Algorithms applied to formal power series

Single power series

Notation. $f_{i,j}$ $(i,j\in\mathrm{I})$ is the sequence of coefficients of the formal power series $f_{i,j}(x)=\sum f_{\nu}^{(i,j)}x^{\nu}$ defined by formula ( ) and derived from the sequence h and the formal power series $g(x)$. $\mathcal{F}$ is the complete array of sequences $f_{i,j}$ $(i,j\equiv\bar{\mathrm{I}})$. $\mathcal{F}_{I,J}[N]\subseteq\mathcal{F}$ $(I,J\in\mathrm{I},N\in\bar{\mathrm{I}})$ means that $\mathcal{F}$ possesses the square block of identical sequences $f_{I+r,J+s}$ $(r,s\equiv\mathrm{I}_{0}^{N})$. $\varkappa\{h'\}=\mu_{\tau}$ means that the nonzero members of the sequence h' are $h'_{\mu(\nu)}$ $(\nu\equiv\mathrm{I}_{0}^{\tau};$ $\tau\in\bar{\mathrm{I}})$ where $\mu(\nu)$ $(\nu\equiv\mathrm{I}_{0}^{\tau})$ is a strictly increasing sequence in $\bar{\mathrm{I}}$. With such an integer sequence prescribed, $\lambda(\nu)$ $(\nu\equiv\mathrm{I}_{0}^{\mu(\tau)-1})$ is the integer sequence defined by setting $\lambda(\nu)=0$ $(\nu\equiv\mathrm{I}_{0}^{\mu(0)})$, $\lambda(\nu)=\nu'+1$ $(\nu'\equiv$

$\underline{\mathbb{I}}_0^{\tau-1}$, $\nu = \underline{\mathbb{I}}_{\mu(\omega')+1}^{\mu(\omega'+1)}$). We set $\varepsilon_{i,j}(x) = f_{i,j}(x) - f(x)$ $(i,j = \underline{\mathbb{I}})$.

$O\{x^r\}$ $(r \in \underline{\mathbb{I}})$ denotes a formal power series in which the coefficients of x^ν $(\nu = \underline{\mathbb{I}}_0^{r-1})$ are zero

The integer function $\lambda(\omega)$ has been introduced to cope with order relationships involving series with gaps. If $\mu(\nu) = \nu$ $(\nu = \overline{\mathbb{I}}_0^{\tau})$, then quite simply $\lambda(\nu) = \nu$ $(\nu = \underline{\mathbb{I}}_0^{\tau})$ also: if $\mu(\nu) = 2\nu$ $(\nu = \overline{\mathbb{I}}_0^{\tau})$, then $\lambda(0) = 0$, $\lambda(2\nu - 1) = \lambda(2\nu) = \nu$ $(\nu = \overline{\underline{\mathbb{I}}}_1^{\tau})$.

Theorem. Set $f = hg$.

(i) a) All series $f_{i,j}(x)$ $(i,j = \underline{\mathbb{I}})$ are well determined.

b) $f_{i,j} = h_{i,j}\, g$ $(i,j = \underline{\mathbb{I}})$.

c) If $H_{I,J}[N] \subseteq H$, then $\mathcal{F}_{I,J}[N] \subseteq \mathcal{F}$.

(ii) a) Let $h_{I,J} \prec h$ and let $h_{I,J}(x)$ be decomposed in the form

$$h_{I,J}(z) = \Sigma_0^m \alpha_\nu z^\nu + \Sigma_1^n \Sigma_0^{n(\omega)-1} A_{\nu,\tau}(1-t_\nu z)^{-\tau-1}.$$

Then $\mathfrak{F}^{\prime}_{I,J}[\infty] = f$, where f is given by formula ( ).

b) Let $g(x)$ be the polynomial $\Sigma_0^\mu g_\nu z^\nu$

(1) Then $f(x) = \Sigma_0^\mu h_\nu g_\nu x^\nu$

(2) Let $\xi(i,j) > \mu$. Then $\mathfrak{F}_{i,j}[\infty] = f$

(3) Let $H_{i,j} \neq 0$ with $i+j+1 > \mu$. Then $\mathfrak{F}_{i,j}[\infty] = f$

(4) Let $H_{i,\mu-i} \neq 0$ $(i = \mathrm{I}_1^{\mu-1})$ Then $f_{i,j} = f$ $(i,j \equiv \mathrm{I}\!\mathrm{I}\ (i+j \geq \mu))$.

(iii) Let $\varkappa\{g\} = \mu_\tau$

a) Let $\xi(i,j) \leq \mu(\tau)$ $[\![> \mu(\tau)]\!]$. Then $\varepsilon_{i,j}(x) = O\{x^{\mu[\lambda\{\xi(i,j)\}]}\}$

$[\![f_{i,j} = f]\!]$.

b) Let $H_{i,j}$, $H_{i+1,j+1} \neq 0$ and $i+j < \mu(\tau)$ $[\![\geq \mu(\tau)]\!]$. Then $\varepsilon_{i,j}(x) =$

$O\{x^{\mu\{\lambda(i+j+1)\}}\}$ $[\![f_{i,j} = f]\!]$

(iv) Let $g_\nu \neq 0$ $(\nu = \mathrm{I}_0^\mu)$

a) Let $\xi(i,j) \leq \mu$. Then $f_{i,j} \neq f$.

b) Let $H_{i,j} \in H_{I,J}[N]$, $H_{i',j'} \in H_{I',J'}[N']$, the two blocks being distinct, with $\xi' = \min(I+J+N+1,\ I'+J'+N'+1) \le \mu$. (1) If $I+J+N+1 \ne I'+J'+N+1$, then $f_{i,j} \ne f_{i',j'}$. (2) Let $I+J+N = I'+J'+N'$ (so that the two blocks have a common backward diagonal); if $I' > I$ $< I > I' >$ let $\mu \geq I'+J$ $< I+J' >$. Then $f_{i,j} \ne f_{i',j'}$.

c) Let $H_{i+1,j}, H_{i,j+1}, H_{i+1,j+1} \ne 0$, and let $i+j+1 < \mu$. Then $f_{i+1,j} \ne f_{i,j+1}$.

(v) If g and h are real, then $f_{i,j}$ is real $(i,j \in \mathbb{I})$

(vi) a) Let $\hat{f}(x)$ be the series obtained from the series $g(x)$ and the sequence ah $(a(\ne 0) \in \mathbb{Z})$ $(i,j \in \mathbb{I})$. Then $\hat{f}_{i,j}(x) = a f_{i,j}(x)$, $(i,j \in \mathbb{I})$

b) Let N be a class of functions defined over $\tilde{\Delta}$ such that $g(y) \in \mathbb{Z}$ $(y \in \tilde{\Delta}, g \in N)$. Let L be an operator applied to functions of N such that $L\{g(\tilde{\Delta})\} \in \mathbb{Z}$ $(g \equiv N)$ and $L\{ag(\tilde{\Delta})\} = aL\{g(\tilde{\Delta})\}$ $(a \equiv \mathbb{Z}, g \equiv N)$. For any series $g(y,x) = \sum g_\nu(y) x^\nu$ with $g_\nu \in N$ $(\nu \equiv \mathbb{I})$, set $L\{g(\tilde{\Delta}, x)\} = \sum L\{g_\nu(\tilde{\Delta})\} x^\nu$. Let $f_{i,j}(y,x)$ be the series derived from the series $g(y,x)$ and the sequence h, and $\tilde{f}_{i,j}(x)$ be derived from $L\{g(\tilde{\Delta}, x)\}$ and h. Then $\tilde{f}_{i,j}(x) = L\{f_{i,j}(\tilde{\Delta}, x)\}$.

Proof. (i)a) For $i,j \equiv \mathbb{I}$, all numbers $\{\alpha_\nu^{(i,j)}\}$, $\{A_\nu^{(i,j)}\}$, $\{t_\nu^{(i,j)}\}$, $\{m(i,j)\}$, $\{n(i,j)\}$, $\{n(i,j;\nu)\}$ occurring in formula ( ) can be determined, the power series $\{\Delta_x^\tau\}x^\tau g(t_\nu^{(i,j)} x)\}$ exist and, $f_{i,j}(x)$ being a finite linear combination of these series, is well defined $(i,j \equiv \mathbb{I})$.

b) From formula ( ), we have for $i,j \equiv \mathbb{I}$

$$h_{\nu'}^{(i,j)} = \alpha_{\nu'}^{(i,j)} + \sum_1^{n(i,j)} \sum_0^{n(i,j;\nu)-1} A_{\nu,\tau} \binom{-\tau-1}{\nu'} (-t_\nu^{(i,j)})^{\nu'}$$

$(\nu' \equiv \mathbb{I})$

where $\alpha_{\nu'}^{(i,j)} = 0$ $(\nu' \equiv \mathbb{I}_{m(i,j)+1})$. Comparing formula (, ,) we see that $\cancel{f}\ f_\nu^{(i,j)} = h_\nu^{(i,j)} g_\nu$ $(\nu = \mathbb{I})$.

c) If $H_{I,J}[N] \subseteq H$, then $P_{I,J}[N] \subset P$, and all $P_{i,j} \in P_{I,J}[N]$ have the same decomposition of the form ( ).

(ii)a) This result follows from 1... and (i)c). b) (1) follows by definition, (2) from the relationships $h_\nu^{(i,j)} = h_\nu$ $(\nu = \xi(i,j))$, $g_\nu = 0$ $(\nu = \mathbb{I}_{\mu+1})$, (2) from 1... and (2), and (3) from 1... and (2).

(iii)a) Since $h_\nu^{(i,j)} = h_\nu$ $(\nu = \mathbb{I}_0^{\xi(i,j)-1})$, $(h_\nu^{(i,j)} - h_\nu) g_\nu = 0$ $(\nu = \mathbb{I}_0^{\mu[\lambda\{\xi(i,j)\}]-1})$ and accordingly the series $f_{i,j}(x) - f(x)$ is devoid of the powers of z accompanying these coefficients. If $\xi(i,j) > \mu(\tau)$, then $f_{i,j}(x) - f(x)$ is the zero formal power series. b) If $H_{i,j}, H_{i+1,j+1} \neq 0$, then $\xi(i,j) = i+j+1$ (1...). The stated result now follows from its predecessor.

~~(iv)a) This result follows as for (iii)b)~~

(iv)a) We have $h_\nu^{(i,j)} g_\nu \neq h_\nu g_\nu$ when $\nu = \xi(i,j)$ and $g_\nu \neq 0$.

b) This result follows as in (iii) b). c) Both results follow directly from those of 1..., as does d) from 1... .

(v) Although the numbers $t_\nu^{(i,j)}$ in formula ( ) may be complex, all numbers with nonzero imaginary parts in this formula occur, if at all, in complex conjugate pairs.

(vi)a) The Padé quotients derived from the series $ah(z)$ are $aP_{i,j}(z)$ (i,j ≡ I) and their series expansions are $ah^{(i,j)}(z)$. b) We have, from (i)b), $f_{i,j}(y,x) = \sum h_\nu^{(i,j)} g_\nu(y) x^\nu$ $(i,j \equiv I, y \equiv \tilde{\Delta})$. Hence $L\{f_{i,j}(\tilde{\Delta},x)\} = L\{h_\nu^{(i,j)} g_\nu(\tilde{\Delta})\} x^\nu = \sum h_\nu^{(i,j)} L\{g_\nu(\tilde{\Delta})\} x^\nu = \tilde{f}_{i,j}(x)$ $(i,j \equiv I)$, as required.

In the simple case in which $g_\nu \neq 0$ $(\nu \equiv I)$ the structures of $\mathcal{F}$ and $\mathcal{P}$ are the same: identical series in f occur only in square blocks of the form $f_{I,J}[N]$, and

~~$f_{I,J}$~~ $f_{I,J}[N] \subseteq f$ if and only if $\mathbb{P}_{I,J}[N] \subset \mathbb{P}$, i.e. if and only if $H_{I,J}[N] \subseteq H$; $f_{I,J}[\infty] \subseteq f$ if and only if $h_{I,J} \in h$, and any $f_{i,j} \neq f_{I,J}[\infty]$ is distinct from f; if $f_{i,j} \in f_{I,J}[N]$ then $f_{i,j}$ agrees with f as far as the term with suffix $I+J+N$ and no further: the results of 1(iv)c) hold with $h_\nu^{(i,j)}$, $h_\nu^{(i',j')}$ replaced by $f_\nu^{(i,j)}$, $f_\nu^{(i',j')}$ respectively.

Difficulties begin to occur when $g(x)$ is a polynomial or g possesses gaps. In the degenerate case in which $g(x) = \sum_0^\mu g_\nu x^\nu$, if the conditions of 2... hold, then all $f_{i,j}$ lying below the backward diagonal $f_{\mu-r,r}$ $(r = \overline{0}^\mu)$ are equal to f, those lying above the diagonal being unequal to f. If these conditions do not hold and, for example, $H_{I,J}[N] \subset H$ with $I+J+N+1 \geq \mu$, then the sequences of $f_{i,j}[N]$ lying above the diagonal $f_{\mu-r,r}$ $(r \in \overline{0}^\mu)$ are also equal to f.

If $I+J+N+1<\mu$, however, those sequences of $\mathfrak{f}_{I,J}[N]$ lying below the diagonal are now unequal to f. If g possesses gaps, anomalous behaviour may present itself. Although $h_\nu^{(i,j)}=h_\nu$ $(\nu=\overline{1}^{\xi(i,j)-1})$ $h_{\xi(i,j)}^{(i,j)}\neq h_{\xi(i,j)}$, nothing, in general, is known concerning the remaining $h_\nu^{(i,j)}$ and h_ν. If $\mu(\nu'-1)$ $\leqslant\xi(i,j)<\mu(\nu')$ and $h_{\mu(\nu')}^{(i,j)}\neq h_{\mu(\nu')}$, then $\varepsilon_{i,j}(x)=O\{x^{\mu(\nu')}\}$ but if $h_{\mu(\nu')}^{(i,j)}=h_{\mu(\nu')}$, as may occur, then $\varepsilon_{i,j}=O\{x^{\mu(\nu'')}\}$ with $\nu''=\nu'+1$ if $h_{\mu(\nu'+1)}^{(i,j)}\neq h_{\mu(\nu'+1)}$ and with higher ν'' if this is not so (in all cases $\varepsilon_{i,j}=O\{x^{\mu[\lambda\{\xi(i,j)\}]}\}$, but if $h_{\mu(\nu')}^{(i,j)}=h_{\mu(\nu')}$, this relationship may, using an obvious notation be replaced by one with O replaced by o). Indeed it is possible that $h^{(i,j)}$ and h fortuitously agree at the very points at which g has nonzero members, when $\mathfrak{f}_{i,j}=f$ for the i,j in question. Such anomalous behaviour is impossible when $g_\nu \neq 0$ ($\nu \equiv \underline{\mathrm{I}}$): we then have $f^{(i,j)}_{\mathfrak{z}(i,j)} \neq f_{\mathfrak{z}(i,j)}$ (as occurs for the Padé table with $g_\nu = 1$ ($\nu \equiv \underline{\mathrm{I}}$) and the matter ends there. Similar considerations relate to the inequality or otherwise of differing sequences $f_{i,j}$. For example, if h is normal and the suffix sets $(i,j), (i',j')$ are distinct, $h^{(i,j)} \neq h^{(i',j')}$, whether $P_{i,j}$ and $P_{i',j'}$ lie on the same backward diagonal or not; if $g_\nu \neq 0$ ($\nu \equiv \underline{\mathrm{I}}$) the same is true of the corresponding $f_{i,j}$ and $f_{i',j'}$. However, if (h being normal) $\varkappa\{g\} = \mu_\tau$, and $f_{i,j}$ and $f_{i',j'}$ lie on the same backward diagonal, being separated by more than $\mu(\tau) - i - j - 2$ intervening sequences, it can occur that $f_{i,j} = f_{i',j'}$ even though $h_{i,j} \neq h_{i',j'}$. (For example, when $h_0 = 1, h_1 = h_2 = 2$, $\overset{\sim}{\cancel{P_{2,0}(z) =}}$ ~~it~~ we have $P_{2,0}(z) = 1/(1 - 2z + 2z^2)$, $P_{0,2}(z) = 1 + 2z + 2z^2$;

now $h_\nu^{(2,0)} = h_\nu^{(0,2)} = h_\nu$ $(\nu \equiv \mathbb{I}_0^2)$, $h_3^{(2,0)} = h_3^{(0,2)} = 0$. If $g(x)$ is the polynomial $\sum_0^3 g_\nu z^\nu$, $f_{2,0} = f_{0,2}$ even though $h_{2,0} \neq h_{0,2}$. If $h_3 g_3 = 0$, $f_{2,0} = f_{0,2} = f$; if $h_3 g_3 \neq 0$, $f_{2,0} = f_{0,2} \neq f(\cdot)$.

The result of clause (vib) has many applications. We may, for example, take $\tilde{\Delta}$ to be a real segment, and L to be defined by $L\{g(\tilde{\Delta})\} = \int_a^b g(y)\,d\sigma(y)$, where all integrals $\int_a^b g_\nu(y)\,d\sigma(y)$ converge. Again we may set $\tilde{\Delta} = \mathbb{I}$, with $L\{g(\tilde{\Delta})\} = \sum a_\nu g(\nu)$, where all series $\sum a_\nu g_{\nu'}(\nu)$ $(\nu' = \mathbb{I})$ are convergent.

Double power series

We now consider the series produced by application of formula ( ) when the variable x in the series $g(x)$ is changed to t, and the coefficients g_ν in this series are taken to be power series ~~of~~ $g_\nu(x)$ in x, so that $f(x)$

and $f_{i,j}(x)$ become double series in t and x (see §1, in particular, formulae ($\frac{4}{5}$)).

Notation . $\mathfrak{F}'(x) = \sum' y_\nu(x) t^\nu$. With $P_{i,j}(z)$ decomposed in the form ( ), $\mathfrak{F}_{i,j}(x)$ is the series given by formula ( ), $\mathcal{E}^t_{i,j}(x) = \mathfrak{F}_{i,j}(x) - \mathfrak{F}'(x)$ $(i,j = \mathbb{I})$.

Theorem 2 . Let $\pi\{y(x)\} = \mu_\tau$, $y_{\mu(\omega)} = O\{x^{\Gamma\{\mu(\omega)\}}\}$ $(\omega = \mathbb{I}_0^\tau)$ with $\Gamma\{\mu(\omega+1)\} > \Gamma\{\mu(\omega)\}$ $(\omega = \mathbb{I}_0^{\tau-1})$.

(i) $\mathfrak{F}'(x)$ may be rearranged in the form $\mathfrak{F}'(x) = \sum' \mathfrak{F}_\nu x^\nu$, where the $\{\mathfrak{F}_\nu\}$ are polynomials in t, with $\mathfrak{F}_\nu$ $(\nu = \mathbb{I}^{\Gamma\{\mu(\omega)\}-1})$, and the degree of $\mathfrak{F}_\nu$ is at most $\omega' - 1$ $(\omega' = \mathbb{I}_1, \nu = \mathbb{I}^{\Gamma\{\mu(\omega')-1\}}_{\Gamma\{\mu(\omega'-1)\}})$

(ii)a) All series $\mathfrak{F}_{i,j}(x)$ $(i,j = \mathbb{I})$ are well determined

b) $\mathfrak{F}_{i,j}(x) = \sum' h_\nu^{(i,j)} y_\nu(x) t^\nu$ $(i,j = \mathbb{I})$ and $\mathfrak{F}_{i,j}(x)$ may be rearranged in the form $\mathfrak{F}_{i,j}(x) = \sum' \mathfrak{F}_\nu^{(i,j)} x^\nu$ where the $\{\mathfrak{F}_\nu^{(i,j)}\}$ are polynomials possesing properties similar to those given for the $\{\mathcal{F}_\nu\}$.

(iii) a) Let $\xi(i,j) \leq \mu(\tau)$ $[\![> \mu(\tau)]\!]$. Then $\mathcal{E}_{i,j}(x) = \mathcal{O}\{x^{\Gamma[\mu\{\lambda[\xi(i,j)]\}]}\}$ $[\![\mathcal{F}^*_{i,j}(x) = \mathcal{F}(x)]\!]$

b) Let $H_{i,j}, H_{i+1,j+1} \neq 0$ and $i+j < \mu(\tau)$ $[\![\geq \mu(\tau)]\!]$. Then $\mathcal{E}_{i,j}(x) = \mathcal{O}\{x^{\Gamma[\mu\{\lambda(i+j+1)\}]}\}$ $[\![\mathcal{F}_{i,j}(x) = \mathcal{F}(x)]\!]$

Proof. (i) When $\mu < \Gamma\{\mu(\nu')\}$, the coefficients of x^ν in the series $\sum h_\nu \psi_\nu(x) t^\nu$ is devoid of constituents of the form $h_{\nu''} t^{\nu''}$ with $\nu'' \geq \mu(\nu')$ and hence, since only powers of t of the form $t^{\mu(\nu)}$ occur in this series, the degree of $\mathcal{F}_\nu(t)$ is at most $\mu(\nu'-1)$. If $h_{\Gamma\{\mu(\nu)\}} \neq 0$, the coefficient of $t^{\mu(\nu)}$ in $\mathcal{F}_{\Gamma\{\mu(\nu)\}}(t)$, namely $h_{\Gamma\{\mu(\nu)\}} \mathcal{E}_{\nu,\Gamma\{\mu(\nu)\}}$ is nonzero $(\nu \in \mathbb{I}_0^\tau)$. Similar considerations hold with regard to clause (iii)

The results of clauses A ii), iv) are direct consequences of the corresponding clauses of Theorem 1. The results of clauses (v–viii) follow by establishing the extent to which the sequences h and $h^{(i,j)}$ agree, and the number of series $\{\xi_k(x)\}$ removed from $\xi^t_{ij}(x)$.

Algorithms applied to functions

General structural results

Notation. $G(\Delta, M) \in \mathbb{V}\{\tau\}$ means that for $\tau \in \mathbb{I}$ the derivatives $\mathfrak{D}_t^{\nu} G(t;x)$ $(\nu = \mathbb{I}_0^{\tau})$ are uniquely defined and finite for $t = \Delta, x = M$; $G(\Delta, M) \in \mathbb{V}\langle\overline{\mathbb{V}}\rangle$ means that $G(\Delta, M) \in \mathbb{V}\{0\} \langle \mathbb{V}\{\tau\}$ with unbounded $\tau\rangle$. $g(M) \in \mathbb{V}$ means that $g(M)$ is uniquely defined and finite. $g(M) \sim x'$ means that, with $g(M) \in \mathbb{V}$, a limit point x' (not necessarily belonging to M) exists such that, with $\varkappa\{g(M)\} = \mu_{\tau}$, $G_{\mu(\omega+1)}(x) = o\{G_{\mu(\omega)}(x)\}$ as $x \to x'$ in M $(\omega = \mathbb{I}_0^{\tau-1})$. $G \sim [\tau; \Delta, g(M) : x']$ means that a) $g(M) \sim x'$, b) with $\varkappa\{g(M)\} = \mu_{\tau'}$, that $G(t,x) - \sum_0^{\nu'} G_{\mu(\omega)} t^{\mu(\omega)} = o\{G_{\mu(\nu')}\}$ as $x \to x'$ in M $(t = \Delta, \nu' = \mathbb{I}_0^{\tau'})$ and c) that a similar system of asymptotic relationships for the functions $\mathfrak{D}_t^{\nu'} G(t;x)$ and the series obtained by successive term by term differentiation of the expression $\Sigma_0^{\tau'} G_{(\omega)} t^{\mu(\omega)}$ also holds (to the extent that such a system exists when τ' is finite) for $\nu'' = \mathrm{I}_1^{\tau'}$. $G \sim [\Delta, \mathfrak{y}(M) : x']$ means that $G \sim [0; \Delta, \mathfrak{y}(M) : x']$. $G(\Delta, M) \in \Upsilon^{\mu}\{\tau'; \hat{\mathfrak{y}}\}$ means that a) $G(\Delta, M) \in \mathbb{V}\{\tau'\}$, b) $\hat{G}_\nu(M) \in \mathbb{V}$ $(\nu = \mathrm{I}_0^{\mu-1})$ and c) that no relationship of the form

$$\Sigma_0^{m-1} \alpha_\nu \hat{G}_\nu(M) + \Sigma_1^{n} \Sigma_0^{n(\nu)-1} A_{\nu,\tau} G(\tau; t_\nu; M) = 0$$

holds, ~~more~~ where $m, n \in \mathrm{I}$, $n(\nu) \in \mathrm{I}_0^{\tau'+1}$ $(\nu = \mathrm{I}_1^{n})$, $m + 2\Sigma_0^{n(\nu)-1} |A_{\nu,\tau}| \neq 0$, $t_\nu \in \Delta$ $(\nu = \mathrm{I}_1^{n})$ are nonzero and distinct, and the function $G(\tau; t; x)$ is defined by formula ( ). $G(\Delta, M) \in \Upsilon^{\mu}\{\tau'\} < \Upsilon^{\mu}\{\hat{\mathfrak{y}}\} > [\overline{\Upsilon}\{\hat{\mathfrak{y}}\}]$ means that $G(\Delta, M) \in \Upsilon^{\mu}\{\tau'; \hat{\mathfrak{y}}\}$ with $m = 0$ in formula ( ) $< \in \Upsilon^{\mu}\{0; \hat{\mathfrak{y}}\} >$ $[\Upsilon^{\mu}\{\infty; \hat{\mathfrak{y}}\}$ with unbounded μ and $\tau']$.

$F_{i,j}(x)$ $(i,j \in \mathbb{I})$ is the function of formula ( ) derived from the functions $G_\nu(x)$ $(\nu \in \mathbb{I})$, $G(t,x)$ and the sequence h_i; $\mathbb{F}$ is the complete array of functions $F_{i,j}$ $(i,j \in \mathbb{I})$. $\mathbb{F}_S$ is a subset of $\mathbb{F}$. $\mathbb{F}_S \Leftrightarrow \mathbb{P}_S$, where $\mathbb{P}_S$ is prescribed, means that $\mathbb{F}_S$ corresponds to $\mathbb{P}_S$ in $\mathbb{P}$. $\mathbb{D}_{m'}$ $\langle \mathbb{O}_{m'} \rangle$ $[\mathbb{L}_{m'}]$ is the subset $F_{i,i+m-1}$ $\langle F_{i,i+2m-1} \rangle$ $[F_{i+m+1,i}]$ $(i \in \mathbb{I}, m \in \mathbb{I}_0^{m'}; m' \in \mathbb{I})$ of $\mathbb{F}$. $\mathbb{D}' \langle \mathbb{O}' \rangle [\mathbb{L}']$ is $\mathbb{D}_{m'} \langle \mathbb{O}_{m'} \rangle [\mathbb{L}_{m'}]$ with arbitrarily large but bounded m'; $\mathbb{F}'$ is the adjunction of $\mathbb{D}'$ and $\mathbb{L}'$. $F_{i,j}(M) \in W$ $(i,j \in \mathbb{I})$ means that $F_{i,j}(x)$ is well determined by formula ( ) for $x = M$; $\mathbb{F}_S(M) \in W$ means that $F_{i,j}(M) \in W$ $(F_{i,j} \in \mathbb{F}_S)$, $F_{\mathbb{I},\mathbb{J}}[M;N] = \hat{F}(M)$ means that with the mapping $\hat{F}(M)$ prescribed, $F_{i,j}(M) = \hat{F}(M)$ $(i,j \in [\mathbb{I},\mathbb{J};N)$. With $F(x)$ prescribed, we set $E_{i,j}(x) = F(x) - F_{i,j}(x)$; furthermore $E_{i,j,i',j'}(x) = F_{i,j}(x) - F_{i',j'}(x)$.

$\hat{F}(x) = O\{\hat{G}(x); M : x'\}$ means that $\hat{F}(x) = O\{\hat{G}(x)\}$ as $x \to x'$ in M; the analogous symbol with o in place of O is similarly defined. $F'(M) \sim \{h', g : x'\}$ means that a) $g(M) \sim x'$, b) with $\varkappa\{h' g(M)\} = \mu_\tau$ and setting $\hat{F}(x; \nu') = F'(x) - \sum_0^{\nu'} h'_\nu G_\nu(x)$ $(\nu' = \mathbb{I}_0^{\mu(\tau)})$, $\hat{F}(x;\nu') = \hat{F}(x; \mu(\tau))$ $(\nu' = \bar{\mathbb{I}}_{\mu(\tau)+1})$ that $\hat{F}(x;\nu') = O\{G_{\mu\{\lambda(\nu'+1)\}}(x), M : x'\}$ $(\nu' = \mathbb{I}_0^{\mu(\tau)-1})$, $\hat{F}(x;\nu') = o\{G_{\mu(\tau)}(x); M : x'\}$ $(\nu' = \mathbb{I}_{\mu(\tau)})$

$g(\Delta) \in \mathbb{V}\{\tau\}$ means that for $\tau \in \mathbb{I}$ the derivatives $\mathcal{D}_x^\nu g(x)$ $(\nu = \mathbb{I}_0^\tau)$ are uniquely defined and finite for $x = \Delta$; $g(\Delta) \in \mathbb{V}\langle\bar{\mathbb{V}}\rangle$ means that $g(\Delta) \in \mathbb{V}\{0\} \langle \mathbb{V}\{\tau\}$ with unbounded $\tau\rangle$. $g \in \mathbb{V}$ means that g is prescribed. $g \sim [\tau; g, \Delta_\phi^\psi]$ means that with $\varkappa\{g\} = \mu_\tau$, we have $g(x) - \sum_0^{\nu'} g_{\mu(\nu)} x^{\mu(\nu)} = o\{x^{\mu(\nu')}\}$ as $x \to 0$ in Δ_ϕ^ψ $(\nu' = \bar{\mathbb{I}}_0^\tau)$ and that a similar system of asymptotic relationships involving the derivatives $\mathcal{D}_x^{\nu''} g(x)$

and analogous to those implied by the symbol $G \sim [\tau; \Delta, \mathfrak{g}(M) : x'_-$ hold for $\nu'' = \overline{1}_0^{\tau}$. $g \sim [g, \Delta_\phi^\psi]$ means that $g \sim [0; g, \Delta_\phi^\psi]$.

$g(\Delta, \hat{\Delta}') \in y^\mu\{\tau'; \hat{g}\}$ means that a) $g(\Delta \times \Delta') \in v\{\tau'\}$, b) $\hat{g}_\nu \in v$ ($\nu = \overline{1}_0^{\mu-1}$) and c) that, for every set of numbers α_ν, $A_{\nu,\tau}$, t_ν restricted as described in connection with formula ( ), at least one $x \in \Delta$ exists such that the equation

$$\sum_0^{m-1} \alpha_\nu g_\nu x^\nu + \sum_1^\mu \sum_0^{n(\nu)-1} A_{\nu,\tau} x^\tau \partial_x^\tau g(t_\nu x) = 0$$

is not satisfied. $g(\Delta, \hat{\Delta}) \in y\{\tau'\} < y\{\hat{g}\} > [\bar{y}\{g\}]$ means that $g(\Delta, \tilde{\Delta}) \in y^\mu\{\tau'; \hat{g}\}$ with $m = 0$ in formula ( ) $< \in y\{0; g$ $[\in y^\mu\{\tau', \hat{g}\}$ with unbounded μ and $\tau']$.

The symbols $f_{i,j}(x)$, $\mathfrak{f}$, $\mathfrak{f}_s$, $u_{m'}$, $v_{m'}$, $z_{m'}$, α', o', γ', δ', $\mathfrak{f}_{i,j}(\Delta) \in W$, $\mathfrak{f}_s(\Delta) \in \mathfrak{W}$ and $\mathfrak{f}_{I,J}[\Delta, N] = \hat{f}(\Delta)$ have meanings with regard to the functions of formula ( ) derived from the function $g(x)$ and the sequences g and h, and defined over the point set Δ, analogous to those of $\bar{F}_{i,j}(x), \dots, \bar{F}_{i,j}\{M; x\}$ $\hat{F}(M)$ given above. $\hat{f}(x) = O\{x^{\prime}; \Delta_{\phi}^{\psi}\}$ means that $\hat{f}(x) = O(x^{\prime})$ as $x \to 0$ in Δ_{ϕ}^{ψ}; the analogous symbol with o in place of O is similarly defined. $f' \sim \mathfrak{f}(\Delta_{\phi}^{\psi})$ means that with $x\{h\} = \mu_{\tau}$, setting $\hat{f}(x;\nu') = f'(x) - \sum_{0}^{\nu'} f'_{\nu} x^{\nu}$ $(\nu' = \underline{I}_{0}^{\mu(\tau)})$, $\hat{f}(x;\nu') = \hat{f}(x;\mu(\tau))$ $(\nu' = \bar{I}_{\mu(\tau)+1})$ we have $\hat{f}(x;\nu') = O\{x^{\mu\{\lambda(\nu'+1)\}}; \Delta_{\phi}^{\psi}\}$ $(\nu' = \underline{I}_{0}^{\mu(\tau)-1})$, $\hat{F}(x;\nu') = o\{x^{\mu(\tau)}; \Delta_{\phi}^{\psi}\}$ $(\nu' = \bar{I}_{\mu(\tau)})$.

We remark that if $\mathfrak{F}\,\mathfrak{G}(M) \sim x'$ $(x\{\mathfrak{G}(M)\} = \mu_{\tau})$ a relationship such as $\hat{F}(x) = O\{G_{\mu(r)}(x); M : x'\}$ $(r \in \bar{I}_{0}^{\tau-1})$ implies that $\hat{F}(x) = o\{G_{\mu(r+1)}(x); M : x'\}$. For the relationship $\hat{f}(x) = O\{x^{\mu(r)}; \Delta_{\phi}^{\psi}\}$ we have the possibly sharper result $\hat{f}(x) = o\{x^{\mu(r)+1}; \Delta_{\phi}^{\psi}\}$. Functions of the class $\mathbb{Y}\{\tau'; \hat{\mathfrak{G}}\}$ are introduced to ensure that certain mappings $F_{i,j}(M)$ with distinct suffix sets are themselves distinct. Functions not of this class, of course, exist; the polynomial $G(t;n) = \sum_0^{\mu} G_\nu(m) t^\nu$ being an exampl

of such a function: we then have $\sum_0^{\mu+1} \binom{\mu+1}{\nu} (-1)^\nu G(t+\nu\kappa, m) =$

($t \equiv \mathbb{Z}$, $k \in \mathbb{Z}$). Again, the exponential function, for example, does

not belong to the class $\mathbb{V}\{\tau!\}$.

Theorem . A. (i) a) Let $t\{P_{i,j}\} \in \Delta$, $n\{P_{i,j}\} \leq \tau \in \mathbb{I}$ ($P_{i,j} = \mathbb{P}_S \Leftrightarrow$

$\mathbb{F}_S$); let $G_\nu(m) \in \mathbb{V}$ ($\nu = \overline{\mathbb{I}}_0^m$) where $m = \max(j-i)$ ($P_{i,j} = \mathbb{P}_S$),

and let $G(\Delta, m) \in \mathbb{V}\{\tau\}$. Then $\mathbb{F}_S(m) \in \mathbb{W}$.

b) Let $H_{i,j} \in H_{I,J}[N]$ and $F_{i,j}(m) \in \mathbb{W}$. Then $F_{I,J}[M,N] \in \mathbb{W}$

c) Let $G(t,x)$ be an entire function of t ($x = m$). Then

$\mathbb{L}(m) \in \mathbb{W}$. If, also, $\mathfrak{G}(m) \in \mathbb{V}$, then $\mathbb{F}(m) \in \mathbb{W}$.

(ii) a) Let $h_{I,J} \in h$. Using the notation of formula ( ), let

$G_\nu(m) \in \mathbb{V}$ ($\nu = \overline{\mathbb{I}}_0^m$), $\mathfrak{D}_t^\tau G(t,m) \in \mathbb{V}$ ($t = t_\nu$, $\tau = n_\nu$, $\nu = \overline{\mathbb{I}}_1^n$).

Set

$$F(m) = \sum_0^m \alpha_\nu G_\nu(m) + \sum_1^n \sum_0^{n_\nu - 1} A_{\nu,\tau} G(\tau; t_\nu; x)$$

$G(\tau; t; x)$ being defined by formula ( ). Then

$F_{I,J}(M,\infty) = F(m)$.

b) Let $G(t,m)$ be the polynomial $\sum_0^\mu G_\nu(m) t^\nu$. Set $F(M) =$

$\sum_0^\mu h_\nu G_\nu(m)$

(1) Let $\xi(i,j) > \mu$. Then $F_{i,j}[M,\infty] = F(m)$

(2) Let $H_{i,j} \neq 0$ with $i+j+1 > \mu$. Then $F_{i,j}[~~\infty~~M,\infty] = F(m)$

(3) Let $H_{i,\mu-i} \neq 0$ $(i = \overline{1}_1^{\mu-1})$. Then $F_{i,j}(M) = F(M)$ $(F_{i,j} \equiv F - F^\wedge$.

(iii) Let $l\{P_{i,j}\} \in \Delta$, $\varkappa\{P_{i,j}\} \leq \tau \in \mathbb{I}$ $(P_{i,j} = P_S \Leftrightarrow F_S)$ and

$G[\tau; \Delta, y(M) : x']$. Then $F_{i,j}(M) \sim \{h^{(i,j)}, y : x'\}$ $(F_{i,j} \equiv F_S)$.

(iv) Let $\varkappa\{y(M)\} = \mu_\tau$

a) Let $F_{i,j}(M) \sim \{h^{(i,j)}, y : x'\}$ and $F(M) \sim \{h, y : x'\}$. If

$\xi(i,j) \leq \mu(\tau)$ then $E_{i,j}(x) = O\{G_{\mu[\lambda\{\xi(i,j)\}]}(x) : M, x'\}$. If

$\xi(i,j) > \mu(\tau) \in \mathbb{I}$, then $E_{i,j} = o\{G_{\mu\{\lambda(\tau)\}}(x) : M, x'\}$. If $\xi(i,j)$

$= \mu(\tau) = \infty$, then $E_{i,j}(x) = o\{G_{\mu(\omega)}(x) : M, x'\}$ for arbitrarily large

ν

b) Let $F_{i,j}(M) \sim \{h^{(i,j)}, \mathcal{Y} : x'\}$, $F_{i',j'}(M) \sim \{h^{(i',j')}, \mathcal{Y} : x'\}$ and $H_{i,j} \in H_{I,J}[N]$, $H_{i',j'}[N']$ the two blocks being distinct with $\xi' = \min(I+J+N+1, I'+J'+N'+1) \leq \mu(\tau)$. (1) If $I+J+N \neq I'+J'+N'$, then $E_{i,j,i',j'}(x) = O\{G_{\mu\{\lambda(\xi')\}}(x); M : x'\}$. (2) Let $I+J+N = I'+J'+N'$; if $I' > I$ $\langle I < I' \rangle$ set $r'' = I'+J$ $\langle I+J' \rangle$ and let $r'' \leq \mu(\tau)$. Then there exists an $\nu \in I_{\xi'}^{r''}$ for which $E_{i,j,i',j'}(x) = O\{G_\nu(x); M : x'\}$.

c) If $\mu(\nu) = \nu$ $(\nu = I_0^{\tau})$ (so that $\mu[\lambda\{\xi(i,j)\}] = \xi(i,j) \leq \tau$) then the order relationships involving the symbol O in a) and b) (1) do not hold with O replaced by o. Furthermore, with regard to b)(2) it is not true that $E_{i,j,i',j'}(x) = o\{G_\nu(x); M : x'\}$ $(\nu = I_{\xi'}^{r''})$

(v) Let $\#\{P_{i,j}\} \in \Delta$, $n\{P_{i,j}\} \leq \tau \in I$ $(P_{i,j} \equiv P_S \Leftrightarrow F_S)$

a) Let $G_\nu(M) \neq 0$ $(\nu = I_0^{\mu})$, $G \sim [\tau; \Delta, \mathcal{Y}(M) : x']$ and

$F(M) \sim \{h, y : x'\}$.

(1) Let $P_{i,j} \in \mathbb{P}_S$ and $\xi(i,j) \leq \mu$. Then $F_{i,j}(M) \neq F(M)$

(2) Let $P_{i,j} \in \mathbb{P}_S$ and $H_{i,j}, H_{i+1,j+1} \neq 0$ with $i+j \leq \mu$. Then

$F_{i,j}(M) \neq F(M)$

(3) Let $P_{i,j}, P_{i',j'} \in \mathbb{P}_S$ and $H_{i,j}, H_{i',j'}$ satisfy the conditions of (iv)b). If $I+J+N \neq I'+J'+N'$, then $F_{i,j}(M) \neq F_{i',j'}(M)$. Let $I+J+N = I'+J'+N'$; if $I' > I$ $\langle I > I' \rangle$ let $\mu \geq I'+J$ $\langle I'+J \rangle$.

Then $F_{i,j}(M) \neq F_{i',j'}(M)$

(4) Let $P_{i+1,j}, P_{i,j+1} \in \mathbb{P}_S$, $H_{i+1,j}, H_{i,j+1}, H_{i+1,j+1} \neq 0$, and $i+j+1 < \mu$. Then $F_{i+1,j}(M) \neq F_{i,j+1}(M)$

b) Let $G(\Delta, M) \in \mathbb{Y}^{\mu}\{\pi, y\}$. Set $F^{\mu}_S(M) = F^{\mu}(M) \cap \mathbb{F}_S(M)$

(1) For any two members of $F^{\mu}_S(M)$, $F_{i',j'}(M) \equiv F_{i'',j''}(M)$ if and only if $P_{i,j} = P_{i'',j''}$; $F_{I,J}[M; N] \subseteq \mathbb{F}^{\mu}_S(M)$ if and only if $\mathbb{P}_{I,J}[N] \subseteq \mathbb{P}_S$; equivalent mappings of

$F_S^{\mu}(M)$ occur only in square blocks (or the portions of square blocks corresponding to the portions of blocks belonging to P_S).

(2) Let $h_{I,J} \leftarrow h_{\bar{I},J}$. Using the notation of formula ( ), let $t_\nu \in \Delta$ $(\nu = \overline{1}_1^N)$ $\not\exists n_\nu < \tau$ $(\nu = \overline{1}_1^n)$ and let $0 \in \Delta$ if $m_\nu 0$. Then $\mathbb{F}_{I,J}[M,\infty] = F(M)$, where $F(M)$ is as defined as in A)iia) and $F_{i,j}(M)$ differs from $F(M)$ $(F_{i,j} \equiv \mathbb{F}_S^{\mu} - F_{I,J}[\infty])$.

(vi)a) Let $\mathbb{t}\{P_{i,j}\} \in \mathbb{D}_\gamma$ $(P_{i,j} \equiv P_S \Leftrightarrow \mathbb{F}_S)$ and $G(t = \mathbb{D}_\gamma, M) \in A_t$ with $G \leftarrow g(M)$. Then $\sum_i h_\nu^{(i,j)} G_\nu(M) \to F_{i,j}(M)$ $(F_{i,j} = \mathbb{F}_S)$

b) If, furthermore, $g(M) \sim x'$, then $F_{i,j}(M) \sim \{h^{(i,j)}, g(M) : x'\}$ $(F_{i,j} = F_S)$.

(vii)a) Let $G(t = \mathbb{N}\{\mathbb{t}\{P_{i,j}\}\}, x = \mathbb{D}) \in A_{t,x}$ $(P_{i,j} \equiv P_S \Leftrightarrow F_S)$ with, if $j \geq i$ for any $P_{i,j} \in P_S$ $g(\mathbb{D}) \in A$. Then $\mathbb{F}_S(\mathbb{D}) \in A$

b) Let $G(t,x)$ be an entire function of t $(x = \mathbb{D})$ and $G(t = B, x = \mathbb{D}) \in A_{t,x}$. Then $\mathbb{L}(\mathbb{D}) \in A$. If $g(\mathbb{D}) \in A$ also, then $\mathbb{F}(\mathbb{D}) \in A$.

