Mapping Def (1) α A, B, …, C, … are all monoid •

aggregates $I(\alpha) := B(\alpha)$ basis set $\qquad bx = a + p$

$M(\alpha, a) := C(\alpha, a)$ coverings $\qquad xc = y + v$

P_Δ covered class $\longrightarrow$ $bxy + bxv = acx + pv$

sol

in $\Omega\{I(\alpha), \Theta\{I(\alpha)\}\}$ etc. Θ necessary? $\qquad bx = a + u$

$\mathbb{B}_m\ \mathbb{B}_s\ \mathbb{B}_u\ \mathbb{B}_f \qquad \mathbb{B}_m(W)$ etc $\quad \mathbb{B}_f = \mathbb{B}_s \cap \mathbb{B}_u \qquad cy = b + v$

$\mathbb{B}\ R(W, I)\langle E(W, I)\rangle \in \mathbb{B}_m \cap \mathbb{B}_f(W) \qquad bcxy = ab + av + bu + uv$

$W \setminus \mathbb{B}_u(W) \subseteq \mathbb{B}_m(W)^3$ If $\mathbb{B} \in \mathbb{B}_u(W)$, $W \setminus \mathbb{B} \in \mathbb{B}$

$iv = b + u$

If $M \in \mathbb{B}_u(W)$ then $W \setminus M \in \mathbb{B}_m(W)$

If $M \in \mathbb{B}_m \cap \mathbb{B}_u(W)$ then $W \setminus M \in \mathbb{B}_m \cap \mathbb{B}_u(W)$

$W \setminus M \notin B_s(W)$

$M \in B_f(W) \to M \in B_u(W) \to W \setminus M \in B_m(W)$ & $W \setminus M \notin B_f(W)$

$M \in B_m \cap B_f(W) \to M \in B_m \cap B_u(W) \to W \setminus M \in B_m \cap B_u(W)$

$M \in B_f(W) \to M \in B_s(W)$: $a \notin W \setminus M \quad b \in W \setminus M \to ab \in W \setminus M$

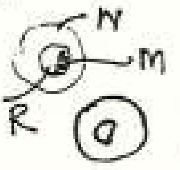
$W \setminus M \notin B_s(W) \quad B_f = B_s \cap B_u \to W \setminus M \notin B_f(W)$ | $bixv = ab + au + bp + up$

$ixv = a \mod I$

- eq $bx = a + u^p$ ∥ $cbx = ca \mod I \to u \in I \to I \in SI(W)$ (num syst additive.)
- eq $dx = c + xq \quad (b \pm d)x = a \pm c + u \pm v \to I$ add perf.
- eq $fy = e + wr \quad$ ~~bf~~ $xy = ae + aw + eu + uw \quad \underline{I}$ add syst $+ SI(W)$
- sol $bz = g + s \quad b(x \pm z) = a \pm g + p \pm s \quad \underline{I}$ add perf ⌈ x can be divided out

$xh = w + t \quad bxw = ahx + phx - bxt \to bw = ah + ph - bt$

$iv = b + u$



$a \phi b \in W \setminus M$

$a \phi b \in W \setminus (R \setminus M)$) ,

$(W \setminus R) \| M$

$bx=a \rightarrow cbx=ca$ eqns

$dy=c \rightarrow bd\,xy=ac$ mult

$bd(x\pm y)=ad\pm bc$ add.

$dx=c \quad (b\pm d)x=(a\pm c)$

$\left.\begin{matrix} bx=a \\ by=c \end{matrix}\right\} \; b(x\pm y)=(a\pm c)$ num syst addve

$\left.\begin{matrix} bx=a+q \\ hx=w \end{matrix}\right\} \; bw=ah$

? $bx=a \quad dx=c \quad (b\pm d)x=a\pm c$

$bx=a \quad xh=w \rightarrow bw=ah$

$bx=a \quad iv=b \quad ixv=a$

$\left.\begin{matrix} bx=a \\ iv=b \end{matrix}\right\} \; i(xv)=a$

eqns ① mult ② add & ③ mult throughout

~~num systems add~~ ① $bx=a \quad dy=c \rightarrow bd\,xy=ac$

② $bx=a \quad dy=c \quad bd(x\pm y)=ad\pm bc$

③ $bx=a \rightarrow cbx=ca$ // (4) div: $cbx=ca \rightarrow bx=a$

num syst ① add $bx=a \quad by=c \rightarrow b(x\pm y)=a\pm c$

$a\le b \quad ab\le c \rightarrow a\le c \quad a\le bc$

$\left.\begin{matrix} bx=a \\ cy=b \end{matrix}\right\} \; \begin{matrix} bc\,xy=ab \\ c\,xy=a \end{matrix}$

$bx=a$

$cy=ab \qquad cz=x \qquad b^2cz=ab$

$cy=b^2x \qquad \cancel{b^2y} \qquad b^2z=y$

$cxy=a^2 \qquad b^2x=cy$

$bc\,xy=a^2b$

$bcz=a$

$cxy=a^2 \rightarrow cz=a$

~~$axy=az$~~ $az=xy$

||

$bx=a \quad bw=y$

$cy=ab$

$cw=a$

$bz=w$

$by=a^2$

$bx=a$

$\bar{b}|a \bmod I \qquad b|a^2 \bmod I \to b|a \bmod I$

$b|a,\ c|a \bmod I \to bc|a^2 \bmod I \to \underline{bc|a \bmod I}$

$b|a \quad c|ab \to c|a$? $\qquad \boxed{c|a^2}$

$b|a \quad c|b \to c|a$

$ab|a^2,\ c|ab \to c|a^2 \to c|a$

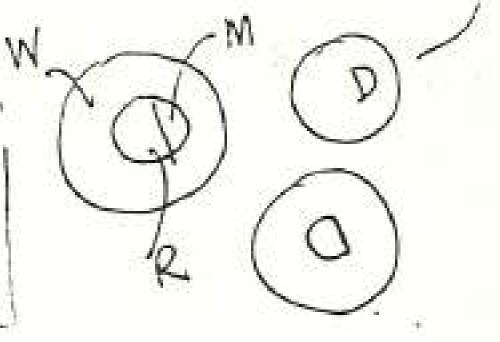
$b|a \quad c|a \to bc|a$

$M\in B_{\phi;s}$:

$a\in M \quad b\in R\setminus M \quad ab\in W\setminus M$

$R\setminus M\in B_{\phi;s} \quad a\in M$

$b\in R\setminus M \quad ab\in W\setminus(R\setminus M)$



$\bar{b}z = a^2+u \qquad \bar{b}x = a+v \qquad z = ax$

$bg\in I \to ag\in I \qquad ba\in I \to a\in I$

$\bar{b}a\in I \to a^2\in I \to a\in I$

$\bar{b}g\in I \to a^2g\in I \to a^2g^2\in I \to ag\in I$

$\bar{b}|a \bmod I \to ag\in I$ whenever $bg\in I$

$\bar{b}|a^2 \bmod I \to ag\in I$ whenever $bg\in I$ ie $a\in EN(b)$

$\mathcal{D}$: either $b+v\in D$ or $ub+v\in D \to a\in EN(b)$

$a\in EN(b) \to b|a \bmod I \mathbin{/\!/} \to bc|\underset{?}{ba} \to bc|a^2 \to bc|a \quad bc|a$

$\to c|a$

$b|a \quad c|ab \longrightarrow bc|ba^2 \to c|a^2 \to c|a$

$b|a = ba|a^2$

$a\le bc \bmod I \to a\le b, a\le c \bmod I \ \|\ bc|ba \to c|ba$

$\to c|a^2 \to c|a$

$a\le bc \quad$ " $\quad a\le b, a\le c \quad$ "

$a\le bc \to \exists\ x\in W,\ u\in I$ s.t. ~~a=~~ $bcx = a+u$ i.e $b(cx) = a+u$

$a\le b \ \|\ a\le bc \to$: $bcg\in I \to ag\in I \quad bg\in I \to bcg\in I \to ag \to I$

$\mathcal{D}(a)$: all $\mathcal{D}(D)$ b such that $\dot{a}\in N(b)$ $\{a\in EN(b)\}$

$\mathcal{D}(a)$ factored system: $a\le bc \to a\le b,\ a\le c$

denominator factored: $bc \in \Delta \rightarrow b, c \in \Delta$

$ubc + v \in E \quad \rightarrow (uc)b + v \in \Delta$

~~$c\{a/b\} \leq \{ca/b\}$~~

~~$b \in \Delta, b \leq c \rightarrow c\{a/b\} = \{ca/b\}$~~

$\{a/b\} \leq \{ca/cb\} \quad b \in \Delta \,;\, b \leq c \rightarrow \{a/b\} = \{ca/cb\}$

$\underline{I} \in \mathbb{I}_{QF}(W)$ note $cb \notin \Delta$ possible

$cbx = ca \mod \underline{I}$

$b \in \Delta \quad cb \overset{?}{\in} \Delta$ no $ub + v \in D$ $\quad c(bx - a) \in \underline{I} \quad b \leq c \rightarrow bx \leq c$

$a \leq b \rightarrow a \leq b$

$\rightarrow bx - a \leq c$

$\underline{I}_{QF} \rightarrow bx - a \in \underline{I}.$

~~bx~~

$c\{a/b\} \leq \{ca/b\}$

$bx = a \rightarrow b$~~$(cx) = ca \mod \underline{I}$~~ $by = ca \quad y = cx$

$by = ca \quad c \in E \quad y = cx \mod \underline{I} \quad bcx = ca \mod \underline{I} \rightarrow bx = a$ $(c \in D)$

$(ub + v)y = uca + vy \quad ua = (ub + v)x \quad v = (ub + v)u'$

$y =$ ~~ba~~ $cx + u' \qquad y \leq b \rightarrow vy \in \underline{I}$

$y = cx \mod \underline{I}, \qquad ag \in \underline{I} \rightarrow uag \in \underline{I} \rightarrow (ub + v)xg \in \underline{I}$

$\therefore \; x \leq a \leq b \qquad \rightarrow xg \in \underline{I} \; (ub + v) \in D$

(without $b \leq u$) $\rightarrow vx \in \underline{I}.$

$ubx = ua \mod \underline{I} \quad a \leq u, x \leq u \rightarrow bx = a \mod \underline{I}$

Have assumed existence of y such that $y \leq b$.

—o—

b: $ub + v \in E \quad u \leq b \quad bv \in \underline{I}$ $I \in \mathbb{I}_{QF}(W)$ ~~$\{a/b\}$~~ $a \leq b \quad \{a/b\} \cap N(b)$ contains nonvoid. if x, x' two members $x = x' \mod \underline{I}$ also $x \leq a$

$ua = (ub+v)x$ mod ~~B~~ $\underline{I}$

$ag \in \underline{I} \rightarrow (ub+v)xg \in \underline{I} \rightarrow xg \in \underline{I}$ $(ub+v \in D)$

$bv \in \underline{I} \rightarrow av \in \underline{I} \rightarrow xv \in \underline{I}$

$ua = ubx \quad a \leq b \rightarrow a \leq u \quad bx \leq u$ | using $a \leq u$ i.e. $b \leq u$

$bx = a$ mod $\underline{I}$ $\quad bx' = a$ mod $I \rightarrow b(x - x') \in \underline{I}$

$x = x'$ mod $\underline{I}$ $\quad (I \in \mathbb{I}_{QR}(W).)$

General solution of $bx = a$ mod I is then

$x = \hat{x} + w \quad \hat{x}$ special solution $\quad w \in \mathcal{O}(b)$

$\{ca/b\} \subseteq c\{a/b\}$ if to each $w' \in \mathcal{O}(b)$ corresponds w also in $\mathcal{O}(b)$ such that $w' = cw$. True if $c \in E$ not otherwise. (e.g. $c = b$ $\quad w' \in \mathcal{O}(b) = cw \in I$.)

* * *

~~$b|a-d$~~ $bx = a, dx = c$ mod $\underline{I} \rightarrow (b \pm d)x = a \pm c$

$\{a/b\} \cap \{c/d\}$ nonvoid $\rightarrow (b \pm d) \,|\, a \pm c$

$\{a/b\} \cap \{c/d\} \subseteq$ ~~[illegible]~~ $\{(pa \pm qc)/(pb \pm qd)\}$

all $p, q \in W$ $\quad$ $pb+qd$ can be 0

$(pb+qd)y = pa+qc$ mod $\underline{I}$ $\quad$ $\supseteq$ untrue; obviously untrue for rationals

a ~~b~~ $\in \Delta \quad b|a$ mod $I \rightarrow b \in \Delta \quad ua+v \in D$

$(ux)b+v \in D \quad$ poss $av \in \underline{I}$ not $bx \in \underline{I}$ also $a \leq u$ but $b \not\leq ux$

$ub+v \in D \quad ad - bc \in \underline{I} \rightarrow \{a/b\} \cap \{c/d\}$ nonvoid?

$bx = a+u \quad bd(x-y) = ad - bc + du - bv$

$dy = c+v$

$bx=a+p$ $p,u\in I$ $bixv=ab+au+pb+up$
$iv=b+u$

~~if $b\in E$ $au+up=bv$ $b\{i$~~

$a\leq b$
$a\leq x$

$b\in D$ $b\{ixv-a\}=au+pb+ub\in I$

$ixv=a+w$ $w\in I$

$bx=a+p$ $x\leq b$

$bx=a+p$ $x=x'+v'$ $bv'\in I$ $x'\leq a$
$cy=b+q$ $y=y'+w'$ c

$bcxcy=ab+aq+bp+pq$ $vcxy=xbv+vqx$

$$(b+v)cxy=a(b+v)+vcxy-\overset{\downarrow}{av}+aq+bp+pq$$

b: either $\exists v$ $b+v\in D$, $bv\in I$ or
$\exists u,v$ $ub+v\in D$ $bv\in I$

$$(ub+v)cxy=(ub+v)a+vcxy-av+u(aq+bq+pq)$$
$$+xbv+xvq$$

$cxy=bx+qx=a+p+qx$ without assumption.

—o—

$bx=a+p$ $by=bcx-bq=ac+cp-pq$
$cx=y+q$

—~

b: $ub+v\in E$ $d\in W$ $(ub+v)d_b=ubd$ $(ub+v)d_v=vd$

$(ub+v)(d_b+d_v)=(ub+v)d$ $d=d_b+d_v$ mod I

$bv\in I$ $(ub+v)bd_v=dbv\in I\to bd_v\in I$ also $vd_v\in I$

$(ub+v)^2d_bd_v=ud^2bv\to d_bd_v\in I$

$d=d'_b+d'_v\underset{I}{\overset{m}{=}}$ $bd'_b\in I$ $(ub+v)d=ubd'_b+vd'_v$ mod I

(I)

$b|a,\ d|c$ mod I $\{a/b\} \cap \{c/d\}$ nonvoid $\to ad-bc \in I$

$\cancel{b}x = a+u \quad dx = c+v \to \cancel{\$}\ 0 = ad-bc+du+bv$ ~~$\{a/d\}+I$~~

$\cancel{b},d \in D \quad b|a,\ d|c$ mod I, $ad-bc \in I \to$ ~~$\{a/d\} \cap \{c/d\}$ nonvoid~~

$\exists x \in \{a/b\}\ y \in \{c/d\}$ s.t ~~$y \equiv x \equiv 0$~~ wc

s.t $x = y$ mod I

$\cancel{b}d(x-y) = ad-bc+du-bv \quad \cancel{\$}$

$\cancel{b}\cancel{d} \in D \to x = y$ mod I

$\cancel{b},d \in E,\ ad-bc \in I \to \nearrow \quad b,d \in E \to b|a,\ d|c$

Structure of $S\{W,I|a/b\}$ $\quad b|a$ mod I $\quad bx' = a$ mod I

$S\{W,I|a/b\} = x' + \mathcal{O}(b)$

$d \leq b \quad a \leq b$

$ad = bc$ mod I $\quad c\underset{0}{g} \in I \to ad\underset{0}{g} \in I$

$a \leq d \to a \leq c : co \in I \to ad\underset{0}{g} \in I \to a\underset{0}{g} \in I$

$a \leq d \to a \leq b$ also

$\cancel{b},d \in \Delta \quad a,c \leq d \leq b$ ~~[illegible]~~ $ad = bc$ mod I $\quad I \in \underline{I}_{\partial F}(W)$

$\to \{a/b\} \cap \{c/d\}$ nonvoid

~~[illegible]~~ x,y minimal solutions in $\{a/b\},\{c/d\}$

$\cancel{\$} \quad x \leq a \leq d \quad y \equiv c \leq d$

$\cancel{b}d(x-y) \in I \to d(x-y) \in I \to x = y$ mod I

$x = y+u \quad y' = y+u \in \{c/d\} \quad x = y' \to \{a/b\} \cap \{c/d\}$ nonvoid

$\{a/b\}\{c/d\} \subseteq \{ac/bd\}$

$bx = a+u, dy = c+v \quad bdz = ac + av + cu + uv$

$bdz = ac + w \qquad z = xy$

$au + v \in D \quad (ub)x + v \in D \quad a \in \Delta \to x \in \Delta$

$x' \leq a \quad av \in I \to x'v \in I \quad a \in \Delta' \to x' \in \Delta' \quad x'$ min soln

$x' \equiv a \quad z \in N(a) \cap \{ac/bd\}$

$x'g \in I \to ag \in I \to zg \in I \to \exists y$ st. $xy = z$ mod I

~~N(a)~~ $a \in \Delta' \quad a \not\leq b \quad d|c$

$a \leq b \to b|a \quad \exists x: bx = a$ mod $\quad au + v \in D$

$(ux)b + v \in D \to b \in \Delta$ not Δ' $\qquad a \in \Delta'' \to b, x \in \Delta''$

x may not be in $N(a) \equiv EN(a)$ $\qquad bdx'y = ac$

∴ assume also $b \in \Delta'$ $\qquad day = ac$

$x' \in \{a/b\} \cap N(a)$ exists & $x' \in N(a)\Delta'$ $\qquad y \leq a$

$z \in \{ac/bd\} \quad b \leq d \quad ag \in I \to bdzg \in I$ $\qquad \to dy = c$

$\to bzg \in I \quad z \leq b \to zg \in I \to z \leq a$ ∥ $dy = c$?∴ ↓

~~$z \equiv$~~ $x'y = z$: $x' \in \Delta' \quad x' \equiv a \to a \leq x' \quad z \leq a$

$\to z \leq x'$ y exists such that $x'y = z$ mod I

~~$\{ac/bd\} \subseteq$~~ $N(b) \cap \{ac/bd\} \subseteq N(a) \cap \{a/b\}\{c/d\}$ mod I

$\{A\} \subseteq \{B\}$ mod I if $B + I \subseteq B \Rightarrow \{A\} \subseteq \{B\}$

$\{a/b\}\{b/d\} \le \{a/d\}$

$bx = a+u \quad dy = b+v \quad dxy = bx+vx = a+u+vx$

$dz = a \quad d\in\Delta' \to z\le a$

$a,b\in\Delta' \quad a\le b \quad d|c \quad bx' = a \quad x'\in\Delta' \,\&\, x'\le a$

~~$\{ab/bd\}\le\{a/d\}$~~ ? no $\quad x'y = z \quad z\le a\le x'$

~~$xy =$~~ $dy = b$? $\quad dx'y = a$ ~~$dx'y$~~

$dy \quad bx' = ba \quad dya = ba$ ‖

$b,d\in\Delta' \quad a\le b\le d$

$\Rightarrow N(d)\cap\{a/d\}\le \overset{N(b)\cap}{} \{a/b\}\{b/d\}$ mod I

assume $b,d\in\Delta' \quad b\le d \quad \exists y\in N(b)\cap\{b/d\} \quad y\in\Delta'$

$dz = a \quad yx = z$

$b,c,d\in\Delta,\ a\le b,\ c\le d$

$N(d)\cap\{ac/bd\}\le N(c)\cap\{a/b\}\{c/d\}$

$\underline{y\equiv b}$

$d\in\Delta' \quad a\le d \to \exists z\le a$ ~~$\exists$~~ ~~$y\in\Delta$~~ $z\le y$

x exists; also, since $y\in\Delta'$ can take $x\equiv z$

~~$y\,bx = bz$~~ $dz = dyx = a \quad dy = b \to bx = a$

~~$\{a/d\}\le$~~ $N(d)\cap\{a/d\}\le\{a/b\}\{b/d\}$ mod I

$z = xy + w \qquad w = (x+v)w'$

$a,b,d\in\Delta,\ a\le b \quad c\le d \to$ ~~$N(b)\cap\le N(a)$~~ $\overset{N(a)}{}\cap\{a/b\}\{c/d\}$ mod I

$N(b)\cap\{ac/bd\}\le$

$b,d\in\Delta,\ ac\le bd \ \exists z : bdz = ac$ mod $I \quad z\equiv ac\le a$

$b\in\Delta \quad a\le b \to \exists x' : bx' = a$ mod I ~~$a\in\Delta$~~ $x'\equiv a$

$a\in\Delta \to x'\in\Delta$ ‖ $x'\in\Delta, \quad z\le a\le x'$

$\exists y' : x'y' = z$ ~~$bdx'y = bdz = ac$~~ and $y'\equiv z\le a$

$bdx'y = bdz \to ady' = ac \quad y'\le a \to dy' = c$

$\{a/b\} + \{c/d\} \subseteq \{(ad+bc)/bd\}$

$bx = a,\ dy = c \mod I \quad bd(x+y) = ad+bc. \mod I$

$bdw \in I \quad br \in I \quad ds \in I \qquad w = pr + q$

$bd(pr+q) \in I \quad bdq \in I \quad s = bq \quad bq = s \quad s \le b$

$rs \in I \qquad\qquad bg \in I \to sg \in I$

$bdz = (ad+bc) \quad bx = a \quad bdx = ad$

$bd(z-x) = bc \quad \begin{matrix} c \le b \\ d \le b \end{matrix} \to d(z-x) = c$

$bd \mid ad+bc \quad b \mid a \quad c, d \le b \quad I \in \mathbb{I}_{QF} \to$ ~~[illegible]~~

$c \mid d$ and $\{a/b\} + \{c/d\} = \{(ad+bc)/bd\}$

$a \le b \quad c \le d \to ad+bc \le bd$

$b, d \in \Delta \quad bx = a \quad dy = c$ show $x+y \le ad+bc$

$x \le a,\ x \le b \to$ ~~$x \le ad$~~ so

$x+y \le dbx + dby = db(x+y)$

rational extension wrt. $E(W, I)$ merely reproduces W?

suffices to show $x+y \le bd$

in rational extension $x+y = a/b + c/d =$
$(ad+bc)/bd$

$bdz = ad+bc \quad bx = a \quad bdx = ad$

$bd(z-x) = bc$

$x \le b \quad bx \le a \to \begin{matrix} x \le ab \\ x \le a \end{matrix}$

$b^2 \le a \to b \le a \quad b \le ab$

$B; I \le ?$ → $A \le B$ mod I → $A \le B$ ‖ $I \in \mathbb{I}_{QF}(W)$ → $a^2 \le b \to a \le b$

Denominator set $D(W, I|a)$ Dimin.. $DD(W, I|a)$

factored, ∈ ind., ∈ sat.

$\Delta''(M)$ $\Delta'(M, J)$ $\Delta(M, J)$

$\Delta'' \in B_m \cap B_f$

$b \in \Delta''\{E(W, J)\} \to N(b) \le Er(b)$

$a \in \Delta''(M) \to D(W, I|a), DD(W, I|a) \in \Delta''(M)$

Δ' ? Δ

$b | a^2 \to b|a$

$a \le b^2 \to a \le b$

$b^2 | a \to b|a$

$a^p \le b^q \to a^r \le b^s$

$b^q | a^p \to$

Structure of $S\{W, I|a/b\}$ $b|a$ $\hat{x} + \delta(b)$

$a \in \Delta''(M) \to S\{W, I|a/b\} \in \Delta''(M)$ all $b \in D(W, I|a)$

Δ' Δ''

$b \in \Delta''\{E(W, I), I\}$ $a \le b \to \hat{x} \le a$ $I \in \mathbb{I}_{QF}(W)$

$\hat{x} \le b$?

$\{a/b\} \le \{ca/cb\}$. $b \in \Delta$ $b \le c$ $I \in \mathbb{I}_{QF} \to \{a/b\} = \{ca/cb\}$

$cb \notin \Delta$ possible ?

$c\{a/b\} \le \{ca/b\}$

$b \in \Delta'$ $a \le b$ $c \in D \to \{ca/b\} \le c\{a/b\}$ mod ?

$c \in E \to$ "

$\{a/b\} \cap \{c/d\} \le \{(pa + qc)/(pb + qd)\}$ all $p, q \in W$

- $b, d \in \Delta$ $a, c \le b$ $ad = bc$ mod I $I \in \mathbb{I}_{QF} \to \{a/b\} \cap \{c/d\}$ nonvoid $\{$ = sets of minimal solutions $\}$

$\{a/b\} \cap \{c/d\}$ nonvoid → $ad = bc$ mod I :

$a \le bc$

$cd \le u$

$b \in \Delta$ $d \in W \to d = d_b + d_v$ $d_b d_v \in I$

$\{a/b\}\{c/d\} \leq \{ac/bd\}$

$b, c, d \in \Delta \quad a \leq b \quad c \leq d \quad N(d) \cap \{ac/bd\} \leq N(c) \cap \{a/b\}\{c/d\}$ mod $\underline{I}$

$(ac \leq bd \quad a \leq b \rightarrow c \leq d\ ?)$

$\{a/b\}\{b/d\} \leq \{a/d\}$

$b, d \in \Delta' \quad a \leq b \leq d \quad N(d) \cap \{a/d\} \leq N(b) \cap \{a/b\}\{b/d\}$ mod $\underline{I}$

$\{a/b\} + \{c/d\} \leq \{(ad+bc)/bd\}$

$bd \mid ad+bc \quad b \mid a \quad c, d \leq b \quad \underline{I} \in \mathbb{I}_{QF} \rightarrow$

$c \mid d$ and $\{a/b\} + \{c/d\} \equiv \{(ad+bc)/bd\}$.

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(1) If $A \leq B$ mod $\underline{I}$ and $B + \underline{I} \leq B$ then $A \leq B$.

(2) If $A \equiv B$ mod $\underline{I}$, ~~and~~ $A + \underline{I} \leq \underline{I}$ and $B + \underline{I} \leq B$, then $A \equiv B$.

Select $a \in A$. If $A \leq B$ mod $\underline{I}$ $b(a) \in B$ and $u(a, b) \in \underline{I}$ ~~exist~~ for which $a = b(a) + u(a, b)$. If $B + \underline{I} \leq B$, B contains b' for which $b(a) + u(a, b) = b'$: ~~a≤b B contains b′ for which a=b′: A≤B.~~ a is present in B: $A \leq B$. The second result is a corollary to the first. ⟨the ~~diminished~~ extended denominator set $D(W, \underline{I} \mid a)$⟩)

Def ( ) The denominator set $D(W, I \mid a)$ of a in W with respect to $\underline{I}$ is the complete system of numbers $b \in W$ for which $b \mid a$ mod $\underline{I}$ ⟨for which $a \leq b$ mod $\underline{I}$)

( ) For each $a\in W$, $D(W,I|a)\langle \overset{E}{\mathfrak{D}} D(W,I|a)\rangle\in B_{fu}(W)$, $B_s(W)$, $B_f(W)$.

If $bc\in D(W,I|a)$, $x\in W$ and $u\in I$ for which $bcx\cancel{\varepsilon}=a+u$ exist and then $b(cx)=a+u$: $b\in D(W,I|a$ and similarly $c\in D(W,I|a)$: $D(W,I|a)\in B_f(W)$. From ( ) $B_f(W)\equiv B_u(W)\cap B_s(W)$.

Since $bc\le b$, $a\le b$ when $a\le bc$: ~~similarly $a\le c$.~~ ~~$b\in DD(a)$ W~~ if $bc\in\overset{E}{\mathfrak{D}}D(W,I|a)$, $b\in\overset{E}{\mathfrak{D}}D(W,I|a)$. Simil $c\in\overset{E}{\mathfrak{D}}D(W,I|a)$: $\overset{E\mathfrak{D}}{\cancel{D}}(W,I|a)\in B_f(W)\ \overset{E}{}\equiv B_u(W)\cap B_s(W)$.

