Numerical experiments in nonassociative algebras

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Distributive (nonassociative) ring $\mathbb{D}$

$\alpha+\beta=\beta+\alpha \quad \alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$

$0 \quad \alpha+0=\alpha \quad \alpha+(-\alpha)=0$

$\alpha\beta \quad \alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma \quad (\beta+\gamma)\alpha=\beta\alpha+\beta\gamma$

Flexible ring $\mathbb{F}$: $\mathbb{D}$ with $\alpha(\beta\alpha)=(\alpha\beta)\alpha$

Alternative ring $\mathbb{A}$: $\mathbb{F}$ with $\alpha(\alpha\beta)=\alpha^2\beta \quad (\beta\alpha)\alpha=\beta\alpha^2$

Ring (associative ring) $\mathbb{R}$: $\mathbb{D}$ with $\alpha(\beta\gamma)=(\alpha\beta)\gamma$

Distributive involution : $\alpha\to\tilde{\alpha}$: $\tilde{\tilde{\alpha}}=\alpha$

$$\widetilde{\alpha\beta}=\tilde{\beta}\tilde{\alpha} \qquad \widetilde{\alpha+\beta}=\tilde{\alpha}+\tilde{\beta}$$

Quaternions $\mathbb{Q}$: $\underset{\sim}{q}=q_1+q_2\underset{\sim}{i}+q_3\underset{\sim}{j}+q_4\underset{\sim}{k}$: $\mathbb{R}$

$\tilde{q}=q_1-q_2\underset{\sim}{i}-q_3\underset{\sim}{j}-q_4\underset{\sim}{k}$

Cayley numbers $\mathbb{C}$: $\underset{\sim}{\alpha}=\sum_{\nu=1}^{8}\alpha_\nu\underset{\sim}{u}_\nu$: $\mathbb{A}$

$\underset{\sim}{u}_1\equiv 1 \quad \tilde{\underset{\sim}{\alpha}}=\alpha_1\underset{\sim}{u}_1-\sum_{\nu=2}^{8}\alpha_\nu\underset{\sim}{u}_\nu \quad n(\underset{\sim}{\alpha})=\underset{\sim}{\alpha}\tilde{\underset{\sim}{\alpha}}=\left(\sum_{\nu=1}^{8}\alpha_\nu^2\right)\underset{\sim}{u}_1$

$\underset{\sim}{\alpha}^{-1}=\tilde{\underset{\sim}{\alpha}}/n(\underset{\sim}{\alpha}) \quad \underset{\sim}{\alpha}^{-1}(\underset{\sim}{\alpha}\underset{\sim}{\beta})=(\underset{\sim}{\beta}\underset{\sim}{\alpha})\underset{\sim}{\alpha}^{-1}=\underset{\sim}{\beta}$

Dixon : $\underset{\sim}{\alpha}\equiv(a, A) \quad a, A\in\mathbb{Q}$

$$\beta=(b, B) \quad \underset{\sim}{\alpha}\pm\underset{\sim}{\beta}\equiv(a\pm b, A\pm B)$$

$$\underset{\sim}{\alpha\beta} = (ab - \tilde{B}A,\ Ba + A\tilde{b}) \qquad (*)$$

$$\tilde{\underset{\sim}{\alpha}} = (\tilde{a}, -A) \qquad (**)$$

Albert: A_0 = real numbers, involution ≡ identity $\tilde{a} = a$

A_i $(i = 0, 1, \ldots)$

$a, A \in A_i \quad \alpha = (a, A) \in A_{i+1}$

Involution over A_{i+1} in terms of inv. over A_i by (**)

Multiplication " " " " " mult " " " (*)

A_1 = complex numbers, A_2 = quaternions, A_3 = Cayley number.

...