(viii) Let $\#\{P_{i,j}\}\in\mathbb{D}$ $(P_{i,j}=\mathbb{P}_S \Leftrightarrow \mathbb{F}_S)$, let $G_\nu(m)\in\mathbb{V}$ be real $(\nu=\underline{I}_0^m)$, where $m=\max(j-i)$ $(P_{i,j}=\mathbb{P}_S)$, let $G(z=\bar{\mathbb{D}}, m)\in A_z$ and let $[a,b]\in\mathbb{D}$ with $G(t,m)$ real for $t=[a,b]$. Let h be real. Then $F_{i,j}(m)$ is real $(F_{i,j}=\mathbb{F}_S)$

(ix) a) Let $F_{i,j}(m)\in\mathbb{W}$. Let $\hat{F}_{i,j}(m)$ be the function defined by formula ( ) in terms of the same functions $G_\nu(x)$ (if they are used), the same function $G(t,x)$, but the sequence ah $(a(\neq 0)\in\mathbb{Z})$. Then $\hat{F}_{i,j}(m)=a\,F_{i,j}(m)$

b) Let $\mathbb{N}'$ be a class of functions defined for $y=\tilde{\Delta}$ such that $g(y)\in\mathbb{Z}$ $(y=\tilde{\Delta}, g=\mathbb{N}')$. Let L be an operator applied to functions of $\mathbb{N}'$ such that $L\{g(\tilde{\Delta})\}\in\mathbb{Z}$ $(g=\mathbb{N}')$ and $L\{ag'(\tilde{\Delta})+bg''(\tilde{\Delta})\}=aL\{g'(\tilde{\Delta})\}+bL\{g''(\tilde{\Delta})\}$ $(a,b=\mathbb{Z},\ g',g''=\mathbb{N}')$. Set $m=\max(j-i)$ $(F_{i,j}=\mathbb{F}_S)$. Let $G_\nu(y;x)\in\mathbb{N}'$ $(\nu=\underline{I}_0^m, x=M)$ and $G(y;t,x)\in\mathbb{N}'$ $(t=\Delta, x=M)$. Let

$F_{i,j}(y,x)$ be the functions defined by formula ( ) in terms of $G_\nu(y;x)$ $(\nu = \mathbb{I}_0^m)$, the function $G(y;t,x)$ and the sequence h, with $F_{i,j}(y,m) \in W$ $(F_{i,j} = \mathbb{F}_s,\ y = \tilde{\Delta})$. Denote by $\tilde{\tilde{F}}_{i,j}(x)$ the functions defined by formula ( ) in terms of $L\{G_\nu(\tilde{\Delta},x)\}$ $(\nu = \mathbb{I}_0^m)$, $L\{G(\tilde{\Delta};t,x)\}$ and h. Then $F_{i,j}(m) = L F_{i,j}(\tilde{\Delta},m)$ $(F_{i,j} = \mathbb{F}_s)$.

B.(i)a) Let $t\{P_{i,j}\} \in \Delta$, $n\{P_{i,j}\} \leq \tau \in \mathbb{I}$ $(P_{i,j} = \mathbb{P}_s \leftrightarrow \mathbb{f}_s)$, let $g(\Delta \times \hat{\Delta}) \in v\{\tau\}$ and let $g \in v$ if $j \geq i$ for any $P_{i,j} \in \mathbb{P}_s$. Then $\mathbb{f}_s(\hat{\Delta}) \in W$.

b) Let $H_{i,j} \in H_{I,J}[N]$ and $f_{i,j}(\hat{\Delta}) \in w$. Then $\mathbb{f}_{I,J}[N] \in w$

c) Let $g(N\{0\}) \in A$. Then $\mathbb{L}(N\{0\}) \in A$. If also $g \in v$, then $\mathbb{f}(N\{0\}) \in A$. If $g \leftarrow g$ then $f_{i,j} \leftarrow h^{(i,j)} g$ $(i,j = \mathbb{I})$

d) Let g be an entire function. Then all members of $\mathbb{L}$ are entire functions. If also $g \in v$, all members of

$\mathfrak{f}$ are entire functions

(ii)a) Let $h_{I,J} \leftarrow h$. Using the notation of formula( ), let $g_\nu \in W$ $(\nu = \underline{\overline{I}}_0^m)$. $\mathfrak{D}_x^{n_\nu - 1}\{g(t_\nu x) \in V$ $(\nu = \underline{\overline{I}}_1^n\}_\nu, x = \hat{\Delta})$. Set

$$f(x) = \Sigma_0^n \alpha_\nu g_\nu x^\nu + \Sigma_1^n \Sigma_1^{n_\nu - 1} A_{\nu,\tau} \mathfrak{D}_x^\tau\{x^\tau g(t_\nu x)\}/\tau! \; (x = \hat{\Delta})$$

Then $\mathfrak{f}_{I,J}[\hat{\Delta}, \infty] = f(\hat{\Delta})$

b) Let $g(x)$ be the polynomial $\Sigma_0^\mu g_\nu x^\nu$. Set $f(\mathbb{Z}) = \Sigma_0^\mu h_\nu g_\nu \mathbb{Z}^\nu$. With the symbols $\mathbb{F}$, $\mathbb{M}$ and F replaced by $\mathfrak{f}$, $\mathbb{Z}$ and f, the results of A(ii)b) (1-2) hold

(iii) Denote the conjunction of the sectors $\Delta_{\phi_\nu}^{\psi_\nu} < \Delta_{\phi_\nu + \phi}^{\psi_\nu + \psi} >$ $(\nu = \underline{\overline{I}}_1^r)$ by $\Delta_{\underline{\Phi}}^{\overline{\Psi}} < \Delta_{\underline{\Phi}+\phi}^{\overline{\Psi}+\psi} >$. Let $t\{P_{i,j}\} \in \Delta_\phi^\psi$, $n\{P_{i,j}\}$ $s \in \underline{\overline{II}}$ $(P_{i,j} \equiv \mathbb{P}_s \Leftrightarrow \mathfrak{f}_s)$ and $g \sim [\tau; g, \Delta_{\underline{\Phi}+\phi}^{\overline{\Psi}+\psi}]$. Then $f_{i,j}(\Delta_\phi^\psi) \sim h^{(i,j)} g$ $(f_{i,j} \equiv \mathfrak{f}_s)$

(iv)a) Let $f_{i,j}(\Delta_\phi^\psi) \sim h^{(i,j)} g$, $f(\Delta_\phi^\psi) \sim hg$ and let $\varkappa\{g\} = \mu_\tau$.

With formulae such as $E_{i,j}(x) = O\{G_\mu\{x(\nu)\}(x) : M, x'\}$

replaced by $e_{i,j}(x) = O\{x^{\mu\{\lambda(r)\}} : \Delta_\phi^\psi\}$, the results of A(iv) hold.

b) Let $g(\mathbb{N}\{0\}) \in A$ with $g \leftarrow g$ and $f(\mathbb{N}\{0\}) \in A$ with $f \leftarrow hg$. Then the results of a) above hold with $\phi = 0, \psi = 2\pi$ (i.e. for all values of $\arg(x)$).

(v) Let $\mathbb{H}\{P_{i,j}\} \in \Delta$, $n\{P_{i,j}\} \leq \tau \in \mathbb{I}$ ($P_{i,j} \equiv \mathbb{P}_S \leftrightarrow \mathfrak{f}_S$).

a) Let Δ_{Φ}^{Ψ} and $\Delta_{\Phi+\phi}^{\Psi+\psi}$ be two composite sectors as described in B.(ii), let $\Delta \in \Delta_{\Phi}^{\Psi}$, $g \sim [\tau; y : \Delta_{\Phi+\phi}^{\Psi+\psi}]$ with $g_\nu \neq 0$ ($\nu = \bar{\mathbb{I}}_0^\mu$). Let $\hat{\Delta} \in \Delta_\phi^\psi$ contain the origin as a limit point, with $g(\Delta \times \hat{\Delta}) \in \mathbb{V}\{\tau\}$. Let $f(\Delta_\phi^\psi) \sim hg$. Then with $\mathbb{F}$, F and M replaced by $\mathfrak{f}$, f and Δ, and $f(\Delta)$ now the function defined in B.(ii)a), the results of A(v)a) (1–4) hold.

b) Let $g(\Delta \times \hat{\Delta}) \in A$, where $\mathbb{N}\{0\} \subseteq \hat{\Delta}$, and let $g \leftarrow g$ with $g_\nu \neq 0$ ($\nu = \bar{\mathbb{I}}_0^\mu$) Let $f\{\mathbb{N}\{0\}\} \in A$, with $f \leftarrow hg$. Then the of a) above hold.

c) Let $g(\Delta, \hat{\Delta}) \in y\{\tau; g\} < y\{\tau\} >$ if $j \geq i$ for any $< j < i$ for all $> P_{i,j} \in \mathbb{P}_S$. Then with $\mathbb{F}$, M and F replaced by $\mathfrak{f}$, Δ and f, and $f(\Delta)$ means the function defined in B(ii)a), the results of A.(v) b) (1,2) hold.

(vi) Let $\#\{P_{i,j}\} \in \mathbb{D}_\gamma$ $(P_{i,j} \equiv \mathbb{P}_S \Leftrightarrow \mathfrak{f}_S)$ and $g(\mathbb{D}_{\gamma}) \in A$ with $g \leftarrow g$. Then $\mathfrak{f}_S(\mathbb{D}_\gamma) \in A$ and $\sum h_\nu^{(i,j)} g_\nu \mathbb{D}_\gamma^\nu \to f_{i,j}(\mathbb{D}_\gamma)$ $(f_{i,j} \equiv \mathfrak{f}_S)$.

(vii) Let $\#\{P_{i,j}\} \in \mathbb{D}$ $(P_{i,j} \equiv \mathbb{P}_S \Leftrightarrow \mathbb{F}_S)$, let g_ν $(\nu = I_0^m)$ be real and finite, where $m = \max(j - i)$ $(P_{i,j} \equiv \mathbb{P}_S)$, let $g(\bar{\mathbb{D}} \times \Delta) \in A$, and let $[a, b] \subset \mathbb{D}$ with $g(x)$ real for $x = [a,b] \times \Delta$. Then $f_{i,j}(x)$ is real $(f_{i,j} = \mathfrak{f}_S)$

(viii) a) Mutatis mutandis, the result of A.(ix)a) holds

b) Let N' and L be as defined in A.(ix) b). Set $m =$

$\max(j-i)$ $(f_{i,j} \equiv \mathfrak{f}_S)$. Let $g_\nu(y) \in N'$ $(\nu \equiv \mathbb{I})$ and $g(y, tx) \in N'$ $(t \equiv \hat{\Delta}, x \equiv \Delta)$. Let $f_{i,j}(y,x)$ be the function defined by formula ( ) in terms of the sequence $g_\nu(y)$ $(\nu \equiv \bar{\mathbb{I}}_0^m)$, the function $g(y,x)$ and the sequence h, and let $f_{i,j}(y,\Delta) \in W$ $(f_{i,j} \equiv \mathfrak{f}_S, y \equiv \tilde{\hat{\Delta}})$. Denote by $\tilde{\hat{f}}_{i,j}(x)$ the functions defined by ~~the~~ formula ( ) in terms of $Lg_\nu(\tilde{\hat{\Delta}})$ $(\nu \equiv \underline{\mathbb{I}}_0^m)$, $Lg(\tilde{\hat{\Delta}}, x)$ and h. The $\tilde{f}_{i,j}(\Delta) = Lf_{i,j}(\tilde{\hat{\Delta}}, \Delta)$ $(f_{i,j} \equiv \mathfrak{f}_S)$.

Proof. A(i)a). The conditions upon the $\{t_\nu^{(i,j)}\}$ and $G(t,x)$ ensure that the requisite derivatives ~~of~~ occurring in formula ( ) are defined for all members of $\overline{\mathbb{H}}_S$; if (as occurs when $j \geq i$) the functions $\{G_\nu(x)\}$ are needed, they are also defined. b) If $H_{I,J}[N] \subseteq H$, then $\mathbb{P}_{I,J} \subseteq P$ (1( ) ..); since one member of $F_{I,J}[N]$ is well determined, the rest are. c) Now all derivatives of $G(t,x)$ are defined for all finite t: $\mathbb{L}(m) \in \mathbb{W}$. If, in addition, $\mathbb{G}(m) \in \mathbb{V}$, then $\mathbb{D}(m) \in \mathbb{W}$ also

(ii) a) Now $P_{i,j} = h_{I,J}$ $(i,j = [I,J,\infty])$; we may use A(i)a).

b) When $G(t,m) = \sum_0^{\mu} G_\nu(m) t^\nu$, then $\mathbb{F}(m) \in \mathbb{W}$ (A.(i)c)) and, as is easily verified, $F_{i,j}(m) = \sum_i h_\nu^{(i,j)} G_\nu(m)$ $(i,j \geq \mathbb{I})$. All results of this clause now follow as in the proof of 2.( )..., replacing x^ν by $G_\nu(m)$

(iii) The conditions imposed upon $G(t,x)$ and $\{G_\nu(x)\}$ ensure that $F_{i,j}(m) \in \mathbb{W}$ (A(i)a)). Furthermore, each term in the second sum on the right hand side of formula ( ) has an asymptotic representation in terms of the functions of the asymptotic sequence $\{G_{\mu(\nu)}(x)\}$; hence $F_{i,j}(x)$, a finite linear combination of such terms, also has the given asymptotic representation.

(iv)a) We now have $F(M) - F_{i,j}(M) \sim \{h - h^{(i,j)}, g(M) : x'\}$. All stated results of the clause follow directly from this relationship. b)(1) We have, (see the proof of 1. (iv)c)) $h_\nu^{(i,j)} = h_\nu^{(i',j')}$ $(\nu = \underline{1}_0^{\xi'-1})$, $h_{\xi'}^{(i,j)} \neq h_{\xi'}^{(i',j')}$. Hence the asymptotic expansion of $E_{i,j,i',j'}(x)$ lacks the functions $G_{\mu(\nu)} < \xi'$. (2) This result follows in a similar fashion from 1.( ). c) If the sequence $G_\nu(x)$ $(\nu = \underline{1}_0^\tau)$ possesses no gaps, a nonzero difference such as $h_{\xi(i,j)} - h_{\xi(i,j)}^{(i,j)}$ $(\xi(i,j) \leq \tau)$ is necessarily accompanied by a corresponding nonzero member of the sequence: we do not have, for example, $E_{i,j}(x) = o\{G_{\xi(i,j)}(x) ; M : x'\}$.

~~(v)a) We now have $F(M) - F_{i,j}(M) \sim \{h - h^{(i,j)}, g(M) : x'\}$.~~

~~All~~

(v)a) $F_{i,j}(M) \sim \{h^{(i,j)}; \mathcal{G}(M) : x'\}$ (A.(iii)) and hence, if $\xi(i,j) \le \mu$

$F(x) - F_{i,j}(x) \sim (h_{\xi(i,j)} - h^{(i,j)}_{\xi(i,j)})\, G_{\xi(i,j)}(x)$ as $x\,(\in M) \to x'$.

Since $h_{\xi(i,j)} \neq h^{(i,j)}_{\xi(i,j)}$, it follows that over that part of M in the neighbourhood of x', $F(x)$ and $F_{i,j}(x)$ do not agree, and $F(M) \neq F_{i,j}(M)$. The demonstration of this result is similar to that of 2( )...) in which we showed that the coefficient of $x^{\xi(i,j)}$ in the series $f(x) - f_{i,j}(x)$ was nonzero. The remaining results of the present clause may be derived by adapting the proofs of those of 2.( ).. in a similar fashion. b)(1) The difference $F_{i,j}(x) - F_{i',j'}(x)$ of any two members of $\mathbb{F}_3$ is expressible in the form exhibited on the left hand side of relationship ( ). Since this relationship is not satisfied for at least one $x \in M$,

$F_{i,j}(m) \neq F_{i,j'}(m)$. The remaining results of this clause follow in the same way.

(vi)a) We now have $G(t,x) = \sum G_\nu(x)t^\nu$ for $t \in \mathbb{D}_\lambda$, $x \in M$, and further series representations for $\mathcal{D}_t^\tau G(t,x)$ $(\tau \in \mathbb{I})$ obtained by term by term differentiation of this series. Replacing t by the appropriate members of the set $\#\{P_{i,j}\}$, each constituent of $F_{i,j}(x)$ in formula ( ) has a convergent series representation of the above form; these constituents (finite in number) may be combined to yield the series $\sum h_\nu^{(i,j)} G_\nu(x)$, converging to $F_{i,j}(x)$ for $x \in m$. b) We shall show that if the series $\sum h_\nu^{(i,j)} G_\nu(x)$ converges to $F_{i,j}(x)$ $(x \in M)$ and $\mathcal{G}(m) \sim x'$, then $F_{i,j}(m) \sim \{h^{(i,j)}\mathcal{G} : x'\}$. Set $\varkappa\{h^{(i,j)}\mathcal{G}(m)\} = \mu_\tau$. We consider the behaviour of the remainder term $R_{\mu(\tau)}(x)$

$= \Sigma_{\nu'} h^{(i,j)}_{\mu(\nu)} G_{\mu(\nu)}(x)$ for an $\nu' \in \mathbb{I}_0^\tau$. Since $\mathfrak{t}\{P_{i,j}\} \in \mathbb{D}_\gamma$, the series $\Sigma h^{(i,j)}_{\mu(\nu)} \gamma^{-\mu(\nu)}$ converges absolutely. Set $\hat{G}_{\mu(\nu)}(M) = \gamma^{\mu(\nu)} G_{\mu(\nu)}(M)$ $(\nu = \mathbb{I}_0^\tau)$. Since $G(t \equiv \mathbb{D}_\gamma, M) \in A_t$, $\hat{G}_{\mu(\nu)}(x) / \hat{G}_{\mu(\nu')}(x)$ is bounded for $\nu = \mathbb{I}^\tau_{\nu'+1}$ and $x \in M_1$ where $M_1 \subseteq M$ possesses x' as a limit point. Hence $R_{\mu(\nu')}(x) = \gamma^{\nu'} G_{\mu(\nu')}(x) \Sigma^\tau_{\nu'} h^{(i,j)}_{\mu(\nu)} \gamma^{-\mu(\nu)} \hat{G}_{\mu(\nu)}(x) / \hat{G}_{\mu(\nu')}(x)$ $= O\{G_{\mu(\nu')}(x)\}$ as $x (\in M_1) \to x'$, and hence $F_{i,j}(M) \sim \cancel{\mathfrak{A}}^{(i,j)}\{h^{(i,j)}, \mathfrak{g}(M) : x'\}$.

(vii) We have $\mathfrak{D}^\tau_t G(t, x \equiv \mathbb{D}) \in A_x$ for $t \equiv \mathbb{N}\{\mathfrak{t}\{P_{i,j}\}\}$ and $\mathfrak{g}(\mathbb{D}) \in A$ if $j \geq i$. All constituents (finite in number) of $F_{i,j}(x)$ are analytic functions of x for $x \equiv \mathbb{D}$, and the same is true of $F_{i,j}(x)$ itself.

(viii) That $F_{i,j}(M) \in W$ $(F_{i,j} \equiv \mathbb{F}_s)$ follows from (i)a). Should values of $A^{(i,j)}_{\nu,\tau}$ and $t^{(i,j)}_\nu$ with nonzero imaginary parts occur in formula ( ), they do so in complex conjugate pairs when h is real. Furthermore, subject to the stated conditions upon $G(t,x)$, complex conjugate values of $t_\nu^{(i,j)}$ induce complex conjugate values of $G(\tau; t_\nu^{(i,j)}, x)$.

(ix) a) See the proof of 2.(vi)a).

b) The proof of this result is similar to that of 2(vi)b. It needs only to be remarked that in that case the operation of L upon a formal power series was defined in such a way as to make the distributive property of L unnecessary. Now L operates upon function consistently, and this property is required to establish the formula $\tilde{F}_{i,j}(m) = L\,F_{i,j}(\tilde{\Delta}, m)$.

B. The results of this part of the theorem follow directly from the corresponding results of the first part. (It is merely necessary to remark, in the case of clause B.(i)c), that if $g(\mathbb{N}\{0\}) \in A$, then $g(tx)$ $(t \in \mathbb{R})$ is also regular at the origin and has the series expansion $\sum g_\nu t^\nu x^\nu$ there.

In later theorems we define a function $F(x)$ related to further functions considered, and it will then transpire that this function then reduces to the forms given in clause 1. A. .. and ... under further conditions equivalent to those of these clauses. Furthermore it will also be shown that under certain conditions $F(M) \sim \{h, g : x'\}$, so that clauses 1. A. .. and ... may be applied.

We wish to point out that two functions can be represented by the same asymptotic series without necessarily being ~~exact~~ identical. Hence the result of 4.A.( )) that $E_{i,j}(x) = o\{G_{\mu(\nu)}(x)\}$ as $x(\in M) \to x'$ for arbitrarily large ν does not imply that $F(x) = F_{i,j}(x)$. In this sense the results of 4.A.( )) and 2... differ; for in the latter ~~case~~ clause, in which formal power series are considered, agreement between all corresponding coefficients means agreement between the power series $f(x)$ and $f_{i,j}(x)$.

In Theorem 2, the variable t may be given a fixed numerical value, unity for example; the polynomi $\mathfrak{F}'_\nu$ and $\mathfrak{F}^{(i,j)}_\nu$ then become numbers and the double series $\mathfrak{F}(x)$ and $\mathfrak{F}_{i,j}(x)$ become simple formal power series expressible as $\Sigma h_\nu g_\nu(x)$ and $\Sigma h^{(i,j)}_\nu g_\nu(x)$ respectively. If the functions $G_\nu(x)$ ($\nu \in \underline{I}$) of the asymptotic sequence considered in 4.A)(iii) are such that the limit point x' is zero and $G_{\mu(\nu)}(x) = O\{x^{\Gamma\{\mu(\nu)\}}\}$ as $x \to 0$ $(\nu = \in \mathbb{I}_0^\infty)$ where Γ is as defined in Theorem 3, then there is an isomorphism between that part of the array of formal power series of Theorem 3 ~~and~~ corresponding to $\mathbb{P}_S$ and $\mathbb{F}_S$. For $F_{i,j} \in \mathbb{F}_S$, $\mathbb{F}_S$ has an asymptotic power series in x with coefficients $\mathfrak{F}_\nu^{(i,j)}$. When $g(\mathbb{N}\{0\}) \in A$ and $g \in \mathfrak{g}$ there is of course another isomorphism between the power series $f_{i,j}(x)$ of the array $\mathfrak{F}$ of Theorem 2 and the functions $f_{i,j}(x)$ of the array $\mathfrak{f}$ of 4.B)(c). In this case $f_{i,j} \leftarrow f_{i,j}$ $(i,j = \mathbb{I})$.

As with Theorem 3, the structure of $\mathbb{F}$ is simplified by introducing constraints upon $G(t,x)$ and the $G_\nu(x)$. For example, let $G(t,x)$ be an entire function of t $(x \in M)$, $G \leftarrow \mathfrak{G}(M)$, $G_\nu(M) \neq 0$ $(\nu = \mathbb{I})$ and $\mathfrak{G}(M) \sim x'$;

then identical mappings in $\mathbb{F}(m)$ occur only in square blocks of the form $\mathbb{F}_{I,J}[\,N]$, and $\mathbb{F}_{I,J}[M,N] \leq \mathbb{F}(m)$ if and only if $H_{I,J}[N] \leq H$; $F_{I,J}[M,\infty] \leq \mathbb{F}(m)$ if and only if $h_{I,J} \leftarrow h$, and any $\mathbb{F}_{i,j}(m) \notin \mathbb{F}_{I,J}[M,N]$ then differs from the mapping $F(m)$ of formula ( ). If a function $F(m)$ exists for which $F(m) \sim \{h, g; x'\}$, then $E_{i,j}(x) = O\{G_{\frac{1}{3}(i,j)}(x): M, x'\}$ $(i, j \equiv \underline{I})$; if $H_{i,j}$, $H_{i',j'}$ satisfy the conditions of A(iv)b) then, in the notation of that clause, if $I+J+N \neq I'+J'+N'$, $E_{i,j,i',j'}(x) = O\{G_{\mathfrak{z}'}(x); M: x'\}$, and if $I+J+N = I'+J'+N' = \mathfrak{z}'-1$, $r' = \max(I'+J, I+J')$, there exists an $s \in \underline{I}_{\mathfrak{z}'}^{\mathfrak{z}'+r'-1}$ for which $E_{i,j,i',j'}(x) = O\{G_s(x); M; x'\}$ (in the first two relationships involving O, this symbol may not be replaced by o; with regard to the last, it is not true that $E_{i,j,i',j'}(x) = o\{G_\nu(x); M : x'\}$). Again, if $g\{N;0\}$ $\in A$ with $\underset{\circ}{g} \leftarrow g$, a function f exists for which $f \leftarrow gh$ and $g_\nu \neq 0$ $(\nu = \underline{\mathrm{I}})$, all the remarks made in Theorem 2 concerning the case in which $g_\nu \neq 0$ $(\nu = \underline{\mathrm{I}})$ may be reformulated in terms of the functions of f. If the sequence $\mathcal{G}(M)$ possesses gaps, however, anomalous behaviour analogous to that described in connection with Theorems 2 and 3 can occur.

As for the result of 2(vi)b) those of A.(ix) b) and B.(viii) b) of the above theorem can be used to establish that a set of functions $F_{i,j}(x)$ derived from convergent integrals of members of the sequence $G_\nu(y,x)$ $(\nu = \underline{\mathrm{I}})$ and of the function $G(y,t,x)$ are well determined; the same can be done with regard to convergent sums of values of these functions.

Functions related to the exponential function

Notation. We set

$$h(z:a,v;I,J;u,t;\phi)=ae^{vz}\{\Pi_1^J(1+u_\nu z)\}/\{\Pi_1^I(1-t_\nu z)\},$$

where $I,J\in\bar{\mathbb{I}}$ and $a(\neq 0),v,u_\nu(\neq 0,\nu=\mathbb{I}_1^J),t_\nu(\neq 0,\nu=\mathbb{I}_1^I)$ $\in\mathbb{Z}$, $\Sigma_1^J|u_\nu|,\Sigma_1^I|t_\nu|<\infty$ with $\phi\in[0,2\pi)$, $\arg(v)=\phi$ if $v\neq 0$ and $\arg(u_\nu)=\phi$ $(\nu=\mathbb{I}_1^J)$ if $J\neq 0$ and $\arg(t_\nu)=\phi$ $(\nu=\mathbb{I}_1^I)$ if $I=0$.

The integral expression in formula ( ) is denoted by $F(x:\mathbb{C};G,h)$.

$\mathbb{F}_s(M)\Rightarrow F(M)$ means that any infinite sequence of functions $F_{i,j}(x)$ belonging to the infinite subset $\mathbb{F}_s$ with distinct suffix sets (i,j) converges to $F(x)$ for $x\in M$. $\mathbb{T}\{\mathbb{F}_s\}$ is the tail of any infinite sequence of functions $F_{i,j}\in\mathbb{F}_s$ with distinct suffix sets $\{i,j\}$, i.e. that part of such an infinite sequence obtained by the removal of at most a terminating initial subsequence.

$f(x:\mathbb{C};g,h)$ is $F(x:\mathbb{C};G,h)$ with $G(t,x)=g(tx)$.

The symbols $\mathfrak{f}_s(\Delta)\to f(\Delta)$ and $\Pi\{\mathfrak{f}_s\}$ have meanings analogous to those given above.

Theorem 4. A. Let $h(z)=h(z:a,v;\theta,J;u,t:\phi), h\in\mathfrak{h}$. Set $F(x)=F(x:\mathbb{C};G,h)$ where $\mathbb{C}$ is a circle with centre at the origin, lying in $N\{0\}$ and is described in an anti-clockwise direction.

a) Let $x\{\mathfrak{h}(m)\}=\mu_{\tau'}$. If $v=0$, let τ be the maximum value of $\nu\in I_0^{\tau'}$ for which $\mu(\nu)\leq J$; otherwise, set $\tau=\tau'$. Then $x\{h\mathfrak{h}(m)\}=\mu_\tau$.

b)(1) $\sum h_\nu G_\nu(m)\to F(m)$

(2) If $\mathfrak{h}(m)\sim x'$, then $F(m)\sim\{h,\mathfrak{h}:x'\}$.

c) Let $v=0$, $J \in \mathbb{I}$. Set $h_A(z) = \sum_0^J h_\nu z^\nu$. Then $F(M) = \sum_0^J h_\nu G_\nu(m)$ and $F_{0,J}[M,\infty] = F(m)$

d) Let $G(t,m)$ be the polynomial $\sum_0^\mu G_\nu(m) t^\nu$. Then $F(m) = \sum_0^\mu h_\nu G_\nu(m)$ and $F_{i,j}(m) = F(m)$ $(F_{i,j} \equiv \mathbb{F} - \mathbb{F}^\mu)$

e) $\mathbb{F}(m) \to F(m)$

f) $\sum h_\nu^{(i,j)} G_\nu(m) \to F_{i,j}(m)$ $(F_{i,j} \equiv \mathbb{T}\{\mathbb{F}\})$

g) Let $y(m) \sim x'$. Then $F_{i,j}(m) \infty \{h_{i,j}, y : x'\}$ $(F_{i,j} \equiv \mathbb{T}\{\mathbb{F}\})$

h) Let $x\{y(m)\} = \mu_\tau$ and assume that the conditions of A.(i)c) do not hold <do hold>.

(1) $E_{i,j}(x) = \cup\{G_{\mu \geq \lambda(i+j+1)}(x); M : x'\}$ $(F_{i,j} \equiv \mathbb{T}\{\mathbb{F}^{\mu(\tau)}\}$ $\langle \mathbb{T}\{\mathbb{F}^{\mu(\tau)} - F_{0,J}[\infty]\rangle$

(2) Let $F_{i,j}, F_{i',j'} \in \mathbb{T}\{\mathbb{F}^{\mu(\tau)}\}$ $\langle \mathbb{T}\{\mathbb{F}^{\mu(\tau)} - F_{0,J}[\infty]\rangle$, with $\xi' = \min(i+j+1, i'+j'+1) \leq \mu(\tau)$. If $i+j = i'+j'$,

then $E_{i,j,i',j'}(x) = O\{G_{\mu}\{\lambda(j')\}(x); M : x'\}$ Let $i+j = i'+j'$.

If $i' > i$ 〚$i > i'$〛 set $r'' = i'+j$ 〚$i+j'$〛 and let $r'' \leq \mu(\tau)$

Then there exists an $\nu \in \mathbb{I}_{j''}^{r''}$ for which $E_{i,j,i',j'}(x)$

$= O\{G_{\nu}(x); M : x'\}$.

(3) If $\mu(\nu) = \nu$ ($\nu \equiv \mathbb{I}_0^{\tau}$), then the order relationships involving the symbol O in (1) and the first result of (2) do not hold with O replaced by o. Furthermore it is not true that $E_{i,j,i',j'}(x) = o\{G_{\nu}(x); M : x'\}$ ($\nu \equiv \mathbb{I}_{j'}^{r''}$)

(i) Let $y(M) \sim x'$ and $G_{\nu}(M) \neq 0$ ($\nu \equiv \mathbb{I}_0^{\mu}$) and assume that the conditions of A(i)c) do not hold <do hold>

(1) $F_{i,j}(M) \neq F(M)$ ($F_{i,j} \equiv \mathbb{F}^{\mu}$ <$\mathbb{F}^{\mu} - F_{0,J}[\infty]$>)

(2) Let $F_{i,j}, F_{i',j'} \in \mathbb{F}^{\mu}$ <$F^{\mu} - F_{0,J}[\infty]$>. If $i+j \neq i'+j'$, then $F_{i,j}(M) \neq F_{i',j'}(M)$

(j) Let $G(\mathbb{N}\{0\}, M) \in \check{Y}$, and assume that the condition of $A(i)c)$ do not hold <do hold>. Then all $F_{i,j}(M)$ are distinct and differ from $F(M)$ $(F_{i,j} = \mathbb{T}\{\mathbb{F}\} \Leftrightarrow < \mathbb{T}\{\mathbb{F}\} - \mathbb{F}_{0,J}[\infty] >)$.

j) Let $G(t = \mathbb{N}\{c\}, x = \tilde{\mathbb{D}}) \in A_{t,x}$ with $G \leftarrow g(\tilde{\mathbb{D}})$. (1) $F(\hat{\mathbb{D}}) \in A$.

(2) Assume that the conditions of $A(i)c)$ do not hold <do hold>. Then $F_{i,j}(\hat{\mathbb{D}}) \in A$ $(F_{i,j} = \mathbb{T}\{\mathbb{F}\} < \mathbb{T}\{\mathbb{F}\} \cup \mathbb{F}_{0,J}[\infty] >)$

(ii) Let $g(\mathbb{N}\{0\}) \in A$, with $g \leftarrow g$. Set $f(x) = f(x; \mathbb{C}; g, h)$ where $\mathbb{C}$ is as in $A(i)$.

a) f is an entire function, and $f \leftarrow hg$.

b) Let the conditions of $A.(i)c)$ hold. Then $f(\mathbb{B}) = \sum_0^J h_\nu g_\nu \mathbb{B}^\nu$ and $f_{0,J}[B, \infty] = f(\mathbb{B})$

c) Let $g(x)$ be the polynomial $\sum_0^\mu g_\nu x^\nu$. Then $f(\mathbb{B}) = \sum_0^\mu h_\nu g_\nu \mathbb{B}^\nu$ and $f_{i,j}(\mathbb{B}) = f(\mathbb{B})$ $(f_{i,j} = f - f^\mu)$

d) $\mathbb{f}(\mathbb{B}) \to f(\mathbb{B})$

e) $f_{i,j}(x)$ is an entire function, and $f_{i,j} \leftarrow h^{(i,j)} g$ $(f_{i,j} = \mathbb{T}\{\mathfrak{f}\}.)$

f) Let $\chi\{g\} = \mu_\pi$ and assume that the conditions of A(i) c) do not hold <hold>. Then with relationships such as $E_{i,j}(x) = O\{G_{\mu\{\lambda(i+j+1)\}}(x); M: x'\}$ replaced by $e_{i,j}(x) = O\{x^{\mu\{\lambda(i+j+1)\}}\}$ and the symbol $\mathbb{F}$ replaced by $\mathfrak{f}$, the results of A(i) h) ~~hold~~ (1-3) hold.

g) Let $g_\nu \neq 0$ $(\nu \in \bar{I}_0^{\mu})$ and assume that the conditions of A(i)c) do not hold <hold>. Then with the symbols $F, \mathbb{F}, M$ replaced by $f, \mathfrak{f}, \mathbb{B}$, the results of A(i)(1,2) hold.

h) Let $g(N\{0\}, \hat{\mathbb{D}}) \in \bar{\mathcal{Y}}$ for a $\hat{\mathbb{D}} \subseteq \mathbb{B}$, and assume that the conditions of A(i)c) do not hold <hold>. Then all $f_{i,j}(\mathbb{B})$ are distinct and differ from $f(\mathbb{B})$ $(f_{i,j} \equiv \mathbb{T}\{\mathfrak{f}\}_j \langle T\{\mathfrak{f}\} - \mathfrak{f}_{0,J}[\infty J] \rangle)$.

B) Let $h(z) = h(z, a, v; I, J; w, t; \phi)$ where $I \in \bar{\mathbb{I}}_1$. Let $h \leftarrow h$. Set $\hat{\mathbb{D}} = N\{0\} \cup N\{t_\nu^{-1}\}$ $(\nu \in \bar{\mathbb{I}}_1^{I})$ and $T^{-1} \in (0,\infty)$

$> |t_\nu| \ (\nu = \overline{\underline{I}}_1^I)$

i) Let $G(t \in \acute{\overline{\mathbb{D}}}, m) \in A_t$, $G(t \in \mathbb{D}_t, m' \subseteq m) \in A_t$. Set $F(x) = F(x : \mathbb{C}; G, h)$ where $\mathcal{C}$ is now a system of $I+1$ circles with centres at $0, t_\nu \ (\nu = \overline{\underline{I}}_1^I)$ and lying in $\acute{\overline{\mathbb{D}}}$

a) Let $\varkappa\{\mathcal{G}(M < m' >)\} \mathcal{B} = \mu_\tau$. Then $\varkappa\{h\mathcal{G}(M < m' >)\}$ also

b) (1) $\sum h_\nu G_\nu(m') \to F(m')$

(2) Let $\mathcal{G}(m') \sim x'$. Then $F(m') \sim \{h, \mathcal{G} : x'\}$

(3) Let $G \sim [\acute{\overline{\mathbb{D}}}, \mathcal{G}(m) : x']$. Then $F(m) \sim \{h, \mathcal{G}; x'\}$.

c) Let $v < 0$, $I \in \overline{\mathbb{I}}_1$, $j \in \overline{\mathbb{I}}$. Set, in this case,

$$h_B(z) = \sum_0^{J-I} \alpha_\nu z^\nu + \sum_1^I A_\nu / (1 - t_\nu z)$$

Then

$$F(m) = \sum_0^{J-I} \alpha_\nu G_\nu(m) + \sum_1^I A_\nu G(t_\nu ; m)$$

and $\mathbb{F}_{I,J}[M, \infty] = F(m)$

d) Mutatis mutandis, the results of A(i)d) hold.

e) $\mathbb{F}(m) \to F(m)$

f) $\sum h_\nu^{(i,j)} G_\nu(M') \to F(M')$ $(F_{i,j} \equiv \mathbb{T}\{\mathbb{F}\})$

g) Replacing $\mathbb{N}\{0\}$ and $\mathbb{F}_{0,J}$ by $\acute{\mathbb{D}}$ and $\mathbb{F}_{J,J}$, the results of A.(i) f)–i) hold

h) (1) Let $y(M') \sim x'$. Then $F_{i,j}(M') \sim \{h_{i,j}, y : x'\}$ $(F_{i,j} \equiv \mathbb{T}\{\mathbb{F}\})$.

(2) Let $G \sim [\acute{\mathbb{D}}, y(M) : x']$. Then $F_{i,j}(M) \sim \{h_{i,j} : y : x'\}$ $(F_{i,j} \equiv \mathbb{T}\{\mathbb{F}\})$

i) Let $x\{y(M')\} \sim x'$ [$G \sim [\acute{\mathbb{D}}, y(M) : x']$] and assume that the conditions of A.(ii)c) do not hold <hold>. Then, replacing $M, \mathbb{F}_{0,J}$ [$\mathbb{F}_{0,J}$] by M', $\mathbb{F}_{J,J}$ [$F_{J,J}$] the results of A.(i) i) hold

j) Let $G(\acute{\mathbb{D}}, M) \in \bar{\Upsilon}$ and assume that the conditions of P.(i)c) do not hold <hold>. Then all $F_{i,j}(M)$ are distinct and differ from $F(M)$ $(F_{i,j} \equiv \mathbb{T}\{\mathbb{F}\} < \mathbb{T}\{\mathbb{F}\} - \mathbb{F}_{J,J}[\infty]>)$.

(ii) Let $g(\mathbb{D}[\omega]) \in A$ with $\omega(\theta) > \Omega \in (0,\infty)$ $(\theta \in [0, 2\pi])$.

and $g \leftarrow g$; set $\hat{\omega}(\theta) = T\omega(\theta+\phi)$ $(\theta = [0, 2\pi])$. Set $f(x) = f(x : \mathcal{C}; g, h)$ where $\mathcal{C}$ is as in B.(i). Set $\mathbb{D} = \mathcal{B}\{\mathbb{D}[\hat{\omega}]\}$

a) $f(\mathbb{D}[\hat{\omega}]) \in A$, $f \leftarrow hg^{?}$ and $\sum_i h_\nu g_\nu (\mathbb{D}_{T,\Omega})^{?} \to f(\mathbb{D}_{T,\Omega})$

b) Let the conditions of B(i) c) hold. Then

$$f(\mathbb{D}) = \sum_0^{J-I} \alpha_\nu g_\nu \mathbb{D}^\nu + \sum_1^n \sum_0^{n_\nu - 1} A_{\nu,\kappa} g(b_\nu \mathbb{D})$$

and $\oint_{I,J}[\mathbb{D}, \infty] = f(\mathbb{D})$

c) Mutatis mutandis, the results of A(ii) c) hold

d) $\oint(\mathbb{D}) \to f(\mathbb{D})$

e) $f_{i,j}(\mathbb{D}) \in A$, $f_{i,j} \leftarrow h^{(i,j)} g$ and $\sum_i h_\nu^{(i,j)} g_\nu (\mathbb{D}_{T,\Omega})^{?} \to f_{i,j}(\mathbb{D}_{T,\Omega})$ $f_{i,j} \in \Pi\{\oint\}$.

f) Let $\varkappa\{g\} = \mu_\pi$ and assume that the conditions of B(i) c) do not hold <hold>. Then with relationships such as $E_{i,j}(x) = O\{G_{\mu\{\lambda(i,j+1)\}}(x) : M : x'\}$ replaced by $e_{i,j}(x) = O\{x^{\mu\{\lambda(i,j+1)\}}\}$ and the symbols $\mathbb{F}, \mathbb{F}_{0,J}$ replaced by

$\mathfrak{f}$, $\mathfrak{f}_{I,J}$ the results of A (i) (1-3) hold.

g) Let $g_\nu \neq 0$ ($\nu = \overline{I}_0^k$) and assume that the condition of B (i) c) do not hold <hold>. Then with the symbols F, $\mathbb{F}$, $\mathbb{F}_{0,J}$ and M replaced by f, $\mathfrak{f}$, $\mathfrak{f}_{J,J}$ and $\mathbb{D}$ the results of A(i) i) (1,2) hold

h) Let $g(\acute{\mathbb{D}}, \tilde{\mathbb{D}}) \in \overline{\mathcal{Y}}$ for a $\tilde{\mathbb{D}} \subseteq \mathbb{D}$, and assume that the conditions of B (i) c) do not hold <hold>. Then all $f_{i,j}(\mathbb{D})$ are distinct and differ from $f(\mathbb{D})$ ($f_{i,j} = \Pi\{\mathfrak{f}\}_j$ < $\Pi\{\mathfrak{f}\}_j - \mathfrak{f}_{I,J}\{\infty J\}$>.

Proof A.(i)a) If $v=0$, $h_\nu > 0$ ($\nu = \overline{I}_0^J$); otherwise $h_\nu > 0$ ($\nu = \overline{I}$). The sequence $h_\nu G_\nu(M)$ ($\nu = \overline{I}_0^{\mu(\kappa)}$) has gaps only where $G_\nu(m)$ ($\nu = \overline{I}_0^{\mu(\kappa)}$) has them.

b)(1) $h_A(z)$ is an entire function, and is therefore bounded for $|z| = \delta_1^{-1}$ where $\delta_1 \in (0,\infty)$ is the radius of $\mathcal{B}$.

$G(z,x)$ is analytic and therefore bounded within and upon $\mathscr{C}$ $(x \equiv M)$. The integral ( ), subject to the stated modification therefore exists. Expanding $h_A(z^{-1})$ in powers of z^{-1} upon $\mathscr{C}$, we have $F(x) = \sum h_\nu G_\nu(x)$ for $x \equiv M$. (2) the method of proof of 4. ... holds for this result also.

c) The stated results follow from the definition of $F(x)$, and from 4. ..

d) If the conditions of c) do not hold <hold> the [ ] $H_{i,j} \neq 0$ $(i,j \equiv \underline{I})$ $\langle i \equiv \underline{I}, j \equiv \underline{I}_0^{J} \rangle$. We may now use c) and 4.

e) It has been shown [ ] that, using a notation similar to that employed by us with regard to the array F, $P(B) \to h_A(B)$ when $\phi = 0$. A simple change of variable leads to a similar result holding when $\phi \neq 0$. Taking B to be $D_{1/\delta'}$, where δ' is again the radius of $\mathscr{C}$,

$P_{i,j}(z)$ is devoid of poles in $\mathbb{D}_{1/\delta'}$ for all $P_{i,j}(\mathbb{D}_{1/\delta'}) \in \mathbb{T}\{\mathbb{P}(\mathbb{D}_{1/\delta'})\}$, i.e. for all such $P_{i,j}$, $t\{P_{i,j}\} \in \mathbb{D}_{\delta'}$. The integral obtained from ( ) by replacing $h_A(z^{-1})$ by $P_{i,j}(z^{-1})$ with $\mathcal{C}$ as described exists for all such $P_{i,j}$ and, from formulae (,), this integral represents $F_{i,j}(x)$.

