chi := [i] \rangle\}^{-1} = UL \left[(-1)^{\frac{z_i - i}{2}} \prod_{\omega} z_\omega \langle \omega := (\omega, z) \rangle \right] \langle z \rangle_{\text{odd}}$

bp 179 (209, 210) (see p. 314)

$$\begin{aligned}\pi(\alpha // m+1, r) &= \sum_i \pi(\alpha // m+\omega+1, r-\omega/\alpha_m) \pi(\alpha // m, \omega) \langle \omega := [r] \rangle \\ \pi(\alpha // m, r) &= \sum_i \pi(\alpha // m, \omega | \alpha_{m+r}) \pi(\alpha // m+\omega+1, r-\omega) \langle \omega := [i] \rangle\end{aligned}$$

from which (212)

$$\begin{aligned}\frac{1}{z - \alpha_m} &= \sum_i \frac{\pi(\alpha // m+\omega+1, r-\omega/\alpha_m)}{\pi(\alpha // m+\omega, r-\omega+1/z)} \langle \omega := [r] \rangle \quad r \in \overline{\mathbb{A}} \\ &= \sum_i \frac{\pi(\alpha // n-\omega, \omega/\alpha_m)}{\pi(\alpha // n-\omega-1, \omega+1/z)} \langle \omega := [n-m] \rangle \quad m \in \mathbb{N}, n \in \mathbb{N}\end{aligned}$$

(first formula implies

$$\begin{aligned}\delta(\pi(\alpha // m+1, r | \langle \alpha \rangle); \alpha // m, i) &= \delta(\theta(\alpha // m+1, r); \alpha // m, i) \\ &= \pi(\alpha // m+i+1, r-i/\alpha_m) \langle i := [1] \rangle\end{aligned}$$

206

$$\left\{ \prod \pi(\alpha // m, \omega | \beta_{k+\omega}) \langle \omega := [n] \rangle \right\} = \left\{ \prod \pi(\beta // k+\omega, n-\omega | \alpha_{m+\omega}) \langle \omega := [n] \rangle \right\} \Rightarrow p_{305}$$

reciprocity $\left\{ \prod \frac{\pi(\alpha // m, n | \beta_{k+\omega})}{\pi(\beta // k, n | \alpha_{m+\omega})} \langle \omega := [n] \rangle \right\} = (-1)^n$

$$\prod \pi(\alpha // m; i, \omega) \langle \omega := [i] \rangle = \overline{\pi}(\alpha // m; i)^{-1} \pi(\alpha // m, i+1)^i \quad \checkmark$$

$$\pi(\alpha // m; i, 0) : \neg \pi(\alpha // m; i, i)$$

product for exponentiation formula for $\pi(\alpha // m; i, \omega)$

$$\mu(\alpha // m; i, \omega) = \frac{1}{\pi(\alpha // m; i, \omega / \alpha_{m\omega})} \mu(\alpha // m; i, 0) \mu(\alpha // m; i, i)$$

(253)

$$\pi(\alpha // m + h + k + l, r - h - k | \alpha_{m+h+k})$$

$$\mu(\alpha // m + h; k, x) = \pi(\alpha // m, h | \alpha_{m+h+x}) \pi(r - h - k; m + h, k + l) \quad \checkmark$$

$$\mu(\alpha // m; r, h + x) ([h, h + k] \subseteq [r])$$

(298)

$$\left\{ \prod \mu(\alpha // m; r, \omega) \langle \omega := [r] \rangle \right\} = (-1)^{\frac{r(r)}{2}} \left\{ \prod \pi(\alpha // m, \omega | \alpha_{m\omega}) \langle \omega := [r] \rangle \right\}^{-2}$$

$$= (-1)^{\frac{r(r)}{2}} \left\{ \prod \pi(\alpha // m + \omega + 1, r - \omega | \alpha_{m+\omega}) \langle \omega := [r] \rangle \right\}^{-2}$$

$$= (-1)^{\frac{r(r)}{2}} \left\{ \prod \pi(\alpha // m + r - \omega + 1, \omega | \alpha_{m+r-\omega}) \langle \omega := [r] \rangle \right\}^{-2}$$

(↑ invariance of $\prod (\mu(\frac{1}{3}; \alpha // \dots))$?) (see p 366)

(from (250))

(243)

Def. 1) The notation $\frac{1}{3} \in O(h, k)$ indicates that $\frac{1}{3} \in \text{seg}(N / \Delta h, k)$ with

$$\frac{1}{3}[h+k] = [h, h+k] \quad \text{let } (*)$$

2) With $\frac{1}{3} \in O(h, k)$ be a formula involving $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots, f_1, f_2, \dots$ where $i, j \in [h, h+k]$ The result obtained from (*) by replacing i by $\frac{1}{3}(i)$, j by $\frac{1}{3}(j)$, ... is the $\frac{1}{3}$ -ordered form of (*) in the special case in which $\frac{1}{3} \in RO(h, k)$, the derived result is called the (h, k) reversed order form of (*)

(2) The notation $\frac{1}{3} \in ROD(h, k)$ indicates that $\frac{1}{3} \in O(h, k)$ and $\frac{1}{3}(\omega) = 2h+k-\omega$ $\langle \omega := [h, h+k] \rangle$

Def: $\alpha \in \text{seq}(K | \geq h+k) \Rightarrow \exists \in O(h, k) \quad [m, m+i] \subseteq [h, h+k]$

$$\pi(\xi : \alpha || m, i | z) = \overline{\Pi} (z - \alpha_{\xi(m+\omega)}) \langle \omega := [i] \rangle$$

$$\exists \in O(h, k) \quad \pi(\xi : \alpha || h, k) = \pi(\alpha || h, k) \quad \text{invariance}$$

$$\exists \in RO(h, k) \quad [m, m+i] \subseteq [h, h+k] \quad \pi(\xi : \alpha || h, i) = \overline{\Pi} (\alpha || 2h+k-m-i+1, i)$$

kp. 176

$$\exists \in O(h, k) : \pi(\xi : \alpha || m; i, \omega | z) = \overline{\Pi} (z - \alpha_{\xi(m+\omega)}) \langle \omega := [i] \rangle$$

$$\exists \in RO(h, k) \quad \pi(\xi : \alpha || m; i, \omega) = \pi(\alpha || 2h+k-m-i; i, i-\omega)$$

$$\exists \in O(h, k) \quad [m, m+i] \subseteq [h, h+k] \quad \omega \in [i]$$

$$\mu(\xi : \alpha || m; i, \omega) = \frac{1}{\{\Pi (\alpha_{\xi(m+\omega)} - \alpha_{\xi(m+\omega)}) \langle \omega := [i] - \omega \rangle\}}$$

$$\exists \in O(h, k) \quad \omega \in [k]$$

$$\mu(\xi : \alpha || h; k, \omega) = \mu(\alpha || h; k, \xi(h+\omega) - h)$$

$$\exists \in RO(h, k) \quad [m, m+i] \subseteq [h, h+k] \quad \omega \in [i]$$

$$\xi \mu(\xi : \alpha || m; i, \omega) = \mu(\alpha || 2h+k-m-i; i, i-\omega)$$

(195) kp. 175

$$\left\{ \mathcal{U} \mathcal{L} \mathcal{D} [z_x] \xi \langle x := [i] \rangle \right\} \left\{ \mathcal{L} [(-1)^{x_\omega}] \overline{\Pi} z_\omega \langle \omega := [j], \tau \rangle \right\} \left\langle z_\omega = [i] \right\rangle = \mathbb{I}[i]$$

$$\left\{ \mathcal{U} \mathcal{L} \mathcal{D} \left[\alpha_{m+x-i} z \right] \langle x := [i] \rangle \right\} \left\{ \mathcal{L} [\pi(\alpha || m+i, \tau - \omega | z)] \right\} \left\langle \omega := [i] \right\rangle = \mathbb{I}[i]$$

-(249)

$$\left\{ \mathcal{U} [\pi(\alpha || m, \tau | \alpha_{m+\omega})] \right\} \left\langle z_\omega = [i] \right\rangle \left\{ \mathcal{U} [\mu(\alpha || m; \omega, \tau)] \right\} \langle .. \rangle = \mathbb{I}[i]$$

(195)

$$\mathcal{L} [\pi(\alpha || m, \tau - \omega | \alpha_{m+i-\omega})] [i] \mathcal{L} [\mu(\alpha || m; \tau - \omega, i - \omega)] = \mathbb{I}[i]$$

$$\mathcal{L}[\pi(\alpha \parallel m + z, i - z \mid \alpha_{m+z})] \mathcal{L}[\mu(\alpha \parallel m + z; i - z)] = I[z] \quad \checkmark$$

(221) by 125

$$\pi(\alpha \parallel m, z) \stackrel{x}{=} (\alpha_{m+z} - \alpha_m) \pi(\alpha \parallel m+1, z-1) \stackrel{x}{=} + \pi(\alpha \parallel m+1, z) \stackrel{x}{=} \langle K \rangle \quad \text{displacement:}$$

$$U \mathcal{L} \mathcal{D} [\alpha_{m+z} - \alpha_m] \text{ col} [\pi(\alpha \parallel m+1, z \mid z)] = \text{ col} [\pi(\alpha \parallel m, z \mid z)] \quad \checkmark$$

$$\text{col} [\pi(\alpha \parallel m+1, z) \stackrel{x}{=}] = \mathcal{L}[\pi(\alpha \parallel m+1, z-1 \mid \alpha_m)] \text{ col} [\pi(\alpha \parallel m, z) \stackrel{x}{=}] \quad \checkmark$$

Def of $\delta(f; \alpha \dots)$, recursion, also diff of $\frac{d}{dx}$ deg. prod. = 0

$$\begin{aligned} \mathcal{L} \mathcal{D} [\alpha_{m+z} - \alpha_m] &\Rightarrow \mathcal{L}[\pi(\alpha \parallel m+z, z-z \mid z)] = I[z] \quad (m = m+z?) \\ &\text{divided diff of prod.} \quad (x(-1) = z) \end{aligned} \quad \textcircled{3}$$

$$\delta(\overset{(w)}{\underset{\omega}{\prod}} \overset{(w)}{\underset{\omega}{\delta}} \langle \omega := [k \dots 1] \rangle; \alpha \parallel m, \vec{x}) = \quad \text{see p. 308}$$

$$\sum_i^z \sum_{x(0)}^{x(i)} \dots \sum_i^{x(k-i)} \delta(\overset{(w)}{\underset{\omega}{\delta}}; \alpha \parallel m, x(k)) \underset{\omega \in \omega}{\prod} \quad \checkmark$$

$$\underset{\omega \in \omega}{\prod} \delta(\overset{(w)}{\underset{\omega}{\delta}}; \alpha \parallel m + x(\omega), x(\omega-1) - x(\omega)) \langle \omega := [k] \rangle$$

(195)

$$\begin{aligned} U[\pi(\alpha \parallel m, z \mid \alpha_z)] \text{ diag}[f_{m+x}] U \left[\overset{(w,z)}{\underset{\omega \in \omega}{\delta}} \right] [\mu(\alpha \parallel m; z, z)] \\ = U[\delta(f; \alpha \parallel m + z, z - z)] \quad \checkmark \end{aligned}$$

$$U[\delta(f; \alpha \parallel m + z, z - z)] U[\pi(\alpha \parallel m, z \mid \alpha_z)] = U[\pi(\alpha \parallel m, z \mid \alpha_z)] \text{ diag}[f_{m+x}]$$

$$\begin{aligned} \mathcal{L}[\pi(\alpha \parallel m, i - z \mid \alpha_{m+i-z})] \text{ diag}[f_{m+i-z}] \mathcal{L}[\mu(\alpha \parallel m; i - z, i - z)] \\ = \mathcal{L}[\delta(f; \alpha \parallel m + i - z, z - z)] \quad \checkmark \end{aligned}$$

bp 172 Laganga

$$\lambda(f; \omega \| m, i) = \sum \lambda(f; \alpha \| m; i, \omega) f_{m(i)} \quad \langle \alpha := [i] \rangle$$

$$\text{Neutron } \quad u = \sum_i S(f; \alpha || m_i, \omega) \pi(\alpha || m_i, \omega) \quad \langle \omega := [i] \rangle$$

$$= \sum_i S(f; \alpha || m_{i-1}, \omega) \pi(\alpha || m_{i-1}, \omega) \quad \dots$$

$$\text{Jacobi } \Lambda(f; \alpha|m; i|z) = \pi(\alpha|m, i) \delta\left(\frac{f}{z - \langle \alpha \rangle}; \alpha|m, i\right)$$

$$(243) \quad \text{diag} \left[\pi(\alpha || m+2i-\omega, \omega | z)^{-1} \right] \left[\delta \left(\frac{f}{z(\alpha)} ; \alpha || m+2i-z-1 \right) \right] \text{diag} \left[\pi(\alpha || m+2i-\omega) \right]$$

✓

$$= \left[\Lambda(f; \alpha || m+2i-2-z | z) \right]$$

(245)

$$[\delta(f(z-\langle\alpha\rangle); \alpha | m \mapsto, 2i - \tau - \bar{\nu})]$$

$$= \left[(\alpha - \alpha_{m+2}) \delta(f; \alpha || m+2, 2i-2-j-1) - \delta(f; \alpha || m+2+1, 2i-2-j-2) \right]$$

$$= \left[(\alpha_{m+2i-z-1}) \delta(f; \alpha || m \alpha, 2i-z-2) - \delta(f; \alpha || m \alpha, 2i-z-2-1) \right]$$

(249)

$$u[\Lambda(f; \alpha || m + \tau, \bar{\alpha} - \tau)] = \quad \text{(exap. 315)}$$

$$\text{diag}[\pi(\alpha||m, \omega)^{-1}] \mathcal{U}[\pi(\alpha||m, \tau|\omega_m)] \text{diag}[f_{mn}] \mathcal{U}[\lambda(\alpha||^{m+2g_2+1, ?})]$$

$$U[\pi(\alpha||m, \tau|\alpha_{mn})] \text{diag} \left[\frac{\delta_{mn}}{\tau - \alpha_{mn}} \right] U[\mu(\alpha||m; \tau)]$$

$$= \mathcal{U} \left[\pi(\alpha|m+z, \rho - \tau H|z)^{-1} \Lambda(f; \alpha|m+z, \rho - \tau|z) \right] \quad (\text{from 1st result p 315})$$

$$U \begin{bmatrix} u(\alpha || m, \pi | \alpha_{m+1}) \end{bmatrix} \text{diag} \begin{bmatrix} f_{m+1,0} \\ \vdots \\ f_{m+1,m} \end{bmatrix} =$$

$$\text{diag}[\pi(\alpha|m, \omega|z)] u[\Lambda(f; \alpha|m+z, \omega-z|z)] \text{diag}[\pi(\omega|m, \omega+1|z)^{-1}] \\ u[\pi(\omega|m, \pi|\alpha_{m+\omega})]$$

$$\Delta_1^{-1}F = \Delta_0^{-1}\Lambda_0\Delta_1^{-1}F = \Delta_0^{-1}F\Lambda_0\Delta_1^{-1} = \Delta_0^{-1}\Lambda_0F'\Lambda_1^{-1} = F'\Delta_1^{-1}$$

374

$$F\Delta_1 = F\Lambda_1\Delta_0\Delta_0^{-1} = \Delta_1\Delta_0F\Delta_0^{-1}$$

$$F\Delta_1 = F\Lambda_1\Delta_0^{-1}\Lambda_0 = \Delta_1\Delta_0^{-1}F\Lambda_0 = \Delta_1\Delta_0^{-1}\Lambda_0F'$$

require F commutes $\Delta_1\Delta_0^{-1} \Leftrightarrow F'$ commutes $\Delta_0^{-1}\Lambda_1 \Rightarrow \Delta_1$ factor

$$\Lambda_0F'\Delta_0^{-1}\Lambda_1\Delta_0^{-1} = \Lambda_1\Delta_0^{-1}\Lambda_0F'\Delta_0^{-1} \quad \Delta_0^{-1}\Lambda_1F' = F'\Delta_0^{-1}\Delta_1$$

$$F\Lambda_0\Delta_1 = \Lambda_0\Lambda_1F = \Lambda_0F'\Delta_1 = \Delta_1F$$

$$F \text{ commutes } \Lambda_0\Delta_1 \Rightarrow \Delta_1^{-1} \text{ factor } \cancel{F'\Delta_0^{-1}\Lambda_0\Delta_1} \Rightarrow \cancel{\Lambda_0\Lambda_1\Delta_0^{-1}\Delta_1}$$

↓

F commutes $\Delta_1\Delta_0$

373.

iv) F commutes multiplicatively with the product $\Lambda_0\Lambda_1$ over $NS(\Lambda_0)$ if and only if F' is related to the product $\Delta_1\Delta_0$ in the same way.

b) If the above conditions hold

$$\Delta_1F = F'\Delta_1 \quad \langle NS(\Lambda_0) \rangle$$

and $\{\Lambda_1\}^{-1}$ is a similarity factor of $\{F, F'\}$ over $NS(\Lambda_0, \Lambda_1)$.

= Submatrix mappings \otimes products \otimes triangular matrix mappings

h, i, j, k, n being suitably presented, \Rightarrow p 212: the formation pp. 210, 211.

(195)

$$\mathcal{L}[\pi(\alpha \parallel m+z+1, i-z \text{th}/\alpha_m)] \text{diag}[f_{mz}] \mathcal{L}[\mu(\alpha \parallel m; i-z, z-z)] \checkmark$$

$$= \mathcal{L}[\delta(f; \alpha \parallel m, z-z)]$$

(222)

$$\mathcal{L}[\pi(\alpha \parallel m-z, z-z/\alpha_m)] \text{col}[\pi(\alpha \parallel m-z+1, z)] = \text{col}[\pi(\alpha \parallel m-z, z)] \checkmark$$

see (8) p 291 with $z = \alpha_{m+i}$

$$\text{With } L_i(\alpha \parallel f_m) = \mathcal{L}[\delta(f; \alpha \parallel m, z-z)]$$

$$\mathcal{L} L_i(\alpha \parallel f_m g_m) = \mathcal{L}(\alpha \parallel f_m) \cdot L_i(\alpha \parallel g_m) \checkmark$$

(244)

$$\delta(\frac{s}{s}; f; \alpha \parallel m, i) = \sum_j \mu(\frac{s}{s}; \alpha \parallel m; i, \omega) f_{\frac{s}{s}(m\omega)} \quad \langle \omega := [i] \rangle$$

$$\frac{s}{s} \in RO(h, k) \quad [m, m+i] \subseteq [h, h+k]$$

$$\delta(\frac{s}{s}; f; \alpha \parallel m, i) = \delta(f; \alpha \parallel 2h+k-m-i, i) \checkmark$$

$$\text{all } \frac{s}{s} \in O(h, k) \quad \delta(\frac{s}{s}; f; \alpha \parallel h, k) = \delta(f; \alpha \parallel h, k) \quad \text{invariance} \checkmark$$

Lagrange multipliers

$$D. \lambda(\frac{s}{s}; \alpha \parallel m; i, \omega) = \overline{\prod} \frac{z - \alpha_{\frac{s}{s}(m\omega)}}{\alpha_{\frac{s}{s}(m+\omega)} - \alpha_{\frac{s}{s}(m)}} \cdot \langle \omega := [i] \rangle$$

$$\frac{s}{s} \in O(h, k) \quad \omega \in [k]$$

$$\lambda(\frac{s}{s}; \alpha \parallel h; k, \omega) = \lambda(\alpha \parallel h; k, \frac{s}{s}(h\omega) - h) \quad \langle k \rangle \checkmark$$

$$\frac{s}{s} \in RO(h, k) \quad [m, m+i] \subseteq [h, h+k]$$

$$\lambda(\frac{s}{s}; \alpha \parallel m; i, \omega) = \lambda(\alpha \parallel 2h+k-m-i; i, i-\omega) \quad . \checkmark$$

$$\lambda(\frac{s}{s}; \alpha \parallel m, i, \omega) = \mu(\frac{s}{s}; \alpha \parallel m; i, \omega) \pi(\frac{s}{s}; \alpha \parallel m; i, \omega)$$

$$\overline{\lambda(\alpha \parallel m; i, h\omega)} = \frac{\pi(\alpha \parallel m, h) \pi(\alpha \parallel m+h+k+1, i-h-k) \lambda(\alpha \parallel m+h, k, \omega)}{\pi(\alpha \parallel m, h/\alpha_{m+h}) \pi(\alpha \parallel m+h+k+1, i-h-k/\alpha_{m+h})} \checkmark$$

(250)

294.

$$\begin{aligned}
 \text{diag}[f_{mn}] &= U[\mu(\alpha||m; z, \omega)] \text{diag}[\pi(\alpha||m, \omega|z)] U[\Delta(f; \alpha||m+z, \omega|z)] \\
 &\quad \text{diag}[\pi(\alpha||m, \omega+1|z)^{-1}] U[\pi(\alpha||m, \omega|\alpha_{mn})] \text{diag}[z - \alpha_{mn}] \\
 &= U[\mu(\alpha||m; z, \omega)] \text{diag}[\pi(\alpha||m, \omega)] U[\Delta(f; \alpha||m+z, \omega|z)] \\
 &\quad U[\pi(\alpha||m, \omega|\alpha_{mn})] \text{diag}[\pi(\alpha||m, \omega)^{-1}] \quad \begin{array}{l} \text{no: } z - \alpha_{mn} \text{ term does} \\ \text{not cancel through } U[\pi\cdot] \end{array} \\
 &=
 \end{aligned}$$

(250, 271)

$\sum_i (\alpha||m; z, \omega)$: sum of all possible products formed from $z-2$ distinct
members of the set $\alpha_m, \dots, \alpha_{m+i-1}$: $(\sum_i (\alpha||m; z, \omega) = \overline{\alpha}(\alpha||m; z, \omega))$

$$\begin{aligned}
 L[(-1)^z \sum_i (\alpha||m; z, \omega)] L[\delta(\langle \alpha \rangle^z; \alpha||m, \omega)] &= I[i] \\
 \frac{z}{\prod_i} \left\{ V^\omega \left(\sum_i U L[\delta(-\alpha_{m+\omega+i}, \omega)] \langle x := [i-\omega] \rangle \right) \right\} \langle \omega := [i-1] \rangle & \\
 L[\delta(\langle \alpha \rangle^z; \alpha||m, \omega)] &= I[i] \quad \text{see p 316}
 \end{aligned}$$

(253)

$$\pi(\alpha||m, h) \pi(\alpha||m+h+k+1, i-h-k) \Delta(f; \alpha||m+h, k) =$$

$$\sum \lambda(\alpha||m; i, h|\omega) \pi(\alpha||m; h|\alpha_{m+h|\omega}) \pi(\alpha||m+h+k+1, i-h-k|\alpha_{m+h|\omega}) f_{mn|\omega} \\
 \langle \omega := [k] \rangle \quad \text{from}$$

(255)

$$\underline{\Delta(f; \alpha||m, i|x) - \Delta(f; \alpha||m, i|y)} =$$

$$\sum \frac{\pi(\alpha||m, \omega|x)}{\pi(\alpha||m, \omega+1|y)} \left\{ \Delta(f; \alpha||m, i|y) - \Delta(f; \alpha||m, \omega|y) \right\} \langle \omega := [i] \rangle$$

(256)

$$(\alpha_{m+i} - \alpha_m) \Lambda(f; \alpha \| m, i+1 | z) =$$

$$(z - \alpha_m) \Lambda(f; \alpha \| m+1, i | z) - (z - \alpha_{m+i+1}) \Lambda(f; \alpha \| m, i+1 | z)$$

$$\frac{\Lambda(f; \alpha \| m+1, i)}{\pi(\alpha \| m+1, i+1 | z)} - \frac{\Lambda(f; \alpha \| m, i+1 | z)}{\pi(\alpha \| m, i+2 | z)} = (\alpha_{m+i+1} - \alpha_m) \frac{\Lambda(f; \alpha \| m, i+1 | z)}{\pi(\alpha \| m, i+2 | z)}$$

$$\frac{\Lambda(f; \alpha \| m+1, i | z) - \Lambda(f; \alpha \| m, i | z)}{\Lambda(f; \alpha \| m+1, i | z) - \Lambda(f; \alpha \| m, i+1 | z)} = \frac{\alpha_m - \alpha_{m+i+1}}{z - \alpha_{m+i+1}}$$

(257) bp 169

$$\exists [m, m+k] \subseteq [h, h+k] \quad \exists \in \mathcal{E}(h, k)$$

$$\Lambda(\frac{1}{z}; f; \alpha \| m, i) := \sum_{\omega} \lambda(\frac{1}{z}; \alpha \| m, i, \omega) f_{\frac{1}{z}(m+\omega)} \quad \langle \omega := [i] \rangle$$

$$\frac{1}{z} \in R_0(h, k) : \Lambda(\frac{1}{z}; f; \alpha \| m, i) = \Lambda(f; \alpha \| 2h+k-m-i, i)$$

$$\frac{1}{z} \in S(h, k) : \Lambda(\frac{1}{z}; f; \alpha \| h, k) = \Lambda(f; \alpha \| h, k)$$

(258) bp 169

$$U\mathcal{L}\mathcal{D}[\alpha_{m+\omega} - z] \circ \mathcal{L}[\pi(\alpha \| m+1, z-\omega | z)] = J[i]$$

$$\left\{ \mathcal{L}[\delta(\pi(\alpha \| m, i | \beta)) ; \beta \| k, \omega] \right\}^{-1}$$

$$\begin{aligned} &= \overline{\prod} \left\{ V^{\frac{i-k-1}{i-\omega-1}} \left\{ U\mathcal{L}\mathcal{D}[\beta_{k+\omega-1} - \alpha_{m+i-\omega-\frac{1}{4}}] \langle x := [\omega+1] \rangle \right\}^{-1} \right\} \langle \omega := [i] \rangle \\ &= \overline{\prod} \left\{ V^{\frac{i-k-1}{i-\omega-1}} \left\{ \mathcal{L}[\pi(\beta \| k+2, z-\omega | m+i-\omega-\frac{1}{4})] \langle z, \omega := [\omega+1] \rangle \right\}^{-1} \right\} \langle \omega := [i] \rangle \end{aligned}$$

see p 321

(237) bp 169

$$\prod \nabla^{\omega} \{ \text{L} \in \mathcal{D} [\beta_{k+\chi-1} - \alpha_{mn}] \langle \chi := \{ \vdash \omega \} \rangle \} \langle \omega := \{ \overset{i}{\underset{j}{\overbrace{\dots}}} \} \rangle \\ = \mathcal{L} [\delta(\pi(\alpha \parallel m, \tau k \beta)) ; \beta \parallel k, \omega]$$

(238)

$$\mathcal{L} [\delta(\pi(\beta \parallel k, \tau \langle \alpha \rangle) ; \alpha \parallel m, \omega)] \mathcal{L} [\delta(\pi(\alpha \parallel m, \tau k \beta) ; \beta \parallel k, \omega)] \\ = \{ \vdash \omega \}$$

(239)

$$\mu(\alpha \parallel m; i, \omega) = \frac{1}{\pi(\alpha \parallel m, \omega | \alpha_{mn}) \pi(\alpha \parallel m_{n+1}, i-n | \alpha_{mn})}$$

(bp 208) bp 168

$$\frac{\pi(\alpha \parallel m, i | x) - \pi(\alpha \parallel m, i | y)}{x - y} = \sum \pi(\alpha \parallel m, \omega | x) \pi(\alpha \parallel m_{m+1}, i-\omega-1 | y) \langle \omega := \{ i \} \rangle \\ = \pi(\alpha \parallel m, i | y) \sum \frac{\pi(\alpha \parallel m, \omega | x)}{\pi(\alpha \parallel m, \omega+1 | y)} \langle \omega := \{ i \} \rangle \\ = \pi(\alpha \parallel m, i | x) \sum \frac{\pi(\alpha \parallel m, \omega | y)}{\pi(\alpha \parallel m, \omega+1 | x)} \langle \omega := \{ i \} \rangle \\ g \in \text{pp}(k | i) \quad f : g \overset{:= g(\alpha[mn])}{\underset{\text{def}}{\overbrace{\dots}}} \rightarrow \bigwedge (\alpha f ; \alpha \parallel m, i) = g$$

bp 168

\mathcal{D}_0 set of r distinct members of K ; $S_0 \{ g_{(i)} \}$ sum of all possible products of r distinct members of \mathcal{D}_0 $\omega := \{ i \} : S_0(g_{(i)}) = 1$

$$\pi(\alpha \parallel m, i | z) = \sum z^{i-r} (-1)^r S_\omega \{ \alpha[m, m+r] \} \langle \omega := \{ i \} \rangle$$

$$\pi(\alpha \parallel m; i, \omega | z) = \sum z^{i-\omega} (-1)^\omega S_\omega \{ \alpha[m, m+i] - \omega \} \langle \omega := \{ i \} \rangle$$

$$\Delta(\pi(\beta \parallel k; h, v | \langle \alpha \rangle); \alpha \parallel m, i)$$

$$= \sum \lambda(\alpha \parallel m; i, \omega) \pi(\beta \parallel k; h, v | \alpha_{m+\omega}) \checkmark$$

$$= \pi(\beta \parallel k; h, v) \text{ if } h \leq i \checkmark$$

(266) bp 167

$$\mathcal{L} \left[\frac{\mu(\alpha \parallel m; i, \omega)}{\pi(\beta \parallel k+i, i-\omega | \alpha_{m+\omega})} \right] \mathcal{L} \left[\frac{\pi(\beta \parallel k+i+1, i-2-\omega | \alpha_{m+\omega}) \times}{\pi(\alpha \parallel m, \omega | \alpha_{m+\omega})} \right] =$$

$$\text{diag}[(\alpha_{m+\omega} - \beta_{k+\omega})^{-1}] \langle \omega := [i] \rangle \quad \checkmark$$

$$\mathcal{U} \left[\frac{\mu(\beta \parallel i-2-\omega, i-\omega)}{\pi(\alpha \parallel m; i+1 | \beta_{k+\omega})} \right]$$

$$\mathcal{U} \left[\frac{\mu(\beta \parallel k+\omega; i-2-1, 2-\omega)}{\pi(\alpha \parallel m; i+1 | \beta_{k+\omega})} \right] \mathcal{U} \left[\frac{\pi(\beta \parallel k+i+1, i-2-\omega | \beta_{k+\omega}) \times}{\pi(\alpha \parallel m, \omega | \beta_{k+\omega})} \right] =$$

$$\text{diag}[(\beta_{k+\omega} - \alpha_{m+\omega})^{-1}] \langle \omega := [i] \rangle \quad \checkmark \quad (\text{interchange } \alpha, m \leftrightarrow \beta, k)$$

"(265, 266) bp 167 ...

Treatment of translation matrix $\Lambda_n(\alpha, m | \beta, k)$

(277, 278)

$$|\Lambda_n(\alpha, m | \beta, k)| = \prod \frac{\pi(\alpha \parallel m, \omega | \alpha_{m+\omega})}{\pi(\beta \parallel k, \omega | \beta_{k+\omega})} \langle \omega := [i] \rangle \quad \checkmark$$

$$= \prod \frac{\pi(\alpha \parallel m+\omega, i-\omega | \alpha_{m+\omega+\omega})}{\pi(\beta \parallel k+\omega, i-\omega | \beta_{k+\omega+\omega})} \langle \omega := [i] \rangle \quad \checkmark$$

$$= \prod \left\{ \prod \frac{\alpha_{m+\omega} - \alpha_{m+\omega}}{\beta_{m+\omega} - \beta_{m+\omega}} \langle \omega := [\omega] \rangle \right\} \langle \omega := [i] \rangle \quad \checkmark$$

(294) bp. 162.

1] Let corresponding members of the two sequences $\alpha[m, m+i]$, $\beta[k, k+i]$ satisfy the relationship

$$\rho(\alpha[m, m+i]) = \mu + \rho \rho(\beta[k, k+i])$$

where $\rho \neq 0$, $\rho \in \text{pp}(K | \frac{h}{\alpha})$ where $h \in i$ and let $\rho(\alpha_{m+z}) - \rho(\beta_{k+z})$ be nonzero for at least one z in $[i]$. Let $k = \lfloor \frac{i/h}{(i-1)/h} \rfloor$. The matrix

$\Lambda_i(\alpha, m | \beta, k)$ possesses the $k+i$ eigenvalue-eigenvector pairs.

$$\rho^0, \text{col}[\{\rho(\alpha_{m+z}) - \rho(\beta_{k+z})\}^T] \underset{\langle z := [i] \rangle}{\longleftrightarrow} \langle \lambda := [k] \rangle,$$

2] Let the sequence $\alpha[m, m+i]$ be a linear function of the sequence $\beta[k, k+i]$ in the sense that

$$\alpha_{m+z} = \mu + \rho \beta_{k+z} \quad \langle z := [i] \rangle$$

where $\rho \neq 0$. $\rho^0, \text{col}[(\alpha_{m+z} - \beta_{k+z})^T] \underset{\langle z := [i] \rangle}{\notin}$ form a complete system of eigenvalue-vector pairs of $\Lambda_i(\alpha, m | \beta, n)$.