Def (1) With $M\le W$, $\Delta''(W;M)$ is the complete system of numbers $b\in W$ for which ~~either~~ W contains either

α) ~~W contains~~ $v(b)$ such that $b+v(b)\in M$ or

β) ~~a~~ $v(b)$ and $u(b,v)$ such that $u(b,v)b+v(b)\in M$

~~In case (α), v(b)~~ In case ($\alpha\langle\beta\rangle$), $v(b)$ is an M a ~~v(b) is called~~

~~complement of b~~

In case ($\alpha\langle\beta\rangle$) $v(b)$ is a direct ⟨a displaced⟩ M complement; in case (β) $u(b,v)$ is ~~a~~ an M lifting factor, and is said to lift b into M!

2) With $M, J\in M$, $\Delta'(W;M,J)$ is the complete system of numbers $b\in W$ for which ~~either~~ W contains eith

α) $v(b)$ such that $b+v(b)\in M$, $bv(b)\in J$ or

β) ~~$u(b,v)$~~ $v(b)$ and $u(b,v)$ such that $u(b,v)+v(b)\in M$ and $bv(b)\in J$.

In case ($\alpha\langle\beta\rangle$) $v(b)$ is a direct $\langle$a displaced$\rangle$

J orthogonal M complement.

3) With $M, J \in$ W, $\Delta(W; M, J)$ is the complete system of numbers $b^{\in W}$ for which W contains either

α) $v(b)$ as in (2α) or

β) $v(b)$ and $u(b,v)$ ~~such that~~ as in (2β) above together with $b \leq u(b,v)$ mod J.

( ) ~~$\Delta \subseteq W$~~ For any sets $M, J \subseteq W$, $\Delta(W; M, J) \subseteq$ ~~Δ'~~ y

$\Delta'(W; M, J) \subseteq \Delta''(W; M)$

~~Each set~~ The members of Δ' are those of Δ'' that satisfy ~~[illegible]~~ an additional restriction; similarly for Δ and Δ'.

( ) ~~[illegible]~~ $D(W, I|a) \subseteq ED(W, I|a)$ for all $a \in W$

If $b|a$ mod I, then $a \leq b$ mod I: all $b \in D(W, I|a)$ belong to $ED(W, I|a)$.

( ) $E(W, I) \subseteq D(W, I|a)$ for all $a \in W$.

If ~~b~~ $b \in E(W, I)$, $x(a,b) \in W$ for which $b\,x(a,b) = a$ $+ u$ ~~where~~ with $u \in I$ exists: $b \in D(W, I|a)$.

$x \in D(+)$

$\Delta'' \in \overset{B_s\,B_u}{B_f^{\gamma}}(W)$. $M \in B_m(W)$, $\Delta'' \in B_m(W)$ $W \setminus \Delta'' \in B_m \cap B_u(W)$

~~Δ''~~ $W \setminus \Delta'' \in B_m(W)$

$M \in B_m(W)$ $J \in B_a \cap B_{si}(W)$: $\Delta' \in B_m$

" " $J \in B_a \cap B_{si} \cap B_{sf}(W)$

$B_{+;c|\overset{f}{x};si|x;sf}$

$B(+;c|x;si,$ ~~[illegible]~~ $sf)$

With $K \subseteq W$, let

Definition. ~~Let~~ γ be a binary operation ~~on~~

$R \times R$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ R

~~$W \times W$~~ $\to W$ whose operation, ~~is exp~~ with $a, b \in$ ~~W~~,

is expressed in the form $a \phi b$.

(1) A system $M \subseteq R$ of numbers which is such that $a \phi b \in M$ for all $a, b \in M$ is a contractive ϕ-system.

~~$B_{\phi;c}$ is the class of contractive ϕ-systems~~

$\qquad\qquad\qquad\qquad\quad R$

(2) A system $M \subseteq$ ~~W~~ of numbers for which

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad b \in R \setminus M \quad a \in R \setminus M,$

$a \phi b \notin$ ~~W~~ $\overset{\in}{} \; W \setminus M$ when either $a \in M$, ~~$b \notin M$~~ or ~~$a \notin M$~~, $b \in M$

$\qquad\qquad\qquad R \setminus M$

$\langle$when $a, b \notin$ ~~W~~$\rangle$ is a saturated $\langle$an independent$\rangle$

$\qquad\qquad\qquad$ in R

ϕ-system. ~~$B_{\phi;s}$ $\langle B_{\phi;u} \rangle$ is the class of saturated $\langle$independent$\rangle$ ϕ-systems~~

(3) A ~~[illegible]~~ system $M \subseteq R$ of numbers for which

for all $a, b \in R$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ in R

$a \phi b \in M$ only when $a, b \in M$ is a factored ϕ-system.

(4) $B_{\phi;c}(R)$, $B_{\phi;s}(R)$, $B_{\phi;u}(R)$ and $B_{\phi;f}(R)$ are ~~the classes~~

$\qquad\qquad\qquad$ ~~sets~~

respectively the ~~classes~~ of contractive, saturated, independent and factored ϕ-systems in R.

~~(5) With $R \subseteq W$, $B_{\phi,c}(R)$ is the class of contractive~~

$B_{\phi;c}(R) \cap B_{\phi;s}(R)$ is expressed as $B_{\phi;c} \cap B_{\phi;s}(R)$; unions of other classes are similarly expressed.

( ) Let $R\subseteq W$ and ϕ be a binary operation $R\times R\to W$.

$B_{\phi;f}(R)=B_{\phi;s}\cap B_{\phi;u}(R)$.

Let $M\in B_{\phi;f}(R)$. If $a\in M$ and $b\in R\setminus M$, then $a\phi b\in W\setminus M$, since if ~~ab~~ $a\phi b\in M$ then $b\in M$, violating the condition that $b\notin M$: $B_{\phi;f}(R)\subseteq B_{\phi;s}(R)$. If ~~$a\in R\setminus M$,~~ $a,b\in R\setminus M$ then $a\phi b\in W\setminus M$, since if ~~$a,b\in$~~ $a\phi b\in M$ a and b are in M: $B_{\phi;f}(R)\subseteq B_{\phi;u}(R)$: $B_{\phi;f}(R)\subseteq B_{\phi;s}\cap B_{\phi;u}(R)$. Let $M\in B_{\phi;s}(R)$. The condition ~~ab~~ $a\phi b\in M$ implies that the twin conditions $a\in M, b\in R\setminus M$ cannot hold: either $a,b\in M$ or $a,b\in R\setminus M$. Let $M\in B_{\phi;u}(R)$ also. If ~~ab~~ $a\phi b\in M$ the conditions ~~a~~ $a,b\in R\setminus M$ do not hold: the single condition $a,b\in M$ holds: $B_{\phi;s}\cap B_{\phi;u}(R)\subseteq B_{\phi;f}(R)$.

( ) Let $R\subseteq W$ and ϕ be a binary operation $R\times R\to W$

(1) ~~If~~ $M\in B_{\phi;u}(R)$ ~~then~~ if and only if $R\setminus M\in B_{\phi;c}(R)$

(2) ~~If~~ $M\in B_{\phi;s}\cap B_{\phi;u}(R)$ ~~then~~ if and only if $R\setminus M\in B_{\phi;c}\cap B_{\phi;u}(R)$ also

(3) If $M\in B_{\phi;f}(W)$ then $R\setminus M\in B_{\phi;c}(W)$ but $W\setminus M\notin B_{\phi;s}(W)$ and $W\setminus M\notin B_{\phi;f}(W)$.

~~If~~ $M \in B_{\phi;u}$ ~~$\Rightarrow$~~ if and only if $a\phi b \in R\setminus M$ for all $a,b \in R\setminus M$. ~~$R\setminus M \in B_{\phi;c}(R)$. If~~ $M \in B_{\phi;u}$ if and only if $R\setminus M \in B_{\phi;o}(R)$.

~~The two results $M \in B_{\phi;u}$ if and not~~

The result of (1) may be reformulated: ~~as~~ $M \in B_{\phi;c}$ ~~$R\setminus M \in B_{\phi;u}$~~ if and only if ~~$M \in B_{\phi;c}$~~ $R\setminus M \in B_{\phi;u}$. Combining this result with the original version, the result of (2) is obtained.

If $M \in B_{\phi;f}(\overset{W}{\cancel{R}})$ then $M \in B_{\phi;u}(\overset{W}{\cancel{R}})$ from ( ) and $\overset{W}{\cancel{R}}\setminus M \in B_{\phi;c}$ from (1). Also $M \in B_{\phi;s}(\overset{W}{\cancel{R}})$ so that for all ~~$a \notin R$~~ $a \in M \equiv \overset{W}{\cancel{R}}\setminus(\overset{W}{\cancel{R}}\setminus M)$ and $b \in \overset{W}{\cancel{R}}\setminus M$, ~~ab~~ $a\phi b \in \cancel{R}\setminus M$. ~~i.e.~~

~~$R\setminus M \notin B_{\phi;s}(R)$ and, since $B_{\phi;f}(R) \equiv B_{\phi;s} \cap B_{\phi;u}(R)$,~~ ~~$a\phi b \notin M \equiv R\setminus(R\setminus M)$~~ ~~$R\setminus M \notin B_{\phi;f}(R)$~~

If $\overset{W}{\cancel{R}}\setminus M \in B_{\phi;s}(\overset{W}{\cancel{R}})$ then for all $a \in \overset{W}{\cancel{R}}\setminus(\overset{W}{\cancel{R}}\setminus M)$ and $b \in \overset{W}{\cancel{R}}\setminus M$, $a\phi b \in W\setminus(\overset{W}{\cancel{R}}\setminus M) \equiv M$. Hence $W\setminus M \notin B_{\phi;s}(W)$ and, since $B_{\phi;f}(W) \equiv B_{\phi;s} \cap B_{\phi;\cancel{u}s}(W)$, $W\setminus M \notin B_{\phi;f}(W)$.

( ) ~~[illegible]~~ With $M \le W$, $\Delta''(W;M) \in B_{X;f}(W)$ and $W \setminus \Delta''(W,M) \in$ $B_{X;c}(W)$.

~~The relation With~~ If $cd \in \Delta''(W;M)$, ~~the relationship~~ $ucd + v \in M$ or a simpler relationship holds. ~~ud and uc function as lifting factors for lift c and d respectively into M.~~ v is an ~~indirect~~ displaced M-complement ~~of~~ with c and d, and ud and uc ~~are M-lifting fo~~ lift c and d respectively into M. Since $\Delta''(W,M) \in$ ~~(1)~~ $B_f(W) \equiv B_s(W) \cap B_u(W) \le B_u(W)$, $W \setminus \Delta''(W,M) \in B_c(W)$, from (1).

$B_{X;c}$?

( ) Let $M \in B_c(W)$. $\Delta''(W;M) \in B_c(W)$ and $W \setminus \Delta'' \in B_c \cap B_u(W)$.

Let $b, c \in \Delta''(W;M)$ with $ub + v \in M$, $sc + t \in M$. If $M \in B_c(W)$, ~~then~~ $us(bc) +$ $w \in M$ ~~$(usc + ubt + scv + vt) \in M$~~. ~~$w = ubt + scv + v$~~ where $w = ubt + scv + vt$, us and w function as M-lifting factor and M-complement of bc: ~~[illegible]~~ $\Delta''(W,M) \in B_c(W)$. From (,), $\Delta''(W,M) \in B_f(W) \in B_u(W)$. Hence, from ( ), $W \setminus M \in B_c \cap B_u(W)$.

( ) Let $M \in B_{x;c}(W)$ and $J \in B_{x;si} \cap B_{+;c}(W)$.

$\Delta'(W;M,J)$, ~~$\in B_{x;c}(W)$~~ $\Delta(W;M,J) \in B_{x;c}(W)$.

Let $b, c \in \Delta'(W;M,J)$ with $ub+v, sc+t \in M$ and bv, ~~s~~ $ct \in J$. Since $M \in B_{x;c}(W)$, $us(bc) + w \in M$ where $w = ubt + scv + vt$. ~~and~~ us and w function as M lifting factor and M-complement of bc, and ~~also $bcw \in I$~~ Also, Since $J \in B_{x;si} \cap B_{+;c}$, $w \in J$ and $bcw \in J$: $\Delta'(W;M,J) \in B_{x;c}(W)$. The two conditions $b \leq u$, $c \leq s$ mod J imply that $bc \leq us$ mod J: $\Delta(W,M,J) \in B_{x;c}(W)$ also.

—∘—

$N(W, I|b) \subseteq EN(W, I|b)$. $a \in EN(W, I|b)$

$a = ye + w$ ~~$e \in E$~~ $ub + v = e \in E$ $bv \in I \to av \in I$

$av = yev + vw$ $\to yv \in I$ $a = buy + vy + w$

~~$a = b$~~ $bx = a + t$ $x = uy$ $t =$

$ye = a + w$ $bx = a + t$ $x = uy$ $t = w - vy$

$ubx = ua + s$ $bv_0 \in I \to av_0 \in I \to yv_0 \in I \to uyv_0 \in I$

$x \leq b$ also $x \leq a$

$(ub+v)x = ua + w$

—∘—

$bx = a + s$

$u'b + v' = e'$ $a = y'e' + w'$ $av' \in I$ $y'v' \in I$

$(ub+v)x = ua$

$bx' = a + t'$ $x' = u'y'$ $t = w' - v'y'$

$+v$

(1c)

( ) ~~Let b~~ If $b \in \Delta'\{E(W,I), \underline{I}\}$ then $N(W,\underline{I}|b) =$ $EN(W,I|b)$

If $b \in \Delta''\{E(\cancel{W,I}), I\}$, $u \in W$ and $v \in I$ for which $ub+v = e \in E(\cancel{W,I})$ and $bv \in I$ exist, or a simpler condition holds. Select $a \in EN(W,I|b)$. Since $e \in E$, $y \in W$ and $w \in I$ for which $a = ye+w$ exist. Since $bv \in E$ and $a \leq b \bmod I$, $av \in I$ also

Thus $yev = av - vw \in I$ and, since $e \in D$, $yv \in I$. Set $x = uy$. Then $bx = (ub+v)y - vy = ey - vy =$ ~~$a - vy - w$~~ $= a + t$ where $t = -vy - w \in I$: $a \in N(W,I|b)$: $EN(W,I|b) \subseteq N(\underline{W,I}|b)$. From ( ), $N(W,I|b) \subseteq EN(W,I|b)$.

$a \in \Delta''$ $bx = a+w$ $ua+v \in M$ $(ux)b+v - w \in M$ $ua+v = e \in M$

$w \in \underline{I}$ $M : M+\underline{I} \subseteq M$ $(ux)b+$

$bx = a+w$ $ua+v = e$ $u(bx-w)+v = e$ $ubx+v = e+uw$

$(ux)b+v \in M$?

$\underline{I} \in B_{x;si}$ $M+\underline{I} \subseteq M$ | $a \in \Delta' \to D(W,I|b) = DD(M,I|b$

$av \in \underline{I} \to bv \in \underline{I}$? no | $a \leq b \to b|a$?

$ua+v = e \in E$ $av \in \underline{I}$ M??

(1) Let $\underline{I} \in B_{x;si}$ $M \in B_{x;c}$ and $M+\underline{I} \subseteq \underline{I}$. If $a \in \Delta''(M)$ ~~Then b and~~ then $D(W,I|a) \in \Delta''(M)$.

(2) ~~Select b∈~~ If $a \in \Delta''(M)$, then $u, v \in W$ such that $ua+v \overset{=e}{\in} M$ exist, or a simpler condition holds. Select

$b \in D(W, I|a)$; so that $x \in W$ and $w \in I$ for which $bx = a + w$ exist. $(ux)b + v = e + uw$. Since $I \in B_{x;si}$, $uw \in I$ and, since $M + I \subseteq M$, $e + uw \in M$. ux and v function as M lifting factor and displaced complement of b: $D(W, I|a) \in \Delta''(M)$. Part (2) is a corollary to the result just proved.

2) Let $I \in \mathbb{I}(W)$. If $a \in \Delta''\{R(W, I)\}$ ~~then~~ then ~~$\langle a \in \Delta''\{E(W, I)\}\rangle$ then~~ $D(W, I|a) \in \Delta''\{R(W, I)\}$. A similar result holds with $R(W, I)$ replaced by $E(W, I)$.

~~With $I \subseteq W$, the solution system of the~~

Definition. $S\{W, I|a/b\}$ of the equation $bx = a \bmod I$ is the complete system of numbers $x \in W$ satisfying the relationship $bx = a + v(a, b; x)$ where $v(a, b; x) \in I$.

( ) Let $b|a \bmod I$.

1) $x = \hat{x}$ being any a particular ~~special~~ solution of the equation $bx = a \bmod I$ ~~number~~ ~~number satisfying the equation $b\hat{x} = a \bmod I$~~, $S\{W, I|a/b\} = \hat{x} + \eth(W, I|b)$.

$b\hat{x} = a + \hat{w}$ $\quad bx = a + w$ $\quad b\{x - \hat{x}\} = w - \hat{w}$

$w, \hat{w} \in I$ $\quad \exists\, t \in I$ s.t. $w = \hat{w} + t$ $\quad x - \hat{x} \in \eth(b) \to x \in \hat{x} + \eth(b)$

* $v \in \eth(b)$ $\quad b\{\hat{x} + v\} = a + \hat{w} + bv$

$b x_0 = a_0 + u_0$ $\quad s = a_0 + t$ $\quad s + w = t$

$(ub+v)x = ua + w$ $\quad x \le a \to xv \in I$

$b \le u \to bx = a \bmod I$ $\quad (+ I \in \mathbb{I}_{3=}(W))$

$(u'b+v')x' = u'a + w'$ $\quad ubx = ua + t$

$uu'b(x-y) =$ $\quad \Delta''(M,J)$ also

$M \in B_{x;c}$

$I \in B_{x;si} \cap B_{+;c}$

$M + I \in B_{x;c}$

$bx = a \Rightarrow xy \in I \to ay$

$a \le x$

Let $b \in \overrightarrow{\Delta}'(W; E(W,I), I)$ where $I \in \mathbb{I}(W)$, and $a \le b \bmod I$.

$S\{W, I \,|\, a/b\}$ ~~contains a set of soln~~

$a \in EN(b) \to a \le b$

$b \in ED(a) \to a \le b$

~~$AN(W, I|a)$ contains a is nonvoid~~

contains a set $\{x', x'', \dots\}$ with $x' = x'' = \dots$ ~~mod I~~

~~all members~~ $\overset{?}{=} a \bmod I$ $\quad S\{W, I|a/b\} \subseteq ED(a)$

Let $b \in \Delta(W; E(W,I), I)$ where $I \in \mathbb{I}_{QF}(W)$ and $a \le b \bmod I$

$S\{W, I \,|\, a/b\}$ contains a set $X = \{x', x'', \dots\}$ with $x' = x'' = \dots = a \bmod I$. All members of $S\{W, I|a/b\}$ contained in $AN(W, I|b)$ belong to this set. With u, v ~~lifting factor~~ $[E(W,I), I]$ lifting factor and displaced orthogonal complement respectively of b, the relationship $(ub+v)x = ua \bmod I$ determines a representative member x of X, and $X = x + I$. The determination of X in this way is independent of the lifting factor-displaced orthogonal complement pair used (and of the direct orthogonal complement used in the same way).

$v = w$

Definition. ~~With $I\subseteq$~~ Let $I\subseteq W$.

1) The solution ~~system~~ space $S\{W, I|a/b\}$ of the equation $bx=a \bmod I$ is the complete system of numbers $x\in W$ to each of which a member $v(a,b;x)$ of I for which $bx=a+v(a,b;x)$ corresponds.

2) The minimal solution space $MS\{W, I|a/b\}$ of the equation $bx=a \bmod I$ is that subset of members x' of $S\{W, I|a/b\}$ for which $x'\equiv a \bmod I$.

( ) Let $b|a \bmod I$.

1) Let $I\in B_{x;si}$ ~~(W)~~ $\cap B_{+;ls}(W)$. $a\leq x$ ~~for~~ mod I for all members x of $S\{W, I|a/b\}$: $S\{W, I|a/b\}\subseteq ED\{W, I|a\}$.

2) Let $I\in B_{x;si}$ ~~(W)~~ $\cap B_{+;c}(W)$. $S\{W, I|a/b\}+I\subseteq S\{W, I|a/b\}$.

3) Let $I\in$ ~~$B_{x;si}\cap$~~ $B_{+;ls}(W)$. $S\{W, I|a/b\}\subseteq S\{W, I|a/b\}+I$ // (4) Let $I\in B_{x;si}$ ~~$\cap B_{+;c}\cap B_{+;ls}(W)$~~ $I(W)$. $S\{W, I|a/b\}\pm S\{W, I|a/b\}+I$.

If $b|a \bmod I$, $S\{W, I|a/b\}$ is nonvoid. Select $x\in S\{W, I|a/b\}$, so that $bx=a+u$ where $u\in I$. If

~~$x_0\in I$ and I_0 $a_0=bx_0-u_0$ and $I\in B_{x;si}\cap B_{+;pc}(W)$~~

~~$a\underset{0}{g} = bx\underset{0}{g} - u\underset{0}{g} \in I$~~ (define $B_{+;\ell_s}$) $\mathbb{N}$ $D \subseteq ED$?

For any $g \in W$, $s = a\underset{0}{g} + t$ where $s = bx\underset{0}{g}$, $t = u\underset{0}{g}$. If $I \in B_{\times;s\ell}(W)$, and $x\underset{0}{g} \in I$, then $s, t \in I$. If $I \in B_{+;\ell_s}(W)$, $w \in I$ for which $s + w = t$ exists and then $a\underset{0}{g} = w \in I$. $a\underset{0}{g} \in I$ for all $\underset{0}{g} \in W$ for which $x\underset{0}{g} \in I$: $a \leq x \bmod \underline{I}_s$. $ED\{W, I|a\}$ is the complete system of numbers y for which $a \leq y \bmod I$: $S\{W, I|a/b\} \subseteq ED\{W, I|a/b\}$.

To prove part (2), ~~select let~~ select $x \in S\{W, I|a/b\}$ again ~~and~~ with $bx = a + u$ and $u \in \underline{I}$. Select $w \in I$. $b(x+w) = a + u + bw$. If $\underline{I} \in B_{\times;s\ell} \cap B_{+;c}(W)$, $u + bw \in I$ and $x + w \in S\{W, I|a/b\}$: $S\{W, I|a/b\} + \underline{I} \subseteq S\{W, I|a/b\}$.

If $I \in B_{+;\ell_s}(W)$, $0 \in I$ and ~~automa~~ without further assumption ~~so~~ $A + I \subseteq A$ for any set $A \subseteq W$. Part (3) has been disposed of. Part (4) follows from it's two predecessors.

( ) Let $b \in \Delta'\{E(W, I), \underline{I}\}$ and $a \leq b \bmod \underline{I}$ where $\underline{I} \in \mathrm{II}(W)$. ''(1) $MS\{W, I|a/b\}$ is nonvoid ~~and~~ ''~~(2)~~ ~~$MS\{W, I|a/b\} = MS\{W, I|a/b\} + \underline{I}$~~ ind. of $b \in \Delta'$ etc.

If $b \in \Delta'\{E(W, I), I\}$, $u \in W$ and ~~be~~ $v \in \overset{?}{I}$ for which $ub + v = e \in E(W, I)$ and $bv \in I$ exist, or a simpler condition holds: $x \in W$ ~~exists~~ ~~such th~~ for which $ua = (ub+v)x + w$ where $w \in I$ exists. Multiplying throughout by b and adding av to both sides of the resulting equation it follows that $(ub+v)(bx-a) = t$ where $t = -bw - va$. Since $a \leq b \bmod I$, and $bv \in I$, $av \in I$: $t \in I$. Since $ub+v \in Ri(W,I)$, $bx = a + s$ for some $s \in I$; $x \in S\{W, I|a/b\}$. If $ag \in I$, $(ub+v)xg \in I$ and again $xg \in I$: $x \leq a \bmod I$. From (1), $a \leq x \bmod I$: ∴ $a \equiv x \bmod I$: $MS\{W, I|a/b\}$ is nonvoid. ~~For any $x \in MS\{W, I|a/b\}$ and $y \in I$, $x+y \in S\{W, I|a/b\}$ and $x+y \equiv a \bmod I$,~~ so ~~that $x+y \in MS\{W, I|a/b\}$: $MS\{W, I|a/b\} + I \subseteq MS\{W, I|a/b\}$. I contains 0: $MS\{W, I|a/b\} \subseteq MS\{W, I|a/b\} + I$ and $MS\{W, I|a/b\} + I = MS\{W, I|a/b\}$.~~

( ) Let $b \in \Delta'\{E(W,I), I\}$ and $a \leq b \bmod I$, where $I \in \Pi_{OF}(W)$.

1) $MS\{W, I|a/b\}$ is nonvoid ~~and $MS\{W, I|a/b\} + I = MS\{W, I|a/b\}$~~.

2) All members of $MS\{W, I|a/b\}$ are equal mod I. x being any member of $MS\{W, I|a/b\}$, $x + I = MS\{W, I|a/b\}$.

3) All members of $S\{W, I|a/b\}$ contained in $N(W, I|b)$ belong to $MS\{W, I|a/b\}$.

4) Let u, v be an $E(W,I), I$ lifting factor and orthogonal

complement pair of b. Since $ub+v \in E(W,I)$, W contain a number x for which $(ub+v)x = ua$ mod I. This x is a representative member of $MS\{W,I|a/b\}$ and $MS\{W,I|a/b\} \equiv x+I$. The determination of $MS\{W,I|a/b$ in this way is independent of the lifting factor-displaced orthogonal complement used (and of the direct orthogonal complement possibly used in a similar way)

Since ~~$\Delta\{E(W,I),I\} \subseteq \Delta'\{E(W,I),I\}$ and~~ $\mathbb{I}_{QF}(W) \subseteq \mathbb{I}(W)$, the first result is a consequence of ( ).

Select $x \in MS\{W,I|a/b\}$ ~~and let x' be any member of $S\{W,I|a/b\}$~~ so that $bx = a$ $+s$ where $s \in I$ ~~mod I~~ and $x \in EN(W,I|a)$ Since ~~$EN(W,I|a) \subseteq EN(W,I$~~ $a \leq b$ mod I, $EN(W,I|a) \subseteq EN(W,I|b)$, from ( ): $x \in EN(W,I|b)$. The set of ~~Select any~~ $x' \in S\{W,I|a/$ contained in $EN\{W,I|b\}$ ~~is nonvoi~~ contains x and is nonvoid. Select an x' from this set, so that $bx' = a$ $+s'$ ~~mod I~~ where $s' \in I$. Since $I \in B_{+;2s}(W)$, $b(x-x') = \overset{\cancel{s}\in I}{0}$ ~~mod I~~. Since, from ( ), $EN\{W,I|b\} \in B_{+;2s}(W)$, $x-x' \in b$ mod I. ~~So~~ From ( ), $x-x' \in I$, ~~from or from~~ since $I \in \mathbb{I}_{QF}(W)$. All members of $S\{W,I|a/b\}$ contained in $EN\{W,I|b)$ are equal to x mod I. This holds in particular for all members of

$MS\{W,I|a/b\}$: $MS\{W,I|a/b\} \subseteq x+\underline{I}$. For any $y\in\underline{I}$, $x+y\in S\{W,I|a/b\}$ and $x+y\equiv a \bmod \underline{I}$, so that $x+y\in MS\{W,I|a/b\}$: $x+\underline{I}\subseteq MS\{W,I|a/b\}$, and $x+\underline{I}\equiv MS\{W,I|a/b\}$. All members of $x+\underline{I}$ are mutually equal mod $\underline{I}$. Part (2) has been disposed of.