Schafer: $A_i = \mathbb{F}$ $(i = 0, 1, \ldots)$

```
begin integer i, j, tp6, tp7;
  real v;
  array alp, bet, gam, ab, bg, no [1:32];
  integer procedure d(i);
  integer i;
    d := 2↑i;
  procedure eq(i, a, b);
  integer i;
  real a, b;
  begin integer di;
    di := d(i); for j := 1 step 1 until di do a := b
  end;
  real procedure conj(no);
  real no;
    conj := if j=1 then no else -no;
  procedure m(i, gam, alp, bet);
  integer i;
  real gam, alp, bet;
```

```
begin if i=0 then
  begin j:=1; gam:=alp * bet
  end else
  begin integer di, did;
    di:=d(i); did:=di÷2;
    begin integer did1, i1, j1;
                                          ,st5
      array st1, st2, st3, st4 [1:did];
      did1:=did+1; i1:=i-1;
      for j:=did1 step 1 until di do
      begin j1:=j-did; st1[j1]:=alp; st2[j1]:=bet
      end;
      m(i1, st3[j], alp, bet);
      m(i1, st4[j], conj(st2[j]), st1[j]);
               st5[j]
      eq(i1, gam, st3[j]-st4[j]);
      m(i1, st3[j], st2[j], alp);
      m(i1, st4[j], st1[j], conj(bet));
       eq(i1, gam, st5[j]);
      for j:=did1 step 1 until di do
      begin j1:=j-did; gam:=st3[j1]+st4[j1]
      end
    end
  end
end;
```

```
real procedure norm (i, no);
integer i;
real no;
begin integer di;
  real sum, comp;
  di:=d(i); sum:=0.0;
  for j:=1 step 1 until di do
  begin comp:= no ; sum := sum + comp * comp
  end;
  norm := sum
end;
procedure inv (i,a,b);
integer i;
real a,b;
begin real n;
  n:= norm (i,b); eq(i,a, conj(b)/n)
end;
procedure op (i,no);
integer i;
real no;
```

```
begin integer dir;
  NLCR; if i<3 then
  begin NLCR; dir:=d(i);
    for j:=1 step 1 until dir do print (no)
  end else
  begin integer r, r1, r2;
    dir:=d(i)÷4; r1:=-3;
    for r:=1 step 1 until dir do
    begin NLCR; r1:=r1+4; r2:=r1+3;
      for j:=r1 step 1 until r2 do print(no)
    end
  end
end;
procedure ms(i, a, b);
integer i;
real a, b;
  m(i, no[j], a, b);
procedure ops(i);
integer i;
  op(i, no[j]);
```

```
procedure pi(i1);
integer i1;
begin NLCR; NLCR; print(i1); NLCR
end;
integer procedure ri;
begin integer i1;
  procedure sq;
  begin v:=v*v; if entier(v)>9 then v:=v/10
  end;
ss: i1:= entier(tp7*v)-10*entier(tp6*v);
   if i1=0 then begin sq; goto ss end;
   sq; ri:=i1
end;
tp6:= 10↑6; tp7:= 10*tp6;
v:= 3.14159 26535 89793 2385;
for j:=1 step 1 until 32 do
begin alp[j]:=ri; bet[j]:=ri; gam[j]:=ri
end;
for i:=2 step 1 until 5 do
```

```
begin pi(i); pi(0);
  op(i,alp[j]); op(i,bet[j]); op(i,gam[j]); pi(1);
  m(i,bg[j],bet[j],gam[j]);
  ms(i,alp[j],bg[j]); ops(i);
  m(i,ab[j],alp[j],bet[j]);
  ms(i,ab[j],gam[j]); ops(i); pi(2);
  ms(i,alp[j],ab[j]); ops(i);
  ms(i,alp[j],alp[j]); ms(i,no[j],bet[j]);
  ops(i); pi(3); ms(i,gam[j],alp[j]);
  ms(i,ab[j],no[j]); ops(i);
  ms(i,alp[j],bg[j]); ms(i,no[j],alp[j]);
  ops(i); pi(4); inv(i,no[j],ab[j]); ops(i);
  inv(i,no[j],bet[j]); inv(i,bg[j],alp[j]);
  ms(i,no[j],bg[j]); ops(i); pi(5);
  ms(i, ab[j],alp[j]); ops(i);
  ms(i, bet[j],alp[j]); ms (i,alp[j],no[j]);
  ops(i); pi(6); NLCR; NLCR;
  print(norm(ab[j]));
  print(norm(alp[j])*norm(bet[j])); NLCR
 end
end
```

$i = 2(1)5$	$i=2$: quaternions, $i=3$: Cayley numbers
0 α β γ	$i=4,5$ two succeeding members of Albert's hierarchy
1 $\alpha(\beta\gamma)$ $(\alpha\beta)\gamma$	Test associative law : numbers equal for $i=2$, unequal thereafter
2 $\alpha(\alpha\beta)$ $\alpha^2\beta$	Test left alternative law : numbers equal for $i=2,3$ unequal for $i=4,5$
3 $(\alpha\beta)(\gamma\alpha)$ $\{\alpha(\beta\gamma)\}\alpha$	Test Moufang identity : as in 2?
4 $(\alpha\beta)^{-1}$ $\beta^{-1}\alpha^{-1}$	Test reverse product law for inverses : as in 2?
5 $(\alpha\beta)\alpha$ $\alpha(\beta\alpha)$	Test flexible law : numbers equal for all i
6 $n(\alpha\beta)$ $n(\alpha)n(\beta)$	Test norm product law : as in 2?