Since $\mathbb{P}(\mathbb{D}_{1/\delta''}) \to h_A(\mathbb{D}_{1/\delta''})$ for some $\delta'' \in (0,\delta')$, $|h_A(z^{-1}) - P_{i,j}(z^{-1}) \le \delta_1$ for all $P_{i,j}(\mathbb{D}_{1/\delta''}) \in \mathbb{T}\{\mathbb{P}(\mathbb{D}_{1/\delta''})\}$ $G(z,x)$ is bounded for all z on $\mathcal{C}$ $(x \in M)$. The length of $\mathcal{C}$ is finite. Hence $|F(x) - F_{i,j}(x)| \le \delta_2$ $(x \in M)$ for all $F_{i,j}(x)$ corresponding to $P_{i,j}(x)$ and $\mathbb{F}(M) \to F(M)$ as stated.

~~e) The sta~~

f) For any $F_{i,j} \in \mathbb{T}\{\mathbb{F}\}$, as has been shown above, $t\{P_{i,j}\} \in \mathbb{D}_{\delta'}$. We may now use 4....

g) This result follows from f) and 4...

h) The results of this clause follow by noting that $H_{i,j} \neq 0$ $(i,j = \mathbb{I})$ $\langle i = \mathbb{I},\ j = \mathbb{I}_0^J \rangle$ [ ] (so that (1...)) $\xi(i,j) = \xi(i+j+1)$ $(i, j = \mathbb{I},\ j = \mathbb{I}_0^{J-1} \rangle \rangle)$ and using 4.

i) These results follow, as for those of the preceding clause, from 4...

j) It has been shown (see the proof of e) above) that $\bar{\pi}\{P_{i,j}\} \in \mathbb{N}\{0\}$ for $F_{i,j} = \mathbb{T}\{F\}$ $\langle \mathbb{T}\{F\} - F_{0,J}\ [\infty] \rangle$ We may now use 4.

The results of clause A(ii) are special cases of those of A.(i).

The results of part B. are derived by an extension of the methods used above, it being merely necessary to remark (1) that now $H_{i,j} \neq 0$ for $i,j = \mathbb{I}$ when the conditions of B)(ii)c) do not hold, and for $i = \mathbb{I}, j = \mathbb{I}_0^J$

and $i \equiv \underline{\mathrm{I}}_0^{I}$, $j \equiv \underline{\mathrm{I}}_{J+1}$ when they do, and (2) that, using a similar notation to that used in the proof of A(i)c), when $\phi = 0$, $\mathbb{P}\{\mathbb{D}\} \to h_B(\mathbb{D})$ where $\mathbb{D} \equiv \mathbb{B}\{\mathbb{E}\{t_{\mathcal{D}}^{-1}(\nu \equiv \underline{\mathrm{I}}_1^{J})\}\}\{$:

Convergence theory

Notation. $h \in DS$ [DHN] ~~($\nu \in I$)~~ means that the Stieltjes moment problem over the interval $[0,\infty]$, i.e. the derivation of a function $\sigma(t)$ which is nondecreasing and real valued for $t \in [0,\infty]$ and satisfies conditions ( ) [the Hamburger-Nevanlinna moment problem] associated with the sequence h_ν ($\nu \in I$) is determinate.

If $h = ms(\sigma \in BND[\alpha,\beta])$ ($[\alpha,\beta] \subseteq [0,\infty]$) $[[-\infty,\infty]]$ and $|d\sigma(t)| = O(\exp(-\eta |t|^{1/2}))$ [$O\exp(-\eta|t|)$] ($\eta \in (0,\infty)$) for large values of $|t|$ in $[\alpha,\beta]$, then $h \in DS$ [DH][ ] (this includes the case in which $[\alpha,\beta]$ is a suitably situated finite interval). Furthermore, if $h = ms(\sigma \in BND[\alpha,\beta])$ ($[\alpha,\beta] \subseteq [0,\infty]$ $[[-\infty,\infty]]$) and the series $\sum_1 |h_\nu|^{-1/2\nu}$ [$\sum_1 |h_{2\nu}|^{-1/2\nu}$] diverges, then $h \in DS$ [DH][ ].

Theorem. Let $h = ms\{\sigma \in BND[\alpha,\beta]\}$ and set $F(x) = F[x:\sigma, G; \alpha, \beta]$

A.(i) Let $\alpha \in [0,\infty)$, $\beta = \infty$, $G(t \in [\alpha,\beta), m) \in \mathbb{C}_t$ and $F(m) \in W$. As $t \to \infty$ let $G(t,x) = O\{G(t)\}$ $(x \in m)$ where $t_0 \in [0,\infty)$ exists such that either a) $G(t) \geq 0$ $(t \in [t_0,\infty))$ and $\mathfrak{D}_t^{2r} G(t) \geq 0$ $(r \in \mathbb{I}_{r'})$ for $t \in [t_0,\infty)$ or b) with $\tilde{G}(t) = \{G(t) - G(t_0)\}/(t - t_0)$, $\tilde{G}(t) \geq 0$ $(t \in [c,\infty))$ and $\mathfrak{D}_t^{2r} \tilde{G}(t) \geq 0$ for $t \in [c,\infty)$ where $c \in [0,\infty)$. Let $g(m) \in V$ and, if $\alpha = 0$, $G \sim g(m)$. Let $h_{m'} \in \mathbb{D}S$. Then $\mathfrak{U}_{m'}(m) \to F(m)$. If c) $G(t) \geq 0$ $(t \in [t_0,\infty))$ and $\mathfrak{D}_t^{r} G(t) \geq 0$ $(r \in \mathbb{I}_{r'},\ t \in [t_0,\infty))$, then $\mathfrak{U}_{m'+1}(m) \to F(m)$.

(ii) Let $[\alpha,\beta] \subseteq [0,\infty)$, $G(t \in (\alpha,\beta), m) \in \mathbb{C}_t$, $F(m) \in W$, $g(m) \in V$ and, if $\alpha = 0$, $G \sim g(m)$. Then $\mathfrak{U}'(m) \to F(m)$

(iii) Let $\alpha \in [-\infty,0)$, $\beta = \infty$, $G^{\pm} \sim g(m)$ and $h_{2m'} \in \mathbb{D}H$

a) Let $G(t \in (\alpha,\beta), m) \in \mathbb{C}_t$. If $\alpha > -\infty$, let $|G(\alpha,x)| <$

∞ $(x \equiv M)$. Let $G(t,x) = O\{G(t)\}$ $(x \equiv M)$ as $t \to \infty$ in $[\alpha,\infty]$, where $\mathfrak{D}_t^{2\tau} G(t) \geq 0$ $(\tau \equiv \mathbb{I}_{\tau'})$ for $t \equiv [-\infty,\infty]$. Let $F(M) \in W$. Then $\overline{\mathbb{O}}_{2m'}(m) \to F(m)$. If, in addition, $\sigma \in \mathcal{S}y$, then $\overline{\mathbb{D}}_{2m'+1}(m) \to F(m)$

b) Let $G(t \equiv [-\infty,\infty], M) \in \mathbb{C}_t$ and let $G(t,x)$ tend to finite limits as $|t| \to \infty$ $(x \equiv M)$. Then $\mathbb{D}_{2m'+1}(m) \to F(m)$

(iv) Let $[\alpha,\beta] < (-\infty,\infty)$ and $G^{\pm} \sim y(m)$

a) Let $G(t \equiv (\alpha,\beta), M) \in \mathbb{C}_t$ and $F(M) \in W$. Then $\overline{\mathbb{O}}'(m) \to F(m)$

b) Let $G(t \equiv (-\infty,\infty), m) \in \mathbb{C}_t$ and let $G(t,x)$ tend to finite limits as $t \to \pm\infty$ $(x \equiv m)$. Then $\mathbb{D}'(M) \to F(m)$

B.(i) Let $\alpha \in [0,\infty)$, $\beta = \infty$ and $G(z \equiv N\{[\alpha,\infty]\}, M) \in A_2$

With

$$h_{r'} = \{h_{r+1} - \Sigma_0^{r-1} h_{r-\nu} \tilde{h}_\nu\}/h_0 \qquad (r \equiv \mathbb{I})$$

let $\tilde{h}^*_{m'} \in \mathbb{D}S$. Then $\overline{\mathbb{I}}_{m'}(m) \to F(m)$

(ii) Let $[\alpha,\beta] \subset [0,\infty)$ and $G(z = \mathbb{N}\{[\alpha,\beta]\}, m) \in A_z$. Then $\mathbb{L}i(m) \to F(m)$

(iii) Let $\alpha \in [-\infty,\infty)$, $\beta = \infty$. With $\hat{h}_r = \{h_0^{-1} h_1 h_{r+1} - h_{r+2} - \sum_0^{r-1} h_{r-\nu} \hat{h}_\nu\}/h_0 \quad (r \in I)$ let $\hat{h}_{2m'} \in \mathbb{D}H$.

a) Let $G(z = \tfrac{2}{8}\mathbb{N}\{[\alpha,\infty]\}, m) \in A_z$. Then $\mathbb{R}_{2m'}(m) \to F(m)$ and, if $\sigma \in \mathcal{S}y$, $\overline{\overline{\mathbb{L}}}_{2m'}(m) \to F(m)$.

b) Let $G(z = \mathbb{N}\{[-\infty,\infty]\}, m) \in A_z$. Then $\mathbb{L}_{2m'+1}(m) \to F(m)$

(iv) Let $[\alpha,\beta] < (-\infty,\infty)$

a) Let $G(z = \mathbb{N}\{[\alpha,\beta]\}, m) \in A_z$. Then $\mathbb{R}'(m) \to F(m)$ and, if $\sigma \in \mathcal{S}y$, $\mathbb{L}'(m) \to F(m)$

b) Let $G(z = \mathbb{N}\{[-\infty,\infty]\}, m) \in A_z$. Then $\mathbb{L}'(m) \to F(m)$

c) Let $\sigma \notin St$ <$\sigma = st\{M_\nu, b_\nu; n\}$ with $\hat{\mathbb{F}}$ as in 6. A(iii)a)>

(i) Let $|\alpha|, |\beta| < \gamma \in (0,\infty)$, $G(z = D_\gamma, m) \in A_z$ and $G \leftarrow \mathfrak{g}(m)$

a) Let $[\alpha,\beta] \subseteq [0,\infty)$ or $\sigma \in \mathcal{S}y$. Then (1) $\sum h_\nu^{(i,j)} G_\nu(m) \to F(m)$ and (2) if, in addition, $\mathfrak{g}(m) \sim x'$, then $F_{i,j}(m) \sim \{h^{(i,j)}; \mathfrak{g} : x'\}$ $(F_{i,j} = \pi\{\mathbb{F}'\})$ <$\pi\{\mathbb{F}'\} \cup \hat{\mathbb{F}}$>

b) Let $0 \in (\alpha,\beta)$. Then the results (1,2) i) a) hold with $\mathbb{F}'$ replaced by $\mathbb{D} \cup \mathbb{R}'$.

~~(ii) Let $[\alpha,\beta] \subseteq [0,\infty)$.~~

(ii) Let $G(z = \mathbb{N}\{[\alpha,\beta]\}, m) \in A_z$ and $G \sim [\mathbb{N}\{[\alpha,\beta]\}; g(m): x']$

a) Let $[\alpha,\beta] \subseteq [0,\infty)$. Either (1) in the notation of formula ( ) let $\tilde{h}_{m'} \in \mathbb{D}S$ or (2) let $\beta < \infty$ and set $m' \in \mathbb{I}$ arbitrarily large. Then $F_{i,j}(m) \sim \{h_{i,j}; g(m): x'\}$ ($F_{i,j} = \mathbb{T}\{L_{m'}\} < \mathbb{T}\{L_{m'}\} \cup \hat{\mathbb{F}} >$).

b) Let $0 \in (\alpha,\beta)$. Either (1) in the notation of formula ( ) let $\hat{h}_{2m'} \in \mathbb{D}H$ or (2) let $|\alpha|, |\beta| < \infty$ and set $m' \in \mathbb{I}$ arbitrarily large. Then the results of a) hold with $L_{m'}$ replaced by $R_{2m'}$. If, in addition $G \sim [\mathbb{N}\{[-\infty,\infty]\},$
? $g(m): x']$, then the results of a) hold with $L_{m'}$ replaced by $L_{2m'+1}$

c) Let $\sigma \in S_y$ and define m' as in b). Then the results of a) hold with $L_{m'}$ replaced by $\overset{?}{T}_{2m'}$

(iii) Let $G(z = \mathbb{N}\{[\alpha,\beta]\}, M) \in A_z$, $G \sim [\mathbb{N}\{[\alpha,\beta]\}; \mathfrak{h}(M) : x']$

and $x\{\mathfrak{h}(M)\} = \mu_\tau$

a) Let $[\alpha,\beta] \subseteq [0,\infty]$. Define m' as in (ii)a). Then the results of 6.A.(vi)a), (vii)a) and (viii)a) hold with $\mathbb{L}^{\mu(\tau)}$ replaced by $\pi\{\mathbb{L}^{\mu(\tau)}_{m'}\}$.

b) Let $0 \in (\alpha,\beta)$. Define m' as in (ii)b). Then the results of 6.A (vi)c), (vii)c), (viii)c) and (xi)b,c) hold with $\bar{\mathbb{O}}^{\mu(\tau)}$ replaced by $\pi\{\mathbb{R}^{\mu(\tau)}_{2m'}\}$. If, in addition, $G(z = \mathbb{N}\{[-\infty,\infty]\}, M) \in A_z$ and $G \sim [\mathbb{N}\{[-\infty,\infty]\}, \mathfrak{h}(M) : x')$. Then the results of 6.A (vi)b) and (viii)b) hold with regard to all $F_{i,j} \in \pi\{\mathbb{L}^{\mu(\tau)}_{2m'}\} < \pi\{\mathbb{L}^{\mu(\tau)}_{2m'} - \hat{\mathbb{F}}\}>$ and those of 6.A.(vii)b), (vii)b), (xi)c,d) hold with $\mathbb{L}^{\mu(\tau)}$ and $\mathbb{O}^{\mu(\tau)}$ replaced by $\pi\{\mathbb{L}^{\mu(\tau)}_{2m'}\}$ and $\mathbb{R}^{\mu(\tau)}$.

c) Let $\sigma \in Sy$. Define m' as in (ii)b). Then the results of 6. A(vi)d), (vii)d) and (viii)d) hold with $\mathbb{U}^{\mu(\pi)}$ replaced by $\mathbb{T}\{\mathbb{L}^{\mu(\pi)}_{2m'}\}$.

(iv) Let $G(z \equiv \mathbb{N}\{[\alpha,\beta]\}, M) \in A_z$ and $G(z \equiv \mathbb{N}\{[\alpha,\beta]\}, m) \in \bar{\Psi}^{\mu}_{1_z}$

a) Let $[\alpha,\beta] \subseteq [0,\infty]$. Define m' as in (ii)a). Then the result of 6.A.(ix)a) holds with $\mathbb{U}^{\mu}$ replaced by $\mathbb{T}\{\mathbb{L}^{\mu}\}$.

b) Let $0 \in (\alpha,\beta)$. Define m' as in (ii)b). Then the results of 6.A.(ix)c) hold with $\mathbb{L}$ replaced by $\mathbb{T}\{R_{2m'}\}$. If, in addition $G(\mathbb{N}\{[-\infty,\infty]\}, M) \in A_z$, $G(\mathbb{N}\{[-\infty,\infty]\}, m) \in \Psi^{\mu}_{1_z}$ then the results of 6.A.(ix) b) hold, mutatis mutandis, for the mappings of $\mathbb{T}\{\mathbb{L}_{2m'}(m)\}$.

c) Let $\sigma \in \mathcal{S}y$. Define m' as in (ii)b). Then the results of 6.A.(ix)d) hold with $\mathbb{L}$ replaced by $\mathbb{T}\{\mathbb{L}_{2m'}\}$.

(v) Let $G(z = \mathbb{N}\{[\alpha,\beta]\}, x = \mathbb{D}) \in A_{z,x}$.

a) Let $[\alpha,\beta]\subseteq[0,\infty]$. Define m' as in (ii)a). Then $F_{i,j}(\mathbb{D})$ $\in A\ (F_{i,j}\equiv T\{L_{m'}\}<\mathbb{T}\{\mathbb{L}_{m'}\}\cup\hat{\mathbb{F}}>)$

b) Let $0\in(\alpha,\beta)$. Define m' as in (ii)b). Then $F_{i,j}(\mathbb{D})\in A$ $(F_{i,j}\equiv\mathbb{T}\{\mathbb{R}_{2m'}\}<\mathbb{T}\{\mathbb{R}_{2m'}\}\cup\hat{\mathbb{F}}>$. If, in addition, $G(z\equiv\mathbb{N}\{[-\infty,\infty]\},x\equiv\mathbb{D})\in A_{z,x}$, then we may replace $\mathbb{R}_{2m'}$ by $\mathbb{L}_{2m'+1}$ in this result.

c) Let $\sigma\in\mathcal{S}y$. Define m' as in (ii)b). Then $F_{i,j}(\mathbb{D})\in A$ $(F_{i,j}\equiv\mathbb{T}\{\mathbb{L}_{2m'+1}\}<\mathbb{T}\{\mathbb{L}_{2m'+1}\}\cup\hat{\mathbb{F}}>)$.

D. Let $M\subset\mathbb{D}$, let $G([\alpha,\beta],x\equiv\mathbb{D})\in A_x$ and let $|G(t,x)|\leq K\in(0,\infty)$ $(t\in[\alpha,\beta],x\in\mathbb{D})$. Let M possess a limit point in $\mathbb{D}$ and $\mathbb{D}'$ be a domain bounded by a contour interior to $\mathbb{D}$.

(i) Let $[\alpha,\beta]\subseteq[0,\infty]$. If $\alpha=0$, let $G\sim g(\mathbb{D})$, otherwise let $g(\mathbb{D})\in\mathbb{V}$. Either (1) let $h_m\in\mathcal{DS}$ or (2)

Let $\beta<\infty$ and set $m'\in\mathbb{I}$ arbitrarily large. Then $\mathbb{U}_{m'}(\mathbb{D}')$ $\to F(\mathbb{D}')$

(ii) Let $0\in(\alpha,\beta)$ and $G^{\pm}\sim\mathfrak{g}(\mathbb{D})$. Either (1) let $h_{2m'}\in\mathbb{D}H$ or (2) let $|\alpha|, |\beta|<\infty$ and set $m'\in\mathbb{I}$ arbitrarily large. Then $\mathbb{U}_{2m'}(\mathbb{D}')\to F(\mathbb{D}')$. If, in addition, $G([-\infty,\infty], x\equiv\mathbb{D})\in A_x$ and $|G(t,x)|\le K \in(0,\infty)$ $(t\equiv[-\infty,\infty], x\equiv\mathbb{D})$, then $\mathbb{U}_{2m'+1}(\mathbb{D}')\to F(\mathbb{D}')$

(iii) Let $\sigma\in\mathfrak{S}y$ and $G\sim\pm\mathfrak{g}(\mathbb{D})$. Define m' as in b). Then $\overline{\mathbb{U}}_{2m'}(\mathbb{D}')\to F(\mathbb{D}')$.

Proof. A. We consistently ignore the possibility that $\sigma\in St$. In such a case (see 6. A.(iii)a)), $\mathbb{F}$ contains an infinite block to which the tail of the sequence under investigation ~~consideration~~ must belong, and questions of convergence hardly arise. We may disregard the case in which

$x\{\mathcal{G}(m)\} = \mu_\tau$ with $\mu(\tau) \in \mathbb{I}$ for the same reason. Furthermore, all subsets of $\mathbb{F}$ to which the sequences considered in this theorem ~~relate~~ are compounded of bounded number of diagonal sequences. It therefore suffices, and it suits our convenience, to restrict proofs to diagonal sequences in $\mathbb{F}$.

(i) a) Let $m \in \mathbb{I}_0^{m'}$. We have $h_m = ms\{\sigma_m \in \mathrm{BND}[\alpha, \infty]\}$. The zeros $t_\nu^{(i,i+m-1)}$ $(\nu = \mathbb{I}_1^i)$ of the orthogonal polynomials $p_i^{(m)}(t)$ $(i = \mathbb{I}_1)$ obtained from h_m are simple and confined to the open interval (α, ∞) (see the proof of 6.A.(iv)). The weight factors $M_\nu^{(i,i+m-1)}$ $(\nu = \mathbb{I}_1)$ associated with these zeros are positive real numbers for which $\sum_1^i M_\nu^{(i,i+m-1)} = h_m$ $(i = \mathbb{I}_1)$. The series $\sum h_{m+\nu} z^\nu$ generates an associated continued fraction ~~of order i~~ whose convergent of order i is given by

$$C_i^{(m)}(z) = \Sigma_1^i M_\nu^{(i,i+m-1)} / (1 - t_\nu^{(i,i+m-1)} z).$$

Hence, from formula ( ),

$$P_{i,i+m-1}(z) = \Sigma_0^{m-1} h_\nu z^\nu + z^m \Sigma_1^i M_\nu^{(i,i+m-1)} / (1 - t_\nu^{(i,i+m-1)} z)$$

$$(i = \mathrm{I}_1)$$

Since $n\{P_{i,i+m-1}\} = 0$ $(i = \mathrm{I}_1)$ we also have from formula ( )

$$P_{i,i+m-1}(z) = \Sigma_0^{m-1} \alpha_\nu^{(i,i+m-1)} z^\nu + \Sigma_1^i A_\nu^{(i,i+m-1)} / (1 - t_\nu^{(i,i+m-1)} z)$$

$$(i = \mathrm{I}_1)$$

Comparing these two forms, we have, for $i = \mathrm{I}_1$

$$\alpha_{\nu'}^{(i,i+m-1)} = h_{\nu'} - \Sigma_1^i A_\nu^{(i,i+m-1)} (t_\nu^{(i,i+m-1)})^{\nu'} \quad (\nu' = \mathrm{I}_0^{m-1})$$

$$A_\nu^{(i,i+m-1)} = (t_\nu^{(i,i+m-1)})^{-m} M_\nu^{(i,i+m-1)} \quad (\nu = \mathrm{I}_1^i)$$

Since $G \in \mathcal{G}(M)$ when $\alpha = 0$, and $\mathcal{G}(M) \in W$ when $\alpha > 0$, the function $G_m(t,x) = \{G(t,M) - \Sigma_0^{m-1} G_\nu(M) t^\nu\} / t^m$ is well defined for $t = [\alpha, \infty)$ (it also possesses the continuity properties of $G(t,x)$ for the same reason $\mathcal{G}(x = M)$).

From formulae (-) we obtain

$$F_{i,i+m-1}(M) = \Sigma_0^{m-1} h_\nu G_\nu(M) + \Sigma_1^i M_\nu^{(i,i+m-1)} G_m(t_\nu^{(i,i+m-1)}, M) \quad (i = \mathrm{I}_1)$$

Setting $\sigma_i^{(m)} = st\{i; M_\nu^{(i,i+m-1)}, t_\nu^{(i,i+m-1)}\}$, $F_{i,i+m-1}(m)$ may be expressed with the help of a Stieltjes integral as

$$F_{i,i+m-1}(m) = \sum_0^{m-1} h_\nu G_\nu(m) + \int_\alpha^\infty G_m(t,m)\,d\sigma_i^{(m)}(t).$$

To prove the convergence of the sequence $F_{i,i+m-1}(m)$ $(i=\overline{I}_1)$ to $F(m)$, we appeal to a result (based upon earlier work of M. Riesz [ ], of Shohat and Tamarkin [ ]. (Since we require this result in its most general form, also accommodating the case in which $[\alpha,\infty] \subseteq [-\infty,\infty]$ we state it in this form now) Let $h' \in mo\{\sigma \in BND[\alpha';\infty]$ $(\alpha'=0$ or $\alpha'=-\infty)$; let the zeros and weight factors of the orthogonal polynomial of degree i derived from h' be $t_\nu^{(i)}, M_\nu^{(i)}$ $(\nu=\overline{I}_1^i)$ respectively; set $\sigma_i = st\{i, M_\nu^{(i)}, t_\nu^{(i)}\}$ $(i=\overline{I}_1)$; let the function $\tilde{G}(t)$ be continuous in every closed finite subinterval in $t \in (\alpha';\infty)$ and such that the integral $\overset{\infty}{F} = \int_{\alpha'}^{\infty\infty} \tilde{G}(t)\,d\sigma(t)$ is well defined in the sense of the notation $F[M : \tilde{G} ; \sigma ; \alpha, \infty] \in W$; as $|t| \to \infty$ in $[\alpha', \infty]$ let $\tilde{G}(t) = O\{G(t)\}$ where, with $t_0 = -\infty$ if $\alpha' \neq -\infty < 0$, $t_0 \in [0, \infty]$ if $\alpha' = 0$, $G(t) \geq 0$ $(t \in [t_0, \infty))$ and, with $\tau(\nu) \in \underset{\sim}{I}$ $(\nu \in I)$ a prescribed strictly increasing integer sequence, $\mathfrak{D}_t^{2\tau(\nu)+2} G(t) \geq 0$ $(\nu \in I, t \in (t_0, \infty))$. Then the sequence of numbers $F_{\tau(\nu)} = \int_{\alpha}^{\infty} \tilde{G}(t)\,d\sigma_{\tau(\nu)}(t)$ $(\nu \in I)$ converges.

Considering now the Stieltjes case in which $\alpha = \alpha' \geq 0$, we remark that if $h' \in \mathbb{D}S$, the distributions σ_i $(i \in I_1)$ converge to σ. Hence the sequence $F_{\tau(\nu)}$ $(\nu \in I)$ converges to F. When $\tau(\nu) = \nu$ $(\nu \in I_{\tau'})$, the sequence F_i $(i \in I_{\tau'})$ converges to F.

If, as $t \to \infty$, $G(t, x) = O\{G(t)\}$ $(x \in M)$ as stated in the theorem, then $G^{(m)}(t, x) = O\{t^{-m} G(t, x)\} = O\{G(t, x)\}$

$= O\{G(t)\}$ also. Since $h_{m'} \in DS$, $h_m \in DS$ also (if the Stieltjes moment problem associated with h_m has two distinct solutions σ_m, that associated with $h_{m'}$ also has two distinct solutions given by formulae ( ) of the form $d\sigma_{m'}(t) = t^{m'-m} d\sigma_m(t)$ $(t = [0, \infty])$).

Applying the convergence result of the penultimate paragraph to the mappings of formula ( ), we see that

$$\lim_{i=\infty} F_{i,i+m-1}(M) = \sum_{0}^{m-1} h_\nu G_\nu(M) + \int_{\alpha}^{\infty} G^{(m)}(t, M) d\sigma^{(m)}(t)$$

$$= \int_{\alpha}^{\infty} \left\{ \sum_{0}^{m-1} G_\nu(M) t^\nu \right\} d\sigma(t) + \int_{\alpha}^{\infty} G^{(m)}(t, M) t^m d\sigma(t)$$

$$= F[M : G, \sigma; \alpha, \infty] = F(M)$$

With regard to the alternative condition b) imposed upon $G(t)$, we remark that the convergence result of Shohat and Tamarkin as described above also holds, in the special case in which $\alpha' \geq 0$, for functions so constrained.

For the last result of the clause, we remark that when $h_m = ms\{\sigma_m \in BND[\alpha,\infty]\}$ $(\alpha \geq 0)$, the series $\sum h_{m+\nu} z^\nu$ generates a corresponding continued fraction whose convergents of odd order are given by

$$C_{2i+1}^{(m)}(z) = \hat{M}_0^{(i,i+m-1)} + \sum_1^i M_\nu^{(i,i+m-1)} / (1 - t_\nu^{(i,i+m)} z) \quad (i = \bar{I}_{\#})$$

where $t_\nu^{(i,i+m)}$ $(\nu = \bar{I}_1^i)$ are the zeros of the i^{th} order orthogonal polynomial $p_i^{(m+1)}(t)$ obtained from the sequence h_{m+1} $(i = \bar{I}_1$. $\hat{M}_\nu^{(i,i+m-1)} \in (0,\infty)$ $(\nu = \bar{I}_0^i)$ and $\sum_0^i \hat{M}_\nu^{(i,i+m-1)} = h_m$. From formula ( ), we now have

$$P_{i,i+m-1}(z) = \sum_0^{m-1} h_\nu z^\nu + z^m \{ \hat{M}_0^{(i,i+m-1)} + \sum_1^i \hat{M}_\nu^{(i,i+m-1)} / (1 - t_\nu^{(i,i+m)} z) \} \quad (i = \bar{I}_1)$$

Analysis similar to that ~~given~~ involving formulae (-) given above allows ~~&~~ us to write

$$\begin{aligned} F_{i,i+m}(M) &= \sum_0^{m-1} h_\nu G_\nu(M) + \hat{M}_0^{(i,i+m-1)} G_m(M) + \sum_1^i \hat{M}_\nu^{(i,i+m-1)} G^{(m)}(t_\nu^{(i,i+m)}, M) \\ &= \sum_0^{m-1} h_\nu G_\nu(M) + \hat{M}_0^{(i,i+m-1)} G_m(M) + \int_\alpha^\infty G^{(m)}(t, M)\, d\hat{\sigma}_i^{(m)}(t) \quad (i = \bar{I}_1) \end{aligned}$$

where $\hat{\sigma}_i^{(m)} = st\{i; \hat{M}_\nu^{(i,i+m-1)}, t_\nu^{(i,i+m-1)}\}$.

Shohat and Tamarkin also gave [ ] a result which may be applied to the numbers $\hat{M}_\nu^{(i,i+m-1)}$. Let $h_\nu' = ms\{\sigma \in BND[0,\infty]\}$, let $\hat{t}_\nu^{(i)}, \hat{M}_\nu^{(i)}$ be the numbers corresponding to $t_\nu^{(i,i+m)}, \hat{M}_\nu^{(i,i+m-1)}$ in formula ( ) associated with the corresponding continued fraction derived from the series $\sum h_\nu' z^\nu$; let the function $\hat{G}(t)$ be continuous in every closed finite subinterval in $t = (0,\infty)$ and such that $F = \int_0^\infty \hat{G}(t)\, d\sigma(t)$ exists; as $t \to \infty$, let $\hat{G}(t) = O\{G(t)\}$ where, with $t_0 \in [0,\infty)$, $G(t) \geq 0$ ($t \in [t_0,\infty]$) and, with $\tau(\nu) \in \mathbb{I}$ ($\nu \in \mathbb{I}$) a prescribed strictly increasing integer sequence, $\mathfrak{D}_t^{2\tau(\nu)+1} G(t) \geq 0$ ($\nu \in \mathbb{I}, t \in [t_0,\infty)$). Letting $\hat{\sigma}^{(i)}(t)$ be the step function with saltus of magnitude $\hat{M}_0^{(i)}$ at $t=0$, and

$\hat{M}_{\nu}^{(i)}$ at $t=\hat{t}_{\nu}^{(i)}$ $(\nu=\underline{I}_1^{\,i})$, the sequence of numbers $\hat{F}_{\tau(\omega)} = \int_0^{\infty}\hat{G}(t)\,d\hat{\sigma}^{\tau(\omega)}(t)$ converges. When $h' \in DS$, the distributions $\sigma^{(i)}$ $(i=\underline{I}_1)$ converges to σ and the sequence $\hat{F}_{\tau(\omega)}$ converges to F.

The further proof of convergence of $F_{i,irm}(m)$ $(i=\underline{I}_1)$ to $F(m)$ is as given above for the sequence $F_{i,irm-\nu}(m)$ $(i=\underline{I}_1)$. We have only to remark that when $\alpha>0$ $\hat{M}_0^{(i,irm-\nu)}$ tends to zero as i increases, and the term involving it may effectively be discarded from formula ( ). When $\alpha=0$, $\hat{\sigma}_i^{(m)}(t)$ may be extended so as to include the saltus of magnitude $M_0^{(i,irm-\nu)}$ at $t=0$, thus incorporating the term referred to in the integral expression.

When $m \in \underline{I}_0^{\,m'-1}$, the above proof of convergence yields no result not otherwise to be derived for a sequence of the form $F_{i,i+m-1}(m)$. However, when $m = m'$, we obtain, by use of the above proof, convergence for the one diagonal sequence of $\mathbb{D}_{m'+1}(m)$ not in $\mathbb{D}_{m'}(m)$.

(ii) In this case $h_m = ms\{\sigma_m \in BND[\alpha, \beta]\} \in \mathbb{DS}$ $(m \in \mathbb{I})$, each series $\sum i h_{m+\nu} z^{\nu}$ $(m \in \mathbb{I})$ generates an associated continued fraction, and the decomposition ( ) holds for all $F_{i,i+m-1} \in \mathbb{D}'$. To establish convergence of a sequence $F_{i,i+m-1}(m)$ $(i \in \mathbb{I})$ to $F(m)$, we use a result of Stekloff [ ] and Fejer [ ]. Let $t_{\nu}^{(i)}, M_{\nu}^{(i)}$ $(\nu \in \mathbb{I}_1^i)$ be the zeros and weight factors of the orthogonal polynomial of degree i derived from the sequence $h' = ms\{\sigma \in BND[\alpha, \beta]\}$ where $[\alpha, \beta] \subset (-\infty, \infty)$ $(i \in \mathbb{I}_1)$; set $\sigma_i = st\{i; M_{\nu}^{(i)}, t_{\nu}^{(i)}\}$ $(i \in \mathbb{I}_1)$; let $F = \int_{\alpha}^{\beta} G(t)\,d\sigma(t)$ exist; set $F_i = \int_{\alpha}^{\beta} G(t)\,d\sigma_i(t)$. Then the sequence F_i

($i \in \mathrm{I}_1$) converges to F. The remainder of the proof is as for the first stated result of (i).

(iii)a) This part of the clause is demonstrated by methods used in the proof of clause (i), the more general version of the result of Shohat and Tamarkin concerning intervals of the form $[x', \infty)$ with $x' = -\infty$ now needed having already been stated. If $\sigma \in S_y$, then all functions of $\overline{\Theta}_{2m'}$ are, of course, to be found in $\overline{\Phi}_{2m'}$. Under the stated conditions $G(t,x) = O(1)$ as $|t| \to \infty$ in $[-\infty, \infty]$ ($x \in M$) and, from (iii)a), $\overline{\Phi}_{2m'}(M) \to F(M)$.

b) The first difficulty that must be overcome in the investigation of sequences of the form $F_{i,i+2m}(M)$ ($i \in \mathrm{I}$) derives from the fact that, when $h = ms\{\sigma \in BND[\alpha, \beta]\}$ with $0 \in (\alpha, \beta)$, the precise form of any associated quotient $P_{i,i+2m}$ is, a priori, unknown, depending upon whether the quotient is or is not a member of a block (see the proof of 6.A...). This difficulty is overcome by associating the denominators of these quotients with a system of quasi-orthogonal polynomials.

Let p_i $(i \in \mathbb{I})$ be the orthogonal polynomials derived from the sequence $h' = ms\{\sigma' \in BND[\alpha,\beta]\}$. Define the quasi-orthogonal polynomials q_i $(i \in \mathbb{I})$ by $q_0(t) = 1$

$$q_i(t) = (1-\omega_i)\,p_i(t) - \omega_i\, p_{i-1}(t) \qquad (i \in \mathbb{I})$$

where $\omega_i \in (-\infty,\infty)$ $(i \in \mathbb{I}_1)$ are prescribed. The zeros $\tilde{t}_\nu$ $(\nu = \mathbb{I}_1^{i'})$ $(i' = i$ if $\omega_i \neq 1$, $i' = i-1$ if $\omega_i = 1)$ of $q_i(t)$ $(i \in \mathbb{I}_2)$ are real and simple. In particular, if $\omega_i = 0$, they lie in the interval (α,β); if $\omega_i/(1-\omega_i) = p_i(0)/p_{i-1}(0)$, one of them is zero. The associated weight factors $\tilde{M}_\nu^{(i)}$ are positive real numbers with $\sum_1^{i'} \tilde{M}_\nu^{(i)} = h_0'$ (i' as above). (The theory as presented in [,] omits the factor $(1-\omega_i)$ from formula ( ); we introduce it, without impairing the analysis, for later convenience.)$_z$

As in the proof of 6. A. (iv), we associate orthogonal polynomials $p_i^{(2m)}$ with the denominators of the quotients $P_{i,i+2m-1}$ $(m \in \underline{I})$, setting $p_i^{(2m)}(t) = \pi^{(i,i+2m-1)}(z)\, z^{-i}$ $(t = z^{-1})$ $(i \in \underline{I})$. If a) $H_{i-1,i+2m-1}, H_{i,i+2m} \neq 0$ $(i \in \underline{I}_1, m \in \underline{I})$ the denominator of $P_{i-1,i+2m-1}$ is given by $\pi^{(i,i+2m)}(z) = z^{i-1} p_{i-1}^{(2m+1)}(t)$, $(t = z^{-1})$, where

$$t p_{i-1}^{(2m+1)}(t) = p_i^{(2m)}(t) - \{p_i^{(2m)}(0) / p_{i-1}^{(2m)}(0)\} p_{i-1}^{(2m)}(t).$$

If b) $H_{i-1,i+2m-1} \neq 0$, $H_{i,i+2m} = 0$ < c) $H_{i-1,i+2m-1} = 0$, $H_{i,i+2m} \neq 0$ > the denominator of $P_{i-1,i+2m-1}$ is simply $\pi^{(i,i+2m-1)}(z)$ < $\pi^{(i-1,i+2m-2)}(z)$ >. In case a) derive $\omega_i^{(2m)}$ from the equation $\omega_i^{(2m)}/(1-\omega_i^{(2m)}) = p_i^{(2m)}(0)/p_{i-1}^{(2m)}(0)$

and in case b) <c)> set $\omega_i^{(2m)} = 0 \langle 1 \rangle$, thereby defining

a sequence of real numbers $\omega_i^{(2m)}$ $(i = \mathbb{I}_1)$ and, in turn,

a sequence of quasi-orthogonal polynomials $q_i^{(2m)}$ $(i = \mathbb{I})$

obtained from a formula of the form ( ) with q_i, ω_i

replaced by $p_i^{(2m)}, \omega_i^{(2m)}$. Let $\tilde{t}_i^{(2m)}, \tilde{M}_i^{(2m)}$ be the zeros and

associated weight factors of the $q_i^{(2m)}$ $(i = \mathbb{I}_1)$. Set $\tilde{\sigma}_i^{(2m)}$

$= st\{i'; \tilde{M}_i^{(2m)}, t_\nu^{(2m)}\}$ where $i' = i$ or $i-1$ depending upon

whether $\omega_i^{(2m)} = 1$ or not $(i = \mathbb{I}_2)$. As in the derivation

of formulae (,) we may show that

$$F_{i,i+2m}(m) = \sum_0^{2m-1} h_\nu G_\nu(m) + \int_{-\infty}^{\infty} G_{2m}(t, m)\, d\tilde{\sigma}_i^{(2m)}(t) \quad (i = \mathbb{I}_2)$$

We now make use of a convergence result, also based

upon the work of M. Riesz [ ] and due to Shohat

and Tamarkin. For $i \in \mathbb{I}_1$, let $\tilde{t}_\nu, \tilde{M}_\nu$ $(\nu = \mathbb{I}_1^{i'})$ be the zeros and weight factors associated with the quasi-orthogonal polynomials q_i deriving from the sequence $h' = ms\{\sigma' \in BND[-\infty,\infty]\}$ and set $\tilde{\sigma}_i = st\{i'; \tilde{M}_\nu, \tilde{t}_\nu\}$, ($i'$ depending upon the degree of i); let $h' \in DH$; let $G(t)$ be continuous in every closed finite subinterval in $t = (-\infty,\infty)$ and tend to finite limits as $|t| \to \infty$ in this interval. Then the sequence $\int_{-\infty}^{\infty} G(t) d\tilde{\sigma}_i(t)$ $(i \in \mathrm{I}_2)$ converges to $\int_{-\infty}^{\infty} G(t) d\sigma'(t)$

The convergence of the sequence $F_{i,i+2m}(m)$ $(i \in \mathrm{I})$ to $\mathfrak{E}F(m)$ follows, for reasons given in connection with earlier clauses, directly from this result.

c) By extending the domain of definition of G as described and extending that of σ by setting $\sigma(t) = \sigma(a)$ $(t \in [-\infty, a])$ this part of the clause is a simple corollary to its predecessor. The mapping to which convergence is proved in this case has the form $F[M:\sigma, G; -\infty; \infty] = F(M:\sigma, G; \alpha; \infty]$.

(iv) a) In this case $h_{2m} = ms\{\sigma_{2m} \in BND[\alpha, s]\} \in DH$ $(m \geq I)$ each series $\sum h_{2m+\nu} z^{\nu}$ $(m \geq I)$ generates an associated continued fraction, and the decomposition ( ) holds for all $F_{i, i+2m-1} \in \mathbb{D}'$. The further steps in the proof of the first stated result are those described in connection with (ii). That $\mathbb{D}'(M) \to F(M)$ when $\sigma \in S_y$ follows for reasons given in the proof of (iii)a).

b) This result follows by an extension of the methods used in the proof of (iii)c).

B.(i). Since $G(z,x)$ is analytic at the point at infinity, $\hat{G}(x) = \lim G(z,x)$ as $z \to \infty$ exists $(x \in M)$. We may set $G(z,x) = \hat{G}(x) + \tilde{G}(z,x)$, where $\tilde{G}(z,x) = O(z^{-1})$ as $z \to \infty$. The convergence behaviour of the $F_{i,j}(x)$ may be investigated examining that of the two arrays of functions obtained by applying our algorithm to the functions $\hat{G}(z) \equiv G$ and $\tilde{G}(z,x)$ separately. The first array presents ~~no~~ no problem: all functions of this array are identically equal to $h_0 \hat{G}(z)$. The following proof concerns the function $\tilde{G}(z,x)$ which we write as $G(z,x)$: we may assume that $\lim G(z,x) = O(z^{-1})$ as $z \to \infty$.

Let the contour $\mathbb{K}$ in the z-plane be compounded of the two lines $Im(z) = \pm \delta'$, $Re(z) \geq 0$ and the semi-circle $|z| = \delta'$ $(Re(z) \leq 0)$ and lie wholly in $\mathbb{N}\{[\alpha,\beta]\}$, and $\mathbb{K}(R)$ be that part of $\mathbb{K}$ lying in $\mathbb{D}_R$, so that $\lim \mathbb{K}(R) = \mathbb{K}$ as $R \to \infty$. Set $h(z) = h[z; \varepsilon; 0, \bar{\infty}]$. It is easily demonstrated that $\lim F(x : G, h; \mathbb{K}(R))$ as $R \to \infty$ exists, this limit being $F(x)$ $(x \in \mathbb{M})$.

Denote by $\mathbb{K}'(R)$ that part of $\mathbb{K}_R$ left after the arc enclosed by $\mathbb{K}$ has been removed. Set $F'(R,x) = F(x: G, h; \mathbb{K}'(R))$. For sufficiently large R_1, $G(z \equiv \{\mathbb{Z} - \mathbb{D}_R\}, x) \in A_z$ and $|F'(R,x)| \leq \frac{1}{3}\delta'$ $(x \equiv M)$ for all $R \geq R_1 \in (0,\infty)$. Also $|F(x) - F(x: G, h; \mathbb{K}(R))| \leq \frac{1}{3}\delta$ for all $R \geq R_2 \in (0,\infty)$. Denote by $\overline{\mathbb{K}}(R)$ the adjunction of $\mathbb{K}(R)$ and $\mathbb{K}'(R)$. Set $F''(x;R) = F(x: G, h: \overline{\mathbb{K}}(R))$. Then $|F(x) - F''(x;R)| \leq \frac{2}{3}\delta'$ $(x \equiv M)$ where $R = \max(R_1, R_2)$.