It was shown above that any $x'\in S\{W,I|a/b\}$ contained in $EN\{W,I|b\}$ belongs to $x+\underline{I}\equiv MS\{W,I|a/b\}$. ~~Since~~ $N(W,I|b)\equiv EN(W,I|b)$ when $b\in\Delta\{E(W,I),I\}$: ~~Par~~ part (3) has been ~~obtain~~ proved.

The number x described in part ④ is the number shown in the proof of ( ) to be a member of $MS\{W,I|a/b\}$. That $MS\{W,I|a/b\}\equiv x+\underline{I}$ follows from the above. Use of another lifting factor-displaced orthogonal complement pair results in the determination of another $x'\in W$ for which equally $MS\{W,I|a/b\}\equiv x'+\underline{I}$.

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om $ye=c+s$ $s\in\underline{I}$

~~$ab+v=e$~~ $ua+v=e\in E$ $bx=a+t$ $(ub)x+v=e+ut$

~~$\underline{I}\in\underline{\underline{I}}(W)$~~. need $I\in B_{x;si}$ $E+\underline{I}\in E$ | $y(e+z)=c+s+yz$

$I\in B_{x;si}\cap B_{+;c}$ $a\leq b \bmod \underline{I}$ $\leftarrow$ | $\underline{I}\in B_{x;si}\cap B_{+;c}$

$a\in\Delta''(E(W,I))\to S\{W,I|a/b\}\in\Delta''(E(W,I))$

$a\vee\in\underline{I}\to x\vee\in I$ for $x\in MS\{W,I|a/b\}$

$a,b\in\Delta'(E(W,I),I)\to MS\{W,I|a/b\}\in\Delta'(E(W,I),I)$

always b|a mod I, a∈Δ′ no.

$ub = (ua+v)y + w$ [illegible] $x = ba$

$v = (ua+v)y$ $av \in I \rightarrow vag \in I$

$b \in \Delta'\{E(W,I), I\}$ $a \leq b \bmod I$ $a \in \Delta\{E(W,I), I\}$

$ubx + v = e + ut$ $ua+v = e$ $av \in I$ $a \leq u$

$bx = a+t$ $u\underset{0}{g} \in I \rightarrow a\underset{0}{g} \in I \rightarrow bx\underset{0}{g} \in I$

( ) Let $I \in B_{x;si} \cap B_{+;c}(W)$ and $b|a \bmod I$. If $a \in \Delta''\{E(W,I)\}$, $S\{W, I|a/b\} \in \Delta''\{E(W,I)\}$

Since $b|a \bmod I$, ~~$x \in W$ and~~ $S\{W, I|a/b\}$ is nonvoid and for any $x \in S\{W, I|a/b\}$, $bx = a + t(x)$ where $t(x) \in I$. If $a \in \Delta''\{E(W,I)\}$ then $u, v \in W$ exist such that $ua + v = e \in E(W,I)$ or a simpler condition holds. Then $ubx + v = e + ut(x)$. When $I \in B_{x;si}(W)$, ~~[illegible]~~ $ut(x) \in I$, and when $I \in B_{x;si} \cap B_{+;c}(W)$, $E(W,I) + I \subseteq E(W,I)$, from ( ). Hence $ubx + v \in E(W,I)$. ub and v function as $E(W,I)$ lifting factor-displaced orthogonal complement pair ~~for a~~ for x: $S\{W, I|a/b\} \in \Delta''\{E(W,I)\}$.

( ) Let $I \in \mathbb{I}(W)$ and $a \leq b \bmod I$. If $a, b \in \Delta'\{E(W,I), I\}$, $MS\{W, I|a/b\} \in \Delta'\{E(W,I), I\}$.

From ( ), the three conditions $I \in I(W)$, $a \leq b \bmod I$

and $b \in \Delta'\{E(W,I), I\}$ suffice to ensure that $MS\{W, I|a/b\}$
and for all $x \in MS\{W, I|a/b\}$, $x \equiv a \bmod I$ ~~where~~ with
is nonvoid: $x \equiv a \bmod I$ ~~exist~~ ~~for which~~ $bx = a + t(x)$ ㉓
where with $t(x) \in I$. ~~exists in W~~. As in the preceding proof,
$ubx + v \in E(W,I)$ ~~Since av~~ where now $av \in I$. Since
$x \le a \bmod I$, $xv \in I$: $x \in \Delta'\{E(W,I), I\}$.

( ) Let $I \in \mathbb{I}(W)$, ~~as~~ $a \le b \bmod I$ and $b \in \Delta'\{E(W,I), I\}$.
If $a \in \Delta\{E(W,I), I\}$, $MS\{W, I|a/b\} \in \Delta^{*}\{E(W,I), I\}$

As in the preceding proof, $MS\{W, I|a/b\}$ is nonvoid and for all $x \in MS\{W, I|a/b\}$, $x \equiv a \bmod I$ with $bx = a + t(x)$ where $t(x) \in I$. Also $ubx + v \in E(W,I)$, where $av \in I$ and now $a \le u \bmod I$. The three conditions $a \le u$, $a \le b$, $x \le a$ $\bmod I$ imply that $x^2 \le ub \bmod I$ and, since $I \in \mathbb{I}_{QF}(W)$, $x \le ub \bmod I$, from ( ): $x \in \Delta'\{E(W,I), I\}$.
Since $av \in I$ and ~~using~~ $x \equiv a \bmod I$, $xv \in I$: $x \in \Delta\{E(W,I), I\}$.

$\{a/b\} \subseteq \{ca/cb\}$ all $\overline{c \in R}$ $I \in B_{x;SI}(W)$ $bx = a + t$ $cbx = ca + ct$

$c \in Ri$ $\{a/b\} = \{ca/cb\}$ $c(bx - a) = s \rightarrow bx - a = t \in I$ | $b\underset{0}{g} \in I \rightarrow a$
$c\underset{0}{y} \in I$

~~$a \le b \bmod I$~~ $cb | ca \bmod I$ $b \le c \bmod I$ $I \in \mathbb{I}_{QF}(W)$

$b \le c \rightarrow bx \le c$ if $I \in B_{x;si}$ | $I \in B_{x;si}$ $a \le \vec{c}$ $a c s c$ $B_{x;qf}(W)$:
$c a \underset{0}{g} \in I$

$cb | ca \bmod I \rightarrow ca \le cb \bmod I$ $a \le c$ $b \le c$ $a\underset{0}{g} = t$ $b\underset{0}{g} =$

$b\underset{0}{g} \in I \rightarrow cb\underset{0}{g} \in I$ $(I \in B_{x;si}) \rightarrow ca\underset{0}{g} \in I \rightarrow a\underset{0}{g} \in I$ $(a\underset{0}{g} \le c)$ ~~$I \in$~~ $I \in B_{x;qf}$

$ba \in I$ $a \le b \rightarrow a^2 \in I$ ~~$I \in$~~ $I \in B_{x;si} \cap B_{x;qf}$

$a, b \le c \bmod I$ $cb | ca \bmod I \rightarrow \{a/b\} = \{ca/cb\}$ I

$c \in Ri$ $I \in B_{X;si}$ $MS\{a/b\} \equiv MS\{ca/cb\}$

$c \in Ri \rightarrow a \equiv ca$ $x \in MS\{a/b\} \rightarrow$ ~~$x \in$~~ $S\{ca/$

$x \in S\{a/b\}$ $x \equiv a \rightarrow x \in S\{ca/cb\}$ & $x \equiv ca$ also $\Leftarrow$.

—o—

$a, b \leq c \bmod I$ $cb|ca \bmod I$ $I \in B_{X;si} \cap B_{X;qf}(W)$

$MS\{a/b\} \equiv MS\{ca/cb\}$

( ) Let $I \in B_{X;si}(W)$. $S\{W, I|a/b\} \subseteq S\{W, I|ca/cb\}$ for all $a, b, c \in W$.

If $S\{W, I|a/b\}$ is void, the result is correct. Otherwise, select x in this set, so that $bx = a + t(x)$ with $t(x) \in I$. The $(cb)x = ca + s(x)$ where $s(x) = ct(x) \in I$, since $I \in B_{X;si}(W)$: $x \in S\{W, I | ca/cb\}$.

( ) Let $I \in B_{X;si}(W)$ and $c \in Ri(W, I)$.

$S\{W, I | a/b\} \equiv S\{W, I|\overset{ca/cb}{\cancel{a/b}}\}$ and $MS\{W, I|a/b\} \equiv MS\{W, I|\overset{ca/cb}{\cancel{a/b}}\}$ for all $a, b \in W$.

From ( ), $S\{W, I|a/b\} \subseteq SW\{W, I|ca/cb\}$. If $S\{W, I|\overset{ca/cb}{\cancel{a/b}}\}$ is void, $S\{W, I|\cancel{a/b}\,ca/cb\} \subseteq S\{W, I|a/b\}$ and the first result is correct. Otherwise, select $x \in S\{W, I|ca/$ so that $c(bx-a) =$ ~~t~~ $s \in I$. Since $c \in Ri(W, I)$, $t \in I$ for which $bx - a = t$ exists: $x \in S\{W, I|a/b\}$ and $S\{W, I|ca/cb\} \subseteq S\{W, I|a/b\}$.

If $MS\{W, I|a/b\}$ is void, $MS\{W, I|a/b\} \subseteq MS\{W, I|ca/cb\}$. Otherwise, all members x of $S\{W, I|a/b\}$ for which

$x\equiv a$ mod I are in $S\{W,I|$ ~~a/b~~ $ca/cb\}$, from ( ), and $x\equiv ca$ mod I also for these x, from ( ): $MS\{W,I|a/b\}\leq MS\{W,I|ca/cb\}$. That $MS\{W,I|ca/cb\}\leq MS\{W,I|a/b\}$ is proved in the same way.

( ) Let $a,b\leq c$ mod I, and $cb|ca$ mod I, where $I\in B_{\times;si}\cap B_{\times;qf}\cap B_{+;\ell s}(W)$. $S\{W,I|a/b\}\equiv SW\{W,I|ca/cb\}$ and $MS\{W,I|a/b\}\equiv MS\{W,I|$ ~~a/b~~ $ca/cb\}$.

Since $cb|ca$ mod I, W contains x for which $c(bx-a)=s\in I$. Since $I\in B_{\times;si}(W)$ and $b\leq c$ mod I, $bx\leq c$ mod I, from ( ). Since $bx,a\leq c$ mod I and $I\in B_{+;\ell s}(W)$, $bx-a\leq c$ mod I, from ( ). Since $I\in B_{\times;qf}(W)$ and $c(bx-a)\in I$, $bx-a\in I$ from ( ): $b|a$ mod I: $S\{W,I/a/b\}$ is nonvoid and $SW\{W,I|ca/cb\}\leq SW\{W,I|a/b\}$. From ( ), $SW\{W,I|a/b\}\leq SW\{W,I|ca/cb\}$: $S\{W,I|a/b\}\equiv S\{W,I|ca/cb\}$.

From ( ), $ac\equiv a$ mod I when $a\leq c$ mod I and $I\in B_{\times;si}\cap B_{\times;qf}(W)$. Using this result, the proof of the second part is as for that of ( ).

$D(W, \underline{I}|a) \subseteq ED(W, \underline{I}|a)$ conditions on $\underline{I}$

(4) $B_{\phi; ls}$ (5) $B_{\phi; si}$ abs pr ; $B_{\phi; sa}$ self absorbed

$M \in B_{x;c}$ $\underline{I} \in B_{x;so} \cap B_{t;c}$ $M + \underline{I} \in B_{x;c}(W)$

conditions on $\underline{I}$ in ths,

(1) $S\{W, \underline{I}|a/b\} \in D\{W, \underline{I}|a\}$

(2) - - - $ED\{$ " $\}$ conditions on $\underline{I}$

proof $D \subseteq ED$

(33, 4, 5) (6) $\underline{I} \in \mathbb{I}(W)$. $MS\{W, \underline{I}|a/b\}$ nonvoid

$MS\{W, \underline{I}|a/b\} \equiv MS\{W, \underline{I}|a/b\} + \underline{I}$

remove " from $b \in \Delta'\{E(W, \underline{I}); \underline{I}\} \ldots$

$=$

$\underline{I} \in \underline{\mathbb{I}}_{QF}(W)$ (1) MS nonvoid

(2) All members of $MS\{W, \underline{I}|a/b\}$ are equal mod $\underline{I}$.

x being any member of $MS\{W, \underline{I}|a/b\}$, $MS\{W, \underline{I}|a/b\} = x + \underline{I}$.

(3) All .. $S\{W, \underline{I}|a/b\}$ .. $N\{W, \underline{I}|b\}$ belong to $MS\{W, \underline{I}$.

$c\{a/b\} \subseteq \{ca/b\}$ $ty = ca + s$ $c \in E$ $y = cx + t$

$S, MS,\ c\{a/b\} = \{ca/b\}$ $I \in B_{x;si} \cap B_{+;ls}$

$y = ca$ $uc + w \in E$ $cw \in I$ ~~$y \leq c \to uy \leq c$~~

$(uc+w)x = uy + z$ $uby = uca + us$ $y \leq ac \to y \leq c$

$\to uy \leq c$

$(uc+w)bx = uby + bz = uca + us + bz$

$= (uc+w)a + us + bz - wa$ $b \leq c$

$\to bw \in I$

$\underbrace{uc}bx = \underbrace{uc}a + us + bz - h$

$c \leq u$ $a \leq b \leq c \leq u$ $bx = a \text{ mod } I$

$c \in \Delta'$ $a \leq c$ $uc + w \in E$ $cw \in I$

$y \in \{ca/b\}$ $by = ca + s$ $s \in I$

$(uc+w)x = uy + z$ $z \in I$

$MS\{ca/b\}$ nonvoid $\exists\, y \leq b \text{ mod } I$

$byw \in I$ $I \in \mathbb{I}_{gf} \to yw \in I$

$(uc+w)xw \in I \to xw \in I$

$ucx = uy \text{ mod } I \,\|\, c \in \Delta : c \leq u$

$cg \in I \to byg \in I \to yg \in I \to y \leq c$

$(uc+w)bx = uby + bz = uca + us + bz \mid \to y \leq c \to cx = y \text{ mod } I$

$= (uc+w)a + us + bz - wa$

$= (uc+w)a \text{ mod } I \ (a \leq c,\ cw \in I)$

$bx = a \text{ mod } I$ $(uc+w) \in Ri$

but ~~[illegible]~~ $cx \neq y$ ‖

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$b \in \Delta'$ $a \leq b$ $a \leq c$ $MS\{a/b\}$ nonvoid also $MS\{ca/b\}$

$(ub+v)y = \overset{uca}{ca}$ mod I $(ub+v)x = \overset{ua}{a}$ mod I $(ub+v)(y - cx) = 0 \text{ mod } I$

$y = cx \text{ mod } I$

$0 \in J \subseteq I$

$a \leq b$ $ca \leq cb$? $cbg \in I \Rightarrow a(cg) \in I$

$b\in\Delta'$ $ca\le b \to b|ca \to by=ca+s$

$b\in\Delta'$ $ca\le b$ mod $\underline{I}$ $c\in\Delta$ $\underline{I}\in\underline{\mathrm{I}}_{QF}$ $a\le c$

$MS\{ca/b\}$ nonvoid $y,y'\in MS\{ca/b\}\to y\equiv y'$ mod $\underline{I}$

~~[illegible]~~ Select $y\in MS\{ca/b\}$

~~[illegible]~~ $by=ca+s$ $s\in\underline{I}$ $y\le b$ (also $y\le ca$)

$c\in\Delta$ $uc+w\in E$, $cw\in\underline{I}$ ~~[illegible]~~ $c\le u$ mod $\underline{I}$

$(uc+w)x=uy+z$ $(uc+w\in E)$

$(uc+w)xw=uwy+z$ $y\le ca\le c, cw\in\underline{I}\to \boxed{yw\in\underline{I}}$

$(uc+w)xw\in\underline{I}\to xw\in\underline{I}$ $(uc+w\in$ ~~[illegible]~~ $Ri)$

$ucx=uy+t$ $t\in\underline{I}$

$cx=y+r$ $r\in\underline{I}$ $(c\le u, y\le c\le u; \underline{I}\in\underline{\mathrm{I}}_{QF}(W))$

$bcx=ca+p$ $p\in\underline{I}$ $(p=s-\underline{br})$

$yg_0\in\underline{I}$ $(uc+w)xg_0=uyg_0$ mod $\underline{I}\to xg_0\in\underline{I}$ $(uc+w\in Ri)$

$x\le y\to x\le ca\le \overset{a,c}{\cancel{ac}}$ mod $\underline{I}$

$a\le c$ mod $\underline{I}$ assumed

$bcx=ca$ mod $\underline{I}\to bx=a$ mod $\underline{I}$

with $x\le a$: $x\in MS\{a/b\}$

$MS\{ca/b\}\le cMS\{a/b\}$

$S\{ca/b\}\equiv y+O(b)$ ~~[illegible]~~ $cS\{a/b\}\equiv cx+cO(b)$
Then $S\{ca/b\}\not\equiv cS\{a/b\}$.

$c=b$ possible: $cO(b)\le\underline{I}$. $O(b)\ne\underline{I}$ $b\in 1(W,\underline{I})$ $b=c$

select $y'\le b$ $y=y'+v\in S\{a/b\}$ $bv\in\underline{I}$ $v\notin\underline{I}$

$b=c$ $by=by'+bv\le b$ possible $y\not\le b$

$bx=a$ mod $\underline{I}$

$cx\le b\to cx=cy$

$by=a$ mod $\underline{I}$ $a\equiv y$

$cx\le a$

$N(c)\cap S\{a/b\}$

$\le cMS\{a/b\}$

$b\in\Delta'(W,\underline{I}), c\in\Delta(W,\underline{I})$

$a\le c, ca\le b$ mod $\underline{I}$

$\underline{I}\in\underline{\mathrm{I}}_{QF}(W)$ MS.. a/b, ca/b nonvoid and

$cMS\{W,\underline{I}|a/b\}\equiv MS\{W,\underline{I}|ca/b\}$

⇒ All members of ? $MS\{W,\underline{I}|a/b\}$, $cMS\{W,\underline{I}|a/b\}$ and $MS\{W,\underline{I}|ca/b\}$ equal mod $\underline{I}$. and these sets constitute the only members of $S\{W,\underline{I}|a/b\}$, ~~$cS\{W,\underline{I}|a/b\}$~~ and $S\{W,\underline{I}|ca/b\}$ respectively to be found in $N\{W,\underline{I}|b'\}$.

$cx\le a$

$x\le a$

$x\le c$

$z\le b$ $bz\le a$

$z\le ab$ $z\le a$

$ac\le b$ $a\le bc$

$a\le bc$ $a\le b$

$cy\le b$ ~~[illegible]~~ $y\in S\{a/b\}\to y\le b$?

$by=a+u$ $bcy=ca+u'$

not only members confined to $cMS\{W,\underline{I}|a/b\}$

$cS\{W,\underline{I}|a/b\}\not\equiv S\{$..

$b=c$: all members of $cS\{W,\underline{I}|a/b\}$ contained in $N\{W,\underline{I}/b\}$

Teoremas de Configuración   Argunov e Skorniakov

La regla en construcciones geometricas

Smogorzhevski

Induccion en la geometria   Golovina e Yaglom

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( ) Let $I \in B_{x;si}(W)$. $cS\{W,I|a/b\} \subseteq S\{W,I|ca/b\}$ and $cMS\{W,I|a/b\} \subseteq MS\{W,I|ca/b\}$ for all $a,b,c \in W$.

Select $a,b,c \in W$. If $S\{W,I|a/b\}$ is void, the first stated relationship holds. Otherwise select any x in this set: I contains $s(a,b;x)$ for which $bx=a+s(a,b;x)$. ~~exists~~ $by=ca+t(a,b;x)$ where $y=cx$ and $t(a,b;x)=cs(a,b;x)$; and, since $I \in B_{x;si}(W)$, $t(a,b;x) \in I$: $y \in S\{W,I|ca/b\}$: ~~cS{W,~~ the first stated relationship again holds.

Intp.

( ) Let ~~c∈E(W,I) where~~ $I \in B_{x;si} \cap B_{+;ls}(W)$, $0 \in J \subseteq I$ and $c \in E(W,J)$. $cS\{W,I|a/b\} \overset{\text{mod } J}{=} S\{W,I|ca/b\}$ and $cMS\{W,I|a/b\} \overset{\text{mod } J}{=} MS\{W,I|ca/b\}$ for all $a,b \in W$.

Select $a,b \in W$. From ( ), $cS\{W,I|a/b\} \subseteq S\{W,I|ca/b\}$ mod J. If $S\{W,I|ca/b\}$ is void, $cS\{W,I|a/b\} = S\{W,I|ca/b\}$ mod J. Otherwise, select $y \in S\{W,I|ca/b\}$, so that $by=ca+t$ for some $t \in I$. Since $c \in E(W,I)$, W contains x such that $y=cx+w$ where $w \in J$ ($J \subseteq I$). $c(bx-a)=z$ where $z=t-bw \in I$, since $I \in B_{x;si} \cap B_{+;ls}(W)$. Since $c \in Ri(W,I)$, I contains s such that $bx=a+s$: $x \in S\{W,I|a/b\}$: $S\{W,I|ca/b\} \subseteq cS\{W,I|a/b\}$ mod J.

i) If $MS\{W, \underline{I}/a/b\}$ is void, $cMS\{W, \underline{I}/a/b\} \subseteq MS\{W, \underline{I}/ca/b\}$. Otherwise select $x \in MS\{W, \underline{I}/a/b\}$. ~~From ( ), $y = cx$ is present in~~ so that $x \equiv a \bmod \underline{I}$. From ( ), $y = cx$ is present in $S\{W, \underline{I}/ca/b\}$. Also $y \equiv ca \bmod \underline{I}$ ~~and, since $c \in Ri\{W, \underline{J}\} \subseteq Ri\{W, \underline{I}\}$,~~ so that $y \in MS\{W, \underline{I}/ca/b\}$: $cMS\{W, \underline{I}/a/b\} \subseteq MS\{W, \underline{I}/ca/b\}$ also when $MS\{W, \underline{I}/a/b\}$ is nonvoid. From ( ), $cMS\{W, \underline{I}/a/b\} \subseteq MS\{W, \underline{I}/ca/b\}$ mod $\underline{J}$. mod $\underline{J}$.

If $MS\{W, \underline{I}/ca/b\}$ is void, $MS\{W, \underline{I}/ca/b\} \subseteq cMS\{W, \underline{I}/a/b\}$. Otherwise select $y \in MS\{W, \underline{I}/ca/b\}$ so that $y \equiv ca \bmod \underline{I}$. From the proof of the first result, $S\{W, \underline{I}/a/b\}$ contains x for which $y = cx + w$ where $w \in \underline{J} \subseteq \underline{I}$ so that $y \equiv cx \bmod \underline{I}$ and $ca \equiv cx \bmod \underline{I}$. From (,) $c \in E(W, \underline{J}) \subseteq E(W, \underline{I}) \subseteq Ri(W, \underline{I})$: $a \equiv x \bmod \underline{I}$ and $x \in MS\{W, \underline{I}/a/b\}$: $MS\{W, \underline{I}/ca/b\} \subseteq cMS\{W, \underline{I}/a/b\}$ mod $\underline{J}$ also when $MS\{W, \underline{I}/ca/b\}$ is nonvoid.

( ) Let $b \in \Delta'(W, \underline{I})$, $c \in \overset{?}{\Delta}(W, \underline{I})$, and $a \leq c$, $ca \leq b \bmod \underline{I}$ where $\underline{I} \in \mathbb{I}_{QF}(W)$.

1) $MS\{W, \underline{I}/a/b\}$ and $MS\{W, \underline{I}/ca/b\}$ are nonvoid.

2) All members of $MS\{W, \underline{I}/a/b\}$ are equal mod $\underline{I}$; the same holds true for the sets $cMS\{W, \underline{I}/a/b\}$ and $MS\{W, \underline{I}/ca/b\}$.

3) The sets $MS\{W, I|a/b\}$, $cMS\{W, I|a/b\}$ and $MS\{W, I|ca/b\}$ constitute the only members of $S\{W, I|a/b\}$, $c\{N(W, I|c) \cap S(W, I|a/b)\}$ and $S\{W, I|ca/b\}$ respectively to be found in $N\{W, I|b\}$.

4) $cMS\{W, I|a/b\} = MS\{W, I|ca/b\}$ mod I ?

The results of (1–3) concerning the sets $MS\{W, I|ca/b\}$ follow directly from ( ), since $b \in \Delta'(W, I)$, $ca \le b$ mod I and $I \in \mathbb{I}_{QF}(W)$. From ( ), the conditions $ac \le b$, $a \le c$ mod I ~~are~~ imply, when $I \in \mathbb{I}_{QF}(W)$, that $a \le b$ mod I. From ( ), the conditions $b \in \Delta'(W, I)$ and $\{a \le b$ mod I imply that $MS\{W, I|a/b\}$ is nonvoid and, from ( ), the further condition $I \in \mathbb{I}_{QF}(W)$ implies both that all members of $MS\{W, I|a/b\}$ are equal mod I and that $MS\{W, I|a/b\}$ contains all members of $S\{W, I|a/b\}$ belonging to $N(W, I|b)$. All pairs of members of $cMS\{W, I|a/b\}$ have the form cx, cx' where $x, x' \in MS\{W, I|a/b\}$. Since $x = x'$ mod I, ~~$cx = cx'$ mo~~ and $I \in I_{x;si}(W)$, $cx = cx'$ mod I: all members of $cMS\{W, I|a/b\}$ are equal mod I. ~~Parts (1-3) have been disposed of.~~

○⇒ ~~Th Select $x \in MS\{W, I|a/b\}$. From ( ), $cx \in S\{W, I|ca/b\}$.~~

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~~Also, since cx $x \equiv a$ mod I, $cx \equiv ca$ mod I: $cx \in MS\{W, I|ca/b\}$.~~

~~$cMS\{W, I|a/b\} \le$~~

From ( ), $cMS\{W, I|a/b\} \le MS\{W, I|ca/b\}$ mod I. Select

$by=ca+t$ $t\in J$ ‖ $c\in\Delta(W,J)$ $uc+w\in E(W,J)$ $cw\in J$ $c\le u \bmod J$

$(uc+w)x=uy+z$ $z\in J$ $(uc+w)xw=uyw+zw$ ‖ $y\le ca\le a$

$y\le ca \bmod \underline{I}$ ‖ $\underline{I}=J$: $cw\in\underline{I}\to yw\in\underline{I}\to xw\in\underline{I}$ ‖ $ag_0\in\underline{I}\to yg_0\in\underline{I}\to xg_0\in\underline{I}$

~~$y=cx+p$~~

$ucx=uy+z-wx$ $z-wx\in\underline{I}$ $c\le u$ $y\le c\le u\to$ ~~$cx=y+p$~~ $cx=y+p$

$p\in\underline{I}$ ~~bx=~~ $yg_0\in I\to xg_0\in\underline{I}\to x\le y\le c$

$\$ c(bx-a)=t+bp$ $x\le c, a\le c\to bx-a\in I$

$a\le b \bmod \underline{I}$ $bg_0\in J\to by\in\underline{I}\to ag_0\in\underline{I}\to ag_0\in I\setminus J$? $J\le\underline{I}$

$y\in MS\{W,I|ca/b\}$ which, from (1), is known to be nonvoid. ~~I cont~~ $\underline{I}$ contains t for which $by=ca+t$, and $y\equiv ca\le c \bmod \underline{I}$. Since $c\in$ ~~$\Delta(W,I)$,~~ $\Delta\{E(W,I),\underline{I}\}$, $u,w\in W$ for which $uc+w\in E(W,I)$, $cw\in\underline{I}$, $c\le u \bmod \underline{I}$ exist, or a simpler condition holds. Since $uc+w\in E(W,I)$, W contains x for which $(uc+w)x=uy+z$ where $z\in\underline{I}$. Since $cw\in\underline{I}$ and $y\in c \bmod \underline{I}$, $yw\in I$ so that $(uc+w)wx=uyw+wz\in\underline{I}$. $(uc+w)\in Ri(W,I)$ and hence $wx\in\underline{I}$: ~~wx~~ $u(cx-y)=z-wx\in\underline{I}$. Since ~~$c\le y$~~ $c\le u$, ~~and~~ $y\le c\le u \bmod I$ and $\underline{I}\in\underline{\mathbb{I}}_{QF}(W)$, $cx=y+p$ where $p\in\underline{I}$. Since $(uc+w)x=uy+z$ where $z\in\underline{I}$, and $(uc+w)\in Ri(W,I)$, the condition $yg_0\in\underline{I}$ implies that $xg_0\in I$: $x\le y\le c \bmod \underline{I}$. Thus ~~$c(bx-a)=t+bp\in\underline{I}$~~ since $by=ca+t$ and $cx=y+p$, it follows that $c(bx-a)=t+bp\in\underline{I}$ and since $bx, a\le c \bmod \underline{I}$, and $\underline{I}\in\underline{\mathbb{I}}_{QF}(W)$, $bx=a \bmod \underline{I}$: $x\in S\{W,I|a/b\}$. Since

$y \in MS\{W, I|ca/b\}$, $y \leq ca \leq a \bmod I$. Also $(uc+w)x = uy+z$ where $uc+w \in Ri(W,I)$ and $z \in I$. Thus if $ag \in I$, ~~yg~~ then $yg \in \underline{I}$ and $xg \in \underline{I}$: $x \leq a \bmod I$. $a \leq x \bmod \underline{I}$ for any x satisfying the equation $bx = a \bmod \underline{I}$: $x \equiv a \bmod \underline{I}$ and $x \equiv MS\{W, I|a/b\}$. Any $y \in MS\{W, I|ca/b\}$ has a representation of the form $y = cx - p$ where ~~§~~ $x \in MS\{W, I|a/b\}$ and $p \in I$. $MS\{W, I|ca/b\} \leq c\,MS\{W, I|a/b\} \bmod \underline{I}$. ~~The result of~~ Part (4) has been dealt with.

