Integral transforms of continued fractions

An array of functions

Notation. A membership sign ($\in$, $\subset$ or $\subseteq$) indicates that the member concerned is fixed; an equivalence sign ($\equiv$) is used to indicate that an accompanying statement holds uniformly of a prescribed set.

$\mathbb{I}$ denotes the set of finite nonnegative integers $0, 1, \ldots$. $\bar{\mathbb{I}}$ is $\mathbb{I}$ augmented by infinity. $i \in \mathbb{I}$ and $j \in \bar{\mathbb{I}}$ mean that i and j are fixed integers in the ranges $0 \leq i < \infty$ and $0 \leq j \leq \infty$ respectively; $i, j \in \mathbb{I}$ means that $0 \leq i, j < \infty$ and so on. $\mathbb{I}_j^k$ $(j, k \in \mathbb{I})$, $\mathbb{I}_j$ $(j \in \mathbb{I})$ denote the sets of integers $j, j+1, \ldots, j+k$ and $j, j+1, \ldots$ respectively. The symbols $\nu \equiv \mathbb{I}$ and $\tau \equiv \bar{\mathbb{I}}_i^j$ are used to indicate that accompanying statements hold for $\nu = 0, 1, \ldots$ and $\tau = i, i+1, \ldots, j$ where $i \in \mathbb{I}$, $j \in \bar{\mathbb{I}}$ respectively. When such symbols are used conjointly, the sequences of integers referred to are understood to be nested; thus $i \equiv \mathbb{I}_1$, $j \equiv \mathbb{I}_i$ means $i = 1$, $j \equiv \mathbb{I}_1$,

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$i=2$, $j=\mathrm{I}_2$, ... ; an abbreviation such as $i,j=\mathrm{I}$ is understood to mean $i=\mathrm{I}$, $j=\mathrm{I}$.

The index of single summation is always ν; it is omitted from the summation sign: $\sum_0^n g_\nu x^\nu$ means $\sum_{\nu=0}^{\infty} g_\nu x^\nu$. If the limits of summation are 0 and ∞ they are omitted from the summation sign: $\sum h_\nu z^\nu$ means $\sum_{\nu=0}^{\infty} h_\nu z^\nu$. The indexes of double summation are always ν and τ respectively; they are omitted from the summation signs: $\sum_1^n \sum_0^{\nu-1} a_{\nu,\tau}$ means $\sum_{\nu=1}^{n} \sum_{\tau=0}^{\nu-1} a_{\nu,\tau}$. The index used in the formation of a product is always ν; it is omitted from the product sign: $\prod_1^n a_\nu$ means $\prod_{\nu=1}^{n} a_\nu$.

$\mathfrak{D}_t$ is the differential operator with respect to t: $\mathfrak{D}_t^2 G(t) = d^2G(t)/dt^2$.

$$h(z) = \{\textstyle\sum_0^J v_\nu z^\nu\}/\{\textstyle\sum_0^I \pi_\nu z^\nu\}$$

then [ ] the table contains the infinite block of identical quotients $P_{i,j}(z) = h(z)$ $(i = J_I, j = J_J)$. ~~Padé at [ ] also established the formal connection between continued fractions derived from the transformation of power series, and diagonal sequences of quotients such as~~ ~~$P_{i,i+m-1}(z)$ $(m \in J, i = J)$~~ ~~and~~ ~~$P_{i+m,i}(z)$ $(m \in J, i = J)$.~~

The structural and convergence properties of the Padé table have been systematically investigated for the two cases in which the series $\sum h_\nu z^\nu$ is simply related to the exponential series, and in which ~~the~~ h_ν ~~are moments with respect to a bounded nondecreasing~~ $(\nu = J)$ is a sequence of moments.[9] With regard to the first of these, when $h_\nu = 1/\nu!$ $(\nu = J)$, the $P_{i,j}(z)$ may be given in simple closed form; they are all distinct, with $\tilde{i} = i$, $\tilde{j} = j$ in formula ( ), and $\zeta(i,j) = i+j+1$ ~~(i,j)~~ $(i,j = J)$[ ]. Using the closed expressions available in this case, Padé showed that any infinite

~~by Padé [ ].~~ He also showed ~~in particular~~ that if ~~$\Sigma h_\nu z^\nu$ is the series reciprocal to $h(z)$, in the~~ $\tilde{P}_{i,j}(z)$ $(i,j \in J)$ are the Padé quotients derived by ~~h~~ $\Sigma \tilde{h}_\nu z^\nu$, the series reciprocal to $h(z)$ defined by the formula $(\Sigma h_\nu z^\nu)(\Sigma \tilde{h}_\nu z^\nu) = 1$, then $\tilde{P}_{i,j}(z) = 1/P_{j,i}(z)$ $(i,j \in J)$.

If $h(z)$ generates an associated continued fraction [ ], ~~the~~ the successive convergents of the latter are ~~$P_{i,i+1}(z)$~~ $P_{i-1,i}(z)$ ~~$(i \in J)$ and if $h(z)$ is the expan~~ itself ~~or $P_{..}$~~, with $i \in J_0^n$ for ~~some~~ suitable $n \in J$ if $h(z)$ is itself generated by a rational function of the form ( ) ~~and $i \in J$ otherwise~~ (the continued fraction then terminates) ~~to~~ and $i \in J$ otherwise. If $h(z)$ generates a corresponding continued fraction [ ] ~~the suc~~ its successive convergents ~~are~~ form ~~taken from the~~ part of or the whole of the sequence $P_{i-1,i}(z), P_{i,i}(z)$ $(i \in J)$ according as to whether $h(z)$ is or is not generated by a rational function. Padé [ ] also established the ~~formal~~ formal connection between the associated and corresponding continued fractions generated by the delayed series $\Sigma h_{m+\nu} z^\nu$ $(m \in J)$ and the diagonal sequences $P_{i-1,i+m}(z)$ $(i \in J)$ and $P_{i-1,i+m}(z), P_{i,i+m}(z)$ $(i \in J)$, and the connection between continued fractions derived from delayed forms of the reciprocal series and diagonal sequences taken from the lower half of the Padé table sequence of quotients with distinct suffix sets $\{i;j\}$ converges uniformly to e^z in any bounded domain of the complex plane. This result has recently been extended by Arms and Edrei [ ] to functions $h(z)$ expressed as a product of $e^{\gamma z}$, ~~$(\gamma \neq 0)$~~ $\gamma \in (0,\infty)$) and a convergent infinite product of linear terms of the form $1-u_\nu z$, ~~γ and~~ $(u_\nu \in (0,\infty))$, ~~where all u_ν have the same argument~~; they have also shown that the structural properties are preserved if ~~$h(z)$~~ the above function $h(z)$ also contains as a factor a convergent infinite product of terms of the form $1/(1-t_\nu z)$ $(t_\nu \in (0,\infty))$, ~~where the t_ν have the same argument as γ and the t_ν~~; convergence in this case holds ~~on~~ in a bounded domain from which ~~all~~ neighbourhoods of all points t_ν^{-1} have been removed.

In the second case in which properties of the Padé table have been extensively investigated, the coefficients of ~~the series~~ $h(z)$ $\sum h_\nu z^\nu$ have the form

$$h_\nu = \int_\alpha^\beta t^\nu d\sigma(t) \qquad (D=J)$$

where $\sigma(t)$ is a bounded nondecreasing real valued function ~~for $t = [\alpha,\beta]$. For a fixed value of $t \in [\alpha,\beta]$, the integrand~~

For $t \in [\alpha,\beta]$. ~~The series~~ $h(z)$ is then associated with the function

$$h(z)=\int_\alpha^\beta d\sigma(t)/(1-zt).$$

For a fixed value of $t\in[\alpha,\beta]$, $(1-zt)^{-1}$ has a singularity at the point $z=t^{-1}$. The singularities of $h(z)$ are confined to the real ~~segments~~ values given by $z=t^{-1}$, $t\in[\alpha,\beta]$, and elsewhere $h(z)$ is analytic. If ~~$0\notin(\alpha,\beta)$~~ $0\notin(\alpha,\beta)$, ~~the points of increase of $\sigma(t)$ are dense everywhere dense in the interval $t\in[\alpha,\beta]$, and $[\alpha,\beta]=$ $0\notin(\alpha,\beta)$~~, the singularities of $h(z)$ are confined to a single cut; if ~~$0\in(\alpha,\beta)$~~ $[\alpha,\beta]=[-\infty,\infty]$ ~~the single cut is the~~ they are confined to the ~~entire complex~~ real axis; otherwise, with $0\in(\alpha,\beta)$, two cuts are involved. If, when ~~$[\alpha,\beta]\subseteq(-\infty,\infty)$~~ and $\gamma=\max(|\alpha|,|\beta|)<\infty$, $h(z)$ converges to $h(z)$ for $|z|<\gamma^{-1}$, and in so doing represents $h(z)$ ~~uniformly~~ asymptotically, as $|z|$ tends to zero, for all values of $\arg(z)$. ~~If $\beta=\infty$, α~~ It was shown by Stieltjes [ ] that when $\beta=\infty$, $\alpha\in[0,\infty)$, $h(z)$ represents $h(z)$ asymptotically for ~~$0<\arg$~~ $\arg(z)=(0,2\pi)$, and by Hamburger [ ] and Nevanlinna [ ] that when $0\in(\alpha,\beta)$, this asymptotic representation holds in the two sectors $(0,\pi)$ and $(\pi,2\pi)$.

was studied by Tschebyscheff [,] who exhibited the ith convergent (i.e. $P_{i-1,i}(z)$) in the notation of the Padé table in the form

$$P_{i-1,i}(z) = \sum_{1}^{i} M_{\nu}^{(i)}/(1-t_{\nu}^{(i)}z)$$

where $t_{\nu}^{(i)}$ and $M_{\nu}^{(i)}$ $(\nu = 1(1)i)$ are the zeros and weight factors respectively of the orthogonal polynomial of degree i generated by the distribution $\sigma(t)$ over the range $t = [\alpha,\beta]$, showing that $t_{\nu}^{(i)} \in (\alpha,\beta)$ $(\nu = 1(1)i)$ with ..., and that the series expansion of $P_{i-1,i}(z)$ and $h(z)$ (both regular at the origin in this case) agree as far as the term involving z^{2i-1}.

Tschebyscheff P., Sur les fractions continues, Jour de math. pures et appl., Sér. II, 3 (1858) 289–328

——— Sur les résidus intégraux qui donnent des valeurs approchées des intégrales, Acta math., 12 (1888–9) 287–322

The structure of the ~~§~~ associated continued fraction generated by $h(z)$ when $[\alpha,\beta] \in (-\infty,\infty)$, ~~and the the degrees of asymptotic equivalence between the convergents and h(z), and the~~ location of the poles of ~~these convergents in this case, were~~

studied by ~~Tschebyscheff and [ ] and Markoff [ ].~~

Similar investigations of the structure and asymptotic properties of the corresponding ~~convergent~~ continued fraction generated by $h(z)$ when $[\alpha,\beta] \subseteq [0,\infty]$ were carried out by Stieltjes [ ], and of the associated continued fraction generated by $h(z)$ when $[\alpha,\beta] \subseteq [-\infty,\infty]$ by Hamburger [ ] and Nevanlinna [ ].

Extending Stieltjes's results to the Padé table generated by $h(z)$ for the case in which $[\alpha,\beta] \subseteq [0,\infty]$, Van Vleck [ ] showed that ~~when~~ ~~then~~ for all $P_{i,j}(z)$ not ~~occur~~ belonging to the infinite block occurring when σ is a simple step function ($h(z)$ is then a rational function) $\tilde{i}(i,j)=i$, $\tilde{j}(i,j)=j$ ~~; furthermore~~ so that all such $P_{i,j}(z)$ are distinct; ~~furthermore, the poles $t_\nu^{(i,i+m-1)}$ $(\nu=1,i)$ of the quotients $P_{i,}$~~ he examined the degrees of asymptotic equivalence between the $P_{i,j}(z)$ and $h(z)$ in this case, and showed ~~that the poles in parts~~ that, in particular, the poles $t_\nu^{(i,i+m-1)}$ $(\nu=1,i)$ of the quotients $P_{i,i+m-1}(z)$ $(i=1,\ldots,\ m=1,\ldots)$ (~~the~~ lying in the upper half of the Padé table) are simple and lie in (α,β). Van Vleck's investigations were extended by Wall [ ] to the case in which $[\alpha,\beta] \subseteq [-\infty,\infty]$ who showed that the set of quotients not belonging to the possible infinite block can containing blocks of at most four identical members, and that the results for the numbers $\xi t_{\nu}^{(i,i+m-1)}$ given above now hold with m replaced by $2m$.