It was shown by Van Vleck{ } that $\tilde{h} = ms\{\tilde{\sigma}; 0, \infty\}$ and by Wall [ ] that if $\tilde{h}_{m''} \in \mathbb{D}S$, all diagonal sequences of the form $P_{i+m,i}(z)$ $(i \equiv \mathbb{I})$ converge uniformly to $h(z)$ for $z \equiv \mathbb{B}\{\mathbb{Z} - \mathbb{N}\{[0,\infty]\}\}$ $(m \equiv \mathbb{I}_0^{m''})$. The closed region $\mathbb{D}(\xi, R)$ bounded by the curve having the same shape as $\mathbb{K}(R)$, assembled from segments of the two lines $\mathrm{Im}(z) = \pm\,\xi \in (0,\infty)$, the small semi circle in the left half plane $|z| = \xi$, $\mathrm{Re}(z) > 0$, and an arc of the larger circle $|z| = R' > \xi$ is a convergence domain for this sequence. Since $h(\mathbb{D}(\xi, R')) \in A$, it follows that $i' \in \mathbb{I}$ exists such that all members of the sequence $P_{i+m,i}(z)$ $(i \in \mathbb{I}_{i'})$ are devoid of poles in $\mathbb{D}(\xi, R)$. When $z \in \mathbb{D}(\xi, R')$, $z' = z^{-1}$ belongs to a simply connected claw shaped region $\mathbb{D}'(\xi, R')$ bounded by the following curves: the large semi-circle $|z'| = 1/\xi$ $(\mathrm{Re}(z') \leq 0)$ in the left half-plane; parts of the two semi circles $|z' \pm 1/(2\xi)| = 1/(2\xi)$ $(\mathrm{Re}(z') \geq 0)$ in the right half-plane, and part of the small circle $|z'| = 1/R'$ (the latter circle intersects the two semi circles in the first and fourth quadrants; $\mathbb{D}'(\xi, R')$ is symmetric about the real axis). If $R' \overset{?}{=} 1/(2\delta')$, $\xi =$

$\xi = \delta'(4R'^2)$, the curve $\overline{\mathbb{K}}(R)$ lies entirely within $\mathbb{D}'(\xi, R')$. Since the quotients $P_{irm,i}(z\xi^{-1})$ $(i \in \mathbb{I}_{i'})$ are devoid of poles in $\mathbb{D}'(\xi, R')$ they are also devoid of poles within and upon $\overline{\mathbb{K}}(R)$.

Consider the quotient $P_{irm,i}(z^{-1})$ $(i \in \mathbb{I}_{i'})$. It is a rational function: a circle $\mathbb{K}_\eta$ exists enclose $\overline{\mathbb{K}}(R)$ and all poles of $P_{irm,i}(z^{-1})$, so that all such poles lie in the region enclosed between $\overline{\mathbb{K}}(R)$ and $\mathbb{K}_\eta$. Hence

$$F_{irm,i}(x) = \frac{1}{2\pi i}\left[\int_{\mathbb{K}(R)} - \int_{\mathbb{K}_\eta}\right] z^{-1} P_{irm,i}(z^{-1}) G(z,x)\,dz$$

$P_{irm,i}(z^{-1})$ is analytic outside $\mathbb{K}_\eta$ and $\lim_{z=\infty} P_{irm,i}(z^{-1}) = k_0$.

$G(z,x)$ is analytic outside $\mathbb{K}_\eta$ and $\lim_{z=\infty} G(z,x) = 0$ $(x \geq m)$

η may be thus be increased indefinitely without changing the value of the second integral in formula ( ) and, furthermore, its limiting values as $z \to \infty$ is zero.

The same reasoning may be applied to all members of the sequence $P_{irm,i}(z^{-1})$ $(i \equiv \mathbb{I}_{i'})$ and we have $F_{irm,i}(M) = F(M: G, P_{irm,i}; \bar{\mathbb{K}}(R))$ $(i \equiv \mathbb{I}_{i'})$. The sequence $P_{irm,i}(z^{-1})$ $(i \equiv \mathbb{I}_{i'})$ converges to $h(z^{-1})$ at every point of $\bar{\mathbb{K}}(R)$ which is of bounded length: we may find $i'' \in \mathbb{I}_{i'}$ such that $|F''(x;R) - F_{irm,i}(x)| \leq \frac{1}{3}\delta'$ $(i \equiv \mathbb{I}_{i''}, x \equiv M)$. Since $|F(x) - F''(x;R)| \leq \frac{2}{3}\delta'$ $(x \equiv M)$, we have $|F(x) - F_{irm,i}(x)| \leq \delta'$, ~~we have~~ $(i \equiv \mathbb{I}_{i''}, x \equiv M)$.

(ii) Let the contour $\mathbb{K}$ in the z-plane be compounded of two lines $\mathrm{Im}(z) = \pm\delta'$, $\alpha \leq \mathrm{Re}(z) \leq \rho$, and the two semi-circles $|z-\alpha| = \delta'$ $(\mathrm{Re}(z-\alpha) \leq 0)$, $|z-\rho| = \delta'$ $(\mathrm{Re}(z-\rho) \geq 0)$ and lie wholly in $\mathbb{N}\{[\alpha,\rho]\}$. Set $h(z) = \mathfrak{h}[z;\sigma;\alpha,\rho]$. It is easily demonstrated that $F(M: G, h; \mathbb{K}) = F(M)$.

Let $h(z) = h_0 / \{1 - z\tilde{h}(z)\}$. Then [ ] $\tilde{h}(z) = h(z: \tilde{\sigma}; \tilde{\alpha}, \tilde{\rho})$

where $\tilde{g} \in BND[\tilde{\alpha}, \tilde{\beta}]$ and $[\tilde{\alpha}, \tilde{\beta}] \subseteq [\alpha, \beta]$. Let $\tilde{h} \leftarrow \tilde{h}$. Let the Padé quotients derived from $\tilde{h}$ be denoted by $\tilde{P}_{i,j}$ $(i, j \in \mathbb{I})$. Applying the result of A(ii) to the sequence $\tilde{h}$ the function $G(t, z) = 1/(1-tz)$ and the functions $G_\nu(z) = z^\nu$ $(\nu \in \mathbb{I})$, we deduce that all diagonal sequences of functions $\tilde{P}_{i, i+m-1}(z)$ $(i \in \mathbb{I})$ converge for $m \in \mathbb{I}$ uniformly to $\tilde{h}(z)$ for $z \in \mathbb{D}$ where $\mathbb{D} = B\{\mathbb{Z} - N\{[\beta^{-1}, \alpha^{-1}]\}\}$. From ....

$P_{i+m, i}(z) = h_0 / \{1 - z \tilde{P}_{i, i+m-1}(z)\}$ $(i, m \in \mathbb{I})$. Since $h(\bar{\mathbb{D}}) \in A$, it follows that all diagonal sequences of functions $P_{i+m, i}(z)$ $(i \in \mathbb{I})$ converge for $m \in \mathbb{I}$ uniformly to $h(z)$ for $z \in \mathbb{D}$ (That such convergence holds for $z \in B\{\mathbb{Z} - N\{[\beta^{-1}, \alpha^{-1}]\}\}$ has been demonstrated by Wall [,])

The remainder of the proof continues as in the proof of ....

(iii)(a) The method of proof is similar to that of (i). We now take $\mathbb{K}$ to be the contour in the z-plane compounded of the two lines $\operatorname{Im}(z) = \pm\delta'$ ($\operatorname{Re}(z) \equiv [\alpha, \infty]$) joined, if $\alpha > -\infty$ by the semi circle $|z - \alpha| = \delta'$ ($\operatorname{Re}(z - \alpha) \leq 0$) and lying in $N\{[\alpha, \infty]\}$. Setting $h(z) = h[z : \varepsilon; \alpha, \infty]$, we again show that $F(M : G, h; K) = F(M)$. When $\alpha = -\infty$, we let $\mathbb{K}(R)$ be two closed contours, the first composed of those parts of $\mathbb{K}$ and $\mathbb{K}_R$, joined at their points of intersection in the upper half plane, the second being the reflection of the first in the real axis. We again show that R can be so chosen that with $F''(x, R) = F(x; G, h : \mathbb{K}(R))$ $|F(x) - F''(x; R)| \leq \frac{2}{3}\delta$ ($x \in M$). If $\alpha > -\infty$, we take $\mathbb{K}(R)$ to be the contour obtained by distending the $\mathbb{K}(R)$ of (i) in such a way that it crosses the negative real axis at the point $z = \alpha - \delta$.

With $h(z) = h[z : \sigma ; \alpha, \beta]$ ($[\alpha,\beta] \subseteq [-\infty, \infty]$). Let $h(z) = h_0 / \{1 - (h_1 z)/h_0) + z^2 \hat{h}(z)\}$. Then [ ] $\hat{h}(z) = h(z : \hat{\sigma} ; \hat{\alpha}, \hat{\beta})$ where $[\hat{\alpha}, \hat{\beta}] \subseteq [\alpha,\beta]$ and $\hat{\sigma} \in BND[\hat{\alpha}, \hat{\beta}]$. In the case under consideration (in which $\beta = \infty$) we have $[\hat{\alpha}, \hat{\beta}] \subseteq [\alpha, \infty]$. Denote the Padé quotients derived from $\hat{h}$ by $\hat{P}_{i,j}(z)$ ($i, j \in \mathbb{I}$) so that (...) $P_{i+m+1,i}(z) = h_0 / \{1 - (h_1 z/h_0) + z^2 \hat{P}_{i,i+m-1}(z)\}$ ($m, i \in \mathbb{I}$). As in the proof of (ii) we deduce from (iii)a) that all diagonal sequences of functions $P_{i+2m+1,i}(z)$ ($i \in \mathbb{I}$) converge for $m \in \mathbb{I}_0^{m'}$ uniformly to $h(z) = h[z : \sigma ; \alpha, \infty]$ for $z \in B\{\mathbb{Z} - N\{[0,\infty]\} - N\{[-\infty, \alpha^{-1}]\}\}$ (This is again a slight extension of a result of Wall [ ], who showed that such convergence holds for $z \in B\{\mathbb{Z} - N\{[-\infty, \infty]\}\}$).

The steps taken in the proof of (i) are now repeated

b) Now the poles of $\hat{P}_{i,i+2m}(z)$, although real and simple may not be confined to the interval (α, ∞). We take $\mathbb{K}(R)$ to be the circumference of the pair of sectors of $\mathbb{K}_R$ as described above, and use the convergence result of (iii)b) to show that all diagonal sequences of functions $P_{i+2m+2,i}(z)$ $(i \in \mathbb{I})$ converge for $m \in \mathbb{I}_0^{m'}$ uniformly to $h(z)$ for $z \in \mathbb{B}\{\mathbb{Z} - N\{[-\infty, \infty]\}\}$.

(iv)a) We may take $\mathbb{K}$ to be as described in the proof of (ii)a). The diagonal sequences of functions $P_{i+2m+1,i}(z)$ $(i \in \mathbb{I})$ now converge for $m \in \mathbb{I}$ uniformly to $h(z) = h(z; \alpha, \beta]$ for $z \in \mathbb{B}\{\mathbb{Z} - N\{[-\infty, \alpha^{-1}]\} - N\{[\beta^{-1}, \infty]\}\}$ (again extending Wall's result [ ])

b) We use the method of proof of (iii)b).

c) In each of the clauses of this part of the theorem, the diagonal sequences making up the subarrays of mappings $F_{i,j}(m)$ converge to $F(m)$. Furthermore, the numbers $t_{\nu}^{(i,j)}$ occurring ~~are to~~ in these functions are known ultimately to lie either in $\mathbb{N}\{[\alpha,\beta]\}$ or, in the special case in which $D \in (\alpha,\beta)$ and the sequence lies in $L_{2m'}(m) - R_{2m'}(m)$ in $\mathbb{N}\{[-\infty,\infty]\}$. Each clause is proved as for its counterpart in theorem 6. ((i) as for 6.A.(v)b), (ii) as for 6.A.(v)a), and (iii) and (iv) as for the counterparts mentioned).

D. As is easily demonstrated, in each of the cases considered $|F_{i,j}(D)|$ is bounded for all $F_{i,j}$ of the sequences considered; the required result follows from the Stieltjes-Vitali theorem.

The result of A.(i) concerns the moment sequence $h = ms\{\sigma \in BND[\alpha,\beta]\}$ with $[\alpha,\beta] \subseteq [0,\infty]$. Trivial changes of variable yield a related result concerning the sequence with $[\alpha,\beta] \subseteq [-\infty,0]$. Trivial changes of variable yield a related result concerning the sequence with $[\alpha,\beta] \subseteq [-\infty,0]$. This remark applies with equal force throughout the theorem.

The result of C.(i)a) holds, if $\sigma = st\{M_\nu, t_\nu; n\}$, also for $F_{i,j} = \hat{F}$. But this result has already been stated in 6.A.(v)b), and hence this extension has been omitted from the above theorem, as have others similar to it. When dealing with the behaviour of $F_{i,j}(\sigma)$ and with a result such as $F_{i,j}(m) \neq F_{i',j'}(m)$ (this part of the theory is covered in C.(iii)(iv)) we do, of course, need to know whether F contains an infinite block, since the functions concerned lie outside it. In this respect it may occur that $N\{[\alpha,\beta]\}$ is so small that the only $F_{i,j}$ concerned for which the $t_\nu^{(i,j)}$ belong to $N\{[\alpha,\beta]\}$ are precisely those belonging to the infinite block; and in such a case the stated result becomes nugatory.

In parts A, B. we impose the condition $F(m) \in W$, and $G(t,x)$ may still tend to infinity as t tends to an endpoint of the interval $[\alpha,\beta]$ without violating this condition. If $\alpha = 0$ in A(i), $G(t,x)$ must, since $G \in \mathfrak{h}(m)$, be finite as $t \to 0$; but if $\alpha > 0$, this condition need not hold. If $\int_0^t s(t')\,dt' = O\{\exp(-\kappa t^{\xi})\}$ $(\kappa \in (0,\infty),\ \xi \in [\tfrac{1}{2},\infty))$ (m' may be arbitrarily large in A.(i)), $G(t,x)$ may be $O\{\exp(\kappa' t^{\xi})\}$ $(\kappa' < \kappa)$ as $t \to \infty$ without violating the condition $F(m) \in W$. In this case $G(t,x)$ is allowed to tend to infinity as $t \to \infty$, rather steeply if ξ is large). In A(iii)a), if $\alpha \in (-\infty, 0)$, it may ~~occur~~ again occur that $G(t,x)$ tends to infinity as $t \to \cancel{\infty}$. If $\int_\alpha^t \sigma(t')dt' = O\{\exp(-\kappa|t|^\xi)\ (\kappa \in (0,\infty), \xi \in [1,\infty])\}$ as $|t| \to \infty$ in $[\alpha, \infty)$ (m' may again be taken arbitrarily large) we again require only, for example, a condition such as $G(t,x) = O\{\exp(\kappa'|t|^{\xi'})\ (\kappa' < \kappa)\}$ as $t \to \infty$, again permitting $G(t,x)$ to tend to infinity ~~with~~ as $|t| \to \infty$. Under the conditions of (iii)b) and (iv)b), $G(t,x)$ must remain finite for $t = [\alpha, \beta]$; under those of (ii) and (iv)a), $G(t,x)$ may become infinite at the end points of this interval.

The functions $G(t,x)$ concerned in B. remain finite for $t = [\alpha, \beta]$. If the functions $F_{i,j}$ are formed from the sum of a function $G(t,x)$ satisfying the condition of the appropriate clause, and a polynomial of the form $\sum_0^{\mu} \hat{G}_\nu t^\nu$, then convergence still holds: the corresponding constituents of $F_{i,j}$ can be treated separately; all $F_{i,j}$ formed from the polynomial part and belonging to an infinite subarray for which $i+j+1 \geq \mu$ have the form $\sum_0^{\mu} h_\nu \hat{G}_\nu(x)$; thus convergence automatically holds with regard to this part. Hence, using complex variable methods as in the proof of part B., we can only demonstrate convergence for functions $G(t,x)$ tending to infinity with no more than polynomial growth as $|t| \to \infty$ in $[\alpha,\beta]$.

Theorem . Let $h = ms\{\sigma \in BND[\alpha,\beta]\}$, let $G(t,x)$ be an entire function of t ($x \in M$) with $G \in \mathfrak{G}(M)$ and such that the series $\sum h_\nu G_\nu(x)$ converges ($x \in M$). Set $F(x) = F[x:\sigma;G;\alpha,\beta]$

(i) $\sum h_\nu G_\nu(M) \to F(M)$

(ii) $\mathbb{D}'(M) \to F(M)$

Proof. (i) Since the sequence $\sum_0^{r-1} G_\nu(x) t^\nu$ $(r = \mathbb{I})$ converges to $G(t,x)$ for any $t \in (\alpha,\beta)$ $(x = M)$, we have

$$\sum h_\nu G_\nu(x) = \lim_{r=\infty} \int_\alpha^\beta \sum_0^{r-1} G_\nu(x) t^\nu d\sigma(t)$$

$$= \int_\alpha^\beta \lim \sum_0^{r-1} G_\nu(x) t^\nu d\sigma(t)$$

$$= F(x).$$

(ii) When $\sigma \in St$, the stated result follows immediately from 6.A.(...) $(x = M)$. We therefore assume that $\sigma \notin St$ and deal first with the case in which $[\alpha,\beta] \subseteq [0,\infty]$. Let $m \in \mathbb{I}$. Then $h_m = ms\{\sigma^{(m)} \in BND[\alpha,\beta]\}$ and $h_{i,i+m-1} = ms\{\sigma_i^{(m)} \in BND[\alpha,\beta]\}$, where $\sigma_i^{(m)} = st[i; M_\nu^{(i,i+m-1)}; t_\nu^{(i,i+m-1)}]$, $t_\nu^{(i,i+m-1)}$ and $M_\nu^{(i,i+m-1)}$ being the zeros and weight factors of the orthogonal polynomial of the ~~f~~ i^{th} degree generated by h_m $(i = \mathbb{I}_1)$ (see the proof of 7.A(i)). If $h_m \in \mathbb{D}S$, the distributions $\sigma_i^{(m)}$ $(i \in \mathbb{I}_1)$ tend to the only solution $\sigma' = \sigma$ of the Stieltjes moment problem associated with h_m: when $h_m \notin DS$, these distributions tend to a solution of this problem (namely the extremal solution σ' for which $d\sigma'(t) = 0$ over the largest possible interval $t \in [0, \gamma]$) $F_{i,i+m-1}(m)$ is expressed in terms of the distribution $\sigma_i^{(m)}$ by means of formula ( ) $(i \in \mathbb{I}_1)$. Whether $\sigma' = \sigma$ or not, we have, as in the derivation of formula ( ), $\lim F_{i,i+m-1}(m) = F(M; \sigma; G; \alpha, \beta] = \sum h_\nu G_\nu(m) = F(m)$ as $i \to \infty$

Let $[\alpha, \beta] \subseteq [-\infty, \infty]$ and again $m \in \mathbb{I}$. Now we have $h_{2m} = ms[\sigma^{(2m)} \in BMD[\alpha, \beta]]$ and $h_{i,i+2m-1} = ms\{\sigma_i^{(2m)} \in BMD[\alpha, \beta]\}$ $(i \in \mathbb{I}_1)$. If $h_{2m} \in DH$, the distributions $\sigma_i^{(2m)}$ $(i \in \mathbb{I}_1)$ tend to the only solution of the Hamburger moment problem associated with h_{2m}, and the argument of the preceding paragraph can be applied. If $h_{2m} \in DS$, the distributions $\sigma_{2i-1}^{(2m)}, \sigma_{2i}^{(2m)}$ $(i = \overline{I}_1)$ each tend to differing extremal solutions of this moment problem. The sequences $F_{2i-1, 2m+2i-2}(m)$, $F_{2i, 2m+2i-1}(m)$ may be treated separately to yield the joint result that the sequence $F_{i, 2m+i-1}(m)$ converges to $F(m)$.

As discussed in the proof of 7.A.(i) b), a sequence of distributions $\tilde{\sigma}_i^{(2m)}$ $(i = \overline{I}_1)$ may be derived from the moment sequence h_{2m}. Whether $h_{2m} \in DH$ or not this sequence tends to $\sigma^{(2m)}[\;\;,\;\;]$. Formula ( ) may be used to show, as above that the sequence $F_{i, 2m+i}(m)$ converges to $F(m)$.

The convergence results of Theorem 7 depend upon the determinacy of the moment problems of certain sequences derived from h, those of the above theorem do not. When $\alpha \in [0,\infty)$, $h = ms\{\sigma \in BND[\alpha,\infty]\}$ and $h_{m'} \notin DS$, the distributions $\sigma_{2i-1}^{(m')}$ and $\sigma_{2i}^{(m')}$ tend to differing extremal solutions of the moment problem associated with $h_{m'}$. Subject to the further conditions imposed upon the function $G(t,x)$ in 7.A.(i), the sequences $F_{i,irm'-1}(m)$ and $F_{i,irm'}(m)$ $(i = \mathbb{I}_1)$ each tend to a limit; but these limits may differ, and differ from $F(m)$ (in the special case in which $G(t,x)$ is an entire function as described in Theorem 7, this does not, however, occur). Indeed Wall [ ] has shown that when $h \in DS$, each diagonal sequence $P_{i,irm-1}(z)$ or $P_{irm,i}(z)$ $(m = \mathbb{I})$ converges uniformly for $z = \mathbb{B}\{\mathbb{Z} - N\{[0,\infty]\}\}$ to a different limit $(i = \mathbb{I})$. If, fortuitously, σ is the extremal solution of the moment problem associated with h for which $d\sigma(t)=0$ over the largest possible interval $t \in [0,\gamma]$, then at least the sequence $P_{i,i-1}(z)$ converges as described to $h[z:\sigma;\alpha,\infty]$. Otherwise, it may occur that none of the diagonal sequences mentioned converge to this limit. Similar considerations of a more complicated nature relate to the case in which $[\alpha,\beta] \subseteq [-\infty,\infty]$ (see [ ] for a description of the convergence behaviour of the Padé table in this case).

Theorem . Let $[\alpha,\beta] \subseteq [0,\infty)$, $h = \text{mo}\{\sigma \in BND[\alpha,\beta]\}$ and let $\sigma(t)$ have salti at the distinct points $t_\nu \in (\beta',\beta]$ ($\nu \in I_1^n$) and no other points of increase for $t \in [\beta',\beta] < (\alpha,\beta]$. Let $G(z \in \hat{D}, M) \in A_z$ where

$\hat{\mathbb{D}} = N\{\mathbb{D}_{\beta'}\} \cup N\{[\beta', \beta]\}$, and let $G \in \mathfrak{H}(M)$. Set $F(x)$ $= F(x : \sigma; G; \alpha, \beta]$, then $\mathbb{F}_{n,n}(M) \to F(M)$, where $\mathbb{F}_{n,n}$ is the set $F_{i,j}$ $(\overline{i,j} = [n, n; \infty])$.

Proof. Take $\mathbb{K}$ to be a simple closed contour surrounding $\bar{\mathbb{D}}_{\beta'}$ and the segment $[\beta', \beta]$ and lying in $\hat{\mathbb{D}}$. Then, as is easily demonstrated, $F(M : G, h; \mathbb{K}) = F(M)$. The author has proved the following result: let h and σ be as described in the theorem; let $\tilde{\mathbb{D}}$ be a closed domain lying in $\mathbb{D}_{1/\beta'}$ cut along the real segment $[1/\beta, 1/\beta']$ and to which no point of either $\mathbb{K}_{1/\beta'}$ or of the cut belongs; then any sequence of quotients $P_{i,j}(z)$ with distinct integer suffix pairs belonging to the set $\overline{i,j} \equiv [n, n; \infty]$ converges uniformly to $h[z; \sigma; \alpha, \beta]$ for $z \in \tilde{\mathbb{D}}$. The remainder of the proof uses complex variable methods as employed, for example, in the proof of A)....

Theorem Let $h = \text{ms}\{\theta \in BND[\alpha,\beta]\}$ and set $f(z) = f[x:\epsilon; \delta; \alpha,\beta]$

A.(i) Let $\alpha \in [0,\infty)$, $\beta = \infty$. If $\alpha > 0$ [$\alpha = 0$] let $\tilde{\Delta} = \Delta[\rho_0, \infty]_\phi^\psi$, $\Delta = \Delta[\rho/|\alpha|, \infty]_\phi^\psi$, $g \in V$ [$\tilde{\Delta} = \Delta\{= \Delta[0,\infty]_\phi^\psi$, $g \sim g$] Let $g(\tilde{\Delta}) \in C$, $f(\Delta) \in W$. Let $g(te^{i\theta}) = O\{G(t)\}$ as $t \to \infty$ ($\theta = [\phi, \psi]$), where $G(t)$ is as described in 7.A.(i) a or b). Let $h_{m'} \in DS$. Then $u_{m'}(\mathbb{B}\{\Delta\}) \to f(\mathbb{B}\{\Delta\})$. If $G(t)$ satisfies the condition of 7.A.(i) c), then $u_{m',1}(\mathbb{B}\{\Delta\}) \to f(\mathbb{B}\{\Delta\})$.

(ii) Let $[\alpha,\beta] \subseteq [0,\infty)$. If $\alpha > 0$ [$\alpha = 0$], let $\tilde{\Delta} = \Delta[\rho, \omega]_\phi^\psi$ where $\beta\rho(\theta) < \alpha\omega(\theta)$ ($\theta = [\phi, \psi]$), $\Delta = \Delta[\rho, \omega]_\phi^\psi$ and $g \in V$ [$\tilde{\Delta} = \Delta[0,\omega]_\phi^\psi$, $\Delta = \Delta[0, \omega/\beta]_\phi^\psi$

and $g \sim g$]]. Let $g(\hat{\Delta}) \in \mathbb{C}$, $f(\Delta) \in W$. Then $\alpha'(\mathbb{B}\{\Delta\}) \to f(\mathbb{B}\{\Delta\})$

(iii) Let $\alpha \in [-\infty, 0)$, $\beta = \infty$, $g(\pm\Delta_{\phi}^{\psi}) \sim g$ and $h_{2m'} \in \mathbb{D}H$

a) If $\alpha > -\infty$ [[$= -\infty$]] let $\tilde{\Delta} = \Delta[0,\infty] \cup (-\Delta[0,\omega]_{\phi}^{\psi})$

$\Delta = \Delta[0, \omega/|\alpha|]_{\phi}^{\bar{\psi}}$ and $|g(|\alpha|\rho(\theta)e^{i\theta})| < \cancel{M} \infty$ $(\theta = [\phi+\pi, \psi+\pi])$

[[$\tilde{\Delta} = \Delta = \pm\Delta[0,\infty]_{\phi}^{\psi}$]]. Let $g(\tilde{\Delta}) \in \mathbb{C}$ and, as $|t| \to \infty$ in $[\alpha, \infty]$ let $g(te^{i\theta}) = O\{G(t)\}$ $(\theta = [\phi, \psi])$, where $G(t)$ is as described in 7. A. (iii) a). Let $f(\Delta) \in W$. Then $\mathbb{D}_{2m'}(\mathbb{B}\{\Delta\}) \to f(\Delta)$. If, in addition $\bar{\sigma} \in Sy$, then $\bar{u}_{2m'+1}(\Delta) \to f(\Delta)$.

b) Let $\Delta = \pm\Delta[0,\infty]_{\phi}^{\psi}$, $g(\Delta) \in \bar{\mathbb{C}}$. Then $u_{2m'+1}(\Delta) \to f(\Delta)$.

(iv) Let $\alpha \in (-\infty, 0)$, $\beta \in (0, \infty)$ and $g(\pm\Delta_{\phi}^{\psi}) \sim g$

a) Let $\tilde{\Delta} = \Delta[0, \hat{\omega}]_{\phi}^{\psi} \cup (-\Delta[0, \tilde{\omega}]_{\phi}^{\psi})$, $\Delta = \Delta[0, \omega']_{\phi}^{\bar{\psi}}$

$\cup(-\Delta[0,\omega'']_\phi^\psi)$ where $\omega'(\theta)=\min\{\hat{\omega}(\theta)/\beta, \tilde{\omega}(\theta)/|\alpha|\}$, $\omega''(\theta)=\min\{\hat{\omega}(\theta)/|\alpha|, \tilde{\omega}(\theta)/\beta\}$ $(\theta=[\phi,\psi])$. Let $g(\tilde{\Delta})\in\mathbb{C}$, $f(\Delta)\in W$. Then $\mathbb{D}'\{B(\Delta)\}\to f\{B(\Delta)\}$. If, in addition, $\sigma\in S_q$, then $\mathbb{u}'\{B(\Delta)\}\to f\{B(\Delta)\}$.

b) Let $\Delta=\pm\Delta[0,\infty]_\phi^\psi$, $g(\Delta)\in\mathbb{T}$, and $f(\Delta)\in W$. Then $\mathbb{u}'\{B(\Delta)\}\to f\{B(\Delta)\}$

B. In the following, let $g(N\{\tilde{\Delta}\})\in A$

(i) Let $\alpha\in[0,\infty)$, $\beta=\infty$ and $\tilde{\Delta},\Delta$ be as in A(i). With $\tilde{h}$ defined by formula ( ), let $\tilde{h}_{m'}\in\mathbb{D}S$. Then $\mathcal{L}_{m'}\{B(\Delta)\}\to f\{B(\Delta)\}$.

(ii) Let $[\alpha,\beta]\subseteq[0,\infty)$ and $\tilde{\Delta},\Delta$ be as in A.(ii). Then $\mathcal{L}'\{B\{\Delta\}\}\to f(B\{\Delta\})$.

(iii) Let $\alpha\in[-\infty,0)$, $\beta=\infty$. With $\hat{h}$ as defined by formula ( ), let $\hat{h}_{2m'}\in DH$.

a) Let $\tilde{\Delta},\Delta$ be as in A(iii). Then $\mathcal{L}_{2m'}(B\{\Delta\})\to f(B\{\Delta\}$

If, in addition, $\sigma \in Sy$, then $\mathcal{U}_{2m'}(\mathbb{B}\{\Delta\}) \to f(\mathbb{B}\{\Delta\})$.

b) Let $\tilde{\Delta} = \Delta = \pm\Delta[0,\infty]^{\psi}_{\phi}$. Then $\mathcal{U}_{2m'+1}(\mathbb{B}\{\Delta\}) \to f(\mathbb{B}\{\Delta\})$.

(iv) Let $\alpha \in (-\infty, 0)$, $\beta \in (0,\infty)$

a) Let $\tilde{\Delta}, \Delta$ be as in A.(iv)a). Then $w'(\mathbb{B}\{\Delta\}) \to f(\mathbb{B}\{\Delta\})$. If, in addition $\sigma \in Sy$, then $\mathcal{U}'(\mathbb{B}\{\Delta\}) \to f(\mathbb{B}\{\Delta\})$.

b) Let $\tilde{\Delta} = \Delta = \pm\Delta[0,\infty]^{\psi}_{\phi}$. Then $\mathcal{U}'(\mathbb{B}\{\Delta\}) \to f(\mathbb{B}\{\Delta\})$.

C. Let $\sigma \notin St\langle \sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{f}$ as in 6 B.(iii)a)$\rangle$

(i) Let $|\alpha|, |\beta| < \infty$, $g(\mathbb{D}_\kappa) \in A$ and $g \leftarrow g$

a) Let $[\alpha,\beta] \subseteq [0,\infty)$ or $\sigma \in Sy$. Then $f_{i,j}(\bar{\mathbb{D}}_{\kappa/\gamma}) \in A$ and $\sum h_\nu^{(i,j)} \xi_{g\nu}(\bar{\mathbb{D}}_{\kappa/\gamma})^\nu \to f(\bar{\mathbb{D}}_{\kappa/\gamma})$ $(f_{i,j} = \pi\{f'\} < \top\{\hat{f}'\} \cup \hat{f})$;

b) Let $0 \in (\alpha,\beta)$. Then the results of a) hold with f replaced by $\mathfrak{o}' \cup w'$.

(ii) Let $g(\mathbb{N}\{[\alpha,\beta]\}\times\Delta)\in A$, $g(z=\mathbb{N}\{[\alpha,\beta]\},\Delta)\in\bar{\mathfrak{Y}}^{\mu}_{z}$

a) Let $[\alpha,\beta]\subseteq[0,\infty]$. Define m' as in 7.C.(ii)a). Then the results of 6.A.(ix)a) holds with F, $\mathbb{U}^{\mu}$ replaced by f, $\mathbb{T}\{\chi^{\mu}\}$.

b) Let $0\in(\alpha,\beta)$. Define m' as in 7.C.(ii)b). Then the results of 6.A.(ix)c) hold with F, $\mathbb{U}^{\mu}$ replaced by $\mathbb{T}\{v^{\mu}_{2m'}\}$

If, in addition, $g(\mathbb{N}\{[-\infty,\infty]\times\Delta)\in A$, $g(z=\mathbb{N}\{[-\infty,\infty]\},\Delta)\in\bar{\mathfrak{Y}}^{\mu}_{z}$ then the result of 6.A.(ix)b) holds, mutatis mutandis, for the mappings of $\mathbb{T}\{\lambda_{2m'}(\Delta)\}$.

c) Let $\sigma\in\mathfrak{S}y$. Define m' as in 7.C.(ii)b). Then the results of 6.A(ix)d) hold with F, $\mathbb{U}$ replaced by f, $\mathbb{T}\{\lambda_{2m'}\}$

(iii) Let any one of the convergence results of part B) be stated as $\tilde{f}(\Delta)\to f(\Delta)$. Then, subject to the stated conditions upon which the result in question is based, $f_{i,j}(\Delta)\in A$ ($f_{i,j}=\mathbb{T}\{f'\}$).

D. Let $\Delta \subseteq \mathbb{D}$, $g([\alpha,\beta]\times\mathbb{D}) \in A$, and $|g(z)| \leq K \in (0,\infty)$

$\mathbb{D}'$: $(z \in [\alpha,\beta]\times\mathbb{D})$. Let Δ possess a limit point in $\mathbb{D}^*$ and $\mathbb{D}'$

be a domain bounded by a contour interior to $\mathbb{D}$.

(i) Let $[\alpha,\beta] \subseteq [0,\infty]$. If $\alpha = 0$ let $g \in \mathcal{G}$; otherwise let $g \in \mathcal{V}$.

Either (1) let $h_{m'} \in \mathbb{D}S$ or (2) let $\beta < \infty$ and set $m' \in I$

arbitrarily large. Then $u_{m'}(\mathbb{D}') \to f(\mathbb{D}')$

(ii) Let $0 \in (\alpha,\beta)$ and $g \in \mathcal{G}$. Either (1) let $h_{2m'} \in \mathbb{D}H$ or

(2) let $|\alpha|, |\beta| < \infty$ and set m' arbitrarily large. Then

$v_{2m'}(\mathbb{D}') \to f(\mathbb{D}')$. If, in addition, $g([-\infty,\infty]\times\mathbb{D}) \in$

A and $|g(z)| \leq K \in (0,\infty)$ $(z \in [-\infty,\infty]\times\mathbb{D})$, then $u_{2m'+1}(\mathbb{D}')$

$\to f(\mathbb{D}')$.

(iii) Let $\alpha \in S_g$ and $g \in \mathcal{G}$. Define m' as in b). Then

$u_{2m'}(\mathbb{D}') \to f(\mathbb{D}')$.

Proof. The results of parts A and B of the above theorem are special cases of the corresponding results given in 7. A,B. The results of C. (i – iii) may be deduced from those of 7.C (i, iv, v) respectively and those of D from 7. D.

The results of 7.C.(ii) may be presented in terms of the functions $g(x)$, $f_{i,j}(z)$ and the sequences $h^{(i,j)}$, g to yield a result of the form $f_{i,j} \leftarrow h^{(i,j)} g$ for $f_{i,j}$ belonging to appropriate subsets of $\mathfrak{F}$. However, when $0 \in [\alpha, \beta]$ the required conditions upon g imply, in particular, that $g(N\{0\}) \in A$, and the results of the type described are already available in ... . It would appear, nevertheless, that there is a further result outstanding for the case in which $g(z)$ is not analytic in the neighbourhood of the origin, and

$[\alpha,\beta] \subset (0,\infty]$. However, in such a case, use is made of a formula such as $f_{i,j}(x) = f(x:g,h;K)$ where K is a contour lying in $N\{[\alpha,\beta]\}$ and within which $g(xz)$ is analytic. An asymptotic result of the form $f_{i,j} \sim h^{(i,j)}g$ concerns the behaviour of $f(x)$ as $x \to 0$. As $x \to 0$, K must be taken to lie indefinitely closely to $[\alpha,\beta]$ and as this occurs, the tail of any sequence of functions $f_{i,j}$ for ~~which~~ which all $t_{\nu}^{(i,j)}$ lie within K becomes progressively attenuated, and no meaningful asymptotic relationship of the form $f_{i,j} \sim h^{(i,j)}g$ holding for all functions of such a tail can be stated. Since special cases of 7.C.(iii) utilise outstanding asymptotic relationships, they also have no analogue not already available in 6....

We also remark, with regard to the result of C(ii) b), that if Δ is a domain containing the origin, the condition, the condition $g(\mathbb{N}\{[-\infty,\infty]\}\times\Delta)$ implies that $g(z)$ is a constant, in which case the stated result becomes nugatory. For similar reasons the additional result of D.(ii) becomes trivial when $\mathbb{D}$ contains the origin.

Theorem . Let $h = m_\sigma\{\sigma \in BND[\alpha,\beta]\}$, let g be an entire function with $g < g$; let the series $\sum h_\nu g_\nu x^\nu$ converge for $x \in \Delta$. Set $f(x) = f[x:\sigma, g:\alpha,\beta]$.

(i) $\sum h_\nu g_\nu \Delta^\nu \rightarrow f(\Delta)$.

(ii) $u'(\Delta) \rightarrow f(\Delta)$.

Proof. The above results follow directly from those of Theorem .

We wish only to remark in connection with the above theorem that Δ will be either the single point $x=0$, or an open disc, or such a disc together with certain points of its boundary.

Theorem . Let $[\alpha,\beta]\subseteq[0,\infty)$, $h=\text{mo}\{\varpi\in BND\,\mathfrak{z}[\alpha,\beta]\}$ and ϖ, β' and n be as described in Theorem . Let $g(D[\omega])\in A$ and $\mathfrak{z}\leftarrow g$. Let $\min\omega(\theta)=\kappa$ $(\theta\in[0,2\pi])$ and $\tilde{\omega}(\theta)=\min\{\kappa/\beta', \omega(\theta)/\beta\}$. Set $\tilde{D}=D[\tilde{\omega}]$ and $f(x)=f[x:\varpi,\mathfrak{z};\alpha,\beta]$. Then $\mathfrak{f}_{n,n}(\hat{\tilde{\Delta}})\to f(\hat{\tilde{\Delta}})$ where $\mathfrak{f}_{n,n}$ is the set $f_{i,j}$ $(i,j\in[n,n;\infty])$.

Proof. With $\mathfrak{z}$ as described and x restricted to the domain $\tilde{D}$, the function $\mathfrak{z}(\varpi x)$ is analytic for $x\in\hat{D}$, where $\hat{D}$ is the domain described in Theorem .. ; the required result now follows from that theorem.

Properties of best approximation

Notation. $y(M) = ms\{\Lambda(M); \alpha', \beta'\}$ means that $c_\nu(x) = \int_{\alpha'}^{\beta'} y^\nu d\Lambda(y,x)$ $(\nu = \mathrm{I}, x = M)$. We set $st\{t: \Lambda(M); \alpha', \beta'\}$ $\int_{\alpha'}^{\beta'} (1-yt)^{-1} d\Lambda(y,M)$. $\Lambda(y,M) \in BND_y[\alpha', \beta']$ means that $\Lambda(y,x)$ is a bounded non-decreasing real valued function for $y = [\alpha', \beta']$ $(x = M)$; $\Lambda(y,M) \in BM_y[\alpha', \beta']$ means that $\Lambda(y,x)$ is a complex valued function $(x = M)$ of the real variable y such that $|\mathrm{Re}\,\Lambda(y,M)|$, $|\mathrm{Im}\,\Lambda(y,M)| \in BND_y[\alpha', \beta']$. $D \in BM[\hat{\alpha}^*, \hat{y}^*]$ means that $|\mathrm{Re}\, D(y)|$, and $|\mathrm{Im}\, D(y)|$ are bounded nondecreasing functions for $y = [\hat{\alpha}, \hat{\beta}]$. For prescribed intervals $[\alpha, \beta]$, $[\alpha', \beta']$ and functions σ, Λ, $\min \Upsilon(r,m; M)$ denotes the minimum of the values of the expression $\int_{\alpha'}^{\beta'} y^m \left[\int_\alpha^\beta (1-yt)^{-1} t^m \{1 + \Sigma_1^r X_\nu (1-yt)^\nu\}^2 d\sigma(t) d\Lambda(y,x)\right.$ (

over the set $X_\nu \in (-\infty,\infty)$ $(\nu = \overline{I}_1^r)$ for $x = M$; for a suitable prescribed function Ψ, $\min \Psi\{\Upsilon(m; M)\}$ denotes the minimum of the corresponding function of the above expression. For prescribed intervals $[\hat{\alpha},\hat{\beta}]$, $[\tilde{\alpha},\tilde{\beta}]$ and functions σ and D, $\min v(r, m, \Delta)$ denotes the minima of the values of the expressions

$$\int_{\hat{\alpha}}^{\hat{\beta}} y^m \left[\int_\alpha^\beta (1-yt)^{-1} t^m \left\{1+\sum_1^r X_\nu (1-yt)^\nu\right\}^2 d\sigma(t)\right] dD(y)$$

over the above set for $x = \Delta$, and we adopt a similar convention regarding functions of this expression.

Theorem . Let $h = ms\{\sigma \notin St; \sigma, \rho\}$ and $m, r = \underline{I}$ in the following results

A. Let $y(M) = ms\{\Lambda(M); \alpha', \rho'\}$, $G(t, M) = st\{t: \Lambda(M); \alpha', \rho'\}$ and set $F(x) = F(x: \sigma, G; \alpha, \rho)$.

(i) Let $\Lambda(y, M) \in BND_y[\alpha'; \beta']$

a) Let $[\alpha, \beta] \subseteq [0, \infty]$.

~~(1) Let $[\hat{\alpha}, \hat{\beta}] \times [\tilde{\alpha}, \tilde{\beta}] \in (-\infty, 0]$. Then $(-1)^m$~~

(1) Let $[\alpha', \beta'] \subset (-\infty, \beta^{-1}]$. Then $E_{r, 2m+r-1}(M) = \min \Upsilon(r, 2m; M) > 0$. (2) If $\beta \neq \infty \langle = \infty \rangle$ let $[\alpha', \beta'] \in (-\infty, 0] \langle (-\infty, 0) \rangle$; then $(-1)^m E_{r, m+r-1}(M) = \min\{(-1)^m \Upsilon(r, m; M)\} > 0$. (3) If $\beta < \infty$, let $[\alpha', \beta'] \in [0, \beta^{-1})$. Then $E_{r, m+r-1}(M) = \min \Upsilon(r, m; M) > 0$. (4) If $\alpha > 0$, let $[\alpha', \beta'] \in (\alpha^{-1}, \infty)$. Then $E_{r, m+r-1}(M) = \min\{-\Upsilon(r, m; M)\} > 0$.

b) Let $0 \in (\alpha, \beta)$, $[\alpha', \beta'] \subset (\alpha^{-1}, \beta^{-1})$. Then $E_{r, 2m+r-1}(M) = \min \Upsilon(r, 2m; M)$.

(ii) Let $\Lambda(t, M) \in BM_t[\alpha'; \beta']$, let $[\alpha, \beta] \subset [0, \infty]$ and $[\alpha', \beta'] \in (-\infty, \beta^{-1})$. Then $|E_{r, 2m+r-1}(M)| = \min |\Upsilon(r, 2m; M)| > 0$. Further results of the same type for complex valued functions $F_{i,j}(x)$ may be derived from the results $g(i)$ by inserting modulus signs in the same way.