~~With regard to part (3) ~~it~~ is pointed out that all members of $cS\{W, I|a/b\}$ contained in $N\{W, I|b\}$ may not be confined to $cMS\{W, I|a/b\}$. With x selected from $MS\{W, I|a/b\}$, $S\{W, I|a/b\} \equiv x + \Theta(W, I|b)$.~~

$bx = a \bmod I$, $cx \leq b$ ~~[illegible]~~

$cx^2 \leq bx \leq a \rightarrow cx \leq a$ $x \leq c \rightarrow x \leq a \rightarrow x \equiv a$

$c\{N(W, I|c) \cap S(W, I|a/b)\} \leq cMS(W, I|a/b)$

$cMS\{W, I|a/b\} \ldots$ $c\{N(W, I|c) \cap S(W, I|a/b)\}$ ~~$\}$~~ ... to be found in $N\{W, I|b\}$

↖ To prove the second result of part (3), select $x \in N(W, I|c) \cap S(W, I|a/b)$ so that $x \leq c \bmod I$ and, since $bx = a \bmod \underline{I}$, $bx \leq a \bmod \underline{I}$. If $cx \in N(W, I|b)$, $cx \leq b \bmod I$, so that $cx^2 \leq bx \leq a \bmod I$ and, since $I \in \underline{\Pi}_{QF}(W)$,

$cx \le a \bmod I$. From ( ), the two conditions $x \le c \bmod I$, $cx \le a \bmod I$ imply, when $I \in \mathbb{I}_{SP}(W)$, that $x \le a \bmod I$. For all $x \in S\{W, I|a/b\}$, $a \le x \bmod I$: $x \equiv a \bmod I$: $x \in MS\{W, I|a/b\}$ and $cx \in cMS\{W, I|a/b\}$. All members of $c\{N(W, I|c) \cap S(W, I|a/b)\}$ belonging to $N(W, I|b)$ are contained in $cMS\{W, I|a/b\}$. Any $x \in MS\{W, I|a/b\}$ belongs to $S\{W, I|a/b\}$. Also $x \equiv a \bmod I$ for such an x : $x \le a \bmod I$ and by assumption $a \le c \bmod I$: $x \le c \bmod I$: $x \in N(W, I|c)$: $cx \in c\{N(W, I|c) \cap S(W, I|a/b)\}$. In conclusion $cMS\{W, I|a/b\}$ ~~$c\{N(W, I|c) \cap S(W, I|a/b\} \equiv c$~~ $\le c\{N(W, I|c) \cap S(W, I|a/b)\}$. The set $cMS\{W, I|a/b\}$ ~~constitutes~~ consists of the only members of $c\{N(W, I|c) \cap S(W, I|a/b)\}$ to be found in $N(W, I|c)$. Parts (1–3) have been disposed of.

With regard to part (4) it is pointed out that the relationship $cS(W, I|a/b) \equiv S\{ca/b\}$ may be false. With ~~$x \in S\{a/b\}$, $\not c S$~~ $x \in S\{W, I|a/b\}$, ~~c~~$S\{W, I|a/b\} \equiv$ ~~c~~$x +$ ~~c~~$O(W, I|b)$ and $cS\{W, I|a/b\} \equiv cx + cO(W, I|b)$~~$V$~~. Similarly, with $y \in S\{W, I|ca/b\}$, $S\{W, I|ca/b\} \equiv y + O(W, I|b)$. It is possible to select $x \in MS\{W, I|a/b\}$ and $y = cx$. It is also possible ~~to set~~ that under suitable conditions in the theorem, $b = c$. Taking $b \in T(W, I)$, $v \in W \setminus I$ exists for which $bv \in I$. $cS(W, I|a/b)$ has the form $cx + I'$ where $I' \equiv bO(W, I|b) \le I$. $S\{$~~$ca/$~~$W, I|ca/b\}$ contains $cx + v$ with $v \notin I$.

$\underline{I} \in B_{x;si} \cap B_{+;c}(W) \to MS\{W, I|a/b\} + \underline{I} \subseteq MS\{W, I|a/b\}$ ㊶

$MS\{W, I|a/b\}$ vid: correct $bx = a + s \quad s \in \underline{I} \quad x \equiv a$

~~$x+t$~~ $b(x+t) = a + s + bt \to x+t \in S\{W, I|a/b\}$

$\underline{x\breve{o} \in \underline{I} \leftrightarrow a\breve{o} \in \underline{I}} \quad (x+t)\breve{o} \in \underline{I}$

$v \in \breve{O}(b) \quad bv \in \underline{I} \quad \underset{\in \underline{I}}{b(v+t) = bv + bt} \quad \underline{I} \in B_{x;si} \cap B_{+;c}(W)$

$\underline{I} \in B_{x;si} \cap B_{+;c}(W) \to \breve{O}(b) + \underline{I} \subseteq \breve{O}(b) \;\|\; \underline{I} \in B_{x;si}(W) \to$

$0 \in \underline{I} \to \breve{O}(b) \subseteq \breve{O}(b) + \underline{I} \qquad 0 \in \underline{I}$

[~~$\underline{I} \in B_{x;si} \cap B_{+;c}(W) \to \breve{O}(b)$~~ $\underline{I} \in B_{x;si}(W) \to \breve{O}(b) \subseteq \breve{O}(b) + \underline{I}$

$\underline{I} \in B_{x;si} \overset{\Downarrow ?}{\cap} B_{+;c}(W) \to \breve{O}(b) + \underline{I} = \breve{O}(b)$ ‖ $\boxed{\breve{O}(B)\text{: all } v \text{ such that } bv \in I \text{ for all } b \in B}$

~~$(b+t)v \in \underline{I} \quad \underline{I} \in B_{x;si} \cap B_{+;ls}(W) \quad b+t \; bv + tv = w$~~

$bv = y \in \underline{I} \quad (b+t)v = y + tv \quad \underline{I} \in B_{x;si} \cap B_{+;c} \to$ ~~$b+t \in \breve{O}$~~

$v \in \breve{O}(b+t) \quad : \underline{I} \in B_{x;si} \cap B_{+;c} \to \breve{O}(b) \equiv \breve{O}(b+\underline{I})$]

$B \subseteq B' \to \breve{O}(B') \subseteq \breve{O}(B) \to \breve{O}(b+\underline{I}) \subseteq \breve{O}(b)$

~~$(b+t)v = y \in \underline{I}$~~ $a \leq b \bmod \underline{I} \quad \breve{O}(b) \subseteq \breve{O}(a) \qquad b = a + c(a,b)$

$a \leq b + \underline{I}$] $\qquad a = b$

$bv = y \in \underline{I} \quad (b+t)v = y + tv \quad \underline{I} \in B_{x;si}(W) \cap B_{+;c}$

~~$v\in$~~ $\Rightarrow v \in \breve{O}(b) \to v \in \breve{O}(b+t) \;|\; \Rightarrow \underset{\text{all } t \in \underline{I}}{\breve{O}(b) \subseteq \breve{O}(b+t)}$

$(b+t)v = z \in \underline{I} \quad bv = z - tv \quad \underline{I} \in B_{x;si}(W) \cap B_{+;ls}$

$\| \; \| \qquad \Rightarrow \breve{O}(b+t) \subseteq \breve{O}(b)$ all $t \in \underline{I}$

$\underline{I}\in B_2(W) \to \tilde{O}(b)\equiv\tilde{O}(b+t)$ all $t\in \underline{I}$

$a\leq b \bmod \underline{I} \Longleftrightarrow \tilde{O}(b)\subseteq\tilde{O}(a)$

=

$a\leq b \bmod \underline{I},\ \underline{I}\in B_{\times;si}(W)\cap B_{+;ls}$

$\Rightarrow a\leq b+t$ all $t\in\underline{I}$: $\tilde{O}(b+t)\subseteq\tilde{O}(b),\ \tilde{O}(b)\subseteq\tilde{O}(a)$

$\longrightarrow \tilde{O}(b+t)\subseteq\tilde{O}(a)$

=

$a\leq b \bmod \underline{I},\ \underline{I}\in B_{\times;si}(W)\cap B_{+;c}$

$\Rightarrow a+t\leq b \bmod \underline{I}$ all $t\in I$: $\tilde{O}(b)\subseteq\tilde{O}(a),\ \tilde{O}(a)\subseteq\tilde{O}(a+t)$

$\longrightarrow \tilde{O}(b)\subseteq\tilde{O}(a+t)$

=

$a\leq b \bmod \underline{I},\ \underline{I}\in B_2(W) \Rightarrow a+s\leq b+t \bmod \underline{I}$ all $s,t\in\underline{I}$

=

$a\leq b\quad b\leq a \quad\Rightarrow a\leq b+t\quad b\leq a+t$

=

$a\equiv b \bmod \underline{I}\quad \underline{I}\in B_2(W) \to a+s\equiv b+t \bmod \underline{I}$ all $s,t\in\underline{I}$

$a\leq b \to a+s\leq b+t\quad b\leq a\to b+t\leq a+s$

=

$\underline{I}\in B_2(W)\quad MS\{W,I|a/b\}+\underline{I}\equiv MS\{W,I|a/b\}$

$MS\{W,I|a/b\}$ void : correct. $x\in MS\{W,I|a/b\}\ bx=a+s\ s\in\underline{I}$

$x\equiv a \bmod \underline{I}$: $b(x+t)=a+s+bt\quad x+t\in S\{W,I|a/b\}$

$x+t\equiv a \to x+t\in MS\{W,I|a/b\}\to MS\{W,\underline{I}|a/b\}+\underline{I}\subseteq$
$MS\{W,\underline{I}|a/b\}$

$\underline{I}\in B_2(W)\to 0\in\underline{I}\to MS\{W,I|a/b\}\subseteq MS\{W,I|a/b\}+\underline{I}$.

=

$I\in B_{+;ls}$ ~~all~~ each pair $a,b\in I\ \exists\ c(a,b)\in\underline{I}$ such that ~~then~~ $b=a+c(a,b)$

=

$S\{W,I|a/b\}S\{W,I|c/d\}\subseteq S\{W,I|ac/bd\}\quad I\in \underline{B}_{\times;si}(W)\cap B_{+;c}$

$bx=a+s\quad dy=c+t\quad bdxy=ac+at+cs+st$

$x\equiv a\quad y\equiv c\quad xy\equiv ac\quad MS\{W,I|a/b\}MS\{W,I|c/d\}\subseteq$
$MS\{W,I|ac/bd\}$

$bdz = ac + w$ $\quad bz = ay + s \quad a \in Ra(W,I) \quad \underset{b}{ay} = c + w$

$day + ds = ac + w \quad a(dy - c) = w - ds$

$dy = c + t \quad ||| \quad b \in Ra(W,I) \quad bx = a + t \quad b(z - xy) = s - yt$

$z = xy + q$

$$\Downarrow \left[\begin{array}{l} b|a \ \text{mod}\ \underline{I} \quad ||| \quad a \in Ra(W,I) \quad c \in W \\ I \in B_{x;si}(W) \cap B_{+;c} \qquad \Downarrow \\ b \in Ra(W,I) \quad S\{W,I|a/b\} \in Ra(W,I) \end{array}\right.$$

$\S \ af = c + h$

$bx = a + t$

$bxf = af + tf = c + h + tf$

$bdz = ac + w \quad x \in S\{W, I|a/b\} \quad z = yx + t$

$bx\,dy = ac + w - bdt$ $\quad$ using $\underline{I} \in B_{+;ls}$

$a(dy - c) = w - bdt \quad dy = c + s$

$S\{W,I|ac/bd\} \underset{\equiv}{\leq} S\{W,I|a/b\}S\{W,I|c/d\} + \underline{I}$

$=$

$a \in Ri(W,I) \quad x_0 \in \underline{I} \quad bx_0 = a_0 + t_0 \ ||| \ \underline{I} \in B_{x;si}(W) \cap B_{+;c} \quad a \in Ri(W,I)$

$D(W,I|\overset{a}{b}) \leq Ri(W,I)$ and ~~for any~~ $\underbrace{S\{W,\underline{I}|a/b\}}_{\equiv MS\{W,\underline{I}|a/b\}} \leq Ri(W,\underline{I})$

all $b \in W$. $\{$true when $b \nmid a$ mod I.$\}$

$bx = a + t \quad x_0 \in \underline{I} \quad bx_0 = a_0 + t_0 \quad \underline{I} \in B_{x;si}(W) \cap B_{+;ls} \quad a_0 \in \underline{I}$

$a \leq x$ mod $\underline{I}$

$a_0 \in \underline{I} \quad bx_0 \in \underline{I}, \quad b \in Ri(W,I) \to x_0 \in \underline{I}$

$bx = a + t \quad bx_0$ ~~t_0~~ $= a_0 + t_0 \ || \ \underline{I} \in B_{x;si}(W) \cap B_{+;c} \quad b \in Ri(W,\underline{I})$

$\Rightarrow S\{W,I|a/b\} \equiv MS\{W,I|a/b\}$.

$=$

$a \in Ra(W,I), \ I \in B_{x;si}(W) \cap B_{+;c} \quad D(W,I|\overset{a}{b}) \underline{\in} Ra(W,\underline{I})$

$S\{W,I|a/b\} \equiv MS\{W,I|a/b\} \leq Ra(W,\underline{I})$

$\equiv$

$b|a$ mod $\underline{I}$, $a \in Ra(W,I) \quad I \in B_2(W)$

$S\{W,I|ac/bd\} \equiv S\{W,I|a/b\}S\{W,I|c/d\} + \underline{I}$

$a, b, d \in \Delta'(\overset{Ra(W,I),I}{\cancel{W,I}})$ $a \leq b \bmod I$ $ac \leq bd \bmod I$

$b \in \Delta'(\overset{Ra(W,I),I)}{W,I)}$ $a \leq b \bmod I \rightarrow b | a \bmod I$ $bx = a \bmod I$

$a \in \Delta'(\overset{R,}{W,I}) \rightarrow x \in \Delta'(W,I)$?

$I \in \mathbb{I}(W)$ $a \leq b \bmod I$ $a, b \in \Delta'\{Ra(W,I),I\} \rightarrow MS\{W,I|a/b\} \in \Delta'(\cancel{W}\ Ra, I)$

$bx = a + s$ $s \in I$ $x \equiv a$ $x \in \Delta'(Ra(W,I),I)$

$a \leq b$ $ab \leq c$
$a \leq bc$

$b, d \in \Delta'(Ra(W,I),I)$ $ac \leq bd \bmod I$ $I \in B_{x;si}(W) \cap B_{+;c}$

$bd \in \Delta'(Ra(W,I),I)$

$bdz = ac + w$ $w \in I$

$MS\{W,I|ac/bd\}$ nonvoid $\rightarrow \exists z:$ $z \equiv ac \leq a \bmod I$

$a \leq b$ $bcg \in I$
$\rightarrow acg \in I$

$bd \in \Delta'(Ra(W,I),I) \rightarrow b, d \in \Delta'(Ra(W,I),I)$? $\ u\,bd + v \in Ra$

$bdv \in I \rightarrow bv \in I$? no

~~z~~ $z \cancel{\leq} \leq a \bmod I$, $x | z \bmod I$ $z \leq a \equiv x$ $ab \leq cd$ $a \leq b \rightarrow a \leq d$

$xy = z + t$ $t \in I$ $\quad bdxy = bdz + bdt$
$\qquad\qquad\qquad\qquad = ac + bdt + w$

$y \equiv z \equiv a$

$ady = ac + bdt + w - sdy$

$$\boxed{\begin{array}{l} ac \leq bd \quad c \leq a \rightarrow c \leq d \\ dg \in I \rightarrow bdg \in I \rightarrow \\ acg \in I \quad cg \in I \rightarrow \\ \qquad c \leq d \end{array}}$$

$c \leq a \bmod I$ + $I \in \mathbb{I}_{QF}(W) \rightarrow dy = c \bmod I$

$c \leq b'$ $b'c \leq a \rightarrow c \leq a$ $\quad c \leq d$ $dc \leq a \rightarrow c \leq a$

$c \leq a \leq b$, $ac \leq bd \bmod I$ $\quad MS\{W,I|ac/bd\}$

$\subseteq x\,MS\{W,I|c/d\} + I$

$a \leq b$, $c \leq d$

$z = x\underline{y} + t'$: $\rightarrow$ $\quad y(x,z) + t(x,y,z)$

$bdg \in I$

$MS\{W,I|ac/bd\}$ $\ adg \in I \rightarrow acg \in I$

~~$xy = z$~~ $\quad xy(x,z) = z + t(x,y,z)$

$= MS\{W,I|a/b\}\,MS\{W,I|c/d\} + I$

( ) Let $a, b, d \in \Delta'\{Ra(W,I),I\}$ and $c \leq a \leq b$, $ac \leq bd$ mod I, where $I \in \mathbb{I}_{QF}(W)$ (1) $MS\{W,I|a/b\}$, (2) $MS\{W,I|c/d\}$ and $MS\{W,I|ac/bd\}$ are nonvoid. ~~each~~ For ~~each~~ a fixed $x \in MS\{W,I|a/b\}$, each $z \in MS\{W,I|ac/bd\}$ contains x as a mod I factor, ~~its~~ the cofactor of x being in $MS\{W,I|c/d\}$: $y(x,z) \in MS\{W,I|c/d\}$ and $t(x,y,z) \in I$ ~~exist~~ such that

$xy(x,z)=z+\overset{t}{\cancel{p}}(x,y,z)$ exist.

(3) For ~~[illegible]~~ each $x\in MS\{W,I|a/b\}$, $MS\{W,I|ac/bd\} \leq xMS\{W,I|c/d\}+I$

(4) $MS\{W,I|ac/bd\} = MS\{W,I|a/b\}MS\{W,I|c/d\}+I$

Since $a\leq b$ mod I and $b\in\Delta'\{$~~[illegible]~~$E(W,I),I\}$, $MS\{W,I|a/b\}$ is nonvoid, from ( ). The conditions $ac\leq bd$, $c\leq a$ mod I and $I\in\mathbb{I}_{QF}(W)$ imply, from ( ), that $c\leq d$ mod I. Since ~~$d\in MS\{W,I|c/d\}$ is~~ $d\in\Delta'\{E(W,I),I\}$, $MS\{W,I|c/d\}$ is nonvoid. Lastly, the conditions $b,d\in\Delta'\{Ra(W,I),I\}$ imply, from ( ), that $bd\in\Delta'\{Ra(W,I),I\}$. Since $ac\leq bd$ mod I, $MS\{W,I|ac/bd\}$ is nonvoid.

Select $x\in MS\{W,I|a/b\}$ so that $bx=a+s$ for some $s\in I$ and $x\equiv a$ mod I. Since $a\in\Delta'\{Ra(W,I),I\}$, $x\in\Delta'\{Ra(W,I),I\}$, from ( ). Select $z\in MS\{W,I|ac/bd\}$ so that $bdz=ac+w$ where $w\in I$ and $z\equiv ac\leq a\equiv x$ mod I. The two conditions $x\in\Delta'\{Ra(W,I),I\}$ and $z\leq x$ mod I imply, from ( ), that ~~$MS\{Ra(W,I),I\}$ is~~ ~~$x\leq z$ mod I~~. $MS\{W,I|z/x\}$ is nonvoid: $MS\{W,I|z/x\}$ contains y for which $xy=z+t$, where $t\in I$ and $y\equiv z\leq a$ mod I. Then $bdxy=bdz+bdt=ac+bdt+w$. Hence $ady=ac+r$ where $r=bdt+w-sdy\in I$, ~~Since $y\leq a$ mod~~ and $a(dy-c)\in I$. Since $y\leq a$ mod I and $c\leq a$ mod I, $dy-c\leq a$ mod I. $I\in\mathbb{I}_{QF}(W)$, so that $dy=c+p$ where $p\in I$: $y\in S\{W,I|c/d\}$. But $y\equiv z\equiv ac$ mod I, so that $y\leq c$ mod I. ~~Since $c\leq y'$ for mod I~~ Since $y\in S\{W,I|c/d\}$, $c\leq y$ mod I, from ( ): $y\equiv c$ mod I and accordingly $y\in MS\{W,I|c/d\}$. z has been ~~expressed in the form~~ shown to satisfy the relationship $xy(x,z)=z+t(x,y,z)$ as described in (2).

But (3) is a corollary to it's predecessor.

From (3), $MS\{W, \underline{I} | ac/bd\} \subseteq MS\{W, I | a/b\} MS\{W, I | c/d\} + \underline{I}$. From ( ), $xy \in MS\{W, \underline{I} | ac/bd\}$ for each pair $x \in MS\{W, \underline{I} | a/b\}$ and $y \in MS\{W, I | c/d\}$. From ( ), $MS\{W, \underline{I} | ac/bd\} + \underline{I} =$ $MS\{W, I | ac/bd\}$ when $I \in B_2(W)$. Hence for each triplet $x \in MS\{W, I | a/b\}$, $y \in MS\{W, I | c/d\}$ and $t \in \underline{I}$, $xy + t \in$ $MS\{W, I | ac/bd\}$: $MS\{W, I | a/b\} MS\{W, \underline{I} | c/d\} + \underline{I} \subseteq MS\{W, I | ac/bd\}$. The result of (4) follows.

—○—

~~( ) If $a \le b$, $ab \le cd$ mod $\underline{I}$, where $I \in$ ~~$B_{x;qf}$~~ $B_{x;si} \cap B_{x;qf}(W)$, then~~ ~~$a \le d$ mod $\underline{I}$~~ ~~$a \le c$, $a \le d$ mod $\underline{I}$.~~

~~For any $\underset{\circ}{g} \in W$ such that $c\underset{\circ}{g} \in \underline{I}$, $cd\underset{\circ}{g} \in \underline{I}$ since $\underline{I} \in B_{x;si}(W)$. Since $ab \le cd$ mod I, $ab\underset{\circ}{g} \in \underline{I}$. If $I \in B_{x;si}(W)$, $a\underset{\circ}{g} \le b$ mod $\underline{I}$ when $a \le b$ mod $\underline{I}$. The~~ ~~two~~ ~~three conditions $ab\underset{\circ}{g} \in I$, $a\underset{\circ}{g} \le b$ mod $\underline{I}$ and $\underline{I} \in B_{x;qf}(W)$ imply, from ( ), that $a\underset{\circ}{g} \in \underline{I}$: $a \le c$ mod I. Similarly $a \le d$ mod $\underline{I}$.~~

( ) If $a \le b$, $ab \le c$ mod $\underline{I}$, where $I \in B_{x;si} \cap B_{x;qf}(W)$ then $a \le bc$ mod I and $a \le c$ mod $\underline{I}$.

From ( ) the condition $a \le b$ mod I implies that $a^2 \le ab$ mod $\underline{I}$. ~~and then, from ( ),~~ Since $ab \le c$ mod $\underline{I}$, $a^2 \le c$ mod $\underline{I}$. From ( ), the two conditions $a^2 \le c$ mod $\underline{I}$ and $\underline{I} \in B_{x;si} \cap B_{x;qf}(W)$ imply that $a \le c$ mod $\underline{I}$. The two conditions $a \le b, a \le c$ mod $\underline{I}$ imply that $a^2 \le bc$ mod $\underline{I}$ and again $a \le bc$ mod $\underline{I}$.

( ) Let $a \le d$ mod $\underline{I}$, where $I \in B_{x;si} \cap B_{x;qf}(W)$. ~~$ab \le cd$~~ $ad \le bc$ mod I if and only if the two conditions $a \le b, a \le c$ mod I hold.

The two conditions $a \le b, a \le c$ mod $\underline{I}$ imply that $a^2 \le bc$ mod I and, when $I \in B_{x;si} \cap B_{x;qf}(W)$, that $a \leq bc$ mod I, from ( ). Then $ad \leq bc$ mod I, from ( ).

For any $g \in W$ such that $bg \in I$, $bcg \in I$ since $I \in B_{x;si}(W)$. If $ad \leq bc$ mod I, $adg \in I$. If $I \in B_{x;si}(W)$, $ag \leq d$ mod I when $a \leq d$ mod I. The three conditions $adg \in I$, $ag \leq d$ mod I and $I \in B_{x;qf}(W)$ imply, from ( ), that $ag \in I$: $a \leq b$ mod I. Similarly $a \leq \overset{c}{\cancel{b}}$ mod I.

—o—

( ) If $I \in B_{x;si}(W) \cap B_{+;c}$. ~~then~~ $S\{W, I|a/b\}$ ~~·~~ $S\{W, I|c/d\} \subseteq$ ~~$S\{W, I|ac/bd\}$ for all a, b, c a,.., d∈W.~~

~~If either S{W, I|a/b} or S{W, I|c/d} is void~~

$S\{W, I|ac/bd\}$ and $\mathfrak{m}S\{W, I|a/b\}\mathfrak{m}S\{W, I|c/d\} \leq \mathfrak{m}S\{W, I|ac/bd\}$ for all $a, .., d \in W$.

If either $S\{W, I|a/b\}$ or $S\{W, I|c/d\}$ is void, the first ~~relationsh~~ stated relationship is correct. Assuming neither of these systems to be void, select x and y for which $bx = a+s$ and $dy = c+t$ where $s, t \in I$. Then $bdxy = ac + w$ where $w = at + cs + st$ and, since $I \in B_{x;si}(W) \cap B_{+;c}$, $w \in I$: $xy \in S\{W, I|ac/bd\}$.

With regard to the second ~~stated~~ relationship stated, let $x \equiv a$, $y \equiv c$ mod I. From ( ), $xy \equiv ac$ mod I: $xy \in \mathfrak{m}S\{W, I|ac/bd\}$.

—o—

~~( ) Let $a \in Ra(W, I)$ and $b|a$ mod I, where $I \in B_{\tau}(W)$.~~

~~(1) For each $x \in S\{W, I|a/b\}$, $S\{W, I|ac/bd\} \equiv$ ~~$S\{W, I|a/b\}$~~ x $S\{W, I|c/d\} + I$.~~

~~(2) $S\{W, I|ac/bd\} \equiv S\{W, I|a/b\}S\{W, I|c/d\} + I$.~~

Select $x \in S\{W, I|a/b\}$ so that $bx = a+s$ where $s \in I$.