The first general inquiry into the convergence behaviour of the continued fractions generated by the series $h(z)$ whose coefficients are given by expressions of the form ( ), with σ as described above, was conducted by Markoff [ ]: if $[\alpha,\beta] \subset (-\infty,\infty)$ the associated continued fraction derived from $h(z)$ converges uniformly to $h(z)$ in any bounded open domain not containing any point z for which $z^{-1} \in [\alpha,\beta]$ as an interior point; if, in addition, $0 \notin (\alpha,\beta)$ this result also holds for the corresponding continued fraction derived from $h(z)$. Stieltjes [ ], who considered the case in which $[\alpha,\beta] \subseteq [0,\infty]$, showed that convergence of the corresponding continued fraction as just described holds if and only if the Stieltjes moment problem associated with the sequence h_ν $(\nu = 0)$ (i.e. the problem of determining a suitably normalised bounded nondecreasing function σ satisfying equations ( ) with

The condition that the Stieltjes or Hamburger moment problems should be determinate imposes a ~~condition~~ limit in each case upon the rate of growth of the coefficients h_ν. In particular, Stieltjes showed the [ ] that if $h_\nu = (\kappa\nu+\lambda)!$ ($\kappa, \lambda \in [0,\infty)$, $\nu = J$), the associated Stieltjes moment problem is determinate if $\kappa \leq 2$ and indeterminate if $\kappa > 2$.

$[\alpha=0, \beta=\infty)$ is determinate. Hamburger [ ] and Nevanlinna [ ] extended the researches of Stieltjes to the case in which $[\alpha,\beta] \subseteq [-\infty,\infty]$ and proved convergence of the associated continued fraction as described subject to the condition that the Hamburger moment problem (i.e. the analogue of the Stieltjes moment problem with $\alpha=-\infty, \beta=\infty$ in formula ( )) should be determinate.

Wall [ ] applied Stieltjes' theory to the Padé table generated by ~~the~~ $h(z)$ when $[\alpha,\beta] \subseteq [0,\infty]$, and, in particular, proved convergence of the diagonal sequences $P_{i,i+m-1}(z)$ $(i=J)$ for $m=J_0^{m'}$ to $h(z)$ as described above, subject to the condition that the Stieltjes moment problem associated with the sequence $h_{m'+\nu}$ ($m' \in J$, $\nu=J$) should be determinate; he also derived an analogous result for sequences of the form $P_{i+m,i}(z)$ $m=J_0^{m'+1}$, $i=J$ in terms of the coefficients of the series reciprocal to $h(z)$. These results were further extended by Wall ~~and~~ [ ] to the case in which $[\alpha,\beta] \subseteq [-\infty,\infty]$. Wall also showed [ ] that when $[\alpha,\beta] \subset (-\infty,\infty)$ with $\gamma=\max(|\alpha|,|\beta|)$, any infinite sequence of quotients with distinct suffix sets i,j converges uniformly to $h(z)$ in the open disc $|z|<\gamma^{-1}$.

Carleman [ ] ~~showed proved~~ provided the simple criterion that the corresponding continued fraction generated by $h(z)$ in the

Stieltjes case $[\alpha,\beta] = [0,\infty]$ converges to $h(z)$ as described in the penultimate paragraph if the series $\sum_1^\infty h/.$. diverges. Carleman's criteri for convergence of the associated continued fraction in the Hamburger–Nevanlinna case $[\alpha,\beta] \subseteq [-\infty,\infty]$ is that the series $\sum_1^\infty$ . . should diverge. These results may be ~~applied~~ extended to prove convergence of diagonal sequence of the Padé tables generated by $h(z)$ in these two cases.

Optimal estimates of the rate of convergence of the sequence of convergents of the associated continued fraction generated by $h(z)$ when $[\alpha,\beta] \subset (-\infty,\infty)$ have been given by Gragg [ ]. Estim of a similar nature for the corresponding continued fraction generated by $h(z)$ when $[\alpha,\beta] \subseteq [0,\infty]$ ~~man~~, and for the associated fraction when $[\alpha,\beta] \subseteq [-\infty,\infty]$, may be ~~derived~~ ~~from~~ deduced directly from Carleman's theory. These estimates may be extended to ~~the~~ diagonal sequences of the Padé tables generated by $h(z)$ in these cases.

It was shown by Stieltjes [ ] that ~~each conv~~ when $z \in (-\infty, 0)$, each convergent of the corresponding continued fraction generated by $h(z)$ when $[\alpha,\beta] = [0,\infty]$ is in a certain sense a best approximation to $h(z)$. This result has been extended by the author [ ] ~~author [ ] to the convergents of the associated continued fraction~~ ~~for real $z \in$~~

~~generated by $h(z)$ when $[\alpha,\beta]\subset(-\infty,\infty)$.~~

firstly for fixed real values of z such that $z^{-1}\notin[\alpha,\beta]$ when $[\alpha,\beta]\subseteq[0,\infty]$, and also to the convergents of the associated continued fraction generated by $h(z)$ when $[\alpha,\beta]\subset(-\infty,\infty)$. These results have been extended in turn [ ] to show that ~~certain~~ ~~if~~ when $[\alpha,\beta]\subseteq[0,\infty]$ and z is real with $z^{-1}\notin[\alpha,\beta]$, each Padé quotient of the set $P_{i,i+m-1}(z)$ $(i,m=\mho)$ is a best approximation to $h(z)$, and that analogous results hold when $[\alpha,\beta]\subset(-\infty,\infty)$ for the set $P_{i,i+2m-1}(z)$ $(i,m=\mho)$.

The results described in the preceding paragraph for $[\alpha,\beta]\subseteq[0,\infty]$, ~~z~~ and real $z\notin[0,\infty]$ have been used by the author [ ] to show that in this case ~~the set~~ the forward and backward diagonal sequences taken from the set $P_{i,i+2m-1}(z)$ $(i,m=\mho)$ are monotonically increasing, while ~~the~~ such sequences taken from from the set $P_{i,i+2m}(z)$ $(i,m=\mho)$ are monotonically decreasing. This result offers a geometrical picture of the complete ensemble of numerical values of the set $P_{i,i+m-1}(z)$ for a fixed ~~to~~ value of z as described: these values lie on two semi-hulls, ~~whose~~ one of which is upturned, the two keels ~~lying~~ ~~coinciding with~~ lying in the ~~above~~ forward diagonal directions of the sequences $P_{i+1,i}(z)$ $(i=\mho)$ and $P_{i,i}(z)$ $(i=\mho)$ respectively; the two ~~half~~ semi-hulls are separated by the ~~constant~~ plane through the value of $h(t)$, they are convex when ~~one is~~ viewed from ~~the~~ this plane. A similar picture has been constructed for the remaining quotients of the Padé table, and the theory has been extended still further [ ] to the two cases in which $[\alpha,\beta] \subset [0,\infty]$ and $[\alpha,\beta] \subseteq [-\infty,\infty]$.

A convergence result of quite another kind, and based in large measure upon Hadamard's theory of functions defined by their Taylor series expansions [ ], was derived by de Montessus de Ballore [ ]: let $h(z)$ converge to the function $h(z)$ for sufficiently small values of z; let $h(z)$ possess n poles (counted according to their multiplicity) within the circle $|z| = r$ and be otherwise regular within this circle; then the row sequence $P_{n,j}(z)$ $(j = J)$ converges to $h(z)$ uniformly in the open disc $|z| < r$ from which neighbourhoods of all poles have been removed ~~in any open domain not containing any point of the circle or any pole as an interior point~~. de Montessus also gave an analogous ~~further~~ result ~~concerning a column sequence~~ involving the zeros of $h(z)$ and concerning a column sequence of Padé quotients. ~~These~~ The author has combined these results of de Montessus ~~have been combined~~ with the theory of the moment problem to derive the following result [ ]: let $h(z)$ be given by formula ( ) where $[\alpha,\beta] \subset [0,\infty)$ and $\sigma(t)$ is a bounded nondecreasing function for $t \in [\alpha,\beta]$ with, in particular, n saltus and no other points of increase in the range $t \in [\hat{\beta},\beta]$ $(\hat{\beta} \in (\alpha,\beta))$; then any sequence infinite sequence of quotients with distinct suffix sets i, j taken from the set of ~~quo~~ Padé quotients $P_{i+n, j+n}(z)$ $(i, j = J)$ generated by $h(z)$ converges to $h(z)$ uniformly in ~~any open domain not containing any point of the circle $|z| = \rho 1/\hat{\beta}$ or any point of the segment $[1/\rho, 1/\hat{\beta}]$ as an interior point.~~ the open disc $|z| < 1/\hat{\beta}$ cut along the real segment $[1/\rho, 1/\hat{\beta}]$.

The Padé table finds its main application as a powerful method for transforming slowly convergent and divergent power series. However, its range of application is severely limited. Firstly, the known convergence results places considerable restrictions upon the rate of growth of the coefficients h_ν of the series being transformed, the allowed variation lying between expressions of the form ~~of~~ $h_\nu = \gamma^\nu/\nu!$ in the case of the exponential function and $\gamma^\nu(2\nu+\mu)!$, in the case of a determinate Stieltjes moment problem (see the above resumé). Secondly, the singularities of the functions which generate the series under transformation are confined to one, or possibly two cuts in the complex plane. Concerning the second of these limitations, which is of course extremely severe, we quote the words of Borel (he ~~writes in~~ is ~~writing~~ speaking, in particular, of the use of the continued fractions of Stieltjes in the solution of differential equation, but might, de ne pas se contenter de la généralisation que nous venons d'en donner et de chercher à les étendre dans d'autres directions. Pour y arriver, la méthode la plus sûre consisterait peut être à chercher à démontrer les propositions même de Stieltjes par des méthodes plus générales. On aura ainsi des démonstrations probablement plus longues que les siennes, qui sont fort ingénieuses et qui paraissent aussi simples que possible pour le cas particulier traité, mais ces démonstrations plus longues auraient sans doute l'avantage de s'étendre sans effort à des cas plus générau

~~Since these~~

Some time after these words were written, an advance towards the more ~~investigation~~ general ~~treatment~~ treatment envisaged by Borel was ~~conducted~~ ~~carried out~~ ~~presented~~ made by Shohat and Tamarkin [ ]. Largely Basing their theory upon work of M. Riesz [ ] they ~~consider~~ ~~examine~~ investigated ~~various~~ conditions to be imposed upon the function $g(t)$ sufficient to ensure that the numbers

$$g_i = \sum_1^i M_{i,\nu}\, g(t_\nu^{(i)}) \qquad (i = 1,)$$

should converge to the value of the integral $\int_a^b g(t)\,d\sigma(t)$, where $\sigma(t)$ is a ~~as described~~ ~~as described~~ bounded nondecreasing function of t for $t \in [a, b]$ and such that the moments ( ) exist, and $t_\nu^{(i)}, M_\nu^{(i)}$ are, as in formula ( ), the zeros and weight factors of the orthogonal polynomial of i^{th} degree generated by $\sigma(t)$ over the interval $[\alpha, \beta]$. Their methods are essentially those of real variable theory; values of the function t upon the segment $t \in [\alpha, \beta]$ only are used, and their results can be applied to functions defined only for such values of t. ~~Nevertheless~~ They apply their results to the function $g(t) = 1/(1-xt)^{\nu}$ with $[\alpha,\beta] = [0, \infty]$ and $x \in (-\infty, 0)$ and en passant ~~so doing~~ pick up much of the theory of Stieltjes continued fractions ~~en passant~~ with real argument.