(292) bp 63 | Proof given in full on pp. 294, 295 of earlier notes

(Example of eigenvalue-vectors arising from sequences other than those considered above?)

$$\alpha_{m+z} = \gamma^z \quad \langle \alpha, \beta_{k+z} = a + b \beta^z \rangle \quad \langle z := [2] \rangle$$

$\Lambda_2(\alpha, m | \beta, k)$ has eigenvalue-vector pair $1 - (1, 1, 1)$ and further

eigenvalues x, y where $xy = \frac{\gamma(\gamma+1)(\gamma-1)^3}{b^3 \beta(\beta+1)(\beta-1)^3}$ and

$$x+y = \frac{b(1+\beta)(\gamma-\beta^2) + \gamma(2a-\gamma-\gamma^2) + \beta(1+\gamma-2a)}{b^2 \beta(1+\beta)(\beta-1)^2} (1-\gamma)$$

$$\beta=2 \quad \delta=3 \quad \alpha=2 \quad b=1 \Rightarrow \alpha_m=1 \quad \alpha_{m+1}=3 \quad \alpha_{m+2}=9$$

$$\beta_k=3 \quad \beta_{k+1}=4 \quad \beta_{k+2}=6$$

$xy=16 \quad x+y=9 \quad x, y$ are roots of $z^2 - 9z + 16 = 0$

$9^2 - 4 \cdot 16 = 17$ is not a perfect square $\Rightarrow x, y$ irrational

(296) bp 162

Cauchy matrix

$$D: J_i(\alpha, m | \beta, k) = [(\alpha_{m+i} - \beta_{k+i})^{-1}] \quad \langle i := 1 \rangle$$

Relationship to L:

$$CaJ_i(\alpha, m | \beta, k) = \text{diag} [\pi(\beta \parallel k, i \parallel \alpha_{m+i})^{-1}] L_i(\alpha, m | \beta, k)$$

$$\text{diag} [\mu(\alpha \parallel k, i \parallel \beta, \omega)^{-1}]$$

LU decomposition

$$J_i(\alpha, m | \beta, k) = \hat{L}_i(\alpha, m | \beta, k) \hat{U}_i(\alpha, m | \beta, k) \quad p 329$$

$$\hat{L}_i(\alpha, m | \beta, k) = \text{diag} [\pi(\beta \parallel k, i \parallel \alpha_{m+i})^{-1}] L_i(\alpha, m | \beta, k)$$

$$\hat{U}_i(\alpha, m | \beta, k) = U_i(\alpha, m | \beta, k) \text{diag} [\mu(\beta \parallel k, i \parallel \beta, \omega)^{-1}]$$

(L_i, U_i ↑ featuring in LU dec. of $L_i(\alpha, m \dots)$)

Inversion

$$CaJ_i(\alpha, m | \beta, k)^{-1} = \text{diag} [\mu(\beta \parallel k, i \parallel \beta, \omega)] L_i(\beta, k | \alpha, m) \quad \hat{U}_i(\beta, k | \alpha_{m+i}) \quad \text{diag} [\pi(\beta \parallel k, i \parallel \alpha_{m+i})]$$

(297)

Determinants

$$|J_i(\alpha, m | \beta, k)| = (-1)^{\frac{i(i+1)}{2}} \frac{\pi(\alpha \parallel m, \omega | \alpha_{m+i}) \pi(\beta \parallel k, \omega | \beta_{k+i})}{\pi(\beta \parallel k, \omega | \alpha_{m+i}) \pi(\alpha \parallel m, \omega | \beta_{k+i})} \quad \langle \omega := \{i\} \rangle$$

$$= (-1)^{\frac{i(i+1)}{2}} \frac{\pi(\alpha \parallel m, \omega | \alpha_{m+i}) \pi(\beta \parallel k, \omega | \beta_{k+i})}{\pi(\alpha \parallel m, i | \beta_{k+i})} \quad \langle \omega := \{i\} \rangle \quad p 331.$$

$$= (-1)^{\frac{i(2m)}{2}} \frac{\pi(\alpha \parallel m, \omega | \alpha_{m+i}) \pi(\beta \parallel k, \omega | \beta_{k+i})}{\pi(\beta \parallel k, i | \alpha_{m+i})} \quad \langle .. \rangle$$

(295) bp 16
Eigenvalues of $C_2(\alpha, \beta, \gamma | \beta, \gamma)$

$$\text{Set } \alpha_0 - \beta_0 = a \quad \alpha_1 - \beta_1 = b \quad \beta_1 - \beta_0 = c$$

$$\text{Eigenvalues satisfy } (\lambda - \frac{1}{a})(\lambda - \frac{1}{b}) - \frac{1}{(a-c)(b+c)} = 0$$

$$\left(\frac{1}{a} + \frac{1}{b}\right)^2 + \frac{4}{(a-c)(b+c)} = \frac{a^3b + a^3c + a^2bc - a^2c^2 + 6a^2b^2 - abc^2 - 2abc^2 + ab^3 - b^3c - b^2c^2}{a^2b^2(a-c)(b+c)} \quad (*)$$

num of (*) not divisible by $(a-c)(b+c)$. (*) may not be perfect square,
 λ may be irrational.

bp 160.

$$A_0 := [\delta(f<\alpha>^2; \alpha \parallel m, 2i-z-1)] \quad \langle z, i \rangle = \langle i, 1 \rangle$$

$$A_1 := [\delta(f<\alpha>^{z+2}; \alpha \parallel m, 2i-1)]$$

$$A_2 := [\delta(f; \alpha \parallel \overset{m+2}{m}, 2i-z-2-1)]$$

$$A_3 := [\delta(f<\alpha>^2; \alpha \parallel \overset{m+2}{m+\tau}, 2i-z-1)]$$

$$A_4 := [\delta(f<\alpha>^2; \alpha \parallel \overset{m+2}{m+\tau-1}, i)]$$

$$A_0^T := [\delta(f<\alpha>^2; \alpha \parallel m, 2i-2-1)]$$

$$A_2^T := [\delta(f; \alpha \parallel m+\tau, 2i-z-2-1)]$$

$$A_3^T := [\delta(f<\alpha>^2; \alpha \parallel \overset{m+2}{m+\tau}, 2i-2-1)]$$

$$A_4^T := [\delta(f<\alpha>^2; \alpha \parallel m+\tau-1, i)]$$

$$\tilde{A}_4 := [\delta(f<\alpha>^2; \alpha \parallel m+z, i)]$$

$$\tilde{A}_4^T := [\delta(f<\alpha>^2; \alpha \parallel m+\tau, i)]$$

$$(1) \mathcal{L}[\delta(\langle \alpha \rangle^{\tau}; \alpha \| m+2i-2-1, \omega)] A_0 = A_1$$

$$(2) A_2 \cup [\delta(\langle \alpha \rangle^{\tau}; \alpha \| m, \omega)] = A_0$$

$$(3) \mathcal{L}[\delta(\pi(\alpha \| m, \tau | \langle \alpha \rangle); \alpha \| m+2i-2-1, \omega)] A_0 = A_3$$

$$(4) \cup [\delta(\pi(\alpha \| m, i-2-1 | \langle \alpha \rangle); \alpha \| m+2i-2-1, \omega)] A_0 = A_4$$

$$(5) A_0^T \mathcal{L}[\delta(\pi(\alpha \| m, i-2-1 | \langle \alpha \rangle); \alpha \| m+2i-2-1, \tau)] = A_4^T$$

$$(6) A_0^T \cup [\delta(\pi(\alpha \| m, \omega | \langle \alpha \rangle); \alpha \| m+2i-2-1, \tau)] = A_3^T$$

$$(7) \mathcal{L}[\delta(\langle \alpha \rangle^{\tau}; \alpha \| m, \omega)] A_2^T = A_0^T$$

$$(8) A_0^T \cup [\delta(\langle \alpha \rangle^{\tau}; \alpha \| m+2i-2-1, \tau)] = A_1$$

$$(9) A_3^T \cup [\delta(\langle \alpha \rangle^{\tau}; \alpha \| m, \tau)] = A_1$$

$$(10) \mathcal{L}[\delta(\langle \alpha \rangle^{\tau}; \alpha \| m+2i-2-1, \omega)] A_2 = A_3^T$$

$$(11) A_3^T \cup [\delta(\pi(\alpha \| m+2i-2, \omega | \langle \alpha \rangle); \alpha \| m, \tau)] = A_0^T$$

$$(12) A_3^T \mathcal{L}[\delta(\pi(\alpha \| m+i+2+1, i-2-1 | \langle \alpha \rangle); \alpha \| m+\tau, \tau)] = A_4^T$$

$$(13) \cup [\delta(\pi(\alpha \| m+i+2+1, i-2-1 | \langle \alpha \rangle); \alpha \| m+\tau, \tau)] A_3 = \tilde{A}_4$$

$$(14) \mathcal{L}[\delta(\pi(\alpha \| m+2i-2, \tau | \langle \alpha \rangle); \alpha \| m, \omega)] A_3 = A_0$$

$$(15) A_2^T \cup [\delta(\langle \alpha \rangle^{\tau}; \alpha \| m+2i-2-1, \tau)] = A_3$$

$$(16) \mathcal{L}[\delta(\langle \alpha \rangle^{\tau}; \alpha \| m, \omega)] A_3 = A_1$$

$$(9, 11, 12) \equiv (16, 14, 13)^T \quad (1-4): A_0 \Rightarrow A_1 \dots A_4 \quad (16) A_1 \Rightarrow A_3$$

$$(5-8): A_0^T \Rightarrow A_1^T, \dots A_4^T \quad (11): A_0^T \Rightarrow A_3^T \quad (9, 10, 15) \quad A_1 \Rightarrow \tilde{A}_4^T$$

$$A_2 \Rightarrow A_3^T \quad A_3 \Rightarrow A_2^T$$

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$$(17) \quad \mathcal{L}[\delta(\langle\alpha\rangle^\tau; \alpha \parallel m+2i-1, \nu)] A_2 \mathcal{U}[\delta(\langle\alpha\rangle^\tau; \alpha \parallel m, \nu)] = A_1$$

$$(18) \quad \mathcal{L}[\delta(\langle\alpha\rangle^\tau; \alpha \parallel m+2i-1, \nu)] \mathcal{L}[\delta(\pi(\alpha \parallel m+2i-2, \tau | \langle\alpha\rangle); \alpha \parallel m, \nu)] A_3 \\ = A_1$$

$$(19) \quad \mathcal{L}[\delta(\langle\alpha\rangle^\tau; \alpha \parallel m+2i-1, \nu)] \mathcal{U}[\delta(\pi(\alpha \parallel m, i-2-1 | \langle\alpha\rangle); \alpha \parallel m+2i-1, \nu)]^{-1} A_4 = A_1$$

$$(20) \quad A_2 \mathcal{U}[\delta(\langle\alpha\rangle^\tau; \alpha \parallel m, \nu)] = \mathcal{L}[\delta(\pi(\alpha \parallel m+2i-2, \tau | \langle\alpha\rangle); m, \nu)] A_3$$

$$(21) \quad \mathcal{U}[\delta(\pi(\alpha \parallel m, i-2-1 | \langle\alpha\rangle); \alpha \parallel m+2i-2-1, \nu)] A_2 \\ \mathcal{U}[\delta(\langle\alpha\rangle^\tau; \alpha \parallel m, \nu)] = A_4$$

$$(22) \quad \mathcal{U}[\delta(\pi(\alpha \parallel m, i-2-1 | \langle\alpha\rangle); \alpha \parallel m+2i-2-1, \nu)]$$

$$\mathcal{L}[\delta(\pi(\alpha \parallel m+2i-2, \tau | \langle\alpha\rangle); \alpha \parallel m, \nu)] A_3 = A_4$$

$$(16+k) + (k+1) \Rightarrow (16+k) \langle k := [3] \rangle : (17) + (k+17) \Rightarrow (19+k) \langle k := [2] \rangle,$$

$$20+21 \Rightarrow 22.$$

$$(16+k) : A_1 \Rightarrow A_{k+1} \langle k := [3] \rangle \quad 20 \quad A_2 \Rightarrow A_3 \quad 21 \quad A_2 \Rightarrow A_4$$

$$22 \quad A_3 \Rightarrow A_4)$$

from p 281

$$\pi(\alpha \parallel m, r | z) = \pi(\alpha \parallel m+1, r | z) - \pi(\alpha \parallel m+1, r-1 | z) \{ z - \alpha_m - z + \alpha_{m+r} \}$$

from p 288

$$\begin{aligned} \pi(\alpha \parallel m+1, i+1) &= \sum_i \pi(\alpha \parallel m+1, i-\omega | \alpha_m) \pi(\alpha \parallel m, \omega | z) \\ &= \sum_i \pi(\alpha \parallel m+\omega, i-\omega | \alpha_m) \{ \pi(\alpha \parallel m, \omega+1 | z) + (\alpha_{m+\omega} - \alpha_{m+\omega}) \pi(\alpha \parallel m, \omega | z) \} \\ &= \sum_i \pi(\alpha \parallel m+\omega, i-\omega-1 | \alpha_m) \pi(\alpha \parallel m, \omega | z) \quad \langle \omega := (i+1) \rangle \\ &\quad + \sum_i \pi(\alpha \parallel m+\omega-1, i-\omega | \alpha_m) \pi(\alpha \parallel m, \omega | z) \quad \langle \omega := [i] \rangle \end{aligned}$$

assume both. Then

$$\begin{aligned} \pi(\alpha \parallel m, i+1 | z) &= \sum_i \pi(\alpha \parallel m+\omega+1, i-\omega | \alpha_m) \{ \pi(\alpha \parallel m, \omega+1 | z) + (\alpha_{m+\omega} - \alpha_m) \pi(\alpha \parallel m, \omega | z) \} \\ &= \sum_i \pi(\alpha \parallel m+\omega, i-\omega+1 | \alpha_m) \pi(\alpha \parallel m, \omega | z) \quad \langle \omega := (i+1) \rangle \\ &\quad - \sum_i \pi(\alpha \parallel m+\omega, i-\omega+1 | \alpha_m) \pi(\alpha \parallel m, \omega | z) \quad \langle \omega := [i] \rangle \Rightarrow \text{lhs} \\ &= \\ \pi(\alpha \parallel m, i | z) &= \sum_i \pi(\alpha \parallel m+\omega, i-\omega | \alpha_m) \pi(\alpha \parallel m, \omega | z) \quad \langle \omega := (i) \rangle \\ &\quad - \sum_i \pi(\alpha \parallel m+\omega, i-\omega | \alpha_m) \pi(\alpha \parallel m, \omega | z) \quad \langle \omega := [i] \rangle \\ &= \sum_i \pi(\alpha \parallel m+\omega, i-\omega | \alpha_m) \pi(\alpha \parallel m, \omega-1 | z) (z - \alpha_{m+\omega-1}) \\ &\quad - \sum_i \pi(\alpha \parallel m+\omega, i-\omega | \alpha_m) \pi(\alpha \parallel m, \omega | z) \end{aligned}$$

$$\begin{aligned} \pi(\alpha \parallel m+1, i | z) &= \sum_i \pi(\alpha \parallel m+\omega+1, i-\omega | \alpha_m) \pi(\alpha \parallel m+\omega, \omega | z) \quad \langle \omega := (i) \rangle \\ &\quad - \sum_i \pi(\alpha \parallel m+\omega+1, i-\omega | \alpha_m) \pi(\alpha \parallel m+\omega, \omega | z) \quad \langle \omega := [i] \rangle \end{aligned}$$

$$\begin{aligned} \text{from p 188, } j=0 \quad \delta(\Theta(\alpha \parallel m+1, r); \alpha \parallel m, i) &= \delta(\Theta(\alpha \parallel m+\omega+1, r-i); \alpha \parallel m, \omega) \\ \text{in } [2] \quad &= \pi(\alpha \parallel m+\omega+1, r-i | \alpha_m) \end{aligned}$$

leading to 1st formula

$$\begin{aligned} \text{2nd formula correct if } \delta(\pi(\alpha \parallel m, r | \langle \alpha \rangle); \alpha \parallel m+\omega, r-\omega) &= \pi(\alpha \parallel m, \omega | \alpha_{m+\omega}) \\ \text{in } [1] \quad j'=r \quad k'=\omega \quad r'=r-\omega \quad r'+k'-j'=0 \quad & \end{aligned}$$

$$\begin{aligned} \delta(\Theta(\alpha \parallel m, r); \alpha \parallel m+\omega, r-\omega) &= \delta(\Theta(\alpha \parallel m, \omega); \alpha \parallel m+r, 0) \\ &= \pi(\alpha \parallel m, \omega | \alpha_{m+r}) \end{aligned}$$

$$\omega' := \omega - \omega \quad r = n - m - 1 \quad \kappa = r + m + 1$$

$$\sum_{\omega=0}^r \frac{\pi(\alpha || m + r - \omega + l, \omega | \omega_m)}{\pi(\alpha || m + r - \omega, \omega + l | z)} \langle \omega := [r] \rangle$$

$$\begin{aligned} & - \sum_{\omega=0}^r z_{\omega} (\omega) z_{\omega} \cdot (-1)^{r-\omega} \pi z_{\omega} \langle \omega := [r, r-1] \rangle \\ & + (-1)^{r-\omega} \pi z_{\omega} \langle \omega := [r, r] \rangle \end{aligned}$$

A in $\mathcal{U}\mathcal{L}\mathcal{A}[z_i]$ $\langle \chi := [i] \rangle$ lower principal diagonal elements are

$$A_{r, r-1} = z_r \quad z := [i]$$

$$\begin{aligned} & \prod_{\omega=0}^i \prod_{\chi=0}^{m-1} (\beta_{k+\omega} - \alpha_{m+\chi}) = \\ & \begin{array}{ccccccc} \omega & & \chi & & & & \\ i & & 0 & & & & \\ 2 & & 0 & 1 & & & \\ i & 0 & \dots & i-1 & & & \end{array} \end{aligned}$$

$$(-1)^{\frac{i(i+1)}{2}} \prod_{\chi=0}^{i-1} \prod_{\omega=m+1}^i (\alpha_{m+\chi} - \beta_{k+\omega}) = \prod_{\omega=0}^i \prod_{\chi=0}^{i-\omega-1} (\alpha_{m+\omega} - \beta_{k+\omega+m+1})$$

$$(-1)^{\frac{i^2}{2}} = (-1)^{\frac{i(i-1)}{2}} (-1)^{\frac{i}{h-1}} = (-1)^{\frac{i}{h}}$$

$$\begin{aligned} & \frac{\pi(\alpha || m, i+1 | z)^{i+1}}{\prod_{\omega=0}^i (z - \alpha_{m+\omega})} \left/ \frac{\prod_{\omega=0}^r (\alpha_{m+h+\omega} - \alpha_{m+\omega})}{\prod_{\omega=0}^r [h+\omega]} \right. \left/ \frac{\prod_{\omega=k+1}^r (\alpha_{m+h+\omega} - \alpha_{m+\omega})}{h+k+1} \right. \\ & \text{m+h+k+1, r-h-k} \end{aligned}$$

$$= \frac{\pi(\alpha || m, h | \alpha_{m+h+x}) \pi(\alpha || m+k+1, \cancel{k} | \alpha_{m+h+x})}{\prod_{\omega=0}^{h+k} [h+\omega]}$$

$$\begin{aligned} & \left/ \frac{\prod_{\omega=0}^k (\alpha_{m+h+\omega} - \alpha_{m+\omega})}{\prod_{\omega=0}^k [h+\omega]} \right. = \left/ \frac{\prod_{\omega=0}^k [\chi]}{\prod_{\omega=0}^k (\alpha_{m+h+\chi} - \alpha_{m+\omega})} \right. \\ & \text{m+h+k+1, r-h-k} \end{aligned}$$

$$\begin{aligned} & \left/ \prod_{\omega=1}^i \prod_{\chi=0}^{m-1} (\alpha_{3(m+\omega)} - \alpha_{3(m+\chi)}) \right. = (-1)^{\frac{i(i+1)}{2}} \left/ \prod_{\chi=0}^{i-1} (\beta_{3(m+\chi)} - \alpha_{3(m+\omega)}) \right. \\ & \left/ \prod_{\omega=1}^i \prod_{\chi=0}^{m-1} (\alpha_{3(m+\omega)} - \alpha_{3(m+\chi)}) \right. = (-1)^{\frac{i(i+1)}{2}} \left/ \prod_{\chi=0}^{i-1} (\beta_{3(m+\chi)} - \alpha_{3(m+\omega)}) \right. \end{aligned}$$

$m, n \in$

$$\exists \in \text{seq}'(N \geq m_{i-1}) \quad \beta \in \text{seq}'(N \geq n_{i-1}) \quad m, n \in \overline{\mathbb{N}}, i \in \mathbb{N}$$

$$\alpha \in \text{seq}'(K \geq \exists[m, n]) \quad \beta \in \text{seq}'(K \geq \beta[m, n])$$

$$\left\{ \prod \pi(\exists : \alpha \parallel m, \omega | \beta_{\beta(m+\omega)}) \langle \omega := (i) \rangle \right\} =$$

$$(1) \frac{i(i+1)}{2} \left\{ \prod \pi(\beta : \beta \parallel m, n_{i+1}, i-\omega | \alpha_{\beta(m+n)}) \langle \omega := [i] \rangle \right\}$$

$$\exists : 2, 0, 1 \quad \prod (\exists : \alpha \parallel m, \omega | \alpha_{\beta(m+\omega)}) \langle \omega := (2) \rangle$$

$$(\alpha_0 - \alpha_2) \cdot (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_0) \parallel \frac{\prod (\exists : \alpha \parallel m, \omega | \beta_{\beta(m+\omega)})}{\text{invariance with w.r.t } \exists} \text{ not preserved.}$$

$$(\alpha_1 - \alpha_0) \cdot (\alpha_2 - \alpha_0)(\alpha_2 - \alpha_1) \left| \begin{array}{l} (\beta_0 - \alpha_2)(\beta_1 - \alpha_2)(\beta_1 - \alpha_0) \\ (\beta_1 - \alpha_0)(\beta_2 - \alpha_0)(\beta_2 - \alpha_1) \end{array} \right.$$

$$\prod \prod_{\omega=0}^{\omega-1} (\alpha_{\beta(m+\omega)} - \alpha_{\beta(m+\omega)}) \langle \omega := (i) \rangle$$

$$\omega' = \beta(m+\omega) - m \quad \chi' = \beta(m+x) - m$$

$$\omega = 1, 2, \dots, i \quad \omega' = \beta(m+1) - m, \beta(m+2) - m, \dots, \beta(m+i) - m$$

$$\omega' : \prod \langle \omega' := [i] - (\beta(m) - m) \rangle$$

if

$$\omega = \beta^{-1}(m+\omega') - m$$

$$\chi' : \beta(m) - m, \beta(m+1) - m, \beta\{\beta^{-1}(m+\omega') - m - 1\} - m$$

=

$$\exists : 2, 3, 0, 1$$

$$(\alpha_3 - \alpha_2)(\alpha_0 - \alpha_2)(\alpha_0 - \alpha_3)(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_0)$$

$$(\alpha_1 - \alpha_0)(\alpha_2 - \alpha_0)(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_0)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)$$

$h, k \in [i]$ $\alpha_h - \alpha_k$ occurs once and only once in prod $\prod (\exists : \dots)$

$h = \beta(m+\omega')$ $k = \beta(m+\omega'')$ if $\omega' > \omega''$ $\alpha_h - \alpha_k$ occurs in

$\prod (\exists : \alpha \parallel m, \omega' | \alpha_{\beta(m+\omega')})$; if $\omega' < \omega''$, $\alpha_h - \alpha_k$ occurs in from

$\alpha_k - \alpha_h = \alpha_{\beta(m+\omega'')} - \alpha_{\beta(m+\omega')}$ in $\prod (\exists : \alpha \parallel m, \omega'' | \alpha_{\beta(m+\omega'')})$

if true

$[m, m+i]$

$h, k \in [i]$ $\alpha_h - \alpha_k$ occurs once and only once in product $\prod_{\substack{h=1 \\ h \neq k}}^n (\alpha_{m+h} - \alpha_{m+k})$ $\langle \omega := \text{LHS} \rangle$

if $h > k$, $\alpha_h - \alpha_k$ occurs in $\prod(\alpha || m, h-m | \alpha_h)$

$\prod_{k=1}^n (\alpha || m, \omega | \alpha_{m+k})$

if $h < k$, $\alpha_h - \alpha_k$ occurs from $\alpha_k - \alpha_h$ in $\prod(\alpha || m, k-m | \alpha_k)$

$$\begin{matrix} \bullet & \circ & \square & \times & \circ & \circ \\ \circ & \square & \times & \circ & \circ & \circ \end{matrix} \quad \begin{matrix} \frac{1}{3}(0) & \frac{1}{3}(1) & \frac{1}{3}(2) & \frac{1}{3}(3) & \cancel{\frac{1}{3}(4)} \\ 0 & 2 & 1 & 3 & \end{matrix}$$

$$\cancel{\frac{1}{3}(0)} < \cancel{\frac{1}{3}(2)} < \cancel{\frac{1}{3}(1)} < \cancel{\frac{1}{3}(0)} \quad \frac{1}{3}(0) < \frac{1}{3}(2) < \frac{1}{3}(1) < \cancel{\frac{1}{3}(2)}$$

$$\frac{1}{3}: \circ \ 2.1 \ \#$$

$$(\alpha_2 - \alpha_0)(\alpha_1 - \alpha_0)(\alpha_1 - \alpha_2)$$

$$(\alpha_1 - \alpha_0)(\alpha_2 - \alpha_0)(\alpha_2 - \alpha_1) \quad \text{sign change: invariance not preserved}$$

$$\text{W} \quad \text{par}(\frac{1}{3}) = \left\{ \sum_i \left\{ \sum_j \left[\frac{1}{3}(m+j) > \frac{1}{3}(m+i) \right] \right\} \langle \omega := \langle i \rangle \rangle \right\}_{m \in \mathbb{Z}}$$

$$\text{W} \quad (-1)^{\text{par}(\frac{1}{3})} \left\{ \prod_{k=1}^n \mu\left(\frac{1}{3} : \alpha || m, \omega | \alpha_{\frac{1}{3}(m+k)}\right) \langle \omega := \langle i \rangle \rangle \right\} =$$

$$\left\{ \prod_{k=1}^n \mu\left(\alpha || m, \omega | \alpha_{m+k}\right) \langle \omega := \langle i \rangle \rangle \right\} \text{ for all } \frac{1}{3} \in \mathbb{O}[m, m+i]$$

$$\left\{ \prod_{k=1}^n \mu\left(\frac{1}{3} : \alpha || m; i, \omega\right) \langle \omega := \langle i \rangle \rangle \right\} = \prod_{k=1}^n \mu\left(\alpha || m; i, \omega\right) \langle \omega := \langle i \rangle \rangle$$

$$= (-1)^{\frac{i(i+1)}{2}} \left\{ \prod_{k=1}^n \mu\left(\alpha || m, \omega | \alpha_{m+k}\right) \langle \omega := \langle i \rangle \rangle \right\}^{-2}$$

invariance (since parity factor α_{m+i} squared.)

$$= \begin{array}{c} \longleftarrow \rightarrow \\ h, h+k \\ \xrightarrow{\alpha_{m+i-1} \rightarrow \alpha} \end{array} \quad \alpha'_{h+\omega} = \alpha_{h+k-\omega} \quad m+i-1 = h + (m+i-h-i) \\ \quad h+k-m-i+h+1 = 2h+k-m-i+1$$

$$\langle 1 \rangle^{z+2} \prod_{\omega=0+1}^z (\alpha_{m+\omega-1} - z) = (-1)^{z+e+z-d} \prod_{\omega=0}^{z-e-1} (z - \alpha_{m+\omega+1})$$

$$\sum \mu(\alpha || m; e, \omega) \prod(\alpha || m, z | \alpha_{m+\omega}) \quad \omega := \left[\frac{z+e}{z-d} \right] = \omega := [\omega] \quad z \leq d$$

Jan 3-251

$$\text{for } p=2k: \quad \delta(g_{\alpha}^{(0)}; \alpha || m, i) = \sum_{\chi^{(k)}=0}^i \delta(g_{\alpha}^{(0)}; \alpha || m, \chi^{(k)}) \delta(g_{\alpha}^{(1)}; \alpha || m + \chi^{(k)}, i - \chi^{(k)})$$

$$\overset{\wedge}{\circ}{}^{(1)} := \overset{\circ}{\circ}{}^{(1)} \overset{\circ}{\circ}{}^{(2)}$$

$$\delta(g^{(0)}) \propto \chi(-i) - \chi(i) = \sum_{\chi(i)=0}$$

$$\sum_{\chi(1)=0} \delta(g^{(1)}; \alpha || m + \chi(0), \chi(1)) \delta(g^{(2)}; \alpha || m + \chi(0) + \chi(1); m + \chi(-1) - \chi(1))$$

$$k=1 \quad \hat{\phi}^{(0)} = \phi^{(0)} \phi^{(2)} - \sum_{\chi(i)=\chi(j)} \delta(\dots || m + \chi(i), \chi(i) - \chi(j)) \delta(\dots || m + \chi(i); m + \chi(-i) - \chi(i) + \chi(j))$$

$$\delta(\overset{\wedge}{\phi}^{(\omega)}; \alpha || m, x(0)) = \sum_{\chi(1)=0}^{\chi(0)} \delta(\overset{\wedge}{\phi}^{(\omega)}; \alpha || m, \chi(1)) \delta(\overset{\wedge}{\phi}^{(\omega)}; \alpha || m + x(1), \chi(\omega) - \chi(1))$$

$$S(\overset{(1)}{g}, \overset{(2)}{g}, \overset{(2)}{g}; \alpha(m, i)) =$$

$$\sum_{\chi(0)=0}^z \sum_{\chi(1)=0}^{x(0)} \delta(g^{(0)}; \alpha || m, \chi(1)) \delta(g^{(1)}; \alpha || m + \chi(0), z - x(0)) \delta(g^{(2)}; \alpha || m + \chi(1), \chi(0) - x(1)) \\ \Rightarrow \prod \delta(g^{(n+1)})$$

$$\hat{g}^{(0)}, \hat{g}^{(0)} g^{(k+2)}$$

$$\delta(\hat{g}^{(0)}; \alpha \| m, \chi(k)) = \sum_{\omega(kn)=0}^{\chi(k)} \delta(\hat{g}^{(0)}; \alpha \| m, \omega(k+1)) \delta(g^{(k+2)}; \alpha \| m + \omega(k+1), \omega(k) - \omega(kn))$$

$$\left\{ \prod \delta(g^{(\omega+1)}; \alpha \| m + \omega(\omega), \omega(\omega-1) - \omega(\omega)) \langle \omega := [k] \rangle \right\} \delta(g^{(k+2)}; \alpha \| m + \omega(k+1), \omega(k) - \omega(kn))$$

-

$$(3) \delta(\prod \hat{g}^{(\chi)} \langle \chi := [kn] \rangle; \alpha \| m, i) =$$

$$\left\{ \sum_{\omega(0)=0}^{\omega} \left\{ \sum_{\omega(1)=0}^{\omega(0)} \dots \left\{ \sum_{\omega(k)=0}^{\omega(k-1)} \delta(g^{(\omega)}; \alpha \| m, \omega(k)) \right. \right. \right.$$

$$\left. \left. \left. \left\{ \prod \delta(g^{(\omega+1)}; \alpha \| m + \omega(\omega), \omega(\omega-1) - \omega(\omega)) \langle \omega := [k] \rangle \right\} \right\} \right\}$$

$$\langle \omega(k) := [\omega(k-1)] \rangle \dots \langle \omega(1) := [\omega(0)] \rangle \dots \langle \omega(0) := [i] \rangle$$

$$\overbrace{k=0}^{\omega(0)=0} \delta(g^{(0)} g^{(1)}; \alpha \| m, i) =$$

$$\sum_{\omega(0)=0}^{\omega} \delta(g^{(0)}; \alpha \| m, \omega(0)) \delta(g^{(1)}; \alpha \| m + \omega(0), i - \omega(0)) \langle \omega(0) := [i] \rangle$$

assume true for ω as expressed.

$$\delta(g^{(\omega)} g^{(k+2)}; \alpha \| m, \omega(k)) =$$

$$\sum \delta(g^{(\omega)}; \alpha \| m, \omega(k+1)) \delta(g^{(k+2)}; \alpha \| m + \omega(k+1), \omega(k) - \omega(kn))$$

replace $g^{(\omega)}$ by $\hat{g}^{(0)} g^{(k+2)}$ in (*)

L.H.S. becomes resulting L.H.S. becomes that started with k replaced by $k+1$

Since

$$\left\{ \prod \delta(g^{(\chi+1)}; \alpha \| m + \omega(\chi), \omega(\chi-1) - \omega(\chi)) \langle \chi := [k] \rangle \right\} \delta(g^{(k+2)}; \alpha \| m + \omega(k+1), \omega(k) - \omega(kn))$$

$$= \left\{ \prod (g^{(kn)}; \alpha \| m + \omega(\chi), \omega(\chi-1) - \omega(\chi)) \langle \chi := [kn] \rangle \right\}$$