(Statement of Theorem, 9 pages on)

If $S\{W,I|c/d\}$ is void, ~~$S\{W,I|a/b\}$~~ x $S\{W,I|c/d\}+I\subseteq S\{W,I|ac/bd\}$. Assuming $SW\{W,I|c/d\}$ to be nonvoid, select ~~$x\in MS\{$ $x\in S\{W,I|a/b\}$. and~~ $y\in S\{W,I|c/d\}$ so that, from ( ), $xy\in S\{W,I|ac/bd\}$. From ( ), $S\{W,I|ac/bd\}$~~$\subseteq$~~$+I\subseteq S\{W,I|ac/bd\}$. For any $w\in I$, $xy+w\in S\{W,I|ac/bd\}$: ~~$S\{W,I|a/b\}$~~ x $S\{W,I|c/d\}+I\subseteq S\{W,I|ac/bd\}$.

If $S\{W,I|ac/bd\}$ is void, $S\{W,I|ac/bd\}\subseteq S$~~$\{W,I|a/b$~~ ~~$S\{W,I|a/b\}$~~ x $S\{W,I|c/d\}+I$. Assuming $S\{W,I|ac/bd\}$ to be nonvoid, select $z\in W$ for which $bdz=ac+w$, where $w\in I$. ~~Select $x\in$~~ ~~Since $a\in Ra(W,I)$ and $b|a \bmod I$, $S\{W,I|a/b\}\subseteq Ra(W,I)$, from ( ). Select Select $x\in S\{W,I|a/b\}$ so that $bx=a+s$ where $s\in I$~~ Since $a\in Ra(W,I)$, $x\in Ra(W,I)$ also, from ( ). W and I contain y and ~~t~~ p respectively for which $z=xy+$~~t~~p. Then $bxdy=ac+w-bd$~~t~~p or $a(dy-c)=r$ where $r=w-bd$~~t~~$p\in I$, since $I\in B_{x;si}(W)\cap B_{+;ci}$ $a\in Ra(W,I)\subseteq Ri(W,I)$, so that I contains t for which $dy=c+t$: $y\in S\{W,I|c/d\}$. To each $z\in S\{W,I|ac/bd\}$, $y\in S\{W,I|c/d\}$ and $p\in I$ for which $z=xy+p$ correspond: $S\{W,I|ac/bd\}\subseteq xS\{W,I|c/d\}+I$. The relevant results of Part (2) has been dealt with. That of Part (3) is a corollary to ~~its predecessor~~, 9 pages on

( ) Let $b\in Ri(W,I)$ where $I\in$ ~~$B_{x;si}(W)\cap B_{+;c}$~~ $B_{z}(W)$. $S\{W,I|a/b\}=MS\{W,I|a/b\}$.

$MS\{W,I|a/b\}\subseteq S\{W,I|a/b\}$ by definition.

If $S\{W,I|a/b\}$ is void, $S\{W,I|a/b\}\subseteq MS\{W,I|a/b\}$. Otherwise, select $x\in S\{W,I|a/b\}$, so that $bx=a+s$ where $s\in I$. ~~From~~

If $ag\in I$, $bxg=ag+tg\in I$ when $I\in B_{x;si}(W)\cap B_{+;c}$ and, since $b \in Ri(W,I)$, $x\underset{\circ}{g} \in I$: $x \le a \bmod I$. From ( ), $a \le x \bmod I$ for all $x \in S\{W, I|a/b\}$ when $I \in B_{x;si}(W) \cap B_{+;ls}$. Hence $a \equiv x \bmod I$: $x \in MS\{W, I|a/b\}$.

( ) Let $a \in Ri(W,I)$ where $I \in B_{x;si}(W) \cap B_{+;ls}$.

(1) $D(W,I\}|a) \subseteq Ri(W,I)$

(2) $S\{W,I|a/b\} \subseteq Ri(W,I)$ for all $b \in W$.

~~Select $b \in D(W,I|a)$ so that~~

If $D(W,I|a)$ is void, part (1) is correct. Otherwise, select $b \in D(W,I|a)$, so that $x \in W$ and $s \in I$ for which $bx = a+s$ exist.

~~If $a\underset{\circ}{g} \in I$ then $bx\underset{\circ}{g} = a\underset{\circ}{g} + s\underset{\circ}{g} \in I$ also, since~~

If $b\underset{\circ}{g} \in I$ then $a\underset{\circ}{g} = bx\underset{\circ}{g} - s\underset{\circ}{g} \in I$, since $I \in B_{x;si}(W) \cap B_{+;ls}$. However, $a \in Ri(W,I)$ so that $\underset{\circ}{g} \in I$: $b\underset{\circ}{g} \in I$ only when $\underset{\circ}{g} \in I$: $b \in Ri(W,I)$.

If $S\{W,I|a/b\}$ is void part (2) is correct. Otherwise, selecting $x \in S\{W,I|a/b\}$, it is shown as above that $x\underset{\circ}{g} \in I$ only when ~~$a\underset{\circ}{g}$~~ $\underset{\circ}{g} \in I$.

( ) Let $a \in Ri(W,I)$ where $I \in B_i(W)$. $S\{W,I|a/b\} = MS\{W,I|a/b\} \subseteq Ri(W,I)$ for all ~~$b \in D(W,I|a)$~~ W

~~If $a \in Ri(W,I)$, be and $b \in D(W,I|a)$, $b \in Ri(W,I)$ from ( ).~~

~~From ( ), $S\{W,I|a/b\} \supseteq MS\{W,I|a/b\}$~~

If for any $b \in W$, $S\{W,I|a/b\}$ is void, the stated result is correct. ~~Otherwise~~ If $S\{W,I|a/b\}$ is nonvoid, $b \in D(W,I|a)$ and $b \in Ri(W,I)$ from ( ). From ( ), $S\{W,I|a/b\} \supseteq MS\{W,I|a/b\}$ ~~as~~ since $I \in B_i(W)$~~,~~ and, since $I \in B_{x;si}(W) \cap B_{+;ls}$, $S\{W,I|a/b\} \subseteq Ri(W,I)$ from ( ).

( ) Let $a \in Ra(W, I)$ where $I \in B_{x;si}(W) \cap B_{+;c}$

(1) $D(W, I|a) \subseteq Ra(W, I)$

(2) $S\{W, I|a/b\} \subseteq Ra(W, I)$ for all $b \in W$.

Since $Ra(W, I)$ is nonvoid, and $Ra(W, I) \subseteq D(W, I|a)$ for all $a \in W$, $D(W, I|a)$ is nonvoid. Select $b \in D(W, I|a)$ so that $bx = a + s$ for some $x \in W$ and $s \in I$. Select any $c \in W$, so that $f \in W$ and $h \in I$ for which $af = c + h$ exist. Then $bxf = c + ~~h+tf~~ w$ where $w = h + tf \in I$, since $I \in B_{x;si}(W) \cap B_{+;c}$: to $c \in W$, $g = xf$ ~~and~~ $\in W$ and $w \in I$ such that $bg = c + w$: $b \in Ra(W, I)$. If $S\{W, I|a/b\}$ is void, part (2) is correct. Otherwise the proof that $x \in S\{W, I|a/b\}$ belongs to $Ra(W, I)$ is as above.

( ) Let $a \in Ra(W, I)$ where $I \in B_i(W)$. $S\{W, I|a/b\} = MS\{W, I|a/b\} \subseteq Ra(W, I)$ for all $b \in W$.

If for any $b \in W$, $S\{W, I|a/b\}$ is void, the stated result is correct. If $S\{W, I|a/b\}$ is nonvoid, $b \in D(W, I|a)$ and $b \in Ra(W, I)$, from ( ). Hence $b \in Ri(W, I)$. From ( ), $S\{W, I|a/b\} = MS\{W, I|a/b\}$ since $I \in B_i(W)$ and, since $I \in B_{x;si}(W) \cap B_{+;c}$, $S\{W, I|a/b\} \subseteq Ra(W, I)$ from ( ).

—o—

( ) Let $I \in B_{x;si}(W) \cap B_{+;ls}$. $a \leq x \bmod I$ for all $x \in S\{W, I|a/b\}$

Select $x \in SW\{W, I|a/b\}$. so that $bx = a + s$ where $s \in I$. If $xg \in I$, $ag = bxg - sg \in I$, since $I \in B_{x;si}(W) \cap B_{+;ls}$: $a \leq x \bmod I$.

( ) Let $I \in B_i(W)$. $MS\{W, I|a/b\} + \underline{I} = MS\{W, I|a/b\}$ for all pairs $a, b \in W$.

If $MS\{W, I|a/b\}$ is void the stated result is correct. Otherwise, select $x \in MS\{W, I|a/b\}$ so that $bx = a+s$ where $s \in \underline{I}$ and $x \equiv a \bmod I$. For any $t \in I$, $b(x+t) = a+s+bt$ where $s+bt \in I$, since $I \in$ ~~$B_{x;si} \cap B$~~ $B_{x;si}(W) \cap B_{+;\cancel{s}c}$: $x+t \in S\{W, I|a/b\}$. From ( ), the two conditions $x \equiv a \bmod I$ and $I \in B_i(W)$ imply that $x+t \equiv a \bmod I$: $x+t \in MS\{\cancel{\in}W, I|a/b\}$ and accordingly $MS\{W, I|a/b\} + \underline{I} \subseteq MS\{W, I|a/b\}$.

Since $I \in B_{x;si}(W)$, $0 \in \underline{I}$ and $MS\{W, I|a/b\} \subseteq MS\{W, I|a/b\} + \underline{I}$.

( ) If $I \in B_{x;si}(W)$, ~~$\theta(b) \subseteq \theta$~~ $\theta(W, I|b) \subseteq \theta(W, I|b) + \underline{I}$ for all $b \in W$.

If $I \in B_{x;si}(W)$, $0 \in I$ and $\theta(W, I|b) \subseteq \theta(W, I|b) + \underline{I}$.

( ) If $I \in B_{x;si}(W) \cap B_{+;c}$, $\theta(W, \underline{I}|b) = \theta(W, I|b) + I$ for all $b \in W$.

From ( ), $\theta(W, I|b) \subseteq \theta(W, I|b) + \underline{I}$, since $\underline{I} \in B_{x;si}(W)$. Select $v \in \theta(W, I|b)$ so that $bv \in \underline{I}$ and $t \in \underline{I}$. Then ~~$(b+t)v = bv + vt \in I$, since $I \in B_{x;si}(W) \cap B_{+;c}$: $b+t$~~ $b(v+t) = bv + bt \in I$, since $I \in B_{x;si}(W) \cap B_{+;c}$: $v+t \in \theta(W, I|b)$ and accordingly $\theta(W, I|b) + \underline{I} \subseteq \theta(W, I|b)$.

(1) If $I \in B_{x;si}(W) \cap B_{+;c}$, $\theta(W, I|b) \subseteq \theta(W, I|b+t)$ for all $b \in W$ and $t \in \underline{I}$.

2) If $I \in B_{x;si}(W) \cap B_{+;ls}$, $\theta(W, \underline{I}|b+t) \subseteq \theta(W, I|b)$ for all $b \in W$ and $t \in \underline{I}$.

3) If $I \in B_{\dot{z}}(W)$, $\tilde{O}(W, I|b) \equiv \tilde{O}(W, I|b+t)$ for all $b \in W$ and $t \in I$

Select $v \in \tilde{O}(W, I|b)$, so that $bv \in I$. $(b+t)v = bv + tv \in I$, if $I \in B_{\times;si}(W) \cap B_{+;c}$: each $v \in \tilde{O}(W, I|b)$ belongs to $\tilde{O}(W, I|b+t)$: part (1) has been dealt with. If $(b+t)v \in I$ then $bv = (b+t)v - tv \in I$ if $I \in B_{\times;si}(W) \cap B_{+;ls}$: part (2) has been proved. Part (3) is a consequence of it's two predecessors.

( ) Let $a \leq b \bmod I$.

(1) If $I \in B_{\times;si}(W) \cap B_{+;c}$ then $a+t \leq b \bmod I$ for all $t \in \underline{I}$

(2) If $I \in B_{\times;si}(W) \cap B_{+;ls}$ then $a \leq b+t \bmod I$ for all $t \in \underline{I}$

(3) If $I \in B_{\dot{z}}(W)$, then $a+s \leq b+t$ for all $s, t \in \underline{I}$.

$a \leq b \bmod I$ if and only if $\tilde{O}(W, I|b) \subseteq \tilde{O}(W, I|a)$.

If $I \in B_{\times;si}(W) \cap B_{+;c}$ then $\tilde{O}(W, I|a) \subseteq \tilde{O}(W, I|a+t)$ from (1) and then $\tilde{O}(W, I|b) \subseteq \tilde{O}(W, I|a+t)$. If $I \in B_{\times;si}(W) \cap B_{+;ls}$ then ~~$\tilde{O}(b+t) \subseteq \tilde{O}$~~ $\tilde{O}(W, I|b+t) \subseteq \tilde{O}(W, I|b)$ from (2) and then $\tilde{O}(W, \underline{I}|b+t) \subseteq \tilde{O}(W, \underline{I}|a)$. Part (3) is a corollary to it's predecessors.

( ) If $I \in B_{\dot{z}}(W)$ and $a \equiv b \bmod \underline{I}$ then $a+s \equiv b+t \bmod I$ for all $s, t \in \underline{I}$.

This result is a corollary to (3).

A set I ~~for which~~ such that corresponding to each pair $a, b \in \underline{I}$, $\underline{I}$ also contains $c(a,b)$ for which $b = a \phi c(a,b)$ is said to be ϕ locally soluble. $B_{\phi;ls}$ is the class of such sets.

$a \leq b$ $b \leq c$ $\rightarrow a \leq c$. ‖ $a \leq b$ $c \leq d$ $ac \leq bd$ ‖ $a \leq b$ $ac \leq b$

$a \leq b$ $a^m \leq b^n$

1 ( ) For any set $I \subseteq W$

(1) the two conditions $a \leq b$, $b \leq c \bmod I$ imply that $a \leq c \bmod I$ a..

(2) the two conditions $a \leq b$, $c \leq d \bmod I$ imply that $ac \leq bd \bmod I$.

If $a \leq b$, $b \leq c \bmod I$ then $\tilde{O}(W, I | b) \leq \tilde{O}(W, I | a)$, $\tilde{O}(W, I | c) \leq \tilde{O}(W, I | b)$; hence $\tilde{O}(W, I | c) \leq \tilde{O}(W, I | a)$ and $a \leq c \bmod I$.

If $a \leq b$, $c \leq d \bmod I$ then for any $g \in W$ such that $bdg \in I$, $adg \in I$ since $a \leq b \bmod I$ and $acg \in I$ since $c \leq d \bmod I$: $ac \leq bd \bmod I$.

( ) Let $a \leq b \bmod I$ where $I \in B_{x;si}(W)$. $ac \leq b \bmod I$ for all $c \in W$.

~~For any~~ If $a \leq b \bmod I$ then for any $g \in W$ such that $bg \in I$, $ag \in I$ also and, if $I \in B_{x;si}(W)$, $acg \in I$: $ac \leq b \bmod I$

( ) Let $a \leq b \bmod I$ where $I \in B_{x;si} \cap B_{x;qf}(W)$. $a^m \leq b^n \bmod I$ for $m, n = 1, 2, \ldots$.

~~When $m \geq n$ the result that $a^n = b^n \bmod I$ follows~~

If $a \leq b \bmod I$ it follows as a corollary to (2) that $a^n = b^n \bmod I$ for $n = 1, 2, \ldots$. When $m > n$, the further result that $a^m = a^n a^{m-n} \leq b^n \bmod I$ follows from ( ), since $I \in B_{x;si}(W)$. To deal with the case in which $n > m$, it is first remarked that the condition $a^m \leq b^{m+r-1} \bmod I$ yields, by use of (2), the relationship $a^m b \leq b^{m+r} \bmod I$. The ~~[illegible]~~ three conditions $a^m \leq b^{m+r-1}$, ~~mod I,~~ $a^m b \leq b^{m+r} \bmod I$ and $I \in B_{x;si} \cap B_{x;qf}(W)$ yield, from ( ), the result that $a^m \leq b^{m+r} \bmod I$. ~~Similarly, the relat~~ The assumed condition holds for $r=1$ and hence for $r = 1, \ldots, n-m$.

$\underline{I} \in B_{x;si}(W) \cap B_{+;c}$ $S\{W,I|a/b\}S\{W,I|b/d\} \leq S\{W,\underline{I}|a/d\}$

~~[illegible]~~ $bx=a+s$ $dy=b+t$ ~~$bdxy=ab+w$ [illegible]~~

$dxy=a+s+xt$ $x \equiv a$ $y \equiv b$ $xy \equiv ab$

$a \leq b$ mod I $I \in B_{x;qf}$ $xy \equiv ab \equiv a$

$b|a$ mod I $bx=a+s$ $bg \in \underline{I} \to bxg \in \underline{I} \to ag+sg \in \underline{I} \to ag \in \underline{I}$

$\underline{I} \in B_{x;si}(W) \cap B_{+;lc}$ $b|a$ mod $\underline{I} \to a \leq b$ mod I

$a \leq b$ mod I $ab \leq a$ mod $\underline{I}$. $I \in B_{x;si}(W)$ $\underline{I} \in B_{x;si} \cap B_{x;qf}(W)$ $a \leq b$ mod I

~~$ab \in I$~~ $\Rightarrow ab \equiv a$ ''

$abg \in \underline{I}$ $ag \leq b$ if $I \in B_{x;si}(W)$ $bag \in \underline{I} \to ag \in \underline{I}$, $\Rightarrow a \leq ab$

$a \leq b$ mod $\underline{I}$ $ab \in I \to a \in \underline{I}$ $ab \in \underline{I} \to a^2 \in \underline{I}$ $\underline{I} \in \mathbb{I}_{x;qf}(W)$

$\underline{I} \in B_i$[illegible]$\cap B_{x;qf}(W)$. $MS\{W,\underline{I}|a/b\}MS\{W,I|b/d\} \leq MS\{W|a/d\}$

$MS\{W,I|a/b\}$ or $MS\{W,I|b/d\}$ void: correct // both nonvoid.

$x \in MS\{W,I|a/b\}$ $y \in MS\{W,I|b/d\}$ ( ): $xy \in S\{W,I|a/d\}$

$b|a$ mod I $\underline{I} \in B_{x;si}(W) \cap B_{+;lc} \to a \leq b$ mod $\underline{I}$

$x \equiv a$ $y \equiv b$ $xy \equiv ab$ ( ): $I \in B_{x;si} \cap B_{x;qf}(W) \overset{a \leq b \text{ mod } I}{\Rightarrow} ab \equiv a$ mod $\underline{I}$

$xy \equiv a$ $xy \in MS\{W,I|a/d\}$.

$a \in Ra(W,I)$ $b|a$ mod $\underline{I}$ $I \in B_2(W)$

(1) $x \in S\{W,I|a/b\}$: $S\{W,I|a/d\} \equiv xS\{W,I|b/d\}+\underline{I}$

(2) $S\{W,\underline{I}|a/d\} \equiv S\{W,I|a/b\}S\{W,I|b/d\}+\underline{I}$

$y \in S\{W,I|b/d\}$ ( ): $xy \in S\{W,I|a/d\}$ ( ) $xy+w \in S\{W,I|a/d\}$ $w \in \underline{I}$

$xS\{W,I|b/d\}+\underline{I} \leq S\{W,\underline{I}|a/d\}$ W $bx=a+s$ $s \in \underline{I}$

$z \in S\{W,I|a/d\}$ $dz=a+w$ $w \in \underline{I}$ $z=xy+p$

~~dz~~ $dxy=$ ~~$w-dt$~~ $a+w-dt = bx+w-s-dt$

$x \in$ ~~Ra~~ $Ra(W,I) \leq Ri(W,I)$ $dy \equiv b$ mod $\underline{I}$ $y \in S\{W,I|b/d\}$

$S\{W,I|a/d\} \leq xS\{W,I|b/d\}+\underline{I}$.

~~$y \equiv b$~~ $a \in Ra(W,I)$ ~~$\to b \in Ra(W,$~~ $\leq Ri(W,I) \to b \in Ri(W,\underline{I})$ $\to d \in Ri(W,\underline{I})$

$S\{W,I|a/d\} \equiv MS\{W,I|a/d\}$ $S\{W,I|b/d\} \equiv MS\{W,I|b/d\}$.

$a\in Ra(W,I)$ $b|a \bmod I$ $I\in B_i(W)$ $S\{W,I|a/b\}\equiv MS\{W,I|a/b\}$?

$x\in S\{W,I|a/b\}$ $MS\{W,I|ac/bd\}\equiv xMS\{W,I|c/d\}+I$??

$a\in Ra(W,I)\to a\in Ri(W,I)\to b\in Ri(W,I)\ (b|a \bmod I, (\ ))$

$\to S\{W,I|a/b\}\equiv MS\{W,I|a/b\}$

$y\in MS\{W,I|c/d\}\to xy+w\in S\{W,I|ac/bd\}$

$x\in Ra(W,I)\leq Ri(W,I)$ $xy\equiv y\equiv c$ $a\in Ri(W,I)\to ac\equiv c$

$xy\equiv ac$ $I\in B_i(W)\Rightarrow xy+w\equiv xc\to xy+w\in MS\{W,I|ac/bd\}$

$z\in MS\{W,I|ac/bd\}$ $z\equiv xy+t$ $z\equiv ac\equiv c$ $xy\equiv z\equiv c$ $x\in Ri(W,I)$

$\Rightarrow y\equiv c$

—o—

$a\leq ab \bmod I$ all $a\in W$. If $I\in B_{x;si}(W)$,

$b\in Ri(W,I)$ $ab\equiv a \bmod I$ all $a\in W$ ~~$I\in B_{x;si}(W)$~~

$I\in B_{x;si}(W)$ $ab\leq a \bmod I$ ( ). $ab\underset{\circ}{g}\in I\to a\underset{\circ}{g}\in I\ (b\in Ri(W,I))$

$\to a\leq ab \bmod I$.

—o—

$a\in Ra(W,I)\to \dot{a}\in Ri(W,I)$ $I\in$??

—o—

$a\leq b$ $b\leq d$ $x\in MS\{W,I|a/b\}$ $a,b,d\in\Delta'\{Ra(W,I),I\}$

$z\in MS\{W,I|a/d\}$ $dz=a+w$ $w\in I$ $z\equiv a \bmod I$ $bx=a+s$

$x\in\Delta'\{Ra(W,I),I\}$ $z\equiv a$ $\exists$ ~~$y\in W$~~ $y\in MS\{W,I|z/x\}$ ‖ $dy=b+v$

$xy=z+t$ $dxy=dz+dt=a+w+dt=bx+w+dt-s$

$y\equiv z\equiv a$ ‖ must assume $b\leq a$

$\frac{a}{b}\cdot\frac{b}{d}=\frac{a}{d}$ $a\leq b$ $b\leq d$ $b,d\in\Delta'\{Ra(W,I),I\}$ $I\in I_{qf}(W)$

$y\in MS\{W,I|b/d\}$ $z\in MS\{W,I|a/d\}$ $dz=a+w$ $z\equiv a$

$y\in\Delta'\{Ra(W,I),I\}$ $y\equiv b$ $xy=z+p$ $dy=b+t$

$dxy=dz+dp$ $bx=a+w+dp-xt$ $x\equiv z\equiv a$

$xy=z+t$ $dy=c+p$ $bdz=ac+w$ $bx=a+s$ $a\leq b$ $c\leq d$

$z\equiv ac$ $bdz=bcx+cs+w$ $z\equiv a$ $x\equiv a$

$bz+bt=ay+ys$ $dz=cx+q$ $dxy=cx+q+dt$

1 ( ) Let $b \in Ri(W, I)$

~~If~~ I being ~~is~~ any set contained in W,

(1) $a \leq ab \bmod I$ for all $a \in W$

(2) If $I \in B_{\times;si}(W)$, $ab \equiv a \bmod I$ for all $a \in W$.

For any $g \in W$ such that $abg \in I$, $ag \in I$, since $b \in Ri(W, I)$: $a \leq ab \bmod I$. ~~If $I \in B_{\times;si}(W)$ and $ag \in I$ then $abg \in I$ and~~ Since $a \leq a \bmod I$, ~~and~~ $ab \leq a \bmod I$ from ( ): Using (1), $ab \equiv a \bmod I$.

( ) Let $b | a \bmod I$ where $I \in B_{\times;si}(W) \cap B_{+;ls}$. $a \leq b \bmod I$.

If $b | a \bmod I$, $x \in W$ and $s \in I$ ~~exis~~ such that $bx = a + s$ exist. ~~If~~ For any $g \in W$ such that $bg \in I$, $bxg \in I$ since $I \in B_{\times;si}(W)$, ~~&~~ and then $ag + sg \in I$ and $ag \in I$, since $I \in B_{\times;si}(W) \cap B_{+;ls}$: $a \leq b \bmod I$.

( ) Let $I \in B_{\times;qf}(W)$. If $a \leq b \bmod I$ and $ab \in I$ then $a \in I$.

If $ab \in I$ then $a^2 \in I$, since $a \leq b \bmod I$, and $a \in I$, since $I \in B_{\times;qf}(W)$.

( ) Let $a \leq b \bmod I$ where $I \in B_{\times;si} \cap B_{\times;qf}(W)$. $ab \equiv a \bmod I$.

~~If $a \leq b$~~ $ab \leq a \bmod I$, since $I \in B_{\times;si}(W)$, from ( ). ~~If~~ ~~$abg \in I$~~ For any $g \in I$, ag ~~[illegible]~~ $\leq a \bmod I$, again ~~from (~~ since $I \in B_{\times;si}(W)$. If $abg \in I$, then $ag \in I$, since $ag \leq a \leq b \bmod I$ and $I \in B_{\times;qf}(W)$, from ( ): $a \leq ab \bmod I$.

( ) Let $a \in Ra(W, I)$ and $b|a \bmod I$, where $I \in B_\ell(W)$

~~(1) For each $x \in S\{W, I|a/b\}$~~

~~$S\{W, I|ac/bd\} = xS\{W, I|c/d\} + I$~~

~~(2) $S\{W, I|ac/bd\} = S\{W, I|a/b\}S\{W, I|c/d\} + I$~~

(1) $S\{W, I|a/b\} = MS\{W, I|a/b\}$

(2) For each $x \in S\{W, I|a/b\}$

$S\{W, I|ac/bd\} = xS\{W, I|c/d\} + I$, ~~$MS\{W, I|ac/bd\}$~~

$MS\{W, I|ac/bd\} = xMS\{W, I|c/d\} + \underline{I}$

(3) $S\{W, I|ac/bd\} = S\{W, I|a/b\}S\{W, I|c/d\} + \underline{I}$

$MS\{W, I|ac/bd\} = MS\{W, I|a/b\}MS\{W, I|c/d\} + \underline{I}$

Since $a \in Ra(W, I)$, $a \in Ri(W, I)$ from ( ), since $I \in$. $b|a \bmod \underline{I}$, so that $b \in Ri(W, I)$, from ( ), since $I \in \dots$. When $b \in Ri(W, I)$, $S\{W, I|a/b\} = MS\{W, I|a/b\}$, since $I \in B_\ell(W)$, from ( ).

The results of the ~~theorem~~ parts (2,3) concerning the sets $S\{W, \underline{I}|c/d\}$ and $S\{W, I|ac/bd\}$ are first proved. Select $x \in S\{W, I|a/b\}$ so that $bx = a + s$ for some $s \in \underline{I}$.

$\sqcup$ 9 pages back.

To prove the results of the parts (2,3) concerning the sets $MS\{W, I|c/d\}$ and $MS\{W, I|ac/bd\}$ it is first remarked that if $MS\{W, I|c/d\}$, $xMS\{W, \underline{I}|c/d\} + \underline{I} \subseteq MS\{W, \underline{I}|ac/bd\}$. Otherwise, select $y \in MS\{W, I|c/d\}$ and $w \in I$, so that $y \equiv c \bmod I$ and $xy + w \in S\{W, I|ac/bd\}$ from the above. Since $x \in Ra(W, I) \subseteq Ri(W, I)$, $xy \equiv y \equiv c \bmod \underline{I}$. ~~Again,~~ from ( ). Again, since $a \in Ri(W, I)$, $ac \equiv c \bmod \underline{I}$. Thus $xy \equiv ac \bmod I$. For any $y \in MS\{W, I|c/d\}$ and $w \in \underline{I}$, $xy + w \in MS\{W, I|ac/bd\}$: $xMS\{W, \underline{I}|c/d\} + \underline{I} \subseteq MS\{W, I|ac/bd\}$.