In this paper we wish to advance still further. We consider ~~[illegible]~~ general functions of two variables ~~[illegible]~~ in formulae ~~[illegible]~~ such as ( ), we extend the inquiry to ~~[illegible]~~ quotients of the Padé table other than those lying on the forward diagonal ~~[illegible]~~ $P_{i-1,i}(z)$ $(i = 0)$ (we are concerned, that is to say, with an array of functions as opposed to a sequence of numbers), ~~[illegible]~~ and we develop ~~extend~~ the investigation still further ~~from auxiliary series whose coefficients are given by expressions of the form ( ) to such~~ so as to concern auxiliary series other than those whose coefficients are given by expressions of the form ( ).

~~Our theory is perhaps most easily motivated~~

F The Padé quotient $P_{i,j}(z)$ may be decomposed in the form

$$P_{i,j}(z) = \sum_0^{m(i,j)} \alpha_\nu^{(i,j)} z^\nu + \sum_1^{n(i,j)} \sum_0^{n(i,j,\nu)-1} A_{\nu,\tau}^{(i,j)} \tau! (t_\nu^{(i,j)})^\tau z^\tau (1-t_\nu^{(i,j)} z)^{-\tau-1} \cancel{A_{\nu,\tau}^{(i,j)} (1-t_\nu^{(i,j)} z)^{-\tau-1}}.$$

We introduce a sequence of functions $G_\nu(x)$ and a function of two variables $G(t,x)$, and define the function $F_{i,j}(x)$ by means of the formula

$$F_{i,j}(x) = \sum_0^{m(i,j)} \alpha_\nu^{(i,j)} G_\nu(x) + \sum_1^{n(i,j)} \sum_0^{n(i,j,\nu)-1} A_{\nu,\tau}^{(i,j)} G(\tau; t_\nu^{(i,j)}; x)$$

where

$$G(\tau; t, x) = \cancel{D_t^\tau G(t,x)/dt}.\ t^\tau D_t^\tau G(t,x)$$

~~Formu~~ ¶ Formula ( ) is perhaps most easily motivated by ~~assuming that~~ letting ~~$G(z,x)$ is analytic at the point $z=0$, with $\sum G_\nu(x) z^\nu$~~ can analytic

$G(z,x)$ be a function of the complex variable z, analytic at the origin, with $\sum G_\nu(x) z^\nu$ as its ~~Maclaurin~~ Maclaurin Taylor series expansion, and letting $\mathbb{C}$ be a closed contour surrounding the origin and the points $t_\nu^{(i,j)}$ and lying in ~~within~~ a region within which $G(z,x)$ is analytic; we then have the integral transform formula

$$F_{i,j}(x) = \int_{\mathbb{C}} z^{-1} P_{i,j}(z^{-1}) G(z,x) dz$$

~~For suitable $P_{i,j}(z)$, $G(z,x)$ and $\mathbb{C}$, such a formula holds for an infin~~

Supposing that the series $h(z)$ which generates the Padé quotients $P_{i,j}(z)$ in question ~~is~~ to be itself ~~generate the Maclaurin ex~~ generated by

If, in formula ( ), C may be taken to be a circle with centre at the origin and including all singularities of $h(z^{-1})$, then $F(x) = \sum h_\nu Q_\nu(x)$. In most cases & with which we deal we establish ~~a connection, either between~~ that this series is either ~~bot~~ a convergent or asymptotic representation of the function $F(x)$.

~~that this series represents F(x), either as a~~

of function $h(z)$, we ~~should~~ may, subject to ~~certain~~ suitable assumptions concerning $h(z)$, define a function $F(x)$ by means of the formula

$$F(x) = \int_C z^{-1} h(z^{-1})\, G(z,x)\, dz.$$

In certain cases, as we shall see, it is possible by means of complex variable methods to show that a prescribed infinite sequence of the $F_{i,j}(x)$ converges to $F(x)$.

However, the ~~conditions~~ assumptions introduced for the above motivation are by no means indispensable. If the coefficients of $h(z)$ are given by formula ( ) it is known (see above) that for certain i,j the $\zeta^{(i,j)}_{\nu}$ are confined to the real segment $[\alpha,\beta]$ and that the poles of the relevant $P_{i,j}(z)$ are simple (i.e. $n(i,j;\nu)=1$ $(\nu = 1, \ldots, n(i,j))$ in formula ( )). In such cases, ~~formula ( )~~ the function $F_{i,j}(x)$ is well defined by formula ( ) even when no more concerning $G(t,x)$ is assumed than that it should be finite for $t \in [\alpha,\beta]$. In certain such cases, as we shall see, the ~~functions~~ appropriate functions $F_{i,j}(x)$ converge to

$$F(x) = \int_\alpha^\beta G(t,x)\, d\sigma(t).$$

It can even occur that ~~the sequence~~ this result still holds when the ~~sequence~~ $G_\nu(x)$ ~~to~~ form an arbitrary ~~finite~~ sequence of finite valued functions, quite unrelated to $G(t,x)$.

In the important special case in which the functions $G_\nu(x)$ have the simple form $g_\nu x^\nu$, and the variables in the function $G(t,x)$ occur only in the form tx, so that this function may be written as $g(tx)$, the imposed conditions can be simplified and results more extensive than those holding for the more general case are readily available. Accordingly, we study the more restricted functions

$$f_{i,j}(x)=\sum_0^{m(i,j)}\alpha_\nu^{(i,j)}g_\nu x^\nu+\sum_1^{n(i,j)}\sum_0^{n(i,j,\nu)-1}A_{\nu,\tau}^{(i,j)}x^\tau D_x^\tau\{g(t_\nu^{(i,j)}x)\}_\xi.$$

The associated function $f(x)$ which takes the place of $F(x)$ in this case is either expressible ~~by means of an integral of the form ( ) with $G(t,x)$ replaced by $g(tx)$~~ either in the form

$$f(x)=\int_C z^{-1}h(z^{-1})\,g(zx)\,dz$$

or in the form

$$f(x)=\int_\alpha^\beta g(tx)\,d\sigma(t).$$

The series in most cases associated with $f(x)$ is $\sum h_\nu g_\nu x^\nu$.

We study the structure of the array of functions $F_{i,j}(x)$ and the asymptotic equivalence of these functions to a suitably defined function $F(x)$, and we investigate the convergence of sequences of these functions to $F(x)$; we do likewise to the functions $f_{i,j}(x)$ and $f(x)$. Our

The approximating function of Jacobi [ ] and Frobenius [ ], or the Padé quotient [ ] of order (i,j) derived from the power series $h(z) = \sum h_\nu z^\nu$ $(h_0 \neq 0)$ is the uniquely determined irreducible rational function

$$P_{i,j}(z) = \sum_0^{\tilde{j}} v_\nu^{(i,j)} z^\nu \Big/ \sum_0^{\tilde{i}} \pi_\nu^{(i,j)} z^\nu \quad (\tilde{i} \in J_0^i, \tilde{j} \in J_0^j, \pi_0^{(i,j)} = 1$$

whose series expansion $h^{(i,j)}(z) = \sum h_\nu^{(i,j)} z^\nu$ agrees with $h(z)$ in the largest number of initial terms: if $h_\nu^{(i,j)} = h_\nu$ $(\nu = J_0^{\xi(i,j)-1})$, $h_{\xi(i,j)}^{(i,j)} \neq h_{\xi(i,j)}$, the $v_\nu^{(i,j)}$ and $\pi_\nu^{(i,j)}$ are so chosen that with $\tilde{i} \in J_0^i$, $\tilde{j} \in J_0^j$, $\xi(i,j)$ is as large as possible. The quotients $P_{i,j}(z)$ $(i,j = J)$ may be set in a two dimensional array, the Padé table, in which i and j correspond to row and column numbers respectively. (We append, for convenience in exposition, the quotient $P_{0,-1}(z) = 0$.)

The structural properties of this array were investigated by Padé, who showed that such quotients determined by the above conditions that are identical occur in square blocks in the Padé table. If $h(z)$ is the series expansion of the irreducible rational function theory includes, that of the Padé table of quotients $P_{i,j}(z)$ as a special case, obtained by setting $g_\nu = 1$ $(\nu = \bar{I})$, $g(tx) = (1-tx)^{-1}$ in formula ( ), and replacing x by z.

We are able to extend to the transformation of series of a far wider range the same power that attends the use of Padé quotients.

As points of detail, we mention that we permit the sequences $G_\nu(x)$ and g_ν $(\nu \equiv \bar{I})$ to possess gaps, with nonzero members $G_{\mu(\nu)}(x)$ and $g_{\mu(\nu)}$ $(\nu \equiv \bar{I}_0^\infty)$ (it is in any case desirable to present the theory in the most general terms possible; furthermore, power series with gaps are of considerable interest in themselves). In this way the user of the theory is permitted another degree of flexibility. Given a series $\sum F_\nu(x)$ to transform, he is required to decompose its terms $F_\nu(x)$ in the form $h_{\mu(\nu)} G_{\mu(\nu)}(x)$ $(\nu = \overline{0, \infty})$ where the numbers h_ν and functions $G_\nu(x)$ occur in one of our convergence results, and the values of the functions $G(t,x)$, which is usually associated with the series $\sum_0^\infty G_{\mu(\nu)}(x) t^{\mu(\nu)}$, may easily be determined; he has considerable latitude in the choice of decomposition, and has added freedom in the choice of the integer function $\mu(\nu)$. The same holds with regard to the restricted functions of formula ( ).

Also the notations used in general allow for the sets of points considered in our theory to be segments of the real axis, or domains in the complex plane; we are thus enabled to consider the transformation of, in particular, Fourier series with real coefficients, as well as general series of functions of a complex variable. Furthermore, when considering questions of ~~convergence~~ asymptotic equivalence, we assume that the functions $G_{\mu(\sigma)}(x)$ form an asymptotic sequence in the sense that $G_{\mu(\sigma+1)}(x) = o\{G_{\mu(\sigma)}(x)\}$ $(\sigma = \overline{\mathbb{I}}_0^{\tau-1})$ as x tends through a prescribed point set to a limit point. These functions can be of the form $g_{\mu(\sigma)} \exp(-\lambda_{\mu(\sigma)} x)$, $(g_{\mu(\sigma)} \neq 0$ $(\sigma = \overline{\mathbb{I}}_0^{\tau})$, $Re\{\lambda_{\mu(\sigma+1)}\} > Re\{\lambda_{\mu(\sigma)}\}$ $(\sigma = \overline{\mathbb{I}}_0^{\tau-1})$, for example occurring in the transformation of power series. When $\mu(\sigma) = \sigma$ and $G_\sigma(x) = x^\sigma$ $(\sigma = \overline{\mathbb{I}})$ our theory of asymptotic equivalence is, of course, simply that holding for the Padé table as described above.

Reverting to Borel's strictures concerning the singularities of the functions represented with the aid of Stieltjes' theory, we remark that the singularities of the functions with which our theory deals are far less constrained. If, for example, the function g in formula ( ) possesses $N \in \mathbb{I}_1$ poles (none at the origin) and is otherwise regular throughout the complex plane, the singularities of f lie on radial cuts; if $\alpha \in (0, \infty)$, $\beta = \infty$, f is regular in a star shaped domain; if $[\alpha, \beta] = [0, \infty]$, f is regular in a system of disjoint sectors. With more complicated functions g in formula ( ) the nature of the singularities of f can be far more varied; this remark holds with even greater force concerning the function $F(x)$ of formula ( ).

Whether our extension of Stieltjes's theory is precisely that envisaged by Borel we cannot say; but plus longue it certainly is. In order to contain the exposition of our theory within reasonable length, we must perforce make use of an integrated system of special notations. For the benefit of the general reader, and to clarify the development of the theory, we now summarize the results subject to minor simplifications, without using special notations.