R.H.S. similarly modified

$$\sum_{\chi(0)=0}^i \delta(g^{(0)}; \alpha // m, i - \chi(0)) \delta(g^{(0)}; \alpha // m + i - \chi(0), \chi(0))$$

$$\delta(g^{(0)} g^{(2)}; \alpha // m, i - \chi(0)) = \sum \delta(g^{(0)}; \alpha // m, \chi(1)) \delta(g^{(2)}; \alpha // m + i - \chi(0) - \chi(1), \\ m + \chi(1), i - \chi(0) - \chi(1))$$

$$\chi(1) := [i - \chi(0)]$$

$$\sum_{\chi(0)=0}^i \delta(g^{(0)}; \alpha // m + \chi(0), i - \chi(0)) \delta(g^{(1)}; \alpha // m, \chi(0))$$

$$\sum_{\chi(0)=0}^i \sum_{\chi(1)=0}^i \delta(g^{(0)}; \alpha // m + \chi(1), i - \chi(0) - \chi(1)) \delta(g^{(1)}; \alpha // m + i - \chi(0), \chi(0))$$

$$\delta(g^{(2)}; \alpha // m, \chi(1))$$

$$k=2 \quad \delta(g^{(\omega)} g^{(1)} g^{(2)}; \alpha // m, i) =$$

$$\sum_{\omega(0)}^i \sum_{\omega(1)}^{\omega(0)} \sum_{\omega(2)}^{\omega(1)} \delta(g^{(0)}; \alpha // m, \omega(2)) \Rightarrow \omega(0) - \omega'(2) \Rightarrow \omega(-1) - \omega'(0) - \omega'(1) - \omega'(2)$$

$$\omega(0) = 0 \quad \omega(1) = \omega(2) = 0$$

$$\delta(g^{(1)}; \alpha // m + \omega(0), \omega(-1) - \omega'(0), \omega'(0)) \delta(g^{(2)}; \alpha // m + \omega(1); \omega(0) - \omega(1)) \\ \Rightarrow m + \omega(0) - \omega'(1), \omega'(1) \\ \Rightarrow m + \omega(-1) - \omega'(0) - \omega'(1), \omega'(1)$$

$$\delta(g^{(3)}; \alpha // m + \omega(2), \omega(1) - \omega(2)) \\ m + \omega(1) - \omega'(2), \omega'(2) \Rightarrow m + \omega(0) - \omega'(1) - \omega'(2), \omega'(2)$$

$$\omega(2) = \omega'(-1) - \omega'(2) \quad \omega(1) = \omega(0) - \omega'(1) \quad \omega(0) = \omega(1) - \omega'(0)$$

$$\sum_{\omega(0)}^i \sum_{\omega(1)}^{\omega(0)} \sum_{\omega(2)}^{\omega(1)} \delta(g^{(0)}; \alpha // m, \omega(-1) - \omega'(0) - \omega'(1) - \omega'(2))$$

$$\delta(g^{(1)}; \alpha // m + \omega(-1) - \omega'(0), \omega'(0)) \delta(g^{(2)}; \alpha // m + \omega(-1) - \omega'(0) - \omega'(1), \omega'(1))$$

$$\delta(g^{(3)}; \alpha // m + \omega(-1) - \omega'(0) - \omega'(1) - \omega'(2), \omega'(2))$$

$$\delta(g^{(0)}; \alpha // m, i - \sum \omega(k) \langle k := [k] \rangle)$$

$$\prod \delta(g^{(x+1)}; \alpha // m + i - \sum \omega(k) \langle k := [x] \rangle, \omega(x)) \langle x := [k] \rangle$$

$$\left\{ \begin{array}{l}
 \delta(\pi_{\circ}^{(x)} \langle x := [k_1] \rangle; \alpha \| m, i) = \\
 \{ \sum \{ \sum \dots \{ \sum \delta(\circ^{(0)}; \alpha \| m, i - \sum \omega(k) \langle k := [k] \rangle) \\
 \{ \prod \delta(\circ^{(x+1)}; \alpha \| m+i - \sum \omega(k) \langle k := [x] \rangle, \omega(x)) \langle x := [k] \rangle \} \\
 \langle \omega(k) := [\omega(k-1)] \dots \langle \omega(1) := [\omega(0)] \rangle \} \langle \omega(0) := [i] \rangle \} \\
 (\text{replace } \omega(x) \text{ by } \omega(x-1) - \omega(x) \text{ in } (*)) \\
 = \\
 m + i - \omega(x-1), \omega(x-1) - \omega(x) \quad m + i - \omega(k), \omega(k) \quad (*) \\
 - \\
 m + \sum \omega(k) \langle k := [x], \omega(x) \rangle \quad m + \sum \omega(k) \langle k := [k] \rangle \quad (***) \\
 \delta^{(0)} m + i - \omega(k), \omega(k) \quad \circ^{(0)}; m, i - \omega(0) \quad \circ^{(1)}; m + i - \omega(0), \omega(0) - \omega(1) \quad (**) \\
 \circ^{(0)} m + \sum \omega(k) \langle k := [k] \rangle, i - \sum \omega(k) \langle k := [k] \rangle \\
 \circ^{(1)} m, \omega(0) \quad \circ^{(2)} m + \omega(0), \omega(1) \quad \circ^{(3)}; m + \omega(0) + \omega(1), \omega(2) \quad (***)
 \end{array} \right.$$

$$\left\{ \begin{array}{l}
 \delta(\prod \circ^{(x)} \langle x := [k_1] \rangle; \alpha \| m, i) = \\
 \{ \sum \{ \sum \dots \{ \sum \delta(\circ^{(0)}; \alpha \| m+i - \omega(k), \omega(k)) \\
 \{ \delta(\circ^{(x+1)}; \alpha \| m+i - \omega(x-1), \omega(x-1) - \omega(x)) \langle x := [k] \rangle \} \\
 \langle \omega(k) := [\omega(k-1)] \} \dots \langle \omega(1) := [\omega(0)] \rangle \} \langle \omega(0) := [i] \rangle \} \\
 \{ \sum \{ \sum \dots \{ \sum \delta(\circ^{(0)}; \alpha \| m + \sum \omega(k) \langle k := [k] \rangle, i - \sum \omega(k) \langle k := [k] \rangle) \\
 \{ \delta(\circ^{(x+1)}; \alpha \| m + \sum \omega(k) \langle k := [x] \rangle, \omega(x)) \langle x := [k] \rangle \} \\
 \langle \omega(k) := [\omega(k-1)] \} \dots \langle \omega(1) := [\omega(0)] \rangle \} \langle \omega(0) := [i] \rangle \} \\
 (\text{[} \underline{\omega_1, m+i} \text{] reverse } \alpha \text{ [} \underline{m, m+i} \text{] } \alpha, \circ^{(0)}, \dots, \circ^{(k+1)} \text{ [} \underline{m, m+i} \text{] reverse order} \\
 \text{forms of } (*) \text{ and } (**))
 \end{array} \right.$$

$$A(1|\tau, \nu)$$

$$\left\{ \sum \alpha(1|\tau, \omega) \alpha(0|\omega, \nu) \quad \langle \omega := [\nu, \tau] \rangle \right\}$$

$$A(2|\tau, \nu)$$

$$\left\{ \sum \alpha(2|\tau, \omega(1)) \left\{ \sum \alpha(1|\omega(1), \omega(0)) \alpha(0|\omega(0), \nu) \quad \langle \omega(0) := [\nu, \omega(1)] \rangle; \right. \right. \\ \left. \left. \langle \omega(1) := [\nu, \tau] \rangle \right\} \right\}$$

$$\left\{ \sum \alpha(3|\tau, \omega(2)) \left\{ \sum \alpha(2|\omega(2), \omega(1)) \left\{ \sum \alpha(1|\omega(1), \omega(0)) \alpha(0|\omega(0), \nu) \quad \langle \omega(0) := [\nu, \omega(1)] \rangle; \right. \right. \right. \\ \left. \left. \left. \langle \omega(1) := [\nu, \omega(2)] \rangle \right\} \langle \omega(2) := [\nu, \tau] \rangle \right\} \right\}$$

$$A(k|\tau, \nu) =$$

$$\left\{ \sum \alpha(k|\tau, \omega(k-1)) \left\{ \sum \alpha(k-1|\omega(k-1), \omega(k-2)) \right\} \left\{ \sum \alpha(k-2|\omega(k-2), \omega(k-3)) \right. \right. \\ \left. \left. \dots \left\{ \sum \alpha(1|\omega(1), \omega(0)) \alpha(0|\omega(0), \nu) \right\} \right\} \langle \omega(0) := [\nu, \omega(1)] \rangle \langle \omega(1) := [\nu, \omega(2)] \rangle \dots \right. \\ \left. \left. \left. \langle \omega(k-2) := [\nu, \omega(k-1)] \rangle \right\} \langle \omega(k-1) := [\nu, \tau] \rangle \right\} \right\}$$

$$= \omega(k) := [\nu, \tau]$$

$$\sum \alpha(3|\tau, \omega(2)) \left\{ \sum \alpha(2|\omega(2), \omega(1)) \left\{ \sum \alpha(1|\omega(1), \nu + \omega(0)) \alpha(0|\nu + \omega(0), \nu) \right. \right. \\ \left. \left. \langle \omega(0) := [\omega(1) - \nu] \rangle \right\} \langle \omega(1) := [\nu, \omega(2)] \rangle \right\} \langle \omega(2) := [\nu, \tau] \rangle$$

$$\sum \alpha(3|\tau, \omega(2)) \left\{ \sum \alpha(2|\omega(2), \nu + \omega(1)) \left\{ \sum \alpha(1|\nu + \omega(1), \nu + \omega(0)) \right. \right. \\ \left. \left. \alpha(0|\nu + \omega(0), \nu) \right\} \right\}$$

$$\langle \omega(0) := [\omega(1)] \rangle \left\{ \langle \omega(1) := [\omega(2) - \nu] \rangle \right\} \langle \omega(2) := [\nu, \tau] \rangle$$

$$\sum \alpha(3|\tau, \nu + \omega(2)) \left\{ \sum \alpha(2|\nu + \omega(2), \nu + \omega(1)) \left\{ \sum \alpha(1|\nu + \omega(1), \nu + \omega(0)) \right. \right. \\ \left. \left. \alpha(0|\nu + \omega(0), \nu) \right\} \right\}$$

$$\langle \omega(0) := [\omega(1)] \rangle \left\{ \langle \omega(1) := [\omega(2)] \rangle \right\} \langle \omega(2) := [\tau - \nu] \rangle$$

$$\tau := \nu + \omega(0)$$

$$A(k | \nu + \omega(k), \nu) :=$$

$$\left\{ \sum_i a(k | \nu + \omega(k), \nu + \omega(k-i)) \left\{ \sum_i a(k-1 | \nu + \omega(k-i), \nu + \omega(k-2)) \dots \right. \right.$$

$$\left. \left. \left\{ \sum_i a(1 | \nu + \omega(1), \nu + \omega(0)) a(0 | \nu + \omega(0), \nu) \right\} \right\} \right\}$$

$$\langle \omega(0) := [\omega(1)] \rangle \langle \omega(1) := [\omega(2)] \rangle \dots \langle \omega(k-1) := [\omega(k)] \rangle \}$$

$$\omega(k) := [i-\nu] \quad \nu := [i]$$

Let $i, \nu \in \overline{\mathbb{N}}^*, n \in \mathbb{N}$ and $a(k) \in \{ \text{L}[k|i] \mid k := [n] \}$. Set

Define $A(k) \in \mathcal{L}[k|i] \mid k := [n]$ by setting

$$A(k) := \overline{T(a(x))} \quad \langle x := [k] \rangle$$

For $\nu := [i]$, $k := [n]$ and $\omega := [i]$

$$A(k | \omega(k), \nu) =$$

$$\left\{ \sum_i a(k | \omega(k), \omega(k-i)) \left\{ \sum_i a(k-1 | \omega(k-i), \omega(k-2)) \right. \right.$$

$$\left. \left. \left\{ \sum_i a(k-2 | \omega(k-2), \omega(k-3)) \dots \left\{ \sum_i a(1 | \omega(1), \omega(0)) a(0 | \omega(0), \nu) \right\} \right\} \right\} \right\}$$

$$\langle \omega(0) := [\nu, \omega(1)] \rangle \langle \omega(1) := [\nu, \omega(2)] \rangle \dots$$

$$\langle \omega(k-2) := [\nu, \omega(k-1)] \rangle \langle \omega(k-1) := [\nu, \omega(k)] \rangle \}$$

for $\omega(k) := [\nu, i]$ and

$$A(k | \nu + \omega(k), \nu) :=$$

$$\left\{ \sum_i a(k | \nu + \omega(k), \nu + \omega(k-i)) \left\{ \sum_i a(k-1 | \nu + \omega(k-i), \nu + \omega(k-2)) \dots \right. \right.$$

$$\left. \left. \left\{ \sum_i a(1 | \nu + \omega(1), \nu + \omega(0)) a(0 | \nu + \omega(0), \nu) \right\} \right\} \right\}$$

$$\langle \omega(0) := [\omega(1)] \rangle \langle \omega(1) := [\omega(2)] \rangle \dots \langle \omega(k-1) := [\omega(k)] \rangle$$

for $\omega(k) := [i-\nu]$.

(Replace $\omega(x)$ by $\nu + \omega(x)$ in first result to obtain second.)

$$\sum \pi(\alpha || m, \tau | \alpha_{m+\omega}) f_{m+\omega} \mu(\alpha || m; \tau, \omega) \quad \langle \omega := [\tau, \omega] \rangle \equiv \langle \omega := [\omega] \rangle$$

$$= \delta(\pi(\alpha || m, \tau | \langle \omega \rangle) \times f; \alpha || m, \omega) = \delta(f; \alpha || m + \tau, \tau - \omega)$$

$$\sum \pi(\alpha || m, i - \tau | \alpha_{m+i-\omega}) f_{m+i-\omega} \mu(\alpha || m; i - \tau, i - \omega) \quad \langle \omega := [\tau, \omega] \rangle \equiv \langle \omega := [\omega, i] \rangle$$

$$\sum \pi(\alpha || m, i - \tau | \alpha_{m+i}) f_{m+i} \mu(\alpha || m; i - \tau, \omega) \quad \langle \omega := [i - \tau] \rangle$$

$$= \delta(\pi(\alpha || m, i - \tau | \langle \omega \rangle) \times f; \alpha || m, i - \tau)$$

$$= \delta(f; \alpha || m + i - \tau, \tau - \omega)$$

$$\sum \pi(\alpha || m + \tau + 1, i - \tau | \alpha_{m+\omega}) f_{m+\omega} \mu(\alpha || m + \tau + 1, i - \tau, \omega - \nu) \quad \langle \omega := [\tau, \omega] \rangle = \langle \omega, i \rangle$$

$$= \pi(\alpha || m + \tau + 1, i - \tau | \langle \omega \rangle) \times f; \alpha || m + \tau + 1, i - \tau$$

$$m + i - \tau + 1, \tau \quad m + \omega \quad 2m + i - m - \tau \quad 2m + i - m - \nu, \tau - \nu$$

$$U[\pi(\alpha || m + i - \tau + 1, \tau | \alpha_{m+i-\omega})] \text{diag}[f_{m+i-\omega}] U[\mu(\alpha || m + \tau + 1, m + \omega - \nu, \tau - \nu)]$$

$$L[\pi(\alpha || m + \tau + 1, i - \tau | \alpha_{m+\omega})] \text{diag}[f_{m+\omega}] L[\mu(\alpha || m + \tau + 1, m + \omega - \nu, \tau - \nu)]$$

$$m + \tau, \dots, m + \omega \quad m + i - \tau, \tau - \nu \quad = L[\delta(f; \alpha || m + \tau, \tau - \nu)]$$

U reverse order from 291 last result p.291 without rearrangement ??

first result of 292 = reverse order from 291 line-3 + rearrangement

$$m + i - \tau - 1 \Rightarrow m + \tau + 1 \quad 2m + i - m - i + 2 \quad i - \tau \quad \tau - \nu \quad m + \omega, \tau - \nu$$

$$L[\pi(\alpha || m + \tau + 1, i - \tau | \alpha_{m+\omega})] \text{diag}[f_{m+\omega}] L[\mu(\alpha || m + \tau + 1, i - \tau, \tau - \nu)]$$

$$= L[\delta(f; \alpha || m + \tau, \tau - \nu)]$$

i.e. first result of 292.

$$\sum \pi(\alpha // m - z, z - \omega | \alpha_m) \pi(\alpha // m - \omega + 1, \omega) \langle \omega := [z] \rangle$$

$z = i + \omega'$

$$\pi(m+i-z, z - \omega | \alpha_{m+i}) \pi(\alpha // m+i - \omega + 1, \omega) \langle \omega := [z] \rangle$$

$$\omega' = i - \omega \quad \omega' := [i - z, i]$$

$$\sum \pi(\alpha // m+i - z, z - i + \omega | \alpha_{m+i}) \pi(\alpha // m + \omega + 1, i - \omega) \langle \omega := [i - z, i] \rangle$$

$$\pi(\alpha // m, i - z | \alpha_{m+i})$$

$$= \text{col} [\pi(\alpha // m + i - z + 1, z)] = \mathcal{ULR} [\alpha_{m+i-\omega} - \alpha_{m+i}] \text{col} [\pi(\alpha // m + \overline{z}, z)]$$

$$(\alpha_{m+i-z} - \alpha_{m+i}) \pi(\alpha // m + \overline{z}, z - 1 / z) + (\alpha_{m+i} - \alpha_{m+i-z}) \pi(\alpha // m + \overline{z}, z / z) =$$

$$\pi(\alpha // m+i-z+1, z/z)$$

$$= (\alpha_{m+i-z} - \alpha_{m+i} + z - \alpha_{m+i-z}) \pi(\alpha // m+i-z+1, z-1/z) =$$

$$\left. \begin{aligned} & \sum \pi(\alpha // m+i-z, z - \omega | \alpha_{m+i}) \pi(\alpha // m+i - \omega + 1, \omega) \langle \omega := [z] \rangle \\ & = \pi(\alpha // m+i-z, z) \end{aligned} \right\} \text{second result on p 288}$$

$$m+i-z = m', z = i' \quad m+i-z + \cancel{i} = z - \omega + 1 \quad \parallel m+i-z, m+i-1$$

$$\sum \pi(\alpha // m', \omega | \alpha_{m+i}) \pi(\alpha // m' + \omega + 1, i - \omega) = \pi(\alpha // m', i) \langle z = i \rangle$$

two results on p 288 $\pi(\alpha // m+1, r) = \dots$ lhs indep. of α_m

$\pi(\alpha // m, r) = \dots$ lhs indep. of α_m

$$h \dots m \quad m+i \quad h+k \quad \alpha_{m+i} = \alpha'_{h+h+k-m-i}$$

$$\frac{\pi(\alpha // m+2, 2i-\omega / z)}{\pi(\alpha // m+2i-z, z / z)} = \pi(\alpha // m+2, 2i-\omega / z)$$

$$u[\Lambda(f; \alpha \| m+z, \bar{z}-\bar{r})] = u\left[\pi(\alpha \| m+z, \bar{z}-\bar{r}+1|z) S\left(\frac{f}{z-\langle \alpha \rangle}; \alpha \| m+z, \bar{z}-\bar{r}\right) \right]^{31}$$

$$\pi(\alpha \| m+z, \bar{z}-\bar{r}+1|z) = \frac{\pi(\alpha \| m, \bar{z}+1|z)}{\pi(\alpha \| m, \bar{z}|z)}$$

$$= \text{diag} [\pi(\alpha \| m, \omega | z)^{-1}] u[S(f; \alpha \| m+z, \bar{z}-\bar{r})] u\left[S\left(\frac{1}{z-\langle \alpha \rangle}; \alpha \| m+z, \bar{z}-\bar{r}\right) \right]$$

$$\text{diag} [\pi(\alpha \| m, \omega+1|z)]$$

$$= \text{diag} [\pi(\alpha \| m, \omega | z)^{-1}] u[\pi(\alpha \| m, \tau | \alpha_{m+\omega})] \text{diag}[f_{m+\omega}] u[\mu(\alpha \| m; \bar{z}, \tau)] \times \\ u\left[\frac{1}{\pi(\alpha \| m+z, \bar{z}-\bar{r}+1|z)} \right] \text{diag} [\pi(\alpha \| m, \omega+1|z)] \Rightarrow u[\pi(\alpha \| m, \tau | z)]$$

from (198) $\sum_i \mu(\alpha \| m; \omega, \tau) \pi(\alpha \| m, \omega) \langle \omega := [z, i] \rangle = \lambda(\alpha \| m; i, \tau)$

$$\sum_i \mu(\alpha \| m; \omega, \tau) \pi(\alpha \| m, \omega | z) \langle \omega := [z, i] \rangle = \lambda(\alpha \| m; i, \tau) z$$

$$\Rightarrow \text{diag} [\pi(\alpha \| m, \omega | z)^{-1}] u[\pi(\alpha \| m, \tau | \alpha_{m+\omega})] \text{diag}[f_{m+\omega}] u[\lambda(\alpha \| m; \bar{z}, \tau)]$$

- also $\sum_i \pi(\alpha \| m, \tau | \alpha_{m+\omega}) f_{m+\omega} \lambda(\alpha \| m; i, \tau) \langle \omega := [z, i] \rangle = \omega := [z]$
 $= \Lambda(\pi(\alpha \| m, \tau | \langle \alpha \rangle) f; \alpha \| m, \bar{z})$

$$\Rightarrow \text{diag} [\pi u[\Lambda(f; \alpha \| m+z, \bar{z}-\bar{r})]] = \text{diag}$$

$$\text{diag} [\pi(\alpha \| m, \omega+1)^{-1}] u[\Lambda(\pi(\alpha \| m, \tau | \langle \alpha \rangle) f; \alpha \| m, \bar{z})]$$

$$u u[\Lambda(f; \alpha \| m+z, \bar{z}-\bar{r})] =$$

$$\text{diag} [\pi(\alpha \| m, \omega)^{-1}] u\left[\frac{1}{\pi(\alpha \| m+z, \bar{z}-\bar{r}+1)} \right] u[f(f; \alpha \| m+z, \bar{z}-\bar{r})] \text{diag} [\pi(\alpha \| m, \omega)]$$

$$u = \text{diag} [\pi(\alpha \| m, \omega)] u[i^{-1}] u[S(f; \alpha \| m+z, \bar{z}-\bar{r})] \text{diag} [\pi(\alpha \| m, \omega+1)]$$

$$\text{diag}[f_{m+\omega}] = u[\mu(\alpha \| m, \tau | \alpha_{m+\omega}) \text{diag} [\pi(\alpha \| m, \omega)] u[\Lambda(f; \alpha \| m+z, \bar{z}-\bar{r})] u[\lambda(\alpha \| m; \bar{z}, \tau)] \\ \left[\frac{\pi(\alpha \| m, \bar{z}+1|z)}{z - \alpha_{m+\omega}} \mu(\alpha \| m; \bar{z}, \tau) \right]^{-1}$$

$$\sum_i (\alpha \parallel m; z | 0) = \alpha_m \dots \alpha_{m+z-1} \quad \sum_i (\alpha \parallel m; z | z-1) = \alpha_m \alpha_{m+1} \dots \alpha_{m+z-1}$$

$$\text{II } \left\{ \sum_i \sum_j (\alpha \parallel m; z | \omega) (-1)^{z+i} z^\omega \langle \omega := [z] \rangle \right\} := \pi(\alpha \parallel m, z | z)$$

$$[\alpha_{m+z}] \not\in [(-1)^{z+1} \sum_i (\alpha \parallel$$

$$L[\delta(\langle \alpha \rangle^z; \alpha \parallel m, \omega)] L[(-1)^{z+1} \sum_i (\alpha \parallel m; z | \omega)] [\alpha_{m+z}] =$$

$$\text{III } L[\delta(\langle \alpha \rangle^z; \alpha \parallel m, \omega)] L[\pi(\alpha \parallel m; z | \alpha_{m+z})] = [\alpha_{m+z}^z] \quad \text{sup p 157}$$

$$(z - \alpha_m) \left\{ \sum_i \sum_j (\alpha \parallel m+1, z-1, \omega) (-1)^{z+i+1} z^\omega \langle \omega := [z-1] \rangle \right\}$$

$$= \sum_i \sum_j (\alpha \parallel m; z | \omega) (-1)^{z+i} z^\omega \langle \omega := [z] \rangle$$

$$(-1)^{z+1} \sum_i (\alpha \parallel m+1, z-1, \omega-1) + \alpha_m (-1)^{z+1} \sum_i (\alpha \parallel m+1; z-1, \omega) \quad \textcircled{2}$$

$$= (-1)^{z+1} \sum_i (\alpha \parallel m; z | \omega) \langle \omega := [z] \rangle \left(\begin{array}{l} \sum_i (\alpha \parallel m; z, z) = 1 \\ \sum_i (\alpha \parallel m; z, \omega) = 0 \quad \text{if } z > \omega \\ \quad \quad \quad \text{if } z < \omega \end{array} \right)$$

$$(z - \alpha_{m+z-1}) \left\{ \sum_i \sum_j (\alpha \parallel m, z-1, \omega) (-1)^{z+i+1} z^\omega \langle \omega := [z-1] \rangle \right\}$$

$$= \sum_i \sum_j (\alpha \parallel m; z | \omega) (-1)^{z+i} z^\omega \langle \omega := [z] \rangle \quad \textcircled{3}$$

$$\text{I } \sum_i (\alpha \parallel m, z-1, \omega-1) + \alpha_{m+z-1} \sum_i (\alpha \parallel m, z-1, \omega) = \sum_i \alpha \parallel (m; z | \omega) \langle \omega := [z] \rangle$$

$$\sum_i (\alpha \parallel m, z, \omega) = (\alpha_{m+z} - \alpha_m) \sum_i (\alpha \parallel m+1, z-1, \omega) = \sum_i (\alpha \parallel m+1, z, \omega) \quad \langle \omega := [z] \rangle$$

from $(\beta : m := m+1) - \textcircled{2}$

$$z - \alpha_m = (\alpha_{m+1} - \alpha_m) (z + z - \alpha_{m-1})$$

$$= V^1 V^0 : \mathcal{ULZ}[-\alpha_{m+z-1}] L[\delta(\langle \alpha \rangle^z; \alpha \parallel m, \omega)]$$

$$- \alpha_{m+z-1} \cdot \delta(\langle \alpha \rangle^{z-1}; \alpha \parallel m, \omega) + \delta(\langle \alpha \rangle^z; \alpha \parallel m, \omega) = \delta(\langle \alpha \rangle^{z-1}; \alpha \parallel m+1, z-1)$$

$$\begin{aligned}\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, v) &= \alpha_m \delta(\langle \alpha \rangle^{\tau-1}; \alpha \parallel m+1, v) + \delta(\langle \alpha \rangle^{\tau-1}; \alpha \parallel m+H, v-1) \\ &= \delta(\langle \alpha \rangle^{\tau-1}; \alpha \parallel m, v-1) + \delta(\langle \alpha \rangle^{\tau-1}; \alpha \parallel m, v) \alpha_{m+1}\end{aligned}$$

$$\vee^0 u \mathcal{L} D[-\alpha_m] \langle (i) \rangle \mathcal{L} [\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, v)] \langle z, v := [i] \rangle$$

$$= \vee^1 \mathcal{L} [\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m+1, v)] \langle z, v := [i-1] \rangle$$

$$\vee^1 u \mathcal{L} D[-\alpha_{m+1}] \langle (i-1) \rangle \vee^1 \mathcal{L} [\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m+1)]$$

$$= \vee^2 \mathcal{L} [\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m+2, v)] \langle z, v := [i-2] \rangle$$

$$\prod \left[\left\{ \vee^{\omega} \{ u \mathcal{L} D[-\alpha_{m+\omega}] \langle (i-\omega) \rangle \} \right\} \mathcal{L} [\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, v)] \langle z, v := [i] \rangle \right] \langle \omega := [k] \rangle$$

$$= \vee^{k+1} \left\{ \mathcal{L} [\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m+k+1, v)] \langle z, v := [i-k-1] \rangle \right\}$$

$$\prod \left[\left\{ \vee^{\omega} \{ u \mathcal{L} D[-\alpha_{m+\omega}] \langle (i-\omega) \rangle \} \right\} \mathcal{L} [\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, v)] \langle z, v := [i] \rangle \right] \langle \omega := [i-k] \rangle$$

$$= [i]$$

\therefore

$$\delta(\langle \alpha \rangle^i; \alpha \parallel m, v)x + \delta(\langle \alpha \rangle^i; \alpha \parallel m, v+1)y = 0$$

$$x \delta(\langle \alpha \rangle^{i-1}; \alpha \parallel m, v) + y \delta(\langle \alpha \rangle^{i-1}; \alpha \parallel m, v) = 0 \quad x, y \text{ ind if } ?$$

$$\begin{aligned}1 &\quad 1 & 1 & 0 & 1 & = \left[\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, v) \right] \\ \cancel{1} & \cancel{1} & \cancel{0} & \cancel{1} & \cancel{0} & \\ \alpha_{m+1} & \alpha_{m+2} & \alpha_{m+1} & \alpha_{m+2} & \alpha_{m+2} & \\ & = \left[\cancel{0} \quad 1 \quad 0 \right] ? \\ & \quad 0 \quad \left[\delta(\langle \alpha \rangle^{\tau+1}; \alpha \parallel m, v) \right] ?\end{aligned}$$

$$\delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, 0) + \alpha_{m+1} \delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, 1) = \delta(\langle \alpha \rangle^{\tau+1}; \alpha \parallel m, 1)$$

$$\{ \delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, v) \} + \alpha_{m+2} \delta(\langle \alpha \rangle^{\tau}; \alpha \parallel m, v+1) = \delta(\langle \alpha \rangle^{\tau+1}; \alpha \parallel m, v+1)$$

$$\begin{aligned}
 & \left[\delta(\langle \alpha \rangle^{j+k}; \alpha \parallel m, \omega) \right] \prod_{i=1}^m \text{U}_{\omega, 2} [\alpha_{m+i}, \omega + k - i] \quad \begin{matrix} \langle \omega := [k] \rangle \\ \langle x_i = i \rangle \end{matrix} \quad 3/3 \\
 & = \left[\delta(\langle \alpha \rangle^{j+k+1}; \alpha \parallel m, \omega + k + 1) \right] \quad j, k \in \mathbb{N} \\
 & \quad [\delta(\langle \alpha \rangle^{j+k+2}; \alpha \parallel m, \omega + k + 2)] \text{U}_{\omega, 2} [\overline{\alpha_{m+k+1}}, \overline{\omega + k + 1}] \\
 & = [\delta(\langle \alpha \rangle^{j+k+2}; \alpha \parallel m, \omega + k + 2)] \\
 & \quad \delta(\langle \alpha \rangle^{j+k+1}; \alpha \parallel m, \omega + k + 1) + \alpha_{m+k+2} \delta(\langle \alpha \rangle^{j+k+1}; \alpha \parallel m, \omega + k + 2) \\
 & = \delta(\langle \alpha \rangle^{j+k+2}; \alpha \parallel m, \omega + k + 2) \quad \therefore !, !, = \text{defn}
 \end{aligned}$$

$$\begin{aligned} & \pi \left[\left\{ \vee^{\omega} \{ \text{URL2}[-\alpha_{m+\omega}] \langle \langle i-\omega \rangle \rangle \} \right\} \left[S(\langle \alpha \rangle^{j+z}; \alpha \parallel m, \omega) \right] \langle z, \omega := [i] \rangle \right. \\ & \quad \left. \langle \omega := [k] \rangle \right] \\ & = \vee^{k+1} \left\{ \left[S(\langle \alpha \rangle^{j+z}; \alpha \parallel m+k-1, \omega) \right] \langle z, \omega := [i-k-1] \rangle \right\} \end{aligned}$$

④ incorrect when $i=0$ since $\delta(\langle \alpha \rangle^{*j}; \alpha || m, 0) 1^{k+1} + \delta(\langle \alpha \rangle^{j+k+1}; \alpha || m, k+1)$