Again if $MS\{W, I|ac/bd\}$ is void, $MS\{W, I|ac/bd\} \subseteq xMS\{W, I|c/d\} + \underline{I}$.

Otherwise, to each $z\in MS\{W,I|ac/bd\}$, $y\in S\{W,I|c/d\}$ and $t\in I$ for which $z=xy+t$ correspond, from the above. Now $z\equiv ac\equiv c \bmod I$, since $a\in Ri(W,I)$. Also $z\equiv xy \bmod I$, since $I\in B_i(W)$, from ( ). Thus $y\equiv xy\equiv c \bmod I$, since $x\in Ri(W,I)$, from ( ); $y\in MS\{W,I|c/d\}$: $MS\{W,I|ac/bd\}\subseteq xMS\{W,I|c/d\}+\underline{I}$.

—o—

((1) If $I\in B_{x;si}(W)\cap B_{+;c}$, $S\{W,I|a/b\}S\{W,I|b/d\}\subseteq S\{W,I|a/d\}$ for all $a,b,d\in I$

(2) If $I\in B_i\cap B_{x;qf}(W)$, $MS\{W,I|a/b\}MS\{W,I|b/d\}\subseteq MS\{W,I|a/d\}$ for all $a,b,d\in I$.

If either $S\{W,I|a/b\}$ or $S\{W,I|b/d\}$ the result of part (1) is correct. Assuming neither set to be void select x,y such that $bx=a+s$, $dy=b+t$ where $s,t\in I$. Then $dxy=a+$ ~~$s+xt$~~ w where $w=s+xt\in I$, since $I\in B_{x;si}(W)\cap B_{+;c}$: $xy\in S\{W,I|a/d\}$.

Assuming neither $MS\{W,I|a/b\}$ nor $MS\{W,I|b/d\}$ to be void, select x and y from these sets. Since $b|a \bmod \underline{I}$ and $\underline{I}\in B_{x;si}(W)\cap B_{+;lc}$, $a\leq b \bmod I$ from ( ). Since $x\equiv a$, $y\equiv b \bmod I$, $xy\equiv ab$ ~~from ( )~~ $\bmod I$, from ( ). But $ab\equiv a \bmod \underline{I}$ when $a\leq b \bmod \underline{I}$ and $I\in B_{x;si}\cap B_{x;qf}(W)$, from ( ). Hence $xy\equiv a \bmod \underline{I}$ and $xy\in MS\{W,I|a/d\}$.

—o—

( ) Let $a\in Ra(W,I)$ and $b|a$ $\underline{\bmod I}$, where $I\in B_i(W)$, and $d\in W$.

(1) $S\{W,I|a/b\}=MS\{W,I|a/b\}$.

(2) The sets $S\{W,I|a/d\}$ and $S\{W,I|b/d\}$ are either both void or both nonvoid

(3) $MS\{W,I|a/d\}=S\{W,I|a/d\}$ and $MS\{W,I|b/d\}=S\{W,I|b/d\}$.

(4) For each $x\in S\{W,I|a/b\}$, $S\{W,I|a/d\}=xS\{W,I|b/d\}+\underline{I}$

(5) $S\{W,I|a/d\}=S\{W,I|a/b\}S\{W,I|b/d\}+\underline{I}$.

From ( ), the conditions $a \in Ra(W, I)$ and $b | a$ mod I imply, ~~that~~ when $I \in B_i(W)$?, that $b \in Ra(W, I)$. When $I \in$ . . . , $Ra(W, I) \subseteq Ri(W, I)$ so that $b \in Ri(W, I)$ and, from ( ), $S\{W, I | a/b\} = MS\{W, I | a/b\}$ since $I \in B_2(W)$.

~~From the above~~ $a, b \in Ra(W, I)$, from the above. From ( ), $S\{W, I | a/d\}$ and $S\{W, I | b/d\}$ are nonvoid if and only if $d \in Ra(W, I)$. $MS\{W, I | a/d\}$ and $MS\{W, I | b/d\}$ are subsets of $S\{W, I | a/d\}$ and $S\{W, I | b/d\}$ respectively. If $d \notin Ra(W, I)$, all four subsets mentioned in part (3) are void. If $d \in Ra(W, I) \subseteq Ri(W, I)$, the results of part (3) follow directly from ( ), since $I \in B_2(W)$.

To prove the result of part (4), select $x \in S\{W, I | a/b\}$, so that $bx = a + s$ where $s \in I$.

If $S\{W, I | b/d\}$ is void, $xS\{W, I | b/d\} + I \subseteq S\{W, I | a/d\}$. Assuming $S\{W, I | b/d\}$ to be nonvoid, select $y \in S\{W, I | b/d\}$ so that, from ( ), $xy \in S\{W, I | a/d\}$. From ( ), $S\{W, I | a/d\} + I = S\{W, I | a/d\}$, since $I \in B_2(W)$??. For any $w \in I$, $xy + w \in S\{W, I | a/d\}$: $xS\{W, I | b/d\} + I \subseteq S\{W, I | a/d\}$.

If $S\{W, I | a/d\}$ is void, $S\{W, I | a/d\} \subseteq xS\{W, I | b/d\} + I$. Assuming $S\{W, I | a/d\}$ to be nonvoid, select $z \in W$ for which $dz = a + w$ where $w \in I$. Since $a \in Ra(W, I)$, $x \in Ra(W, I)$ also, from ( ). W and I contain y and p respectively for which $z = xy + p$. Then $dxy = dz - dp = a + w - dp = bx$ ~~$+s$~~ $+ w - s - dp$. ~~so #~~ Since $x \in Ra(W, I) \subseteq Ri(W, I)$, $dy = b$ mod I and $y \in S\{W, I | b/d\}$. To each $z \in S\{W, I | a/d\}$, $y \in S\{W, I | b/d\}$ and $p \in I$ for which $z = xy + p$ correspond: $S\{W, I | a/d\} \subseteq xS\{W, I | b/d\} + I$.

Part (4) has been dealt with. Part (5) is a corollary to its predecessor.

( ) Let $a, b, d \in \Delta'\{Ra(W,I), I\}$ and $a \equiv b \leq d$ mod I, where $I \in \mathbb{I}_{QF}(W)$ ~~$x|z$ mod I~~

(1) $MS\{W,I|a/b\}$, $MS\{W,I|b/d\}$ and $MS\{W,I|a/d\}$ are nonvoid

(2) For a fixed $x \in MS\{W,I|a/b\}$, ~~each $z \in MS\{W,I|a/d\}$~~ $x|z$ mod I for each $z \in MS\{W,I|a/d\}$, ~~contains x as a mod I factor~~, the cofactor of x being in $MS\{W,I|b/d\}$: $y(x,z) \in MS\{W,I|b/d\}$ and $t(x,y,z) \in I$ for which $xy(x,z)$ ~~$\leq$~~ $= z + t(x,y,z)$ exist $\overset{?}{\leq}$

(3) For each $x \in MS\{W,I|a/b\}$, $MS\{W,I|a/d\} \equiv x\,MS\{W,I|b/d\} + I$

(4) $MS\{W,I|a/d\} = MS\{W,I|a/b\}\,MS\{W,I|b/d\} + I$.

~~Since $a \overset{\text{mod } I}{\leq} b$, $b \in \Delta'\{Ra(W,I), I\}$, $b \leq d$ mod I, $d \in \Delta'\{R$~~

Since $a \leq b$, $b \leq d$, $a \leq d$ mod I and $b, d \in \Delta'\{Ra(W,I), I\}$, $MS\{W,I|a/b\}$, $MS\{W,I|b/d\}$ and $MS\{W,I|a/d\}$ are all nonvoid, from ( )

Select $x \in MS\{W,I|a/b\}$, so that $bx = a + s$ for some $s \in I$ and $x \equiv a$ mod I. Since $a \in \Delta'\{Ra(W,I), I\}$, $x \in \Delta'\{Ra(W,I), I\}$, from ( ). Select $z \in MS\{W,I|a/d\}$ so that $dz = a + w$ where $w \in I$ and $z \equiv a$ mod I. The two conditions $x \in \Delta'\{Ra(W,I), I\}$ and $z \leq x$ mod I imply, from ( ), that $MS\{W,I|z/x\}$ is nonvoid: $MS\{W,I|z/x\}$ contains y for which $xy = z + t$ where $t \in I$ and $y \equiv z \equiv a$ mod I. Then $dxy = dz + dt = a + w + dt = bx + w + dt - s$, ~~Since~~ or $x(dy - b)$ ~~$\equiv$~~ $= w + dt - s \in I$. Since $y \equiv a \leq x$ mod I, $dy \leq x$ mod I from ( ). $b \leq a$ ~~$\equiv x$~~ mod I by assumption, so that $b \leq x$ mod I: $dy - b \leq x$ mod I and, since $I \in \mathbb{I}_{QF}(W)$,

$dy = b + p$ where $p \in I$. Also $y \equiv a \equiv b$ mod I, so that $y \in MS\{W, I | b/d\}$; z has been shown to satisfy the relationship $xy(x, z) = z + t(x, y, z)$ as described in (2).

Part (3) is a corollary to its predecessor.

fol. page

$MS\{W, I | b/d\} + I$

From (3), $MS\{W, I | a/d\} \subseteq MS\{W, I | a/b\} + I$. From ( ), $xy \in MS\{W, I | a/d\}$ for each pair $x \in MS\{W, I | a/b\}$ and $y \in MS\{W, I | b/d\}$. From ( ), $MS\{W, I | a/d\} + I = MS\{W, I | a/d\}$, when $I \in B_I(W)$: for each triplet $x \in MS\{W, I | a/b\}$, $y \in MS\{W, I | b/d\}$ and $t \in I$, $xy + t \in MS\{W, I | a/\overset{d}{\cancel{b}}\}$: $MS\{W, I | a/b\} MS\{W, I | b/d\} + I \subseteq MS\{W, I | a/\overset{d}{\cancel{b}}\}$. The result of (4) follows.

—o—

( ) Let $b, d \in \Delta'\{Ra(W, I), I\}$ and $a \leq b \leq d$ mod I, where ~~$I \in P_{QF}(W)$~~ ??? $B_I(W)$.

(1) $MS\{W, I | a/b\}$, $MS\{W, I | b/d\}$ and $MS\{W, I | a/d\}$ are nonvoid.

(2) For a fixed $y \in MS\{W, I | b/d\}$, ~~each $z \in MS\{W, I | a/d\}$ contain~~ $y | z$ mod I for each $z \in MS\{W, I | a/d\}$, the cofactor of y being in $MS\{W, I | a/b\}$: $x(y, z) \in MS\{W, I | a/b\}$ and $p(x, y, z) \in I$ for which $x(y, z)\, y = z + p(x, y, z)$ exist.

(3) For each $y \in MS\{W, I | b/d\}$, $MS\{W, I | a/d\} \subseteq y\, MS\{W, I | a/b\} + I$

(4) $MS\{W, I | a/d\} = MS\{W, I | a/b\} MS\{W, I | b/d\} + I$

Since $a \leq b$, $b \leq d$, $a \leq d$ mod I and $b, d \in \Delta'\{Ra(W, I), I\}$, $MS\{W, I | a/b\}$, $MS\{W, I | b/d\}$ and $MS\{W, I | a/d\}$ are all nonvoid, from ( ).

Select $y \in MS\{W, I | b/d\}$, so that $dy = b + t$ for some $t \in I$ and $y \equiv b$ mod I. Since $b \in \Delta'\{Ra(W, I), I\}$, $y \in \Delta'\{Ra(W, I), I\}$, from

( ). Select $z\in MS\{W,I|a/d\}$ so that $dz=a+w$ where $w\in I$ and $z\equiv a$ mod I. The two conditions $y\in\Delta'\{Ra(W,I),I\}$ and $z\leq y$ mod I imply from ( ) that $MS\{W,I/z/y\}$ is nonvoid: $MS\{W,I/z/y\}$ contains x for which $xy=z+q$ where $q\in I$ and $x\equiv z\equiv a$ mod I. Then $dxy=dz+dq=a+w+dq$ and $bx=a+r$ where $r=w+dq-xt\in I$: $x\in S\{W,I|a/b\}$. Since $x\equiv a$ mod I $x\in MS\{W,I|a/b\}$. z has been shown to satisfy the relationship $x(y,z)y\neq z+p(x,y,z)$ as described in (2).

The result of (2) may be rephrased by stating that ~~each $z\in MS\{W,I|a/b\}$ may~~ for a fixed $y\in MS\{W,I|b/d\}$, each $z\in MS\{W,I|a/d\}$ may be expressed in the form $z=yx(y,z)+p'(x,y,z)$ where $x(y,z)\in MS\{W,I|a/b\}$ and $p'(x,y,z)=-p(x,y,z)\in I$: $MS\{W,I|a/d\}\subseteq y\,MS\{W,I|a/b\}+I$. From ( ), $xy'\in MS\{W,I|a/d\}$ for each pair $x\in MS\{W,I|a/b\}$ and $y'\in MS\{W,I|b/d\}$. From ( ), $MS\{W,I|a/d\}+I=MS\{W,I|a/d\}$ when $I\in B_2(W)$: for each triplet $x\in MS\{W,I|a/b\}$, $y'\in MS\{W,I|b/d\}$ and $t\in I$, $xy'+t\in MS\{W,I|a/d\}$: $MS\{W,I|a/b\}MS\{W,I|b/d\}+I\subseteq MS\{W,I|a/d\}$. Hence, in particular, $y\,MS\{W,I|a/b\}+I\subseteq MS\{W,I|a/d\}$, leading to the result of (3). Also $MS\{W,I|a/d\}\subseteq MS\{W,I|a/b\}MS\{W,I|b/d\}+I$ from the above, leading to the result of (4).

—o—

(termination of proof of prev. theorem)

The result of (2) may be rephrased by stating that for a fixed $x\in MS\{W,I|a/b\}$, each $z\in MS\{W,I|a/d\}$ may be expressed in the form $z=xy(x,z)+t'(x,y,z)$ where $y(x,z)\in MS\{W,I|b/d\}$ and $t'(x,y,z)=-t(x,y,z)\in I$: $MS\{W,I|a/d\}\subseteq x\,MS\{W,I|b/d\}+I$. From ( ), $x'u\in MS\{W,I|a/d\}$ for each pair $x'\in MS\{W,I|a/b\}$ and

$y\in MS\{W,I|b/d\}$. From ( ), $MS\{W,I|a/d\}+I\equiv MS\{W,I|a/d\}$ when $I\in B_2(W)$: for each triplet $x\in MS\{W,I|a/b\}$, $y\in MS\{W,I|$ and $t\in I$, $x\cdot y+t\in MS\{W,I|a/d\}$: $MS\{W,I|a/b\}MS\{W,I|b/d\}+I$ $\le MS\{W,I|a/d\}$. Hence, in particular, $xMS\{W,I|b/d\}+I\subseteq MS\{W,I|a$ leading to the result of (3). Also $MS\{W,I|a/d\}\le MS\{W,I|a/b\}$ $MS\{W,I|b/d\}+I$ from the above, leading to the result of (4).

$I\in B_{x;si}(W)\cap B_{+;c}$ $S\{W,I|a/b\}+S\{W,I|c/d\}\le$

$S\{W,I|(ad+bc)bd\}$

$bx=a+s$ $dy=c+t$ $bd(x+y)=ad+bc+\cancel{ds+bt}w$

$w=ds+bt\in I$.

$b,d\in R_i(W,I)$ $I\in B_i(W)$ $MS\{W,I|a/b\}=S\{W,I|a/b\}$

c/d $(ad+bc)/bd$

$z\in S\{W,I|(ad+bc)/bd\}$ $bdz=ad+bc+r$

$bd(z-x-y)=r-ds-bt$ $bd\in R_i(W,I)$

$a,b\in I\rightarrow\exists c\in I$

$a=b+c$

~~$a=b$~~

$z-x-y=p\in I$ $x\in S\{W,I|a/b\}$ fixed

~~x fixed~~ $S\{W,I|(ad+bc)/bd\}=x+S\{W,I|\cancel{b}/d\}$

$S\{W,I|(ad+bc)\}=S\{W,I|a/b\}+S\{W,I|c/d\}$

$c,d\le b$ mod I, $b|a$ mod I $bd|ad+bc$ mod I $I\in I_{SF}(W)$

$x\in S\{W,I|a/b\}$ fixed

$S\{W,I|(ad+bc)/bd\}=x+S\{W,I|c/d\}$

~~$z\in$~~ z: ~~$bdz(ad+$~~ $bdz=ad+bc+w$ $bx=a+s$ $bdx=da+ds$

$bd(z-x)=bc+w-ds$ $c\le b, d\le b$ $I\in I_{SF}$ $x\le a$ $y\le c$

$d(z-x)=c+p$ ~~$z=x+y$~~

$dy=c+p$ $z-x\in S\{W,I|c/d\}$ $a\le d$

$x\equiv a$ $z\equiv ad+bc\le b$ $a\le b$ $a\le c\rightarrow ad\le bc$

$a\le c\rightarrow z\le c$, $x\le c\rightarrow y\le c$

$\Rightarrow MS\{W,I|(ad+bc)/bd\}\overset{\subseteq}{\cancel{=}}x+MS\{W,I|c/d\}$?

1 $\left[b\in\Delta'\{Ra(W,I),I\}\right.$ $a\le b \bmod I$ $I\in B_{1}(W) \Rightarrow MS\{W, I|a/b\}$ monoid

$b\in\Delta'\{Ra(W,I),I\}$ $a\le b \bmod I$ $I\in I_{QF}(W)$. $MS(a/b)$ monoid.

$x\in MS(a/b)$ $x+I\equiv MS(a/b)$ $S(a/b)\cap N(b)\subseteq MS(a/b)$

All members of $S\{W,I|a/b\}$ in $N(b)$ belong to $MS\{W,I|a/b\}$

~~$x+r$~~ $ub+v\in Ra$ $(ub+v)x = ua \bmod I$ · $x\in MS(a/b)$ $x+I\equiv MS(a/b)$

$I\in \mathbb{I}(W)$ $a\le b$ $a,b\in\Delta'\{Ra(W,I)I\}$ $MS(a/b)\in\Delta'(..)$

$I\in I(W)$ $a\le b$ $b\in\Delta'\{Ra(W,I),I\}$ $a\in\Delta\{Ra(W,I),I\}$, $MS(a/b)\le\Delta$

=

$(ub+v)x = ua+\cancel{q}p$ $(sd+t)y = sc+\cancel{v}q$ $I\in I_{QF}$

$a\le b$ $c\le d$ $ad\le bd$ $bc\le bd$ $\to ad+bc\le bd$ $b,d\in\Delta'(W,I)$

$(ub+v)(sd+t)z = us(ad+bc)+\cancel{p}r$

$ua\,t+sc\,v$

$(ub+v)(sd+t)x = uasd+uat+p(sd+t)$ $+uav+sct$

$(ub+v)(sd+t)y = usbc+scv+q(ub+v)$ $(ua+sc)(v+t)$

$(ub+v)(sd+t)z = sd\{(ub+v)x-p\}+ub\{(sd+t)y-q\}+r$

$b\equiv d \bmod I$ $dt\in I\to bt\in I\to at\in I$ $bv\in I\to cv\in I$

$z=x+y+h$ $h\in I$ $MS\{(ad+bc)/bd\}\le x+MS\{c/d\}$

$b,d\in Ri$

$b|a$

either $d|c$

or $bd|ad+bc$

or both conditions false

==

$ax+by=e$ $adx+bdy=de$ $(ad-bc)x=de-bf$

$cx+dy=f$ $bcx+bdy=bf$ $(ad-bc)y=af-ec$

$acx+bcy=ec$

$acx+ady=af$

$ax:$ $(ad-bc)(ax)=a(de-bf)$

$(ad-bc)(by)=b(af-ec)$

$(ad-bc)(ax+by)=\cancel{e}\,e(ad-bc)$

~~ba~~ $a\equiv b$

$\mathcal{O}(a)=\mathcal{O}(b)$

$p\le r$ $pr\le q$ $q\in\Delta'$ $r\in\Delta$ $rMS(p/q)\equiv MS\{pr/q)$

$e,f\le(ad-bc)$

$bz=eb$

bz

$(ax'+by')v=0$ x'',y'' further solution

$(cx'+dy')v=0$

$a(x''-x)+b(y''-y)\in I \to (ad-bc)(x''-x)\in I$

$c(x''-x)+d(y''-y)\in I$

$x''-x\in\mathcal{O}\{ad-bc\}$

$x''-x = xv$ $bv\in I$

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}1&1\\1&1\end{pmatrix} \qquad \begin{matrix}2\\0\end{matrix}$$

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix} \qquad \begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}2-t\\t\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix} \qquad \begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}\cancel{z}-z\\z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d-b(c/a) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$c \leq a \quad a \in \Delta'$

$$\begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$ $g = e$

$e(c/a) + h = f \quad h = f - e(c/a)$

$$\begin{pmatrix} a & b \\ 0 & d-b(c/a) \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix} = \begin{matrix} e \\ f - e(c/a) \end{matrix}$$

$l = \{f - e(c/a)\} / \{d - b(c/a)\}$

$ak + bl = e$

$k = (e - bl)/a$ ||| $l = (af - ec)/(ad - bc)$

$k = \{e(ad - bc) - b(af - ec)\}/(ad - bc)a$

~~$ead - bcf$~~ $\Rightarrow (ed - bf)/(ad - bc)$

$(ad/a) \equiv d$

$ax \equiv ad$

$x \leq a \quad d \leq a \Rightarrow x \equiv d$

$ad \equiv ad \quad a(x-d) \in I$

$b(c/a) \equiv bc/a$

$I \in I_{qf}(W) \quad b \equiv d \Leftrightarrow bd \equiv b \equiv d \quad a \leq b \to a \leq bd$

$c \leq d \to c \leq bd$

—o—

( ) Let $I \in B_{x;si}(W) \cap B_{+;c}$. $S\{W, I \mid a/b\} + S\{W, I \mid c/d\} \subseteq S\{W, I \mid (ad+bc)/bd\}$ for all $a, .., d \in W$.

If either $S\{W, I \mid a/b\}$ or $S\{W, I \mid c/d\}$ is void, the stated result is correct. Otherwise, let $bx = a + s$, $dy = c + t$ where $s, t \in I$. Then $bd(x+y) = ad + bc + w$ where $w = ds + bt \in I$, since $I \in B_{x;si}(W) \cap B_{+;c}$.

( ) Let $b, d \in Ri(W, I)$ where $I \in B_i(W)$ and $a, c \in W$.

(1) ~~$MS\{a/b\}$~~ $MS\{W, I \mid a/b\} \equiv S\{W, I \mid a/b\}$, $MS\{W, I \mid c/d\} \equiv S\{W, I \mid c/d\}$ and $MS\{W, I \mid (ad+bc)/bd\} \equiv S\{W, I \mid (ad+bc)/bd\}$.

(2) Let $b \mid a \bmod I$ and x be a fixed member of $S\{W, I \mid a/b\}$. $S\{W, I \mid (ad+bc)/bd\} \equiv x + S\{W, I \mid c/d\}$.

(3) $S\{W, I|a/b\} + S\{W, I|c/d\} \equiv S\{W, I|(ab+cd)/bd\}$

The results of (1) concerning the sets $MS\{W, I|a/b\}$ and $MS\{W, I|c/d\}$ are consequences of ( ), since $b, d \in Ri(W, I)$; that concerning the set $MS\{W, I|(ad+bc)/bd\}$ follows from (,) since $bd \in Ri(W, I)$ ~~then $b, d \in Ri(W, I)$~~.

From ( ), $S\{W, I|c/d\}$ and $S\{W, I|(ad+bc)/bd\}$ are either void or nonvoid together. In the former case the result of (2) is correct. In the latter case it is first remarked that, from ( ), $x + S\{W, I|c/d\} \leq S\{W, I|(ad+bc)/bd\}$. ~~Select $z \in S\{W, I|(ad+bc)/bd\}$, so that bdz~~ Let $bx = a+s$ ~~where $s \in I$~~. ~~Select $z \in S\{W, I|(ad+bc)/bd\}$~~ and $dy = c+t$ where $s, t \in I$. Select $z \in S\{W, I|(ad+bc)/bd\}$ so that, ~~bdz~~ $bdz = ad+bc+r$ where $r \in I$. Then ~~d~~ $\cdot\, bd(z-x-y) = p$ where $p = r - ds - bt \in I$, since $I \in B_{x;ri}(W) \cap B_{+;ls}$. Since $bd \in Ri(W, I)$, $z - x - y = q \in I$. But, from ( ), $S\{W, I|c/d\} + I \equiv S\{W, I|c/d\}$ when $I \in B_i(W)$. Since $y \in S\{W, I|c/d\}$, $z = x + y'$ where $y' = y + q \in S\{W, I|c/d\}$.

From ( ), $S\{W, I|a/b\} + S\{W, I|c/d\} \leq S\{W, I|(ab+cd)/bd\}$ and from (2), $S\{W, I|(ab+cd)/bd\} \leq S\{W, I|a/b\} + S\{W, I|c/d\}$.

( ) Let $c, d \leq b \bmod I$ and $b|a$, $bd|ad+bc \bmod I$ where $I \in I_{QF}(W)$.

(1) ~~$S\{W, I|c/d\}$ is~~ $d|c \bmod I$.

~~( ) Let x be a fixed member of $S\{W, I|$~~

(2) For each $x \in S\{W, I|a/b\}$, $S\{W, I|(ad+bc)/bd\} \equiv x + S\{W, I|c/d\}$.

(3) $S\{W, I|a/b\} + S\{W, I|c/d\} \equiv S\{W, I|(ad+bc)/bd\}$.

(4) If in addition $a \leq c \bmod I$, $MS\{W, I|(ad+bc)/bd\} \leq x + MS\{W, I|c/d\}$, for each $x \in MS\{W, I|a/b\}$.

Select $x \in S\{W, I|a/b\}$ and $z \in S\{W, I|(ad+bc)/bd\}$ so that $bx = a+s$ and $bdz = \{ad+bc\} + w$ where $s, w \in I$. Then $b\{d(z-x)-c\} = w - ds \in I$. Since $c; d \leq b \bmod I$, $d(z-x) - c \in I$ and, since $I \in I_{QF}(W)$, $d(z-x) = c+p$ where $p \in I$. $S\{W, I|c/d\}$ contains $y = z-x$ and is nonvoid: $d|c \bmod I$. Also $z = x+y$; ~~leading to the~~ $S\{W, I|(ad+bc)/bd\} \leq x + S\{W, I|c/d\}$. From ( ), $x + S\{W, I|c/d\} \leq S\{W, I|(ad+bc)/bd\}$. The proof of (3) is as for (3).

If either $MS\{W, I|a/b\}$ or $MS\{W, I|(ad+bc)/bd\}$ is void, the result of (4) is correct. Otherwise, take $z \equiv ad+bc \bmod I$ in the above. ~~If $a \leq c \bmod I$,~~ ~~then~~ $bc \leq c \bmod I$ and, if $a \leq c$, $ad \leq c \bmod I$, so that $z \leq c \bmod I$. Also $x \equiv a \leq c \bmod I$, so that $y = z-x \leq c \bmod I$. Since $c \leq y \bmod I$ for all $y \in S\{W, I|c/d\}$, $y \equiv c \bmod I$: $y \in MS\{W, I|c/d\}$.

( ) Let $a \leq b, c \leq d, b \equiv d \bmod I$ and $b, d \in \Delta'\{Ra(W, I), I\}$ where $I \in I_{QF}(W)$.