We show how the Padé quotient is derived and assemble, in Theorem 1, its known properties and some others which we have had to derive for the occasion.

The structural theory of the arrays of functions $\{F_{i,j}(x)\}$ and $\{f_{i,j}(x)\}$ is presented in terms of formal series. We treat the series $g(x) = \sum g_\nu x^\nu$, for example, as an infinite sequence of coefficients g_ν $(\nu \equiv \mathbb{I})$ and deal with the derivation of one such sequence from others; questions of convergence do not at this stage concern us. ~~$g(tx)$ is simply the series $\sum (g_\nu t^\nu) x^\nu$~~. linear magnification of the argument is defined by the formula $g(tx) = \sum (g_\nu t^\nu) x^\nu_0 \, (t \in \mathbb{Z})$ differentiation of series by $\mathcal{D}^\tau_x g(x) = \sum \{(\tau+\nu)! g_{\tau+\nu} / \nu!\} x^\nu$ ($\tau \in \mathbb{I}$), and displacement of series by $x^\tau g(x) = \sum g'_\nu x^\nu$ where $g'_\nu = 0$ $(\nu = \mathbb{I}_0^{\tau-1})$, $g'_{\tau+\nu} = g_\nu$ $(\nu = \mathbb{I})$. We assume first that the sequences g_ν, h_ν $(\nu = \mathbb{I})$ are prescribed $(h_0 \neq 0)$ and consider the power series $f_{i,j}(x)$ given by

$$f_{i,j}(x) = \sum_0^{m(i,j)-1} \alpha_\nu^{(i,j)} g_\nu x^\nu + \sum_1^{n(i,j)} \sum_0^{n(i,j;\nu)-1} A^{(i,j)}_{\nu,\tau} x^\tau \mathcal{D}^\tau_x g(t_\nu^{(i,j)} x).$$

We set $f(x) = \sum h_\nu g_\nu x^\nu$. In theorem 2 (i) we show that for ~~all~~ $i,j = \mathbb{I}$ all series $f_{i,j}(x)$ can be constructed, and that their coefficients are $h_\nu^{(i,j)} g_\nu$ $(\nu = \mathbb{I})$. (ii) We consider the degenerate form of this array of the series in the cases a) in which $h(x)$ is the series expansion of a rational function and b) in which $g(x)$ is the polynomial $\sum_0^\tau g_\nu x^\nu$ (iii) We consider the degree of agreement between the $f_{i,j}(x)$ and

$f(x)$ (i.e. the extent to which $f(x) - f_{i,j}(x)$ is deficient in initial powers of x). (iv) Under given conditions we examine the extent to which the $f_{i,j}(x)$ can be shown to be distinct.

The results of the above theorem reveal some properties which the array $\{f_{i,j}(x)\}$ has in common with the Padé table. If $f(x)$ may be expressed as

$$f(x) = \sum_0^{\kappa-1} \alpha_\nu g_\nu x^\nu + \sum_1^n \sum_0^{n(\nu)-1} A_{\nu,\tau} \mathfrak{D}_x^\tau \{x^\tau g(t_\nu x)\}$$

the $\{t_\nu\}$ being distinct, then all series $f_{i,j}(x)$ $(i \equiv \mathbb{I}_I, j \equiv \mathbb{I}_J)$ for some $I, J \in \mathbb{I}$ are equivalent to $f(x)$ (I and J are determined by the condition that

$$h(z) = \sum_0^{\kappa-1} \alpha_\nu z^\nu + \sum_1^n \sum_0^{n(\nu)-1} A_{\nu,\tau} (1 - t_\nu z)^{-\tau-1}$$

should be the irreducible rational function [ ] and $h(z)$ is then the series expansion of this function. In this case use of the algorithm producing the series $f_{i,j}(x)$

does no more than reconstruct $f(x)$. (When $g(x) = \sum x^\nu$ we obtain the result for the Padé table concerning the rational function ( )). Again, if $g_\nu \neq 0$ ($\nu \equiv \mathbb{I}$) then identical series $f_{i,j}(x)$ occur only in square blocks, and these occur where and only where the Padé table derived from $h(z)$ has them. However, if $g_\nu \neq 0$ ($\nu \equiv \mathbb{I}_0^\tau$) $g_\nu = 0$ ($\nu \equiv \mathbb{I}_{\tau+1}$) and certain restrictions are imposed upon the $\{h_\nu\}$, all series lying on and below the backward diagonal containing the $f_{i,j}(x)$ with $i+j=\tau$ are equal to $f(x)$; if the restrictions upon the $\{h_\nu\}$ are dropped, the shape of the domain containing series equal to $f(x)$ is more ragged. Furthermore, with the $\{g_\nu\}$ restricted as above, identical series $f_{i,j}(x)$ are not confined to square blocks. When the sequence g is permitted to contain gaps still greater variety in the behaviour of the $f_{i,j}(x)$ is exhibited; in this case, it may even occur that one such series is fortuitously equal to $f(x)$, although no other series of the entire array achieves complete agreement with $f(x)$; equivalence between differing series $f_{i,j}(x)$ can also, in this case, be more unsystematic.

When the variable x in the algorithm of formula ( ) is changed to t, and the numbers $\{g_\nu\}$ are taken to be formal power series $\{\mathcal{G}_\nu(x)\}$ in a new variable x, the series $g(x)$ becomes a double series $\mathcal{G}(t,x) = \sum_\nu \mathcal{G}_\nu(x)t^\nu$, the series $f_{i,j}(x)$ become double series $\mathcal{F}_{i,j}(x) = \sum_\nu h_\nu^{(i,j)}\mathcal{G}_\nu(x)t^\nu$ defined by

$$\mathcal{F}_{i,j}(x) = \sum_0^{m(i,j)} \alpha_\nu^{(i,j)}\mathcal{G}_\nu(x)t^\nu + \sum_0^{n(i,j)}\sum_0^{n(i,j;\nu)-1} A_{\nu,\tau}^{(i,j)}\mathcal{G}(\tau;t,x)$$

where

$$\mathcal{G}(\tau;t,x) = \mathcal{D}_t^\tau \mathcal{G}(t,x)/\tau!$$

and the associated series $\mathcal{F}(x)$ is $\sum_\nu h_\nu \mathcal{G}_\nu(x)t^\nu$. Naturally, all the results based upon the behaviour of the new series with respect to t, in particular those concerning the degenerate cases in which $h(z)$ is the series expansion of a rational function, and in which the $l_\nu(x)$ are identically zero for $\nu = \overline{I}_{\tau+1}$, hold as before. New results, however, may be derived from the assumption that the nonzero members of the sequence $\{l_\nu(x)\}$ are $l_{\mu(\nu)}(x)$ $(\nu = \overline{I}_0^{\tau})$ and that these series are successively deficient in powers of x lower than $F\{\mu(\nu)\}$ $(\nu = \overline{I}_0^{\tau})$, this latter sequence being strictly increasing. In Theorem 3 we show (i) that $\mathcal{F}(x)$ may be rearranged in the form $\sum_\nu \mathcal{F}_\nu x^\nu$ where the $\{\mathcal{F}_\nu\}$ are polynomials in t, and (ii) that the same holds for all series $\mathcal{F}_{i,j}(x)$ $(i,j = \overline{I})$; (iii) we also examine the degrees of equivalence between $\mathcal{F}(x)$ and the $\mathcal{F}_{i,j}(x)$ regarded as series in x.

The Padé table of quotients $P_{i,j}$ and the arrays of series $f_{i,j}(x)$ and $\mathcal{F}_{i,j}(x)$ are complete in the sense that entries for all values of $i,j = \mathbb{I}$ can be constructed. The functions $F_{i,j}(x)$ of formula ( ) involve values of the function $G(t,x)$ for $t = t_\nu^{(i,j)}$, and possibly of the derivatives of this function at these points. If $G(t,x)$ is prescribed only ~~at~~ over a limited set $t \in \Delta$, it is clear that the array of functions $F_{i,j}(x)$ may be incomplete in the sense that those functions defined in terms of numbers $t_\nu^{(i,j)}$ not in Δ are not well determin[ed]. Nevertheless it is possible to construct an extensive theory of the functions $F_{i,j}(x)$ even when Δ is considerably restricted. Firstly, as described above, it is known that when the h_ν are moments of the form ( ), then when $\alpha \geq 0$ all $\{t_\nu^{(i,j)}\}$ with $j = i+m-1$ $(i, m = \mathbb{I})$ belong to $[\alpha, \beta]$; furthermore the roots of the denominators $\tilde{\pi}^{(i,j)}(z)$ for these values of i and j

are simple. For general positions of $[\alpha,\beta]$, these considerations still hold for the subsets corresponding to $j = i+2m-1$. Thus if it is assumed only that $G(t,x)$ is uniquely defined and finite for $t \in [\alpha,\beta]$, we know that the infinite subsets of functions $F_{i,2m-1}(x)$ or $F_{i,i+2m-1}(x)$ $(i, m \in \mathbb{I})$ can be constructed. Further knowledge of the location of the numbers $t_\nu^{(i,j)}$ derives from the convergence theory of the Padé table. If it is known that a prescribed sequence of quotients $P_{i,j}(z)$ converges uniformly over a domain $\mathbb{D}$ in the z-plane, the quotients must ultimately be devoid of poles in this domain, i.e. for all quotients belonging to the tail of such a sequence, $t_\nu^{(i,j)^{-1}} \notin \mathbb{D}$. By inversion of $\mathbb{D}$ we obtain an inclusion domain for the numbers $t_\nu^{(i,j)}$. Certain results are readily available without any knowledge of the $t_\nu^{(i,j)}$. For example, if $g(x)$ is analytic in the neighbourhood of the origin, the same is true of $g(t_\nu^{(i,j)}x)$, and hence this is also true of all functions $f_{i,j}(x)$ of formula ( ); again, if $g(x)$ is entire, all $f_{i,j}(x)$ are entire.

As a preliminary to the derivation of special convergence results we give a number of existence and structural results based upon assumptions fulfilled in various ways in subsequent theorems. The notation used in general allows for the sets of points involved to be segments of the real axis (or, segments of rays) or domains in the complex plane. In Theorem 4 A.(i/a) we make assumptions concerning the numbers $t_\nu^{(i,j)}$ associated with a subset $\mathcal{P}_S$ of the quotients $P_{i,j}$ and concerning $G(t,x)$, and deduce the existence of a corresponding subset $\mathcal{F}_S$ of functions $F_{i,j}(x)$; b) we show that if $F_{i,j}(x)$ exists and $P_{i,j}$ belongs to a block in the Padé table derived from

$h(z)$, then a corresponding block of well defined and equal functions exists in the array $\{F_{i,j}(x)\}$; c) we show that if $G(t,x)$ is an entire function of t, and all $G_\nu(x)$ are uniquely defined and finite, then the array $\{F_{i,j}(x)\}$ is complete. (ii) We discuss the degenerate structures of the arrays $\{F_{i,j}(x)\}$ in the cases in which a) $h(z)$ is the expansion of a rational function and b) $G(t,x)$ is a polynomial in t. (iii) Assuming the functions $F(x)$ and $F_{i,j}(x)$ to be asymptotically represented by the series $\sum h_\nu G_\nu(x)$ and $\sum h_\nu^{(i,j)} G_\nu(x)$, we examine the asymptotic equivalence between these two functions. (iv) a) Making assumptions similar to those of (iii) we show that certain functions of the array $\{F_{i,j}(x)\}$ are mutually distinct and differ from $F(x)$, and b) do the former assuming ~~$G(t,x)$~~ that $G(t,x)$ does not satisfy a functional equation of a certain form. (vi) We give conditions upon $G(t,x)$ sufficient to ensure that the series $\sum h_\nu^{(i,j)} G_\nu$ converges to $F_{i,j}(x)$ and (vii) give further conditions to ensure that $F_{i,j}(x)$ is analytic over a prescribed domain B. We represent the foregoing results in slightly simpler terms with reference to the functions $f_{i,j}(x)$.