(**) holds for $\omega := (i)$

$$\text{when } \beta = i \quad \delta(\langle \alpha \rangle^{j+k+2+1}; \alpha || m, i+k+1) + \delta(\langle \alpha \rangle^{j+k+2+2}; \alpha || m, i+k+2)$$

if $j > 0$ is correct when $j + K + \cancel{z} \leq i + K$

i.e. concerning $[\delta(\langle\alpha\rangle^{j+z}; \alpha \parallel m, \nu)] \langle z := [h], \nu := [i] \rangle$

where h_{i-j} result to be obtained when $j=0$ by stripping away first j rows

$$= - \sum_{i=1}^m \begin{pmatrix} 1 & 0 \\ -\alpha_m & 1 \end{pmatrix} \begin{pmatrix} \alpha_m^j & \frac{\alpha_{m+1}^j - \alpha_m^j}{\alpha_{m+1}^j - \alpha_m^j} \\ \alpha_m^{j+1} & \frac{\alpha_{m+1}^{j+1} - \alpha_m^{j+1}}{\alpha_{m+1}^{j+1} - \alpha_m^{j+1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{m+1}^j \end{pmatrix} \quad \text{again only when } j=0$$

$$\begin{aligned} & \mathcal{L}\left[(-1)^{\omega} \sum_{\alpha \parallel m; \tau, \omega} \right] \langle z, \omega := [i] \rangle = \\ & \overline{\prod} \left\{ \nu^{\omega} \left\{ \mathcal{U} \mathcal{L} \mathcal{Z}[-\alpha_{m+\omega}] \langle (i-\omega) \rangle \right\} \right\} \langle \omega := [i] \rangle \\ & \left\{ \mathcal{U} \mathcal{L} \mathcal{Z}[-\alpha_{m+\omega}] \langle (i-\omega) \rangle \right\}^{-1} = \mathcal{U} \mathcal{L} \left[(-1)^{\omega} (-1)^{\omega} \alpha_{m+\omega}^{z \rightarrow \tau} \right] \langle \tau, \omega := [i-\omega] \rangle \end{aligned}$$

$$\begin{matrix} i=1 & 1 & \cdots & 1 \\ & \omega_m & 1 & \cdots & \omega_m & 1 \end{matrix}$$

$$\begin{aligned}
 & \left[\delta(\langle \alpha \rangle^{j+z}; \alpha \parallel m, z) \right] \xleftarrow{\zeta := \lfloor \frac{i}{k} \rfloor} \mathcal{U} \left[\zeta (\alpha \parallel m; z / \alpha_{m, z}) \right] \langle z, \zeta := \lfloor \frac{i}{k} \rfloor \rangle \\
 & \quad \langle z := \lfloor \frac{i}{k} \rfloor \rangle \\
 & = \left[\alpha_{m, z}^{j+z} \right] \xleftarrow[\zeta := \lfloor \frac{i}{k} \rfloor]{\zeta := \lfloor \frac{i}{k} \rfloor}
 \end{aligned}$$

$$\left\{ \text{UL2} \left[\alpha_{m+k-\omega+\chi} \right] \langle x := (i) \rangle \right\}^{-1}$$

$$uL\left[(-1)^{z\omega} \overline{P}_{m+k-\omega+\chi}^{\chi} \langle \chi := (2, z) \rangle\right] = \overline{\prod \alpha_{\chi} \langle \chi := (m+k-\omega+\omega, m+k-\omega+\bar{\omega}) \rangle}$$

$$\left\{ \mathcal{L} \left[\delta(\langle \alpha \rangle^{k+z+1}; \alpha \parallel m, d+k+1) \right] \langle z, d := \langle i, j \rangle \right\}$$

$$\prod_{k=1}^m \text{U}\mathcal{L} \left[(-1)^{\alpha_k} T \right] \alpha_k < \chi := (m+1), m+2 \dots \rangle \left[\begin{matrix} m+1 \\ m+2 \end{matrix} \right] \left\langle \begin{matrix} m+1 \\ m+2 \end{matrix} \right\rangle = \left\langle \begin{matrix} m+1 \\ m+2 \end{matrix} \right\rangle$$

$$= \mathcal{L}[\delta(\langle\alpha\rangle^z; \alpha \parallel m, \nu)] \langle z, \nu = [i] \rangle$$

$$\pi(\alpha \parallel m, i+1 | y) \sum \frac{\pi(\alpha \parallel m, i | x)}{\pi(\alpha \parallel m, \omega+1 | y)} \langle \omega := [i] \rangle$$

$$= (y - \alpha_{m+i}) \left\{ \frac{\pi(\alpha \parallel m; i | x) - \pi(m; i | y)}{x - y} \right\} + \pi(\alpha \parallel m, i | x)$$

$$= \frac{(y - \alpha_{m+i} + x - y) \pi(\alpha \parallel m; i | x) - (y - \alpha_{m+i}) \pi(\alpha \parallel m; i | y)}{x - y}$$

—

$$\sum \delta(f; \alpha \parallel m, \omega) \left\{ \sum \frac{\pi(\alpha \parallel m, \omega | x)}{\pi(\alpha \parallel m, \omega+1 | y)} \langle \omega := [\omega] \rangle \right\} \xrightarrow{\text{def}} \pi(\alpha \parallel m, \omega | y) \langle \omega := [\omega] \rangle$$

$$\sum_i \frac{\pi(\alpha \parallel m, \omega | x)}{\pi(\alpha \parallel m, \omega+1 | y)} \left\{ \sum_j \pi(\alpha \parallel m, \omega | y) \delta(f; \alpha \parallel m, \omega) \langle \omega := [\omega], i \rangle \right\} \langle \omega := [i] \rangle$$

$$i=1 \text{ l.h.s.} = \delta(f; \alpha \parallel m, 1) = \text{r.h.s}$$

$$\mathcal{L}[\pi(\beta \parallel k, \omega, \tau - \omega | \alpha_m)] \vdash \mathcal{L}[\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega)]$$

$$\begin{aligned} & \sum \delta(\pi(\alpha \parallel m, \omega)) \delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega) \pi(\beta \parallel k, \omega - \omega | \alpha_m) \\ & \quad \langle \omega := [\omega, \tau] \rangle \\ & \pi(\beta \parallel k, \omega | \alpha_m) \pi(\beta \parallel k, \omega - \omega | \alpha_m) = \pi(\beta \parallel k, \omega | \alpha_m) \\ & \therefore = \frac{\pi(\alpha \parallel m, \tau | \alpha_m) - \Delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \tau - \omega | \alpha_m)}{\pi(\beta \parallel k, \omega | \alpha_m)} \end{aligned}$$

$$= - \frac{\pi(\beta \parallel k, \omega | \alpha_m)}{\pi(\beta \parallel k, \omega | \alpha_m)} \delta\left(\frac{\pi(\alpha \parallel m, \tau | \langle \beta \rangle)}{\alpha_m - \langle \beta \rangle}; \beta \parallel k, \tau - \omega | \alpha_m\right)$$

$$= \delta(\pi(\alpha \parallel m, \tau - \omega | \langle \beta \rangle); \beta \parallel k, \tau - \omega | \alpha_m)$$

$$\mathcal{L}[\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega)] \xrightarrow{\text{replace by mult p322}} \langle \tau, \omega := [i] \rangle \quad \text{④}$$

$$\prod \left\{ V^{i-\omega-1} \left\{ \mathcal{L}[\pi(\beta \parallel k, \tau - \omega | \alpha_{m+i-\omega-1})] \langle \tau, \omega := [\omega+1] \rangle \right\} \langle \omega := [j, i] \rangle; \right.$$

$$= V^{i-j} \mathcal{L}[\delta(\alpha \parallel \cancel{m+i-\omega-1}, \tau | \langle \beta \rangle); \beta \parallel k, \omega] \quad \begin{aligned} \langle \omega := [i-j, i] \\ [i-j, i] \end{aligned}$$

$$\langle j := [i] \rangle \uparrow$$

$$j = [s, e]$$

$$j' = i - j$$

$$\langle \omega := [j', i] \rangle$$

$$j' = [i]$$

$$\text{II ④ with } \langle \omega := [i] \rangle = \bar{I}[i]$$

$$A \in [K|i] \quad V^0 A = \bar{I}[i] \quad \omega \Delta i \quad \parallel \quad B = \bar{\prod} A(\omega) \quad \langle \omega := [k] \rangle$$

$$V^i A = \bar{I}[i] \text{ iff } A_0 = 1$$

$$\bar{B}' = \bar{\prod} A(k-\omega)^{-1} \quad \langle \omega := [k] \rangle$$

$$\mathcal{L}[\pi(\beta \parallel k, \tau - \omega | \alpha_{m+i-\omega-1})]^{-1} = \cancel{\mathcal{U}\mathcal{L}\mathcal{D}} \left[\beta_{k+x} - \alpha_{m+i-\omega-1} \right] \langle x := (\omega+1) \rangle$$

$\langle \tau, \omega := [\omega+1] \rangle$

$$\begin{aligned} \omega' &= i - \omega - 1 \\ \omega + 1 &= i - \omega' \end{aligned}$$

$$\left\{ \mathcal{L}[\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega)] \langle \tau, \omega := [i] \rangle \right\} = \quad \text{replace by mult p322}$$

$$\prod \cancel{V^\omega} \left\{ \mathcal{U}\mathcal{L}\mathcal{D} \left[\beta_{k+x-1} - \alpha_{m+\omega} \right] \langle x := \cancel{[\omega+1]} \rangle \right\} \cancel{\langle \omega := [i] \rangle}$$

$$= \sum_{\omega=0}^D \delta \llcorner \pi(\beta \parallel k, \tau | \langle \alpha \rangle; \alpha \parallel m, \omega) \sum_{x=0}^D \mu(\beta \parallel k, \tau | \beta_{k+x}) \pi(\alpha \parallel m, \omega | \beta_{k+x})$$

$$\sum_{x=0}^D \mu(\beta \parallel k, \tau | \beta_{k+x}) \left\{ \pi(\beta \parallel k, \tau | \beta_{k+x}) - \Lambda(\pi(\beta \parallel k, \tau | \langle \alpha \rangle; \alpha \parallel m, D-1)) \right\}$$

$$= 0 \quad (\text{first term} = 0, \text{second term} = \text{pr. deg. } D-1)$$

$$D = n, 1 \times 1$$

$$\mathcal{L}[\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega)]$$

$$= \vee^{i-j} \mathcal{L}[\delta(\beta \parallel m+i-j, \tau | \langle \beta \rangle) \parallel k, \omega)] \times$$

use rule p 323

$$\prod \vee^{\omega} \{ \mathcal{U} \mathcal{L} \mathcal{D} [\beta_{k+x-1} - \alpha_{m+n}] \langle x := (i-\omega) \rangle \} \langle \omega := \overbrace{j \dots}^{[i-j]} \rangle$$

 $j \in [i]$

$$= \prod \vee^{\omega} \{ \mathcal{U} \mathcal{L} \mathcal{D} [\beta_{k+x-1} - \alpha_{m+n}] \langle x := (i-\omega) \rangle \} \langle \omega := [i] \rangle$$

$$= \left\{ \prod \vee^{i-\omega-1} \{ \mathcal{L}[\pi(\alpha \parallel m+\omega, \tau-\omega | \beta_{k+i-\omega-1})] \langle z, \omega := [\omega+1] \rangle \}_{j \in [i]} \right\} \langle \omega := [i] \rangle$$

$$\vee^{i-j} \mathcal{L}[\delta(\beta \parallel k+i-j, \tau | \langle \alpha \rangle), \alpha \parallel m, \omega]$$

use rule p 325

 $j \in [i]$

$$= \prod \vee^{i-\omega-1} \{ \mathcal{L}[\pi(\alpha \parallel m+\omega, \tau-\omega | \beta_{k+i-\omega-1})] \langle z, \omega := [\omega+1] \rangle \} \langle \omega := [i] \rangle$$

$$= \sum_{\omega=0}^i \delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega) \sum_{x=0}^{\omega} \mu(\gamma \parallel h, \omega, x) \pi(\beta \parallel k, \omega | \gamma_{h+x})$$

$$\sum_{x=0}^{\omega} \mu(\gamma \parallel h, \omega, x) \left\{ \sum_{\omega=0}^{\omega} \delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle), \pi(\beta \parallel k, \omega) \parallel (\beta \parallel k, \omega | \gamma_{h+x})) \right.$$

$$- \left. \sum_{\omega=0}^{\omega-1} \right\}$$

$$\{..\} = \sum_{x=0}^{\omega} \mu(\gamma \parallel h, \omega, x) \pi(\alpha \parallel m, \tau | \gamma_{h+x})$$

$$- \sum_{x=0}^{\omega} \mu(\gamma \parallel h, \omega, x) \Delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega-1 | \gamma_{h+x})$$

$$= \delta(\pi(\alpha \parallel m, \tau | \langle \gamma \rangle); \gamma \parallel h, \omega)$$

$$\mathcal{L}[\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega)] \mathcal{L}[\delta(\pi(\beta \parallel k, \tau | \langle \gamma \rangle); \gamma \parallel h, \omega)]$$

$$= \mathcal{L}[\pi(\alpha \parallel m, \tau | \langle \gamma \rangle); \gamma \parallel h, \omega]$$

$$\mathcal{L}[\dots_m] = \vee \mathcal{L}[\dots_{m+1}] \cup \mathcal{L}[\dots_{m+1}]$$

$$\mathcal{I}[\dots_m] = \vee \mathcal{L}[\dots_{m+2}] \cup \mathcal{L}[\dots_{m+2}]$$

$$\mathcal{L}\left[\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega)\right] \langle \tau, \omega \rangle := [i-\omega] \rangle$$

$$= \vee \mathcal{L}\left[\delta(\pi(\alpha \parallel m+\omega, \tau | \langle \beta \rangle); \beta \parallel k, \omega)\right] \langle \tau, \omega \rangle = [i-\omega] \rangle$$

$$\rightarrow \cdot \mathcal{U} \mathcal{L} \mathcal{D} [\beta_{k+\chi-1} - \alpha_{m+\omega}] \langle \chi := (i-\omega) \rangle \quad \underline{\omega \in [i]}$$

$$d=0 : \pi(\alpha \parallel m+\omega, \tau | \beta_k) = (\beta_k - \alpha_{m+\omega}) \pi(\alpha \parallel m+\omega, \tau-1 | \beta_{k-1})$$

$$d>0 \quad \text{true} \quad \delta(\pi(\alpha \parallel m+\omega, \tau | \langle \beta \rangle); \beta \parallel k, \omega) =$$

$$\delta(\pi(\alpha \parallel m+\omega, \tau-1 | \langle \beta \rangle) \parallel k, \omega-1) + (\beta_{k+d} - \alpha_{m+\omega}) \\ + (\beta_{k+d} - \alpha_{m+\omega}) \delta(\pi(\alpha \parallel m+\omega, \tau-1 | \langle \beta \rangle) \parallel k, \omega)$$

$$\mathcal{L}[\dots_k] = \mathcal{L}[\dots_k] \vee \mathcal{L}[\dots_{k+1}]$$

$\Delta \circ \Delta$

$$\mathcal{L}\left[\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega)\right] =$$

$$\mathcal{L}\left[\pi(\alpha \parallel m+\omega, \tau-\omega | \beta_k)\right] \vee \mathcal{L}\left[\delta\left(\pi(\alpha \parallel m+\omega, \tau | \langle \beta \rangle); \beta \parallel k, \omega\right)\right]$$

rhs involves β_{k+d}

$$\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel k, \omega) =$$

$$\sum \pi(\alpha \parallel m+\omega, \tau-\omega | \beta_k) \delta(\beta \parallel k, \omega-1 | \langle \alpha \rangle); \alpha \parallel m, \omega-1$$

$$= \pi(\alpha \parallel m, \tau | \beta_k) \sum \delta\left(\frac{1}{\beta_k - \langle \alpha \rangle}; \alpha \parallel m, \omega-1\right) \sum_{\chi=0}^k \mu(\alpha \parallel m, \omega-1, \chi)$$

$$\pi(\beta \parallel k, \omega-1 \setminus \alpha_{m+\omega}, \omega+1)$$

$$\sum \pi(\alpha \parallel m+\omega, \tau-\omega | \beta_k) \delta(\pi(\alpha \parallel m+\omega, \tau-\omega-1 | \langle \beta \rangle); \beta \parallel k, \omega)$$

$$\mathcal{I}\left[\delta(\pi(\beta \parallel k, \tau | \langle \alpha \rangle); \alpha \parallel m-i, \omega)\right] \Rightarrow \left[\delta(\pi(\alpha \parallel m, \tau | \langle \beta \rangle); \beta \parallel m-i, \omega)\right]$$

$$\pi(\alpha \parallel m, z \mid \beta_k) \sum_{\omega=0}^{\infty} \delta\left(\frac{1}{\beta_{k-\langle\alpha\rangle}}; \alpha \parallel m, \omega-1\right) \sum_{x=0}^{D-1} \mu(\beta \parallel k+1, D-1, x) \\ \pi(\alpha \parallel m, \omega-1 \mid \beta_{k+x+1})$$

$$= \pi(\alpha \parallel m, z \mid \beta_k) \sum_{x=0}^{D-1} \mu(\beta \parallel k+1, D-1, x)$$

$$\Lambda\left(\frac{1}{\beta_{k-\langle\alpha\rangle}}; \alpha \parallel m, z-1 \mid \beta_{k+1}\right) - \Lambda\left(\frac{1}{\beta_{k-\langle\alpha\rangle}}; \alpha \parallel m, D-2 \mid \beta_{k+1}\right)$$

$\uparrow \sum \mu \cdot = 200$
since $\deg D-2$

$$= \pi(\alpha \parallel m, z \mid \beta_k) \sum_{x=0}^{D-1} \mu(\beta \parallel k+1, D-1, x)$$

$$\sum_{k=0}^{z-1} \mu(\alpha \parallel m, z-1, k) \frac{1}{\beta_k - \alpha_{m+k}} \cdot \frac{1}{\beta_{k+z+1} - \alpha_{m+k}} \cdot \pi(\alpha \parallel m, z \mid \beta_{k+z+1})$$

$$\pi(\alpha \parallel m+1, 0 \mid \beta_k) \cdot \delta(\pi(\alpha \parallel m, 0 \mid \beta_k); \beta \parallel k+1, 0) = 1$$

$$\text{if } \theta := 1 : \sum_i \pi(\alpha \parallel m \omega, z-\omega \mid \beta_k) \pi(\alpha \parallel m, \omega-1 \mid \beta_{k+1}) \quad \omega := [z]$$

$$= \frac{\delta(\pi(\alpha \parallel m, z \mid \beta_{k+1}) - \pi(\alpha \parallel m, z \mid \beta_k))}{\beta_{k+1} - \beta_k}$$

$$= \text{show } \sum_i \pi(\alpha \parallel m \omega, z-\omega \mid \beta_k) \delta(\pi(\alpha \parallel m, \omega-1 \mid \beta_k); \beta \parallel k+1, j-1) \leq \omega := [j, i] \quad (*)$$

$$= \delta(\pi(\alpha \parallel m, z \mid \beta_k); \beta \parallel k, j) \quad (j \in [i])$$

$$j := 1 \quad \frac{\pi(\alpha \parallel m, z \mid \beta_{k+1}) - \pi(\alpha \parallel m, z \mid \beta_k)}{\beta_{k+1} - \beta_k} = \sum_i \pi(\alpha \parallel m \omega, z-\omega \mid \beta_k) \pi(\alpha \parallel m, \omega-1 \mid \beta_{k+1}) \quad \omega := [i]$$

suppose this is stated

$$\sum_i \pi(\alpha // m, i - \omega / \beta_k) \left\{ \sum_j \mu(\beta // k+1; j-1, \chi) \pi(\alpha // m, \omega-1 / \beta_{k+\chi+1}) \langle \chi = [j] \rangle \right\} \\ \langle \omega := [j, i] \rangle$$

$$= \sum_i \mu(\beta // k+1; j-1, \chi) \left\{ \sum_i \pi(\alpha // m, i - \omega / \beta_k) \pi(\alpha // m, \omega-1 / \beta_{k+\chi+1}) \langle \omega := [j, i] \rangle \right. \\ \left. \langle \chi = [j] \rangle \right\}$$

$$= \sum_i \mu(\beta // k+1; j-1, \chi) \left\{ \frac{\pi(\alpha // m, i / \beta_{k+\chi+1}) - \pi(\alpha // m, i / \beta_k)}{\beta_{k+\chi+1} - \beta_k} \right. \\ \left. - \cancel{\pi(\alpha // m, j-1 / \beta_{k+\chi+1}) + \pi(\alpha // m, j-1 / \beta_k)} \right\} \\ \langle \chi = [j] \rangle \\ \langle \chi = [j] \rangle$$

$$= \sum_i \mu(\beta // k; j, \chi) \left\{ \pi(\alpha // m, i / \beta_{k+\chi+1}) - \pi(\alpha // m, j-1 / \beta_{k+\chi+1}) \right. \\ \left. - \pi(\alpha // m, i / \beta_k) + \pi(\alpha // m, j-1 / \beta_k) \right\} \langle \chi = [j] \rangle$$

$$m = \delta(\pi(\alpha // m, i / \langle \beta \rangle); \beta // k, j) \\ \langle \chi = [j] \rangle = \langle \chi = [j] \rangle \\ \text{since } \chi = 0 \Rightarrow \{ \dots \} = 0$$

$$= \omega \in [i]$$

$$\mathcal{L}[\delta(\pi(\alpha // m, \tau / \langle \beta \rangle); \beta // k+i-\omega-1, \omega)] \langle \tau, \omega := [\omega_H] \rangle$$

$$\cancel{\mathcal{L}[\pi(\alpha // m, \tau / \langle \beta \rangle); \beta // k+i-\omega-1, \omega]} \cancel{\mathcal{L}[\delta(\pi(\alpha // m, \tau / \langle \beta \rangle); \beta // k+i-\omega-1, \omega)]}$$

$$\mathcal{L}[\pi(\alpha // m+\omega, \tau - \omega / \beta_{k+i-\omega-1})] \vee \cancel{\mathcal{L}[\delta(\pi(\alpha // m, \tau - 1 / \langle \beta \rangle); \beta // k+i-\omega-1, \omega)]}] \\ \langle \tau, \omega := [\omega_H] \rangle \quad \langle \tau, \omega := [\omega] \rangle$$

$$\omega = 0 \rightarrow \omega = 0$$

$$\delta(\pi(\alpha // m, \tau / \langle \beta \rangle); \beta // k+i-\omega-1, \omega) = \text{from result p 824}$$

$$\sum_i \pi(\alpha // m, \tau - i / \beta_{k+i-\omega-1}) \delta(\pi(\alpha // m, \tau - 1 / \langle \beta \rangle); \beta // k+i-\omega-1, \omega-1) \langle \chi := [j, \tau] \rangle$$

$$\mathfrak{I}_\omega [\alpha \parallel m; i, i-d] = \sum (\alpha \parallel m; i, i-d)$$

$$\sum \mu(\alpha \parallel m; z, \omega) \frac{\pi(\beta \parallel k+z; i-d-1 | \alpha_{m+\omega}) \pi(\alpha \parallel m; d | \alpha_{m+\omega})}{\pi(\beta \parallel k+z; i-d | \alpha_{m+\omega})} \quad \langle \omega := [z, \omega] \rangle$$

\uparrow

$$\pi(\beta \parallel k+z+1, i-d-1 | \alpha_{m+\omega}) \pi(\alpha \parallel m; d | \alpha_{m+\omega}) \quad \equiv \langle \omega := [z] \rangle$$

\downarrow

$$\pi(\alpha \parallel m, \omega | \alpha_{m+\omega}) \pi(\alpha \parallel m; d-1, z-\omega | \alpha_{m+\omega}) \quad \alpha_{m+d} - \beta_{k+z}$$

= 0 since of deg z-1

$$= \mathfrak{S}(\pi(\beta \parallel k+z+1, i-d-1 | \langle \omega \rangle) \times \pi(\alpha \parallel m; d | \langle \omega \rangle); \alpha \parallel m; z)$$

$$z' = d \quad z'+d = z \quad d' = z-d$$

$$= \mathfrak{S}(\pi(\beta \parallel k+z+1, i-d-1 | \langle \omega \rangle); \alpha \parallel m+d, z-\omega) = 0 (z > d) \quad \langle \omega \rangle$$

$$\sum \mu(\alpha \parallel m; z, \omega) \frac{1}{\beta_{k+z} - \alpha_{m+\omega}} \quad \frac{1}{\pi(\alpha \parallel m, z+1 | \beta_{k+z})}$$

$$z = d : \cancel{\mathfrak{S}(\pi(\alpha \parallel m; d | \langle \omega \rangle); \alpha \parallel m, d)}$$

$$\mu(\alpha \parallel m; d, d) \frac{\pi(\alpha \parallel m; d | \alpha_{m+\omega})}{\beta_{k+z} \alpha_{m+\omega} - \beta_{k+z}} = \frac{1}{\alpha_{m+d} - \beta_{k+z}} \quad \langle \omega := [z, \omega] \rangle$$

$$\sum \mu(\beta \parallel k+z; i-d-1, \omega - \tau) \frac{\pi(\alpha \parallel m, d | \beta_{k+z}) \pi(\beta \parallel k+z+1, i-d-1 | \beta_{k+z})}{\pi(\alpha \parallel m, z+1 | \beta_{k+z})} \pi(\beta \parallel k+z+1, i-d-1 | \beta_{k+z})$$

$$\text{so } z = d : \mu(\beta \parallel k+z; i-d-1, 0) \frac{1}{\beta_{k+z} - \alpha_{m+z}} \pi(\beta \parallel k+z+1, i-d-1 | \beta_{k+z}) -$$

$$z < d \quad \sum \mu(\beta \parallel k+z; i-d-1, \omega - \tau) \pi(\alpha \parallel m+z+1, d-z-1 | \beta_{k+z}) \pi(\beta \parallel k+z+1, i-d-1 | \beta_{k+z})$$

$$= \sum \mu(\beta \parallel k+z; i-d-1, \omega) \pi(\alpha \parallel m+z+1, d-z-1 | \beta_{k+z}) \pi(\beta \parallel k+z+1, i-d-1 | \beta_{k+z})$$

$\langle \omega := [z-d, \omega] \rangle = \langle \omega := [i-d-1] \rangle$ since $\pi(\beta \parallel \beta) = 0 \quad \Rightarrow \langle \omega := [z-d, i-d] \rangle$ since deg $= i-d-2$

$$\prod \lambda([\bar{\alpha}, \bar{\beta}]_{\omega} \| 0; i, \omega) \alpha_{m+n}$$

$$\underline{\Phi}^D = u \underline{L} \therefore |L| = \prod \lambda([\bar{\alpha}, \bar{\beta}]_{\omega} \| 0; i, \omega | \alpha_{m+n}) \quad \langle \omega := [i] \rangle$$

$$\alpha_m \dots \alpha_{m+n-1} \beta_{n+n} \dots \beta_{n+1}$$

$$\alpha_{m+n}$$

$$\prod \frac{\pi(\alpha \| m, \omega | \alpha_{m+n}) \pi(\beta \| n+m, i-\omega | \alpha_{m+n})}{\pi(\alpha \| m, \omega | \beta_{n+n}) \pi(\beta \| n+\omega+1, i-\omega | \beta_{n+n})} \quad \langle \omega := [i] \rangle$$

$$\prod \pi(\alpha \| m, \omega | \beta_{n+n}) \quad \langle \omega := [i] \rangle = (-1)^{\frac{i(i+1)}{2}} \prod \pi(\beta \| n+\omega+1, i-\omega | \alpha_{m+n})$$

$$\alpha_1 - \alpha_0 \quad (\alpha_1 - \alpha_0)(\alpha_2 - \alpha_0)(\alpha_2 - \alpha_1)$$

$$|L(\bar{\alpha}/\bar{\beta})| = \prod \frac{\pi(\alpha \| m, \omega | \alpha_{m+n})}{\pi(\beta \| n, \omega | \beta_{n+n})} \quad \langle \omega := [i] \rangle$$

$$= \prod \left\{ \prod \frac{\alpha_{m+n} - \alpha_{m+x}}{\beta_{n+n} - \beta_{n+x}} \quad \langle \omega := [x] \rangle \right\} \langle \omega := [i] \rangle$$

Ca

$$\tilde{\Phi}_{\alpha\beta}(\bar{\alpha}, \bar{\beta}) = [(\alpha_{m+i} - \beta_{n+\omega})^{-1}] \quad \langle \omega := [i] \rangle$$

$$L(\bar{\alpha}, \bar{\beta}) = \text{diag} [\pi(\bar{\beta}; i+1); \bar{\alpha}, [i]] \quad \text{Ca} \quad \tilde{\Phi}_{\alpha\beta}(\bar{\alpha}, \bar{\beta}) \text{diag} [\alpha(\bar{\beta}, i), [i]]$$

$$\hat{\tilde{\Phi}}_{\alpha\beta}(\bar{\alpha}, \bar{\beta}) = \hat{\tilde{\Phi}}(\bar{\alpha}, \bar{\beta}) D(\bar{\alpha}, \bar{\beta}) \hat{\tilde{\Psi}}(\bar{\alpha}, \bar{\beta})$$

$$\hat{\tilde{\Phi}}(\bar{\alpha}, \bar{\beta}) = \text{diag} [\pi(\bar{\beta}; i+1); \bar{\alpha}, [i]]^{-1} \hat{\tilde{\Phi}}(\bar{\alpha}, \bar{\beta})$$

$$\hat{\tilde{\Psi}}(\bar{\alpha}, \bar{\beta}) = \hat{\tilde{\Psi}}(\bar{\alpha}, \bar{\beta}) \text{diag} [\alpha(\bar{\beta}, i), [i]]^{-1}$$

$$\hat{\tilde{\Phi}}: \quad \frac{\lambda([\bar{\alpha}, \bar{\beta}]_{\omega} \| 0; i, \omega | \alpha_{m+n})}{\pi(\beta \| n, i+1 | \alpha_{m+n})}$$

$$\alpha_m \dots \alpha_{m+2-1} \beta_{n+2} \dots \beta_{n+i}$$

$$\frac{\pi(\alpha|m, 2|\alpha_{m+2})\pi(\beta|n+2+1, i-2|\alpha_{m+2})}{\pi(\alpha|m, 2|\beta_{n+2})\pi(\beta|n+2+1, i-2|\beta_{n+2})\pi(\beta|n, i+1|\alpha_{m+2})}$$

$$-\frac{\pi(\alpha|m, 2|\alpha_{m+2})\pi(\beta|n+2+1, i-2|\beta_{n+2})\pi(\beta|n, i+1|\alpha_{m+2})}{\pi(\beta|n, 2+1|\alpha_{m+2})}$$

$$\frac{1}{\Phi D} \frac{\lambda([\bar{\alpha}, \bar{\beta}|2+1]//0; i, 2|\alpha_{m+2})}{\pi(\beta|n, i+1|\alpha_{m+2})}$$

$\alpha_m \dots \alpha_{m+2} \beta_{n+2} \dots \beta_{n+i}$

$$\frac{\pi(\alpha|m, 2|\alpha_{m+2})\pi(\beta|n+2+1, i-2|\alpha_{m+2})}{\pi(\alpha|m, 2|\alpha_{m+2})\pi(\beta|n+2+1, i-2|\beta_{n+2})\pi(\beta|n, i+1|\alpha_{m+2})}$$

$$\frac{\pi(\alpha|m, 2|\alpha_{m+2})\pi(\beta|n+2+1, i-2|\beta_{n+2})\pi(\beta|n, i+1|\alpha_{m+2})}{\pi(\beta|n, 2+1|\alpha_{m+2})}$$

$$\frac{\lambda([\bar{\alpha}, \bar{\beta}|2]//0; i, 2|\alpha_{m+2})}{\mu(\beta|n; i, 2)}$$

$\alpha_m \dots \alpha_{m+2-1} \beta_{n+2} \dots \beta_{n+i}$

$i \geq 2$

$$\frac{\pi(\alpha|m, 2|\alpha_{m+2})\pi(\beta|n+2, 2-2|\alpha_{m+2})\pi(\beta|n+2+1, i-2|\alpha_{m+2})}{\pi(\alpha|m, 2|\beta_{n+2})\pi(\beta|n+2, 2-2|\beta_{n+2})\pi(\beta|n+2+1, i-2|\beta_{n+2})} \times$$

$$\pi(\alpha|m, 2|\beta_{n+2})\pi(\beta|n+2, 2-2|\beta_{n+2})\pi(\beta|n+2+1, i-2|\beta_{n+2})$$

$$\frac{\pi(\beta|n, 2|\beta_{n+2})\pi(\beta|n+2+1, i-2|\beta_{n+2})}{\pi(\beta|n, 2|\beta_{n+2})}$$

$$\pi(\beta|n, 2|\beta_{n+2})$$

$$\text{Ca } \tilde{\Phi}(\bar{\alpha}, \bar{\beta}) = \tilde{\Phi}(\bar{\alpha}, \bar{\beta}) \tilde{D}(\bar{\alpha}, \bar{\beta}) \tilde{\Psi}(\bar{\alpha}, \bar{\beta})$$

$$\frac{\tilde{\Phi}}{\Phi} : \frac{\pi(\alpha|m, 2|\alpha_{m+2})}{\pi(\beta|n, 2+1|\alpha_{m+2})} \quad \frac{\tilde{\Psi}}{\Psi} : \frac{\pi(\beta|n+2, 2-2|\alpha_{m+2})\pi(\beta|n+2+1, i-2|\alpha_{m+2})}{\pi(\alpha|m, 2|\beta_{n+2})\pi(\beta|n+2, 2-2|\beta_{n+2})}$$

$$= \frac{\pi(\beta|n+2, i-2+1|\alpha_{m+2})}{\pi(\alpha|m, 2+1|\beta_{n+2})\pi(\beta|n, 2|\beta_{n+2})}$$

$$= \frac{\pi(\beta|n+2, i-2+1|\alpha_{m+2})}{\pi(\alpha|m, 2+1|\beta_{n+2})\pi(\beta|n, 2|\beta_{n+2})}$$

$$\text{D: } \frac{\pi(\alpha/m, \omega | \alpha_{m+n})}{\pi(\alpha/m, \omega | \alpha_{mn})}$$

$$\pi(\beta/n, \omega | \alpha_{mn}) \pi(\beta/n+\omega+1, i-\omega | \alpha_{mn})$$

$$\frac{\psi}{\psi} = \frac{1}{\pi(\alpha/m, \omega+1 | \beta_{mn}) \pi(\beta/n, i-\omega | \beta_{mn})}$$

$$\text{D}' = \frac{\pi(\beta/n+\omega, i-\omega+1 | \alpha_{mn})}{\pi(\beta/n+\omega+1 | \alpha_{mn})} = \beta_{mn} - \alpha_{mn}$$