B. Let $g = ms\{D; \hat{\alpha}, \hat{\beta}\}$, $g(x) = st\{x : D; \hat{\alpha}, \hat{\beta}\}$ and set $f(x) = f[x : \sigma, g; \alpha, \beta]$, $\Delta = [\tilde{\alpha}, \tilde{\beta}]$.

(i) Let $D \in BND[\hat{\alpha}, \hat{\beta}]$

a) Let $[\alpha, \beta] \subseteq [0, \infty]$. (1) Let $[\hat{\alpha}, \hat{\beta}] \times [\tilde{\alpha}, \tilde{\beta}] \subset (-\infty, \beta^{-1})$. Then $e_{r,2m+r-1}(\Delta) = x^{2m} \min v(r, 2m; \Delta)$. (2) Let $[\hat{\alpha}, \hat{\beta}] \times [\tilde{\alpha}, \tilde{\beta}] \subset (-\infty, 0]$. Then $(-1)^m e_{r,m+r-1}(\Delta) = \min\{(-\Delta)^m v(r, m; \Delta)\} > 0$. (3) If $\beta < \infty$, let $[\hat{\alpha}, \hat{\beta}] \times [\tilde{\alpha}, \tilde{\beta}] \subset [0, \beta^{-1})$. Then $e_{r,m+r-1}(\Delta) = \min\{v(r, m; \Delta)\} > 0$. (4) If $\alpha > 0$, let $[\tilde{\alpha}, \tilde{\beta}] \times [\hat{\alpha}, \hat{\beta}] \subset (\alpha^{-1}, \infty)$. Then $-e_{r,m+r-1}(\Delta) = \min\{(-\Delta)^m v(r, m; \Delta)\}$.

(ii) Let $D \in BM[\hat{\alpha}, \hat{\beta}]$. Let $[\alpha, \beta] \subseteq [0, \infty]$ and $[\hat{\alpha}, \hat{\beta}] \times [\tilde{\alpha}, \tilde{\beta}] \subset (-\infty, \beta^{-1})$. Then $|e_{r,m+r-1}(\Delta)| =$

$\Delta^{2m} \min|\vartheta\{r, 2m; \Delta\}|$. Further results of the same type for complex valued functions $f_{r,j}$ may be derived from the results of (i) in the same way.

Proof. A.(i)a) Set

$$\mu\left[\{X_\nu\}:\sigma; m,r\right]_\alpha^\beta = \int_\alpha^\beta (1-yt)^{-1}\left\{1+\sum_1^r X_\nu(1-yt)^\nu\right\}^2 t^m d\sigma(t)$$
$$(m,r\in \mathrm{I})$$

The author has shown [ ] that when $[\alpha,\beta]\leq[0,\infty]$, $\sigma\in St$ and $y\in(-\infty,\beta^{-1})$, $y^{-m}\{h(y) - P_{r,m+r-1}(y)\}$ $= \min \mu\left[\{X_\nu\}:\sigma; m,r\right]_\alpha^\beta > 0$ $(m,r\in\mathrm{I})$ where the minimum is taken over the set $X_\nu\in(-\infty,\infty)$ $(\nu=\mathrm{I}_1^r)$. The values of the $\{X_\nu\}$ yielding this minimum are determined solely by α,β and σ, and are the same for all $y\in(-\infty,\beta^{-1})$. Hence, if $\Delta(y,x)$ is a bounded nondecreasing function of y for $y=[\alpha',\beta']<(-\infty,\beta^{-1})$ $(x\equiv M)$ we have, replacing m by $2m$ and adopting the special decomposition known to hold for the quotients

$P_{r,m+r-1}(y)$ in this case (see the proof of ...)

$$\int_{\alpha'}^{\beta'} h(y)\,d\Lambda(y,m) - \sum_0^{2m-1} \alpha_\nu^{(r,2m+r-1)} \int_{\alpha'}^{\beta'} y^\nu d\Lambda(y;m)$$

$$-\sum_1^r A_\nu^{(r,2m+r-1)} \int_{\alpha'}^{\beta'} (1-t_\nu^{(r,2m+r-1)} y)^{-1} d\Lambda(y,m)$$

$$= \min \gamma\left[\{X_\nu\} : \sigma, \Lambda(m); m, r\right]_{\alpha,\alpha'}^{-1,\beta,\beta'}$$

The first integral on the left hand side of this relationship is easily shown to be $F(x:\sigma;G;\alpha,\beta)$, and the second and third integral expressions are by definition $G_\nu(m)$ ($\nu = \overline{I}_0^{2m-1}$) and $G(t_\nu^{(r,2m+r-1)}, m)$ ($\nu = \overline{II}_1^r$) respectively the expression upon the left hand side of relationship ( ) is $F(m) - F_{r,2m+r-1}(m) = E_{r,2m+r-1}(m)$.

The remaining results of this clause and its successor also follow from results for Padé quotients derived by the author [ ]: when $\sigma \notin St$, $[\alpha,\beta] \leq [0,\infty]$

and $r, m \in \underline{\mathbb{I}}$, then for $y \in (\alpha^{-1}, \infty)$ (if $\alpha > 0$)

$$y^{-m}\{P_{r,mr-1}(y) - h(y)\} = \min\{-\mu[\{X_\nu\}:\sigma;m,r]_\alpha^\beta\} > 0$$

and if $0 \in (\alpha, \beta)$ and $y \in (\alpha^{-1}, \beta^{-1})$

$$y^{-2m}\{h(y) - P_{r,2mr-1}(y)\} = \min \mu[\{X_\nu\}:\sigma;2m,r]_\alpha^\beta > 0$$

(ii) Under the stated conditions, we have, whether $\operatorname{Re}\Delta(y,M)$ $\operatorname{Im}\Delta(y,M)$ are nondecreasing or nonincreasing, $|\operatorname{Re}\langle\operatorname{Im}\rangle E_{r,2mr-1}(M)| = \min|\operatorname{Re}(\Delta)\langle\operatorname{Im}(\Delta)\rangle\Upsilon(r,2m;M)|$, where $\min \operatorname{Re}(\Delta)\langle\operatorname{Im}(\Delta)\rangle\Upsilon(r,2m;M)$ is $\min\Upsilon(r,2m,M)$ with $\Delta(y,M)$ replaced by $\operatorname{Re}\langle\operatorname{Im}\rangle\Delta(y,M)$. The values of $\{X_\nu\}$ occurring in these two minimal expressions are the same in both cases; they yield the minimum value of $|\Upsilon(r,2m;M)|$ itself: we have $|\operatorname{Re} E_{r,2mr-1}(M)| + i|\operatorname{Im} E_{r,2mr-1}(M)| = \min$ $|\Upsilon_{\operatorname{Re}(\Delta)}(r,2m;M) + i\min|\Upsilon_{\operatorname{Im}(\Delta)}(r,2m;M)$, i.e. $|E_{r,2mr-1}(M)$

$\min|\mathfrak{X}(r, 2m; M)|$

B. In the proof of A(i), we let $\Lambda(y, x)$ take the special form $\Lambda(y; x) = D(y/x)$ and have $\int_{\alpha'}^{\beta'} y^\nu d\Lambda(y/x) = \int_{\alpha'/x}^{\beta'/x} y^\nu dD(y) \} x^\nu = \{ \int_{\hat{\alpha}}^{\hat{\beta}} y^\nu dD(y) \} x^\nu$ $(\nu \equiv I)$ for $x = [\tilde{\alpha}, \tilde{\beta}]$ where $[\tilde{\alpha}, \tilde{\beta}] \times [\hat{\alpha}, \hat{\beta}] = [\alpha', \beta'] \in (\alpha^{-1}, \beta^{-1})$, i.e. $G_\nu(x) = g_\nu x^\nu$ $(\nu \equiv I)$ where $g = ms\{D : \alpha', \beta'\}$. In a similar way, we show that $G(t, x)$ takes the special form $g(tx) = st\{tx : D; \hat{\alpha}, \hat{\beta}\}$ for $t = [\alpha'^{-1}, \beta'^{-1}]$, $x = [\tilde{\alpha}, \tilde{\beta}]$ and that $\int_{\alpha'}^{\beta'} h(y) d\Lambda(y, x)$ can be expressed as $f(x : \sigma; G; \alpha, \beta)$. All results of B. now follow as special cases of those of A.

If $\sigma \in St$, all results of the above theorem hold with regard to the mappings not in the blocks $\hat{F}_\xi$ of A.. or $\hat{f}$ of B. as is appropriate. For the mappings of these blocks the minimum values of the integrals concerned are all zero.

The numbers X_ν ($\nu = \overline{1,r}$) yielding the minimum value considered in each case of part A are determined solely by α, β, σ, m and r: they are the same for every function $\Delta(y, x)$ considered, and for every x of the point set M concerned. The same holds true with regard to the functions D and the interval $[\hat{\alpha}, \hat{\beta}]$.

The original results for Padé quotients upon which the proof of the above theorem is based are obtained from those of B by letting $[\hat{\alpha}, \hat{\beta}]$ be an arbitrarily small interval containing the point $y=1$ and letting $D(y)$ be a simple step function with a saltus of magnitude unity at the point $y=1$.

It is clear that every function $F_{r,2mr-1}(x)$ considered for example in A.(i)a) (1) is a best approximation to $F(x)$ in the sense that it is the function which reduces the value of the integral ( ) with m replaced by $2m$ to a minimum. For fixed intervals $[\alpha,\beta]$, $[\alpha',\beta']$ and generating functions $\sigma(t)$ and $\Delta(y,x)$, m determines the weight function with respect to which the error function $E_{r,2mr-1}(x)$ is a minimum and r the number of available parameters. The same holds with regard to the error functions $e_{i,j}(x)$ considered in B.

When $|\alpha|$ or $|\beta|$ is infinite, it may occur that the sequence $h = ms\{\sigma;\alpha,\beta\}$ is associated with an indeterminate moment problem; we have me sequence h corresponding to many functions σ. In this case each $F_{r,m+r-1}(M)$, for example, considered is the same (since h is fixed) but we have many mappings $F(M:\sigma; G; \alpha, \rho)$ and many corresponding minima min $\gamma(r, m; M)$.

Monotonic sequences

Notation. $a_r(m)\big|_0^{r'} \in MI\{A(m)\} < MD\{A(m)\} > [\![MI'(m)]\!]$

means that the sequence $a_r(x)$ $(r = \overline{\underline{I}}_0^{\,r'})$ is monotonically increasing <decreasing> [[is nondecreasing with no three members equal]] with upper <lower> [[upper]] bound $A(x)$ $(x = m)$.

Theorem. Let $h = mo\{\sigma \notin St; \alpha, \beta]$ and $m \in \mathbb{I}$.

A) Let $g(m) = ms\{\Delta(m); \alpha', \beta'\}$, $G(t, m) = st\{t : \Delta(m); \alpha', \beta'\}$ and set $F(x) = F[x : \sigma; G; \alpha, \beta]$

(i) Let $\Delta(y, m) \in BND_y[\alpha', \beta']$

a) Let $[\alpha, \beta] < [0, \infty]$

(1) Let $[\alpha', \beta'] < (-\infty, \beta^{-1})$. Then $0 < F(m) < \infty$, $F(m) > F_{i,j}(m)$ $(F_{i,j} \neq 0)$, $F_{r,2m+r-1}(m)\big|_0^{\infty} \in MI\{F(m)\}$, $F_{r,2m-r+1}\big|_0^{m+1} \in MI\{F_{m+1,m}(m)\}$.

(2) If $\beta \neq \infty$ $\langle -\infty \rangle$ let $[\alpha', \beta'] \subset (-\infty, 0]$ $\langle (-\infty, 0) \rangle$. Then $0 < F(m) < F_{i,j}(m) < \infty$ $(F_{i,j} \equiv \mathbb{E})$, $F_{r,2m+r}(m) \big|_0^\infty \in MD\{F(m)\}$

$F_{r,2m-r}(m) \big|_0^m \in MD\{F_{m,m}(m)\}$.

(3) If $\beta > 0$, let $[\alpha', \beta'] \subset [0, \beta^{-1})$. Then $0 < F_{i,j}(m) < F(m) < \infty$ $(F_{i,j} \equiv \mathbb{U})$, $F_{r,m+r}(m) \big|_0^\infty \in MI\{F(m)\}$, $F_{r,2m-r}(m) \big|_0^m \in MI\{F_{m,m}(m)\}$, $F_{r,2m-r+1}(m) \big|_0^{m+1} \in MI\{F_{m+1,m}(m)\}$.

(4) If $\alpha > 0$, let $[\alpha', \beta'] \in (\alpha^{-1}, \infty)$. Then $-\infty < F(m) < 0$, $F_{i,j}(m) > F(m)$ $(F_{i,j} \equiv \mathbb{U})$, $F_{r,m+r}(m) \big|_0^\infty \in MD\{F(m)\}$, $F_{r,2m-r}(M) \big|_0^m \in MD\{F_{m,m}(m)\}$, $F_{r,2m-r+1}(m) \big|_0^{m+1} \in MD\{F_{m+1,m}(m)\}$

b) Let $0 \in (\alpha, \beta)$ and $[\alpha', \beta'] \subset (\alpha^{-1}, \beta^{-1})$. Then $0 < F(m) < \infty$, $F_{i,j}(m) < F(m)$ $(F_{i,j} \equiv \mathbb{D})$ and $F_{r,2m+r-1}(m) \big|_0^\infty \in MI'\{F(m)\}$, $F_{r,2m+r-1}(m) \big|_0^{m+1} \in MI'\{F_{m+1,m}(m)\}$

(ii) Let $\Lambda_{\#}(y, m) \in BM_y[\alpha', \beta']$

a) The results of (i) a) (2-3) hold with $F(m)$ and on the relevant $F_{i,j}(m)$ replaced by $|\operatorname{Re} F(m)|$ and $|\operatorname{Re} F_{i,j}(m)|$ respectively, by $|\operatorname{Im} F(m)|$ and $|\operatorname{Im}\{F_{i,j}(m)\}|$, and by $|F(m)|$ and $|F_{i,j}(m)|$.

b) When, as in (i)a)(1) $[\alpha,\beta] \subseteq [0,\infty]$ and $[\alpha',\beta'] \subset (-\infty,\beta^{-1}]$; $|E_{r,2m+r-1}(m)|\,|_0^\infty \in MD\{0\}$, $|E_{m+1,m,r,2m-r+1}(m)|\,|_0^{m+1} \in MD\{0\}$. Further, results of the same type hold for all remaining sequences considered in (i).

B) Let $g = mo\{D;\hat{\alpha},\hat{\beta}\}$, $g(x) = st\{x:D;\hat{\alpha},\hat{\beta}\}$ and set $f(x) = f[x:\sigma:g;\alpha,\beta]$, $[\hat{\alpha},\hat{\beta}] \times [\tilde{\alpha},\tilde{\beta}] = [\alpha',\beta']$

(i) Let $D \in BND[\hat{\alpha},\hat{\beta}]$. With F, M, U and E replaced by f, $[\tilde{\alpha},\tilde{\beta}]$, u and e the results of A(i) hold ;

(ii) Let $D \in BM[\hat{\alpha},\hat{\beta}]$. With modifications similar to those just given, the results of A.(ii) hold.

c) All infinite real sequences for ~~results~~ which results are given in A, B(i) converge, as do the complex sequences considered in A(ii)a) and its counterpart in B(ii).

Proof A.(i) Since $G(t,x)>0$ $(t\in[\alpha,\beta], x\in M)$ in this case, it follows that $F(x)=F[x:\sigma;G;\alpha,\beta]>0$ $(x\in M)$ also.

Let $P_0 \Leftrightarrow$ (1). Under the conditions stated in a)(1) concerning h, all quotients of P_0 are distinct (see the proof of A..). For $m\in I$, the function $\mu[\{X_\nu\}:\sigma$ $m,r]_\alpha^\beta$ $(r\in I_1)$ of formula ( ) has one more disposable parameter than the corresponding function with r replaced by $r-1$. Thus if $y\in(-\infty,\beta^{-1})$, $0<h(y)\leq$ $-P_{r,2m+r-1}(y)\leq h(y)-P_{r-1,2m+r-2}(y)$. But when $y\neq 0$, $P_{r,2m+r-1}(y)\neq P_{r-1,2m+r-2}(y)$ (1..). Hence $0<$

$h(y) - P_{r,2m+r-1}(y) < h(y) - P_{r-1,2m+r-2}(y)$ $(r \equiv \mathrm{II}_1)$. Multiplying the terms of these inequalities by $d\Lambda(y,x)$ and integrating with respect to y over the interval $[\alpha', \beta']$ we obtain the inequalities $0 < F(x) - F_{r,2m+r-1}(x) < F(x) - F_{r-1,2m+r-2}(x)$ $(r \equiv \mathrm{II}_1)$ and hence the result $F_{r,2m+r-1}(m)]_0^x \in MI\{F(m)\}$.

Formula ( ) concerns a minimum over all polynomial $p_r(t)$ of the r^{th} degree in t such that $p_r(y^{-1})^2 = 1$. Formula ( ) with r and m replaced by $r-1$ and $m+1$ respectively concerns a corresponding minimum with the same weight function $t^{2m} d\sigma(t)/(1-yt)$ over all polynomials $\hat{p}_r(t) = yt + \sum_0^{r-1} X_\nu yt(1-yt)^\nu$ of the r^{th} degree in t such that $p_r(+y^{-1})^2 = 1$. The latter formula has one disposable parameter less than the original formula, and hence when $y \in (-\infty, \beta^{-1})$ we deduce that $0 < h(y) - P_{r,2m+r-1}(y) \leq h(y) - P_{r-1,2m+r}(y)$ $(m \in \mathbb{I}, r \in \mathbb{I}_1)$. We derive, as for the preceding results

$$F_{r,2m-r+1}(m)\Big|_0^{m+1} \in MI'\{F_{m+1,m}(m)\} \quad (m \in \mathbb{I}).$$

The remaining results of this clause are deduced from the theory outlined in the proof its predecessor in the same way. With regard to A(i) b), we have only to remark that it may occur (see the proof of .. A..) that, under the conditions of this clause, one or more pairs of quotients of the form $P_{i+1,j}, P_{i,j+1}$ belonging to $\mathbb{P}_0$ may be identical (no three quotients in $\mathbb{P}_0$ being identical). This leads to the special formulation of the result concerning the sequence $F_{r,2m+r-1}(m)$ $(r = \mathbb{I}_0^{m+1})$.

(ii) With the new condition imposed upon $\Delta(y,m)$, a result in (i) such as $0 < F(m) < F_{i,j}(m) < \infty$ leads in particular, to $0 < |\mathrm{Re}\{F(m)\}| < |\mathrm{Re}\{F_{i,j}(m)\}| < \infty$ in (ii)a). Where, however, it is known in (i) only that $0 < F(m) < \infty$, $F(m) > F_{i,j}(m)$, it may not be inferred from the new conditions that $|\mathrm{Re}\{F(m)\}| > |\mathrm{Re}\{F_{i,j}(m)\}|$ (this is false for example, when $\mathrm{Re}\{F_{i,j}(m)\}$ is large and negative). But in all cases, as described in (ii)b), $|E_{i,j}(m)|$ decreases for the sequences considered (whether the value zero is ultimately attained depends, ~~upon~~ of course, upon the convergence behaviour of the $F_{i,j}(m)$, as described in Theorem). (The decreasing sequence $E_{m+1,m,r,2m-r+1}^{(m)}$ $(r = \mathrm{I}_0^{m+1})$ does of course attain zero with its last member)

B. These results are deduced from those of A as in the proof of B.

C. The sequences of A. B.(i) are either increasing or decreasing with upper or lower bound respectively. Those of A(ii)a) and its counter-part in B (ii) are two such sequences.

The position of the function $F_{i,j}(x)$ may be referred to coordinates in the x', y'-plane, and the values of $F(x)$ and $F_{i,j}(x)$ may be measured in the z' direction, x', y' and z' being rectangular coordinates. Under the stated conditions, the results of A.(i)a)(1) indicate that, viewed from the positive z'-direction, the functions of (I) lie on an upturned convex semi hull, with the functions $F_{m+1,m}(x)$ $(m = \mathrm{I\!I})$ lying on the keel. If $[\alpha', \beta'] \times [\alpha, \beta] \subset (-1, 1)$, the sequence $F_{0,m}(x)$ $(m = \mathrm{I\!I})$ converges to $F(x)$: the semi hull has a pronounced prow (it resembles, for example, the hull of a gondola cut down its keel). If -1 or $1 \in (\alpha', \beta') \times (\alpha, \beta)$ and $\sigma(t)$ and $\Delta(y, x)$ have points of increase for some $t \in [\alpha', \beta']$ $y \in (\alpha, \beta)$ such that $|yt| > 1$, then the sequence $F_{0,m}(m)$ diverges: the semi-hull has a receding prow (the gunwales of the ship from which the hull is taken become progressively higher as one retreats from the bow); The plane $z' = F(x)$ lies above the semi-hull. Under the conditions of A. (i) & (a) (2) we are concerned with two semi-hulls. Since, in this case $[\alpha', \beta'] \subset (-\infty, \beta^{-1}$ still, the functions of ① lie on an upturned convex semi-hull as described. Those of Ⓔ lie on a convex semi-hull which has not been upturned, with the functions $F_{m,m}(x)$ lying on its keel. The plane $z' = F(x)$ separates the two semi-hulls. Under the conditions of A(i)a) (3), all functions of $\mathbb{D}$ lie on an upturned convex semi-hull, with the functions $F_{m+1,m}(x)$ $(m \equiv \mathbb{I})$ on its keel. The plane $z' = F(x)$ lies above the semi-hull. Under the conditions of A.(i)a) (4) all functions of $\mathbb{D}$ lie on a similar semi-hull which has not been upturned. The plane $z' = F(x)$ now lies beneath the semi-hull. If the sequence $F_{0,m}(x)$ converges <diverges> (the conditions for this to occur are derived as in the preceding paragraph) the semi-hull concerned has a pronounced <receding> prow.

Under the conditions of A(i) b), the functions of $\mathbb{D}$ lie on an upturned semi-hull as described in connection with A (i) a) (1). Now, however, where $\mathbb{O}$ has identical pairs of functions (lying on backward diagonals) corresponding to identical quotients in $\mathbb{P}$, the semi-hull has dents (If $\sigma \in Sy$, of course, the semi-hull is systematically dented.)

The results of part B. of the theorem may be given a similar geometrical interpretation.

When $\sigma \in St$, the results of the above theorems hold for those functions $F_{i,j}(x)$ and $f_{i,j}(x)$ concerned not belonging to the blocks $\hat{\mathbb{F}}$ and $\hat{\mathfrak{f}}$ considered in .. A and ... B respectively. In this case the semi-hulls described above are hacked away at the water level which coincides with the planes $z' = F(x)$ and $z' = f(x)$.

It is clear that, under the conditions of A.(i) a) (1), i) all the approximations to the numerical value of $F(x)$ offered by a subset of the form $\mathbb{F}^\mu \cap \mathbb{O}$, $F_{m'+1,m'}(x)$, where $m' = [(\mu-2)/2]$, is the best (it lies on the diagonal $F_{m+1,m}(x)$ $(m \equiv \mathrm{II})$ and either on or near to the backward diagonal bounding $\mathbb{F}^\mu$). Similar considerations relate to the other subsets considered in the theorem. In particular, under the conditions of A (i) a) (2), the sequences $F_{m,m}(x)$ $F_{m+1,m}(x)$ $(m \equiv \mathrm{II})$ offer optimal inclusion segments of the form $[F_{m,m-1}(x), F_{m,m}(x)]$ and $[F_{m+1,m}(x), F_{m,m}(x)]$ for $F(x)$. Mutatis mutandis, these remarks hold for the functions $f_{i,j}(x)$.

When $\alpha \in [0,\infty)$, $\beta = \infty$ it may occur that the sequence $h = ms\{\sigma; \alpha, \beta\}$ is associated with an indeterminate moment problem. The functions of $\mathbb{D}$ are then the same but the functions $F(x:\sigma;G;\alpha,\beta)$ differ for each solution σ of the moment problem. Under the conditions of A.(i)a)(1) the upturned semi-hull upon which the functions of $\mathbb{D}$ lie are bounded from above by the minimum of $F(x:\sigma;G;\alpha,\beta)$ as σ ranges over all solutions of the moment problem. Under the conditions of A.(i)a)(2) the two semi-hulls upon which the functions of $\mathbb{D}$ and $\mathbb{E}$ lie are separated by the space defined by $z' = F(x:\sigma;G;\alpha,\beta)$ as σ ranges over these solutions. Similar remarks may be made concerning the remaining semi-hulls associated with the results of A.B(i).

We remark that the result of C holds independently of the determinateness of the moment problem associated with h.

Rates of convergence

Theorem. Let $h = ms(\sigma \in BND_{\alpha}^{\beta})$, $m \in \mathbb{I}$ and $r \in \mathbb{I}_1$.

A. Set $F(x) = F[x : \sigma, G; \alpha, \beta]$ $(x \in \mathbb{Z})$

(i) Denote by $\overline{\mathbb{D}}(\eta; \phi_1, \phi_2)$ the domain in the complex plane containing the points $\eta + \rho e^{i\theta}$ $(\theta \equiv [\phi_1, \phi_2], \rho \equiv [0, \infty])$

Let $G(z, x) = \sum_0^{\mu-1} \hat{G}_\nu(x) z^\nu + \hat{G}(z, x)$ $(\mu \in \mathbb{I}, G_\nu(x) \in \mathbb{Z}$ $(\nu \equiv \mathbb{I}_0^{\mu-1}))$ with $\hat{G} \in \mathcal{G}$. Set $G_m(z, x) = \{\hat{G}(z, x) -$

$\cancel{\sum_0^m G_\nu} - \sum_0^{m-1} G_\nu(x) z^\nu\}. z^m$.

a) Let $[\alpha, \beta] \subseteq [0, \infty]$, $\phi_1 \in (-2\pi, 0)$, $\phi_2 \in (0, \phi_1 + 2\pi)$, $\eta \in (-\infty, 0)$ and $\hat{G}(z \equiv \overline{\mathbb{D}}(\eta : \phi_1, \phi_2), x) \in A_z$. Set $\Omega_m(\eta, \phi) =$

$\int_0^\infty G_m(\eta + te^{i\phi}, x) t^{-3/2} dt / (\sin \frac{1}{2}\phi)^3$.

(1) $|E_{r, m+r+\tau-1}(x)| \leq \{e h_m \sum_1^2 \Omega_m(\eta, \phi_\nu)\} / \{\pi \sum_1^{2r+\tau} (h_m / h_{m+\nu})^{1/2\nu}\}$

$(\tau \equiv \mathbb{I}_0^1, 2r > \mu - m)$

(2) Let $\tilde{h}$ be the sequence defined by formula ( ). Then

$|E_{m+r+\tau+1, r}(x)| = O\{(\sum_1^{2r+\tau} \tilde{h}_{m+\nu}^{-1/2\nu})^{-1}\}$ $(2r \geq \mu - m - 2)$

b) Let $[\alpha,\beta] \subseteq [-\infty,\infty]$, $\phi_1, -\phi_2 \in (-\pi, 0)$, $\phi_1' \in (\phi_2, \pi)$, $\phi_2' \in (-\pi, \phi_1)$, $\eta, -\eta' \in (-\infty, 0)$ and $\hat{G}(z = \{\overline{\mathbb{D}}(\eta; \phi_1, \phi_2) \cup \overline{\mathbb{D}}(\eta'; \phi_1', \phi_2')\}, x) \in A_z$. With ~~$\hat{\eta}(\hat{\phi}, \tilde{\phi})$~~

$$\hat{\eta}(\hat{\phi}, \tilde{\phi}) = \frac{\{\eta' \sin\hat{\phi} \cos\tilde{\phi} - \eta \cos\hat{\phi} \sin\tilde{\phi}\} + i(\eta' - \eta) \sin\hat{\phi} \sin\tilde{\phi}}{\sin(\hat{\phi} - \tilde{\phi})}$$

set $\eta_1 = \hat{\eta}(\phi_1', \phi_2)$, $\eta_2 = -\hat{\eta}(\phi_2', \phi_1)$ and set

$$\hat{\Omega}_m(\tilde{\eta}, \phi) = \int_0^\infty |\hat{G}_m(\tilde{\eta} + te^{i\phi}, x)| \, t^{-2} dt \,/\, (\sin\phi)^2$$

(1)
$$|E_{r+1,2m+r-1}(x)| \; (m>0), \; |E_{r,2m+r-1}(x)|, \; |E_{r,2m+r}(x)| \leq \frac{e h_{2m} \sum_1^2 \{\hat{\Omega}_{2m}(\eta_\nu, \phi_\nu) + \hat{\Omega}_{2m}(\eta_\nu', \phi_\nu')\}}{\pi \sum_1^r (h_{2m}/h_{2m+2\nu})^{1/2\nu}} \quad (2m+2r \geq \mu)$$

(2) Let $\hat{h}$ be the sequence defined by formula ( ).

Then $|E_{2m+r+1,r+1}(x)|$ $(m>0)$, $|E_{2m+r+1}(x)|$, $|E_{2m+r+2,r}(x)| = O\{(\sum_1^r \hat{h}_{2m+2\nu}^{-1/2\nu})^{-1}\}$ $(2r \geq \mu - 2m - 2)$.

(ii) Let $[\alpha,\beta] \subseteq (-\infty,\infty)$. Let $\mathcal{L}_k(\alpha,\beta)$ be the ellipse with foci at α, β and eccentricity $2k/(k^2+1)$ $(k \in (0,1))$ and

$\overline{D}_k(\alpha,\beta)$ be the closed domain with boundary $\mathcal{L}_k(\alpha,\beta)$. Set $K_k = 1$ [[$(1+k^2)/\{4k(1-k^2)\}$]] if $k^2+2k-1 \leq 0$ [[>0]] and $L_k = 4(1+k^6)/\{(\beta-\alpha)(1-k^2)^4\}$. Let $G(z \in \overline{D}_k(\alpha,\beta), x) \in A_2$. Set $M_{k,m}^{(\alpha,\beta)} = \max |G(z,x) - \sum_0^{m-1} G_\nu(x) z^\nu| / |z|^m$ $(z \in \mathcal{L}_k(\alpha,\beta))$.

a) Let $0 \notin (\alpha,\beta)$ and $G \leftarrow g(x)$. Then $|E_{r,mr-1}(x)| \leq K_k L_k M_{k,m}^{(\alpha,\beta)} |h_m| k^{2r+1}$ $(r \in \mathbb{I})$.

b) Let $\alpha > 0$ or $\beta < 0$ and $g(x) \in V$. Then the result of a) holds only for such k that the origin lies outside $\mathcal{L}_k(\alpha,\beta)$ ($\{|\alpha+\beta| - 2(\alpha\beta)^{1/2}\}/(\beta-\alpha)$ is then a lower bound for the possible values of k.

c) Let $0 \in (\alpha,\beta)$ and $G \leftarrow g(x)$. Then $|E_{r,2mr-1}(x)| \leq K_k L_k M_{k,2m}^{(\alpha,\beta)} |h_{2m}| k^{2r+1}$ $(r \in \mathbb{I})$

d) Let $0 \notin \langle 0 \in \rangle (\alpha,\beta)$. Then $E_{mr+r-1}(x) \langle E_{2mr+r-1}(x) \rangle = O(k^{2r}$ as $r \to \infty$.

(iii) Let $[\alpha,\beta] < (-\infty,\infty)$, $0 \notin \langle 0 \in \rangle (\alpha,\beta)$ and $G(z,x)$ be an entire function of z. Then $E_{m+r+1,r}(x)$ $\langle E_{2m+r+1,r}(x) \rangle$ tends to zero as $r \to \infty$ faster than the terms of any geometric progression. If, in addition, $G \leftarrow \mathfrak{H}(x)$, this result holds for the functions $E_{r,m+r-1}(x)$

B. Set $f(x) = f[x : \sigma, g; \alpha, \beta]$ $(x(\neq 0) \in \mathbb{Z})$.

(i) Let $\overline{\mathbb{D}}(\eta;\phi_1,\phi_2)$ be as described in A(i). Let $g(x) = \sum_0^{\mu-1} \hat{g}_\nu x^\nu + \hat{g}(x)$ $(\mu \in \mathbb{I}, \hat{g}_\nu \in \mathbb{Z}$ $(\nu = \mathbb{I}_0^{\mu-1})$ with $\hat{g} \leftarrow g$. Set $g_m(z) = \{\hat{g}(z) - \sum_0^{m-1} g_\nu z^\nu\}/z^m$. Let $\psi = \arg(z)$

a) Let $\alpha, \beta, \phi_1, \phi_2, \eta$ be as in A(i)(a), and let $g(\overline{\mathbb{D}}(\eta x; \phi_1+\psi, \phi_2+\psi) \in A$. Set $\omega_m(\eta;\phi,\psi) = \int_0^\infty |g_m(\eta x + te^{i(\phi+\psi)})| t^{-3/2} dt / (\sin \tfrac{1}{2}\phi)^3$.

$$(1) \quad |e_{r,m+r+\tau-1}(x)| \leq e h_m |x|^{m+\frac{1}{2}} \sum_1^2 \omega_m(\eta x; \phi_\nu, \psi) / \left\{\prod \sum_1^{2r+\tau} (k_m/h_{m+\nu})\right\}^{\frac{1}{2\nu}}$$

$$(\tau = \mathbb{I}_0^1,\ 2r \geq \mu - m)$$

(2) With E replaced by e, the result of A(i)a)(2) holds

b) Let $\alpha, \beta, \phi_1, \ldots, \phi_2', \eta_1$ and η_2 be as in A(i)b), and let

$\hat{g}(\overline{D}(\eta x; \phi_1+\psi, \phi_2+\psi) \cup \overline{D}(\eta' x; \phi_1'+\psi, \phi_2'+\psi)) \in A$. Set

$\hat{\omega}_{2m}(\eta; \phi, \psi) = \int_0^\infty |g_{2m}(\hat{\eta} x + te^{i(\phi+\psi)})| t^{-2} dt / (\sin\phi)^2 = \omega_{2m}(\hat{\eta}; \phi, \psi)$

(1) $|e_{r+1, 2m+r-1}(x)|$ $(m>0)$, $|e_{r, 2m+r-1}(x)|$, $|e_{r, 2m+r}(x)| \leq$

$$\frac{e h_{2m} |x|^{2m+1} \Sigma_1^2 \{\omega_{2m}(\eta_\nu; \phi_\nu, \psi) + \omega_{2m}(\eta_\nu'; \phi_\nu', \psi)\}}{\pi \Sigma_1^r (h_{2m}/h_{2m+2\nu})^{1/2\nu}}$$

(2) With E replaced by e, the results of A(i)b)(2) holds

(ii) Let $[\alpha, \beta] \subset (-\infty, \infty)$, let $\mathcal{L}_k(\alpha, \beta)$, $\overline{D}_k(\alpha, \beta)$, K_k and L_k be as in A(ii), and let $g(\overline{D}_k(\alpha x, \beta x)) \in A$. Set $\hat{M}_{k,m}^{(\alpha,\beta)} =$

$\max |g(z) - \Sigma_0^{m-1} g_\nu z^\nu| / |z|^m$ $(z \in \mathcal{L}_k(\alpha x, \beta x))$

? a) Let $0 \in (\alpha, \beta)$ and $g \in g$. Then $|e_{r, 2m+r-1}(x)| < \frac{K_k L_k \hat{M}_{k,2m}^{(\alpha,\beta)} |x|^{2m}}{} k^{2\nu}$

~~$(r=1)$~~ $K_k L_k \hat{M}_{k,m}^{(\alpha,\beta)} h_m |x|^m k^{2\nu+1}$

b) Let $\alpha > 0$ or $\beta < 0$ and ~~$g \in$~~ $g \in V$. Then mutatis mutandis the result of A(ii)b) holds

c) Let $0 \in (\alpha, \beta)$ and $g \leftarrow g$. Then $|e_{r,2m+r-1}(x)| \leqslant K_k L_k \hat{m}_{k,2m}^{(\alpha,\beta)} |x|^{2m} k^{2r}$

$(r \equiv \mathrm{II})$

d) Let $0 \notin \langle 0 \in \rangle (\alpha, \beta)$. Then $e_{m+r+1,r}(x) \langle e_{2m+r+1,r}(x) \rangle = O(k^{2r})$;

as $r \Rightarrow \infty$

(iii) Let $[\alpha, \beta] \subset (-\infty, \infty)$, $0 \in \langle 0 \in \rangle (\alpha, \beta)$ and g be an entire function.

Then $e_{m+r+1,r}(x) \langle e_{2m+r+1,r}(x) \rangle$ tends to zero as $r \to \infty$ faster than the terms of any geometric progression. If, in addition, $g \leftarrow g$, this result holds for the functions $e_{r,m+r-1}(x)$ $\langle e_{r,2m+r-1}(x) \rangle$.

Proof. The proof of the stated results make use of the relationships between Padé quotients and convergents $C_r^{(m)}(z)$, etc of certain continued fractions. These convergents are defined for $r \in \mathbb{I}$ when $\sigma \notin St$ and for $r \in \mathbb{I}_0^n$ when $\sigma = st\{n; M_\nu, t_\nu\}$. For simplicity in exposition we assume that $\sigma \notin St$. When $\sigma \in St$ the same methods suffice for all functions $F_{i,j}(x)$ not belonging to the block $\hat{\mathbb{F}}$ described in ... A ... ; for $F_{i,j} \in \hat{\mathbb{F}}$, we have $E_{i,j}(x) = 0$ and the results of the theorem are trivially true.

A(i)a)(1) Denoting by $\hat{F}_{i,j}(x)$ the functions of formula ( ) derived from the function $\hat{G}(z,x)$ and the sequence $h_\nu(x)$ alone, we have $F_{i,j}(x) = \hat{F}_{i,j}(x) + \sum_0^{\mu-1} h_\nu G_\nu(x)$ $(i+j \geq \mu-1)$. Setting $\hat{F}(x) = F(x:\sigma; \hat{G}; \alpha, \rho)$, we have $\hat{E}_{i,j}(x) = E_{i,j}(x)$ $(i+j \geq \mu-1)$. We may confine our attention to the functions $\hat{F}_{i,j}(x)$ and $\hat{E}_{i,j}(x)$.

Set $h(z) = st\{z:\sigma; \alpha, \rho\}$ and let $\mathbb{C}$ be the two rectilinear boundaries of $\overline{\mathbb{D}}(\eta; \phi_1, \phi_2)$. $G_m(z \in \overline{\mathbb{D}}(\eta; \phi_1, \phi_2), x) \in A_2$. Hence

$$\hat{F}(x) = (2\pi i)^{-1}\int_{\mathbb{C}} z^{-1}h(z^{-1})\,\hat{G}(z,x)\,dz$$

$$= (2\pi i)^{-1}\int_{\mathbb{C}} z^{-1}h(z^{-1})\left\{\sum_0^{m-1} G_\nu(x)z^\nu + z^m G_m(z,x)\right\}dz$$

$$= \sum_0^{m-1} h_\nu G_\nu(x) + (2\pi i)^{-1}\int_{\mathbb{C}} z^{m-1}h(z^{-1})\,G_m(z,x)\,dz$$

Since $h_\nu^{(r,m+r-1)} = h_\nu$ $(\nu = \mathbb{I}_0^{m-1})$, we have

$$\hat{F}_{r,m+r-1}(x) = \sum_0^{m-1} h_\nu G_\nu(x) + (2\pi i)^{-1}\int_{\mathbb{C}} z^{m-1}P_{r,m+r-1}(z^{-1})\,G_m(z;x)\,dz$$

and

$$\hat{E}_{r,m+r-1}(x) = (2\pi i)^{-1}\int_{\mathbb{C}} z^{m-1}\{h(z^{-1}) - P_{r,m+r-1}(z^{-1})\}\,G_m(z,x)\,dz$$

We examine the behaviour of $z^{m-1}\{h(z^{-1}) - P_{r,m+r-1}(z^{-1})\}$ on $\mathbb{C}$. Let $h' \in \text{ms}(\sigma' \in \text{BND}_{\alpha'}^{\beta'})$. Then $h'_{2m} \in \text{ms}(\sigma'_{2m} \in \text{BND}_{\alpha'}^{\beta'})$ also. The series $\sum h'_{2m+\nu} z^\nu$ generates an associated continued fraction, whose r^{th} convergent we denote by $C_r^{(2m)}(z)$. Then $P'_{r,2m+r-1}(z) = \sum_0^{2m-1} h'_\nu z^\nu + z^{2m} C_r^{2m}(z)$, where $\{P'_{i,j}\}$ are the Padé quotients derived from h'. Let $p_r^{(2m)}(z)$ be the r^{th} degree orthogonal polynomial derived from the moment sequence h'_{2m}, and $q_r^{(2m)}(z)$ its associated orthogonal polynomial. Then $z^{-1}C_r^{(2m)}(z^{-1}) = q_r^{(2m)}(z)/p_r^{(2m)}(z)$. For $\mathrm{Im}(z) \neq 0$

$$\{q_{r+1}^{(2m)}(z) - t q_r^{(2m)}(z)\} / \{p_{r+1}^{(2m)}(z) - t p_r^{(2m)}(z)\}$$

$(t \equiv [-\infty, \infty])$ represents a circle $K_r^{(2m)}(z)$ upon which $z^{-1}C_r^{(2m)}(z^{-1})$ lies. Let $h^{(2m)}(z) = st\{z : \sigma_{2m}; \alpha'; \rho'\}$. It has been shown by Hamburger [ ] and Nevanlinna [ ] that either $z^{-1}h^{(2m)}(z^{-1})$ is equal to one of ~~the~~ $z^{-1}C_{r+\tau}^{(2m)}(z^{-1})$ $(\tau \equiv I_0^1)$ (the associated continued fraction derived from $\sum h'_{2m+\nu} z^\nu$ then terminates with the convergent $C_{r+\tau}^{(2m)}(z)$ in question) or $z^{-1}h^{(2m)}(z^{-1})$ lies inside $K_r(z)$. It may be deduced from theory given by Carleman [ ] that the diameter $\rho_r^{(2m)}(z)$ of $K_r^{(2m)}(z)$ satisfies the inequality

$$\rho_r^{(2m)}(z) \leq 2e h'_m / \{\mathrm{Im}(z)^2 \sum_1^r (h'_{2m}/h'_{2m+2\nu})^{1/2\nu}.$$

Since $h'(z) = st\{z : \sigma'; \alpha', \beta'\} = \sum_0^{2m-1} h'_\nu z^\nu + z^{2m} h^{(2m)}(z)$, we have $|z^{2m-1}\{h'(z^{-1}) - P_{r,2m+r-1}(z^{-1})\}| \leq \rho_r^{(2m)}(z)$, where $\rho_r^{(2m)}(z)$ itself satisfies the inequality( ).

We apply the ~~inequality~~ above theory to the case in which $\sigma' \in S_y$, $-\alpha' = \beta' = \beta^2$, $d\sigma'(t) = 0$ $(t \equiv [0, \alpha^2])$ if $\alpha > 0$, and $\frac{1}{2} d\sigma'(-t) = \frac{1}{2} d\sigma'(t) = d\sigma(t^{\frac{1}{2}})$ $(t \equiv [\alpha^2, \beta^2])$. We then have $h'(z^{\frac{1}{2}}) = h(z)$, $h'_{2\nu} = h_\nu$, $h'_{2\nu+1} = 0$ $(\nu \equiv I)$. Furthermore, $P'_{2r,2m+2r-1}(z^{\frac{1}{2}}) = P_{r,m+r-1}(z)$, $P_{2r+1,2m+2r}(z^{\frac{1}{2}}) = P_{r,m+r}(z)$ $(r, m \equiv I)$. The above inequalities involving $h'(z^{-\frac{1}{2}})$ and $P'_{r,2m+r-1}(z^{-\frac{1}{2}})$ may be presented in terms of $h(z^{-1})$ and $P_{r,m+r-1}(z^{-1})$ or $P_{r,m+r}(z^{-1})$ as is appropriate; they are valid for $Im(z^{\frac{1}{2}}) \neq 0$, in particular for $z \in C$. The points belonging to the upper branch of C are given by $z = \eta + te^{i\varphi_2}$ $(t = [0, \infty])$ and then $|z^{\frac{1}{2}}|\, Im(e^{\frac{1}{2}})^2$

$\geqslant t^{\frac{3}{2}}\{\sin(\frac{1}{2}\phi_2)\}^3$. The stated results follow directly.