(1) $MS\{W, I|a/b\}$, $MS\{W, I|c/d\}$ and $MS\{W, I|(ad+bc)/bd\}$ are nonvoid

(2) For each $x \in MS\{W, I|a/b\}$, $MS\{W, I|(ad+bc)/bd\} \equiv x + MS\{W, I|c/d\}$ and $S\{W, I|(ad+bc)/bd\} \equiv x + S\{W, I|c/d\}$,

(3) $MS\{W, I|(ad+bc)/bd\} \equiv MS\{W, I|a/b\} + MS\{W, I|c/d\}$ and $S\{W, I|(ad+bc)/bd\} \equiv MS\{W, I|a/b\} + S\{W, I|c/d\}$.

Since $a \leq b, c \leq d$ ~~and~~ and $b, d \in \Delta'\{Ra(W, I)|I\}$, $MS\{W, I|a/b\}$ and $MS\{W, I|c/d\}$ are nonvoid, from ( ) since $I \in B_2(W)$. Since $b, d \in \Delta'\{Ra(W, I), I\}$, ~~&~~ $bd \in \Delta'\{Ra(W, I), I\}$, from ( ). ~~The relationships $a \leq b, d \leq d \bmod I$ imply that $ad \leq bd$~~

From ( ), $ad \leq bd$, $bc \leq bd \bmod I$ since $a \leq b$, $c \leq d \bmod I$ and $I \in B$.. . Hence $ad+bc \leq bd \bmod I$ and, from ( ), $MS\{W, I \mid (ad+bc)/bd\}$ is nonvoid.

Since $b \in \Delta'\{Ra(W,I), I\}$, W contains u and v for which $ub+v \in Ra(W,I)$ and $bv \in I$, or a simpler condition holds. From ( ), one member x' of $MS\{W, I \mid a/b\}$ satisfies the ~~equat~~ relationship $(ub+v)x' = ua+p'$ where $p' \in I$. Select $x \in MS\{W, I \mid a/b\}$. From ( ), $x = x'+p''$ where $p'' \in I$ and accordingly $(ub+v)x = ua+p'$ where $p = p'+(ub+v)p'' \in I$. Since $d \in \Delta'\{Ra(W,I), I\}$, W contains s and t for which $sd+t \in Ra(W,I)$ and $dt \in I$ or a simpler condition holds. From ( ), one member y of $MS\{W, I \mid c/d\}$ satisfies the relationship $(sd+t)y = sc+q$ where $q \in I$. ~~One member z' of $MS\{W, I \mid (ad+bc)/bd\}$ satisfies the relationship $(ub+v)(sd+t)z' = us(ad+bc)+r'$ where $r' \in I$~~ From the proof of ( ), $us(bd) + ubt + sdv + vt \in Ra(W,I)$ and $bd\{ubt+sdv+vt\} \in I$. One member z' of $MS\{W, I \mid (ad+bc)/bd\}$ satisfies the relationship $(ub+v)(sd+t)z' = us(ad+bc)+r'$ where $r' \in I$. Select $z \in MS\{W, I \mid (ad+bc)/bd\}$. From ( ), $z = z'+r''$ where $r'' \in I$ and accordingly $(ub+v)(sd+t)z = us(ad+bc)+r$ where $r = r'+(ub+v)(sd+t)r''$ $\in I$. From the above $(ub+v)(sd+t)x = uasd+uat+p(sd+t)$ and $(ub+v)(sd+t)y = usbc+scv+q(ub+v)$. Since $b=d$, ~~mod I~~ ~~as~~ . $a \leq b \bmod I$ and $dt \in I$, $bt \in I$, ~~and~~ $at \in I$ and $uat \in I$. Similarly $scv \in I$: $(ub+v)(sd+t)(z-x-y) \in I$ and, since $(ub+v)(sd+t) \in Ri(W,I)$, $z = x+y+w$ where $w \in I$. $y \in MS\{W, I \mid c/d\}$ and, from ( ), $y+w \in MS\{W, I \mid c/d\}$: $MS\{W, I \mid (ad+bc)/bd\} \leq x + MS\{W, I \mid c/d\}$.

Selecting $x \in MS\{W, I|a/b\}$ and $y \in MS\{W, I|c/d\}$, it follows from ( ) that $x+y \in S\{W, I|(ad+bc)/bd\}$. From ( ), ~~$bd \equiv ab \equiv d$ m~~ ~~$b \equiv bd$, and d~~ $b \equiv bd$ ~~and~~ mod I and $d \equiv bd$ mod I when $b \equiv d$ mod I. $x \equiv a \leq b \equiv bd$ mod I. Similarly $y \leq bd$ mod I. Thus $x+y \leq bd$ mod I From ( ), all members ~~z~~ z of $S\{W, I|(ad+bc)/bd\}$ for which $z \leq bd$ mod I belong to $MS\{W, I|(ad+bc)/bd\}$: $x + MS\{W, I|a/b\} \subseteq MS\{W, I|(ad+bc)/bd\}$. The first result of (2) has been obtained.

With z selected from $MS\{W, I|(ad+bc)/bd\}$, any member z' of $S\{W, I|(ad+bc)/bd\}$ has the form $z' = z+h$, where $h \in \bar{O}(W, I|bd)$ From (,), $\bar{O}(W, I|bd) = \bar{O}(W, I|d)$ when $b \equiv d$ mod I. Since $z = x+y$ where $y \in MS\{W, I|c/d\}$ and $y+h \in S\{W, I|c/d\}$, since $h \in \bar{O}(W, I|d)$, $z' = x+y+h \in x+S\{W, I|c/d\}$: $S\{W, I|(ad+bc)/bd\} \subseteq x + S\{W, I|c/d\}$. With y selected from $MS\{W, I|c/d\}$, any member y' of $S\{W, I|c/d\}$ has the form $y' = y+h$ where $h \in \bar{O}(W, I|d)$. Since $x+y \in$ ~~O~~ $MS\{W, I|(ad+bc)/bd\}$, ~~$x+y+h$~~ . $x+y' = x+y+h \in S\{W, I|(ad+bc)/bd\}$, since $\bar{O}(W, I|d) = \bar{O}(W, I|bd)$: $x + S\{W, I|c/d\} \subseteq S\{W, I|(ad+bc)/bd\}$. The second result of (2) has been dealt with.

From the frist result of (2), $MS\{W, I|(ad+bc)/bd\} \subseteq MS\{W, I|a/b\} + MS\{W, I|c/d\}$. It was shown above that for any $x \in MS\{W, I|a/b\}$ and $y \in MS\{W, I|c/d\}$, $x+y \in MS\{W, I|(ad+bc)/bd\}$: $MS\{W, I|a/b\} + MS\{W, I|c/d\} \subseteq MS\{W, I|(ad+bc)/bd\}$. The first result of part (3) has been derived. The second is obtained in a similar way.

$e, f \in Ra$ $e \equiv f$? $c : W$ $ex = c + s$

$e, f \in Ri(W, I) \to e \equiv f$ mod I; $I \in B_{\times;si}(W)$

$e \in Ri$ $eg \in I$ all $g \in I$ $eg \in I$ only when $g \in I$ $\tilde{O}(e) \equiv I$

$e + I$ $e + s$ $s \in I$ $(e+s)g \in I$ all $g \in I$ $(e+s)g = t \in I \Rightarrow eg = t - sg$

$I \in B_{+;ls}(W) \to t - sg \in I \to g \in I$ | $I \in B_{\times;si}(W) \cap B_{+;lc}$ $e \in Ri \to e + I \subseteq Ri$

$Ri(W, I) + I \equiv Ri(W, I)$ $(0 \in I \to Ri(W, I) \subseteq Ri(W, I) + I)$

$e \in Ra$ $c \in W$ $ex = c + s$ $s \in I$ $(e+t)x = c + s + tx$ $I \in B_{\times;si}(W) \cap B_{+;c}$

$\to e + t \in Ra$ $Ra(W, I) + I \subseteq Ra(W, I)$ $0 \in I \to Ra(W, I) + I \equiv Ra(W, I)$

$ex = c + s$ $fy = c + t$ ‖ not true that with $e \in Ra$ $Ra \equiv e + I$

eg W rational numbers $I = \{0\}$ $Ra \equiv W \setminus I$ $e \in W \setminus I$ ~~$W \setminus I \not\equiv e + I$~~

$e + I \equiv e$ $W \setminus I \not\equiv e$

—o—

$e \in Ra$ $ef = e + s$ $s \in I$ $fg \equiv g + t$ $t \in I$ all $g \in W$?

$ex = g + w$ $fex = fg + fs$ $ex + sx = fg + fs$

$g + w + sx = fg + fs$ $fg = g + w + sx - fs$ $I \in B_{z}(W)$,

—o—

Ri nonvoid $fg = g + t$ $t \in I$ $g + t = s \in I$ if $I \in B_{+;ls}(W)$

If W contains unit multiplier f : $fg = g + t(g)$ $t(g) \in I$ all $g \in W$

and $I \in B_{+;lc}$ $Ri(W, I)$ nonvoid.

—o—

Any $I \subseteq W$: $a \equiv b$ mod I if and only if $\tilde{O}(W, I|a) \equiv \tilde{O}(W, I|b)$

if from definition ‖ let $a \equiv b$ mod I $v \in \tilde{O}(b) \to bv \in I \to av \in I$

$a \equiv b \to v \in \tilde{O}(a)$ $\tilde{O}(b) \subseteq \tilde{O}(a)$ $\tilde{O}(a) \subseteq \tilde{O}(a)$ simly.

—o—

Let $b, d \in Ri(W, I)$ ~~where $I \in$~~ and $b|a$ mod I where $I \in B_{z}(W)$

Either $d|c$ mod I and ~~$(ad+bc)$~~ $bd|(ad+bc)$ mod I or both

these conditions are false

(m, n two +ve integers $I \in B_{\times;qf} \overset{?}{\cap} B_{\times;c}$ $a_r \equiv a_1$ $b_r \equiv b_1$ mod I $(r \geq 2)$

$\prod_{r=1}^{m} a_r \equiv \prod_{r=1}^{n} b_r$ iff $a_1 \equiv b_1$ mod I)

$bx = a+s$ $s\in\underline{I}$ $dy = c+t$ $bdxy = ac+at+cs+ts$ $\underline{I}\in B_{x;si}(W)\cap B_{+;c}$

$d|c$ mod I ~~monoid~~ $\to bd|(ad+bc)$ $bd\nmid(ad+bc)\to d\nmid c$

$bdz = \overset{ad+bc}{\cancel{ac}}+w$ $bdx = ad+ds$ $bd(z-x) = bc+w-ds$

$b\{d(z-x)-c\} = w-ds$

$b\in Ri(W,I)\to \cancel{d\in}\ dy = c+r$ $y = z-x$ $r\in\underline{I}\to d|c$

$bd|(ad+bc)\to d|c$ $d\nmid c\to bd\nmid(ad+bc)$

—o—

$\underline{I}\in B_{x;si}(W)\cap B_{+;c}$ $d|c$ mod $I\to bd|ad+bc$

$I\in B_{x;si}(W)\cap B_{+;lc}$ $bd|ad+bc\to d|c$ mod I

$\underline{I}\in B_i(W)\to d|c$ iff $bd|ad+bc$

$Ra\in Ri$ $Ra(W,I)$ monoid $f\to Ri(W,I)$ monoid $J\in\cancel{B_{+;lc}}\ B_i(W)$

$\overset{h}{\cancel{g}}\in Ri$ $e\in Ra$ $ex = \overset{h}{\cancel{g}}+s$ $s\in I$ $\cancel{e\in}\ e\underset{\circ}{g}\in I$

$ex\underset{\circ}{g} = h\underset{\circ}{g}+s\underset{\circ}{g}$ $h\underset{\circ}{g} = ex\underset{\circ}{g}-s\underset{\circ}{g}\in I$ $h\in Ri\to\underset{\circ}{g}\in I$ $e\underset{\circ}{g}\in I$ only when $\underset{\circ}{g}\in\underline{I}$

Ri monoid $I\in B_{\cancel{si},x;si}(W)\cap B_{+;lc}$ $Ra\le Ri$

$I\in B_i$ $Ra\le Ri$

—o—

( ) Let $I\in B_{x;si}(W)$. $O(W,I|e)\equiv I$ for all $e\in Ri(W,I)$

Since $I\in B_{x;si}(W)$, $I\subseteq O(W,I|e)$. Also $e\underset{\circ}{g}\in I$ only when $\underset{\circ}{g}\in\underline{I}$.

) ~~If~~ Let $I\in B_{x;si}(W)\cap B_{+;lc}$. ~~$e+I\subseteq Ri(W,I)$ for all $e\in Ri(W,I)$~~ //(1) $Ri(W,I)+\underline{I}\subseteq Ri(W,I)$

(2) ~~If $0\in I$, Ri~~ If $0\in I$, $Ri(W,I)+I\equiv Ri(W,I)$ // Let $e\in Ri(W,I)$

If $(e+s)\underset{\circ}{g} = t\in\underline{I}$ where $s\in I$, Then $e\underset{\circ}{g} = t-s\underset{\circ}{g}\in I$, since $I\in B_{x;si}(W)\cap B_{+;lc}$. Since $e\in Ri(W,I)$, $\underset{\circ}{g}\in I$: $(e+s)\underset{\circ}{g}\in I$ only when $\underset{\circ}{g}\in\underline{I}$: $e+s\in Ri(W,I)$. If $0\in I$, $Ri(W,I) = Ri(W,I)+0$ $\subseteq Ri(W,I)+\underline{I}$.

) Let $I\in B_{x;si}(W)\cap B_{+;c}$

1) $Ra(W,I)+I\subseteq Ri(W,I)$

2) If $0\in I$, $Ra(W,I)+I\equiv Ra(W,I)$.

Let $e \in Ra(W,I)$. Select $c \in W$. W and I contain x and s respectively for which $ex = c + s$. Select $t \in I$. $(e+t)x = c + s + tx$. If $I \in B_{x;si}(W) \cap B_{+;c}$, $s + tx \in I$: $e + t \in Ra(W,I)$. If $0 \in I$, $Ra(W,I) = Ra(W,I) + 0 \subseteq Ra(W,I) + I$.

It is not true that $Ra(W,I)$ is generated by a single element, in the sense that if ~~R(W,I)~~ $e \in Ra(W,I)$, ~~Rf~~ $Ra(W,I) = e + I$. For example, if W is the system of rational numbers, and $I = \{0\}$, $Ra(W,I) = W \setminus I$. $e + I = e$ but $W \setminus I \neq e$.

( ) Let $I \in B_i(W)$ and $Ra(W,I)$ be nonvoid. ~~Rf~~ $Ra(W,I)$ contains a unit element f for which $fg = g + t(g)$ with $t(g) \in I$ for all $g \in W$.

Select $e \in$ ~~Ra~~ $Ra(W,I)$, so that W contains f for which $ef = e + s$ where $s \in I$. Select $g \in W$. W contains x for which $ex = g + w$. $fex = fg + fs$ so that, since $ef = e + s$, ~~and~~, $ex + sx = fg + fs$ and, since $ex = g + w$, $fg = g$ ~~w+sx~~ $+ \cancel{s}\,t(g)$, where ~~s~~ $t(g) = w + sx - fs$, since $I \in B_i(W)$. The selection of $g \in W$ is arbitrary: $f \in Ra(W,I)$.

( ) Let W contain a unit element f for which $fg = g + t(g)$ with $t(g) \in I$ for all $g \in W$, where $I \in B_{+;lc}$. f belongs to $Ri(W,I)$ which is accordingly nonvoid.

If ~~fg = gn~~ $fg = w \in I$, then $g = w - t(g) \in I$ since $I \in B_{+;lc}$: $f \in Ri(W,I)$.

( ) Let $Ri(W,I)$ be nonvoid, where $I \in B_{x;si}(W) \cap B_{+;lc}$. ~~Rfa~~ $Ra(W,I) \subseteq Ri(W,I)$.

(2) Let $I \in B_i(W)$. $Ra(W,I) \subseteq Ri(W,I)$.

If ~~Ra~~ $Ra(W,I)$ is void, the result of part (1) is correct. Otherwise (1; let $e \in Ra(W,I)$ and $h \in Ri(W,I)$. W and I contain x and s for which ~~eg~~ $ex = h + s$. If $e\underset{\circ}{g} \in I$, $h\underset{\circ}{g} = ex\underset{\circ}{g} - s\underset{\circ}{g} \in I$, since $I \in B_{x;si}(W) \cap B_{+;ls}$ and $\underset{\circ}{g} \in I$, since $h \in Ri(W,I)$: $e \in Ri(W,I)$.

Part (2) is a corollary to (;).

( ) Let I be any subset of W. ~~[illegible]~~ For all $a, b \in W$, $a \equiv b \bmod I$ if and only if $\bar{O}(W, I|a) = \bar{O}(W, I|b)$.

If $\bar{O}(W, I|a) = \bar{O}(W, I|b)$, then $\bar{O}(W, I|a) \leq \bar{O}(W, I|b)$ and by definition $b \leq a \bmod I$,s Similarly $a \leq b \bmod I$. Hence $a \equiv b \bmod I$.

If $a \equiv b \bmod I$ then $a \leq b \bmod I$. ~~If $v \in \bar{O}(W, I|b)$ then $bv \in I$; hence, [illegible] since $a \leq b \bmod I$, $av \in I$ and $v \in \bar{O}(W, I|a)$. $\bar{O}(W, I|b) \leq \bar{O}(W, I|a)$,~~ and $b \leq a \bmod I$. Hence $\bar{O}(W, I|b) \leq \bar{O}(W, I|a)$ and $\bar{O}(W, I|a) \leq \bar{O}(W, I|b)$.

(The above proofs remain valid when I is void. $\bar{O}(W, I|a)$ is then void for all $a \in W$, and $a \equiv b \bmod I$ for all $a, b \in W$)

( ) Let $b|a \bmod I$.

(1) Let $I \in B_{x;si}(W) \cap B_{+;c}$. If $d|c \bmod I$ then $bd|ad+bc \bmod I$

(2) Let $I \in B_{x;si}(W) \cap B_{+;ls}$ and $b \in Ri(W,I)$. If $bd|ad+bc \bmod I$ then ~~b~~$d|c \bmod I$

~~(3)~~

(3) Let $I \in B_z(W)$ and $b \in Ri(W, I)$. $d|c \bmod I$ if and only if $bd|ad+bc \bmod I$.

If $b|a \bmod I$, W and I contain x and s respectively for which $bx = a + s$. If $d|c \bmod I$, ~~x~~$y \in W$ and $t \in I$ for which $dy = c + t$ exist. Then $bd(x+y) = ad + bc + w$, where $w = ds + bt \in I$ if

$I \in B_{x;si}(W) \cap B_{+;c}$: $bd \mid ad+bc \bmod I$. If $bd \mid (ad+bc) \bmod I$ then W and I contain z and w respectively for which $bdz = ad + bc + w$. Then $b\{d(z-x) - c\} = r$ where $r = w - ds \in I$ if $I \in B_{x;si}(W) \cap B_{+;ls}$. If also $b \in Ri(W, I)$, $q \in I$ ex for which $d(z-x) = c+q$ exists : $d \mid c \bmod I$. Part (3) is a corollary to parts (1,2).

—०—

$a^m \leq b \quad a^m b \leq b^{m+r} \rightarrow a^m \leq b^{m+r} \qquad a^{m-1} \leq a \quad a^{m-1} a \leq b \rightarrow a^{m-1} \leq ab$

$a^m \leq b^n \rightarrow a^m \leq b \quad a^{m-1} \cdot a$ ~~$ab \leq a$~~ $a^{m-1} \leq b$

( ) let $a_\nu \equiv a_1$, $b_\nu \equiv b_1 \bmod I$ for $\nu \geq 2$, where $I \in B_{x;si} \cap B_{x;qf}(W)$, and let i, j, m and be four positive integers. ~~$\prod_{\nu=1}^{i} a_\nu \leq \prod_{\nu=1}^{j} b_\nu \bmod I$ if and only if~~

(1) $\prod_{\nu=1}^{i} a_\nu \leq \prod_{\nu=1}^{j} b_\nu$ mod I if and only if $\prod_{\nu=1}^{m} a_\nu \leq \prod_{\nu=1}^{n} b_\nu \bmod I$

(2) $\prod_{\nu=1}^{i} a_\nu \equiv \prod_{\nu=1}^{j} b_\nu$ mod I if and only if $\prod_{\nu=1}^{m} a_\nu \equiv \prod_{\nu=1}^{n} b_\nu \bmod I$

The result of (1) is first proved with $m=n=1$. Let $a_1 \leq b_1 \bmod I$,

~~It follows as a corollary to (2) that $\prod_{\nu=1}^{i} a_\nu$~~

so that $a_\nu \leq b_\nu \bmod I$ ($\nu = 1, 2, \ldots$). It follows as a corollary to (2) that $\prod_{\nu=1}^{z} a_\nu \leq \prod_{\nu=1}^{z} b_\nu \bmod I_s$ ($z = 1, 2, \ldots$). If $i = j$ the stated result is correct. When $i > j$, the further result that $\left\{\prod_{\nu=1}^{i-j} a_\nu\right\}\left\{\prod_{\nu=1}^{j} a_\nu\right\} \leq \prod_{\nu=1}^{j} b_\nu \bmod I$ follows from ( ), since $I \in B_{x;si}(W)$. To deal with the case in which $i < j$, it is first remarked that ~~the conditions $a\, \prod_{\nu=1}^{i} a_\nu \leq \prod_{\nu=1}^{j+z-1} b_\nu \bmod I$ yields, by use of (2), the relationship $\left\{\prod^{i} a_\nu\right\} b_{j+z} \leq \prod_{\nu=1}^{j+z} b_\nu \bmod I$~~

~~$a_\nu \equiv a_1$ mod I ($\nu \ge 1$), and $b_\nu \equiv b$.~~

when $i=1$, $\prod_{\nu=1}^{i} a_\nu \le b_{j+z}$ mod I for $z=1,2,\ldots$ When $i>1$, $\prod_{\nu=1}^{i} a_\nu = \{\prod_{\nu=2}^{i} a_\nu\} a_1 \le b_{j+z}$ mod I for $z=1,2,\ldots$, from ( ).

From ( ), $c \le e$ mod I if $c \le d$, $cd \le e$ mod I when $I \in B_{x;si} \cap B_{x;qf}(W)$. ~~The condition $\prod_{\nu=1}^{i} a_\nu \le \prod^{j+z-1}$~~ Assuming the condition $\prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{i+z-1} b_\nu$ mod I to hold for some $z \ge 1$, (2) yields the result that $\{\prod_{\nu=1}^{i} a_\nu\} b_{j+z} \le \prod_{\nu=1}^{j+z} b_\nu$ mod I. Setting $\prod_{\nu=1}^{i} a_\nu = c$, $b_{j+z} = d$, $\prod_{\nu=1}^{j+z} b_\nu = e$ in the result just quoted, it then follows that $\prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{j+z} b_\nu$ mod I: if for some $z \ge 1$, $\prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{i+z-1} b_\nu$ mod I, then $\prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{i+z} b_\nu$. The stated assumption ~~is~~ holds when $z=1$ and hence for $z=1,2,\ldots$. In particular $\prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{j} b_\nu$ mod I when $j>i$.

It is now supposed that $\prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{j} b_\nu$ mod I, for some ~~pair i,j~~ integer pair $i,j \ge 1$. If $j=1$, $\prod_{\nu=1}^{i} a_\nu \le b_1$ mod I. If $j>1$, $\prod_{\nu=1}^{j} b_\nu = b_1\{\prod_{\nu=2}^{j} b_\nu\} \le b_1$ mod I, from ( ), and hence from ( ), $\prod_{\nu=1}^{i} a_\nu \le b_1$ mod I again. If $i=1$, $a_1 \le b_1$ mod I. To deal with the case in which $i>1$, it is first remarked that when $z>1$, $\prod_{\nu=1}^{z-1} a_\nu \le a_z$ mod I since ~~$a_\nu = a_z$~~ $a_{z-1} \le a_z$ mod I. Assuming that $\prod_{\nu=1}^{z} a_\nu \le b_1$ mod I it follows from the result quoted above, setting $c = \prod_{\nu=1}^{z-1} a_\nu$, $d = a_z$, $e = b_1$ that $\prod_{\nu=1}^{z-1} a_\nu \le b_1$ mod I. The stated assumption holds when $z=i$ and hence for $z=1$: $a_1 \le b_1$ mod I.

It has been shown that $\prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{j} b_\nu$ mod I if and only if $a_1 \le b_1$ mod I. It follows that $\prod_{\nu=1}^{m} a_\nu \le \prod_{\nu=1}^{n} b_\nu$ mod I if and only if $a_1 \le b_1$ mod I. The result of (1) has been obtained.

If $\prod_{\nu=1}^{i} a_\nu \equiv \prod_{\nu=1}^{j} b_\nu$ mod I then $\prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{j} b_\nu$, $\prod_{\nu=1}^{j} b_\nu \le \prod_{\nu=1}^{i} a_\nu$ mod I and hence $\prod_{\nu=1}^{m} a_\nu \le \prod_{\nu=1}^{n} b_\nu$, $\prod_{\nu=1}^{n} b_\nu \le \prod_{\nu=1}^{m} a_\nu$ mod I, or $\prod_{\mu=1}^{m} a_\nu \equiv \prod_{\nu=1}^{n} b_\nu$ mod I. The converse assertion is proved in the same way, leading to the result of part (2).

(Taking the a_ν, ~~to~~ b_ν to be equal to a, b respectively, it follows from the above that $a^i \le b^j$ mod I if and only if $a^{in} \le b^n$ mod I; a similar special formulation of part (2) may be given)

$a \le d \parallel ad \le bc \Leftrightarrow a \le b \quad a \le c$ ~~$a \le a'd'$~~ $a'e' \le b'c' \quad a'e' \le c'd'$

$a'e' \le$

$a'f' \le b'c' \quad a'f' \le d'e' \qquad a'f'd \le b'c'd'e'$

$a'f'g'h' \le b'c'd'e' \Leftrightarrow a'f' \le b'c' \quad a'f' \le d'e' \parallel a'f' \le g'h'$

$a' \le f' \quad a' \le b' \quad a' \le c' \le d', e'$

$a' \le g' \quad a' \le h'$ $\quad a+b \in I \quad a \equiv b$ mod I $\quad ag + bg \in I \quad ag \in I \rightarrow bg \in I$

$ab \le gh \parallel abgh \le cdef \Leftrightarrow ab \le cd \quad ab \le ef$

$a \le b \parallel ab \le cd \Leftrightarrow a \le c \quad a \le d \qquad ab \le gh \Leftrightarrow a \le g \ast a \le h.$

$ab \le ef \Leftrightarrow a \le e \quad a \le f$

$a \le b, g, h \parallel abgh \le cdef \Leftrightarrow a \le c, d, e, f.$

$a \le b, g, h \Rightarrow ab \le gh \quad abgh \le cdef \Leftrightarrow ab \le cd \quad ab \le ef$

$a \le b \quad ab \le cd \Rightarrow a \le c \quad a \le d \qquad ab \le ef \Rightarrow a \le g \quad a \le h$

+ conv.