Ca

$$\mathcal{Z}_{\alpha\beta}(\bar{\alpha}, \bar{\beta}) = \mathcal{L} \left[\frac{\pi(\alpha/m, \omega | \alpha_{m+n})}{\pi(\beta/n, \omega+1 | \alpha_{m+n})} \right] \text{diag}[\beta_{mn} - \alpha_{mn}] \\ \mathcal{U} \left[\frac{\pi(\beta/n, i-\omega | \beta_{mn})}{\pi(\alpha/m, \omega+1 | \beta_{mn})} \right]$$

$$i=0 \quad \frac{1}{\alpha_m - \beta_n} \cdot (\beta_n - \alpha_m) \cdot \frac{1}{\beta_n - \alpha_m}$$

$$i=1 \quad \begin{pmatrix} \frac{1}{\alpha_m - \beta_n} & & & & \frac{1}{\beta_n - \alpha_m} & \frac{1}{\beta_{n+1} - \alpha_m} \\ & \frac{1}{\alpha_{m+1} - \beta_n} & \frac{\alpha_{m+1} - \alpha_m}{(\alpha_{m+1} - \beta_n)(\alpha_{m+1} - \beta_{n+1})} & & & \frac{1}{(\beta_{n+1} - \alpha_m)(\beta_{n+1} - \beta_{m+1})} \\ & & & \beta_{n+1} - \alpha_{m+1} & & (\beta_{n+1} - \beta_n) \\ & & & & & \end{pmatrix}$$

$$\frac{1}{\alpha_m - \beta_n} \quad \frac{1}{\alpha_m - \beta_{n+1}}$$

$$\frac{1}{\alpha_{m+1} - \beta_n} \quad \leftarrow \frac{\beta_n - \alpha_m}{\alpha_{m+1} - \beta_n} \cdot \frac{1}{\beta_{n+1} - \alpha_m} + \frac{(\alpha_{m+1} - \alpha_m)(\beta_{n+1} - \alpha_{m+1})}{(\alpha_{m+1} - \beta_n)(\alpha_{m+1} - \beta_{n+1})} \cdot \frac{\beta_{n+1} - \beta_n}{(\beta_{n+1} - \alpha_m)(\beta_{n+1} - \beta_{m+1})}$$

$$\frac{(\beta_n - \alpha_m)(\alpha_{m+1} - \beta_{n+1}) + (\alpha_{m+1} - \alpha_m)(\beta_{n+1} - \beta_n)}{(\alpha_{m+1} - \beta_n)(\beta_{n+1} - \alpha_m)(\alpha_{m+1} - \beta_{n+1})} \parallel \cancel{\beta \alpha \cancel{\alpha} \cancel{\beta}}$$

$$\begin{aligned} \alpha_{mn} = & \underbrace{\alpha_m \beta_n - \alpha_n \beta_m}_{\alpha_{mn}} - \underbrace{\alpha_m \beta_{nn}}_{\alpha_{mn} \beta_{nn}} + \underbrace{\alpha_n \beta_{mm}}_{\alpha_{mn} \beta_{mm}} \\ & + \underbrace{\alpha_m \beta_{nn}}_{\alpha_m \beta_{nn}} - \underbrace{\alpha_n \beta_{mm}}_{\alpha_{mn} \beta_{mm}} + \alpha_m \beta_n \end{aligned}$$

$$(\alpha_{m+1} - \beta_n) \beta_{nn} - \alpha_m (\alpha_{m+1} - \beta_n) = (\alpha_{m+1} - \beta_n)(\beta_{nn} - \alpha_m)$$

$$\text{fraction} = \frac{1}{\alpha_{mn} - \beta_{nn}}$$

$$|\mathcal{C}_n(\bar{\alpha}, \bar{\beta})| = \frac{\prod_i \pi(\alpha \parallel m, \omega / \alpha_{mn})}{\prod_i \pi(\beta \parallel n, \omega / \beta_{nn})} \langle \omega := [i] \rangle$$

$$= \prod_i \frac{1}{\pi(\beta \parallel n, \overset{i+1}{\cancel{\beta}} / \alpha_{mn})} \langle \omega := [i] \rangle \prod_i \pi(\beta \parallel n, \omega / \beta_{nn}) \pi(\beta \parallel n+\omega+1, i-\omega / \beta_{nn})$$

$$= \prod_i \frac{\pi(\alpha \parallel m, \omega / \alpha_{mn}) \pi(\beta \parallel n+\omega+1, i-\omega / \beta_{nn})}{\pi(\beta \parallel n, i+1 / \alpha_{mn})} \langle \omega := [i] \rangle$$

$$i=0 \quad \frac{\cancel{\alpha_m - \beta_{nn}}}{\alpha_m - \beta_n} \cdot \frac{1}{\alpha_m - \beta_n} \cdot \frac{(\alpha_{nn} - \alpha_m)(\beta_n - \beta_{nn})}{(\alpha_m - \beta_n)(\alpha_m - \beta_{nn})(\alpha_{m+1} - \beta_n)(\alpha_{mn} - \beta_{nn})}$$

$$i=1 \quad \frac{1}{(\alpha_m - \beta_n)(\alpha_{mn} - \beta_{nn})} - \frac{1}{(\alpha_{mn} - \beta_n)(\alpha_m - \beta_{nn})}$$

$$- \begin{bmatrix} \alpha_m \alpha_{m+1} - \alpha_m \beta_{nn} - \alpha_{mn} \beta_n + \beta_n \beta_{nn} & (\alpha_{mn} - \alpha_m)(\beta_{nn} - \beta_n) \\ - \alpha_m \alpha_{mn} + \alpha_m \beta_n + \alpha_{mn} \beta_{nn} - \beta_n \beta_{nn} & \end{bmatrix}$$

$$+ \frac{\prod_i \pi(\alpha \parallel m+\omega+1, i-\omega / \alpha_{mn}) \pi(\beta \parallel n, \omega / \beta_{nn})}{\pi(\beta \parallel n, i+1 / \alpha_{mn})}$$

$$(-1)^{\frac{i(i+1)}{2}} \pi(\alpha \parallel m, \omega)$$

$$\left| C_{\alpha}(\bar{\alpha}, \bar{\beta}) \right| = \prod \frac{\pi(\alpha|m, \omega | \alpha_{mnw}) \pi(\beta|n_{m+1}, i-w | \beta_{mnw})}{\pi(\beta|n, i+1 | \alpha_{mnw})}$$

$$= \prod \frac{\pi(\alpha|m_{m+1}, i-w | \alpha_{mnw}) \pi(\beta|n, \omega | \beta_{mnw})}{\pi(\beta|n, i+1 | \alpha_{mnw})}$$

$$= (-1)^{\sum_{i=1}^{i(m)}} \prod \frac{\pi(\alpha|m, \omega | \alpha_{mnw}) \pi(\beta|n, \omega | \beta_{mnw})}{\pi(\beta|n, i+1 | \alpha_{mnw})}$$

$$= (-1)^{\frac{i(i-1)}{2}} \prod \frac{\pi(\alpha|m, \omega | \alpha_{mnw}) \pi(\beta|n, \omega | \beta_{mnw})}{\pi(\alpha|m, i+1 | \beta_{mnw})}$$

$$p(\alpha_{m+r}) = \mu + \rho p(\beta_{k+r}) \quad p(\alpha_{m+r}) - p(\beta_{k+r}) = \mu + (\rho-1)p(\beta_{k+r})$$

$$\nu \leq \#h \quad \{ \mu + (\rho-1)p(\beta_{k+r}) \}^d \neq \deg \nu h \quad \nu h \leq i \rightarrow \nu \leq [k/\nu]$$

$$L(\bar{\alpha}/\bar{\beta}) \subset \left[\{ \mu + (\rho-1)p(\beta_{m+r}) \}^d \right] = \\
 \{ p(\alpha_{m+r}) - p(\beta_{m+r}) \}^d =$$

$$c \left[\{ \mu + (\rho-1)p(\alpha_{m+r}) \}^d \right] = c \left[\{ \mu + \rho p(\alpha_{m+r}) - \mu - p(\beta_{m+r}) \}^d \right] \\
 = \rho^d c \left[\{ \dots \}^d \right]$$

$$\alpha[f; \alpha || m, j, [k, i]] =$$

$$\left[\mu(\alpha || m; z+1, \omega) \{ f_{m\omega} - \Lambda(f; \alpha || m+i+1, z-i | \omega_{m\omega}) \} \right]_{z := [j, j+k]}^{j := [i]} \\$$

$$A[f; \alpha || m, j, [k, i]] = \left[S(f; \alpha || m+2, z-2+1) \right]_{z := [j, j+k]}^{j := [i]}$$

$$\alpha[f; \alpha || m, j, [k, i]] = A[f; \alpha || m, j, [k, i]] S(\bar{\alpha})$$

$$S(\bar{\alpha}) = M[\bar{\alpha}, [i]] := \mathcal{L}[\mu(\alpha || m; z, \omega)]$$

$$P[\bar{\alpha}]^{[i]} := \text{row}[\lambda(\alpha || m; i, \omega)]^{j := [i]}$$

$$P[\bar{\alpha}]^{[i]} := \text{row}[\bar{\pi}(\alpha || m, \omega)]^{j := [i]}$$

$$P[\bar{\alpha}]^{[i]} = S(\bar{\alpha}) P[\bar{\alpha}]^{[i]}$$

$$S(\bar{\alpha}; \bar{\beta}) := \Delta[\bar{\nu}, P[\bar{\beta}]^{[i]} || [i]] = [\delta(P[\bar{\beta} | \langle \alpha \rangle]; \alpha || m, z)]$$

$$P[\bar{\alpha}]^{[i]} S(\bar{\alpha}; \bar{\beta}) = P[\bar{\beta}]^{[i]}$$

$$q[f; \alpha || m, j]^{[i]} :=$$

$$\text{row} \left[\begin{aligned} & [\lambda(\alpha || m; h, \omega) \{ f_{m\omega} - \Lambda(\alpha; f || m+i+1, j-i-1 | \omega_{m\omega}) \} \\ & + \lambda(\alpha || m; i, \omega) \Lambda(\alpha; f || m+i+1, j-i-1)]^{j := [h]} \end{aligned} \right] O^{[i-j]}]$$

$$\bar{q}[f; \alpha || m, j]^{[i]} :=$$

$$\text{row} \left[\begin{aligned} & O^{[i-j]} | \lambda(\alpha || m+i-j; j-h, \omega) \{ f_{m+i-h\omega} - \Lambda(\alpha; f || m+i-j, j-i-1 | \omega_{m\omega}) \} \\ & + \lambda(\alpha || m; i, \omega) \Lambda(\alpha; f || m+i-j, j-i-1)]^{j := [h]} \end{aligned} \right]$$

$$\bar{P}[\bar{\alpha}; j]^{[i]} = \text{row}[\bar{\pi}(\alpha || m, \omega + 1, i, j-1)]^{j := [i]} \quad (j=0 \text{ always?}} \parallel \frac{\bar{P}[\bar{\alpha}; j]}{P[\alpha; i, j]}^{[i]}$$

$$Q[\alpha; f || m, j]^{[i]} := \text{row}[\pi(\alpha || m, \omega) \Lambda(\alpha; f || m\omega, j-2)]^{j := [i]}$$

$$\bar{Q}[\alpha; f || m, j]^{[i]} := \text{row}[\pi(\alpha || m\omega H, i-2) \Lambda(\alpha; f || m+i-j, j-i+2)]^{j := [i]}$$

$$\bar{A}[f; \alpha || m, j, [k, i]] := \left[S(f; \alpha || m+i-z, 2-i+z) \right]_{z:= [j, k]}^{z:= [i]} ;$$

$$\bar{S}(\bar{\alpha}; \bar{\beta}) = \bar{\Delta}[p[\bar{\beta}], \bar{\alpha} || \bar{E}i]$$

$$= [\delta(-p[\bar{\beta}] \langle \alpha \rangle); \alpha || m+z, i-z] \langle z, 2 := [i] \rangle$$

$$\bar{P}[\bar{\alpha}]^{[i]} \bar{S}(\bar{\alpha}, \bar{\beta}) = p[\bar{\beta}]^{[i]}$$

$$Q[f; \bar{\alpha} || j]^{[i]} S(\bar{\alpha}) = q[f; \bar{\alpha} || j]^{[i]}$$

$$\bar{Q}[f; \alpha || j]^{[i]} \bar{S}(\bar{\alpha}) = \bar{q}[f; \bar{\alpha} || j]^{[i]}$$

in p52 α defined as above with $z+1 = z$

$$\text{then } q[f; \alpha || m, j]^{[i]} = P[\alpha || m]^{[i]} \alpha[f; \alpha || m, 0, [j, i]]$$

$$\text{with } \bar{\alpha}[f; \alpha || m, j', [k, i]] :=$$

$$[\mu(\alpha || m+i-k+j'+z, k-z, 2+k-i-z)]$$

$$\{ f_{mz} - \Delta(f; \alpha || m+i-k+j'+z, k-i-j'-z-1) \}_{z= [j', j+k]}^{z= [i]}$$

$$\bar{q}[f; \alpha || m, j]^{[i]} = \bar{P}[\alpha || m+i-j]^{[i]} \bar{\alpha}[f; \alpha || m, 0, [j, i]]$$

$$Q[f; \alpha || m, j]^{[i]} = P[\alpha || m]^{[i]} [\delta(f; \alpha || m, z-2)]_{z= [j]}^{z= [i]}$$

$$\bar{Q}[f; \alpha || m, j]^{[i]} := \text{row}[\bar{\alpha}(f; \alpha || m+i-j+1, j-2)] [\delta(f; \alpha || m+i-j+z, j-i-z)]$$

$$= A[f; \alpha || m, [i]] := [\delta(f; \alpha || m, z-2)] \langle z, 2 := [i] \rangle$$

=

$k = i-1 : A[\dots, j, [k, i]]$ should involve f_{m+i-j} & no further

$$m+j+2-1+1 \checkmark \quad \bar{A}[\dots, f_{m-j}] = f_{m+i-j-i}$$

$$m+i-z, m+z \quad \begin{matrix} m+j, m \\ m+y, m \end{matrix} \quad m+i-j, m \quad m+i-k, m \quad m+i-k, m \quad m+i-k, m$$

$$\alpha_m = \alpha'_{m+i} \quad \alpha_{m+j} = \alpha'_{m+i-j}$$

$$m+2, m+z+1 \quad m+i-z-1, m+i-2$$

use, with $k=i$ in special application

$$a[f; \alpha \| m, j, [k, i]] = [\mu(\alpha \| m; z, 2) \{ f_{m+2} - A(f; \alpha \| m+i, z-i-1) \}]_{z=(j, j+k)}^{D=[i]}$$

$$A[f; \alpha \| m, j, [k, i]] = [\delta(f; \alpha \| m+2, z-2)]_{z=(j, j+k)}^{D=[i]}$$

$$A[f; \alpha \| m, j, [k, i]] := [\delta(f; \alpha \| m+i-z, z-2)]_{z=(j, j+k)}^{D=[i]} \quad \begin{array}{l} z' = i-z \\ z' = i-j-1 \\ z' = i-j-k \end{array}$$

$$\text{or } A[f; \alpha \| m, j, [k, i]] := [\delta(f; \alpha \| m+z, 2-z)]_{z=[i-j-k, i-j]}^{D=[i]}$$

$$\text{now reversal in } A_2 \quad [\delta(f; \alpha \| m+2, i+z-2)]_{z=[i]}^{D=[i]}$$

$$= A[f; \alpha \| m, i-1, [i, i]]$$

$$\text{Set } A_0[f; \alpha \| m, j, [k, i]] := [\delta(f<\alpha>^2; \alpha \| m, z)]_{z=(j, k)}^{D=[i]}$$

$$A[f; \alpha \| m, j, [i, k]] \cup [\delta(<\alpha>^2; \alpha \| m, z)] = A_0[f; \alpha \| m, j, [k, i]]$$

$$\sum \delta(f; \alpha \| m+\omega, z-\omega) \delta(<\alpha>^2; \alpha \| m, \omega) \quad \langle \omega := [z] \rangle$$

$$= \delta(f<\alpha>^2; \alpha \| m, z) \quad (j, k)$$

$$\mathcal{L}[\delta(<\alpha>^2; \alpha \| m+x-2, 2)]_{z=(j+k)}^{D=[i]} [\delta(f<\alpha>^2; \alpha \| m, z)]_{z=(j+k)}^{D=[i]}$$

$$\not\in [\delta(f<\alpha>^2; \alpha \| m, z)]_{z=(j+k)}^{D=[i]} \quad A_1$$

$$\sum \delta(<\alpha>^2; \alpha \| m+x-\omega, \omega) \delta(f<\alpha>^2; \alpha \| m, \omega) \quad \langle \omega := (j, j+k) \rangle$$

$$= \dots \langle \omega := (j+k) \rangle -$$

$$\min(z, j+k) = z$$

$$\left[\delta \left(\langle \alpha \rangle^{y(z)} ; \alpha \parallel m+z, x=z \right) \right]_{z := (j, j+k)}^{\mathcal{D} := \{j, j+k\}} \quad \begin{matrix} \text{D:=l} \\ \text{z:=(j,j)} \end{matrix}$$

$$\sum \delta(\langle \alpha \rangle^{y(z)} ; \alpha \parallel m+\omega, x(z)-\omega) \delta(f \langle \alpha \rangle^z ; \alpha \parallel m, \omega) \quad \langle \omega := (j, j+k) \rangle$$

$$x(z) = j+k \quad y(z) < x(z) \quad x(z) - j+1 \geq y(z) \quad k+1 \geq y(z)$$

$y(z) = z-j$

$$\left[\delta(\langle \alpha \rangle^{z-j-1} ; \alpha \parallel m+z, j+k-\omega) \right]_{z := (j, j+k)}^{\mathcal{D} := \{j, j+k\}} \left[\delta(f \langle \alpha \rangle^z ; \alpha \parallel m, z) \right]_{z := (j, j)}^{\mathcal{D} := l}$$

$$\Rightarrow \sum \delta(\langle \alpha \rangle^{z-j-1} ; \alpha \parallel m+\omega, j+k-\omega) \delta(f \langle \alpha \rangle^z ; \alpha \parallel m, \omega) \quad \langle \omega := (j, j+k) \rangle$$

$\equiv \langle \omega := [j+k] \rangle$ since $j+k-\omega \geq z-j-1$ when $z \leq j+k$
and $\omega \leq j \quad j+k-\omega \geq k-1$

$$z = z' + j \quad z' := [k] \quad \omega \leq \omega' + j$$

$$\left[\delta(\langle \alpha \rangle^{z+j} ; \alpha \parallel m+j+k, k-\omega) \right]_{z := [k]}^{\mathcal{D} := \{k\}} \left[\delta(f \langle \alpha \rangle^z ; \alpha \parallel m, z) \right]_{z := (j, j+k)}^{\mathcal{D} := \{z\}}$$

$$= \left[\delta(f \langle \alpha \rangle^{z+j-k-1} ; \alpha \parallel m, j+k) \right]_{z := (j, j+k)}^{\mathcal{D} := \{z\}} \Rightarrow \cancel{\text{not possible}}$$

$$A_1[f; \alpha \parallel m, j, [k, i]] \quad z' = j+1+z'$$

$$= \left[\delta(f \langle \alpha \rangle^{z+k} ; \alpha \parallel m, j+k) \right]_{z := [k]}^{\mathcal{D} := \{z\}} \quad \cancel{\text{not possible}}$$

$$\sum \delta(f \langle \alpha \rangle^z ; \alpha \parallel m, \omega) \delta(\pi(\alpha \parallel m, z | \langle \alpha \rangle) ; \alpha \parallel m+z-\omega, \omega) \quad \langle \omega := (j, j+k) \rangle$$

$$= \delta(\pi(\alpha \parallel m, z | \langle \alpha \rangle) ; \alpha \parallel m+z-\omega, \omega) \quad \text{required: } \delta(\pi(\alpha \parallel m, y(z) | \langle \alpha \rangle) ; \alpha \parallel m+j+k-\omega, \omega)$$

$= 0 \text{ when } \omega \leq j$

$$m(z) \leq m+k \quad m(z)+y(z)-1 \geq m+j+k \quad z \in (j, j+k]$$

take $m(z) = m+k$ $y(z) = z$, then $\omega := [j+k]$ in sum

$$\pi(\alpha \parallel m+k, z | \alpha_{m+j+k-\omega+x}) = 0 \quad x := [\omega] \quad \omega \in [j] \quad z \in (j, j+k]$$

$$m+k, j+1 \quad m+k, \dots, m+k+j \quad \omega = 0 \quad x = 0 \Rightarrow m+k+j$$

$$\omega = j \quad x = 0 \quad m+k$$

$$\omega = j \quad x = j \quad m+k+j$$

$$\Rightarrow S(f \langle \alpha \rangle^D \pi(\alpha \parallel m+k, z | \langle \alpha \rangle); \alpha \parallel (m, j+k)) \quad z \in (j, j+k]$$

$$= \sum S(f \langle \alpha \rangle^D; \alpha \parallel \cancel{m+j+k-\omega})$$

$$m+\omega, j+k-\omega) S(\pi(\alpha \parallel m+k, z | \langle \alpha \rangle); \alpha \parallel (m, \omega))$$

$$\sum_i u(\alpha \parallel m, j+k, \omega) f_{m+\omega} \alpha_{m+\omega}^D \pi(\alpha \parallel m+k, z | \alpha_{m+\omega}) \quad \omega := [j+k]$$

$$\frac{\prod_{\substack{z=1 \\ x=0}}^{j+k} (\alpha_{m+\omega} - \alpha_{m+k+x})}{\prod_{\substack{z=0 \\ x=0}}^{j+k} (\alpha_{m+\omega} - \alpha_{m+x})} \quad \frac{k+z-1}{\prod_{x=k}^{k+z-1}} \quad \frac{k+z-1}{\prod_{\substack{x=j+k+1 \\ x=0}}^{k-1} (\alpha_{m+\omega} - \alpha_{m+x})}$$

$$S(f \langle \alpha \rangle^D \pi(\alpha \parallel j+k+1, z | \langle \alpha \rangle); \alpha \parallel (m, k-1))$$

$$z := \bar{z}$$

$$A_3[f; \alpha \parallel m, j, [k, \bar{e}]] = [S(f \langle \alpha \rangle^D \pi(\alpha \parallel m+k, z | \langle \alpha \rangle); \alpha \parallel (m, j+k))]$$

$$= [S(f \langle \alpha \rangle^D \pi(\alpha \parallel j+k+1, z | \langle \alpha \rangle); \alpha \parallel (m, k-1))]^{z := \bar{e}}_{z := [k]}$$

$$m(z) = m \quad y(z) = k+z$$

$$m+j+k-\bar{e}, m+j+k$$

$$m, \dots, m+k+j$$

$$S(f \langle \alpha \rangle^D \pi(\alpha \parallel m, k+z | \langle \alpha \rangle); \alpha \parallel (m, j+k)) = 0? \text{ since } m+k+z \geq m+j+k$$

$$\sum S(\pi(\alpha \parallel m, k+z | \langle \alpha \rangle); \alpha \parallel (m, \omega))$$

$$y(z)-1$$

\leq

$$\prod_{\alpha=0}^{j+k} \frac{(\alpha_{m+\omega} - \alpha_{m+k})}{m(z)+\alpha}$$

$$m(z) \leq m+k \quad m(z)+y(z)-1 \leq m+j+k$$

$$y(z) =$$

$$j+k+1 > k+y(z)-1$$

$$\prod_{\alpha=0}^{j+k} [w] (\alpha_{m+\omega} - \alpha_{m+k})$$

$$m(z) + y(z) - 1 = m$$

$$m \leq m(z) = m$$

$$\int_{m+\omega}^{\infty} \alpha_m d\alpha$$

$$y(z) < j-2$$

$$m(z) n_j(z) - m-1 = j+k$$

$$m(z) + y(z) - 1 = m+j+k \quad z = j+m$$

$$z \leq j+k$$

$$m+\omega, j+k-\omega$$

$$z \in (j, j+k]$$

$$m(z) = m+z-j$$

$$m+\omega, m, m+z$$

$$y(z) = y(j+k+1) - y(z)$$

$$z > j \quad m(z) = m+k-z+j+$$

$$m+j+k-$$

$$y(z) =$$

$$y(j+k+y-z-k+z-j-1)$$

$$\deg g \leq k \quad \deg g = j+k-z$$

$$\delta(f<\alpha>_g; \alpha || m, j+k) = \sum \delta(g|m, \omega) \delta(f<\alpha>_g; m+\omega, j+k-\omega)$$

$$g = \pi(\alpha || m, j+k-z) = 0 \quad \omega = 0, \dots, j+k-z-1$$

$$\delta(f<\alpha>_g; m+j+k-z, \omega)$$

$$\left[\delta(g(\alpha || m, j+k-z | \langle \alpha \rangle); \alpha || m) \right]_{z=0}^{m+j+k-z} \stackrel{m+j+k-z}{D} := [j, j+k] \quad \left[\delta(g(\omega); \alpha || m, z) \right]_{z=j+k}^{m+j+k-z} \stackrel{j+k-z}{D} := [i]$$

$$\sum \delta(g(\omega); \alpha || m, \omega) \delta(\pi(\alpha || m, j+k-z | \langle \alpha \rangle); \alpha || m+j+k-\omega, \omega) \quad \omega = (j, j+k)$$

$$\times [j+1; \alpha || m, [k]]^{-1} \stackrel{-1}{\text{cat}} [\delta(f<\alpha>_g; m, j+k-z)]_{z=j+k}^{m+j+k-z} \stackrel{m+j+k-z}{D} := [i]$$

$$= [\delta(f<\alpha>_g; m+z, j+1)]_{z=[k]} \stackrel{D := [i]}{=} A_3 [f; \alpha || m, j, [k, i]]$$

$$\times [j+1, \alpha || m, [k]] [\delta(f<\alpha>^z; m+z, j+1)]_{z:= [k]}^{D:= [\bar{e}]} =$$

$$= [\delta(f<\alpha>^z; \alpha || m+z, j+k-z)]_{z:= [k]}^{D:= [\bar{e}]} = A_4[f; \alpha || m, j, [k, \bar{e}]]$$

$$A_0[f; \alpha || m, j, [\bar{e}, k]] = [\delta(f<\alpha>^z; \alpha || m, z)]_{z:= (j, j+k)}^{D:= [\bar{e}]}$$

$$A_1[f; \alpha || m, j, [k, \bar{e}]] = [\delta(f<\alpha>^{z\theta}; \alpha || m, j+k)]_{z:= [k]}^{D:= [\bar{e}]}$$

$$A_2[f; \alpha || m, j, [k, \bar{e}]] = [\delta(f; \alpha || m+z, z-\omega)]_{z:= (j, j+k)}^{D:= [\bar{e}]}$$

$$A_3[f; \alpha || m, j, [k, \bar{e}]] = [\delta(f<\alpha>^z; m+z, j+1)]_{z:= [k]}^{D:= [\bar{e}]}$$

$$A_4[f; \alpha || m, j, [k, \bar{e}]] = [\delta(f<\alpha>^z; \alpha || m+z, j+k-z)]_{z:= [k]}^{D:= [\bar{e}]}$$

$$A_2[f; \alpha || m, j, [k, \bar{e}]] = \mathcal{U}[\delta(<\alpha>^z; \alpha || m, z)] = A_0[f; \alpha || m, j, [\bar{e}, k]]$$

$$[\delta(<\alpha>^z; \alpha || m+j+z, k-\omega)]_{z:= [k]}^{D:= (k)} A_0[f; \alpha || m, j, [\bar{e}, k]] =$$

$$A_1[f; \alpha || m, j, [k, \bar{e}]]$$

$$\times [j+1; \alpha || m, [k]]^{-1} A_0[f; \alpha || m, j, [k, \bar{e}]] = A_3[f; \alpha || m, j, [k, \bar{e}]]$$

$$\times [j+1; \alpha || m, [k]] A_3[f; \alpha || m, j, [k, \bar{e}]] = A_4[f; \alpha || m, j, [k, \bar{e}]]$$

$$\tilde{X} = \tilde{X} X^{-1}$$

$$\tilde{X} [j+1; \alpha || m, [k]] A_0[f; \alpha || m, j, [k, \bar{e}]] = A_4[f; \alpha || m, j, [k, \bar{e}]]$$

express $\delta(f; \alpha || m+z, z-\omega)$ as $\delta(f \# (\dots); \dots)$ and investigate

$$\mathcal{L}[\delta(<\alpha^z>; \alpha || m, z)] A_2$$

$$\delta[\delta(h(z); \alpha \| m, \omega)]_{z:=(j, k)}^{D:=(j+k)} A_2[f; \alpha \| m, j, [k, i]]$$

339.

$$\left\langle \delta(h(z); \alpha \| m, \omega) \right\rangle \delta(\pi(\alpha \| m, \omega) \times f \| m, \omega) \quad \langle \omega := (j, j+k) \rangle$$

$$[\delta(h(z) \times f; \alpha \| m, j+k)]$$

$$= \delta[\Delta(h(z), \alpha \| m, j+k) - \Delta(h(z), \alpha \| m, j)] \times f \| (m, \omega)$$

$$h(z) = \pi(\alpha \| m, z) : \delta(f; \alpha \| m+z, \omega - \tau) = 0 \text{ when } \tau < 0 \Rightarrow \\ \tau > j : \text{all when } j > i$$

$$M_0: A_0 \quad z_0 = j+k-z \quad M_1: A_1 \text{ have } M_4: A_3 \quad z_0 = k-z$$

$$M_2: A_2 \quad M_3: A_4 \quad \text{(6) } M_3 \subset M_1 \quad A_4 \subset A_1$$

$$\sum \delta(h(z); \alpha \| m, \omega) \delta(f \langle \alpha \rangle; \alpha \| m+\omega, j+k-\omega) \quad \langle \omega := (j+k) \rangle$$

$$h(z) := \langle \alpha \rangle^z$$

$$\mathcal{L}[\delta(\langle \alpha \rangle^z; \alpha \| m, \omega)]_{z:=(j+k)} \rightarrow A_4[\dots] = A_1[\dots]$$

$$A_2[f \dots] \text{ is section } A[f]_{z:=(j, j+k)}^{D:=(i)} \text{ having properties} \\ A[\epsilon \times f] = A[\epsilon] A[f]$$

$$\text{determine } U[-]^{-1} A_0, U[-] = kA_0' \quad (\text{i.e. } A_0' = U[-]^{-1} A_0)$$

$$\text{is } A_0' \text{ section } E[A_0'[-]_{z:=(j, j+k)}^{D:=(i)}] \text{ with } A_0' \text{ having multiplication property}$$

$$\text{Similarly } A_1', A_3', A_4'$$

$$\begin{aligned} &= \{p(\alpha_{m+z}) - p(\beta_{k+z})\}_{z:=(j, j+k)}^{D:=(i)} = \{u + (\rho-1)p(\beta_{k+z})\}_{z:=(j, j+k)}^{D:=(i)} \quad \text{as } k \text{ has first. of deg } \leq i \\ &L(\alpha/\beta) \in \text{col}[\{p(\alpha_{m+z}) - p(\beta_{k+z})\}_{z:=(j, j+k)}^{D:=(i)}]_{z:=(j, j+k)} = \text{col}[\{u + (\rho-1)p(\alpha_{m+z})\}_{z:=(j, j+k)}^{D:=(i)}] \\ &= u\epsilon[\{p(\alpha_{m+z}) - p(\beta_{k+z}) + (\rho-1)p(\alpha_{m+z})\}_{z:=(j, j+k)}^{D:=(i)}] \\ &= \rho^2 \text{col}[\{p(\alpha_{m+z}) - p(\beta_{k+z})\}_{z:=(j, j+k)}^{D:=(i)}] \end{aligned}$$

Defn. \sim

$K \sim K'$
 $\text{seq}(K), \dots$
 $\text{seq}(K'), i \in N \quad \text{seq}(K|i) \quad \text{seq}(K|\geq i)$

$[h] \in [h, k] \quad [h, k] \in [h, k] \quad [h, k] \sim [h, k]$

$\text{seq}'(K) \quad \text{seq}'(K|i) \quad \text{seq}'(K|\geq i) \quad \text{seq}'(K|_{\exists [m, m+i]}) \quad \text{seq}'(K|_{[m, m+i]})$