(2) With $\overset{\pi}{h}{}'$ as defined above, derive the sequence $\hat{h}$ by means of a formula analogous to ( ). We then have $\hat{h} = ms\{\hat{\sigma} \in BND_A^B\}$, where $[A, B] \subset [\alpha', \beta']$. Again $\hat{\sigma}_{2m} \in BND_A^B$. Denote the rth convergent of the associated continued fraction generated by the series $\sum \hat{h}_{2m+\nu} z^\nu$ by $\hat{C}_r^{(2m)}(z)$. Set $\hat{h}^{(2m)}(z) = st\{z : \hat{\sigma}_{2m}; A, B\}$. As above, $z^{-1}\hat{h}^{(2m)}(z^{-1})$ lies within or upon a circle upon which $z^{-1}\hat{C}_r^{(2m)}(z^{-1})$ lies, and we may derive an inequality involving the difference of these functions and the diameter of the circle. Furthermore,

$$z^{-1}h(z^{-1}) = h_0' / \{z - \textstyle\sum_0^{2m-1} \hat{h}_\nu z^\nu - z^{-2m-1}\hat{h}^{(2m)}(z^{-1})\}$$

and

$$z^{-1}P_{2m+r+1,r}(z^{-1}) = h_0' / \{z - \textstyle\sum_0^{2m-1} \hat{h}_\nu z^\nu - z^{-2m-1}\hat{C}_r^{(2m)}(z^{-1})\}$$

Using the above inequality, we may derive another involving

$z^{-1}h'(z^{-1})$ and $z^{-1}P'_{2m+r+1,r}(z^{-1})$. Again we take $\sigma \in S_{y_3}$ and related to the distribution σ of the current clause as described above. The numbers $\hat{h}_{2\nu}$ $(\nu \geq \mathbb{I})$ derived from $h'_{2\nu} = h_\nu$, $h'_{2\nu+1} = 0$ $(\nu \geq \mathbb{I})$ are then the numbers $\tilde{h}_\nu$ $(\nu \geq \mathbb{I})$ derived from h_ν $(\nu \geq \mathbb{I})$ by means of formula ( ). We may, in short, derive an inequality involving the difference $z^{m-1}\{h(z^{-1}) - P_{m+r+1,r}(z^{-1})\}$ occurring in the formula similar to ( ) for $E_{m+r+1,r}(x)$. However, this inequality also involves the functions $z^{-1}h(z^{-1})$ and $z^{-1}P_{2m+r+1,r}(z^{-1})$ explicitly, and although they are bounded upon $\mathbb{C}$, their values are not readily available, and we cannot give as precise a result as for (1).

?

b) (1) We now take the contour $\mathbb{C}$ to be composed of two parts: the upper and lower boundaries of the union of $\overline{\mathbb{D}}(\eta;\phi_1,\phi_2)$ and $\overline{\mathbb{D}}(\eta';\phi'_1,\phi'_2)$. The upper part of $\mathbb{C}$ has the form of a (possibly assymetric) letter v, and the lower part that of an inverted v; the vertices of these two letters are the points η_1, η_2 respectively. We derive a formula similar to ( ) with m replaced by $2m$, and are able, simply by dropping the dashes attached to the symbols occurring in the proof of a)(1), to derive the inequality $|z^{2m-1}\{h(z^{-1}) - P_{r,2m+r-1}(z^{-1})\}| \leq \rho_r^{(2m)}(z)$, where $\rho_r^{(2m)}(z)$ satisfies the inequality ( ) with the dashes removed. The points z lying on one branch of that part of $\mathbb{C}$ lying in the upper half-plane $Im(z)>0$ are given by $z = \eta_1 + te^{i\phi_2}$ $(t \in [|\eta_1|, \infty])$ and for such points $Im(z) = t\sin\phi_2$. Similar considerations relate to the other three branches of $\mathbb{C}$. We derive the stated inequality for $|E_{r,2m+r-1}(x)|$.

If $p_r^{(2m)}(0)=0$ (i.e., since $z^r p_r^{(2m)}(z^{-1}) = \pi_r^{(r,2m+r-1)}(z)$, $\pi_{r,r}^{(r,2m+r-1)}=0$, or $H_{r+1,2m+r}=0$ (see ...)), then $P_{r,2m+r}(z^{-1}) = P_{r,2m+r-1}(z^{-1})$. In this case $F_{r,2m+r}(x) = F_{r,2m+r-1}(x)$, and the inequality derived for $|E_{r,2m+r-1}(x)|$ holds for $|E_{r,2m+r}(x)|$.

If $p_r^{(2m)}(0)\neq 0$, we set $P_{r,2m+r}(z) = \sum_0^{2m} h_\nu z^\nu + z^{2m+1}\hat{v}_r^{(r,2m+r)}(z)/\pi_r^{(r,2m+r)}(z)$ and define the two polynomials $p_r^{(2m+1)}(z)$, $q_r^{(2m+1)}(z)$ by $p_r^{(2m+1)}(z) = z^r \pi_r^{(r,2m+r)}(z^{-1})$, $q_r^{(2m+1)}(z) = z^{r-1}\hat{v}_r^{(r,2m+r)}(z^{-1})$. Setting $t = p_{r+1}^{(2m)}(0)/p_r^{(2m)}(0)$, expression ( ) reduces to $h_{2m} + z^{-1} q_r^{(2m+1)}(z)/p_r^{(2m+1)}(z)$, i.e. this function lies upon the circle $K_r^{(2m)}(z)$ derived from the numbers h_ν ($\nu = \mathrm{I}_{2m}^{2m+2r+1}$). Again we derive the inequality $|z^{2m-1}\{h(z^{-1}) - P_{r,2m+r}(z^{-1})\}| \leq \rho_r^{(2m)}(z)$, and hence the stated inequality for $|E_{r,2m+r}(z)|$.

The inequality for $|E_{r+1,2m+r-1}(x)|$ is dealt with in the same way.

(2) Discarding the dashes from formulae ( ) and ( ), we may derive an inequality for the difference $|z^{2m-1}\{h(z^{-1}) - P_{2m+r+1,r}(z^{-1})\}|$. As in a)(2), this inequality may be used to estimate $|E_{2m+r+1,r}(x)|$; but the resulting formula involves values of $z^{-1}h(z^{-1})$ and $z^{-1}P_{2m+r+1,r}(z^{-1})$ explicitly and cannot therefore be used to derive results more precise than that stated.

(ii)a) When $[\alpha,\beta]\subseteq[0,\infty]$, $\sigma_m\in BND_{\alpha}^{\beta}$. The series $\Sigma h_{m+\nu}z^{\nu}$ generates an associated continued fraction whose convergents we may denote by $C_r^{(m)}(z)$. Setting $h^{(m)}(z)=\mathcal{S}\{z:\sigma_m;\alpha,\beta\}$, we have $h(z)=\Sigma_0^{m-1}h_\nu z^\nu+$

$z^m h^{(m)}(z)$, $P_{r,mr+r-1}(z) = \sum_0^{m-1} h_\nu z^\nu + z^m C_r^{(m)}(z)$. We derive a formula similar to ( ) for $E_{r,mr+r-1}(x)$, where now $G_m(z,x) = \{G(z,x) - \sum_0^{m-1} G_\nu(x) z^\nu\}/z^m$ and $\mathbb{C}$ is a contour enclosing the real segment $[\alpha,\beta]$ and lying in a domain in which $G_m(z,x)$ is analytic. Using the two relationships just given, we obtain

$$E_{r,mr+r-1}(x) = (2\pi i)^{-1} \int_{\mathbb{C}} z^{-1}\{h^{(m)}(z^{-1}) - C_r^{(m)}(z^{-1})\} G_m(z,x)\,dz$$

It may be deduced from results given by Gragg [ ] that when $z \notin [\alpha,\beta] \subset (-\infty,\infty)$

$$|z^{-1}h^{(m)}(z^{-1}) - z^{-1}C_r^{(m)}(z^{-1})| \le h_m\, k(z;\alpha,\beta)^{2r} / \{|z-\alpha||z-\beta|\, k_1(z;\alpha,\beta)\}$$

where, with $\theta = |\arg\{(z-\alpha)/(z-\beta)\}|$, $k_1(z;\alpha,\beta) = \cos\frac{1}{2}\theta$ when $\theta \in [0,\frac{1}{2}\pi]$ $\langle \cos\frac{1}{2}\theta \sin\theta$ when $\theta \in [\frac{1}{2}\pi,\pi]\rangle$ and $k(z;\alpha,\beta) = |\{(z-\alpha)^{\frac{1}{2}} - (z-\beta)^{\frac{1}{2}}\}/\{(z-\alpha)^{\frac{1}{2}} + (z-\beta)^{\frac{1}{2}}\}|$. Since $G_m(z,x)$ is analytic within and upon $\mathcal{L}_k(\alpha,\beta)$ and the singularities of $z^{-1}h^{(m)}(z^{-1})$ and $z^{-1}C_r^{(m)}(z^{-1})$ are confined to $[\alpha,\beta]$, $\mathbb{C}$ may be taken to coincide with $\mathcal{L}_k(\alpha,\beta)$. $L = (\beta-\alpha)\pi(k^4-k^2+1)/\{2k(k^2-1)\}$ is an upper bound for the length of the circumference of $\mathcal{L}_k(\alpha,\beta)$. $1/\min\{|z-\alpha||z-\beta|\}$ $(z \equiv \mathcal{L}_k(\alpha,\beta))$ is $[4k/\{(\beta-\alpha)(k^2-1)\}]^2$. $1/\min \kappa(z;\alpha,\beta)$ $(z \equiv \mathcal{L}_k(\alpha,\beta))$ is $(1+k^2)K_k/(1-k^2)$ where K_k is as described. Upon $\mathcal{L}_k(\alpha,\beta)$, $k(z;\alpha,\beta)$ is a constant, namely k. We derive the stated inequality.

b) If $G(z \equiv \mathbb{N}\{0\}, x) \in A_z$, $G \leftarrow \mathcal{G}'(x)$ and $G'_{m-1}(x) \neq G_{m-1}(x)$ $(m = I_1)$ then $G_m(z,x)$ ceases to be analytic at the origin. Under these circumstances, $\mathbb{C}$ may not be stretched to coincide with $\mathcal{L}_k(\alpha,\beta)$ when the origin lies upon or within this curve, and we must content ourselves with an $\mathcal{L}_k(\alpha,\beta)$ for which the origin is an exterior point. $\mathcal{L}_k(\alpha,\beta)$ passes through the origin when k has the given limiting value.

c) When $0 \in (\alpha,\beta)$, we have $\sigma_m \in BND_\alpha^\beta$ only for even values of m.

d) These results are similar, and derived in an analogous fashion to the corresponding results of (i)a,b).

(iii) When $G(z,x)$ is an entire function of z, the ellipse $\mathcal{L}_k(\alpha,\beta)$ of (ii) may be taken arbitrarily large, i.e. k may be taken arbitrarily small.

B.(i) When $\hat{G}(z,x) = \hat{g}(z,x)$, $G_\nu(x) = g_\nu x^\nu$ $(\nu = \overline{I}_0^{m-1})$, $\psi = \arg(x)$, the conditions $\hat{G}(z = \mathcal{D}(\eta : \phi_1, \phi_2); x) \in A_z$, $\lim \hat{G}(z,x) = 0$ $(z \in \Delta_{\phi_1}^{\phi_2}) \to \infty$ of A(i)a) are equivalent to $\hat{g}(\overline{\mathcal{D}}(\eta x; \phi_1 + \psi, \phi_2 + \psi) \in A$, $\lim \hat{g}(z) = 0$ $(z (\in \Delta_{\phi_1+\psi}^{\phi_2+\psi}) \to \infty)$. Also, in the notation of A(i), $\Omega_m(\eta, \phi) = |x|^{m+\frac{1}{2}} \omega(x\eta, \phi, \psi)$ and $\Omega(\tilde{\eta}; \phi) = |x|^{2m+1} \hat{\omega}(\tilde{\eta}x; \phi, \psi)$.

The results of (i) now follow from A.(i).

(ii, iii) When $G(z,x) = g(zx)$, the condition that, for $x(\neq 0) \in Z$ $G(z,x)$ should be analytic within and upon $\mathcal{L}_k(\alpha,\beta)$ means that $g(z)$ is analytic within and upon $\mathcal{L}_k(\alpha x, \beta x)$. Furthermore, $\max|\{g(zx) - \sum_0^{m-1} g_\nu x^\nu z^\nu\}/x^m z^m|$ for $z \equiv \mathcal{L}_k(\alpha,\beta)$ is $|x|^m \max|\{g(z) - \sum_0^{m-1} g_\nu z^\nu\}/z^m|$ for $z \equiv \mathcal{L}_k(\alpha x, \beta x)$. The stated results now follow from those of A(ii, iii).

In the convergence proof of ... A.(i)b) we imposed upon the function $\hat{G}(z,x)$ the condition that it should be analytic in a uniform neighbourhood of the nonnegative real axis; in A(i)a) of the current theorem we require that $\hat{G}(z,x)$ should be analytic in a sectorial region containing this part of the real axis. The two conditions are equivalent. They both require that $\hat{G}(z,x)$ should be analytic at the point at infinity. Under the first condition it is impossible that the singularities of $\hat{G}(z,x)$ can extend to infinity along a line of the form $\mathrm{Im}(z)=\delta$, for if this were to be true, $\hat{G}(z,x)$ would no longer be analytic at infinity. Hence the first condition implies the existence of the sectorial region occurring in the first; trivially, such a sectorial region includes a uniform neighbourhood of the nonnegative real axis.

? The result of A (i) a) (1) holds whether the series $\sum_1 (h_m/h_{m+2})^{1/2}$ diverges or not. In the former case, it follows immediately that the sequence $F_{r,m+r-1}(x)$ $(r=\mathrm{I})$ converges for the value of x in question. In this case, also, the sequence h_m is associated with a determinate Stieltjes moment problem, so that convergence also follows from ... A(i)b). (Whether cases of convergence covered by ... A(i)b) but not by ... A(i)a) exist is an open question: Carleman's condition that the divergence of the series $\sum_1^\infty h_{m}^{-1/(2m)}$ ensures that the Stieltjes moment problem associated with h_m is determinate is a sufficient one; but no example is known, at least to the author, in which the series converges and the moment problem is indeterminate.) The above remarks apply, mutatis mutandis, to the results of A.(i)b) and B(i).

The methods used in the proof of A.(i)a) do not allow us to pay special attention to the case in which $\alpha > c$, and $f_j(x) \in V$ is an arbitrary sequence. Carleman's inequality ( ) can only be used directly for values of $z \notin [0, \infty]$. The origin must be an interior point of $\overline{\mathbb{D}}(\eta;\phi_1,\phi_2)$. If $G_m(z,x)$ is to be analytic in this region in particular at the origin, we must take $g(x)$ to be such that $G \leftarrow g(x)$. The same considerations hold with regard to $\mathcal{B}(i)a)$.

When $x=0$ under the conditions of $\mathcal{B}(i)a)$ or $b)$, we have $f_{i,j;\xi}(x) = h_0(g_0 + \hat{g}_0)$ $(i,j;\xi \equiv \mathrm{I})$ and naturally all diagonal sequences of such functions converge. However, if $\sigma(t)$ increases for unbounded values of t, convergence in a uniform neighbourhood of the point $x=0$ does not obtain. As $x \to 0$ under the conditions of $\mathcal{B}(ii)$, k may be taken to be $O(|x|)$ and then, under the further conditions of $\mathcal{B}(ii)a,b)$, we have $|e_{r,m+r-1}(x)| = O(|x|^{m+2r})$. Analogous results hold with regard to the other subclauses of $\mathcal{B}(ii)$.

Finally, we remark that when, under the conditions of A(i;a), the singularities of $G(z,x)$ lie outside a region of the form $D(\eta;\phi_1,\phi_2)$ and in all consist of simple poles at $z=\zeta_\nu$ ($|\zeta_\nu|<M\in(0,\infty)$) with corresponding residues b_ν ($\nu=\overline{0,\tau}$) ($\sum_0^\tau |b_\nu|<\infty$) and a finite number of further poles of higher order, then use of the integral $\Omega_m(\eta,\phi)$ may be dispensed with. In formula ( ) C may be replaced by a system of circles surrounding the $\{\zeta_\nu\}$. We have derived an inequality for $|z^{-1}\{h(z^{-1})-P_{r,2m+r-1}(z^{-1})\}z^{2m}|$ in the proof of A(i;a). The residue of $G_m(z,x)$ at $z=\zeta_\nu$ is b_ν/ζ_ν^{m} ($\nu=\overline{0,\tau}$). We derive

$$|E_{r,m+r-1}(x)|=\frac{2eh_m\sum_0^\tau |b_\nu|/\{|\zeta_\nu|^m \operatorname{Im}(\zeta_\nu)^2\}}{\sum_1^{2r}(h_m/h_{m+\nu})^{1/2\nu}} \quad (2r>\mu-m)$$

Similar special results may be derived for the other functions $E_{i,j}(x)$ considered in A(i), and also, using a different type of inequality, for those considered in A(ii). Further results may be derived for the functions $e_{i,j}(x)$ of B. In particular, taking $g(x) = 1/(1-x)$, we derive results for the Padé table itself.

Results deriving from the theory of the moment problem

Structural theory

Notation. $\sigma \in BND[\alpha,\beta]$ means that $\sigma(t)$ is a bounded nondecreasing real valued function for $t \in [\alpha,\beta]$. $h = ms\{\sigma \in BND[\alpha,\beta]\}$ means that the numbers h_ν $(\nu \in \underline{I})$ are given by formula ( ), where $\sigma \in BND[\alpha,\beta]$. σ_m $(m \in \underline{I})$, where $\sigma(t)$ is prescribed for $t \in [\alpha,\beta]$, is the distribution given by $d\sigma_m(t) = t^m d\sigma(t)$ $(t \in [\alpha,\beta])$. $\sigma = s\{M_\nu, t_\nu; n\}$ $(n \in \underline{I}_1)$ where $[\alpha,\beta]$ is prescribed, means that $\sigma(t)$ is the simple step function with salti of magnitude $M_\nu \in (0,\infty)$ at the distinct points $t = t_\nu \in [\alpha,\beta]$ $(\nu = \underline{I}_1^n)$; $\sigma \in Sy$ means that σ is symmetric over a prescribed interval $[\alpha,\beta]$, i.e. that $\alpha = -\beta$ and $d\sigma(-t) = d\sigma(t)$ $(t \in [0,\beta])$. $F(x:\sigma; G:\alpha,\beta) < f[x:\sigma; g:\alpha,\beta]>$

denotes the integral expression in formula ( )<( )>.

$\mathbb{O}$ <$\bar{\mathbb{O}}$> is the subset of functions of $\mathbb{U}$ given by $F_{i,i+2m-1}$ ($i, m \in \underline{\mathbb{I}}$ ($i+2m \geq 1$)) <augmented by the functions $F_{0,2m}$ ($m \in \underline{\mathbb{I}}$)>. $\bar{\mathbb{U}}$ <$\bar{\bar{\mathbb{U}}}$> is $\mathbb{U}$ augmented by the sequences of functions $F_{i+2,i}$ ($i \in \underline{\mathbb{I}}$) and $F_{2i+3,2i}$ < and $F_{i+3,i}$ > ($i \in \underline{\mathbb{I}}$). $E(x) = \hat{O}\{G_\nu(x); M : x'\}$ means that the relationships $E(x) = O\{G_\nu(x); M : x'\}$ means that the relationships $E(x) = O\{G_\nu(x); M : x'\}$ and $E(x) \neq o\{G_\nu(x); M : x'\}$ hold simultaneously; $e(x) = \hat{O}\{x^\nu; \Delta^\phi_\phi\}$ and $e(x) = \hat{O}\{\bar{x}^\nu\}$ have similar meanings.

Theorem . In the following, let $h = ms\{\varpi \in BND[\alpha, \beta]\}$

A) Set $F(x) = F(x : \varpi, G; \alpha, \beta]$

(i) Let $g(m) \in \mathbb{V}$

a) If $\sigma = st\{M_\nu, t_\nu ; 1\}$ with $t_1 = 0$, then all components of

$h\mathfrak{y}(m)$ with suffix $\nu \geq 1$ are zero

b) Otherwise (1) if $0 \notin (\alpha, \beta)$, then $\varkappa\{h\mathfrak{y}(m)\} = \varkappa\{\mathfrak{y}(m)\}$; (2) if $0 \in (\alpha, \beta)$ it can occur that $h_{2\nu+1} = 0$ ($\nu \in \mathrm{I\!I}$) in which case, if $G_{2\nu+1}(m) \neq 0$ for the ν in question, $h\mathfrak{y}(m)$ possesses a gap where $\mathfrak{y}(m)$ does not; (3) if $\sigma \in \mathfrak{S}y$, then $h_{2\nu+1} = 0$ ($\nu \equiv \mathrm{I\!I}$) and the components of $h\mathfrak{y}(m)$ with suffix $2\nu+1$ are all zero ($\nu \equiv \mathrm{I\!I}$).

(ii) a) Let $G \sim [(\alpha, \beta), \mathfrak{y}(m) : x']$ and $F(m) \in W$. Then $F(m) \sim \{h, \mathfrak{y} : x'\}$.

b) Let $|\alpha|, |\beta| < \gamma \in (0, \infty)$ and $G(t \equiv \mathrm{I\!D}_\gamma, M) \in A_t$ with $G \leftarrow \mathfrak{y}(m)$. (1) $\sum h_\nu G_\nu(m) \rightarrow F(M)$. (2) If, in addition, $\mathfrak{y}(m) \sim x'$, then $F(m) \sim \{h, \mathfrak{y} : x'\}$

c) Let $[\alpha, \beta] < (-\infty, \infty)$, $\sigma \in st\{M_\nu, t_\nu; n\}$ and $G([\alpha, \beta], M) \in V$; let none of the t_ν be zero $\ll$ let one, namely

t_1, is the zero, and let $G_0(m) \in V$ ≫. Then $F(m) = \sum_1^n M_\nu G(t_\nu; m)$

≪ $M_1 G_0(m) + \sum_2^n M_\nu G(t_\nu; m)$ ≫

(iii) Let $[\alpha, \beta] \subset (-\infty, \infty)$, $G([\alpha, \beta], m) \in V$, $\sigma \notin Sy$ ⟦$\in Sy$⟧ and $\sigma = st\{M_\nu, t_\nu; n\}$ with none {one} is the t_ν zero {and $G_0(m) \in V$}. Then, with $F(m)$ as ~~described~~ defined in (ii)c), $F_{i,j}(m) = F(m)$ for $F_{i,j} \equiv \hat{F}$, where $\hat{F} = F_{n,n-1}[\infty]$ ⟦$F_{1,n-2}[\infty]$⟧ {$F_{n-1,n-1}[\infty]$}.

(iv) Let $G(t, m) = \sum_0^\mu G_\nu(m) t^\nu$ ($\mu \in I$)

a) $F(m) = \sum_0^\mu h_\nu G_\nu(m)$

b) Let $0 \notin (\alpha, \beta)$. Then $F_{i,j}(m) = F(m)$ ($F_{i,j} \equiv F - F^\mu$)

c) Let $0 \in (\alpha, \beta)$ and $\sigma \notin Sy$. If μ is odd, then $F_{i,j}(m) = F(m)$ ($F_{i,j} \equiv F - F^\mu$) and if $H_{i+1,\mu-i} = 0$ then $F_{i,\mu-i-1}(m) = F(m)$ also ($i = I_0^{\mu-1}$). If μ is even then $F_{i,j}(m) = F(m)$ ($F_{i,j} \equiv F - F^{\mu+1}$), and if $H_{i,\mu-i} \neq 0$ then $F_{i,\mu-i}(m) =$

$F(M)$ $(i = \mathbb{I}_0^{\mu})$

d) Let $\sigma \in Sy$. The results of c) hold with the provision that in this case $H_{2i+1,2j+1} = 0$ $(i, j = \mathbb{I})$.

(v) Let $G([\alpha,\beta], M)$, $\mathfrak{y}(M) \in \mathbb{V}$, $\sigma \notin St \langle \sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{\mathbb{F}}$ as in (iii)a)$\rangle$ and either $0 \notin (\alpha,\beta)$ or both $[\alpha,\beta] = [-\infty,\infty]$ and $\sigma \notin Sy$ $[0 \in (\alpha,\beta)$ and $\sigma \notin Sy$ $]$ $\{\sigma \in Sy\}$. Then $\tilde{\mathbb{F}}(M) \in \mathbb{W}$, where $\tilde{\tilde{\mathbb{F}}} = \mathbb{U} \langle \mathbb{U} \cup \hat{\mathbb{F}} \rangle$ $[\bar{\mathbb{O}} \langle \mathbb{O} \cup \hat{\mathbb{F}} \rangle]$ $\{\bar{\mathbb{U}} \langle \bar{\mathbb{U}} \cup \hat{\mathbb{F}} \rangle\}$.

(vi) Let $\sigma \notin St \langle \sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{\mathbb{F}}(M)$ as defined in (iii)a)$\rangle$. In the following clauses, $F_{i,j}(M) \in \mathbb{W}$ for $F_{i,j} = \acute{\mathbb{F}}$, where $\acute{\mathbb{F}}$ is as defined in each case.

a) Let $[\alpha,\beta] \in [0,\infty)$ and $G(z = \{\bar{\mathbb{D}}_{2\gamma} - [-2\beta, -\beta], M)$ $\in A_z$. Then $\acute{\mathbb{F}} = \mathbb{L} \langle \mathbb{L} \cup \hat{\mathbb{F}} \rangle$

b) Let $|\alpha|, |\beta| < \gamma \in (0,\infty)$ and $G(z = \mathbb{D}_{2\gamma}, M) \in A_z$.

Then $\acute{\mathbb{F}} \equiv \mathbb{R}'' < \mathbb{R}'' \cup \hat{\mathbb{F}} >$. If, in addition, $G_0(m) \in V$, then $\acute{\mathbb{F}} \equiv \mathbb{R} < \mathbb{R} \cup \hat{\mathbb{F}} >$.

c) Let $|\alpha|, |\beta| < \gamma \in (0, \infty)$ and $G(z \equiv \{ \mathbb{D}_{2\gamma} \cup \{\pm [2\gamma, \infty]^{\pi/3}_{-\pi/3}\}\}\}, m) \in A_2$. Then $\acute{\mathbb{F}} \equiv \mathbb{L}'' < \mathbb{L}'' \cup \hat{\mathbb{F}} >$. If, in addition $G_0(m) \in V$, then $\acute{\mathbb{F}} \equiv \mathbb{L} < \mathbb{L} \cup \hat{\mathbb{F}} >$.

d) Let $\sigma \in Sy$, $\beta \in (0, \infty)$ and $G(z \equiv \overline{\mathbb{D}}_{2\beta} - \{ i * \{\beta, 2\beta]_{\S} \cup [-2\beta, -\beta]\}\}, m) \in A_2$. Then $\acute{\mathbb{F}} \equiv \overline{\mathbb{L}}'' < \overline{\mathbb{L}}'' \cup \hat{\mathbb{F}} >$. If, in addition, $G_0(m) \in V$, then $\acute{\mathbb{F}} \equiv \overline{\mathbb{L}} < \overline{\mathbb{L}} \cup \hat{\mathbb{F}} >$.

(vii) a) Let $0 \in (\alpha, \beta)$. It can occur that, when $|i' - j'| = 2m$ $(i', j', m \in \mathbb{I})$, $H_{i'+1, j'+1} = 0$; and if when this occurs, $F_{i'', j''}(m) \in W$ for some $\overline{i'', j''} \in [i', j'; 1]$, then $F_{i,j}(m) \in W$ $(\overline{i, j} \equiv [i', j'; 1])$

b) Let $\sigma \in Sy$ and $F_{i'', j''}(m) \in W$ for some $i'', j'' \in [2i', 2j'; 1]$ $(i', j' \in \mathbb{I})$. Then $F_{i,j}(m) \in W$ $(\overline{i, j} \equiv [2i', 2j'; 1])$.

(vii) Let $\sigma \in St \langle \sigma = st\{M_\nu, t_\nu ; n\} \rangle$ and $\hat{\mathbb{F}}, \tilde{\hat{\mathbb{F}}}$ be as defined in (iii)a),(iv)a)

a) Let $G \sim [(\alpha,\beta), \mathfrak{h}(m) : x']$, $F(m) \in W$, and either $0 \notin (\alpha,\beta)$ or both $[\alpha,\beta] \equiv [-\infty,\infty]$ and $\sigma \notin Sy$ $[\![0 \in (\alpha,\beta)$ and $\sigma \notin Sy]\!]$ $\lfloor \sigma \in Sy \rfloor$. Then $F_{i,j}(m) \sim \{h_{i,j}, \mathfrak{h} : x'\}$ $(F_{i,j} \equiv \tilde{\hat{\mathbb{F}}})$

b) Let $|\alpha|, |\beta| < \gamma \in (0,\infty)$, $G(z \equiv \mathbb{D}_\gamma, M) \in A_z$, $G \leftarrow \mathfrak{h}(m)$ and $0 \in (\alpha,\beta)$ $[\![0 \in (\alpha,\beta)$ and $\sigma \notin Sy]\!]$ $\lfloor \sigma \in Sy \rfloor$. (1) $\sum h_\nu^{(i,j)} G_\nu(m) \to F(m)$ $(F_{i,j} \equiv \tilde{\hat{\mathbb{F}}})$. (2) If, in addition $\mathfrak{h}(m) \sim x'$, then $F_{i,j}(m) \sim \{h_{i,j}, \mathfrak{h} : x'\}$ $(F_{i,j} \equiv \mathbb{F})$

c) Let $|\alpha|, |\beta| < \gamma \in (0,\infty)$, $G(z \equiv \mathbb{D}_{2\gamma}, M) \in A_z$ and $G \leftarrow \mathfrak{h}(m)$. (1) If $0 \notin (\alpha,\beta)$ or $\sigma \in Sy$ then the results of §b)(1,2) hold with $\tilde{\mathbb{F}}$ replaced by $\mathbb{F}$. (2) If $0 \in (\alpha,\beta)$ they hold with $\tilde{\hat{\mathbb{F}}}$ replaced by $\mathbb{I} \cup \mathbb{R} \langle \mathbb{I} \cup \mathbb{R} \cup \hat{\mathbb{F}} \rangle$.

(ix) Let $G \sim [(\alpha,\beta), \mathfrak{h}(m) : x']$ with $x\{\mathfrak{h}(m)\}_j^? = \mu_\pi$, let

$F(M) \in W$, and let $\mathcal{B} \notin St$ $\langle \sigma = st\{M_\nu, t_\nu : x'\}$ with $\hat{\mathbb{F}}$ as in (iii)α)$\rangle$

a) Let $o \in (\alpha, \beta)$. Then $E_{i,j}(x) = \hat{O}\{G_{\mu\{\lambda(i+j+1)\}}(x); M : x'\}$ $(F_{i,j} \equiv \mathbb{D}^{\mu(\tau)} \langle \mathbb{U}^{\mu(\tau)} - \hat{\mathbb{F}} \rangle)$

b) Let $o \in (\alpha, \beta)$. (1) $E_{i,j}(x) = O\{G_{\mu\{\lambda(i+j+1)\}}(x); M : x'\}$ $(F_{i,j} \equiv \mathbb{O}^{\mu(\tau)} \langle \mathbb{O}^{\mu(\tau)} - \hat{\mathbb{F}} \rangle)$. (2) If $G_{2\nu}(M) \neq 0$ for any $2\nu \in \Pi^{\mu(\tau)}_{\mu\{\lambda(i+j+1)\}}$ then $E_{i,j}(x) \neq o\{G_{2\nu}(x); M : x'\}$; in particular, (3) if $G_{i+j+1}(M) \neq 0$, then $E_{i,j}(x) = \hat{O}\{G_{i+j+1}(x); M : x'\}$ $(F_{i,j} \in \mathbb{O}^{\mu(\tau)} \langle \mathbb{O}^{\mu(\tau)} - \hat{\mathbb{F}} \rangle)$.

c) Let $o \in (\alpha, \beta)$ and let $F_{i,j} \in \bar{\bar{\mathbb{D}}}^{\mu(\tau)} - \mathbb{O} \langle \bar{\bar{\mathbb{D}}}^{\mu(\tau)} - \mathbb{O} - \hat{\mathbb{F}} \rangle$ be such that if $i, j \in [s+2, s; 1]$ then $H_{s+3, s+1} = 0$ in which case we set $\xi(i,j) = 2s+4$, or $H_{i,j} \ll H_{i+1, j+1} \gg = 0$ in which case we set $\xi(i,j) = i+j \ll i+j+2 \gg$.

(1) $E_{i,j}(x) = O\{G_{\mu[\lambda\{\xi(i,j)\}]}(x); M : x'\}$. (2) If $G_{2\nu}(M) \neq$

0 for any $2\nu \in \underline{\Pi}^{\mu(\tau)}_{\mu}[\lambda \S \xi(i,j)\}]$ then $E_{i,j}(x) \neq 0 \{G_{2\nu}(x); M : x'\}$;

in particular, (3) if $G_{\xi(i,j)} \neq 0$, then $E_{i,j}(x) \neq 0 \{G_{\xi(i,j)}(x);$

$M : x'\}$

d) Let $[\alpha,\beta] \equiv [-\infty,\infty]$ and $F_{i,j} \in \mathbb{U}^{\mu(\tau)} - \mathbb{O} < \mathbb{U}^{\mu(\tau)} - \mathbb{O} - \hat{\mathbb{F}} >$

If $H_{i,j} \ll H_{i+1,j+1} \gg = 0$ set $\xi(i,j) = i+j \ll i+j+2 \gg$;

otherwise (i.e. if $H_{i,j}, H_{i+1,j+1} \neq 0$) set $\xi(i,j) = i+j+1$. The

results of c) (1,3) hold.

(x) Let $|\alpha|, |\beta| < \gamma \in (0,\infty)$, $G(z = D_{2\gamma}, M) \in A_z$, $G \in$

$\mathcal{G}(m)$, $x\{\mathcal{G}(m)\} = \mu_\tau$, $\mathcal{G}(m) \sim x'$ and let $\sigma \notin St < \sigma =$

$st\{M_\nu, t_\nu; n\}$ with $\hat{\mathbb{F}}$ as in (iiia)>

a) The results of (ix) b, c) hold as stated

b) Let $0 \notin (\alpha,\beta)$ The result of (ix) a) holds with

$\mathbb{U}$ replaced by $\mathbb{F}$.

c) Let $0 \in (\alpha,\beta)$. The results of (ix) b) (1,3) hold with

$\mathbb{O}$ replaced by $\mathbb{R}$; those of (ix) c) (1,3) hold with $\overline{\overline{\mathbb{D}}}^{\mu(\tau)} - \mathbb{O}$ replaced by $\mathbb{L}^{\mu(\tau)} - \mathbb{R}$.

d) Let $\sigma \in Sy$. The results of (ix) e) hold with $\mu(\tau)$ replaced by $\mathbb{F}^{\mu(\tau)}$.

(xi) Let $G \sim [(\alpha,\beta), lg(M) : x']$ with $x\{lg(M)\} = \mu_\pi$ and $\sigma \in St$ $\langle \sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{\mathbb{F}}$ as in (iii)n)$\rangle$

a) Let $0 \notin (\alpha,\beta)$ $j'-i' \geq j-i$, $\overline{i,j} \neq \overline{i',j'}$ and $F_{i,j}, F_{i',j'} \in \mathbb{D}^{\mu(\tau)} \langle \mathbb{D}^{\mu(\tau)} - \hat{\mathbb{F}} \rangle$. (1) If $i+j \geq i'+j'$ then $E_{i,j,i',j'}(x) = \hat{O}\{G_{\mu\{\lambda(i'+j'+1)\}}(x); M : x'\}$ (2) If $i+j < i'+j'$ then $E_{i,j,i',j'}(x) = O\{G_{\mu\{\lambda(i+j+1)\}}(x); M : x'\}$ and if there exists an $\nu \in I_{i+j+1}^{i'+j'}$ for which $G_\nu(M) \neq 0$, then $E_{i,j,i',j'}(x) \neq o\{G_\nu(x); M : x'\}$

b) Let $0 \in (\alpha,\beta)$, $j'-i' \geq j-i$ and $F_{i,j}, F_{i',j'} \in \mathbb{O}^{\mu(\tau)} \langle \mathbb{O}^{\mu(\tau)} - \hat{\mathbb{F}} \rangle$ be such that if $\overline{i,j}, \overline{i',j'} \in [r, r+2m; 1]$

then $H_{r+1,r+2m+1} \neq 0$. (1) If $i+j > i'+j'$ then $E_{i,j,i',j'}(x) = O\{G_{\mu\{\lambda(i'+j'+1)\}}(x); M : x'\}$ and (2) if there exists also $\lambda\tau \in \mathbb{I}^{\mu(\tau)}_{i'+j'+1}$ for which $G_{\lambda\tau}(M) \neq 0$, then $E_{i,j,i',j'}(x) \neq o\{G_{\lambda\tau}(M); M : x'\}$. In particular, if $G_{i'+j'+1}(M) \neq 0$ then $E_{i,j,i',j'} = \hat{O}\{G_{i'+j'+1}(x) : M; x'\}$. If $i+j = i'+j'$ and either $|i-i'| > 1$ or $|i-i'| = 1$ and $H_{i,j+1} \neq 0$, then the results of (1,2) also hold. The result of (a) 2) holds.

c) Let $\sigma \in (\alpha, \beta)$. If $F_{i,j} \in \mathbb{O}^{\mu(\tau)} < \mathbb{O}^{\mu(\tau)} - \hat{\mathbb{F}} >$ set $I = i$, $J = j$; if $\overline{i,j} \in [s+2, s; 1]$ < and $F_{s+2,s} \notin \hat{\mathbb{F}}$ > with $H_{s+3,s+1} = 0$ set $I = s+2$, $J = s+1$; if $F_{i,j} \in \mathbb{U}^{\mu(\tau)} - \mathbb{O}^{\mu(\tau)} < \mathbb{U}^{\mu(\tau)} - \mathbb{O}^{\mu(\tau)} - \overline{\mathbb{F}} >$ with $H_{i,j} \ll H_{i+1,j+1} \gg = 0$, set $I = i-1$, $J = j \ll I = i$, $J = j+1 \gg$. Determine integers I', J' from $F_{i',j'}$ in the same way. If $J'-I'$

$\geq J-I \ll J-I \geq J'-I' \gg$ the results of b) hold with i,j,i',j' replaced by $I,J,I',J' \ll I',J',I,J \gg$.

d) Let $[\nu,\rho] = [-\infty,\infty]$ and $i \geq i'$; let $F_{i,j}, F_{i',j'} \in \bar{\bar{\mathbb{U}}}^{\mu(\tau)} < \bar{\bar{\mathbb{U}}}^{\mu(\tau)} - \hat{\mathbb{F}} >$ be such that (1) if $\overline{i,j}, \overline{i',j'} \in [i'',j'';1]$ $(|i''-j''| = 2m,\ i'',j'' \in \mathbb{I},\ m \in \mathbb{I})$ then $H_{i''+1,j''+1} \neq 0$ and (2) if $\overline{i,j} \in [s+2,s;1]$ then $H_{s+3,s+1} = 0$ with $F_{i',j'}$ similarly qualified. Determine integers I_0, J_0 from $F_{i,j}$ as follows: if $F_{i,j} \in \bar{\bar{\mathbb{U}}}^{\mu(\tau)} - \bar{\bar{\mathbb{O}}}^{\mu(\tau)} < \bar{\bar{\mathbb{U}}}^{\mu(\tau)} - \bar{\bar{\mathbb{O}}}^{\mu(\tau)} - \hat{\mathbb{F}} >$ and $H_{i,j} \ll H_{i+1,j+1} \gg = 0$, set $I_0 = i-1,\ J_0 = j \ll I_0 = i,\ J_0 = j+1 \gg$, otherwise set $I_0 = i,\ J_0 = j$; if $F_{i,j} \in \bar{\bar{\mathbb{O}}}^{\mu(\tau)} < \bar{\bar{\mathbb{O}}}^{\mu(\tau)} - \hat{\mathbb{F}} >$ and $H_{i,j+1} \neq 0 \ll = 0 \gg$, set $I_0 = i,\ J_0 = j \ll I_0 = i-1,\ J_0 = j+1 \gg$. Determine integers I_1, J_1 from $F_{i',j'}$ as follows: If $F_{i',j'} \in \bar{\bar{\mathbb{U}}}^{\mu(\tau)} - \bar{\bar{\mathbb{O}}}^{\mu(\tau)} < \bar{\bar{\mathbb{U}}}^{\mu(\tau)} - \bar{\bar{\mathbb{O}}}^{\mu(\tau)} - \hat{\mathbb{F}} >$ and $H_{i',j'} \ll H_{i'+1,j'+1} \gg = 0$ set $I_1 =$

i', $J_1' = j'-1$ $\ll I_1 = i'+1$, $J_1 = j' \gg$, otherwise, set $I_1 = i'$

$J_1 = j'$; if $F_{i',j'} \in \overline{\overline{\mathbb{II}}}^{\mu(\tau)} \langle \overline{\overline{\mathbb{II}}}^{\mu(\tau)} - F \rangle$ and $H_{i'+1,j'} \neq 0$

$\ll = 0 \gg$ set $I_1 = i'$, $J_1 = j'$ $\ll I_1 = i'+1$, $J_1 = j'-1 \gg$. Set

$\xi = \min(I_0 + J_0 + 1, I_1 + J_1 + 1)$. (3) $E_{i,j,i',j'}(x) = 0\{$

$G_{\mu\{\lambda(\xi')\}}(x); M : x'\}$. (4) If $I_0 + J_0 \neq I_1 + J_1$ and

$G_{\xi'}(M) \neq 0$, then $E_{i,j,i',j'}(x) = 0\{G_{\xi'}(x); M : x'\}$.

(5) If $I_0 + J_0 = I_1 + J_1$, $r \leq \mu(\tau)$ where $r = \max(I_0 + J_1,$

$I_1 + J_0)$ and $G_\nu(M) \neq 0$ $(\nu = \mathbb{I}_\xi^r)$, then $E_{i,j,i',j'}(x)$

$\neq 0\{G_r(x); M : x'\}$

e) Let $\varkappa \in Sy$. Set $\hat{G}_{2\nu}(M) = G_{2\nu}(M)$ $(\nu = \mathbb{I}_0^{[\mu(\tau)/2]})$,

$\hat{G}_{2\nu-1}(M) = 0$ $(\nu = \mathbb{I}_1^{[\mu(\tau)/2]})$, $\varkappa\{\hat{f}(M)\} = \hat{\mu}_\tau$, and let

$\hat{\lambda}(\nu)$ be the integer function derived from $\hat{\mu}(\nu)$ in

the same way that $\lambda(\nu)$ is derived from $\mu(\nu)$. Let

$F_{i,j}, F_{i',j'} \in \overline{\mathbb{II}}^{\hat{\mu}(\hat{\tau})} \langle \mathbb{II}^{\hat{\mu}(\hat{\tau})} - \hat{F} \rangle$ where $\overline{i,j} \in [\{i_0,$

$2j_0;1]$, $\overline{i',j'} \in [2i_1, 2j_1; 1]$ $(\overline{i_0,j_0} \neq \overline{i_1,j_1})$ and $j_1 - i_1 \geq j_0 - i_0$. (1) If $i_0 + j_0 \geq i_1 + j_1$ then $E_{i,j,i',j'}(x) = \hat{O}\{G_{\hat{\mu}[\hat{\lambda}\{2(i_1+j_1+1)\}]}(x); M : x'\}$. (2) If $i_0 + j_0 < i_1 + j_1$ then $E_{i,j,i',j'}(x) = O\{G_{\hat{\mu}[\hat{\lambda}\{2(i_0+j_0+1)\}]}(x); M : x'\}$ and (3) if there exists a $\nu \in \underline{I}^{i_1+j_1}_{i_0+j_0+1}$ for which $G_\nu(m) \neq 0$, then $E_{i,j,i',j'}(x) = o\{G_\nu(x); M : x'\}$.