$a \le b, f, g \quad \parallel abfg \le c^2de \equiv cde \Leftrightarrow a \le c, d, e$

$a \le f, g \qquad a^2fg \equiv afg \le bcde \Leftrightarrow a \le b, c, d, e$

$a \le b_\nu$ $\nu=1,\dots,i$ $a\left\{\prod_{\nu=1}^{i} b_\nu\right\} \le \prod_{\nu=1}^{j} c_\nu$ iff in conjunction

$a \le c_z\ (z=1,\dots,j)$

$a \le c_\nu\ (\nu=1,\dots,j)$ $a^j \le \prod_{\nu=1}^{j} c_\nu$ ( ) $\rightarrow a \le \prod_{\nu=1}^{j} c_\nu$ ( ) $\rightarrow a\left\{\prod_{\nu=1}^{i} b_\nu\right\} \le \prod_{\nu=1}^{j} c_\nu$

$a\left\{\prod_{\nu=1}^{i} b_\nu\right\} \le \prod_{z=1}^{j} c_z \Rightarrow a\left\{\prod_{\nu=1}^{i} b_\nu\right\} \le c_z \quad 1 \le z \le j$

$a^i \le \prod_{\nu=1}^{i} b_\nu \rightarrow a \le \prod_{\nu=1}^{i} b_\nu$ $ab \le c$ $a \le b \rightarrow a \le c \rightarrow a \le c_z$

—o—

$\left\{\prod_{\nu=1}^{i} a_\nu\right\}\left\{\prod_{\nu=1}^{j} b_\nu\right\} \le \prod_{\nu=1}^{k} c_\nu$ $a_\nu \le b_z\ (\nu=1,\dots,i;\ z=1,\dots,j)$

$a_z^k \le \prod_{\nu=1}^{k} c_\nu \rightarrow a_z \le \prod_{\nu=1}^{k} c_\nu \Rightarrow \prod_{z=1}^{i} a_z \le \left\{\prod_{\nu=1}^{k} c_\nu\right\}^i$

$\rightarrow \prod_{z=1}^{i} a_z \le \prod_{\nu=1}^{k} c_\nu \rightarrow \prod_{\nu=1}^{i} a_\nu \le \prod_{\nu=1}^{j} b_\nu \le \prod_{\nu=1}^{k} c_\nu$

$\left\{\prod_{\nu=1}^{i} a_\nu\right\}\left\{\prod_{\nu=1}^{j} b_\nu\right\} \le \prod_{\nu=1}^{k} c_\nu \rightarrow \le c_z \quad 1 \le z \le k$

$a_\nu \le b_z \rightarrow a_\nu^j \le \prod_{z=1}^{j} b_z \rightarrow a_\nu \le \prod_{z=1}^{j} b_z \rightarrow \prod_{\nu=1}^{i} a_\nu \le \left\{\prod_{z=1}^{j} b_z\right\}^i$

$\prod_{\nu=1}^{i} a_\nu \le \prod_{z=1}^{j} b_z \rightarrow \prod_{\nu=1}^{i} a_\nu \le c_z$?

$ab^2 \le bc$ $ab \le bc$ $a^2 \le ab$ $a^2 \le bc$ $a \le bc$

$\left\{\prod_{\nu=1}^{i} a_\nu\right\} b \le c$ $a_\nu \le b$ $\nu=1,\dots,i$

$\left\{\prod_{\nu=1}^{i} a_\nu\right\} b \le bc$ | $a\overline{\prod_{\nu=1}^{i} b_\nu} \le \prod_{\nu=1}^{j} c_\nu$ ~~$\Leftrightarrow$~~, $a \le b_\nu$ $\nu=1,\dots,i$

$a \le b_\nu c_z$ $\nu=1,\dots,i;\ z=1,\dots,j$

$a\prod_{\nu=1}^{i} b_\nu \le c$ $a \le b_\nu\ (\nu=1,\dots,i) \Leftrightarrow a \le \prod_{\nu=1}^{i} b_\nu\}$

$a \le b_\nu\ (\nu=1,\dots,i) \rightarrow a^i \le \overset{b}{\prod_{\nu=1}^{i} b_\nu}$ $a \le b$ $b = \prod_{\nu=1}^{i} b_\nu$

$a \le b$ $ab \le c$ ↑

( ) ~~If a~~ Let i be a positive integer and $a \leq b_\nu$ mod I $(\nu = 1, \ldots, i)$ where $I \in B_{x;si} \cap B_{x;qf}(W)$. $a \prod_{\nu=1}^{i} b_\nu \leq c$ mod I if and only if $a \leq c, \prod_{\nu=1}^{i} b_\nu$ mod I and if these conditions hold, $a \leq c$.

~~Set~~ Set $b = \prod_{\nu=1}^{i} b_\nu$. If $a \leq cb$ mod I then, from ( ), since $I \in B_{x;si}(W)$ $a \leq c$ mod I and $ab \leq c$ mod I. If $a \leq b_\nu$ mod I $(\nu = 1, \ldots, i)$ then, from (2), $a^i \leq b$ mod I and, from (1), $a \leq b$ mod I. since $I \in B_{x;si} \cap B_{x;qf}(W)$. The two conditions $ab \leq c$, $a \leq b$ mod I imply, from (2), that ~~a^2~~ $ab^2 \leq bc$, ~~$ab \leq bc$~~ $a^2 \leq ab$ mod I. From ( ), $ab \leq bc$, $a \leq ab$ mod I, ~~since~~ and, from ( ), $a \leq bc$ mod I. Since $I \in B_{x;si}(W)$, $a \leq c$ mod I, from ( ).

( ) Let i, j be positive integers with $i \leq j$ and $a \leq b_\nu$ mod I $(\nu = 1, \ldots, i-1)$ where $I \in B_{x;si} \cap B_{x;qf}(W)$. $a \left\{ \prod_{\nu=1}^{i-1} b_\nu \right\} \leq \prod_{\nu=i}^{j} b_\nu$ if and only if $a \leq b_\nu$ mod I $(\nu = i, \ldots, j)$. (The first conditions and the term $\prod_{\nu=1}^{i-1} b_\nu$ are to be omitted when $i = 1$.)

The case in which $i > 1$ is dealt with; that in which $i = 1$ is simpler. Let $a \leq b_\nu$ mod I $(\nu = i, \ldots, j)$. From (2), $a^{j-i+1} \leq \prod_{\nu=i}^{j} b_\nu$ mod I and, from (1), $a \leq \prod_{\nu=i}^{j} b_\nu$ mod I since $I \in B_{x;si} \cap B_{x;qf}(W)$. From ( ), $a \left\{ \prod_{\nu=1}^{i-1} b_\nu \right\} \leq \prod_{\nu=i}^{j} b_\nu$ mod I since $I \in B_{x;si}(W)$. Let $a \left\{ \prod_{\nu=1}^{i-1} b_\nu \right\} \leq \prod_{\nu=i}^{j} b_\nu$ mod I. For τ fixed in the range $i \leq \tau \leq j$, $a \left\{ \prod_{\nu=1}^{i-1} b_\nu \right\} \leq b_\tau$ mod I, since $I \in B_{x;si}(W)$. ~~Setting~~ Replacing i by $i-1$ and setting $c = b_\tau$, ( ) yields the result that $a \leq c_\tau$ mod I.

$a+b\in I$ ~~$a\leq b$ mod I~~ ~~$\underset{\circ}{a}$~~ $\underset{\circ}{a}+\underset{\circ}{b}=s\in I$ $I\in B_{x;si}(W)$

$\underset{\circ}{a}\in I$ $\underset{\circ}{b}=s-\underset{\circ}{a}\in I$ $I\in B_{+;ls}$ $b\leq a$ simly ~~b~~ $a\leq b$ i.e $a\equiv b$

$a-b\in I$ $\underset{\circ}{a}-\underset{\circ}{b}=t\in I$ $\underset{\circ}{b}\in I\rightarrow \underset{\circ}{a}=t+\underset{\circ}{b}\in I$ $I\in B_{+;c}$

$a\leq b$ mod I $\underset{\circ}{a}\in I$ ~~$I\in B_{+;ls}\rightarrow a\equiv b$ mod I ($\underset{\circ}{a}\in$~~

$\underset{\circ}{a}\in I$ $\underset{\circ}{b}=\underset{\circ}{a}-t\rightarrow \underset{\circ}{b}\in I$ $I\in B_{+;ls}$ $b\leq a$ mod I

$I\in B_i(W)$ $a-b\in I\rightarrow a\equiv b$ mod I

~~p~~$a+$~~q~~$b\in I$ ~~p~~$,q\in Ri(W,I)$ $I\in B_{x;si}(W)\cap B_{+;ls}$ $a\equiv b$

$pa-b\in I$ $I\in B_{x;si}(W)\cap B_{+;c}\rightarrow a\leq b$

$a-qb\in I$ $I\in B_{x;si}(W)\cap B_{+;ls}\rightarrow b\leq a$

$p\in Ri$ $\vartheta(a)=\vartheta(pa)$ $a\leq p$ $a=pa$

$a\equiv pa$

( ) 1) Let $I\in B_{x;si}(W)\cap B_{+;c}$. If $a-b\in I$ then $a\leq b$ mod I

2) Let $I\in B_{x;si}(W)\cap B_{+;ls}$. If $a+b\in I$ then $a\equiv b$ mod I

If $a-b \overset{\in I}{\cancel{\text{mod } I}}$ then $b\leq a$ mod I

3) Let $I\in B_i(W)$. If $a-b\in I$ then $a\equiv b$ mod I

Let ~~Let I~~ $a=b+t$ where $t\in I$. ~~$\underset{\circ}{a}=\underset{\circ}{b}+\underset{\circ}{t}$~~ If $\underset{\circ}{b}\in I$, ~~$\underset{\circ}{b}$~~ ~~$\underset{\circ}{b}$~~ $\underset{\circ}{a}=\underset{\circ}{b}+\underset{\circ}{t}\in I$ when $I\in B_{x;si}(W)\cap B_{+;c}$: $a\leq b$ mod I

Let $a+b=s\in I$. If $\underset{\circ}{b}\in I$, $\underset{\circ}{a}=\underset{\circ}{s}-\underset{\circ}{b}\in I$ when $I\in B_{x;si}(W)\cap B_{+;ls}$: $a\leq b$ mod I. Similarly $b\leq a$ mod I. Let $a=b+t$ again, where $t\in I$. If $\underset{\circ}{a}\in I$, $\underset{\circ}{b}=\underset{\circ}{a}-\underset{\circ}{t}\in I$ when $I\in B_{x;si}(W)\cap B_{+;ls}$: $b\leq a$ mod I

Part 3) follows from (1) and the second result of (2).

( ) Let $b\in Ri(W,I)$ where $I\in B_{x;si}(W)$. For all $a\in W$, $ab\equiv a$ mod I.

From ( ), $ab\leq a$ mod I when $I\in B_{x;si}(W)$. If $\underset{\circ}{ab}\in I$, $\underset{\circ}{a}\in I$ since $b\in Ri(W,I)$: $a\leq ab$ mod I.

1 ( ) Let $a \leq b$ mod I where $I \in B_{x;si} \cap B_{x;qf}(W)$. $ab \equiv a$ mod I

From ( ) $ab \leq a$ mod I, since $I \in B_{x;si}(W)$. For all $\underset{0}{g} \in W$, $a\underset{0}{g} \leq a$ mod $\underline{I}$ from ( ), since $I \in B_{x;si}(W)$ and $a\underset{0}{g} \leq b$ mod I. If $ab\underset{0}{g} \in \underline{I}$ then $a\underset{0}{g} \in \underline{I}$ from ( ), since $\underline{I} \in B_{x;si} \cap B_{x;qf}(W)$: $a \leq ab$ mod I.

* * *

$bx = a+s$ $\quad by = c+t$ $\quad bz = a+c+w$

$b(x+y) = a+c+s+t$ $\quad I \in B_{+;c} \Rightarrow S\{W,\underline{I}\,|\,a/b\} + S\{W,\underline{I}\,|\,c/b\}$
$\leq S\{W,\underline{I}\,|\,(a+c)/b\}$

$x \in S\{W,I\,|\,a/b\}$ fixed : $x + S\{W,\underline{I}\,|\,c/b\} \leq S\{W,\underline{I}\,|\,(a+c)/b\}$
$I \in B_{+;c}$

$b(z-x) = c+w-s$ $\quad \underline{I} \in B_{+;c} \cap B_{+;ls}$ $\quad x + S\{W,I\,|\,c/b\} \equiv S\{W,I\,|\,(a+c)/b\}$
and $S\{W,I\,|\,a/b\} + S\{W,I\,|\,c/b\} =$
$S\{W,I\,|\,(a+c)/b\}$

$x \geq a$ $\quad y \geq c$

$b, d \in \Delta'\{E(W,I), I\}$ $\quad a \leq b$ $\quad c \leq d$

$x \in MS\{W,I\,|\,a/b\}$ $\quad y \in MS\{W,I\,|\,c/d\}$ (both nonvoid)

$b(x+y) = a+c+s+t$ $\quad x+y \in S\{W,I\,|\,(a+c)/b\}$

$x, y \leq b \to x+y \leq b$ all members^{z} of $S\{(a+c)/b\}$ for which $z \leq b$ belong to $MS\{W,I\,|\,(a+c)/b\}$ $\quad x + MS\{W,I\,|\,c/b\} \leq MS\{W,I\,|\,(a+c)/b\}$

$bz = a+c+w$ $\quad z \geq a+c$ $\quad z \leq b$ $\quad z-x \leq b$ $\qquad \to =$

* * *

$I \in B_{+;ls}$ $\quad S\{W,I\,|\,(a+c)/b\} \leq x + S\{W,\underline{I}\,|\,c/b\}$

( ) Let $b|a$ mod $\underline{I}$ and x be a fixed member of $S\{W,I\,|\,a/b\}$.

(1) Let $\underline{I} \in B_{+;c}$. $x + S\{W,I\,|\,c/b\} \leq S\{W,I\,|\,(a+c)/b\}$ and $S\{W,I\,|\,a/b\} + S\{W,I\,|\,c/b\} \leq S\{W,I\,|\,(a+c)/b\}$.

2) Let $\underline{I} \in B_{+;ls}$. $S\{W,I\,|\,(a+c)/b\} \leq x + S\{W,I\,|\,c/d\}$ and $S\{W,I\,|\,(a+c)/b\} \leq S\{W,I\,|\,a/b\} + S\{W,I\,|\,c/d\}$.

3) Let $I \in B_{+;c} \cap B_{+;ls}$. $x + S\{W, I | c/b\} \equiv S\{W, I | (a+c)/b\}$ and $S\{W, I | a/b\} + S\{W, I | c/b\} \equiv S\{W, \underline{I} | (a+c)/b\}$.

Let $bx = a+s$ where $s \in \underline{I}$.

If $S\{W, I | c/b\}$ is void, the ~~first~~ results of (1) are correct. (~~corresponding~~ cases in which void sets feature in the further results can be dismissed in the same way.), Otherwise, let $by = c+t$ where $t \in I$. Then $b(x+y) = a+c+w$ where $w = s+t \in \underline{I}$ ~~since~~ when $I \in B_{+;c}$: $x+y \in S\{W, I | (a+c)/b\}$. This result holds with regard to arbitrary selection of $x \in S\{W, I | a/b\}$ and $y \in S\{W, I | c/b\}$; The second result of (1) follows.

Let $bz = a+c+w$ where $w \in I$. Then $b(z-x) = c+t$ where $t \in w-s \in I$ when $I \in B_{+;ls}$: ~~$y = z - x \in S\{W, I | c/b\}$~~ $z = x+y$ where $y \in S\{W, I | c/b\}$. The second result of (2) follows from the remark that $x \in S\{W, I | a/b\}$.

The results of (3) are corollaries to those of (1,2).

( ) Let $b \in \Delta'\{$~~$E(W)$~~$Ra(W, I), I\}$ and $a \le b, c \le b \bmod \underline{I}$ where $\underline{I} \in B_{+} \cap B_{x;qf}(W)$

(1) $MS\{W, I | a/b\}$, $MS\{W, I | c/b\}$ and $MS\{W, I | (a+c)/b\}$ are nonvoi

(2) For each x in $MS\{W, I | a/b\}$, $x + MS\{W, I | c/b\} \equiv MS\{W, I | (a+c)$

(3) $MS\{W, I | a/b\} + MS\{W, \underline{I} | c/b\} \equiv MS\{W, \underline{I} | (a+c)/b\}$.

~~If~~ If $a \le b, c \le b \bmod I$, then, from ( ), $a+c \le b \bmod I$, since $I \in B_{+;c}$. The three results of ( ) are consequences of ( ).

Select $x \in MS\{W, I | a/b\}$ ~~$bx = a+s$ where $s \in I$ and $x \equiv a \bmod I$. With $y \in MS\{W, I | c/b\}$, $by = c+t$ where~~ and $y \in MS\{W, I | c/d\}$. From (1), $x+y \in S\{W, I | (a+c)/b\}$. Since $x \equiv a \geqq b$, $y \equiv c \le b \bmod I$, $x+y \le b \bmod I$, since $I \in B_{+;c}$, from ( ). ~~The only members z of~~ S~~$\{W, I | (a+c)/b\}$~~ When $b \in \Delta'\{Ra(W, I), I\}$, $(a+c) \le b \bmod I$ and

$I \in B_{\cdot} \cap B_{x;qf}(W)$, the only members z of $S\{W, I \mid (a+c)/b\}$ for which $z \le b \bmod I$ belong to $MS\{W, I \mid (a+c)/b\}$: $x+y \in MS\{W, I \mid (a+c)/b\}$. Select $z \in MS\{W, I \mid (a+c)/b\}$. From ( ), $y = z - x \in S\{W, I \mid c/d\}$. Since $x \le b$, $z = \{a+c\} \le b \bmod I$, $y \le b \bmod I$ from ( ), since $I \in B_{+;ls}$. Again $y \in MS\{W, I \mid c/b\}$. The result of (3) follows from (2) and ( ).

—o—

( ) Let $A, B \le, C \le W$.

(1) If $a + B \le C$ for each $a \in A$, $A + B \le C$

(2) If $C \le a + B$ ~~for each~~ for one $a \in A$, $C \le A + B$

(3) If $a + B = C$ for each $a \in A$, $A + B = C$.

Under the conditions of (1), $a + b \in C$ for all a ~~$b\in$~~ A, $b \in B$. (2) follows from the remark that $a \le A$. (3) follows from (1, 2).

—o—

( ) Let $a, b \le c \bmod I$

(1) If $I \in B_{+;c}$, $a + b \le c \bmod I$

(2) If $I \in B_{+;ls}$, $a - b \le c \bmod I$.

Since $a, b \le c \bmod I$, $ag \in I$ and $bg \in I$ for all $g \in W$ such that $cg \in I$. For all such g, $ag + bg \in I$ when $I \in B_{+;c}$ and $ag - bg \in I$ when $I \in B_{+;ls}$

—o—

~~$x\in$~~ $S\{W, I \mid a/b\} \cap S\{W, I \mid c/d\}$ ~~$\Leftrightarrow$~~ [illegible] $\rightarrow ad - bc \in I$

$bx = a + s$ ~~dz~~ $= c + t$ $s, t \in I$ ~~bdz~~ $0 = ad - bc + ds - bt$

$ad - bc = w \Leftarrow w = bt - ds \in I$ when $I \in B_{x;si}(W) \cap B_{+;ls}$

—o—

$ad - bc \in I \rightarrow ad \equiv bc \rightarrow a \equiv c$ $a \le d$ $c \le b$ ~~$a \le$~~ $a, c \le b, d$

$a \le b \rightarrow bc \le bd$ $c, d \le b \rightarrow c \le d$ $a, b, c \le d$ $a \le d$ $a^{*} \le ad$ $a \le bc$

$a \le d$ $ad \le bc \rightarrow a \le b$ $a \le c$

$bx = a + s \quad dy = c + t \qquad bd(x-y) = ad - bc + ds - bt$

$\in \underline{I}$

$(ub+v)x = ua + s \qquad (u'd+v')y = u'c + t$

$(ub+v)(u'd+v')(x-y) = ua(u'd+v') - u'c(ub+v) + w$

$= uu'(ad-bc) + uav' - u'cv + w$

$b \in Ri(W, I) \quad b|a \text{ mod } \underline{I} \quad ad - bc \in \underline{I} \quad I \in B_{x;si}(W) \cap B_{t;c}$

$bdx = ad + ds = bc + w + ds \quad b(dx - c) \in I$

$dx = c + r \quad r \in \underline{I} \rightarrow x \in S(c/d)$

$MS\{W, I|a/b\} = S\{W, I|a/b\} \leq S\{W, I|c/d\}$

$dy = c + t \quad ad \underset{0}{g} \quad ad \leq bc \equiv c \qquad y \equiv c \rightarrow y \leq a$

$d \in Ri(W, I) \quad a \leq c \rightarrow x \leq c \rightarrow S\{W, I|a/b\} \leq MS\{W, I|c/d\}$

$bdy = bc + t = ad - w + t \rightarrow by = a \text{ mod } \underline{I} \quad I \in B_{t;ls}$

$ad - bc \in \underline{I}$

$b \in Ri(W, I), \; b|a \text{ mod } \underline{I} \quad I \in B_{x;si}(W) \cap B_{t;c} \quad \rightarrow d|c \text{ mod } \underline{I}$

$MS\{W, I|a/b\} = S\{W, I|a/b\} \leq S\{W, I|c/d\}$

$d \in Ri(W, I) \quad d|c \text{ mod } \underline{I} \quad I \in B_{x;si}(W) \cap B_{t;ls} \rightarrow a|b \text{ mod } \underline{I}$

$MS\{W, I|c/d\} \equiv S\{W, \underline{I}|c/d\} \leq S\{W, I|a/b\}$

$b, d \in Ri(W, I)$ either $b|a \text{ mod } I$ or $d|c \text{ mod } \underline{I} \quad I \in B_i(W)$

both $b|a, d|c \text{ mod } \underline{I}$

$MS\{W, \underline{I}|a/b\} \equiv S\{W, I|a/b\} = MS\{W, I|c/d\} \equiv S\{W, I|c/d\}$

$ad - bc \in \overline{I} \quad d \in Ra(W)$ ~~and~~ $dz = c + t \quad bdz = bc + bt$

$ad - bc = s \qquad ad + bt = bdz + s \quad d(bz - a) = bt - s$

$bz = a + w \quad ad - bc \in \underline{I} \quad d \in Ra(W, I) \quad \underline{I} \in B_{x;si}(W) \cap B_{t;ls}$

$\rightarrow b|a \text{ mod } \underline{I} \quad d \in Ra(W, I) \rightarrow$ ~~d∈Ri(W~~, $d|c \text{ mod } \underline{I} \rightarrow b|a$ from (2)

$\cancel{t\in X}$ $a\le b$ $tdx=bc+w+ds$ $MS\{W,\underline{I}|a/b\}$ nonvoid $\underline{I}\in\underline{I}_{qF}$

$b\in\Delta'$ $c\le b$ $dx=c+r$ $x\le c$? $a\le c\le b$ $x\equiv a$

$MS\{W,\underline{I}|a/b\}\le MS\{W,\underline{I}|c/d\}$ $dy=c+t$ $ad\equiv bc\equiv c$

$cb\le ad$ $c\le b$ $c\le a$ $c\le d$

$uad-(ub+v)c+vc=s\in\underline{I}$ $c\le b$

$\cancel{d}$ $(ub+v)z=d+t$ $ua(ub+v)$

$(ub+v)x=ua+s$ $ad-bc=t$

$d(ub+v)x=uad+ds$

$=ubc+ut+ds$

$(ub+v)(dx-c)=w$

$w=ut+ds-vc$

$(ub+v)x=ua+t$ $d(ub+v)x-(ub+v)c=s-vc+\cancel{\quad}+dt$

$ad-bc\in\underline{I}$ $b\in\Delta'\{R\mathring{a}(W,I),\underline{I}\}\to d|c \bmod \underline{I}$ $I\in B_i(W)$

$c\le b$

$\cancel{\Longleftarrow}$ $a\le b$ $I\in B_{x;qf}(W)\to MS(a/b)$ nonvoid

$MS(a/b)\le S(c/d)$

$ad-bc\in I$ $c\le b$ $I\in B_i\cancel{\quad}\cap I_{x;sF}(W)\to c\le a$ $c\le d$

$a\equiv c\to MS\{W,I|a/b\}\le MS\{W,\underline{I}|c/d\}$ $\S$ $a\equiv c\le b$

$(ub+v)f=e+h$ $h\in\underline{I}$ $e\in W$

$(ud+v)a-(ub+v)c+vc-va=s\in\underline{I}$

$(ud+v)af+fvc-fva=fs+ce+ch$

$b,d\in\Delta'\{Ra(W,I),\underline{I}\}$ $a\equiv c\le b$ $ad-bc\in\underline{I}$

$MS\{W,I|a/b\}\underline{=}MS\{W,I|c/d\}$

—o—

( ) Let $I\in B_{x;si}(W)\cap B_{+;ls}$. If $S\{W,I|a/b\}\cap S\{W,I|c/d\}$ is nonvoid, $ad-bc\in I$.

~~Let bx~~ For some $x\in W$, $bx=a+s$ and $dx=c+t$ where $s,t\in I$. ~~for some~~ Thus $ad-bc=w$ where $w=bt-ds\in\underline{I}$ when $I\in B_{x;si}(W)\cap B_{+;ls}$.

—o—

( ) Let $ad-bc\in\underline{I}$.

(1) If $b\in Ri(W,I)$ and $b|a \bmod \underline{I}$ where $\underline{I}\in B_{x;si}(W)\cap B_{+;c}$ then $d|c \bmod \underline{I}$ and $MS\{W,I|a/b\}=S\{W,I|a/b\}\le S\{W,I|c/d\}$

(2) If $d\in Ri(W,I)$ and $d|c \bmod \underline{I}$ where $I\in B_{x;si}(W)\cap B_{+;ls}$ then $a|b \bmod I$ and $MS\{W,I|c/d\}=S\{W,I|c/d\}\le S\{W,I|a/b\}$.

(3) If $b, d \in Ri(W, I)$ and either $b|a$ mod I or $d|c$ mod I, where $I \in B_i(W)$, then both $b|a$, $d|c$ mod I and

$$MS\{W, I|a/b\} \equiv S\{W, I|a/b\} \equiv MS\{W, I|c/d\} \equiv S\{W, I|c/d\}.$$

Let $ad - bc = w$ where $w \in I$.

Under the conditions of (1), $MS\{W, I|a/b\} \equiv S\{W, I|a/b\}$ from ( ). Select $x \in S\{W, I|a/b\}$, so that $bx = a + s$ where $s \in I$. Then $bdx = ad + d$ ~~$= bc + w + ds$~~ so that $b(dx - c) = w + ds \in I$ since $I \in B_{x;si}(W) \cap B_{+;c}$ Since $b \in Ri(W, I)$, $dx - c \in I$ and $x \in S\{W, I|c/d\}$. In particular $d|c$ mod I. The proof of part (2) is similar, now with $dy = c + t$ where $t \in I$. Then ~~$b(dy$~~ $d(by - a) = bt - w \in I$ since $I \in B_{x;si}(W) \cap B_{+;ls}$. Part (3) is a corollary to its predecessors.

1) Let $ad - bc \in I$, $c \leq b$ mod I and $b \in \Delta'\{Ra(W, I), I\}$ here $I \in B_i(W)$. Then $d|c$ mod I.

) If in addition $a \leq b$ mod I, ~~and $I \in B_{x;qf}(W)$~~, then $S\{W, I|a/b\}$ is nonvoid ~~and~~ $MS\{W, I|a/b\} \leq S\{W, I|c/d\}$ if, furthermore, $I \in B_{x;qf}(W)$

Let $ad - bc = w \in I$. Since $b \in \Delta'\{Ra(W, I), I\}$, W contains nd v such that $ub + v \in Ra(W, I)$ and $bv \in I$. ~~Since $ub + v \in Ra(W, I)$~~, a simpler condition holds. Since $ub + v \in Ra(W, I)$, W and I contains x and for which $(ub + v)x = ua + s$. ~~then~~ Then $d(ub + v)x = ubc + uw + ds$ d $(ub + v)(dx - c) = r$ where $r = uw + ds - vc \in I$ since $\in I$ and $c \leq b$ mod I and also $I \in B_i(W)$. $\{ub + v\} \in Ri(W, I)$, $- c \in I$: $d|c$ mod I.

If $a \leq b$ mod I, ~~and~~ $b \in \Delta'\{Ra(W, I), I\}$ ~~and~~ and $I \in B_i(W)$, $\{W, I|a/b\}$ is nonvoid, from ( ).

If furthermore, ~~any member of~~ $I \in B_{x;qf}(W)$ and x' is member of $MS\{W, I|a/b\}$, $p \in I$ exists for which $x' = x + p$, from ( ). ( ) $S\{W, I|c/d\} + I \equiv S\{W, I|c/d\}$ when $I \in B_i(W)$: $x' \in S\{W, I|c/d\}$.