We remark that many of the structural results derived in Theorem 2 for formal power series have counter parts in Theorem 7.

The first special convergence result concerns the case in which $h(z)$ is the series expansion of the product $h(z)$ of an exponential function $e^{\nu z}$ and a convergent product of linear functions of z; $h(z)$ is an entire function $z^{-1}h(z^{-1})$ has a Laurent expansion converging for all z^{-1} bounded away from the origin. If $G(z,x)$ is regular at the origin, the integral

$$F(x) = \frac{1}{2\pi i}\int_C z^{-1} G(z,x) h(z^{-1}) dz$$

where $\mathbb{C}$ is a sufficiently small circle enclosing the origin and described in an anticlockwise direction, exists. If $\sum G_\nu(x) z^\nu$ is the Maclaurin expansion of $G(z,x)$, then the series $\sum h_\nu G_\nu(x)$ converges to $F(x)$, which is now represented both by the integral ( ) and this series. At this point the first motif occurring in the convergence proofs based on complex variable theory is introduced. Any infinite sequence of Padé quotients $P_{i,j}(z)$ derived from $h(z)$ with distinct integer suffix sets converges uniformly to $h(z)$ in, in particular, any circular region of bounded radius [ ]. For sufficiently large $i+j$, the poles of all functions $P_{i,j}(z^{-1})$ relating to such a sequence lie within $\mathbb{C}$. The conditions imposed upon $G(z,x)$ suffice to ensure that the corresponding functions $F_{i,j}(x)$ are given by the integral ( ) with $h(z^{-1})$ replaced by $P_{i,j}(z^{-1})$, and as such are well defined by formula ( ). Furthermore, since the circumference of $\mathbb{C}$ is of bounded length, the convergence of the $P_{i,j}(z^{-1})$ implies that of the corresponding $F_{i,j}(x)$ to $F(x)$. One of the sequences under consideration is that obtained by traversing the entire set of functions $F_{i,j}(x)$ in systematic order: we may say that the array itself converges to $F(x)$, and the notation used suggests this fact.

With reference to the functions $f_{i,j}(x)$, we assume that $g(x)$ is analytic in the neighbourhood of the origin, and set

$$f(x) = \frac{1}{2\pi i}\int_{\mathbb{C}} z^{-1} g(zx) k(z^{-1}) dz$$

where $\mathbb{C}$ is as described above. Now $f(x)$ is an entire function and $f(x) = \sum h_\nu g_\nu x^\nu$. All $f_{i,j}(x)$ for sufficiently large $i+j$ are analytic within a prescribed bounded domain containing the origin, and are well defined over this domain by formula ( ). Convergence of the $f_{i,j}(x)$ to $f(x)$ follows as above.

In Theorem 5.A.(i)(x) we prove convergence of $\sum h_\nu G_\nu(x)$ to $F(x)$ and b) give the special forms of $F(x)$ and $\{F_{i,j}(x)\}$ when $h(z)$ is a polynomial in z and c) when $G(t,x)$ is a polynomial in t. d) We prove convergence of the array $\{F_{i,j}(x)\}$ to $F(x)$, ~~and~~ e) and of the series $\sum h_\nu^{(i,j)} G_\nu(x)$ to $F_{i,j}(x)$ for sufficiently large $i+j$. f) Assuming $G(z,x)$ to be represented asymptotically with respect to x uniformly for all z in the neighbourhood of the origin by the series $\sum G_\nu(x) z^\nu$, we investigate the asymptotic equivalence of $F(x)$ and the $F_{i,j}(x)$, g) by imposing further conditions upon the $G_\nu(x)$, show that certain of the $F_{i,j}(x)$ are mutually distinct and differ from $F(x)$, and h) do the former assuming that $G(t,x)$ does not satisfy a functional equation of a certain form. §(i) We give conditions upon $G(t,x)$ sufficient to ensure that the

$F_{i,j}(x)$ with sufficiently large i, j are analytic. (ii) We apply the foregoing to the functions $f(x)$ and $f_{i,j}(x)$. B. The preceding theory is extended to the case in which $h(z)$ contains as a factor also the reciprocal of a convergent product of linear terms ($h(z)$ is now meromorphic in any bounded domain, and additional conditions upon $G(z,x)$ and $g(x)$ must be imposed).

When the h_ν are moments of the form ( ), much, as has been remarked above, can be said concerning the location of the numbers $\{t_\nu^{(i,j)}\}$ and their multiplicity, without any knowledge of the convergence behaviour of the $P_{i,j}(z)$. We may give an extensive structural theory of the arrays $\{F_{i,j}(x)\}$ and $\{f_{i,j}(x)\}$ before investigating convergence, and in view of the extent of this theory are wise to do so. With these arrays of functions, we associate the functions

$$F(x) = \int_{\alpha}^{\beta} G(t,x)\,d\sigma(t)$$

and

$$f(x) = \int_{\alpha}^{\beta} g(t,x)\,d\sigma(t)$$

In theorem 6(i)a) we assume that $G(t,x)$ is finite for $t=[\alpha,\beta]$ and is asymptotically represented with respect to x uniformly for $t=(\alpha,\beta)$ by the series $\sum G_{\nu}(x)t^{\nu}$ and show that $F(x)$ is asymptotically represented by the series $\sum h_{\nu}G_{\nu}(x)$; b) under other conditions upon $G(t,x)$, we show that this series converges to $G(t,x)$. (ii) We give the special forms of $F(x)$ and $\{F_{i,j}(x)\}$ a) when σ is a simple step function with a finite number of salti and b) when $G(t,x)$ is a polynomial in t. (iii) Assuming only that $G(t,x)$ is uniquely defined and finite for $t=[\alpha,\beta]$ we give the subsets of $\{F_{i,j}(x)\}$ which are then known to be well determined by formula ( ). (iv) We give further conditions ~~upon $G(t,x)$~~ similar to those of (i a) and b) upon $G(t,x)$ sufficient to ensure that the functions $F_{i,j}(x)$ of these subsets are a) asymptotically and b) directly represented by the series $\sum_\nu h_\nu^{(i,j)} \varphi_\nu(x)$ Using the results of (iv)a) we (v) investigate the asymptotic equivalence of $F(x)$ and the $F_{i,j}(x)$ concerned and (vi)a) in turn using these results establish that certain of these $F_{i,j}(x)$ are mutually distinct and differ from $F(x)$ and b) we also do the former assuming that $G(t,x)$ does not satisfy a certain functional equation (vii) We describe the form of the $\{F_{i,j}(x)\}$ when $G(t,x)$ is an entire function of t, and extend the results of (v) and (vi) in this case (viii) We give conditions upon $G(t,x)$ sufficient to ensure that the subsets of $\{F_{i,j}(x)\}$ above are analytic. B. More extensive results for the function $f(x)$ and array $\{f_{i,j}(x)\}$ following the scheme of part A are given.

The second motif recurring in the proofs of convergence of the functions $F_{i,j}(x)$ is revealed in the investigation of the functions introduced above. Letting $[\alpha,\beta]=[0,\infty]$, simplifying the discussion by assuming that σ is not a simple step function with a finite number of salti, denoting the zeros of the successive orthogonal polynomials $p_i(t)$ $(i=\overline{1}_1)$ derived in this case from the moments h_ν by $t_\nu^{(i,i-1)}$ $(\nu=\overline{1}_1^i)$ (these lie in $(0,\infty)$ and are distinct) and the corresponding weights by $M_\nu^{(i,i-1)}$ $(\nu=\overline{1}_1^i)$, the functions $F_{i,i-1}(x)$ $(i=\overline{1}_1)$ may be expressed as

$$F_{i,i-1}(x)=\sum_0^i M_\nu^{(i,i-1)}G(t_\nu^{(i,i-1)},x)$$

It is a consequence of a theorem of Riesz [ ], that if the Stieltjes moment problem associated with the sequence $\{h_\nu\}$ is determinate, and the function $G(t)$ is continuous over every bounded open subinterval of $[0,\infty]$ and tends to a finite limit, as t tends to infinity, then the sequence of numbers

$$F_i = \sum_{\nu=1}^{i} m_\nu^{(i,i-1)} G(t_\nu^{(i,i-1)})$$

converges to the value of the integral $F = \int_0^\infty G(t)\,d\sigma(t)$. If $G(t,x)$ possesses the properties just attributed to $G(t)$, we immediately have a convergence result for the functions $F_{i,i-1}(x)$. If we assume that the series $\sum G_\nu(x) t^\nu$ represents $G(t,x)$ asymptotically now with ~~with~~ respect to t as t tends to 0 through positive real values, we obtain convergence results for further diagonal sequences $F_{i,i+m-1}(x)$ $(m \geq 1)$. If $[\alpha,\beta] \subset (0,\infty]$, the further assumptions can be dropped (the series $\sum G_\nu(x) t^\nu$ and the function $G(t,x)$ are now unrelated); if $[\alpha,\beta] \subset [0,\infty)$, the Stieltjes moment problem associated with the m_ν is automatically determinate, and assumptions concerning the limiting behaviour of $G(t,x)$ as t tends to infinity can be discarded. Riesz' result can also be extended to the theory of the Hamburger moment problem, where now $[\alpha,\beta]\subseteq[-\infty,\infty]$ and in the same way yields convergence results for the diagonal sequences $F_{i,i+2m-\nu}(x)$ ($\nu \in \mathrm{II}$). Convergence results derived in this way belong to the province of real variable theory. Assumptions concerning the behaviour of $G(t,x)$ over the real interval $t\in[\alpha,\beta]$ are made. That this function may also be defined for complex values of t and x is dispensable.

For proofs of convergence concerning the $F_{i,j}(x)$ with $i>j$, however, we must fall back upon the complex variable methods outlined earlier. When $[\alpha,\beta]=[0,\infty]$, the contour $\mathbb{C}$ in formula ( ) is composed of two lines, between which the positive real axis lies, joined by a small semi-circle about the origin; the series $\sum G_\nu(x)z^\nu$ now represents $G(t,x)$ for small values of $\frac{1}{z}$. For the integral ( ) to exist, it is necessary to impose the condition that $G(z,x)$ tends to zero in a certain way as z tends to infinity on $\mathbb{C}$. It is shown that the functions $F(x)$ and $F_{i,j}(x)$ may be uniformly approximated by integrals of the form ( ) and ( ) in which $\mathbb{C}$ is a contour of finite length, and the remainder of the convergence proof follows lines sketched earlier. Modifications of this argument are used for other positions of $[\alpha,\beta]$.

In Theorem 7.A, we use the real and complex variable methods described above to prove convergence of diagonal sequences of functions $F_{i,j}(x)$ in the cases in which (i) $[\alpha,\beta] = [0,\infty]$, (ii) $[\alpha,\beta] \subset (0,\infty]$, (iii) $[\alpha,\beta] \subset [0,\infty)$ (iv) $[\alpha,\beta] \subset (0,\infty)$ (v) $[\alpha,\beta] \subseteq [-\infty,\infty]$ (vi) $[\alpha,\beta] \subset (-\infty,\infty)$. B. We derive some structural results further to those of Theorem 6 using knowledge of the location of the numbers $\{t_\nu^{(i,j)}\}$ deriving from convergence theory. The results of Theorem 7 are applied to the functions $f(x)$ and $f_{ij}(x)$ of formulae ( ) and ( ) and in Theorem 8.

The completely worked out theory of this paper is original; nevertheless hints at it are already contained in the literature.

The first of these is provided by the convergence result of Riesz quoted above. In this, we have introduced an extra variable into the function $G(t)$ in such a way that the result now concerns functions $F_{i,i-1}(x)$ related to the Padé quotients $P_{i,i-1}(z)$, we have extended this new convergence result to related functions associated with general diagonal sequences of quotients, and we have shown how to deal with the cases in which the roots of the orthogonal polynomials concerned (i.e. the poles of the quotients) are complex and multiple.