(xii) Let $|\alpha|, |\beta| < \gamma \in (0, \infty)$, $G(z \in \mathbb{D}_{2\gamma}, M) \in A_z$, $G \leftarrow g(m)$; $g(M) \sim x'$, $x\{g(M)\} = \mu_z$ and $\sigma \notin St$ $\langle \sigma = st\{M_\nu, t_\nu; n\}\rangle$ with $\hat{\mathbb{F}}$ as defined in (iii)$\rangle$

a) The results of (ix) a, b, c, e) hold as stated

b) Let $0 \notin (\alpha, \beta)$ and $\overline{i,j} \neq \overline{i',j'}$. (1) If $F_{i,j}, F_{i',j'} \in \mathbb{L}^{\mu(\tau)} \langle \mathbb{L}^{\mu(\tau)} - \mathbb{F}\rangle$ and $i' - j' \geq i - j$ then the results of (xi) a) (1, 2) hold. Let $F_{i,j}, F_{i',j'} \in \mathbb{F}^{\mu(\tau)} \langle \mathbb{F}^{\mu(\tau)} - \hat{\mathbb{F}}\rangle$ and $\xi = \min(i+j+1, i'+j'+1)$; then $E_{i,j,i',j'}(x) =$

$O\{G_\xi(m)\}; M; x'\}$; if $i+j = i'+j'$ and $G_\xi(M) \neq 0$ then $E_{i,j,i',j'}(x) = \hat{O}\{G_\xi(x); M : x'\}$; if $i+j = i'+j'$ and $G_\nu(M) \neq 0$ $(\nu = \underline{II}_\xi^{r-1})$ where $r = \max(i+j', i'+j)$ then $E_{i,j,i',j'}(x) \neq o\{G_\nu(x); M : x'\}$.

c) Let $\sigma \in (\omega, \rho)$, $\overline{i,j} \neq \overline{i',j'}$ and $i \geq i'$; let $F_{i,j}, F_{i',j'} \in \mathbb{O}^{\mu(\sigma)} \cup \mathbb{R}^{\mu(\sigma)} <\{\mathbb{O}^{\mu(\sigma)} \cup \mathbb{R}^{\mu(\sigma)}\} - \hat{F}>$ be such that if $\overline{i,j}, \overline{i',j'} \in [r, r+2m; 1]$ $\ll [r+2m, r; 1] \gg$ $(r, m \in \underline{II})$ then $H_{r+1, r+2m+1}$ $\ll H_{r+2m+1, r+1} \gg \neq 0$. If $H_{i,j+1} \neq 0$ $\ll = 0 \gg$ set $I_0 = i$, $J_0 = j$ $\ll I_0 = i-1, J_0 = j+1 \gg$. If $H_{i'+1,j'} \neq 0$ $\ll = 0 \gg$ set $I_1 = i'$, $J_1 = j'$ $\ll I_1 = i', J_1 = j'-1 \gg$. Set $\xi = \min(I_0+J_0+1, I_1+J_1+1)$. The results of (xi)d) (3–5) hold.

d) Let $\sigma \in Sy$. Determine the integer functions $\hat{\mu}(\nu)$ and $\hat{\lambda}(\nu)$ and the integers $i_0, \ldots, j_1$ as in (x)e).

If $F_{i,j}, F_{i',j'} \in \mathbb{L}^{\mu(\tau)} < \mathbb{L}^{\mu(\tau)} - \hat{\mathbb{F}} >$, the results of (x)d)(1-3) hold.

(2) Let $F_{i,j}, F_{i',j'} \in \mathbb{F}^{\mu(\tau)} < \mathbb{F}^{\mu(\tau)} - \hat{\mathbb{F}} >$ and $\xi = \min(i_0+j_0+1, i_1+j_1+1)$; then $E_{i,j,i',j'}(x) = O\{G_{2\xi}(x); M : x'\}$. If $i_0+j_0 \neq i'+j'$ and $G_{2\xi}(m) \neq 0$, then $E_{i,j,i',j'}(x) = \hat{O}\{G_{2\xi}(x); M : x'\}$; if $i+j = i'+j'$ and $G_{2\nu}(m) \neq 0$ ($\nu \equiv \xi_{2\xi}^{2r-2}$), where $r = \max(i_0+j_1, i_1+j_0)$, then $E_{i,j,i',j'}(x) \neq o\{G_{2\nu}(x); M : x'\}$.

(xiii) Let $G \sim [(\alpha,\beta), \mathcal{Y}(M) : x']$ with $\varkappa\{\mathcal{Y}(m)\} = \mu_\tau$, let $\mathbb{F}(m) \in W$, and let $\sigma \in St < \sigma = st\{M_\nu, t_\nu; n\} >$ with $\hat{\mathbb{F}}$ as in (iii)a)>.

a) Let $0 \in (\alpha,\beta)$. Then $F_{i,j}(m) \neq F(m)$ ($F_{i,j} \equiv \mathbb{U}^{\mu(\tau)} < \mathbb{U}^{\mu(\tau)} - \hat{\mathbb{F}} >$)

b) Let $0 \in (\alpha,\beta)$ and $F_{i,j} \in \mathbb{O}^{\mu(\tau)} < \mathbb{O}^{\mu(\tau)} - \mathbb{F} >$. (1) If $G_{2\nu}(m) \neq 0$ for any $2\nu \in \mathbb{I}^{\mu(\tau)}_{\mu\{\lambda(i+j+1)\}}$, then $F_{i,j}(m) \neq$

$F(m)$. In particular (2), if $G_{i+j+2}(m) \neq 0$, then $F_{i,j}(m) \neq F(m)$

c) Let $\sigma \in (\alpha, \beta)$ and $F_{i,j}$ satisfy the conditions imposed in (ix) c) with $\xi(i,j)$ as there defined. (1) If $G_{2\nu}(m) \neq 0$ for any $2\nu \in \mathbb{I}^{\mu(\tau)}_{\mu[\lambda\{\xi(i,j)\}]}$ then $F_{i,j}(m) \neq F(m)$; in particular, (2) if $G_{\xi(i,j)}(m) \neq 0$, then $F_{i,j}(m) \neq F(m)$.

d) Let $[\alpha, \beta] \equiv [-\infty, \infty]$ and $F_{i,j} \in \mathbb{W}^{\mu(\tau)} \underline{\quad} \mathbb{O} < \mathbb{W}^{\delta\mu(\tau)} \underline{\quad} \mathbb{O} - \mathbb{F} >$. Determine $\xi(i,j)$ as in (ix). If $G_{\frac{1}{2}\xi(i,j)}(m) \neq 0$ then $F_{i,j}(m) \neq F(m)$.

e) Let $\sigma \in Sy$. Let $F_{i,j} \in \overline{\mathbb{W}}^{\mu(\tau)} < \mathbb{W}^{\mu(\tau)} \underline{\quad} \hat{\mathbb{F}} >$ and $\xi(i,j)$ be as defined in (ix). Then the results of (c) 1,2) hold.

(xiv) Let $|\alpha|, |\beta| < \gamma \in (0, \infty)$, $G(z \in \mathbb{D}_\gamma, m) \in A_z$, $G \leftarrow h(m)$, $\varkappa\{h(m)\} = \mu_\tau$, $h(m) \sim x'$ and $\sigma \notin St < \sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{\mathbb{F}}$ as in (iii)a)>

a) The results of (xiii) b, c) hold as stated

b) Let $0 \in (\alpha,\beta)$. Then the result of (xiii)(a) holds with $\mathbb{U}$ replaced by $\mathbb{F}$; that of (xiii)(b) 2) holds with $\mathbb{O}$ replaced by $\mathbb{R}$.

c) Let $0 \in (\alpha,\beta)$ and $F_{i,j} \in \mathbb{L}^{\mu(\tau)} - \mathbb{R} < \mathbb{L}^{\mu(\tau)} - \mathbb{R} - \hat{\mathbb{F}}$ > and $H_{i,j} \ll H_{i+1,j+1} \gg = 0 \}$; set $\xi(i,j) = i+j \ll i+j+2 \gg$. The result of (xiii)(c) 2) holds

d) Let $\sigma \in Sy$. The result of (xiii)(a) holds with $\mathbb{L}$ replaced by.

(xv) Let $G \sim [(\alpha,\beta), y(m) : x']$ with $x\{y(m)\} = \mu_\sigma$, and $\sigma \notin St < \sigma = st\{m_\sigma, b; n\}$ # with $\hat{\mathbb{F}}$ as in (iii)(a)>

a) Let $0 \notin (\alpha,\beta)$, $j'-i' \geq j-i$, $\overline{i,j} \neq \overline{i',j'}$ and $F_{i,j}$, $F_{i',j'} \in \mathbb{L}^{\mu(\tau)} < \mathbb{L}^{\mu(\tau)} - \hat{\mathbb{F}}$>. If either (1) $i+j \geq i'+j'$ or (2) $i+j < i'+j'$ and $s \in \mathbb{I}_{i+j+}^{i'+j'}$ exists for which $G_s(m) \neq 0$, then $F_{i,j}(m) \neq F_{i',j'}(m)$.

b) Let $0 \in (\alpha,\beta)$, $j'-i' \geq j-i$ and $F_{i,j}, F_{i',j'} \in \mathbb{O}^{\mu(\tau)} <$

$\in \bar{\mathbb{I}}^{\mu(\epsilon)}_{i',j',r_1}$

$\mathbb{U}^{\mu(\epsilon)} - \mathbb{F}>$. If a Z_ν^ν exists for which $G_{\nu}(M) \neq 0$ and either (1) $i+j > i'+j'$ or (2) $i+j = i'+j'$ and $|i-i'| > 1$ or (3) $|i-i'| = 1$ and $H_{i,j+1} \neq 0$, then $F_{i,j}(M) \neq F_{i',j'}(M)$. The result of (a) 2) also holds

c) Let $0 \in (\alpha, \beta)$. If $F_{i,j}$, $F_{i',j'}$ satisfy the conditions imposed in (x) c) determine integers $I, \ldots, J'$ as therein described. If $J'-I' \geqslant J-I$ << $J-I \geqslant J'-I'$>> the results of b) hold with i, j, i', j' replaced by I, J, I', J' << I', J', I, J>>

d) Let $[\alpha, \beta] \supseteq [-\infty, \infty]$. Let $F_{i,j}, F_{i',j'} \in \overline{\overline{\mathbb{U}}}^{\mu(\epsilon)} < \overline{\overline{\mathbb{U}}}^{\mu(\epsilon)} - \hat{\mathbb{F}}>$ satisfy the conditions imposed in (x) d) and determine the integers $I_0, \ldots, J_1, \mathfrak{z}$ as therein described. If either (1) $I_0 + J_0 \neq I_1 + J_1$ and $G_{\mathfrak{z}}(M) \neq 0$ or (2) $I_0 + J_0 = I_1 + J_1$ and $G_\nu(M) \neq 0$

$(\nu \equiv \overline{\mathbb{I}}_{\xi}^{r})$ where $r = \max(I_0 + J_1, I_1 + J_0) \leq \mu(\tau)$, then $F_{i,j}(m)$ $\neq F_{i',j'}(m)$.

e) Let $\sigma \in S_y$. Determine the integer function $\hat{\mu}(\nu)$ $(\nu \equiv \overline{\mathbb{I}}_0^{\hat{\tau}})$ as in (x)e). Let $F_{i,j}, F_{i',j'}, i_0, \ldots, j_1$ be as described in (x)e). If either (1) $i_0 + j_0 \geq i_1 + j_1$ or (2) $i_0 + j_0 < i_1 + j_1$ and a $\nu \in \overline{\mathbb{I}}_{i_0+j_0+1}^{i_1+j_1}$ exists for which $G_{\nu\nu}(m)$ $\neq 0$ then $F_{i,j}(m) \neq F_{i',j'}(m)$.

(xvi) Let $|\alpha|, |\beta| < \gamma \in (0, \infty)$, $G(z \in \mathbb{D}_{2\gamma}, M) \in A_z$, $G \in \mathfrak{h}(m)$, $\mathfrak{h}(M) \sim x'$, $x\{\mathfrak{h}(m)\} = \mu_\tau$, $\sigma \notin St < \epsilon = st\{M_\nu, b_\nu; n\}$ with $\hat{F}$ as in (iii)a)$>$, and $\overline{i,j} \neq \overline{i',j'}$

a) The results of (xv) a,b,c,e) hold as stated

b) Let $0 \in (\alpha, \beta)$. (1) If $i' - j' \geq i - j$, $i + j \geq i' + j'$ and $F_{i,j}, F_{i',j'}$ $\in \mathcal{L}^{\mu(\tau)} < \mathbb{L}^{\mu(\tau)} - \hat{\hat{\mathbb{F}}} >$ then $F_{i,j}(m) \neq F_{i',j'}(m)$ (2) If $F_{i,j}$, $F_{i',j'} \in \mathbb{F}^{\mu(\tau)} < \mathbb{F}^{\mu(\tau)} - \hat{\mathbb{F}} >$ $i + j \neq i' + j'$ and $G_{\xi}(m) \neq 0$, where

$\xi = \min(i+j+1, i'+j'+1)$, then $F_{i,j}(m) \neq F_{i',j'}(m)$

c) Let $0 \in (\alpha, \beta)$ and $i \geq i'$. Let $F_{i,j}, F_{i',j'}$ satisfy the conditions imposed in (xii)c) and determine the integers $I_0, \ldots, J_1, \xi$ as in that clause. If $I_0 + J_0 \neq I_1 + J_1$ and $G_\xi(m) \neq 0$, or if $I_0 + J_0 = I_1 + J_1$ and $G_\nu(m) \neq 0$ $(\nu = \overline{I}_\xi^r)$ where $r = \max(I_0 + J_1, I_1 + J_0)$, then $F_{i,j}(m) \neq F_{i',j'}(m)$

d) Let $\sigma \in Sy$. Determine the integer function $\hat{\mu}(\overset{\ast}{\underset{0}{\beta}})$ $(\nu = \overline{I}_0^{\hat{\varepsilon}})$ and the integers $i_0, \ldots, j_0$ as in (xi)e). (1) Let $F_{i,j}, F_{i',j'} \in \mathbb{L}^{\mu(\tau)} \langle \mathbb{L}^{\mu(\tau)} \underline{\quad} \hat{\mathbb{F}} \rangle$ and $j_0 - i_0 \geq j_1 - i_1$. Subject to either of conditions (xi) e)(1 or 2), $F_{i,j}(m) \neq F_{i',j'}(m)$

(2) If $F_{i,j}, F_{i',j'} \in \mathbb{F}^{\mu(\tau)} \langle \mathbb{F}^{\mu(\tau)} \underline{\quad} \hat{\mathbb{F}} \rangle$ and, with $\xi = \min(i_0 + j_0 + 1, i_1 + j_1 + 1)$, either $i_0 + j_0 \neq i_1 + j_1$ and $G_{2\xi}(m) \neq 0$ or $i_0 + j_0 = i_1 + j_1$ and $G_{2\nu}(m) \neq 0$ $(\nu = \overline{I}_{2\xi}^{2r})$ where $r = \max(i_0 + j_1, i_1 + j_0)$ then $F_{i,j}(m) \neq F_{i',j'}(m)$

(xvii) Let $G([\alpha,\beta], m) \in \Upsilon^{\mu}\{y\}$, and let $\sigma \in St$ <$\sigma = st\{M_\nu, t_\nu : \infty\}$ with $\hat{\mathbb{F}}$ as in (iii)a)>

a) Let $0 \notin (\alpha,\beta)$. Then all $F_{i,j}(m) \in \mathbb{L}^{\mu}(m)$ <$\mathbb{L}^{\mu}(m) - \hat{\mathbb{F}}(m)$> are distinct <and differ from $F(m)$>

b) Let $0 \in (\alpha,\beta)$. (1) <Then $F_{i,j} \neq F(m)$ $(F_{i,j} \in \mathbb{O}^{\mu}(m) - \hat{\mathbb{F}}(m))$> (2) If $F_{i,j}, F_{i',j'} \in \mathbb{O}^{\mu}$ <$\mathbb{O}^{\mu} - \hat{\mathbb{F}}$> and $(i-i')(j-j') \neq 1$ then $F_{i,j}(m) \neq F_{i',j'}(m)$. (3) If $H_{i'+1, i'+2m+1} = 0$ $(F_{i', i'+2m} \in \mathbb{O}^{\mu})$, then $F_{i', i'+2m}[M,1] \in W$. (4) If $H_{i'+1, i'+2m+1} \neq 0$ $(F_{i', i'+2m} \in \mathbb{O}^{\mu}$ <$\mathbb{O}^{\mu} - \hat{\mathbb{F}}$>$)$ then $F_{i'+1, i'+2m}(m) \neq F_{i', i'+2m}(m)$

c) Let $[\alpha,\beta] \equiv [-\infty,\infty]$. <(1) The result of b)(1) holds with $\mathbb{O}$ replaced by $\mathbb{L}$>. (2) All $F_{i,j}(m) \in \mathbb{L}^{\mu}(m)$ <$\mathbb{L}^{\mu}(m) - \hat{\mathbb{F}}(m)$> with suffix pairs belonging to differing subsets of the form $[i', i'+2m; 1]$ are distinct. (3) If $H_{i'+1, i'+2m+1} \neq 0$ $(F_{i', i'+2m} \in \mathbb{O}^{\mu}$ <$\mathbb{O}^{\mu} - \hat{\mathbb{F}}$>$)$ then the four mappings with suffix pairs belonging to $[i', i'+2m; 1]$ are distinct

d) Let $\sigma \in S_y$. < The result of b)(1) holds with $\mathbb{O}$ replaced by $\mathbb{D}$>. That of c)(2) holds with $\mathbb{D}$ replaced by $\bar{\mathbb{D}}$ and i' by $2i'$.

(xviii) Let $|\alpha|, |\beta| < \gamma \in (0, \infty)$ and $\sigma \notin St$ < $\sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{\mathbb{F}}$ as in iii a) >

a) Let $[\alpha, \beta] \subset [0, \infty)$, $G(z = \overset{\infty}{\mathbb{D}}, M) \in A_z$, and $G(z = \hat{\mathbb{D}}, M) \in \bar{\Psi}^\mu$ where $\overset{\infty}{\mathbb{D}} \equiv \mathbb{D}_{2\gamma} - [-2\beta, -\beta]$. (1) The results of (xvi)a) hold with $\mathbb{D}$ replaced by $\mathbb{L}$. (2) If in addition, $G(z = \hat{\mathbb{D}}, M) \in \bar{\Psi}^{\mu}\{y\}$, they hold with $\mathbb{D}$ replaced by $\mathbb{F}$.

b) Let $\sigma \in (\alpha, \beta)$, $G(z = \mathbb{D}_{2\gamma}, M) \in A_z$ and $G(z = \mathbb{D}_{2\gamma}, M) \in \tilde{\Psi}^\mu\{y\}$. The results of (xvii) b) hold with $\mathbb{O}$ replaced by $\mathbb{R}$. If, in addition, $G(z = \mathbb{D}_{2\gamma}, M) \in \bar{\Psi}^\mu\{y\}$, then these results hold with $\mathbb{O}^\mu$ replaced by $\mathbb{O}^\mu \cup \mathbb{R}^\mu$.

c) Let $\sigma \in (\alpha, \beta)$, $G(z \equiv \hat{\mathbb{D}}, M) \in A_z$ and $G(z \equiv \hat{\mathbb{D}}, M) \in \tilde{\Psi}^{\mu}\{y\}$ where $\hat{\mathbb{D}} = \mathbb{D}_{2\gamma} \cup \{ \not{Z} \cap \{ \pm \Delta [2\gamma, \infty]_{-\pi/3}^{\pi/3} \} \}$. <(1) The result of (xvii) b)(1) holds with $\mathbb{O}$ replaced by $\mathbb{L}$>. That of (xvii) c) (2) holds with $\mathbb{U}$ replaced by $\mathbb{L}$, and that of (xviii) c) (3) holds with $\mathbb{O}$ replaced by $\mathbb{R}$. If, in addition, $G(z \equiv \hat{\mathbb{D}}, M) \in \bar{\Psi}^{\mu}\{y\}$, then these results hold with $\mathbb{O}$, $\mathbb{U}$ and $\mathbb{O}$ replaced by $\mathbb{F}$, $\mathbb{F}$ and $\mathbb{O} \cup \mathbb{R}$ respectively.

d) Let $\sigma \in Sy$, $G(z \equiv \mathbb{D}_{2\gamma}, M) \in A_z$ and $G(z \equiv \mathbb{D}_{2\gamma}, M) \in \tilde{\Psi}^{\mu}\{y\}$ < The result of (xvii) b)(1) holds with $\mathbb{O}$ replaced by $\mathbb{L}$>. That of (xvii) c) (2) holds with $\mathbb{U}$ replaced by $\mathbb{L}$. If, in addition, $G(z \equiv \mathbb{D}_{2\gamma}, M) \in \bar{\Psi}^{\mu}\{y\}$, then these results hold with $\mathbb{O}$ and $\mathbb{U}$ replaced in both cases by $\mathbb{F}$.

(xix) Let $\sigma \in (\alpha, \beta)$, $G[(\alpha, \beta), y(M) : x']$ with $*\{y(M)\} = \mu_x$ and $\sigma \in St < \sigma = st\{M_j, b_j; n\}$ with $\hat{\mathbb{F}}$ as in (iii) a)>.

a) Let $[\alpha,\beta]=[-\infty,\infty]$, $F(m)\in\mathbb{N}$ and $F_{i,j}\in\mathbb{U}^{\mu(\tau)}-\mathbb{O}<\mathbb{U}^{\mu(\tau)}-\mathbb{O}-\hat{\mathbb{F}}>$. (1) $E_{i,j}(x)=O\{G_{\mu\{\lambda(i+j)\}}(x);M:x'\}$. (2) If $G_\nu(M)\neq 0$ $(\nu=\mathbb{I}_{i+j}^{i+j+2})$ then $E_{i,j}(x)\neq o\{G_{i+j+2}(x);M:x'\}$ and $F_{i,j}(m)\neq F(m)$.

b) Let $F_{i,j}, F_{i',j'}\in\mathbb{O}^{\mu(\tau)}<\mathbb{O}^{\mu(\tau)}-\hat{\mathbb{F}}>$ with $i+j=i'+j'$, $|i-i'|>1$ and let $r=\max(i+j'+2,i'+j+2)\leq\mu(\tau)$. If $G_\nu(M)\neq 0$ $(\nu=\mathbb{I}_{i+j+1}^{r})$ then $E_{i,j,i',j'}(x)\neq o\{G_r(x);M:x'\}$ and $F_{i,j}(M)\neq F_{i',j'}(M)$.

c) Let $[\alpha,\beta]=[-\infty,\infty]$ and $F_{i,j}, F_{i',j'}\in\mathbb{U}^{\mu(\tau)}<\mathbb{U}^{\mu(\tau)}-\hat{\mathbb{F}}>$. If $F_{i,j}\in\mathbb{O}^{\mu(\tau)}<<\mathbb{U}^{\mu(\tau)}-\mathbb{O}>>$, set $\xi_0=\xi_1=i+j+1<<\xi_0=i+j$, $\xi_1=i+j+2>>$; obtain ξ_0', ξ_1' from i',j' similarly. Set $\xi'=\min(\xi_0,\xi_0')$. (1) If either $\xi_0>\xi_1'$ or $\xi_0'>\xi_1$ then $E_{i,j,i',j'}(x)=O\{G_{\mu\{\lambda(\xi')\}}(x);M:x'\}$ and if $G_\nu(M)\neq 0$ $(\nu=\mathbb{I}_{\xi'}^{\xi'+2})$ then $E_{i,j,i',j'}\neq o\{G_{\xi'+2};M:x'\}$ and $F_{i,j}(m)\neq F_{i',j'}(m)$. (2) Otherwise, let $(i-i')(j-j')\neq 1$

and $r = \max(i+j'+2, i'+j+2) \leq \mu(\tau)$. If $G_\nu(m) \neq 0$ $(\nu = \mathbb{I}^r_{j,1})$ then $E_{i,j,i',j'}(x) \neq 0 \{G_\nu(x); M : x'\}$ and $F_{i,j}(m) \neq F_{i,j'}(m)$.

(xx) Let $0 \in (\alpha, \beta)$, $|\alpha|, \beta < \gamma \in (0, \infty)$, $G(z = \mathbb{D}_{2\gamma}, m) \in A_z$ $G \leftarrow \mathcal{Y}(m)$, $\mathcal{Y}(m) \sim x'$, $x\{\mathcal{Y}(m)\} = \mu_\tau$ and $\sigma \notin St < \sigma = st\{M_\nu, t_\nu : n\}$ with $\hat{\mathbb{F}}$ as defined in (iii)a)>.

a) Let $F(m) \in \mathbb{W}$ and $F_{i,j} \in \mathbb{F}^{\mu(\tau)} - \{\mathbb{D} \cup \mathbb{R}\} < \mathbb{F}^{\mu(\tau)} - \{\mathbb{D} \cup \mathbb{R}\} - \hat{\mathbb{F}}>$. Then the results of (xix)a)(1,2) hold.

b) Those of (xix)b) hold with $\mathbb{D}^{\mu(\tau)}$ replaced by $\mathbb{D}^{\mu(\tau)} \cup \mathbb{R}^{\mu(\tau)}$

(xxi) Let $G(z, x)$ be an entire function of z $(z = M)$, $G \leftarrow \mathcal{Y}(m)$, $x\{\mathcal{Y}(m)\} = \mu_\tau$ and $\mathcal{Y}(m) \sim x'$.

a) Let $F(m) \in \mathbb{W}$. The results of (ix) b, c), (x) b, c, d), (xiii) b, c) and (xiv) b, c, d) hold as stated; those of (ix)d), (xiii)d) and (xix)a) hold with the condition $[\alpha, \beta] = [-\infty, \infty]$ relaxed to $0 \in (\alpha, \beta)$ and $\mathbb{W}^{\mu(\tau)} - \mathbb{D}$ replaced by $\mathbb{F}^{\mu(\tau)} - \{\mathbb{D} \cup \mathbb{R}\}$ in each case.

b) The results of (xi) a, b, c, e), (xii) b, c, d), (xv) a, b, c, e) and (xvi) b, c, d) hold as stated; with the condition $[\alpha,\beta]\ni[-\infty,\infty]$ relaxed to $0\in(\alpha,\beta)$, (1) those of (xv)d) holds with $\overline{\mathbb{D}}, \overline{\mathbb{I}}$ replaced by $\mathbb{F}$, $\mathbb{D}\cup\mathbb{R}$, (2) those of (xi)d) hold with the preceding two modifications and condition (2) of that clause discarded, and (3) those of (xix)c) hold with $\mathbb{D}$, $\mathbb{I}^{\mu(\tau)}$, $\mathbb{I}$ replaced by $\mathbb{F}$, $\mathbb{I}^{\mu(\tau)}\cup\mathbb{R}^{\mu(\tau)}$, $\mathbb{I}\cup\mathbb{R}$. Then results of (xix) b) hold with $\mathbb{I}^{\mu(\tau)}$ replaced by $\mathbb{I}^{\mu(\tau)}\cup\mathbb{R}^{\mu(\tau)}$

(xxiii) a) Let $G([\alpha,\beta], x\equiv\mathbb{D})\in A_x$. Then $F(\mathbb{D})\in A$

b) Let $G([\alpha,\beta], x\equiv\mathbb{D})\in A_x$ and $g(\mathbb{D})\in A$. Subject to the various conditions imposed in (v) upon $[\alpha,\beta]$ and, when appropriate, upon σ, $\tilde{\mathbb{F}}(\mathbb{D})\in A$, where $\tilde{\mathbb{F}}$ is as defined in each case in (v).

c) With the conditions such as $G(z \equiv \{\bar{\mathbb{D}}_{2\beta} - [-2\beta, -\beta]\}, M) \in A_z$ and, where appropriate, $G_0(M) \in V$ imposed in (vi) replaced by $G(z \equiv \{\bar{\mathbb{D}}_{2\beta} - [-2\beta, -\beta], x \equiv \mathbb{D}) \in A_{z,x}$ and $G_0(\mathbb{D}) \in A$ respectively, $\acute{\mathbb{F}}(\mathbb{D}) \in A$, where $\acute{\mathbb{F}}$ is as defined in each case in (vi).

(xxiii) Let $G([\alpha, \beta], x \equiv \mathbb{D}) \in A_x$

a) $\mathbb{F}(\mathbb{D}) \in A$

b) Let $g(\mathbb{D}) \in A$; let $\sigma \in St$ $\langle \sigma = st\{M_\omega, t_\omega; n\}$ and let either $0 \in (\alpha, \beta)$ or both $[\alpha, \beta] \equiv [-\infty, \infty]$ and $\sigma \in St$ ⟦ $0 \in (\alpha, \beta)$ and $\sigma \notin St$ ⟧ $\{\sigma \in St\}$ with $\tilde{\mathbb{F}}$ as in A(iv)a). Then $\tilde{\mathbb{F}}(\mathbb{D}) \in A$.

B. Set $f(x) = f[x : \sigma ; g : \alpha, \beta]$

(i) With $g(M)$ replaced by g, the results of A.(i) hold

(ii) a) Let $\alpha \in [0, \infty)$ $\langle \beta \in (-\infty, 0] \rangle$ ⟦ $0 \in (\alpha, \beta)$ ⟧ and let $g(\langle - \rangle [\![\pm]\!] \Delta^{\psi}_{\phi}) \wedge g$; if $\beta = \infty$ ($[\alpha = -\infty]$) let $g(-\Delta[0, \infty]^{\psi}_{\phi}) \in$

iv. Then $f(\mathbb{I} \pm \mathbb{B}\Delta_\phi^\psi) \sim gh$.

b) Let $\kappa > 1$, $|\beta| < \gamma \in (0, \infty)$ and $g(\mathbb{D}_\kappa) \in A$. Then $\sum h_\nu g_\nu (\mathbb{D}_{\kappa/\beta})$ $\rightarrow f(\mathbb{D}_{\kappa/\gamma})$

c) Let $g([\alpha, \beta] \times \Delta) \in \mathbb{V}$ and $\sigma = st\{M_\nu, t_\nu; n\}$; let none of the $\{t_\nu\}$ be zero [let one, namely t_1, of the t_ν be zero, and let $g_0 \in \mathbb{R}$]. Then $f(\Delta) = \sum_1^n M_\nu g(t_\nu \Delta)$ $[M_1 g_0 + \sum_2^n M_\nu g(t_\nu \Delta)]$

(iii) a) Let $g([\alpha, \beta] \times \Delta) \in \mathbb{V}$, $\sigma \notin S$ $[\in S]$ and $\sigma = s\{M_\nu, t_\nu; n$ with none [one] of the $\{t_\nu\}$ zero. Then with $f(\Delta)$ as defined in (ii) c), $f_{i,j}(\Delta) = f(\Delta)$ for $f_{i,j} = \hat{\mathbb{f}}$, where $\hat{\mathbb{f}} = \mathbb{f}_{n,n-1}[\infty]$ $[\mathbb{f}_{n,n-2}[\infty]$ $\mathbb{I}$ $[\mathbb{f}_{n-1,n-1}[\infty]]]$

b) Let $g(x) = \sum_0^\mu g_\nu x^\nu$ $(\mu \in \mathbb{I})$. Then $f(\mathbb{B}) = \sum_0^\mu h_\nu g_\nu \mathbb{B}^\nu$ and, replacing the symbols F, $\mathbb{F}$ and M by f, $\mathbb{f}$ and $\mathbb{B}$ the modified results of A(iii) b) (2-4) hold

(iv) a) Let $g([\alpha, \beta] \times \Delta)$, $g \in \mathbb{V}$, $\sigma \notin St$ $\langle \sigma = st\{M_\nu, t_\nu; n\}$

with $\hat{f}$ as in B.(iii)a)> and either $0 \in (\alpha, \beta)$, or both $[\alpha, \beta] =$ $[-\infty, \infty]$ and $\sigma \in S_y$ [[$0 \in (\alpha, \beta)$ and $\sigma \notin S_y$]] [$\sigma \in S_y$]. Then $f(\Delta) \in w$, where $\tilde{f} = u < u \cup \hat{f} >$ [[$\bar{v} < \bar{v} \cup \hat{f} >$]] [$\bar{u} < \bar{u} \cup \hat{f} >$]

b) With F, M replaced by f, Δ, the modified results of A.(iv)b, c) hold.

(v) a) Let $\sigma \notin St$ <$\sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{f}$ as in (iii)a)>; let $g(\langle\langle - \rangle\rangle \Delta^{\psi}_{\phi})$ ag if $\beta \in (0, \infty)$ <<$\alpha \in (-\infty, 0)$>>; let $\sigma \notin S_y$ [[$\sigma \in S_y$]]. Then $f_{i,j}(\Delta^{\psi}_{\phi}) \sim h^{(i,j)} g$ and, if $0 \in (\alpha, \beta)$, $f_{i,j}(-\Delta^{\phi}_{\psi})$ $\sim h^{(i,j)} g$ also ($f_{i,j} = \acute{f}$) where $\acute{f} = u < u \cup \hat{f} >$ [[$\bar{u} < \bar{u} \cup \hat{f} >$]].

b) Let $|\kappa|, |\rho| < \gamma \in (0, \infty)$, $g(D_\kappa) \in A$ with $g \prec g$, $\sigma \notin S$ <$\sigma = st\{M_\nu, t_\nu; n\}$> and $0 \notin (\alpha, \beta)$ [[$0 \in (\alpha, \beta)$ with $\sigma \notin S_y$]] [$\sigma \in S_y$] with $\tilde{f}$ as in B.(iv)a). Then $\sum h^{(i,j)}_{\nu} g_\nu (D_{\kappa,\rho})^\nu \to f_{i,j}(D_{\kappa,\gamma})$ ($f_{i,j} = \tilde{f}$).

(vi) Let $\sigma \notin St$ <$\sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{f}$ as in (iii)a)>. If

$\beta \in (0,\infty]$ $\ll \alpha \in [-\infty, 0) \gg$ let $g(\ll - \gg \overset{\downarrow}{\Delta}_\phi) \sim g$, with

$\kappa\{g\} = \mu_\kappa$

a) If $\beta = \infty$ ($[\alpha = -\infty]$), let $g(([-])\Delta[0,\infty]^{\downarrow}_\phi) \in V$

Replacing relationships such as $E_{i,j}(x) = \hat{O}\{G_{\mu\{\lambda(i+j+1)\}}(x) M : x'\}$ in A.(vi) by $e_{i,j}(x) = \hat{O}\{x^{\mu\{\lambda(i+j+1)\}}; \overset{\downarrow}{\Delta}_\phi\}$ and the symbols F, $G_\nu(m)$, U by f, g_ν, u (1) if $0 \notin (\alpha,\beta)$ the modified results of A.(vi)a) hold for $f_{i,j} \equiv u\langle u - \hat{f}\rangle$, (2) if $0 \in (\alpha,\beta)$ and $\sigma \notin S$, those of A(vi)b)(1,2) hold for $f_{i,j} \equiv u\langle u - \hat{f}\rangle$ and (3) if $\sigma \in S$, those of A(vi)d) hold for $f_{i,j} \equiv \bar{u}\langle \bar{u} - \hat{f}\rangle$.

b) Replacing relationships such as $E_{i,j,i',j'}(x) = O\{G_{\mu\{\lambda(\xi')\}}; M : x'\}$ in A.(vii) by $e_{i,j,i',j'}(x) = O\{x^{\mu\{\lambda(\xi')\}}; \Delta^{\downarrow}_\phi\}$, and the symbols F, $G_\nu(m)$, U by f, g_ν, u, the modified results of A.(vii) a, b, d) hold as for a) above

c) If $\beta = \infty$ ([$\alpha = -\infty$]), let g ($(C[-])\Delta[0,\infty]^{\psi}_{\phi}$). Let $\Delta \supseteq \Delta[0,\delta]^{\psi}_{\phi}$ possess a limit point at the origin with $g([\alpha,\beta] \times \Delta) \in V$. With the symbols F, $\mathbb{F}$, $\mathfrak{U}$, $\mathbb{O}$ and $G_{\nu}(m)$ replaced by f, $\mathfrak{f}$, u, o and g_{ν}, the modified results of A(viii) hold.

(vii) Let $\sigma \notin St$ <$\sigma = st\{M_{\nu}, t_{\nu}; n\}$ with $\hat{\mathfrak{f}}$ as in (iii)a)>. Let $|\alpha|, |\beta| < \infty$. Subject to the stated conventions the modified ~~res~~ results of (vi)a, b) hold as do those of (vi)c) where now $\Delta \supset N\{0\}$.

(viii) Let $g([\alpha,\beta],\Delta) \in \mathcal{Y}'\{g\}$, and let $\sigma \notin St$ <$\sigma = st\{M_{\nu}, t_{\nu}; n\}$ with $\hat{\mathfrak{f}}$ as in (iii)a). With the symbols $\mathbb{F}$, $\mathfrak{U}$, $\mathbb{O}$ and M replaced by f, $\hat{\mathfrak{f}}$, u, o and Δ, the modified results of A.(ix) hold.

(ix) Let $g(N\{0\}) \in A$ with $g \leftarrow g$, $x\{g\} = \mu_x$, and let

$\sigma \notin St \langle \sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{\mathfrak{f}}$ as in (iii)a)$\rangle$

a) If $|\alpha|, |\beta| < \infty$, set $\phi = 0, \psi = 2\pi$; otherwise if $\beta = \infty$ ($[\alpha - -\infty]$) let $g(([-]) \Delta [0,\infty] \Delta^{\psi}_{\phi}) \in \mathbb{V}$. Then with relationships such as $E_{i,j}(x) = O\{G_{\mu\{\lambda(i+j+1)\}}(x); M: x'\}$ replaced by $e_{i,j}(x) = O\{x^{\mu\{\lambda(i+j+1)\}}; \Delta^{\psi}_{\phi}\}$, and the symbols $G_\nu(m)$, F, $\mathbb{F}$ replaced by $g_\nu, f, \mathfrak{f}$, the modified results of A(x)a) hold.

b) With relationships such as $E_{i,j,i',j'}(x) = O\{G_{\mu\{\lambda(j')\}}(x); M: x'\}$ replaced by $e_{i,j,i',j'}(x) = O\{x^{\mu\{\lambda(j')\}}\}$ and the symbols $G_\nu(m)$, F, $\mathbb{F}$ replaced by $g_\nu, f, \mathfrak{f}$, the modified results of A(x)b) hold.

(x) Let g be an entire function, with $g \leftarrow g$, $x\{g\} = \mu_r$ and let $\sigma \notin St \langle \sigma = st\{M_\nu, t_\nu; n\}$ with $\hat{\mathfrak{f}}$ as in (iii)a)$\rangle$.

a) Prescribe ϕ, ψ as in (ix)a), and let $\Delta \in \Delta_\phi^\psi$ possess a limit point at the origin. Then with the symbols F, $\mathbb{F}$, $G_0(M)$ and M replaced by f, $\mathfrak{f}$, g_0 and Δ, the modified results of A(x)c) hold

b) Let Δ possess a limit point at the origin. Then with symbols replaced as in a), the modified results of A(x)d) hold.

(xi) Let $\sigma \in (\alpha, \beta)$ and $\sigma \notin Sy$. Let $g(\pm\Delta_\phi^\psi) \sim g$ with $\varkappa\{g\} = \mu_\sigma$. Let $\sigma \notin St < \sigma = \sigma t\{M_\sigma, t_\sigma; n\}$ with $\hat{\mathfrak{f}}$ as in (iii)a)>. Replace the symbols $G_0(M)$, F and $\mathbb{F}$ in A.(xi) by g_0, f and $\mathfrak{f}$

a) Replace relationships such as $E_{i,j}(x) = O\{G_{\mu(\lambda|i+j)\}}(x); M : x'\}$, in A(xi) by $e_{i,j}(x) = O\{x^{\mu\{\lambda|i+j)\}}; \Delta_\phi^\psi\}$.

(1) If $\beta = \infty$ ($[\alpha = -\infty]$), let $g(([-])\Delta[0,\infty]_\phi^\psi) < V$. The

modified results of A.(xi)a) (1,2,4) and b)(1) hold; that of A.(xi)a) (2) also holds if $f_{i,j} \in \bar{\mathbb{D}}^{\mu(\kappa)} \langle \bar{\mathbb{D}}^{\mu(\kappa)} - \hat{f} \rangle$. (2) The modified results of A (xi) c) (1,2,4) hold. ((1) and (2) of the present clause hold whether $[\alpha,\beta] = [-\infty,\infty]$ or not.)

b) Let $\Delta \in \Delta_\phi^\psi$ possess a limit point at the origin, with $g([\alpha,\beta]\times\Delta) \in V$. Replacing M by Δ (now taking cognisance of whether $[\alpha,\beta] = [-\infty,\infty]$ or not) the modified results of A.(xi)a) (3,5), b)(2), (c) (2,5) hold and those of c) (3,5) hold when $[\alpha,\beta] \neq [-\infty,\infty]$ and $f_{i,j}, f_{i',j'} \in \mathbb{D}^{\mu(\kappa)} \langle \mathbb{D}^{\mu(\kappa)} - \hat{\mathbb{f}} \rangle$.

(xii) Let $0 \in (\alpha,\beta)$ and $\sigma \notin St \langle \sigma = st\{M_\nu, t_{\nu}, n\}$ with $\hat{\mathbb{f}}$ as in (iii)a)$\rangle$.

a) If $|\alpha|, |\beta| < \infty$ set $\phi = 0$, $\psi = 2\pi$. Otherwise, if $\beta = \infty$ ($[\alpha = -\infty]$), let $g(([-\infty])\Delta[0,\infty]_\phi^\psi) \in V$. Replace relationships such as $E_{i,j}(x) = O\{G_{i+j+1}(x); M : x'\}$ in A.(xi) by $e_{i,j}(x) = O\{x^{i+j+1}; \Delta_\phi^\psi\}$. Let $f_{i,j} \in \mathfrak{f}^{\mu(\kappa)}$. (1) If $|i-j| = 2m$ ($m \in \mathbb{I}$) then the modified results of A. (xi)a) (1,4) hold. (2) if $|i-j| = 2m+1$ ($m \in \mathbb{I}$) then the modified result of A.(xi)b) (1) holds and if $g_\mu\{\lambda(i+j+1)\} \neq i$ then $e_{i,j}(x) \neq o\{x^{\mu\{\lambda(i+j+1)\}}; \Delta_\phi^\psi\}$.

b) Let $f_{i,j}, f_{i',j'} \in \mathfrak{f}^{\mu(\kappa)}$ and derive $\xi_0, \dots, \xi_1', \xi'$ as in A.(xii) b) (1). The result of A.(xi) c)(1) (modified as in a) of the current clause) holds and, under the additional conditions of A.(xii) b)(1), $e_{i,j,i',j'}(x) \neq o\{x^{\xi''}; \Delta_0^{2\pi}\}$. (2) If neither $\xi_0 > \xi_1'$ nor $\xi_0' > \xi_1$, then the modified result of A.(xi) c)(4) holds.