$K \rightarrow K$

$\frac{i \in K}{h \in K} \rightarrow h, k \in K \quad [K] \xrightarrow{K \rightarrow K} [h, k] \quad [K] \not\sim [h, k]$

$[K|h, k] \quad [K|h]$

$L[K|h] \quad uL[K|h, k] \quad u[K|h, k] \quad uu[K|h]$

$\text{diag}[K|h] \quad K \rightarrow K|h, k]$

$[K \rightarrow K|h, h] \sim [K \rightarrow K|h], \dots, \text{diag}[K \rightarrow K|h]$

$\text{row}[K] \quad \text{row}[K|h] \quad \text{col}[K|h] \quad \text{wt}[K|h]$

$\{K \rightarrow [K]\} \quad A: K \rightarrow [K] \quad \{K \rightarrow [K|h, k]\}, \dots, \{K \rightarrow \text{col}[K|h]\}$

Allocation A :

$\Theta, \phi: K \rightarrow K \quad B \subseteq K \quad \oplus = \phi \langle B \rangle$

Allocation $A_{z, \alpha}: K \rightarrow K \quad z := [h], \alpha := [k] \quad A: K \rightarrow [K|h, k]$

$A := [A_{z, \alpha}]_{z:=[h]}^{\alpha:=[k]} \quad k = h \quad A := [A_{z, \alpha}]_{z:=[h]}^{\alpha:=[k]}$

$b_z: K \rightarrow K \langle z := [k] \rangle \quad b: K \rightarrow \text{col}[K|h] \quad b := [b_z]_{z:=[h]}$

$c_\alpha: K \rightarrow K \langle \alpha := [k] \rangle \quad c \quad \text{row} \quad c := [c_\alpha]_{\alpha:=[k]}$

$| a_{[h], [k]} \quad A: K \rightarrow L[K|h, k] \dots \quad A: K \rightarrow \text{uu}[K|h, k]$

$d: K \quad d_x: K \rightarrow K \quad x := [h] \quad d := d: K \rightarrow \text{diag}[K|h]$

$d := \text{diag}[d_x] \langle x := [h] \rangle \quad \text{wt}[A]$

$A := u \text{ad}[d_x] \langle x := [h] \rangle$

Sequences

Matrices

Mappings

Allocation

The asymmetric allocation relationship ":-" is used for the purpose of definition. Thus $B, C: K \rightarrow K$ being prescribed, the ^{product} mapping $A: K \rightarrow K$ may be defined by setting $\parallel A := BC$

Multiple use of the above allocation relationship is made in the definition of matrix mappings

\parallel If $\text{seq}(\bar{N})$ is the complete system of integer sequences of the form $\bar{s}_0, \bar{s}_1, \dots$ with, where $\bar{s}_0, \bar{s}_1, \dots \in \bar{N}$, with finitely many numbers, with $i \in \bar{N}$

\parallel With $\bar{s} \in \text{seq}(\bar{N})$ prescribed, $\{\bar{s}\}_{i=1}^k$ is the k terms $\bar{s}_0, \bar{s}_1, \dots, \bar{s}_k$, $|s|$ is their key. With $i \in \bar{N}$ prescribed, $[s]_i = s_i$... With $m, n \in \bar{N}$

\parallel $\text{seq}(\bar{N}| \geq m)$ prescribed $\text{seq}(\bar{N}| \bar{s}[m, mi])$ is $\text{seq}(\bar{N}| \geq i)$ in the case in which $\bar{s}[m, mi] \equiv [m, mi]$, $\text{seq}(\bar{N}| \dots)$

$(\text{seq}(\bar{N}), \text{seq}(\bar{N}|i), \dots, \text{seq}(\bar{N}), \text{seq}(\bar{N}|i), \dots, \text{seq}(K|i) \dots)$

$\text{seq}'(\bar{N})$ is the complete system $\dots, \bar{s}_0, \bar{s}_1, \dots$, a subsystem of $\text{seq}''(\bar{N})$ consisting of distinct members. With $i \in \bar{N}$, $\text{seq}'(K|i)$ - 1 $\text{seq}'(K|i)$ is the similar subsystems of $\text{seq}(K|i)$ - 1 $\text{seq}(K|i)$ comp. seq with which $\bar{s} \in \dots \text{seq}'(\bar{N}| \bar{s}[m, mi])$ is ... ~~for all~~ the complete subsystem of $\alpha \text{seq}(\bar{N}| \bar{s}[m, mi])$ for which $\alpha(s(m)), \dots, \alpha(s(mi))$ are distinct. $\text{seq}'(\bar{N}|[m, mi])$ is similarly defined ^{related to} in terms of $\text{seq}(\bar{N}|[m, mi])$

Definitions

1] Prescribed mathematical systems

- i) $\bar{\mathbb{N}}$ and $\bar{\mathbb{R}}$
 $\bar{\mathbb{N}}$ is the sequence ω and $\bar{\mathbb{N}}$ are the integer sequences $0, 1, \dots$, and
 $\omega, -1, 0, 1, \dots$ respectively. The integers possessing both ordering and meaning
 properties, and those of the first two, are used for the purpose of meaning.
 $\bar{\mathbb{R}}$ is a prescribed field which, in certain applications of the
 theory, is assumed to possess required properties (e.g., with $i \in \bar{\mathbb{R}}$ given,
 that of possessing at least i distinct members).

2] Sequences

- ii) $\text{seq}(\bar{\mathbb{R}})$ is the complete system of integer sequences ζ of the form $\zeta(0), \zeta(1), \dots$,
 where $\zeta(0), \zeta(1), \dots \in \bar{\mathbb{N}}$, with finitely many members. With $\{h, k \in \bar{\mathbb{R}}\}$ and $b \in \bar{\mathbb{R}}$,
 $[h, k]$ is the integer sequence $h, h+1, \dots, k$ and when $h < k$, the $\zeta(h, k)$ is the sequence
 $h+1, \dots, k$; and $[h, k] \subset (h, k)$ are similarly defined.

- c) With $\zeta \in \text{seq}(\bar{\mathbb{R}}, \bar{\mathbb{N}})$ prescribed, ζ being $\zeta(0), \zeta(1), \dots, \zeta(k)$, $|\zeta| = k$.
 d) With $i \in \bar{\mathbb{R}}$ prescribed, $\text{seq}(\bar{\mathbb{R}} | \geq i)$ and $\text{seq}(\bar{\mathbb{R}} | i)$ are the complete
 subsystems of $\zeta \in \text{seq}(\bar{\mathbb{R}})$ for which $|\zeta| \geq i$ and $|\zeta| = i$ respectively.
 e) With $m, i \in \bar{\mathbb{R}}$ and $\zeta \in \text{seq}(\bar{\mathbb{R}})$ prescribed, and $j = \max \{n \mid \zeta(n) \leq m\}$,
 $\text{seq}(\bar{\mathbb{R}} | \zeta[m, m_i])$ is $\text{seq}(|\zeta| \geq j)$. In the case in
 $\text{seq}(\bar{\mathbb{R}} | \zeta(m, m_i))$ is similarly defined, as are, with $\zeta \in \text{seq}(\bar{\mathbb{R}} | \geq m_{i-1})$,
 $\text{seq}(\bar{\mathbb{R}} | \zeta[m, m_i])$ and $\text{seq}(\bar{\mathbb{R}} | \zeta[m, m_i])$.
 f) $\text{seq}'(\bar{\mathbb{R}})$ is the complete subsystem of $\text{seq}(\bar{\mathbb{R}})$ consisting of
 distinct members of $\bar{\mathbb{R}}$. With $i \in \bar{\mathbb{R}}$, $\text{seq}'(\bar{\mathbb{R}} | \geq i)$ and $\text{seq}'(\bar{\mathbb{R}} | i)$
 are the similar subsystems of $\text{seq}(\bar{\mathbb{R}} | \geq i)$ and $\text{seq}(\bar{\mathbb{R}} | i)$
 respectively. With $i, m \in \bar{\mathbb{R}}$ and $\zeta \in \text{seq}(\bar{\mathbb{R}} | \geq m_i)$ prescribed,
 $\text{seq}'(\bar{\mathbb{R}} | \zeta[m, m_i])$ is the complete subsystem of $\alpha \in \text{seq}'(\bar{\mathbb{R}} | \zeta[m, m_i])$.

for which the members of $\alpha \in \alpha(\frac{1}{3}(m)), \dots, \alpha(\frac{1}{3}(m_i))$ are distinct. In the case in which $\frac{1}{3}[m, m_i] = [m, m_i]$, $\text{seq}'(\overline{\mathbb{N}} | \frac{1}{3}[m, m_i])$ is written as $\text{seq}'(\overline{\mathbb{N}} | [m, m_i])$

h) $\text{seq}(\overline{\mathbb{N}})$ is the complete system of integer sequences of the form $\frac{1}{3}, \frac{1}{3}, \dots \in \overline{\mathbb{N}}$ with finitely many members. $\text{seq}(\overline{\mathbb{N}} | \geq i), \dots, \text{seq}(\overline{\mathbb{N}}), \dots, \text{seq}(\mathbb{K}), \dots, \text{seq}'(\mathbb{K} | [m, m_i])$ are defined in analogy with the above.

e) With $i, m \in \overline{\mathbb{N}}$ and $\frac{1}{3} \in \text{seq}(\overline{\mathbb{N}} | \geq m_i)$, $\frac{1}{3}[m, m_i]$ is the subsequence $\frac{1}{3}, \dots, \frac{1}{3}$ of $\frac{1}{3}$. $\frac{1}{3}[m, m_i]$ is similarly defined as α , with $\frac{1}{3} \in \text{seq}(\overline{\mathbb{N}} | \geq m_{i-1})$, $\frac{1}{3}[m, m_i]$ and $\frac{1}{3}(m, m_i)$.

i) Subsequences of $\text{seq}(\overline{\mathbb{N}})$ play a special role in the formation of nested subsequences: with $m, i, m \in \overline{\mathbb{N}}$, and $\frac{1}{3} \in \text{seq}(\overline{\mathbb{N}} | \geq m_i)$, and $\alpha \in \text{seq}(\overline{\mathbb{N}} | \frac{1}{3}[m, m_i])$, $\alpha[\frac{1}{3}[m, m_i]]$ is the subsequence $\alpha(\frac{1}{3}(m)), \dots, \alpha(\frac{1}{3}(m_i))$ of α . $\alpha[\frac{1}{3}[m, m_i]], \dots, \alpha[\frac{1}{3}(m, m_i)]$ are similarly defined. With $\alpha \in \text{seq}(\overline{\mathbb{N}} | \frac{1}{3}[m, m_i])$ or... or $\alpha \in \text{seq}(\mathbb{K} | \frac{1}{3}[m, m_i])$, the above subsequences are defined analogously.

3] Matrices

a) $[\mathbb{K}]$ is the complete system of matrices with elements in \mathbb{K} ,
 b) $L[\mathbb{K}]$ is the subsystem of $[\mathbb{K}]$ of members A of $[\mathbb{K}]$ for which
 which $A_{z,y} = 0$ for all elements whose column suffix z
 exceeds the row suffix y). $U[\mathbb{K}]$ is the complete subsystem of
 upper triangular members of $[\mathbb{K}]$ (now $A_{z,y} = 0$ when $z > y$).

[f) $\text{diag}[\mathbb{K}]$ is the complete system of intersection of $\mathcal{L}[\mathbb{K}]$ and $\mathcal{U}[\mathbb{K}]$ (note $A_{z,z}^{(1)} = 0$ when $z \neq z$). $\mathcal{U}\mathcal{L}[\mathbb{K}]$ is the complete system of all unit lower triangular members of $\mathcal{L}[\mathbb{K}]$ (now $A_{z,z}^{(1)} = 1$ for all diagonal members $A_z^{(1)}$ of $A \in \mathcal{L}[\mathbb{K}]$). $\mathcal{U}\mathcal{U}[\mathbb{K}]$ is the system of all unit upper triangular members of $\mathcal{U}[\mathbb{K}]$ (again a condition of the form $A_{z,z}^{(1)} = 1$ holds).] g) $\mathcal{U}\mathcal{L}2[\mathbb{K}]$ is the subsystem of all unit lower diagonal matrices in $\mathcal{U}[\mathbb{K}]$ (now conditions of the form $A_{z,z}^{(1)} = 0$ when $z > z$, $A_{z,z}^{(1)} = 1$

(a) $A_{z,z}^{(1)} = 0$ if either $z > z$ or $z > z+1$ and $A_{z,z}^{(1)} = 1$ holds)

h) $\mathcal{U}\mathcal{U}2[\mathbb{K}]$ is the subsystem of all unit upper diagonal matrices in $\mathcal{U}[\mathbb{K}]$ (now the relevant conditions have the form (a) $A_{z,z}^{(1)} = 0$ if either $z < z$ or $z < z+1$ and $A_{z,z}^{(1)} = 1$)

iii) With $h, k \in \mathbb{N}$ prescribed, $[\mathbb{K}|h,k]$ is the subsystem of all matrices in $[\mathbb{K}]$ with $h+1$ rows and $k+1$ columns. $\mathcal{L}[\mathbb{K}|h,k], \dots, \mathcal{U}\mathcal{U}2[\mathbb{K}|h,k]$ are similarly defined. b) There tasks $[\mathbb{K}|h,k]$ is written simply as $[\mathbb{K}|h]$, and correspondingly abbreviated notations $\mathcal{L}[\mathbb{K}|h], \dots, \mathcal{U}\mathcal{U}2[\mathbb{K}|h]$ also being used and adopted and col $[\mathbb{K}]$ are

iv) $\text{row}[\mathbb{K}]$ is the complete system of row vectors and column vectors with elements in \mathbb{K} .

b) With $h \in \mathbb{N}$ prescribed, $\text{row}[\mathbb{K}|h]$ is the subsystem of all row vectors in $\text{row}[\mathbb{K}|h]$ with $h+1$ elements. $\text{col}[\mathbb{K}|h]$ is similarly defined.

v) With $i \in \mathbb{N}$ prescribed, $I[i]$ is the unit matrix in $\text{row}[\mathbb{K}|i]$.

$\underline{I}^{[i]}$ and $\bar{I}_{[i]}$ are

$\underline{L}[i]$ and $\bar{L}[i]$ are the members of $L[K|i]$ and $\bar{L}[K|i]$ respectively all of whose members not defined to be zero are unity (thus for the elements $A_{\alpha\beta}$ of $A \in L[i]$, $A_{\alpha\beta}=0$ when $\alpha < \beta$ and $A_{\alpha\beta}=1$ when $\alpha \leq \beta$). $I^{[i]}$ and $\bar{I}_{[i]}$ are the members of $\text{row}[K|i]$ and $\text{col}[K|i]$ respectively whose elements are all unity.

(iv) With $h, k \in \bar{\mathbb{N}}$ prescribed $O_{[h]}^{[k]}$ is the zero matrix in $[K|h, k]$. $O_{[h]}^{[k]}$ is written as $O[h]$. $O_{[h]}$ and $O^{[k]}$ are the zero vectors in $\text{row}[K|h]$ and $\text{row}[K|k]$ respectively.

c) Where convenience dictates, the zero matrix in $[K|h-1, k]$ is written either as $O_{[h]}^{[k]}$ or $O^{[h]}$, and that in $[K|h-2, k]$ as $O_{(h)}^{(k)}$, the selection of these alternatives being made to preserve conformity with adjacent formulae in which similar contracted formulations feature. The notations $O_{[h]}^{(k)}$, $O_{(h)}^{(k)}$, ..., $\underline{I}[i]$, $\bar{I}[i]$, ... have ^{corresponding} _{similar} meanings.

v) The formation of compound matrices by adjunction of columns is indicated by the use of a vertical bar. Thus with ~~the~~ $k_0, k_1, \dots, k_2^{(2)} \in \bar{\mathbb{N}}$ prescribed and $A \in [K|h, k_0]$, $B \in [K|h, k_1]$, $C \in [K|h, k_2^{(2)}]$ the compound matrix

$A | B | C$

in $[K|h, k_0 + k_1 + k_2^{(2)} + 2]$ contains in order the ^{successive} columns of A

followed by those of B followed those of C . This adjunction is indicated by the use of vertical double bars. Thus, with suitably declared matrices D, E, F the successive rows of

$D \parallel E \parallel F$

is a comp one those of D followed by those of E followed by those

of

(there exist neither discernment incapable of further refinement nor susceptibility)
so blunted that cruder measure is impossible ??

of F

c) The above conventions are used in conjunction, ^{feature in} ^{auxiliary} possible use of square brackets being made for clarity. Thus, the compound matrix

$$\begin{array}{c} \cancel{[h+k]} | [O[h)] \parallel I[k]] \\ [h+k] | [O^{[k]}_{[h]} \parallel I[k]] \end{array} \quad \begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

is in $[K | h+k, h+2k]$

d) The convention that void constituent submatrices may occur in compound matrix constructions is adopted. (Thus, in the above example, it is permitted that $k=0$; the compound matrix then reduces to $I[h] + I[h]$). In all cases in which this convention is adopted, the resulting compound matrix is nevertheless nonvoid.

e) Determinants of square matrices are indicated by the use of vertical bars. Thus, for suitably defined A , $|A|$ is the determinant of A .

To avoid possible confusion square brackets are introduced into the rep when dealing with determinants of compound matrices. Thus the determinant of $A|B$, A and B being suitably defined, is expressed as $|(A|B)|$

4] Mappings

single
functions occurring in mappings with source and target domain \bar{N} over as 34%.

- i) $\{\bar{R} \rightarrow \bar{N}\}$ is the complete system of ~~functions~~^{functions A occurring in} mappings of the form $A: \bar{R} \rightarrow \bar{R}$. Further complete systems of ~~functions~~^{functions A occurring in} mappings (for example, $\{\bar{R} \rightarrow K\}$, $\{K \rightarrow K\}, \dots$) are denoted in the same way.
- ii) With $i \in \bar{R}$ prescribed $\{K \rightarrow \text{seq}(K[i])\}$ is the complete system of ~~functions~~^{functions A occurring in} ν -fixed type mappings of the form $A: K \rightarrow \text{seq}(K[i])$. Again further systems of ν -fixed type mappings are denoted in a similar way. For example, with $h, k \in \bar{R}$ prescribed $\{K \rightarrow [K[h, k]]\}$ is the complete system of ν -mappings of the form $A: K \rightarrow [K[h, k]]$.

- iii) Equivalence of two mappings over a common ^{sub}source domain is indicated by enclosing the subdomain within triangular brackets. Thus, with $K', K'' \subseteq K$, ~~and~~^{and} $\Theta: K' \rightarrow K$, $\phi: K'' \rightarrow K$ and $B \subseteq K' \cap K''$, the notation

$$\Theta = \phi \langle B \rangle$$

indicates that ~~def~~ $\Theta(z) = \phi(z)$ for each $z \in B$.

5] Allocation

- i) The asymmetric allocation relationship " \coloneqq " is used for the purposes of definition. Thus $B, C: K \rightarrow K$ being prescribed, the product mapping $A: K \rightarrow K$ may be defined by setting

$$A := BC$$

(a) Multiple use
ii) \therefore

\Rightarrow depending on z ??

" $:$ "

ii) The above relationship is used within triangular brackets to indicate the range of running variables in assignment statements. Thus, with $i \in \mathbb{N}$ presented, the notation $A_x : K \rightarrow K \langle x := [i] \rangle$

means that $i+1$ mappings A_x ~~here~~ are presented. Nested $x := [i]$

The use of this notation is made: with $h, k \in \mathbb{N}$ presented, the notation $B_{z,2} : K \rightarrow K \langle z := [h], 2 := [k] \rangle$ has a similar meaning. In the case in which $k = k$, the notation just given is contracted to the form $B_{z,2} : K \rightarrow K \langle z, 2 := [h] \rangle$ fixed type

iii) Conjoint use of the above conventions is made in matrix mapping allocations.

a) With $h, k \in \mathbb{N}$ and $A_{(z,2)} : K \rightarrow K \langle z := [h], 2 := [k] \rangle$ presented,

the ^{fixed type} matrix mapping $A : K \rightarrow [K | h, k]$ may be defined by setting

$$A := [A_{(z,2)}]_{z := [h]}$$

The row variable z featuring ⁱⁿ a suffix declaration, the column variable 2 in a superscript declaration. In the case in which $h = k$, $A : K \rightarrow [K | h]$ is defined by setting

$$A := [A_{(z,2)}] \langle z, 2 := [h] \rangle$$

b) The ~~matrix~~ mapping $A : K \rightarrow \mathcal{L}[K^h, k]$ is defined by an allocation of the form

$$A := \mathcal{L}[A_{(z,2)}]_{z := [k]}$$

- b) With $h, k \in \mathbb{N}$ and $A(z, \bar{z}): K \rightarrow K \langle z := [h], \bar{z} := [\min(z, h)] \rangle$ available prescribed the mapping $\underline{A}: K \rightarrow \underline{K}[h, k]$ is also defined by use of the above notation & the requisite zero mapping $\underline{A} := \underline{\underline{L}}[A(z, \bar{z})]_{\bar{z} = [h]}$ $\check{A}_{z, \bar{z}, h} := 0$ for with $\bar{z} > z$ being assumed to be implied by the specification of A . When $h = k$ the notation is contradicted as above. Similar considerations relate to the definition of mappings $A: K \rightarrow \underline{K}[h, h]$, $A: K \rightarrow \underline{K}[h, k]$, $A: K \rightarrow \underline{K}[k, h]$ and $A: K \rightarrow \underline{K}\underline{K}[h, k]$.
- c) With $i \in \mathbb{N}$, and $d(x): K \rightarrow K \langle x := [i] \rangle$ prescribed the mapping $D: K \rightarrow \text{diag}[K[i]]$ may be for which $D_{x, i, i} = d_i(x)$ $\langle x := [i] \rangle$ is defined by setting

$$D := \text{diag}[d(x)] \langle x := [i] \rangle$$

With zeros h, k, i as above and $\underline{d}x: K \rightarrow K \langle x := (i) \rangle$ prescribed, the mapping $\underline{D}: K \rightarrow \underline{K}\underline{K}[h, k]$ for which $\underline{D}_{x, i, j} = \underline{d}_j(x)$ $\langle x := (i) \rangle$ is defined by setting

$$\underline{D} := \underline{K}\underline{K}[d(x)] \langle x := (i) \rangle$$

and the mapping $F\underline{D}: K \rightarrow \underline{K}\underline{K}\underline{K}[K|h, k]$ for which $F\underline{D}_{x, i, j, l} = d_l(x)$ $\langle x := (i) \rangle$ is defined by setting

$$F\underline{D} := \underline{K}\underline{K}\underline{K}[d(x)] \langle x := (i) \rangle$$

- d) With $i \in \mathbb{N}$ and $\overset{c}{d}(x): K \rightarrow K \langle x := [i] \rangle$ prescribed, a fixed type column mapping $\overset{c}{B}: K \rightarrow \text{col}[K|i]$ may be defined by setting

$$A := \text{col} \left[\overset{c}{\underset{\omega}{\Phi}}(z) \right]_{z := [i]} \quad \text{B: } K \rightarrow \text{row}[K[i]]$$

Similarly, a fixed type row mapping may be defined by setting

$$B := \text{row} \left[c(\omega) \right]^{D := [i]}$$

- e) ~~After~~ Integer intervals that are open at the upper limit also feature in matrix mapping allocation. Thus, with $h \in \mathbb{N}$, $k \in \mathbb{N}$ and $A(z, \omega) : K \rightarrow K \langle z := [h], \omega := [k] \rangle$ prescribed, a fixed type matrix mapping $A : K \rightarrow [K | h, k-1]$ may be defined by setting

$$A := \left[A(z, \omega) \right]_{z := [h]}^{D := [k]}$$

The remaining

~~All of the notations described above are subject to modification in the same way.~~

6] Sums and products $[h, k]$

i) With $\alpha(\omega) : K \rightarrow K \langle \omega := [i] \rangle$ prescribed the sum mapping

$A : K \rightarrow K$ for which

$$A = \alpha(i) + \dots + \alpha(k) \quad \langle K \rangle$$

is indicated by the use of the notation

$$A := \sum_i \alpha(\omega) \quad \langle \omega := [\underline{i}] \rightarrow [h, k] \rangle$$

and the product mapping $B : K \rightarrow K$ for which

$$B = \alpha(h) \alpha(h+1) \dots \alpha(k) \quad \langle K \rangle$$

by use of the notation

$$B := \prod_i \alpha(\omega) \quad \langle \omega := [h, k] \rangle$$

ii) Similar conventions are adopted with regard to well defined 351.

matrix sums mapping sums and products with the additional stipulation that in the latter case of the factors corresponding to lower values of the running suffice occur upon the right. Thus the matrix of mapping product B indicated by use of the above notation has the ordering indicated in the preceding relationship.

$a(\omega) \langle \omega := [h, k] \rangle$ being suitably defined matrix mappings, the product mapping $B' : K \rightarrow K$ for which

$$B' = a(h) \dots a(k-1)a(k) \langle K \rangle$$

may be defined
is denoted by setting

$$B' := \prod a(k-\omega) \langle \omega := [k-h] \rangle$$

iii) As just indicated in the last example, zero lower limits of intervals are omitted. For brevity, semi-open and open interval ranges are used (thus allusion to the sum $\langle \omega := [h, k] \rangle$, $\langle \omega := (h, k] \rangle$ and $\langle \omega := (h, k) \rangle$ may feature in sum and product specifications).

iv) Empty ^{mapping} sums (for example mappings (for example those as described in (i) above with $k < h$) are given the ^{constant mapping} value zero; and empty products mapping products are given the value unity.

v) Empty matrix mapping sums are given the zero-matrix constant value of the zero matrix of dimension equal to that of the constituent terms. In the case in which all factors specified in a matrix mapping product are square

native mappings, an empty product is given the constant value of the unit matrix of dimension corresponding to that of the factors

p343. $A^{\frac{1}{2}} A_B : \underset{\text{delimiters}}{B \subseteq K} \underset{\text{constants in } K'}{\rightarrow} \underset{\text{functions}}{B}$, constants in K' as mapping

3ia) contd.

The elements of $A \in [K]$ are denoted by $A_z^{\frac{1}{2}}$, z being indicated in the row

The elements of a member of $[K]$ are denoted by the use of a suffix-superscript notation, the suffix denoting a denoting a row number form and superscript functioning as row and column indexes respectively (thus the elements of $A \in [K]$ may be denoted by $A_z^{\frac{1}{2}}$, z second ω indicating row and column numbers respectively). Ordering begins at zero (thus, in the example just given, all elements in the first row of A have suffix $z=0$, all elements in the first column have superscript $\omega=0$).

3ii) The elements of a member of $\overset{\text{in the target domain}}{[K]}$

4ia) The value assumed by $\phi: \overset{\text{in the target domain}}{K} \rightarrow \overset{\text{in the target domain}}{K}$ where ~~is~~ i is selected corresponding to i in the source domain $\overset{\text{in the target domain}}{K}$ is denoted by $\phi(i)$. A similar convention is adopted with regard to the other simple mappings mentioned in (i) above.

b) The value in the target domain $\text{seq}(K|i)$ assumed by $\psi: K \rightarrow \text{seq}(K|i)$ corresponding to z in the source domain K is denoted by $\psi(z)$. With $\omega \in [i]$, the prescribed, the mapping in $\{K \rightarrow K\}$ defined by the component of ψ with index ω is

35:

denoted by $\psi(\omega_1)$ (thus $\psi: K \rightarrow \text{seq}(K|k)$ determines its component mappings of the form $\psi(\omega_1): K \rightarrow K$). The component value in K assumed by $\psi(\omega_1)$ corresponding to z in the source domain K is denoted by $\psi(\omega_1|z)$. Similar conventions are adopted with regard to other sequence mappings.

c) The matrix value in the target domain $[K|h,k]$ assumed by $A: K \rightarrow [K|h,k]$ corresponding to z in the source domain K is denoted by $A(z)$. With $z \in [h]$ prescribed, A_z denotes the $A_z: K \rightarrow \text{row}[K|k]$ is the row mapping defined by with index z determined by A ; A with $z \in [k]$, $A^z: K \rightarrow \text{col}[K|h]$ is a corresponding column mapping; with z, ϑ as specified $A_{\vartheta}^z: K \rightarrow K$ is a component mapping. The values in $\text{row}[K|k]$, $\text{col}[K|h]$ and K corresponding to $z \in K$ are denoted by $A_z(z)$, $A^z(z)$ and $A_{\vartheta}^z(z)$ respectively. Similar conventions with regard to other sequence mappings are adopted.

4(v) Sums, differences, and products and, where well defined, quotients of simple mappings are defined pointwise. Thus, with $A, B, C: K \rightarrow K$ ^{being} prescribed, the relationship

$$R = BC \langle K \rangle$$

implies and is implied by the condition that $A(z) = B(z)C(z)$ for each $z \in K$.

^{p354} b) ^{at} Sums & differences of sequence mappings are treated in the same way, as are also products of simple mappings

and sequence mappings (thus, with $i \in \bar{N}$ presented, the above relationship³⁵ has meaning when $A, C: K \rightarrow \text{seq}(K|i)$ and $B: K \rightarrow K$).

d) Matrix functions of matrices are also treated in the same way. Functions involving ~~expressions~~^{mappings} an integer exponent represented by the use with the help of braces (thus the square of ~~the~~^a $A: K \rightarrow [K|i]$ is denoted by $\{A\}^2$; if A is a column mapping, the inverse of suitably defined A is denoted by $\{A\}^{-1}$).

d) Members of the target domain are treated as constant functions^{function} occurring in the ~~functions~~^{function} (thus $0 \in \bar{N}$ is treated as the mapping $\phi: \bar{N} \rightarrow \bar{N}$ for which $\phi(i) = 0$ for each $i \in \bar{N}$, the mapping being denoted by 0) represented by the same symbol, $\sqrt{}$

b) The sum of $\phi, \psi: \bar{N} \rightarrow \bar{N}$ is defined by letting $\phi + \psi$ be the mapping $\Theta(\phi, \psi): \bar{N} \rightarrow \bar{N}$ for which $\Theta(i)$ is composed of $\phi(i)$ followed by the members of $\psi(i)$. The difference $\phi - \psi$ is constructed by removing, for each $i \in \bar{N}$, those members of $\phi(i)$ which also feature in $\psi(i)$, the ordering of the remaining members of $\phi(i)$ being preserved.

With $h \in \bar{N}, k \in [h]$ and $\phi: \bar{N} \rightarrow \text{seq}(\bar{N}|h), \psi: \bar{N} \rightarrow \text{seq}(\bar{N}|k)$ the product $\phi\psi$ is the mapping $\Theta: \bar{N} \rightarrow \text{seq}(\bar{N}|k)$ constructed in the following way. $\text{ord } \psi: \bar{N} \rightarrow \text{seq}(\bar{N})$ is constructed by letting, for each $i \in \bar{N}$ and each $j \in [k]$, $\text{ord } \psi(j|i)$ be the number of members of $\psi(i)$ less than $\psi(j|i)$; for each $i \in \bar{N}$ the successive

members of $\Theta(i)$ are $\phi(\neg \exists x \text{ord} \psi(j \mid i) \mid i)$ as j ranges through $[k]$. 350

Sums, differences and products of mappings with target domains $\text{seq}(\mathbb{R})$ and $\text{seq}(\tilde{\mathbb{R}})$ are treated in the same way.

c) ~~Sums and differences~~ With $i \in \overline{\mathbb{N}}$, sums and differences of $b, c: K \rightarrow \text{seq}(K| i)$ are defined in terms of operations in K (thus the sum ~~base near~~ base is the mapping $a: K \rightarrow \text{seq}(K| i)$ for which, for each $z \in K$, $a(w| z) = b(w| z) + c(w| z)$ as w ranges through $[i]_z$). ~~With~~ The product of $b: K \rightarrow K$ and $c: K \rightarrow \text{seq}(K| i)$ is defined to be the mapping ~~and~~ $a: K \rightarrow \text{seq}(K| i)$ for which, for each $z \in K$, $a(w| z) = b(z)c(w| z)$ as w ranges through $[i]$).

Sums, differences and scalar products of ^{further} mappings with
^{composition} target domains in $\text{seq}(\mathbb{K})$ are treated in the same way

1
p 354.

With $h, k \in \bar{N}$ and $A : K \rightarrow [K|h, k]$ presented, and $\mathcal{B}_{\text{seq}}(\bar{N})$ formed by row selection from A , it is composed from the members of $[h]$, $A_{\frac{h}{z}}$ is that member of $\{K \rightarrow [K|1^z, k]\}$ for which $B_z = A_{\frac{h}{z}(z)}$ for all z as z ranges through $\mathcal{E}(\bar{N})$ corresponds, with $\mathcal{B}_{\text{seq}}(\bar{N})$ composed from the members of $[k]$, A^B is formed by column selection: it is that member of $\{K \rightarrow [K|h, 1^B]\}$ for which $C^B = A^{\frac{h}{B}(z)}$ $\langle h e \rangle$

as α ranges through $[S]$. The above notations are combinable:

$A_{\bar{\alpha}}^{\bar{\beta}}$ is in $\{K \rightarrow [K]^{[1/\bar{\beta}], [1/\bar{\beta}]} \}$.