A second special case of our theory has been given by Ferris Prabhu and Withers [ ], who use results for the Laplace integral formally expressed by the relationships

$$\mathcal{L}\{x^\nu\} = z\int_0^\infty e^{-x/z} x^\nu dx = \nu! z^\nu \qquad (\nu \in \mathbb{I})$$

and $\mathcal{L}^{-1}\{(1-tz)^{-1}\} = \exp(tx)$. Given the series $\Sigma f_\nu x^\nu$, they obtain $\mathcal{L}\Sigma f_\nu x^\nu = \Sigma h_\nu z^\nu$ where $h_\nu = \nu! f_\nu (\nu \in \mathbb{I})$, construct the quotients $P_{i,j}(z)$ with $i > j$, assume that these may be decomposed in the form

$$P_{i,j}(z) = \Sigma_1^i A_\nu^{(i,j)} (1 - t_\nu^{(i,j)} z)^{-1}$$

and derive the exponential sums $\Sigma_1^i A_\nu^{(i,j)} \exp(t_\nu^{(i,j)} x)$. No convergence or other theory is given, but numerical examples relating to the series $\Sigma (1/\nu!)^2 x^\nu$ of Bessel type indicate that the exponential sums produced in the manner described provide excellent approximations to the sum of this series.

In our notation, they are using the function $g(x) = \exp(x)$ (with ~~$g(x) =$~~ $g_\nu = 1/\nu!$ $(\nu \geq 0)$) to construct the functions $f_{i,j}(x)$ of formula ( ) in this case. We dispense with the assumption that $P_{i,j}(z)$ must have a simple decomposition of the form ( ), derive approximations for unrestricted values of i and j and, in Theorem 4 A (ii) (with $h(z) = \exp(z)$) give a considerable structural and convergence theory relating to their example. A recent paper of Gearhart and Stenger [ ] contains ideas closely related to those of Farris Prabhu and Withers.

A third special case has been given by Baker [ ] who considers the functions $F_{i,j}(x)$ with $j \geq i-1$ when the h_ν have the form ( ) with $[\alpha, \beta] = [0, \infty]$ (again the poles of the associated quotients $P_{i,j}(z)$ are simple and the definition of the $F_{i,j}(x)$ is correspondingly restricted). He gives a convergence result based on the use of complex variable methods, assuming that $G(z,x)$ is an analytic function of z in the neighbourhood of the ~~origin~~ positive real axis with $G(z,x) = \sum G_\nu(x) z^\nu$ at the origin, and that $G(z,x)$ tends to zero relatively sharply as z tends to infinity. As remarked above, convergence in this case can be demonstrated by real variable methods, using Riesz theorem, making relatively mild assumptions concerning the behaviour of $G(t,x)$ (Theorem 7. A (i) a)). Complex variable methods, and the added assumptions which their use entails, are only required for the functions $F_{i,j}(x)$ with $i > j$ (Theorem 7. A(i) b)), and these are not considered in the paper referred to.

To facilitate the reader's task we give, at the end of the paper a glossary of the notations used.

Some general notations

Groups of special notations are introduced at points in the text immediately befor the first usage of ~~such a~~ member of such a group. However, apart from the general notations described at the commencement of § , others are also used systematically throughout the paper, and these we now give.

Notation 2. In a manner made obvious by the context, angular brackets are used to combine two propositions in one: much <nothing> is gained <lost> thereby. ~~Addition~~ Additionally full face square brackets [..] and double brackets ⟦..⟧ are used to accommodate further propositions.

α and β are real numbers (possibly infinite in magnitude unless otherwise indicated) with $\alpha<\beta$. $\bar{Z}\langle Z\rangle$ is the complete < the finite part of the > complex plane. $M\times M'$ is the set of points formed by taking all possible products of one element of M and one of M'. M^{-1} is the ~~set of~~ point set whose elements are the reciprocals of those of M. $N\{\Delta\}$ is the domain of points z in $\mathbb{Z}$ for which, $\delta \in (0,\infty)$ arbitrarily small being prescribed, $|z-z'| \leq \delta$ $(z' \equiv \Delta)$. $\mathbb{D}$ is always a closed domain in $\bar{\mathbb{Z}}$. $\mathbb{B}\{\mathbb{D}\}$ is any bounded open domain which, together with its boundary, is contained in $\mathbb{D}$. $\mathbb{B}$ is simply $\mathbb{B}\{\mathbb{Z}\}$. Δ_{ϕ}^{ψ}, where it is tacitly assumed that $\phi \leq \psi \leq \phi + 2\pi$ is the closed sector in the complex plane containing the points z for which $\arg(z) \in [\phi, \psi]$. $\Delta_{\phi'}^{\psi'} \geq \Delta_{\phi''}^{\psi''}$ means that $\psi' - \phi' \geq \psi'' - \phi''$. $\Delta[\rho, \omega]_{\phi}^{\psi}$, where it is tacitly assumed that ~~$-\infty, \phi \leq$~~ $-\infty < \phi \leq \psi \leq \phi + 2\pi < \infty$ and that $\rho(\Theta)$ and $\omega(\Theta)$ are respectively nonnegative and positive single valued real functions for $\Theta \equiv [\phi, \psi]$ is a set of points defined as follows: a) if $\phi = \psi$, then $\rho(\phi) < \omega(\phi)$ and the set of points consists of all points x of the ray $\arg(x) = \phi$, $\rho(\phi) \le |x| \le \omega(\phi)$; b) if $\phi < \psi < \phi + 2\pi$, $\rho(\theta)$ and $\omega(\theta)$ are assumed to be piecewise continuous with $\max\{\rho(\theta-), \rho(\theta+)\} < \min\{\omega(\theta-), \omega(\theta+)\}$ for $\theta = (\phi, \psi)$ where, for example, $\omega(\theta-)$ is the limit as $\delta (>0)$ tends to zero of $\omega(\theta - \delta)$ and $\rho(\phi) \le \omega(\phi)$, $\rho(\psi) \le \omega(\psi)\}$, and the set is then the domain of points x in the complex plane for which $\arg(x) = \phi$, $\rho(\phi) \le |x| \le \omega(\phi)$; $\arg(x) = \theta$, $\min\{\rho(\theta-), \rho(\theta+)\} \le |x| \le \max\{\omega(\theta-), \omega(\theta+)\}$ $(\theta = (\phi, \psi))$; $\arg(x) = \psi$, $\rho(\psi) \le |x| \le \omega(\psi)$; c) if $\psi = \phi + 2\pi$, it is assumed further that $\rho(\phi) < \omega(\phi + 2\pi)$, $\rho(\phi + 2\pi) < \omega(\phi)$, the members of the set belonging to the ray $\arg(x) = \phi$ are determined by the condition $\min\{\rho(\phi + 2\pi), \rho(\phi)\} \le |x| \le \max\{\omega(\phi + 2\pi), \omega(\phi)\}$, the members of the set $\arg(x) = \psi$ are the same as these and need not be specified, and the members of the set belonging to the rays $\arg(x) \equiv (\phi, \psi)$ are as described in b); 0 is written in place of ρ if $\rho(\theta)$ is identically zero, and ∞ in place of ω if $\omega(\theta)$ is identically infinite. When use is made of the symbol $\Delta[\rho, \omega]_{\phi}^{\psi}$ the convention is adopted that $|x|\rho$ is the function $|x|\rho(\theta)$, and that further symbols of this kind have similar meanings. $\mathbb{D}[\omega]$ is, using the previous notation, the closed domain $\Delta[0, \omega]_{0}^{2\pi}$. $\bar{\mathbb{D}}_{\lambda}$ is the closed disc $\mathbb{D}[\omega]$ where $\omega(\theta) = \lambda \in (0, \infty)$ $(\theta \equiv [0, 2\pi])$; $\mathbb{D}_{\lambda}$ is the open disc corresponding to $\bar{\mathbb{D}}_{\lambda}$.

The function f being prescribed, $f(M)$ is the mapping $f(x)$ $(x \equiv M)$. A similar convention is ~~adde~~ adopted with regard to arrays and sequences of functions: if $\mathbb{F}$ represents $F_{i,j}$ $(i, j \equiv \mathbb{I})$, $\mathbb{F}(M)$ represents $F_{i,j}(x)$ $(i, j \equiv \mathbb{I}, x \equiv M)$.

The lower case script letters $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}, \mathfrak{f}_{i,j}, \mathfrak{h}_{i,j}, \mathfrak{h}'$ denote the sequences of elements of Z whose members are $f_\nu, g_\nu, h_\nu, f_\nu^{(i,j)}, h_\nu^{(i,j)}, h'_\nu$ $(\nu \equiv I)$ respectively. In particular, it is tacitly assumed with regard to $\mathfrak{h}$ that $h_0 \neq 0$. $\mathfrak{G}$ is the sequence of functions G_ν $(\nu \equiv \underline{I})$. Products of sequences are defined in terms of corresponding elements: the elements of $\mathfrak{h}\mathfrak{g}$ and $\mathfrak{h}\mathfrak{G}(M)$ are $h_\nu g_\nu$ and $h_\nu G_\nu(M)$ ~~respectively~~ $(\nu \equiv \underline{I})$ respectively.

$g(\mathbb{D}) \in A$ means that $g(z)$ is an analytic function of the complex variable z for $z \equiv \mathbb{D}$. This notation is applied to sequences and arrays of functions: $\mathfrak{G}(\mathbb{D}) \in A$ means that $G_\nu(\mathbb{D}) \in A$ $(\nu \equiv \underline{I})$. Analyticity with respect to a variable of functions of more than one ~~variable~~ argument is indicated by indicating the argument in question and subscripting the Gothic letter A appropriately: $G(t \equiv \mathbb{D}, M) \in A_t$ means that $G(t,x)$ is an analytic function of the variable t for $t \equiv D$ for $x \equiv M$; $G(t \equiv D, x \equiv D') \in A_{t,x}$ means that $G(t \equiv D, D') \in A_t$ and $G(D, x \equiv D') \in A_x$ conjointly.

$g \leftarrow g$ where $G(N\{0\}) \in A$ means that $\sum g_\nu x^\nu$ is the Maclaurin expansion of the function $g(x)$ in the neighbourhood of the origin. $G \leftarrow \mathcal{G}(M)$, where $G(t \equiv N\{0\}, M) \in A_t$, means that $\sum G_\nu(x) t^\nu$ is the Maclaurin expansion in powers of t of the function $G(t,x)$ for $x \equiv M$.

Convergence over a point set is ~~..~~ indicated by use of a forward arrow and by replacing the dummy argument with respect to which convergence holds by the point set in question: $\sum h_\nu^{(i,j)} g_\nu D^\nu \to f^{(i,j)}(D)$ means that the series $\sum h_\nu^{(i,j)} g_\nu x^\nu$ converges to the sum function $f^{(i,j)}(x)$ for $x \equiv D$.

$\overline{i,j}$ denotes a pair of integer suffixes, as, for example, in the symbol $P_{i,j}$; $[I,J;N]$, where $I, J \in \mathbf{I}$, $N \in \bar{\mathbf{I}}$ denotes the block of suffix pairs $i \equiv \mathbf{I}_I^{I+N}$, $j \equiv \mathbf{I}_J^{J+N}$; $\overline{i,j} \in [I,J;N]$ means that $i \in \mathbf{I}_I^{I+N}$, $j \in \mathbf{I}_J^{J+N}$.

If symbols are defined above subject to certain accompanying restrictions, it is tacitly assumed during use of such a symbol that these restrictions hold; thus use of the symbol $[\alpha,\beta]$ implies that $\alpha<\beta$.

With regard to the symbol $\Delta[\rho,\omega]_\phi^\psi$, we remark that this can, when $\phi=\psi$, be a segment of a ray (for example, $\Delta[0,\infty]_0^0$ is the nonnegative real axis). If $\phi=0$, $\psi=2\pi$, $\rho(\theta)=0$ ($\theta\in[0,2\pi]$) (when it reduces to $D[\omega]$) the symbol, by appropriate choice of ω, can represent a star shaped domain. If, in the preceding, ρ assumes nonzero values, the symbol then represents a star shaped domain from which a central region has been removed

Lastly, if the numbers of junctions concerned are arranged in an array in which i and j denote row and column numbers respectively, $[I, J; N]$ refers to entries in this array occurring in a square block of $(N+1)^2$ members.

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The Padé table

Notation. $H_{i,j}$ ($i \in \mathbb{I}_1$, $j \in \mathbb{I}$) is the value of the Hankel determinant [ ] whose τ^{th} row, for $\tau = \mathbb{I}_1^i$, consists of the elements $h_{j-i+\nu}$ ($\nu = \mathbb{I}_\tau^{\tau+i-1}$) where we set $h_\nu = 0$ when $\nu < 0$; we set $H_{0,j} = 1$ ($j = \mathbb{I}$)

Given the formal power series $h(z) = \sum h_\nu z^\nu$ or, equivalently, the sequence h, the Padé quotient $P_{i,j}$ of order (i,j) ($i,j \in \mathbb{I}$) is derived in the following way: a set of coefficients $\pi_\nu^{(i,j)}$ ($\nu = \mathbb{I}_0^j$) (not all zero) is derived from the equations

$$\sum_0^i h_{\tau-\nu} \pi_\nu^{(i,j)} = 0 \qquad (\tau = \mathbb{I}_{j+1}^{j+i};\ h_\nu = 0\ (\nu < 0))\ (\ )$$

(such a set may always be determined: equations ( ) constitute a homogeneous system of i equations in $i+1$ unknowns); thereafter, a further set $v_\nu^{(i,j)}$ ($\nu = \mathbb{I}_0^j$) is constructed by use of the formula

$$v_\tau^{(i,j)} = \sum_0^\tau \pi_\nu^{(i,j)} h_{\tau-\nu}. \qquad (\tau = \mathbb{I}_0^j;\ \pi_\nu^{(i,j)} = 0\ (\nu > i))(\ )$$

The two polynomials $\sum_0^j v_\nu^{(i,j)} z^\nu$ and $\sum_0^i \pi_\nu^{(i,j)} z^\nu$ may have common factors. The rational function $P_{i,j}(z)$ is the irreducible quotient $v^{(i,j)}(z) / \pi^{(i,j)}(z)$ with $\pi^{(i,j)}(0) = 1$ equivalent to $\sum_0^j v_\nu^{(i,j)} z^\nu / \sum_0^i \pi_\nu^{(i,j)} z^\nu$; the degree of $v^{(i,j)}(z)$ $\langle \pi^{(i,j)}(z) \rangle$ is thus $\leq j \langle i \rangle$.

$P_{i,j}$ is the ordered pair of sets of coefficients $\{v_\nu^{(i,j)}\}$ $\{\pi_\nu^{(i,j)}\}$. Whether or not the series $h(z)$ converges for nonzero values of z, we always have $P_{i,j}(N\{0\}) \in A$, and $P_{i,j}(0) = v_0^{(i,j)} = h_0$. The quotients $P_{i,j}$ $(i,j = \mathbb{I})$ may be set in a two dimensional array, the Padé table, in which $i \langle j \rangle$ denotes a row $\langle$column$\rangle$ number.

Notation. H is the complete array of Hankel determinants $H_{i,j}$ $(i,j = \mathbb{I})$. $H_{I,J}[N] \subseteq H$ $(I, J \in \mathbb{I};\ N \in \overline{\mathbb{I}})$ means that ~~$H_{I\geq r, J\geq r}$~~ H possesses the square block of $(N+1)^2$

determinants $H_{i,j}$ $(\overline{i,j} = [I,J;N])$ of which $H_{I+r,J+s} = 0$ $(r,s = \overline{\mathbb{I}}_1^N)$ and $H_{I,J}$, $H_{I,J+1}$, $H_{I+1,J}$, $H_{I+N+1,J+N+1} \neq 0$. $\boxed{N \in \mathbb{I}}$.

$P_{i,j}$ $(i,j \in \mathbb{I})$ is the Padé quotient of order (i,j) derived from the series $\sum h_\nu z^\nu$. $\mathbb{P}$ is the complete array of quotients $P_{i,j}$ $(i,j = \mathbb{I})$. $P_{I,J}[N] \leq \mathbb{P}$ means that $\mathbb{P}$ possesses the square block of $(N+1)^2$ identical quotients $P_{i,j}$ $(\overline{i,j} = [I,J;N])$ with $P_{I-1,J} \neq P_{I,J}$ $(I \in \mathbb{I}_1)$, $P_{I,J-1} \neq P_{I,J}$ $(j \in \mathbb{I}_1)$, $P_{I+N+1,J+N+1} \neq P_{I,J}$ $(N \in \mathbb{I})$. $v^{(i,j)}(z)$ $\langle \pi^{(i,j)}(z) \rangle$ is the numerator $\langle$denominator$\rangle$ of $P_{i,j}(z)$ (see above). $t\{P_{i,j}\}$ is the set of distinct nonzero numbers $t_\nu^{(i,j)}$ $(\nu = \overline{\mathbb{I}}_1^{n(i,j)})$ occurring in the factorisation $\pi^{(i,j)}(z) = \prod_1^{n(i,j)} (1 - t_\nu^{(i,j)} z)^{n(i,j;\nu)}$; $\overline{t}\{P_{i,j}\}$ is $t\{P_{i,j}\}$ augmented by zero if $j \geq i$; $n\{P_{i,j}\}$ is $\max\{n(i,j;\nu) - 1\}$ $(\nu = \overline{\mathbb{I}}_1^{n(i,j)})$. We set $P_{i,j} \leftarrow h^{(i,j)}$ $(i,j = \mathbb{I})$. $h_{I,J}(z)$ is the rational function of formula ( ).

$\xi(i,j)$ $(i,j \in \mathrm{I})$ is the nonnegative integer defined by the relationships $h_\nu^{(i,j)} = h_\nu$ $(\nu = \mathrm{I}_0^{\xi(i,j)-1})$, $h_{\xi(i,j)}^{(i,j)} \neq h_{\xi(i,j)}$. Ξ is the array of integers $\xi(i,j)$ $(i,j = \mathrm{I})$. $\Xi_{I,J}[N] \subseteq \Xi$ $(I,J \in \mathrm{I}, N \in \overline{\mathrm{I}})$ means that Ξ possesses the square block of $(N+1)^2$ equal integers $\xi(i,j)$ $(\overline{i,j} = [I,J;N])$ with $\xi(I-1,J) \neq \xi(I,J)$ $(I \in \mathrm{I}_1)$, $\xi(I,J-1) \neq \xi(I,J)$ $(J \in \mathrm{I}_1)$ $\xi(I+N+1, J+N+1) \neq \xi(I,J)$ $(N \in \mathrm{I})$ with $\xi(I,J) = \infty$ if $N = \infty$.

We remark that $H_{I,J+1}$, $H_{I+1,J}$ only belong to $H_{I,J}[N]$ if $N \in \overline{\mathrm{I}}_1$, and that $H_{I+N+1,J+N+1}$ $(N \in \mathrm{I})$ does not belong to $H_{I,J}[N]$. The integer $\xi(i,j)$ is, of course, a measure of the extent to which the series expansion $\sum h_\nu^{(i,j)} z^\nu$ of $P_{i,j}(z)$ agrees with $\sum h_\nu z^\nu$. Ξ provides a complete picture of the extent to which the ensemble of these series agree with

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$\sum h_v z^v$.

Theorem .(i)a) If $H_{I,J}[N] \subseteq H$, then $H_{I+r,J} \neq 0$ $(r = \mathbb{I}_0^N)$, $H_{I,J+s} \neq 0$ $(s = \mathbb{I}_1^N)$ and if $N \in \mathbb{I}$, $H_{I+r,J+N+1} \neq 0$ $(r = \mathbb{I}_0^{N+1})$, $H_{I+N+1,J+s} \neq 0$ $(s = \mathbb{I}_0^N)$

b) H consists of ~~zero~~ square blocks of the form $H_{I,J}[N]$, some possibly of zero order, one possibly of infinite order (i.e. zero elements in H, if they occur at all, are to be found in square sub-blocks of N^2 members $\in H_{I,J}[N]$, bordered on all sides by nonzero elements, the elements belonging to two of the sides meeting at $H_{I,J}$ ~~pt~~ together with the zero elements constituting $H_{I,J}[N]$.

c) $H_{I,J}$ belongs to its own block of zero order if and only if $H_{I,J}, H_{I,J+1}, H_{I+1,J}, H_{I,J+1} \neq 0$.

d) Two zero determinants $H_{i',j'}, H_{i'',j''} \in H$ are either members of the same block as described in c), or are distant from each other by at least two steps (i.e. either $|i'-i''| \geq 2$ or $|j'-j''| \geq 2$).

e) $H_{I,J}[\infty] \subseteq H$ if and only if $h_{I,J} = h$.

(ii) a) If $H_{i,j} \neq 0$, equations (,) serve directly to determine the numerator and denominator polynomials of $P_{i,j}$ ($v^{(i,j)}(z)$ and $\pi^{(i,j)}(z)$ have no common polynomial factor); furthermore, in this case $\pi_i^{(i,j)} = (-1)^i H_{i,j+1}/H_{i,j}$, $v_j^{(i,j)} = (-1)^i H_{i+1,j}/H_{i,j}$.

b) $P_{I,J}[N] \subseteq P$ if and only if $H_{I,J}[N] \subseteq H$

c) P consists of square blocks of the form $P_{I,J}[N]$, some possibly of zero order, one possibly of infinite order.

d) If $P_{I,J}[N] \subseteq P$, then $P_{I,J}$ (to which all $P_{i,j} \in P_{I,J}[N]$ are equal) is an irreducible quotient whose numerator is of degree J and whose denominator is of degree I.

f) Let $P_{I,J}[N] \in \mathbb{P}$ with $m(I,J) = J - I + 1 \in \mathbb{I}$. Then $P_{i,j}$ has the decomposition

$$P_{i,j}(z) = \sum_{0}^{m(I,J)-1} h_\nu z^\nu + \frac{z^{m(I,J)} \hat{v}^{(I,J)}(z)}{\pi^{(I,J)}(z)} \quad (P_{i,j} \equiv P_{I,J}[N])$$

~~where $\hat{v}^{(i,j)}(z) = \hat{v}^{(I,J)}(z)$, $\pi^{(i,j)}$~~

where $\hat{v}^{(I,J)}(z)$ is a polynomial of degree at most $I-1$, and $\pi^{(I,J)}(z)$ is a polynomial of degree I, and $\hat{v}^{(I,J)}(0) = h_{m(I,J)}$.

g) Let $P_{I,J}[N] \subseteq \mathbb{P}$. Then $P_{i,j}(z)$ has the decomposition ( ) $(P_{i,j} \equiv P_{I,J}[N])$ where the $m(i,j), n(i,j), \{n(i,j;\nu)\}$ $\{\alpha_\nu^{(i,j)}\}$ $\{A_\nu^{(i,j)}\}$ all have the values obtaining for $i = I$ $j = I$, and $m(I,J) = 0$ for $I > J$ (i.e. the first sum in ( ) is missing if $I > J$).

(iii) a) If $H_{i,j} \neq 0$, then $\xi(i,j) \geqslant i+j$

b) If $H_{i,j}, H_{i+1,j+1} \neq 0$, then $\xi(i,j) = i+j$

c) $\xi(i,j) = I + J + N + 1$ for all $\xi(i,j) \in \Xi_{I,J}[N]$

d) $\xi(i,j)$ is a nondecreasing function of $i+j$

e) $\Xi_{I,J}[N] \subseteq \Xi$ if and only if $H_{I,J}[N] \in H$

f) $\Xi_{I,J}[\infty] \subseteq \Xi$ if and only if $h_{I,J} \ll h$.

g) If $\xi(i,j) = \xi(i',j')$ then either $i+j = i'+j'$ (i.e. $\xi(i,j)$ and $\xi(i',j')$ belong to the same backward diagonal in Ξ) or $\xi(i,j), \xi(i',j') \in \Xi_{I,J}[N]$ (i.e. they belong to the same block) or $\xi(i,j), \xi(i',j') \in \Xi_{I,J}[N], \Xi_{I',J'}[N']$ with $I+J+N = I'+J'+N'$ (i.e. they belong to two differing blocks possessing a common backward diagonal).