\overline{p}_{342} (2ic) contd.

$[1/\bar{\beta}]$ is written as $[1/\bar{\beta}]$, $[1/\bar{\beta}]$ as $[1/\bar{\beta}]$ and so on.

Delimiters other than the conventional comma, semicolon and colon are used. Thus, with D a presented domain, the notation $L(\bar{\alpha}/\bar{\beta})$ is used to draw attention to the fact that, with T a suitable target domain, $L: D \times D \rightarrow T$ transforms according to the law

$$L(\bar{\alpha}/\bar{\beta}) L(\bar{\beta}/\bar{\gamma}) = L(\bar{\alpha}/\bar{\gamma})$$

for all $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ in \mathbb{D} .

The choice of parentheses is also extended. Thus the notation $V(L(\bar{\beta}))$ is adopted to emphasize the emphasis to the transformation properties

$$L(\bar{\alpha}/\bar{\beta}) V(L(\bar{\beta})) = V(L(\bar{\alpha}))$$

$$\{ L(\bar{\alpha}/\bar{\beta}) V(L(\bar{\beta})) = V(L(\bar{\alpha}))$$

With $A: K \rightarrow [K]$ and $\bar{\beta} \in \text{seq}(\bar{N})$ presented, $A_{-\bar{\beta}}: K \rightarrow [K]$ is formed by removing from A those rows A_z whose index z feature in the sequence $\bar{\beta}$ (the row dimension of $A_{-\bar{\beta}}$ is determined partly by the row dimension of A and partly by the number of distinct members of $\bar{\beta}$ occurring

357.

in the range covered by this row dimension). Correspondingly, with $f \in \text{seq}(\bar{K})$ prescribed, $A^{-\bar{f}} : K \rightarrow [K]$ is obtained by column removal from A . $A_{-\bar{g}}^{-\bar{f}} : K \rightarrow [K]$ is defined by carrying out both row and column removal.

In all cases in which the above instructions are used in connection with isolated matrix mappings A , the resulting submatrices are well determined (i.e. all columns or rows are not removed from A in the formation of $A_{-\bar{g}}$). It may occur that constituents of compound matrix mappings formed in the above way reduce to void become void.

(In most cases in which the lower triangular matrix mapping A is allocated like as above, the formula expression giving formula expressing $A(z, z)$ with $z \leq z$ may also be evaluated for $z < z$ and, by chance, yields the value zero. In this case the symbol "L" in the above declaration features merely as a comment upon the structure of A ; and is without operative significance. A similar comment may be made concerning allocation of the form

$$A := U[A(z, z)]_{z := [k]}^{-1} := [k]$$

triangular

In the very few cases in which lower and upper matrix mapping allocations have this form by declaration and not as a consequence of the form of $A(z, z)$, a suitable statement drawing attention to the anomaly is made.)

Components of structures are, in ways which depend upon the structures in question, interpreted regarded as structures of simple form

a) Members of Integers in \bar{N} are also taken to be sequences single member sequences in $\text{seq}(\bar{N})$. Thus 2 is the $\{ \}_{\phi} \in \text{seq}(\bar{N}|0)$ where $\{ \}_{\phi}(0) = 2$. (In this way, with $h, k \in \bar{N}$, $k \geq 2$ and $A \in [K|h, k]$, A^2 is the column of A with column superscript 2. Alternatively, with $k \geq 1$, A^{-1} is the submatrix obtained by removing the column with superscript 1 from A (for suitably declared A , $\{A\}^2$ and $\{A\}^{-1}$ are the square and inverse of A). Components Integers in \bar{N} and \bar{N} are regarded as being and members of K are regarded as being single member sequences in $\text{seq}(\bar{N})$, $\text{seq}(\bar{N})$ and $\text{seq}(K)$ respectively.

b) In expressions involving a member ϕ of the ordered product of a member ϕ^2 of $\{K\}^{2 \times 2}$ and a matrix mapping $A \in \{K|h, k\}$, $h, k \in \bar{N}$ being prescribed, ϕ^2 is regarded as being in $\text{diag} \{K\}^{2 \times 2}$ having diagonal elements $\phi^2 \in \{K\}^{2 \times 2}$ for $\langle w := [h] \rangle$. Where ϕ occurs as a post factor, ϕ^2 is regarded as being in $\{K\}^{2 \times 2}$ $\text{diag} \{K|h\}^{2 \times 2}$.

(d) Members of a target domain T are treated as constant functions in $\{D \rightarrow \bar{\mathbb{R}}\}$, D being a prescribed source domain, one represented by the same symbol and given the same name. (Thus $O \in \bar{\mathbb{R}}$ is treated as the function $O: \mathbb{R} \rightarrow \bar{\mathbb{R}}$ for which $O(i) = O$ for each $i \in \mathbb{R}$)

This convention is observed in conjunction conjointly with that also described above. (Thus $\lambda \in K$ is regarded as occurring as a premultiplying factor of a mapping $A: K \rightarrow A[K, h]$ may be regarded as a constant diagonal matrix in $\text{diag}[K|h]$ and then treated as a mapping constant function in $\{K \rightarrow [K|h]\}$

$h, k \in \bar{\mathbb{R}}$ being prescribed,

c) Let expressions involving the product of a matrix $\Phi \in [K|h]$ and a matrix mapping $A: K \rightarrow [K|h, k]$ $k, h, k \in \bar{\mathbb{R}}$ being is subsumed within operations upon matrix mappings by treating Φ as prescribed. Φ is treated as a constant mapping $\Phi: K \rightarrow [K|h]$. A similar convention is observed regarding post multiplicative factors and components of sums and differences.

Source and Target Domains Let $N' = \mathbb{R}$, $B \in \text{dom}(K, \bar{\mathbb{R}})$ and $T \in \text{sh}(K, \bar{\mathbb{R}})$

Extension of source and target domains In constructive mappings, ^{types} B and T are prescribed domains

i) The mapping $e: \mathbb{R} \rightarrow \{B \rightarrow T\} \dots$ p.38.

ii) \Leftrightarrow (i) p.38

A mapping of the form $e: \bar{N} \rightarrow \bar{N}$ may immediately be extended to the form $\bar{e}: \text{seq}(\bar{N}) \rightarrow \text{seq}(\bar{N})$ simply by letting $e(s)$ defining $e(s)$ to be the sequence $e(s(i))$, $e(s(1)), \dots, e(s(i))$ corresponding to where $s \in \text{seq}(\bar{N})$ where $i \in \bar{N}$. Various extensions of other mappings may be described; some finding subsequent use are now described.

Let $\bar{N}' := N$, $\text{Dedom}(K, \bar{N})$ and $\text{Tset}(K, \bar{N})$.

i) The mapping $e: \bar{N} \rightarrow \{B \rightarrow T\}$... p 38.

b) (= in p. 38)

ii) Define $\text{Dedom}(K, \bar{N}')$..

a) (= in p 37). The mapping $g: K$

$\langle B \rangle$

(p 38) The mapping $g: K$... declaration

b) Let $\bar{N}'' := N$.

The mapping $w: K \times \bar{N}'' \rightarrow \{B \rightarrow \text{seq}(T | \bar{N}'')\}$ is extended to the form $w: D \times \bar{N}'' \rightarrow \{B \rightarrow \text{seq}^2(T | \bar{N}', \bar{N}'')\}$ by setting

$$w(\bar{\beta}, k) := w(\beta | n, [i, k])$$

$$:= \{w(\beta | \beta_{n+z}) \langle B \rangle; z := [k], i := [i] \}$$

(b) p 38) \Rightarrow

The first extensions considered are, apart from a slight modification of the target domain, over the primitive form of that just described: the arguments in the extended source

361.

domain are sequences; the length of the ~~function~~ value in the sequence in the target domain is that of the argument sequence. In the second extensions, the source domains selection of a subsequence of the argument sequence is ~~possible~~, in the source domain is possible, the source domain being modified so as to include reference to integers which specify the position and length of the subsequence selected.

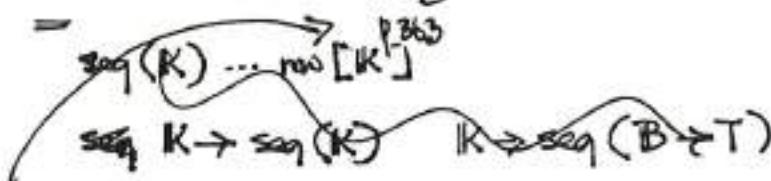
$$\begin{aligned}
 v_1 + v_2 &= \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} & (v_1 + v_2) + v_3 &= \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} + v_3 & v_1 + v_2 + v_3 + \frac{v_1 v_2 v_3}{c^2} \\
 && & \hline & \\
 &= v_1 + v_2 + v_3 + \frac{v_1 v_2 v_3}{c^2} & & 1 + \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \cdot v_3 & 1 + \frac{v_1 v_2}{c^2} + \frac{(v_1 v_2) v_3}{c^2} \\
 &\hline & & \hline & \\
 & v_1 \circ (v_2 \circ v_3) & v_1 + \frac{v_2 + v_3}{1 + \frac{v_2 v_3}{c^2}} & = \frac{v_1 + v_2 + v_3 + \frac{v_1 v_2 v_3}{c^2}}{1 + \frac{v_2 v_3}{c^2} + v_1 \frac{(v_2 + v_3)}{c^2}} \\
 & & \hline & &
 \end{aligned}$$

Type Define sequence as mapping with source domain \mathbb{N} ;
multiplication possible if target domain measurable ordered
(e.g. measure = number of elements less than given element
with respect to ordering); define $\text{inv } \xi$ when $\xi \in \text{seq}(\mathbb{N}, T)$

The elements of a member of row $[LK]$ form a sequence;
the member in question takes its place in a scheme of
operations of linear algebra. It is useful to have available

a type conversion operator which sets the successive members 362.
 of a subsequence of a member of $\text{seq}(K)$ into a row vector
 in $\text{row}[K]$. The source domain over which the conversion operator
 functions possesses three component subdomains, the two integers from
 which are selected two integers specifying the position and
 size of the subsequence together with the sequence α from
 which the subsequence in question; the latter last or last subdomain
 is dependent, since the sequences dealt with must be long
 enough to accommodate the subsequences specified by the
 integers selected from the first two subdomains. The preceding
 considerations relate in equal measure to the conversion
 of subsequences of sequences taken, not from $\text{seq}(K)$, but

from sequences of the form $\text{seq}(B \rightarrow T)$ where B is a
 defined at T a specified set
 specified domain. They also relate to the conversion of
 subsequences into column vectors in $\text{col}[K]$ and diagonal
 matrices in $\text{diag}[B \rightarrow T]$.



For a fixed value of $\alpha_m \in K$, the function v
 occurring in the mapping $v: K \rightarrow \text{seq}(B \rightarrow T)$ assumes
 a value $v(\alpha_m) \in \text{seq}(B \rightarrow T)$ whose components have
 the form $v(\omega | \alpha_m): B \rightarrow T \langle \omega := [v(\alpha_m)] \rangle$. If $|v(\alpha_m)|$
 is sufficiently large these components may be set into

363.

successive positions of a vector in row $[B \rightarrow T]$. By letting α assume the values of the successive constituents of a subsequence $\alpha[m:m+i]_A$ of $\alpha \in \text{seq}(K)$, a matrix mapping in $[B \rightarrow T]$ ^{presented} is defined. The type conversion operator which converts sets the α fixed initial subsequences of the ^{sequence} mapping,

$v(\alpha_{m+z}) \in \text{seq}(B \rightarrow T) \langle z := [i] \rangle$ into successive row positions of a matrix mapping in $[B \rightarrow T]$ is described

362. below.

Since the elements of a row vector form a sequence it is possible type conversion from row vectors to column vectors or diagonal matrices, and other forms of type conversion is feasible. Other forms of type conversion of a similar nature may be defined.

Let $N' := N$ and $B \in \text{dom}(K, \bar{N}^*)$ p 33

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Annihilatory and permanent map matrix mappings

- Some of the matrix mappings
in subsequent work two ^{classes} types of matrix mappings that transform other matrix mappings are encountered. The transforming matrix mappings are those for which all row sums except one are zero (the exceptional row sum taking the value unity) and those for which all row sums are unity. Some properties of those such matrix mappings are stated (p 266 - 268)

D being a suitable domain, and T and A' being ~~mappings~~
~~both from having~~
functions between which multiplication is possible, and with
source domains $D \times D$, the source domain $D \times D$, T transforms
 A' according to the law of the form

$$(*) \quad T(\bar{\alpha}, \bar{\beta}) A'(\bar{\alpha}, \bar{\beta}) = B'(\bar{\alpha}, \bar{\beta})$$

holding for $\bar{\alpha}, \bar{\beta} \in D$. Indeed $B'(\bar{\alpha}, \bar{\beta})$ may simply be
defined by use of the above formula to ensure that the
transformation law holds. In many important cases it is
found, however, that $A'(\bar{\alpha}, \bar{\beta})$ is independent of $\bar{\beta}$ and
 B' $\left.\begin{array}{l} \text{the above law is satisfied unless for a function,} \\ T \text{ is such that } TA' \text{ for which } A'(\bar{\alpha}, \bar{\beta}) \end{array}\right.$
is independent of $\bar{\alpha}$ and B' for which $B'(\bar{\alpha}, \bar{\beta})$ is independent
of $\bar{\beta}$: both exist satisfying the above law exist the transformation
law then takes the form

$$T(\bar{\alpha}, \bar{\beta}) A(\bar{\beta}) = B(\bar{\alpha})$$

holding for $\bar{\alpha}, \bar{\beta} \in D$, where A and B having the common
domain D . There are even cases in which A and B is
the same function, the transformation law D
the form

$$T(\bar{\alpha}, \bar{\beta}) A(\bar{\beta}) = A(\bar{\alpha})$$

exists holding for $\bar{\alpha}, \bar{\beta} \in D$ exists. may also be formulated

Versions of The above remarks concerning a post
multiplying factor T may also be formulated. may be

also be formulated in terms of a post multiplying factor
 T . The special transformation laws then become

$$C(\bar{\alpha})T(\bar{\alpha}, \bar{\beta}) = D(\bar{\beta})$$

and

$$C(\bar{\alpha})T(\bar{\alpha}, \bar{\beta}) = C(\bar{\alpha})$$

both holding for $\bar{\alpha}, \bar{\beta} : \mathbb{D}$.

In certain important cases it occurs that T transforms itself according to a law of the form

$$T(\bar{\alpha}, \bar{\beta})T(\bar{\beta}, \bar{\gamma}) = T(\bar{\alpha}, \bar{\gamma})$$

$\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$ being restricted to certain domains. Subject to suitable conditions, the a simple transformation law involving T and A alone (or such a law involving T and C alone) induces the above cancellation property of T .

→ Displacement of T by A . Transformation Laws?

367. Definition let $i, j, k \in \mathbb{N}$, $D' = D'' = \mathbb{D} \subseteq \text{dom}(K, \bar{N})$, $B: D \times D \subseteq K$,

$\hat{B}, \hat{B}' : D \subseteq K$ and $B: D' \times D'' \subseteq K$ be such that

$$\hat{B} \hat{B}' \quad B(\alpha, \beta) \subseteq \hat{B}'(\alpha) \cap B'(\alpha, \beta) \cap \hat{B}(\beta) \quad (\alpha, \beta : \mathbb{D})$$

In this case

Let $T: D' \times D'' \rightarrow \{ \hat{B}'(D', D'') \rightarrow [K | i, j] \}$.

i) With $A: D \rightarrow \{ \hat{B}(D) \rightarrow [K | j, k] \}$ and $B: D \rightarrow \{ \hat{B}(D) \rightarrow [K | i, k] \}$

ii) $[K | i, k]$, the notation $\{A, B\} \in \text{push}\{T, D, B | i, j; k\}$
 i, j, k indicates that

$$T(\alpha, \beta) A(\beta) = B(\alpha)$$

for $\alpha, \beta \in D$. A is then said to be pushed by T into B

In the special cases in which $i=j$ and $i=j=k$ throughout D over B. In the contracted symbols push $\{T, D, B[i;j]\}$ and pull $\{T, D, B[i;k]\}$

b) In the special case in which $i=j$ and $i=k$ are used in the above notation
If, in the preceding,
 $A = B \langle D \rangle$

The notation $A \in \text{push}\{T\}$

A is said to be pushed around by T throughout D over B and the notation $A \in \text{push}\{T, D, B[i;k]\}$ is used, this notation being abbreviated by use of the symbol push $\{T, D, B[i;k]\}$ in the special case in which $i=k$.

ia) With $C: D \rightarrow \{\hat{B}(D) \rightarrow [K/k, i]\}$ and $D: D \rightarrow \{\tilde{B}(D) \rightarrow [K/k, j]\}$,

the notation $\{C, D\} \in \text{pull}\{T, D, B[i, j; k]\}$ indicates that

$$C(\alpha) T(\alpha, \beta) = D(\beta)$$

for $\alpha, \beta \in D$. C is then said to be pulled into D by T throughout D over B. The contracted symbols pull $\{T, D, B[i;k]\}$ and pull $\{T, D, B[i;k]\}$ are used in circumstances similar to those described in (ia) above.

b) If $\hat{B} = \tilde{B} \langle D \rangle$, $i=j$ and

$$C = D \langle D \rangle$$

C is said to be pulled about by T throughout D over B and the notation $C \in \text{pull}\{T, D, B[i;k]\}$ together with a contracted form when $i=k$, are used.

366. Introducing suitable assumptions concerning their associated source domains, the above ~~theory~~ may be extended to mappings A, \dots, D , and T . It is in this form that the following theory is presented. Definitions and propositions are presented.

hp 50. Multiplicative domains

The existence of multiplicative domains

hp 49

45 The intersection of multiplicative domains

hp 49 The Mapping systems with cancellation

The existence of mapping systems with cancellation

hp 48 The inversion of mapping systems with cancellation

The extension of classes of mapping systems with cancellation

=

Define $NS\{\dots\}$ in preliminary definition with D fixed

= Remark that matrix ^{function} mappings in $\{D \rightarrow [K|i]\}$ from a ring but mappings \mathcal{D} from $A \times D_A \rightarrow [K|i]$ (same domain ranges with A) yield ring only one intersection \mathcal{D} all D_A .

Transformation systems and mapping systems with cancellation

() Let $i \in \bar{\mathbb{N}}, D' := D'' := \mathbb{D} \in \text{dom}(K, \mathbb{N}), B : \mathbb{D} \times \mathbb{D} \subseteq K$ and

$T : D' \times D'' \rightarrow \{B'(D', D'') \rightarrow [K|i]\}$

i) Let $A \in \text{push}\{T, D, B'|i\}$ and $B : \mathbb{D} \times \mathbb{D} \subseteq K$ exist such that

a) for each $\alpha, \beta \in \mathbb{D}$

$$B(\alpha, \beta) \subseteq NS\{A(\beta)\} \cap B'(\alpha, \beta)$$

and

b) $B \in \mathcal{N}(D)$

The above ~~ass~~ conditions imply that $T \in \mathcal{M}(D, B | i)$

ii) Let $C \in \text{pull}\{T, D, B' | i^2\}$ and $B: D \times D \subseteq K$ exist such that the ~~above~~ conditions (a, b) with the relationship of α replaced by

$$B(\alpha, \beta) \subseteq NS\{C(\alpha)\} \cap B'(\alpha, \beta)$$

hold. Again $T \in \mathcal{M}(D, B | i)$

Hyp. \Rightarrow The extension of transformation systems

Let $i, j, k \in \bar{N}, D':=D'' := D \subseteq \text{dom}(K, \bar{N}), B: D \times D \subseteq K$

$\hat{B}, \tilde{B}: D \subseteq K$, and $B: D \times D \subseteq K$ be such that

and $B(\alpha), B(\beta) \subseteq B'' \subseteq \langle \alpha := D \rangle \cap B' \cap \langle \beta := D \rangle$

Let $T: D' \times D'' \rightarrow \{B'(D', D'') \rightarrow [K | i, j]\}$

i) ~~Let $A: D \rightarrow \{\hat{B}(D) \rightarrow [K | j, k]\}$, and $B: D \rightarrow \{\tilde{B}(D) \rightarrow [K | i, k]\}$. Let $F: D' \times D'' \rightarrow \{B''(D', D'') \rightarrow [K | k, h]\}$ be constant over $D' \times D''$. Define $a: D \rightarrow \{\hat{B}(D) \rightarrow [K | j, h]\}$ by setting, for each $\beta \in D$,~~

$$a(\beta) := A(\beta)F(\alpha, \beta) \quad \langle \beta := D \rangle$$

~~for some $\alpha \in D$, and $b: D \rightarrow \{\tilde{B}(D) \rightarrow [K | i, h]\}$ by setting, for each $\alpha \in D$,~~

$$b(\alpha) := B(\alpha)F(\alpha, \beta)$$

for some $f \in D$.

If $\{A, B\} \in \text{push}\{\bar{T}, D, B | i, j; k\}$ then $\{a, b\} \in \text{push}\{\bar{T}, D, B | i, j; h\}$
 $\Rightarrow \text{push}\{\bar{T}, D, B | i, j; k\} F \subseteq \text{push}\{\bar{T}, D, B | i, j; h\}$.

iii) Let $C: D \rightarrow \{\hat{B}(D) \rightarrow [k | k, i]\}$, $D: D \rightarrow \{\tilde{B}(D) \rightarrow [k | k, j]\}$,
and $G: \tilde{B}'' \rightarrow [k | h, k]$. Define $c: D \rightarrow \{\hat{B}(D) \rightarrow [k | h, i]\}$
by setting, for each $\alpha \in D$ and $d: D \rightarrow \{\tilde{B}(D) \rightarrow [k | h, j]\}$
by setting, for each $\alpha \in D$,

$$c(\alpha) := G C(\alpha), \quad d(\alpha) := G D(\alpha).$$

If $\{C, D\} \in \text{pull}\{\bar{T}, D, B | i, j; k\}$ then $\{c, d\} \in \text{pull}\{\bar{T}, D, B | i, j; h\}$

$i, j; h\}$

b) $\text{pull}\{\bar{T}, D, B | i, j; k\} \subseteq \text{pull}\{\bar{T}, D, B | i, j; h\}$

ii) Let $A: D \rightarrow \{\hat{B}(D) \rightarrow [k | j, k]\}$, $B: D \rightarrow \{\tilde{B}(D) \rightarrow [k | i, k]\}$
and $F: \tilde{B}'' \rightarrow [k | k, h]$. Define $a: D \rightarrow \{\hat{B}(D) \rightarrow [k | j, h]\}$
and $b: D \rightarrow \{\tilde{B}(D) \rightarrow [k | i, h]\}$ by setting, for each $\alpha \in D$,

$$a(\alpha) := A(\alpha) F, \quad b(\alpha) := B(\alpha) F$$

If $\{A, B\} \in \text{push}\{\bar{T}, D, B | i, j; k\}$ then $\{a, b\} \in \text{push}\{\bar{T}, D, B | i, j; h\}$.

(b) $\text{push}\{\bar{T}, D, B | i, j; k\} F \subseteq \text{push}\{\bar{T}, D, B | i, j; h\}$

Multiplying relation (a) throughout by a suitable post-factor
 $F'(\alpha, \beta)$, a similar transformation γ_{ab} involving \bar{T}, a' and b' ,
where

$$a'(\alpha, \beta) = A'(\alpha, \beta) F'(\alpha, \beta), \quad b'(\alpha, \beta) = B'(\alpha, \beta) F'(\alpha, \beta)$$

is obtained. Transformation γ_{ab} involving \bar{T} as a post multiplying
factor may be extended in a similar way. Imposing appropriate
conditions upon F , an ordered pair in $\{\text{push}\{\bar{T}, D, B | i, j; k\}\}$
be transformed into an ordered pair in a related system.

The result of clause (ia) of the above theorem naturally holds when $i=j$ and $\hat{B} = \tilde{B}(\mathbb{D})$ and A and B are the same function; the result also concerns single functions that are pushed around. ~~Since~~ Clause (ii) also concerns functions that are pulled about.

If may occur in subclause (ia) that A and $A(\alpha)$ and $B(\beta)$ are members ~~simply~~ ^{functions over their} ~~members~~ constant ^{functions over their} mappings ~~with~~ ^{with} source domains $\hat{B}(\alpha)$, $\tilde{B}(\beta)$ respectively — they are simply members of $[K|_{j,k}]^{\text{al}}_{\text{al}}, [K|i,k]$. F is in general a function ~~also with~~ ^{$\mathbb{P}^{(2)}$} with $z \in \mathbb{B}$ defined for all $z \in \mathbb{B}$.

$A(\alpha), B(\beta)$ then become, under transformation, no longer constants but functions $a(\omega, z), b(\varsigma, z)$ defined for all $z \in B(\alpha, \beta)$. In this way a law involving the transformation of functional forms (e.g. polynomial forms) is derived from a law involving constants. The same consideration hold regarding clause (ii).

Transformation laws and permanent mapping systems.

Given a suitable matrix mapping ~~and~~ T and ~~and~~ two column mappings satisfying a relationship of the form

$$T(\alpha, \beta) a'(\alpha, \beta) = b'(\alpha, \beta) \quad \langle B(\alpha, \beta) \rangle$$

for all $\alpha, \beta \in \mathbb{D}$, where \mathbb{D} and $B(\alpha, \beta)$ are domains, the latter depending ~~on~~ α , defined for $\alpha, \beta \in \mathbb{D}$, a permanent mapping may easily be obtained from T, a' and b' . The same is

true with regard to \bar{T} and two row mappings satisfying a relationship in which \bar{T} occurs as a post-multiplying factor

Let $i, j \in \bar{N}$, $\bar{D}' = D' = \text{Dedom}(K, N)$ and $\hat{B}, \tilde{B}, B' : D \times D \subseteq K$.

Let $T' : D' \times D'' \rightarrow \{B'(D', D'') \rightarrow [K|i, j]\}$ and define $B'' : D \times D \subseteq K$ by setting

$$B''(\alpha, \beta) := \hat{B}(\alpha, \beta) \cap B'(\alpha, \beta) \cap \tilde{B}(\alpha, \beta)$$

Let $a' : D' \times D'' \rightarrow \{\hat{B}(D', D'') \rightarrow \text{col}[K|i, j]\}$ and $b' : D' \times D'' \rightarrow \{\tilde{B}(D', D'') \rightarrow \text{col}[K|i, j]\}$ satisfy the relationship

$$\bar{T}(\alpha, \beta) a'(\alpha, \beta) = b'(\alpha, \beta) \quad \langle B''(\alpha, \beta) \rangle$$

for each $\alpha, \beta \in D$. Define $B : D \times D \subseteq K$ by setting

$$B(\alpha, \beta) := B''(\alpha, \beta) \cap \text{NS}\{b'(\alpha, \beta)\}$$

and $\bar{T}^* : D' \times D'' \rightarrow \{B(D', D'') \rightarrow [K|i, j]\}$ by setting

$$\bar{T}^*(\alpha, \beta) := \{\text{diag}[b'(\alpha, \beta)]\}^{-1} \bar{T}(\alpha, \beta) \text{diag}[a'(\alpha, \beta)]$$

For each pair $\alpha, \beta \in D$, $\bar{T}^*(\alpha, \beta) \in \text{perm}(B(\alpha, \beta)/i, j)$

Special versions of the above result concern mapping functions mapping functions a', b' for which $a'(\alpha, \beta)$ and $b'(\alpha, \beta)$ are independent of α and β respectively, the associated target source domains \hat{B} and \tilde{B} being correspondingly restricted and, further, such pairs for which $i = j$ of identical mapping functions. A result corresponding to the above concerning a mapping function which features as a post-multiplicative factor may be derived for a mapping function involving the transpose of the result stated.

With $i \in \mathbb{N}$ and $K' \subseteq K$, two mapping systems $F, F': K' \rightarrow [K|i]$ for which a third mapping system $\Delta: K'' \rightarrow [K|i]$ exists connected by a relationship of the form

$$F' = \Delta F \{\Delta\}^{-1} \quad \langle K'' \rangle$$

where $K'' \subseteq K'$ and $\Delta: K'' \rightarrow [K|i]$ is a third mapping system, are said to be similar over $\langle K \rangle \rightarrow K''$, and Δ is called a similarity factor of the ordered pair $\{F, F'\}$ over K'' .

~~Existence~~ It is evident that for any other mapping system $X: \hat{K} \rightarrow [K|i]$, where for which $\{NS(X) \cap NS(F), \{NS(X) \cap NS(F')\} \subseteq K$,

~~$X\Delta$~~ is also a similarity factor of $\{F, F'\}$ over $K \cap NS(X)$, no condition (in this case involving F and F' being imposed upon X). It is ~~not~~ possible to derive one similarity factor from another of $\{F, F'\}$ from another by use of the conditions imposed upon ~~either~~ F or F' .

Let $K' \subseteq K$ and $F, F', \Delta_0: K' \rightarrow [K|i]$ be such that

~~$F\Delta_0 = \Delta_0 F' \quad \langle K' \rangle$~~

so that Δ_0 is a similarity factor of $\{F, F'\}$ over $NS(\Delta_0)$.

~~F and F' are similar over $NS(\Delta_0)$.~~

~~(1)~~ Let $\Delta_1: NS(\Delta_0) \rightarrow [K|i]$.

~~(iiia)~~ F commutes with the product $\{\Delta_1\}_{\Delta_0}$ over $NS(\Delta_0)$ if and only if F' does so (\Leftrightarrow F' commutes with the product $\{\Delta_1\}_{\Delta_0}$ in the same way). ~~$F\Delta_1\Delta_0 = \Delta_1\Delta_0 F \quad \langle NS(\Delta_0) \rangle$~~

~~if and only if~~

~~$F'\Delta_1\Delta_0 = \Delta_1\Delta_0 F' \quad \langle NS(\Delta_0) \rangle$~~

b) If the above condition holds,

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$$F\Delta_1 = \Delta_1 F' \quad \langle NS(\Delta_0) \rangle$$

and Δ_1 is a similarity factor of $\{F, F'\}$ over $NS(\Delta_0, \Delta_1)$

391. \Rightarrow (In special cases it may be shown that Δ_1 defined over by the source domain K' satisfies the relationship of (iii b) above over K' ;

in this case Δ_1 is a similarity factor of $\{F, F'\}$ over $NS(\Delta_1)$.)

A similar remark may be made concerning Δ_0 .

392. i) If F commutes multiplicatively with Δ_1 over $NS(\Delta_0)$, i.e. if

$$F\Delta_1 = \Delta_1 F \quad \langle NS(\Delta_0) \rangle$$

then $\Delta_1\Delta_0$ is a similarity factor of $\{F, F'\}$ over $NS(\Delta_0, \Delta_1)$

then

$$F\Delta_1\Delta_0 = \Delta_1\Delta_0 F' \quad \langle NS(\Delta_0) \rangle$$

and $\Delta_1\Delta_0$ is a similarity factor of $\{F, F'\}$ over $NS(\Delta_0, \Delta_1)$

ii) If F' commutes multiplicatively with Δ_1 over $NS(\Delta_0)$, then

$$F\Delta_0\Delta_1 = \Delta_0\Delta_1 F' \quad \langle NS(\Delta_0) \rangle$$

and $\Delta_0\Delta_1$ is a similarity factor of $\{F, F'\}$ over $NS(\Delta_0, \Delta_1)$

$$\underline{F'\Delta_1 = F'\Delta_1\Delta_0\Delta_0^{-1} = \Delta_1\Delta_0 F'\Delta_0^{-1} = \Delta_1 F A_0 \Delta_0^{-1} - \Delta_1 F}$$

$$F\Delta_1 = F\Delta_1\Delta_0\Delta_0^{-1} = \Delta_1\Delta_0 F\Delta_0^{-1} = \quad \parallel \quad F\Delta_1\Delta_0 = \Delta_1 F\Delta_0$$

$$F\Delta_0\Delta_0^{-1}\Delta_1 = \Delta_0 F'\Delta_0^{-1}\Delta_1 \quad \parallel \quad = \Delta_1\Delta_0 F'$$

$$\cancel{\Delta_0 F'\Delta_0^{-1}\Delta_0\Delta_1} = \cancel{\Delta_0\Delta_1\Delta_0 F'\Delta_0^{-1}} \quad \underline{\Delta_0\Delta_1 F' = \Delta_0 F'\Delta_1^* = F\Delta_0\Delta_1}$$

$$F'\Delta_1\Delta_0 = \Delta_1\Delta_0 F' \quad F\Delta_1 = F\Delta_0\Delta_0^{-1}\Delta_1 = \Delta_0 F'\Delta_0^{-1}\Delta_1$$

$$\cancel{\Delta_0 F'\Delta_0^{-1}\Delta_0\Delta_1} = \cancel{\Delta_0\Delta_1^{-1}\Delta_0 F'\Delta_0^{-1}} \quad F'\Delta_1\Delta_0 = \Delta_1^{-1}\Delta_0 F'$$

$$F \text{ comm } \Delta_0\Delta_1^{-1} \text{ iff } F' \text{ comm } \Delta_1^{-1}\Delta_0$$

$$\Delta_1^{-1}F = \Delta_0^{-1}\Lambda_0\Delta_1^{-1}F = \Delta_0^{-1}F\Lambda_0\Delta_1^{-1} = \Delta_0^{-1}\Lambda_0 F' \Delta_1^{-1} = F'\Delta_1^{-1}$$

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$$F\Delta_1 = F\Delta_1\Delta_0\Delta_0^{-1} = \Delta_1\Delta_0 F\Delta_0^{-1}$$

$$F\Delta_1 = F\Delta_1\Delta_0^{-1}\Lambda_0 = \Delta_1\Delta_0^{-1}F\Lambda_0 = \Delta_1\Delta_0^{-1}\Lambda_0 F'$$

require F common $\Delta_1, \Delta_0^{-1} \Leftrightarrow F'$ common $\Delta_0^{-1}\Delta_1 \Rightarrow \Delta_1$ factor

$$\Lambda_0 F' \Delta_0^{-1} \Delta_1 \Delta_0^{-1} = \Delta_1 \Delta_0^{-1} \Lambda_0 F' \Delta_0^{-1} \quad \Delta_0^{-1} \Delta_1 F' = F' \Delta_0^{-1} \Delta_1$$

$$F\Delta_0 \Delta_1 \cancel{F'} = \Lambda_0 \Delta_1 F \cancel{F'} = \Lambda_0 F' \Delta_1 \cancel{F'} \quad \Delta_1 F$$

$$F \text{ common } \Delta_0 \Delta_1 \Rightarrow \Delta_1^{-1} \text{ factor } \cancel{\Lambda_0 F' \Delta_0^{-1} \Lambda_0 \Delta_1} \Rightarrow \Lambda_0 \Delta_1 \Delta_0 F' \Delta_0^{-1}$$

↓

F common Δ_1, Δ_0

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(v)

iv) F commutes multiplicatively with the product $\Delta_0\Delta_1$ over $NS(\Delta_0)$ if and only if F' is related to the product $\Delta_1\Delta_0$ in the same way.

b) If the above conditions hold

$$\Delta_1 F = F' \Delta_1 \quad \langle NS(\Delta_0) \rangle$$

and $\{\Delta_1\}^{-1}$ is a similarity factor of $\{F, F'\}$ over $NS(\Delta_0, \Delta_1)$.

= Submatrix mappings \otimes products \otimes triangular matrix mappings

h, i, j, k, n being suitably prescribed, \Rightarrow p 212 : the formation pp. 210, 211.

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Continued products of lower triangular matrix mappings $A(1)$

With $\xi \in \mathbb{R}$, the elements of the product $\bar{A} \in \mathcal{L}[K|i]$ of the two matrices $a(1), a(0) \in \mathcal{L}[K|i]$ in that order the given order may be expressed in terms of the elements ξ of the factor matrices by means of the formulae

$$A(1)_\omega^D = \sum_i a(1)_\omega^{D+i} a(0)_{D+i}^{\omega} \quad \langle \omega := [D, \omega] \rangle$$

holding for $D := [i]$, $\omega := [D, i]$ and

$$A(1)_{D+\chi}^D = \sum_i a(1)_{D+i}^{D+\omega} a(0)_{D+\omega}^{\chi} \quad \langle \omega := [\chi] \rangle$$

holding for $D := [\omega]$, $\chi := [i - \omega]$.

In both formulae the superscript of $a(1)$ is ~~the~~ and the suffix of $a(0)$ have the same value. These laws are preserved in the formation of continued products of lower triangular matrices, as is shown in the following theorem which is presented in terms of matrix mappings

(*) Let $i \in \mathbb{R}$, $n \in \mathbb{N}$ and $a(k): K' \rightarrow \mathcal{L}[K|i] \quad \langle k := [n] \rangle$.

Define $A(k): K' \rightarrow \mathcal{L}[K|i] \quad \langle k := [n] \rangle$ by setting

$$A(k) := \prod a(k) \quad \langle k := [k] \rangle$$

With $\omega := [k]$ in both cases, for $\omega := [D, i]$

For $k := [n]$, $\omega := [i]$ and $\omega := [D, i]$

$$A(k)_\omega^D = \left\{ \sum_i a(k)_\omega^{D+i} \right\} \left\{ \sum_{i_1} a(k-1)_{\omega+i_1}^{D+i_1} \right\} \left\{ \sum_{i_2} a(k-2)_{\omega+i_2}^{D+i_2} \right\} \dots \left\{ \sum_{i_n} a(1)_{\omega+i_n}^{D+i_n} \right\} a(0)_{\omega+i_n}^D \quad \langle \omega := [D, \omega] \rangle$$

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$$\langle \omega(k-3) := [\omega, \omega(k-2)] \rangle \} \langle \omega(k-2) := [\omega, \omega(k-1)] \rangle \} \langle \omega(k-1) := [\omega, z] \rangle \}$$

and for $k := [n]$, $\omega :=$

$\langle K' \rangle$

and for $\chi := [z \rightarrow]$

$$A(k)_{2+\chi}^{\omega} = \{ \sum_{\alpha(k)}^{2+\omega(k-1)} \{ \sum_{\alpha(k-1)}^{2+\omega(k-2)} \{ \sum_{\alpha(k-2)}^{2+\omega(k-3)} \{ \sum_{\alpha(1)}^{2+\omega(0)} \alpha(0)_{2+\omega(0)}^{\omega} \langle \omega(0) := [\omega(1)] \rangle \}$$

$$\dots \{ \sum_{\alpha(1)}^{2+\omega(0)} \alpha(0)_{2+\omega(0)}^{\omega} \langle \omega(0) := [\omega(1)] \rangle \}$$

rob. $\langle \omega(k-3) := [\omega(k-2)] \rangle \} \langle \omega(k-2) := [\omega(k-1)] \rangle \} \langle \omega(k-1) := [x] \rangle \}$

In the first formula, summation runs from the value of the superscript of $A(1)$ to that of the suffix of $\alpha(1)$.
 In the second summation runs from zero to the difference between the suffix of $\alpha(1)$ and the superscript of $A(1)$.

(The second result is obtained by replacing $\omega(k)$ by $2+\omega(k)$ in the first)

Add 2iii a) on operations on sequences in $\text{seq}(\bar{N})$ $\text{seq}(N)$ $\text{seq}(\tilde{N})$
 ↑ take from p 354 4vb
 p 2ii b) on operations on sequences in $\text{seq}(K)$ ← take from p 355 add x
 Rebin 4i ii iiiabc iv va) d:=b transfer 4vbc

5ii := with ω depending on z Simple nested use dependent nested use
 as dummy statements to be ignored eg $\chi := [e_i]$ when $i=0$

use of $\chi := \exists A\chi : K \rightarrow K \langle \chi := \exists \rangle$ usually with $\exists = [e_i] \circ [e_i] \circ$

6 p 360 redraft sums & products in terms of \exists & transfer

Locate sections of matrix mappings pp 355 356

Delimiters & parentheses

Structural extension pp 358, 359

Extension of source and target domain types p 360 361 362

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Define sequence as mapping with source domain \mathbb{N} .

Annullability and permanent matrix mappings p³⁶³

In introduction give another example of $M[\text{fr.}]$ & please example

Transformation operators? p 364 Similarity factors p 372	$[h, k], \dots [k]; [l] [s] \frac{1}{s} [m, mii]$ $\text{seq}(T \geq j) \quad \text{seq}(T j) \quad \text{seq}(T \frac{1}{s} [m, mii])$ $\text{seq}'(T) \dots \text{seq}'(T \frac{1}{s}) \quad \alpha[\frac{1}{s} [m, mii]]$ $f \in \mathbb{Z}, \quad g \in \mathbb{Z}, \quad d \in \mathbb{Z}$
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Submatrix mappings & products of triangular matrix mappings p 377

Continued products \Rightarrow more triangular mappings p376.

Define $NS(\dots)$

2] Sequences

(ii) A sequence is a mapping whose source domain consists of n elements $\{1, 2, \dots, n\}$ depending on i .

b) The function occurring in (b) is called a sequential function.

c) T being a ~~single~~^{set} domain, $\text{seq}(T)$ is the complete

c) A long string system of sequential functions whose values are in $\overline{1}$ and

a) a T -sequence $\{z_i\} \in \text{seq}(T)$ and $|z_i| = i$, i.e.

b) with $\beta \in \text{seq}(\Gamma)$ the source domain of β being the succession $\beta(0), \beta(1), \dots, \beta(i)$ is called a Γ -sequence.

d) A subsequence of a \bar{t} -sequence

With $h, k \in \mathbb{N}$ and $h \leq k$, the \mathbb{N} -sequence $h, h+1, \dots, k$

is denoted by $[h, k]$. When $h < k$, $(h, k]$ denotes the

Sequence $h+1, \dots, k$; $[h, k)$ and (h, k) are defined similarly.

If, in the foregoing, the symbol "h," is omitted in the foregoing when $h=0$; thus $[k]$ denotes the sequence $0, 1, \dots, k$ and (h, k) is written as (k) .

- iii) With \tilde{z} being a sequence and $[\tilde{z}]$ are sequential functions.
25. c) \tilde{z} being a sequence, $[\tilde{z}]$ is written simply as $[\tilde{z}]$ and $[\tilde{z}]$ is often with $\tilde{z} \in \text{seq}$, and $\tilde{z} \in \text{seq}$.
26. i) (i) \tilde{z} cont. p) Let $\tilde{z} \in \text{seq}$. $\tilde{z} \in \text{seq}$
- (ii) With \tilde{z} a sequence, $|\tilde{z}| = i$, $\{\tilde{z}\}$ is the set $\{\tilde{z}(0), \tilde{z}(1), \dots, \tilde{z}(i)\}$.
- a) With $\tilde{z} \in \text{seq}$, the notation $\tilde{z} \subseteq \tilde{z}'$ indicates that either
- $\tilde{z} \in \text{seq}$ also with $\forall n \in \mathbb{N} \exists m \in \mathbb{N}$ and
 - $\{\tilde{z}\} \subseteq \{\tilde{z}'\}$
- b) With $\tilde{z} \in \text{seq}$, the notation $\tilde{z} \leq \tilde{z}'$ indicates that $\tilde{z} \leq \tilde{z}'$ and that, the ordering of the elements of \tilde{z} is preserved and that, if \tilde{z} is nonvoid, in \tilde{z} (so that $\tilde{z}(0) = \tilde{z}(a), \tilde{z}(1) = \tilde{z}(b), \dots, \tilde{z}(j) = \tilde{z}(c)$ where $|\tilde{z}| = j$ and a strictly increasing sequence \tilde{z} sequence a, b, \dots, c exists for which these relationships are satisfied and \tilde{z} being the sequential function in $\tilde{z} = [a, b, \dots, c]$ in \tilde{z} (so that a strictly increasing \tilde{z} sequence exists $\tilde{z}(0) = \omega(a), \tilde{z}(1) = \omega(b), \dots, \tilde{z}(j) = \omega(c)$ can be determined for which $\tilde{z}(0) = \tilde{z}(a), \tilde{z}(1) = \tilde{z}(b), \dots, \tilde{z}(j) = \tilde{z}(c)$) $\tilde{z}(j) = \tilde{z}(e)$ can be determined, where $|\tilde{z}| = j$)
- c) The inclusion relationship membership relationship $\omega \in \{\tilde{z}\}$ is written simply as $\omega \in \{\tilde{z}\}$.
- d) $[h, k] \subseteq [\tilde{z}]$, $\tilde{z}[h, k]$ is the ~~seq~~ T_2 sequence $\tilde{z}(h), \tilde{z}(h+1), \dots, \tilde{z}(k)$ (\Rightarrow that, \tilde{z} being the sequential function defined by this T_2 sequence, $\tilde{z} \subseteq \tilde{z}$). $\tilde{z}[h, k]$, $\tilde{z}[h, k]$ and $\tilde{z}(h, k)$ are similarly defined.

- (iii) A sequence ... $i \in \overline{\mathbb{N}}$;

(iv) seq is the complete system of sequences

(v); it has a represent it is represented in the form
represented

(vi) A sequence is displayed in the form a, b, \dots, c where a is the value in member of the target domain corresponding to 0 in $\overline{\mathbb{N}}$,
b to 1 and ... and c to i, where the source domain is $0, 1, \dots, i$.

Let T be a set

c) ... $\text{seq}(T)$ is the complete system of all sequences with T as
such a sequence is called a T sequence

target domain and seqf T is the complete ... values are in T

Relationship $\subseteq \equiv \leq \leq^+$ working for sequences, not sequence function
|| \hookrightarrow \hookleftarrow $\overline{\equiv}$?

$$\text{iii} \quad \Xi \subseteq \Sigma \quad \exists \subseteq \Sigma \quad \lambda \in \Sigma \quad \text{product}$$

Ex Let $\exists \in \text{seqf}(\mathbb{T})$ where \mathbb{T} is a set

a) Let $[h,k] \subseteq [\xi]$. $\exists [h,k]$... defined

c) The L-shaped When $k=0$, in the above the abbreviated notation

$\S(k]$, $\S(k]$, $\S[k)$ and $\S((k))$ are used in place of those given above
 \S -sequence $\S(0), \S(1), \dots, \S(I\#)$

Let us off

a) Let $\underline{\Phi} \subseteq [\underline{s}]$, & $[\underline{\Phi}]$ have the representation $\underline{\Phi}(0), \underline{\Phi}(1), \dots, \underline{\Phi}(\underline{s})$
 $(\underline{1}\underline{\Phi})$

$\mathfrak{F}[\Psi]$ is the T-sequence $\mathfrak{F}(\Psi(0)), \mathfrak{F}(\Psi(1)), \dots, \mathfrak{F}(\Psi(n))$
with \mathfrak{F} as the \mathfrak{F} -operator.

def b) ^{With} Let $[h, k] \subseteq [\underline{s}]$, $\underline{s}[[h, k]]$ is written simply as $\underline{s}[h, k]$

$|\Psi\rangle$, $\tilde{g}(h,k]$, $\tilde{g}[h,k)$ and $\tilde{g}(h,k)$ are similarly defined.

 c) When $h=0$ ↑
the notations $\text{segf}(T|j)$ and $\text{desegf}(T|j)$ mean that

vii) Let T be a set.

~~Next it is T₁ which is the nucleus of all the~~

b) Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of functions $f_k : \mathbb{N} \rightarrow \mathbb{R}$. Define $\tilde{f}_k(x) = f_k(x)$ if $x \leq k$ and $\tilde{f}_k(x) = 0$ if $x > k$. Then $\tilde{f}_k \in \text{seq}(\mathbb{N})$ for all $k \in \mathbb{N}$.

~~seq~~($T \mid \S[h, k]$) is ~~seq~~. The notations ~~are~~ and $j = \max \S(\omega)$ for $h \leq j \leq k$ ~~are~~ seqf($T \mid \S[h, k]$) and seqf($T \mid \S[h, k]$) ~~interpreted as~~ interpreted as $\S[h, k]$) are ~~seq~~($T \mid j$) and ~~seq~~($T \mid \geq j$) respectively. When $h=0$, $k=0$ ~~the notation indicates that~~ indicates that \S is seqf(T) and that via) $\S \in \text{seqf}(T)$ is the set of all sequential functions \S for which the sequence $\S(0), \S(1), \dots, \S(|\S|)$ consists of distinct members of T . With $j \in \bar{\mathbb{N}}$, ~~seq~~

b) Let $k \in \mathbb{N}$, $\alpha \in \text{seq}(\bar{\mathbb{N}} \mid \leq k)$ $\alpha[k]$ and $\S \in \text{seq}(\bar{\mathbb{N}} \mid \geq k)$.

The notation $\S[\S]$ indicates that \S is the system of all $\alpha \in \text{seqf}(T \mid \S[h, k])$ and that $\S[h, k]$ for which $\alpha[\S[h, k]]$ consists of distinct members of T . When $\S[h, k] = [h, k]$ the notation $\text{seqf}'(T \mid \S)$ is used. When $h=0$ the preceding notations are abbreviated to the forms $\text{seqf}'(T \mid \S[k])$ and $\alpha \in \text{seqf}'(T \mid k)$ respectively.

The notations $\text{seqf}(T \mid \S[k])$ and $\text{seqf}(T \mid \geq \S[k])$ are used. $\text{seqf}(T \mid \S[h, k])$ is the set of all $\alpha \in \text{seqf}(T \mid \S[h, k])$. mean $\text{seqf}(T \mid \geq \S[h, k]), \text{seqf}(T \mid \S[k])$ and $\text{seqf}(T \mid \geq \S[k])$ have similar Σ Mapping $\Sigma : [S \mid \S \parallel T]$ S source set \S function T target domain

$\text{seqf}(T \mid j)$ complete system of sequential functions featuring \equiv for which $\equiv \equiv \S$ in sequences whose source set contains has the form $\{0, 1, \dots, j\}$ With $\S \in \text{seqf}$ prescribed, $|\S| = \S$, where the source set in \equiv is the sequence in which \S features $\{0, 1, \dots, j\}$

With $\Sigma_i \in \text{seq}$ prescribed $\equiv \equiv i$, where $\{0, 1, \dots, i\}$ is the source set featuring in Σ .

Operations upon sequences in $\text{seq}(\bar{\mathbb{N}})$ mapping $\bar{\mathbb{N}} \rightarrow \text{seq}(\bar{\mathbb{N}})$??

354. $\phi + \psi$ $\phi - \psi$ $\phi\psi$ le hasard fait bien les choses

Operations upon sequences in $\text{seq}(\mathbb{N})$

addition subtraction as above but denoted by $\phi + \psi, \phi \setminus \psi$

(since K may not be ordered, multiplication may not be defined)

are $\text{seq}(K|j)$ for some $j \in \mathbb{N}$ ($\Phi \in \dots j \quad \Psi \in \dots j'$ $\Phi + \Psi$ in $\dots \text{min}(j, j')$
 $\Phi + \Psi \quad \Phi - \Psi \quad \Phi \times \Psi$ (\equiv is displayed as $\vdash(0), \vdash(1), \dots, \vdash(1 \pm 1)$)

iv) Operations upon sequence mappings pointwise? In the following, where relevant
 A finite $A \otimes T$ -based sequence (from now on, σ - a sequence) is a presented target domain.

A sequence is a mapping whose source set has the form

$\{0, 1, \dots, i\}$ where $i \in \mathbb{N}$ depends upon the sequence in question

b) and all integers from 0 to i feature in the set.
~~b) The mapping $\Xi \in \text{seq}$ indicates that Ξ is such a mapping
 seq is the complete system of such mappings.~~

8) With $\Xi \in \text{seq}^{\text{prescribed}}$, $|\Xi| = i$, where the source set of Ξ has the form just as just described

The function \mathfrak{f} occurring in a mapping in seq. is called a sequential function.

prob. a sequential function
prob. seqf is the complete system of all such presented
to each 3 features as a function in a mapping in's
prob. a function \rightarrow a function is a sequential function.

b) The notation $\exists \text{ seqf}$ indicates that \exists is a sequential function.

(a) \vdash being a target domain, the instruction $\Xi \in \text{seq}(\mathcal{T})$ indicates the

- that $\exists \epsilon_{\text{seq}}$ has the \bar{T} as a T is ϵ target domain ϵ in
 $\exists \epsilon_{\text{seq}} \forall \bar{T} \text{ the relation } \exists \epsilon_{\text{seq}} f(\bar{T}) \text{ indicates that } \exists \epsilon_{\text{seq}}$ has

T is a possible target domain (\equiv is then said to be a T sequence)

⑦ The statement $\exists x \text{ seqf} \text{ having been made, } |x| \text{ is } |E|$

where Ξ is the sequence in which ξ features Θ

8) A sequence $\Xi \in \text{seq}$ is displayed in the form a, b, \dots, c where a in the target domain corresponds to 0 in \mathbb{N} , b to 1, ..., and c to $| \Xi |$.
 $\Xi = \{a_1, a_2, \dots, a_{|\Xi|}\}$ is the target set \mathcal{S} in Ξ (i.e. $\{a, b, \dots, c\}$) in the example.

With $\Xi \in \text{seq}$, $\{\Xi\}$ is the "target set" in Ξ (i.e. $\{a, b, c\}$ in the example just given)

(viii) let \bar{T} be either \mathbb{N} , \mathbb{N} or \mathbb{R}

383

a) The sum of $\underline{\Phi}, \underline{\Psi} \in \text{seq}(\bar{T})$ is defined by letting $\underline{\Phi} + \underline{\Psi}$ be the sequence composed of $\underline{\Phi}$ followed by the members of $\underline{\Psi}$.

$\underline{\Phi}$ followed by $\underline{\Psi}$ (thus if Θ is the sequential function in this sum $\Theta[\underline{\Phi}] = \underline{\Phi}$ and $\Theta(\underline{\Phi}, |\underline{\Phi}| + |\underline{\Psi}| + 1) = \underline{\Psi}$)

$$\begin{cases} \underline{\Phi}, \underline{\Psi} \leq \underline{\Phi} + \underline{\Psi} \\ \underline{\Phi} - \underline{\Psi} \leq \underline{\Phi} \end{cases}$$

b) The difference $\underline{\Phi} - \underline{\Psi}$ is constructed by removing from $\underline{\Phi}$ all terms occurring in $\underline{\Psi}$, the ordering of the remaining members of $\underline{\Phi}$ being preserved; $\underline{\Phi} - \underline{\Psi}$ is a void sequence if such exist.

c) With $|\underline{\Psi}| \leq |\underline{\Phi}|$, the product $\underline{\Phi}\underline{\Psi}$ is the \bar{T} sequence constructed in the following way. $\text{ord}\underline{\Phi} \in \text{seq}(\bar{N})$ is constructed (def. subsequence, examine repetition) of elements in $\underline{\Psi} \leq \underline{\Phi}$)

by letting, for each $i \in [\underline{\Psi}]$, ord

$\text{ord}\underline{\Psi}: [\underline{\Psi}] \rightarrow [\underline{\Psi}]$ is constructed

by letting setting $\text{ord}\underline{\Psi}(i)$ letting $\text{ord}\underline{\Psi}(i)$ be the number

Let $\underline{\psi}, \underline{\phi}$ be the sequential functions,

in $\underline{\Psi}$ and construct $\text{ord}\underline{\Psi}: [\underline{\Psi}] \rightarrow [\underline{\Psi}]$ by letting $\text{ord}\underline{\Psi}(i)$ be

the number of members of $\underline{\Psi}$ less than $\underline{\psi}(i)$.

The source set of $\underline{\Phi}\underline{\Psi}$ is $\{[\underline{\Psi}]\}$ and

its sequential function is $\underline{\phi}[\text{ord}\underline{\Psi}]$

$\underline{\Phi}\underline{\Psi}$ is then $\underline{\phi}(\text{ord}\underline{\Psi}(0)), \underline{\phi}(0)$

$(\bar{N} | |\underline{\Psi}|)$

in $\underline{\Psi}$ and $\underline{\Phi}$ respectively and construct $\text{ord}\underline{\Phi} \in \text{seq}^{\bar{T}}$ by letting, in the \bar{N} -sequence, $\text{ord}\underline{\Phi}(0)$ in its representation $\text{ord}\underline{\Phi}(0), \text{ord}\underline{\Phi}(1), \dots, \text{ord}\underline{\Phi}(|\underline{\Phi}|)$, $\text{ord}\underline{\Phi}(i)$ be the number of members of $\{[\underline{\Phi}]\}$ less than $\underline{\phi}(i)$ (such comparison is available i.e. over \mathbb{N} , \mathbb{N} or \mathbb{R}).

$\underline{\Phi}\underline{\Psi}$ is then $\phi[\text{ord } \underline{\Psi}]$.

\Rightarrow § 385.

b) Addition, subtraction and, if K is totally ordered, multiplication of sequences in $\text{seq}(K)$ are as defined above, but ~~as~~ the sum difference and product of $\underline{\Phi}$ and $\underline{\Psi}$ are denoted by $\underline{\Phi} + \underline{\Psi}$, $\underline{\Phi} - \underline{\Psi}$ and $\underline{\Phi} \times \underline{\Psi}$ respectively.

c) The sum, difference and product of $\underline{\Phi}, \underline{\Psi} \in \text{seq}(K)$ are also defined in terms of operations over K . Let $\underline{\Phi}$ and $\underline{\Psi}$ be $\phi(0), \phi(1), \dots, \phi(|\underline{\Phi}|)$ and $\psi(0), \psi(1), \dots, \psi(|\underline{\Psi}|)$ respectively, and set $j = \min\{|\underline{\Phi}|, |\underline{\Psi}|\}$. $\underline{\Phi} + \underline{\Psi}$ is $\phi(0) + \psi(0), \phi(1) + \psi(1), \dots, \phi(j) + \psi(j)$. $\underline{\Phi} - \underline{\Psi}$ and $\underline{\Phi} \times \underline{\Psi}$ are defined termwise in the same way.

\Rightarrow § 385

(Sums, differences and products of sequences in $\text{seq}(\bar{K})$, $\text{seq}(N)$ and $\text{seq}(\tilde{N})$ occur frequently in superscript and suffix expressions, in which it may not be easy to detect the difference between dotted addition and multiplication signs and their naked counterparts. For this reason dotted symbols are not used in the definitions of part a). On the other hand it is desirable to use undotted symbols to denote operations over K . The context dependent interpretation of undotted symbols is therefore adopted.)

ii) In addition to the proper sequences considered in the preceding clause, null T sequences are postulated. They ~~exist~~ may arise from t in the subtraction of one sequence from another and also feature in operations upon sequences (both contexts are dealt with below).

subclause (2a)

(from now on - or void sequences)

385.

ii) A void \bar{N} -based sequence $\bar{\nu}$ is a mapping whose source set is a void set in \bar{N} . A void \bar{T} -sequence is a void sequence with target domain \bar{T} . (Void sequences may arise in the subtraction of one sequence from another and may also feature in operations upon sequences (both contexts are dealt with below).)

384. = under the condition Φ is a void T -sequence and $\Psi \in \text{seq}(T)$

\Leftarrow (via 5) If Φ is a void T -sequence (via 5), $\Phi + \Psi$ is replaced by a void T -sequence, $\Phi + \Psi$ is simply Ψ ; a corresponding result concerns the case in which $\Phi - \Psi$ and $\Psi - \Phi$ being a void T -sequence and Ψ respectively. Ψ is void. $\Phi - \Psi$ and $\Psi - \Phi$ being a void T -sequence and Ψ respectively. $\Phi + \Psi$ and $\Psi + \Phi$ are simply Ψ ; if both Φ and Ψ the sum of two void T -sequences is another. $\Phi \Psi$ is a void T -sequence.

The sum, difference and product of two void T -sequences are all void T -sequences.

384. = constructed from either a K -sequence factor and (or two void K -sequence factors)

\Leftarrow (via 8) The sum, difference and product involving a void K -sequence as one factor constructed from a K -sequence are all void K -sequences.

p.380 =
 \Leftarrow (vd) If Ψ is a void \bar{N} -sequence, $\S[\Psi]$ is a void \bar{T} -sequence

\Leftarrow When $k \neq h$, $\S[h,k]$ is a void T -sequence; similar conventions are adopted with regard to $\S(h,k)$, $\S[h,k]$ and $\S(h,k)$ and also with regard to $\S(k)$ and $\S[0]$ when $k=0$ and to $\S((k))$ when $k \leq 1$

=

p.389 a) Σ is a void
(or

b) Σ is a void sequence whose target domain is that of Ω

c) With $h, k \in \bar{N}$ and $k \leq h$, $[\bar{h},\bar{k}]$ is the void \bar{N} -sequence. In analogous

circumstances, the symbols $[h, k]$, $[h, k)$ and (h, k) are similarly interpreted as are $[k]$ and $[k)$ when $k=0$ and $((k))$ when $k=1$. When Ξ is a void sequence $[\Xi]$ is a void \bar{N} sequence

2]. i) seq seqf T etc. 382 386

ii) void seq > 385

iii) $[h, k] \dots$ 378, 379 385

iv) $\Xi \leq \Omega$ $\Xi \leq \Omega$ 379 385

v) $\exists [\Psi]$ 380 385

vi) $\text{seqf}(T|j), \dots (T| \geq j) \text{ seqf}(T| \exists [h, k]) \dots$ 380, 381

vii) $\Phi \pm \Psi$ 383-385

seqf(i. as set)

ibp.) seqf is the set of all $\xi \in \text{seqf}$.

icp.) seqf(T) is the set of all $\xi \in \text{seqf}(T)$

360. 6] $\omega := \Omega$ $\Omega \in \text{seq}(\bar{N})$ = displayed as
 \Leftarrow i) With the \bar{N} -sequence $x(0), x(1), \dots, x(n), x(1), \dots, x(1 \equiv 1)$
 and the members of $K \alpha(\omega)$ of $K \langle \omega : = \Rightarrow$ available,
 presented, the sum $A \in K$
 and the mapping $\alpha(\omega) : K \rightarrow K \langle \omega : = \Xi \rangle$ presented, the sum
 mapping $A := K \rightarrow K$ for which

$$A = \alpha(x(1 \equiv 1)) + \dots + \alpha(x(1)) + \alpha(x(0)) \langle K \rangle$$

is indicated by use of the notation

$$A := \sum a(\omega) \langle \omega : = \Xi \rangle$$

and the product mapping $B : K \rightarrow K$ for which

$$B := \alpha(x(1 \equiv 1)) b(x(1 \equiv 1)) \dots \alpha(x(1)) \alpha(x(0)) \langle K \rangle$$

by use of the notation

$$B := \prod a(\omega) \langle \omega : = \Xi \rangle$$

302. vi) Sums and products of members of $[K]$ and of matrices 887.

in $[K]$ are treated as special cases of the above in which the mappings concerned are constant over the same domain K .

305. —

305. vii) Void matrices (i.e. matrices with zero rows or zero columns) also feature in theory to be given, later. They may arise in the processes of row or column removal from given matrices V and occur in compound matrix expressions which, nevertheless, represent proper matrices. Void matrices do not feature independently in matrix sums, differences, or products; they are ~~introduced~~ by a recourse of convenience. (Instead of giving one expression which involves proper matrices when, for example $b_{i,j}^k$ are such that $b_{i,k}^j$, and another reduced expression to accommodate the case in which $b_{j,i}^k$, the first expression is used consistently; removal of ~~the~~ void submatrices, when they arise, yields the relevant reduced expressions.)

305. —

305. viii) In addition to the proper matrices and vectors defined above, void versions are also encountered as constituents of compound matrix expressions. (When $h = \underset{at}{0} \in K$, for example, $O^{[k]}_{[h]}$ is a void matrix.)

306. —

306. viii) Operations upon matrices in $[K]$ are denoted indicated in to conventional way. Thus the ^{ordinary} product of suitably defined A and B is written as AB . Powers of matrices are, however, indicated indicated expressed into the help of braces: the square and inverse of suitably defined A are written as $\{A\}^2$ and $\{A\}^{-1}$ respectively.

- 4(iv) Structural operations upon mappings are effected pointwise:
 (i.e. for each argument value in the source domain, operations defined in the target domain are carried out upon function values.
 (Thus with $h, k: K \rightarrow \bar{N}$ and $\psi: K \rightarrow \text{seq}(T)$)
 (Thus with $i \in \bar{N}$ and $h, k: K \rightarrow \bar{N}$ such that $k \leq h \leq i < K$, $h, k \subseteq i < K$ and $\psi: K \rightarrow \text{seq}(\bar{N}|i)$, $\psi[h, k]$ is the $\psi[h, k]: K \rightarrow \text{seq}(T)$ is the mapping $\phi: K \rightarrow \text{seq}(T)$ defined by ~~setting~~ for which $\phi(z) = \psi(z)[h(z), k(z)]$ for each $z \in K$. (It may occur that $k(z) < h(z)$ for certain $z \in K$; ~~and~~ for these values $\phi(z)$ is void.))

With $j, h, k \in \bar{N}$, ~~and~~ $\Xi: K \rightarrow \text{seq}(\bar{N}|j)$ such that ~~that~~ $\Xi \subseteq [h] < K$ and $A: K \rightarrow [K|h, k]$ all prescribed,
 $A_{\Xi}: K \rightarrow [K|j, k]$ is the mapping $B: K \rightarrow [K|j, k]$ for which $B(z) = A(z)_{\Xi(z)}$ for each $z \in K$. The interpretation of other submatrix mappings is analogous.

Compound matrix mappings of fixed type are dealt with in the same way.

552. b) With $i \in \bar{N}$ fixed and $\psi: K \rightarrow \text{seq}(T|i)$, the sequential function value in the target domain $\text{seq}(T|i)$ corresponding to z in the source domain K is denoted by ~~by~~ the ~~the~~ sequential function $\psi(z|): [i] \rightarrow T$; when $w \in [i]$, the term with index w yielded by that this sequential function is denoted by $\omega \psi(z|w)$. Again when $w \in [i]$, $\psi(w): K \rightarrow T$

is the mapping defined by the values of this fixed term $\psi(z|\omega)$
with index ω as z ranges through K .

Inequalities