(iv) a) Let $H_{i,j}, H_{i+1,j+1} \neq 0$. Then $h^{(i,j)}_{i+j+1} \neq h^{(i+1,j+1)}_{i+j+1}$

b) Let $H_{i,j} \in H_{I,J}[N]$, $H_{i',j'} \in H_{I',J'}[N']$, the two blocks being distinct, with $\xi' = \min(I+J+N+1, I'+J'+N'+1)$.

(1) If $I+J+N \neq I'+J'+N'$, then $h^{(i,j)}_{\xi'} \neq h^{(i',j')}_{\xi}$. (2) If $I+J+N = I'+J'+N'$ (so that the two blocks have a common backward

diagonal) set $r' = \max(I'+J, I+J')$ ($r' - \xi' + 1$ is then the number of quotients lying on the backward diagonal and lying between the two blocks). Then for at least one $\nu \in \mathbb{I}_{\xi'}^{\xi'+r'-1}$, $h_\nu^{(i,j)} \neq h_\nu^{(i',j')}$.

c) Let $H_{i+1,j}, H_{i,j+1}, H_{i+1,j+1} \neq 0$. Then $h_{i+j+2}^{(i,j+1)} \neq h_{i+j+2}^{(i+1,j)}$.

Proof. (i) a, b) It may be shown (see [ ] Ch) by use of the recursion $H_{j-1,i} H_{j+1,i} - H_{j,i}^2 = H_{j-1,i-1} H_{i,j+1}$ $(i, j = \mathbb{I}_1)$ for Hankel determinants that contiguous zero elements in H occur only in square ~~blocks~~ sub-blocks of the form $H_{i,j}$ $(i, j = [I+1, J+1; N-1]$ belonging to the block $H_{I,J}[N]$ $(N \in \overline{\mathbb{I}}_1)$. c) This result follows from the definition of $H_{I,J}[0]$.

~~e~~d) According to a well known result of Kronecker [ ] ~~h~~ h where $h(z)$ is a rational function if and only if $H_{j+1,j} = 0$ $(j = \mathbb{I}_J)$ for some $J \in \mathbb{I}$. The sharper result of this clause is demonstrated by the use of methods similar to those employed by Kronecker.

(ii)a) If $H_{i,j} \neq 0$, equations ( ) in conjunction with the condition $\pi_0^{(i,j)} = 1$ serve uniquely to determine the set $\{\pi_\nu^{(i,j)}\}$; thereafter, the set $\{v_\nu^{(i,j)}\}$ is uniquely determined by formula ( ). (The polynomials $\sum_0^j v_\nu^{(i,j)} z^\nu$, $\sum_0^i \pi_\nu^{(i,j)} z^\nu$ then have no common polynomial factor ([ ][ ]Ch)). Denoting by $H_{i+1,j}[\phi_\nu]$ the determinant whose value is $H_{i+1,j}$ with the first row $t_{j-i+\tau}$ $(\tau = \overline{I}_0^j)$ replaced by the elements ϕ_τ $(\tau = \overline{I}_0^i)$, it is easily demonstrated that when $H_{i,j} \neq 0$, $v^{(i,j)}(z) = (-1)^i H_{i+1,j}\left[\sum_0^{\tau+j-i} h_\nu z^{i-\tau+\nu}\right] / H_{i,j}$, $\pi^{(i,j)}(z) = (-1)^i H_{i+1,j}[z^{i-\tau}] / H_{i,j}$, and examining the coefficient of the relevant power of z in each polynomial $v_j^{(i,j)}$ and $\pi_i^{(i,j)}$ are as stated. b) and c) are fundamental results of Padé ([ ], [ ]Ch). d) follows from (i)a) and (iib), and e) from (i)d) and (ii)b.

~~result of this clause is demonstrated by the use of methods similar to those employed by Kronecker.~~

8§) $P_{i,j} \equiv P_{I,J}$ $(i = \mathbb{I}_I^{I+N}, j = \mathbb{I}_J^{J+N})$. By division, we obtain $P_{I,J}(z) = \alpha^{(I,J)}(z) + \tilde{\upsilon}^{(I,J)}(z)/\pi^{(I,J)}(z)$ where $\alpha^{(I,J)}(z) = \sum_0^{J-1} \alpha_\nu^{(I,J)} z^\nu$, $\tilde{\upsilon}^{(I,J)}(z) = \sum_0^{I'-1} \tilde{\upsilon}_\nu^{(I,J)} z^\nu$ $(I' \in \mathbb{I}_1^I)$. If $I > J$, $P_{I,J}(z)$ is already available in the required form with $\tilde{\upsilon}^{(I,J)}(z) = \upsilon^{(I,J)}(z)$. If $J \geqslant I$, the $\{\alpha_\nu^{(I,J)}\}$ are obtained recursively from the equations $\sum_0^\tau \pi_{I-\tau+\nu}^{(I,J)} \alpha_{J-I-\nu}^{(I,J)} = \upsilon_{J-\tau}^{(I,J)}$ $(\tau \equiv \mathbb{I}_0^{J-I})$ (since $\pi_I^{(I,J)} \neq 0$, this is always possible); the coefficients $\{\tilde{\upsilon}_\nu^{(I,J)}\}$ are then determined from the formulae $\tilde{\upsilon}_\tau^{(I,J)} = \upsilon_\tau^{(I,J)} - \sum_0^\tau \pi_{\tau-\nu}^{(I,J)} \alpha_\nu^{(I,J)}$ $(\tau \equiv \mathbb{I}_0^{I'-1})$. Thereafter $\tilde{\upsilon}^{(I,J)}(z)/\pi^{(I,J)}(z)$ is decomposed into partial fractions.

(iii)a) If $H_{i,j} \neq$ then ((ii)a above) equations ( ) directly determine the set $\{\pi_\nu^{(i,j)}\}$ with $\pi_0^{(i,j)} = 1$. Furthermore

$\sum_0^i h_{\tau-\nu}^{(i,j)} \pi_\nu^{(i,j)} = v_\tau^{(i,j)}$ $(\tau = \overline{I}_0^j)$, $= 0$ $(\tau = \overline{I}_{J+1})$. From equations (,) it follows that $h_\nu^{(i,j)} = h_\nu$ $(\nu = \overline{I}_0^{i+j})$. Thus, by definition, $\xi(i,j) \geq i+j+1$ in this case. b) If $h_{i+j+1}^{(i,j)} = h_{i+j+1}$ also, then the conditions $\sum_0^i h_{\tau-\nu}^{(i,j)} \pi_\nu^{(i,j)} = \sum_0^i h_{\tau-\nu} \pi_\nu^{(i,j)} = 0$ $(\tau = \overline{I}_{j+1}^{j+i+1})$ imply that $H_{i+1,j+1} = 0$, contrary to assumption. Hence, in this case, $\xi(i,j) = i+j+1$. c) $\mathbb{P}$ consists of square blocks of the form $\mathbb{P}_{I,J}[N]$ containing $(N+1)^2$ identical members (possibly with $N=0$)((ii)c) above). The number $\xi(i,j)$ associated with each quotient of such a block is the same for each quotient. Since $\mathbb{P}_{I,J}[N] \subseteq \mathbb{P}$ implies $H_{I,J}[N] \subseteq H$((ii)b), $H_{I,J+N}, H_{I+1,J+N+1} \neq 0$; hence $\xi(i,j) = \xi(I, J+N) = I+J+N+1$ ((ii)b)) for all such $\xi(i,j)$. Ξ consists of square blocks, $\Xi_{I,J}[N]$, corresponding to $P_{I,J}[N]$, containing $(N+1)^2$ integers all equal to $I+J+N+1$. d) However the blocks

Ξ are situated, $\xi(i,j)$ is a nondecreasing function of $i+j$. e) $\Xi_{I,J}[N] \leq \Xi$ if and only if $P_{I,J}[N] \leq P$, i.e. if and only if $H_{I,J}[N] \leq H$ ((ii)b)). f) $\Xi_{I,J}[\infty] \leq \Xi$ if and only if $P_{I,J}[\infty] \leq P$, i.e. if and only if $h_{I,J} \leftarrow h$ as stated ((ii)d)). f) This result follows directly from the above.

(iv)a) If $H_{i,j}, H_{i+1,j+1} \neq 0$, then $\xi(i,j) = i+j+1$, $\xi(i+1,j+1) \geq i+j+2$ ((ii)aξ, b)). Hence $h^{(i,j)}_{i+j+1} \neq h_{i+j+1}$, $h^{(i+1,j+1)}_{i+j+1} = h_{i+j+1}$.

b) $\xi(i,j) = I+J+N+1$, and $\xi(i',j') = I'+J'+N'+1$. (1) If these two numbers are unequal one of $h^{(i,j)}_{\xi'}$, $h^{(i',j')}_{\xi'}$ is equal to $h_{\xi'}$, the other is not. 2) Let $I' > I$ (the case in which $I > I'$ is dealt with in a similar manner) so that $\xi' + r' - 1 = I' + J$. We shall show that we cannot have $h^{(i,j)}_{\nu} = h^{(i',j')}_{\nu}$, i.e. $h^{(I,J)}_{\nu} = h^{(I',J')}_{\nu}$ for $\nu = \overline{I}^{I'+J}_{I+J+N+1}$. Since $h^{(I,J)}_{\nu} = h^{(I',J')}_{\nu} = h_{\nu}$ $(\nu = \overline{I}^{I+J+N}_{0})$ we may set $h^{(I,J)}_{\nu} = h^{(I',J')}_{\nu} = h'_{\nu}$ $(\nu = \overline{I}^{I'+J}_{0})$. Denote the Hankel determinants derived from the numbers $\{h_\nu^{(I,J)}\}$, $\{h_\nu^{(I',J')}\}$ and $\{h'_\nu\}$ by $\{H_{i,j}^{(I,J)}\}$, $\{H_{i,j}^{(I',J')}\}$ and $\{H'_{i,j}\}$ respectively. Since $H_{I+r,J+s}^{(I,J)}=0$ $(r,s=\mathbb{I}_1)$ (( ) with $h_{I,J}$ replaced by $P_{I,J}$) we have $H'_{I+r,J+1}=0$ $(r=\mathbb{I}_1^{I'-I})$. Similarly, since $H_{I'+r,J'+s}^{(I',J')}=0$ $(r,s=\mathbb{I}_1)$, $H'_{I'+1,J'+s}=0$ $(s=\mathbb{I}_1^{J-J'})$. However, these two conditions imply that $H_{I'+r,J'+s}=0$ $(r,s=\mathbb{I}_1^{I'-I})$ (..), in particular, that $H'_{I',J'+N'+1}=0$, i.e. that $H_{I',J'+N'+1}=0$. Since $P_{I',J'}[N']\subset\mathbb{P}$, $H_{I',J'}[N']\subset\mathbb{H}$ (..) so that $H_{I',J'+N'+1}\neq 0$. Hence at least one member of the set $h_\nu^{(i,j)}$ $(\nu=\mathbb{I}_0^{I'+J})$ is unequal to its counterpart in $h_\nu^{(i',j')}$ $(\nu=\mathbb{I}_0^{I'+J})$. But $h_\nu^{(I,J)}=h_\nu^{(I',J)}=h_\nu$ $(\nu=\mathbb{I}_0^{\xi'-1})$ so that inequality between corresponding numbers is confined to the sets $h_\nu^{(i,j)}, h_\nu^{(i',j')}$ $(\nu=\mathbb{I}_{\xi'}^{I'+J})$

$P_{i+1,j}$, $P_{i,j+1}$ now satisfy the conditions of , with $P_{i,j}$ replaced by $P_{i+1,j}$, $P_{i',j'}$ replaced by $P_{i,j+1}$, $\xi'=i+j+2$, $r'=1$