(xiii) Let $\vartheta \in (\alpha, \beta)$ and $\sigma \in S_\vartheta$. Let g be an entire function with $g \neq g$, $\kappa\{g_\vartheta\} = \mu_\kappa$, and let $\sigma \in St\langle\sigma =$

$\mathrm{st}\{M_\nu, t_\nu; n\}$ with $\hat{\mathfrak{f}}$ as in (iii)(a)>

a) If $|\alpha|, \beta < \infty$ set $\phi = 0, \psi = 2\pi$; otherwise if $\beta = \infty$ ($[\alpha = -\infty]$) let $g(([-])\Delta[0,\infty]_\phi^\psi) \in V$. Let $\Delta \in \Delta_\phi^\psi$ ($\Delta \in \mathcal{B}$) possess a limit point at the origin. If $|i-j| = 2m$ ($m \in \underline{I}$) and $g_\nu \neq 0$ ($\nu = \underline{I}_{i+j}^{i+j+2}$) then $f_{i,j}(\Delta) \neq f(\Delta)$ ($f_{i,j} \in \mathfrak{f}^{\mu(\tau)-1}$), and if $|i-j| = 2m+1$ ($m \in \underline{I}$) and $g_{i+j+1} \neq 0$ then $f_{i,j}(\Delta) \neq f(\Delta)$ ($f_{i,j} \in \mathfrak{f}^{\mu(\tau)}$).

b) Let $\Delta \in \mathcal{B}$ possess a limit point at the origin. Let $f_{i,j}, f_{i',j'} \in \mathfrak{f}^{\mu(\tau)}$ and derive $\xi_0, \dots, \xi_1', \xi'$ as in A.(xii) b)(1). If either $\xi_0 > \xi_1'$ and $g_\nu \neq 0$ ($\nu = \underline{I}_{\xi_0'}^{\xi_1'}$) or $\xi_0' > \xi_1$ and $g_\nu \neq 0$ ($\nu = \underline{I}_{\xi_0}^{\xi_1}$) then $f_{i,j}(\Delta) \neq f_{i',j'}(\Delta)$. If neither $\xi_0 > \xi_1'$ nor $\xi_0' > \xi_1$, let $g_\nu \neq 0$ ($\nu = \underline{I}_{\xi}^{r'}$) where $r' = \max(i'+j'+2, i'+j+2)$; then $f_{i,j}(\Delta) = f_{i',j'}(\Delta)$

(xxiv) Let $g([\alpha,\beta] \times \bar{\mathbb{D}}) \in A$

a) $f(\mathbb{D}) \in A$

b) Let $\sigma \notin St \langle \sigma = st\{M_\nu, t_\nu; n\} \rangle$ and $0 \notin (\alpha, \beta)$ $[\![0 \in (\alpha,\beta)$ and $\sigma \notin S]\!]$ $[\sigma \in S]$ with $\tilde{\tilde{ff}}$ as in (iii)a). Then $f_{i,j}(\bar{\mathbb{D}}) \in A$ ($f_{i,j} = \tilde{\tilde{ff}}$).

Proof A(i)a) If $\sigma = st\{M_\nu, t_\nu; 1\}$ with $t_1 = 0$, then $h_\nu = 0$ ($\nu \in \mathbb{I}_1$)

b)(1) If $[\alpha,\beta] \subseteq [0,\infty]$ $\langle\langle [-\infty, 0] \rangle\rangle$ then h_ν $\langle\langle (-1)^\nu h_\nu \rangle\rangle > 0$ ($\nu \geq 0$)

the remaining results are proved in the same way

(ii)a) Subject to the stated conditions, $G(t,x)$ is bounded upon the interval $t \in [\alpha,\beta]$ for $x \in M$, and since $\sigma(t)$ is bounded and nondecreasing, the integral for $F(x)$ exists ($x \in M$). The functions $G_{\mu(\nu)}(x)$ form an asymptotic sequence; $G(t,x) \sim \sum G_{\mu(\nu)}(x) t^{\mu(\nu)}$ as $x \to x'$ uniformly for $t \in (\alpha,\beta)$. $G(t,x)$ for each $x \in M$ and $t^{\mu(\nu)}$ for each $\nu \in \mathbb{I}$ are measurable functions of t and each of the integrals ( ) exists.

Hence ([I] J§ 1.4) $F(x) \sim \sum h_{\mu(\nu)} G_{\mu(\nu)}(x)$ as $x(\in M) \to x'$. b)(1)

Expanding $G(t,x)$ in ascending powers of t under the integral sign in formula ( ), we derive the required result.

c) Formula ( ) yields $F(x) = \sum_1^n M_\nu (t_\nu; x)$ if no t_ν is zero and $F(x) = M_1 G_0(x) + \sum_2^n M_\nu G(t_\nu; x)$ if $t_1 = 0$.

? (ii) a) We now have $h(z) = \sum_1^n M_\nu / (1 - t_\nu z)$. If $\sigma \notin St$ [S_y] and none [one] of the t_ν is zero $h(z)$ reduces to the form ( ) with $I = n$, $J = n-1$ [$I = n$, $J = n-1$] [we have $n = 2n'$ [$2n'+1$] with $t_{2\nu} = -t_{2\nu-1} \neq 0$, $M_{2\nu} = M_{2\nu-1}$ ($\nu = \overline{1}_1^{n'}$) [and $t_{2n'+1} = 0$] and $h(z)$ has the form ( ) with $I = n$, $J = n-2$ [$I = J = n$]. The stated results follow from 4.A.( ). b)(1) Formula ( ) now yields $F(x) = \sum_0^n h_\nu G_\nu(x)$.

(2) We first remark that under the conditions of (iii)a) it follows from 1.(i)..) that for the Hankel determinants

of the infinite block of H in question, we have $H_{n,r}, H_{r,n-1}$ $[\![H_{r,n-2}, H_{n,r-1}]\!] < [\![H_{n-1,r-1}, H_{r,n-1}]\!] > \neq 0$ $(i = \bar{\mathbb{I}}_n)$ and $H_{i+1,j}$ $[\![H_{i+1,j-1}]\!] < [\![H_{i,j}]\!] > = 0$ $(i,j = \mathbb{I}_n)$.

The structures of the Padé tables generated by the moment sequence $h = ms\{\sigma; \alpha, \beta\}$ when $\sigma \notin St$ and in the cases in which $[\alpha, \beta] \subseteq [0, \infty]$ and $0 \in (\alpha, \beta)$ were investigated by Van Vleck [ ]. It may be shown by systematic use of the result of 1.(i)... that in the first case $H_{i,i+m-1} > 0$ $(m, i = \underline{\mathbb{I}})$ and $(-1)^m H_{j+2m+1,j} > 0$, $(-1)^m H_{j+2m,j} > 0$ $(j, m = \underline{\mathbb{I}})$. A simple change of variable yields the result that when $\sigma \notin St$ and $[\alpha, \beta] \subseteq [-\infty, 0]$ then $H_{i,j} \neq 0$ $(i, j = \mathbb{I})$ (i.e. this result holds for $\overset{?}{0} \notin (\alpha, \beta]$: the precise nature of the signs attached to the $H_{i,j}$ does not concern us). It is a matter of detail to extend Van Vleck's analysis to the case in which $\sigma =$ st$\{M_\nu, t_\nu; n\}$: when $0 \notin (\alpha, \beta)$, if none [one] of the t_ν is zero, then $H_{i,j} \neq 0$ ($i = \mathbb{I}_0^n$ [$\mathbb{I}_0^{n-1}$], $j = \mathbb{I}$; $i = \mathbb{I}$, $j = \mathbb{I}_0^{n-1}$) with zero $H_{i,j}$ as given in the preceding paragraph. The result (iii) b) (2) now follows from 4... .

(3) When $\sigma \notin$ St and $0 \in (\alpha, \beta)$, we have [ ] $H_{i,i+2m}$ > 0 ($i = \mathbb{I}$, $m = \underline{\mathbb{I}}$) and $(-1)^m H_{j+2m+1,j} > 0$ ($j, m = \underline{\mathbb{I}}$), i.e. when $|i-j| = 2m+1$ ($m \in \mathbb{I}$) $H_{i,j} \neq 0$; but it can occur that when $|i-j| = 2m$ ($m \in \underline{\mathbb{I}}$), $H_{i,j} = 0$, although (4(i)..; in view of the stated distribution of nonzero elements of H, two such determinants with zero values cannot be neighbours. Van Vleck's analysis is again easily extended to the case in which $\sigma \notin S$ and $\sigma =$ st$\{M_\nu, t_\nu; n\}$: if none [one] of the t_ν is zero, denote

the set $H_{i,j}$ with $\overline{i,j} \equiv [n, n-1; \infty]$ $\left[[n-1, n-1; \infty] \right]$ by $\hat{H}$ (so that, from an earlier paragraph§, $\hat{H}$ corresponds to a block of the form $H_{J,J}[\infty]$ in H; for all $H_{ij} \in H - \hat{H}$, when $|i-j| = 2m+1$ $(m \in \mathbb{I})$ $H_{i,j} \neq 0$; and when $|i-j| = 2m$ $(m \in \mathbb{I})$ it can occur that $H_{i,j} = 0$. The result of (iii) b)(3) now follows from 4....

(4) When $\sigma \in S$, then, by inspection, $H_{2i+1,2j+1} = 0$ $(i,j \in \mathbb{I})$. If $\sigma \in S$, H and Ξ are composed of blocks of the form $H_{2i,2j}[1]$, $\Xi_{2i,2j}[1]$ $(i,j \in \mathbb{I})$ respectively. If $\sigma = st\{M_\nu, t_\nu; n\}$ with none of the t_ν zero, $H - H_{n,n-2}[\infty]$, $\Xi - \Xi_{n,n-2}[\infty]$, $[H - H_{n-1,n-1}[\infty], \Xi_{n-1,n-1}[\infty]]$ are composed of such blocks. The required result again follows from 4...

(iv)a) If $h = ms\{\sigma \in BND[\omega,\rho]\}$ with $\sigma \notin St$ and $[\omega,\rho]$

$\subseteq [0,\infty]$, then $h_m = ms\{\sigma_m \in BND[\alpha,\beta]\}$ with $\sigma_m \in St$ also. Since $H_{i,i+m-1} \neq 0$, equations ( ) determine the denominator of $P_{i,i+m-1}$ $(i \in \mathbb{I}_1, m \in \mathbb{I})$ directly (1...).
These equations are orthogonality conditions: setting $z^{-1}\pi^{(i,i+m-1)}(z) = p_i^{(m)}(t)$ $(t = z^{-1})$, equations ( ) become

$$\int_\alpha^\beta t^r p_i^{(m)}(t)\, d\sigma_m(t). \qquad (r = \mathbb{I}_0^{i-1})$$

An elementary and well known argument shows that the zeros of $p_i^{(m)}(t)$, the orthogonal polynomial of degree i derived from the moment sequence h_m, are confined to the segment (α,β). i.e. $n\{P_{i,i+m-1}\} = 0$, $t\{P_{i,i+m-1}\} \in (\alpha,\beta)$ for $i = \mathbb{I}_1$, $m = \mathbb{I}$. (The above analysis is due to Van Vleck [ ]; we need the details later.).

It is a matter of detail to extend the above reasoning to the cases in which $\sigma \in st\{M_\nu, t_\nu; n\}$: with none [one] of the $\{t_\nu\}$ zero and $i \in \underline{I}_0^n$ [$\underline{I}_0^{m-1}$] $m = \underline{I}$ (for $i = \underline{I}_{n+1}$ [$\underline{I}_n$], $m = \underline{I}$, we have, of course, $n\{P_{i,i+m\to}\} = 0$ $t\{P_{i,i+m\to}\} = \{t_\nu\}$). A simple change of variable yields the same result for $\beta \in (-\infty, 0]$. Hence, if $0 \notin (\alpha, \beta)$ and $G([\alpha,\beta], M)$, $\mathbb{D}(M) \in W$, and if $\sigma \in S$, $\hat{F} \in W$ also, as required.

We now consider the case in which $0 \in (\alpha, \beta)$. The argument concerning the polynomials $p_i^{(2m)}(t)$ given above can be repeated in its entirety: for $i \in \underline{I}_1$, $m \in \underline{I}$, $n\{P_{i,i+2m\to}\} = 0$ and $t\{P_{i,i+2m\to}\} \in (\alpha, \beta)$. If $G(M) \in W$ the mappings $F_{0,j}(M)$ $(j = \underline{I})$ in $\bar{\mathbb{D}}(M)$ are well determined as stated. If $0 \in (\alpha, \beta)$ and $\sigma(t)$ has points of increase for both negative and positive values of t, however,

$d\sigma_{2m+1}(t)$ $(m \in \mathbb{I})$ does not preserve its sign for $t \in [\alpha, \beta]$. A modification of the argument referred to above reveals that, when $H_{i,i+2m} \neq 0$, $i-1$ of the zeros of $p_i^{(2m+1)}(t)$ are confined to the interval (α, β), that since the coefficients of $p_i^{(2m+1)}(t)$ are real, the remaining zero is real but, if either α or β is finite may lie outside this interval, and that all zeros are simple. (When $H_{i,i+2m} = 0$ $(i \in \mathbb{I}_1, m \in \mathbb{I})$, which can occur, then $H_{i-1,i+2m-1}[1] < H$; we then have $n\{P_{i-1,i+2m-1}\} = n\{P_{i,i+2m}\} = n\{P_{i,i+2m-1}\} = 0$ and $t\{P_{i-1,i+2m-1}\} = t\{P_{i,i+2m-1}\} = t\{P_{i,i+2m-1}\} \in (\alpha, \beta)$ from the above results concerning the $\{P_{i,i+2m-1}\}$.) If $[\alpha, \beta] = [-\infty, \infty]$ then, of course, all zeros of $p_i^{(2m+1)}(t)$, i.e. all $t\{P_{i,i+2m}\}$ belong to (α, β) and once again, as stated in the theorem $\mathrm{II}'(m) \in W$. However, if $0 \in (\alpha, \beta)$ and $[\alpha, \beta] \neq [-\infty, \infty]$, one of the $\{t_\nu^{(i, i+2m)}\}$ may lie outside the range of t over which $G(t, m)$ is defined, and it may then occur that $F_{i, i+2m}(m) \notin W$.

When $\sigma \in St$, all $H_{i,j}$ not belonging to the infinite block in H which occurs when $\sigma \in St$ belong to blocks of the form $H_{2i', 2j'}$ [1]. We have $F_{2i', 2i'+2m}(m) = F_{2i'+1, 2i'+2m+1}(m) = F_{2i'+1, 2i'+2m_1}(m)$ for the $\{F_{i,j}(m)\}$ in question. The last of these three mappings belongs to $\mathrm{O}(m)$ and is well determined, which implies that the first two also belong to $W(m)$. Since, in particular, $F_{2i, 2i-1}(m) \in W$ $(i \equiv \mathrm{I}_1)$ it follows for similar reasons that $F_{2i-1, 2i-1}(m), F_{2i, 2i-1}(m), F_{2i, 2i}(m) \in W$ $(i \equiv \mathrm{I}_1)$ also, i.e. the mappings which $\mathrm{IV}(m)$ possesses in addition to those of $\mathbb{L}X(m)$ are well determined. (iii) b) is, of course, a simple consequence of those remarks.

(v) a) For all $F_{i,j}$ in question, the corresponding $t\{P_{i,j}\}$ belong to $[\alpha, \beta]$, and $n\{P_{i,j}\} = 0$ (see the proof of A)(iii), 4. A) (iii) with $\Delta' = [\alpha, \beta]$, may now be applied

b)(1) Now all $t_i\{P_{i,j}\}$, in belonging to $[\alpha, \beta]$, also belongs to $\mathbb{D}_\gamma$ and 4. A$\S$. ( ) may be used. (2) This result follows from 4. $\mathcal{K}$.).

(vi) a) We now have $F_{i,j} \sim \{h, \mathcal{L} : x'\}$ and $F_{i,j}(m) \sim \{h_{i,j}, \mathcal{L} : x'\}$ with $\xi(i,j) = i+j+1$ for all $F_{i,j}$ belonging to the set referred to. Hence (4...), $E_{i,j}(x) = O\{C_{i,\mu\{\lambda(i+j+1)\}}(x) : M : x'\}$ for these values of i, j. We must now show why O may be replaced by $\hat{O}$ in this relationship, i.e. why the corresponding relationship with O replaced by o is false. It was shown by Grommer [ ] (see also [ ]) that when $\sigma \notin St$, $0 \in (\alpha, \beta)$, and $h = ms\{\sigma \in BND[\alpha,\beta]\}$, then $h_{2\nu}^{(i,i-1)}$ ($i = \underline{I}_1$, $\nu = \underline{I}_i$). Use of the same methods leads to the result that when $[\alpha,\beta] \leq [0,\infty]$ the more comprehensive set of inequalities $h_\nu^{(i,i-1)} < h_\nu$ ($i = \underline{I}_1$, $\nu = \underline{I}_i$) holds. Denote the Padé quotients derived from h_m ($m \in \underline{I}$) by $P_{i,j}^{(m)}$ with $P_{i,j}^{(m)} \leftarrow h_{i,j}^{(m)}$, $h_{i,j}^{(m)}$ being $h_\nu^{(m;i,j)}$ ($\nu = \underline{I}$). Since, when $[\alpha,\beta] \leq [0,\infty]$, $h_m = ms\{\sigma \in BND[\alpha,\beta]\}$, we have $h_\nu^{(m;i,i-1)} < h_{m+\nu}$ ($m = \underline{I}$, $i = \underline{I}_1$, $\nu = \underline{I}_{2i}$). But (1...) $h_\nu^{(m;i,i-1)} = h_{m+\nu}^{(i,i+m-i)}$ ($m, i, \nu = \underline{I}$, $m+i \geq 1$) and hence in this case $h_\nu^{(i,i+m-1)} = h_\nu$ ($\nu = \underline{I}_0^{2i+m-1}$) $h_\nu^{(i,i+m-1)} < h_\nu$ ($\nu = \underline{I}_{2i+m}$) for $i, m = \underline{I}$ ($i+m \geq 1$). Thus, if $i+j+1 \leq \mu(\sigma)$, $E_{i,i+m-1}(x) \sim (h_\nu - h_\nu^{(i,i+m-1)})G_\nu(x)$ as $x (\in M) \to x'$, where $\nu = \mu\{\lambda(i+j+1)\}$ and the bracketed difference is nonzero, i.e. O may not be replaced by o in the relationship referred to above. A simple change of variable accommodates the case in which $[\alpha, \beta] \subseteq [-\infty, 0]$

b) In this case $F_{i,j}(m) \in W$ $(F_{i,j} \equiv \mathbb{U})$ (see the proof of (iv)a)) and $F_{i,j}(m) \sim \{h_{i,j} ; b_j : x'\}$ for these mappings, and again $F(m) \sim \{h, b_j : x'\}$. When $F_{i,j} \in \mathbb{U}^{\mu(t)} < U^{\mu(t)} \underline{\ } \hat{\mathbb{F}} >$ $\xi(i,j) = i+j+1$, and for $F_{i,j} \in \mathbb{E}^{\mu(t)} < E^{\mu(t)} \underline{\ } \hat{\mathbb{F}} >$, $\xi(i,j)$ takes one of the possible values $i+j+1, i+j, i+j+2$ depending upon whether $H_{i,j}$ is a member of a block of order zero or unity, and upon the position of $F_{i,j}$ in the latter case. The result (1) now follows straightforwardly from 4... , and (2), as described above, from Grommer's result.

c) New results hold only for $F_{i,j} \in \overline{\mathbb{U}}^{\mu(t)} < \mathbb{U}^{\mu(t)} \underline{\ } \hat{\mathbb{F}} >$

systematically, and for those $F_{i,j} \in \overline{\mathbb{U}}^{\mu(\kappa)} - \mathbb{O}^{\mu(\kappa)} < \mathbb{U}^{\mu(\kappa)} - \mathbb{O}^{\mu(\kappa)} - \hat{F} >$ for which ~~F~~ $H_{i,j}$ fortuitously belongs to a block of ~~zero~~ order unity in H (see (iv) b)).

d) In this case $F_{i,j}(m) \in W$ ($F_{i,j} \equiv \overline{\mathbb{U}}^{\mu(\kappa)} < \mathbb{U}^{\mu(\kappa)} - \hat{F} >$) and the distribution of zero elements in H, and hence the precise ~~truncation~~ character of Ξ, is known completely.

(vii) a) We have $F_{i,j}(m), F_{i',j'}(m) \in W$, $F_{i,j}(m) \sim \{h_{i,j}\}$ by (ix) with a similar expansion for $F_{i',j'}$, and $\xi(i,j) = i+j+1$, $\xi(i',j') = i'+j'+1$ for $F_{i,j}, F_{i',j'} \in \mathbb{U}^{\mu(\kappa)} < \mathbb{U}^{\mu(\kappa)} - \hat{F} >$. The result of (1) follows directly from 4... . In this case also, one of the sequences $h_{i,j}$, $h_{i',j'}$ has its components with $\nu = \mathrm{II}_{\xi'}^{\xi''}$ equal to those of h, those of the other, as described in the proof of (vi) a) being unequal to the corresponding h_ν. The result of (2) follows immediately.

b) $\hat{I},\hat{J}$ << $\overset{\infty}{I},\tilde{J}$ >>, as determined in this clause, are the subscripts of the lower left << upper right >> hand member of the block, possibly of zero order to which $H_{i,j}$ << $H_{i',j'}$ >> belongs. The result (7) follows from 4. ... and (8) as a)(2), from Grommer's result; (9, 10) follow from 4 ..., and (11) deals with those $F_{i,j} \in \overline{\overline{U}}^{\mu(r)} < \overline{\overline{U}}^{\mu(r)} - \hat{\mathbb{F}} >$ for which $F_{i,j}(m) \in W$.

(viii) The results of this clause follow from those of (vi) and (vii), as 4... does from 4...

(ix) The results of this clause follow from 4... and what is known (see the proof of A.(iii) b)(2–4) concerning the block structure of H in each case <if $\sigma = st\{M_\nu, t_\nu; n\}$, $F(x)$ has a representation as a linear combination of values of $G(t,x)$, possibly together with a multiple of $G_0(m)$

(see (ii) c)>.

(x) If $G(t,x)$ is an entire function, then $F(m) \in W$ (4...) and, under the stated conditions, $F_{i,j}(m) \sim \{h_{i,j} : l_y, x'\}$ ($F_{i,j} \in \mathbb{F}$). The special results of (vi) – (viii) still hold for the relevant $F_{i,j}$ for which $j > i-1$, but Grommer's result may not be applied to the remaining $F_{i,j}$. We must fall back upon use of 4... – ... and what has to be shown (see the proof of A (iii) b) (2 – 4)) concerning the block structure of H in each case.

(xi/a) When $0 \notin (\alpha, \beta)$ or $x \in S_y$, the block structure of H is known completely, and precise results, such as those of the relevant clauses of (vi) – (x) may be given. When $[\alpha, \beta] = [-\infty, \infty]$ however, although it is known that $E(m) \in W$, the exact value of $\xi(i,j)$

for an $F_{i,j}$ of this set is unknown without reference to $H_{i,j}$. Nevertheless it is known to be $i+j$ if $H_{i,j}=0$, $i+j+2$ if $H_{i+1,j+1}=0$ and $i+j+1$ otherwise. Hence results such as (1-4), not requiring detailed knowledge of H can under conditions slightly more extensive than those depending upon knowledge of H, still be given.

b) When $[\alpha,\beta] \neq [-\infty,\infty]$, it can only be stated, without reference to H, that $F_{i,j}(m) \in W$ for $F_{i,j} = \mathbb{D} < \mathbb{D} \cup \hat{\mathbb{F}} >$

c,d) The remarks made in connection with a,b) hold in these cases. Furthermore, the precise locations of $H_{i,j}$, $H_{i',j'}$ within the blocks to which they possibly belong, is now unknown. Nonetheless an upper bound to the distance between the nearest corners of those blocks (which is needed to deal with the case in which the blocks share a common backward diagonal, can be given; no more than $r'-\xi'$ elements of $W^{(\mu|\tau)}$ lie on this diagonal and separate the two corners).

(xii)a) As in (x), $F(m) \in W$; but as in (xi)a) it can only be stated that $i+j \leq \xi(i,j) \leq i+j+2$ when $|i-j| = 2m$ $(m \in \mathrm{I\!I})$. b) Now the remarks made in connection with (xi)b) also hold.

(xiii)a) follows from the definition of $F(x)$, and b) from the remark that for all $F_{i,j}$ concerned, $F_{i,j}(m)$ is a finite linear combination of elements of $f(m)$ and values of $f(t,m)$.

B). The results of this part of the theorem follow from their counterparts in the preceding part, with few points of difference covered by the following remarks.

Firstly, with regard to (ii)a), if $\beta = \infty$ some restriction must be replaced upon $g(x)$ in order that $f(x)$ should exist, and the condition $g(\Delta[0,\infty]^{\psi}_{\phi}) \in V$ suffices for this purpose. If, however, $[\alpha,\beta] \subset [0,\infty)$ and $g(\Delta^{\psi}_{\phi}) \sim g$, no additional restriction is needed, for this second condition implies that $g(tx)$ is finite for sufficiently small values of $x \in \Delta^{\psi}_{\phi}$ and for $t \in [\alpha,\beta]$. Similar considerations hold with regard to the cases in which $\alpha = -\infty$ and $g(-\Delta^{\psi}_{\phi}) \sim g$. Again, the conditions $[\alpha,\beta] \subseteq [0,\infty]$ and $g(\Delta^{\psi}_{\phi}) \sim g$ alone imply that $f_{i,j}(\Delta^{\psi}_{\phi}) \sim h_{i,j}g$, whether $\beta = \infty$ or not, for $f_{i,j} \equiv u$ (see B.(v)a),(vi)a)(2),(vi)b), (ix)a),(ix)b)). For in this case $f_{i,j}(x)$ is a linear combination of expressions of the form $g(t^{(i,j)}_{\nu} x)$ with

$t_{\rho}^{(i,j)} \in (0,\infty)$. Further, if $g(\pm\Delta_{\phi}^{\psi}) \sim g$, then $f_{i,j}(\pm\Delta_{\phi}^{\psi}) \sim h_{i,j}g$ for all $f_{i,j} \equiv \mathfrak{U}$ (and not $\mathfrak{O}$ alone) even when $[\alpha,\beta] \neq [-\infty,\infty]$ (see B.(v)a), (vi)a)(2), (vi)b), (ix)a), (xi)a, (xii)a, b)). (We remark that in the special cases considered in B(v)a), (vi)c) the compound sectors of 4.B.(iii)(v)a) reduce to one or possibly two segments of the real axis). When, however, a fixed set Δ over which the $f_{i,j}(x)$ are to be defined is prescribed, then it is necessary to restrict the behaviour of $g(x)$ for $x \in [\alpha,\beta] \times \Delta$ (see B.(iii)a), (iv)a, b), (v) b), (vi)c), (vii), (viii), (x), (xi) b), (xiii)). Lastly, if $g(N\{0\}) \in A$, then $\mathfrak{f}(N\{0\}) \in A$ (4.B..) and asymptotic relationships involving the function $\mathfrak{f}_{i,j,i',j'}(x)$ are not restricted to the subsets $\mathfrak{U}$ and $\mathfrak{O}$ (see (ix), (xii) b)); if g is an entire function then $\mathfrak{f}(\Delta) \in W$

for all $\Delta \in B$. and less restricted inequalities between mappings $f(\Delta)$, $f_{i,j}(\Delta)$ and $f_{i,j}(\Delta)$, $f_{i',j'}(\Delta)$, determined solely by the structure of H, are also available (see (x), and (xiii)).

Having proved Theorem 6, we remark that the condition $G([\alpha,\beta],M) \in \mathbb{V}$ implied in the notation $G \sim [[\alpha,\beta], g(M): x']$ ensures that $F(x)$ exists $(x \in M)$; but, taking $\beta = \infty$ for example, $F(x)$ may also exist (depending upon the rate of increase of $d\sigma(t)$ when $G(t,x)$ ceases to be finite as $t \to \infty$, and the result $F(M) \sim \{h, k; x'\}$ may still hold in such cases (furthermore, the condition $G([\alpha,\beta],M) \in \mathbb{V}$ is then not required for A.(v)a), (vi–viii), (x, xii)). A similar remark may be made concerning the functions $g(x)$ and $f(x)$. However in later convergence theorems, this condition must be imposed and, to simplify the exposition, we impose it at this point.

When $\sigma \in Sy$, $\sigma \notin St$ and $h = ms\{\sigma \in BND[\alpha, \beta]\}$, we have $H_{2i+1,2j+1} = 0$ $(i,j = \underline{I})$, and since $H_{2i,2j+1}$, $H_{2i+1,2j} \neq 0$ $(i,j = \underline{I})$ it immediately follows (see 1...) that in this case $H_{2i,2j} \neq 0 \, (i,j = \underline{I})$. When the condition $\sigma \in Sy$ is dropped, however, it can occur that a member of the set $H_{2i,2j}$ $(i,j = \underline{I})$ assumes the value zero. For example, let $\sigma(t)$ be compounded of a saltus of magnitude unity at $\alpha \in (-\infty, 0)$ and a distribution for which $d\sigma(t) = dt$ for $t \in [0,1]$, i.e. $h_\nu = \alpha^\nu + (\nu+1)^{-1}$ $(\nu = \underline{I})$. We then find that $H_{2,m} = \eta_m + m\alpha^{m-1} |\alpha - m + i\eta_m^{\frac{1}{2}}|^2$ where $\eta_m = 1/\{m(m+1)^2(m+2)\}$ $(m = \underline{I}_1)$; for even values of $m \in \mathrm{I}_2$ we may determine α such that $H_{2,m} = 0$.

When $h \in ms\{\sigma \in BND[\alpha,\beta]\}$, it is known (see the proof of A.(iv)) that ~~for~~ the $t_\nu^{(i,j)}$ corresponding to the members of $\mathfrak{V}$ are real. For other values of i, j this ceases to be true. For example, taking $\alpha = -1, \beta = 1$ and $\sigma(t) = t$ (so that $h_{2\nu} = 2/(2\nu+1)$, $h_{2\nu+1} = 0$ $(\nu \in \mathrm{I})$) we find that two of the $t_\nu^{(4,0)}$ have the values $\pm i/\{(15+3\sqrt{105})/8\}$.

We wish also to comment that use of Grommer's result enables us to say far more ~~then~~ about the structure of $\mathbb{F}$, assuming far less about $f(m)$ than is required, for example, for the analogous results of Theorem 5. Assuming that $0 \notin (\alpha,\beta)$ and

$\overset{?}{\sigma} \notin St$, it is no longer necessary to assume in A.(viii)(i) that $G_{i,j+1}(m) \neq 0$ to ensure that $F_{i,j}(m) \neq F(m)$ ($F_{i,j} = U$): the existence of one nonzero component $G_{\nu}(m)$ of $\mathcal{G}(m)$ ensures that $F_{i,j}(m) \neq F(m)$ for all $F_{i,j} = U^{k'}$. Furthermore, it is no longer necessary to assume that $G_{i,j+1}(m) \neq 0$ in order to ensure that $F_{i,j}(m) \neq F_{i',j'}(m)$: that $G_{\nu}(m) \neq 0$ for any $\nu \in I^{i',j'}_{i,j+1}$ suffices for this purpose. Similar considerations hold with regard to the case in which $\sigma \in \overset{?}{S}$, and for other dispositions of the interval $[\alpha, \beta]$.

Assuming $\sigma \notin St$, no $H_{i,j} = 0$ when $\sigma \in (\alpha, \beta)$; also, when $\sigma \in S_y$, the disposition of zero $H_{i,j}$ is completely determined a priori. In these cases the degrees of asymptotic agreement between $F(x)$ and ~~F~~ the $F_{i,j}(x)$, and between the $F_{i,j}(x)$ themselves, and also the structure of part of $\mathbb{F}(m)$, can largely be determined by inspecting the integer suffixes of the nonzero components of $\mathfrak{g}(m)$ (see A. (vi) a, d), (vi) a, d), (viii) a, d), (x) a) (1,3) b) (1,3), c) (1,2), d) (1.3)) Precise results of a similar nature for the case in which $O \in (\alpha, \beta)$ require, however the preliminary construction of $\mathbb{H}$, or part of $\mathbb{H}$ (see the remaining results ~~of~~ in the clauses just listed). Nevertheless, without knowledge of $\mathbb{H}$ it is still possible to assert for an $F_{i,j} \in \mathbb{U}^{\mu(k)} - \mathbb{O}^{\mu(k)}$, for example, that $\xi(i,j) \in \mathbb{I}_{i+j}^{i+j+2}$, and to assert that even when $H_{i,j}$ belongs to a block, the order of the block is no more than unity. Thus, a number of results involving slightly more restrictive conditions upon the $G_\nu(n)$ can still be formulated without reference to the $H_{i,j}$ (they are given in A.(xi3, xii)).

The remarks of the preceding two paragraphs hold, of course, with equal force for the functions $f(x)$ and $f_{i,j}(x)$.

In B.(ii) c),(iv) a),(vi) c) (viii)(xxiv), we gave results for the functions $f(x)$ and $f_{i,j}(x)$ holding for values of x belonging to a point set Δ, it being assumed that $\underset{\circ}{g}([\alpha,\beta] \times \Delta) \in \mathbb{V}$. We wish to describe the permissible forms of Δ.

Suppose to begin with, that $g(x)$ is uniquely determined and bounded except in the neighbourhood of points belonging to the radial cuts $x = e^{i\phi_\nu}$,

$0 \leq \rho_\nu \leq |x| \leq \omega_\nu \leq \infty$ $(\nu = \overline{\underline{\mathrm{I}}}_0^r)$ (we include thereby the case in which $\rho_\nu = \omega_\nu$ $(\nu = \overline{\underline{\mathrm{I}}}_0^r)$ and $y(x)$ has poles at $x = \omega_\nu e^{i\phi_\nu}$ $(\nu = \overline{\underline{\mathrm{I}}}_0^r)$ in the complex plane, and also that in which $r = \phi_0 = 0$ or $r = 0, \phi_0 = \pi$ or $r = 1, \phi_0 = 0, \phi_1 = \pi$ and functions of a real variable x only are being considered). If $[\alpha, \beta] \subseteq [0, \infty]$, Δ may be taken to be any set not including a point in the neighbourhood of the radial cuts $x = e^{i\phi_\nu}$, $\rho_\nu/\beta \leq |x| \leq \omega_\nu/\alpha$ $(\nu = \overline{\underline{\mathrm{I}}}_0^r)$; (if $\alpha = 0$, Δ in the complex plane may then be a star shaped domain; if $\beta = \infty$, Δ may then be a region with radial cuts emanating from the origin; if $\alpha = 0, \beta = \infty$, Δ may be the adjunction of a number of sectors; if $[\alpha, \beta] \leq [-\infty, 0]$ the excluded cuts lie along the lines

$x = -e^{i\phi_\nu}$ ($\nu = \overline{\mathrm{II}}_0^r$); if $0 \in (\alpha,\beta)$ they are given by $x = e^{i\phi_\nu}$, $\rho_\nu/\beta \le |x| \le \infty$, $x = -e^{i\phi_\nu}$, $\omega_\nu/\alpha \le |x| \le \infty$ ($\nu = \overline{\mathrm{I}}_0^r$). When the values of ϕ_ν become arbitrarily close, the extended cuts become when $[\alpha,\beta] \subseteq [0,\infty]$ a set of the form $\Delta[\rho,\omega]_{\phi'}^{\psi'}$. We may then take Δ to be a set of the form $\Delta[0,\infty]_\phi^\psi - \Delta[\rho/\beta,\omega/\alpha]_{\phi'}^{\psi'}$, where $\phi \le \phi' \le \psi' \le \psi$. Setting $\rho(\theta) = \omega(\theta) = \infty$ ($\theta \in [\phi,\phi') \cup (\psi',\psi]$) Δ may be written in the form $\Delta[0,\rho/\beta]_\phi^\psi \cup \Delta[\omega/\alpha,\infty]_\phi^\psi$; $[\alpha,\beta]\times\Delta$ is then $\Delta[0,\rho]_\phi^\psi \cup \Delta[\omega,\infty]_\phi^\psi$. When $[\alpha,\beta] \subseteq [-\infty,0]$, Δ may be taken to be $(-\Delta[0,\rho/|\alpha|]_\phi^\psi) \cup (-\Delta[\omega/|\beta|,\infty]_\phi^\psi)$; $[\alpha,\beta]\times\Delta$ is then as just described. When $0 \in (\alpha,\beta)$, Δ may be taken to be $\Delta[0,\rho/\beta]_\phi^\psi \cup (-\Delta[0,\rho/|\alpha|]_\phi^\psi)$; $[\alpha,\beta]\times\Delta$ is then $\Delta[0,\hat{\rho}]_\phi^\psi \cup (-\Delta[0,\hat{\omega}]_\phi^\psi)$, where $\hat{\rho}(\theta) = \rho(\theta) \ll \beta\rho(\theta)/|\alpha| \gg$, $\hat{\omega}(\theta) = |\alpha|\rho(\theta)/\beta$ $\ll \rho(\theta) \gg$ $(\theta = [\phi, \psi])$ if $\beta/|\alpha| \leq 1 \ll > 1 \gg$ (when $[\alpha, \beta] = [-\infty, \infty]$, the formulae determining $[\alpha, \beta] \times \Delta$ become indeterminate, but in this case Δ reduces to a single point, and the results stated in connection with this set of points then becomes nugatory).

If $g(\Delta[\rho, \omega]_{\phi}^{\psi}) \in \mathbb{V}$ ($\Delta[\rho, \omega]_{\phi}^{\psi}$ may (see §) be a sector of a star shaped domain from which a central portion has been removed, or may be a segment of the real axis), $[\alpha, \beta] \subseteq (0, \infty)$ and $\rho(\theta)/|\alpha| \leq \omega(\theta)/|\beta|$ $(\theta = [\phi, \psi])$, Δ may be taken to be $\Delta[\rho/\alpha, \omega/\beta]_{\phi}^{\psi}$, while if $[\alpha, \beta] \subseteq (-\infty, 0]$, we may set $\Delta = -\Delta[\rho/|\alpha|, \omega/|\beta|]_{\phi}^{\psi}$. If $g(\Delta[\omega]_{\phi}^{\psi})$ $\in \mathbb{V}$ (again $\Delta[\omega]_{\phi}^{\psi}$ may be a sector of a star shaped domain or a segment (with one end point at the origin of the real axis) and $[\alpha,\beta]\subseteq[0,\infty]$ we may take $\Delta = \Delta[\omega/\beta^-]^{\psi}_{\phi}$ ($\omega(\phi)/\beta = \infty$ when $\omega(\phi) = \infty$ whether $\beta = \infty$ or not); when $[\alpha,\beta]\subseteq[-\infty,0]$ we may take $\Delta = -\Delta[\omega/|\alpha|^-]^{\psi}_{\phi}$, and when $0\in(\alpha,\beta)$ we may take $\Delta = \Delta[\omega/\beta]^{\psi}_{\phi} \cup \{-\Delta[\omega/|\alpha|]^{\psi}_{\phi}$ (the latter representations for Δ may again be sectors of star shaped domains or segments of the real axis).

The regions Δ and $[\alpha,\beta]\times\Delta$ described above are of two types: the first is derived by exclusion of a region over which g has singularities, the second by linear extension of a region ϕ over which g is free of singularities. Examination of the formulae given reveals ~~that~~ in every case except that in which $[\alpha,\beta]$

$\subseteq (0,\infty)$ (or $[\alpha,\beta] \subseteq (-\infty,0)$) the regions of the two types have similar forms.

Recursive application

Notation. $P_{i,j}^{[s]}$ $(s = \overline{I}_0^{\tau}; i,j = I)$ are the Padé quotients derived from the prescribed sequences $h^{[s]}$ $(s = \overline{I}_0^{\tau})$. We set $P_{i,j}^{[s]} \leftarrow h_{i,j}^{[s]}$ $(s = \overline{I}_0^{\tau}; i,j = I)$, where $h_{i,j}^{[s]}$ is $h_\nu^{[s;i,j]}$ $(\nu = I)$.

Let the number $\mathfrak{b} \in \mathbb{Z}$, the function $\hat{G}(t,x)$, the sequence $\hat{\mathfrak{h}}(x)$ the sequences $h^{[s]}$ $(s = \overline{I}_0^{\tau})$ and the functions $F^{[s]}(\mathfrak{b},x)$ $(s = \overline{I}_0^{\tau})$ be prescribed; construct the sequences $G_\nu^{[0]}(\mathfrak{b},x) = \mathfrak{b}^\nu G_\nu(x)$ $(\nu = I)$, $G_\nu^{[s+1]}(\mathfrak{b},x) = h_\nu^{[s]} G_\nu^{[s]}(\mathfrak{b},x)$ $(s = \overline{I}_0^{\tau-1}; \nu = I)$ and the functions $G^{[0]}(\mathfrak{b};t,x) = G(\mathfrak{b}t,x)$, $G^{[s+1]}(\mathfrak{b};t,x) = F^{[s]}(\mathfrak{b}t,x)$. $F_{i,j}^{[s]}(x)$ is the function defined by formula ( ) with the $\alpha_\nu^{(i,j)}, \dots, t_\nu^{(i,j)}$ derived from $P_{i,j}^{[s]}$ and $G_\nu(x), G(t,x)$ replaced by $G_\nu^{[s]}(\mathfrak{b};x)$, $G^{[s]}(\mathfrak{b};t,x)$ respectively $(s = \overline{I}_0^{\tau})$. $\mathbb{F}^{[s]}$ is the complete array of such functions $(s = \overline{I}_0^{\tau})$; $\mathbb{W}_m^{[s]}(M), \dots, \mathbb{F}_m^{[s]}(M)$

have meanings with respect to $\mathbb{F}^{[s]}$ analogous to those given

? in Notations... . $\mathfrak{h}^{[s]}(\xi, m)$ is the sequence $G_\nu(x)\xi^\nu$ $(\nu \equiv \mathrm{I})$. $F(x:\sigma^{(s)}; G^{(s)}(\xi):\alpha(s), \beta(s))$ is $F[x:\sigma^{(s)}; G:\alpha(s); \beta(s)]$ with the function $G(t,x)$ replaced by $G^{[s]}(\xi; t, x)$.

Let the function $\hat{g}$, the sequence $\hat{g}$, the sequences $h^{[s]}$ $(s \equiv \bar{\mathrm{I}}_0^{\tau})$ and the functions $f^{[s]}(s \equiv \bar{\mathrm{I}}_0^{\tau})$ be prescribed; construct the sequences $g^{[0]} = g$, $g^{[s+1]} = h^{[s]} g^{[s]}$ $(s \equiv \bar{\mathrm{I}}_0^{\tau-1})$ and the functions $g^{[0]} = \hat{g}$, $\hat{g}^{[s+1]} = f^{[s]}$ $(s \equiv \bar{\mathrm{I}}_0^{\tau-1})$. $f_{i,j}^{[s]}$ is the function defined by formula ( ) with the $\alpha_\nu^{(i,j)}$, $t_\nu^{(i,j)}$ as above, and g_ν, $g(x)$ replaced by $g_\nu^{[s]}$, $g^{[s]}(x)$ respectively. $\mathfrak{f}^{[s]}, \ldots, \mathfrak{f}^{[s]\prime}(\Delta)$ have meanings analogous to those of $\mathbb{F}^{[s]}, \ldots, \mathbb{F}^{[s]\prime}(m)$ above.

In all of the theorems given in this section, the conditions imposed upon $h^{(s)}$, $h^{(s)}$, $[\alpha(s), \beta[s]]$ (where relevant) and the results stated, except where otherwise indicated, all hold for $s = \bar{\bar{I}}_0^{\infty}$.

In the theorems concerning the functions $F_{i,j}^{[s]}$, $f_{i,j}^{[s]}$ which we shall give, each function $F^{[s]}(\zeta, x)$, $f^{[s]}(x)$ is the limit of a convergent sequence of the $F_{i,j}^{[s]}(\zeta, x)$, $f_{i,j}^{[s]}(x)$: these functions are constructed by means of recursive application of the algorithm using formulae ( ) and ( ). Accepting, at each stage, $F_{i(\nu),j(\nu)}^{[\nu]}(\zeta, x)$ to be a sufficiently good approximation to $F^{[\nu]}(\zeta, x)$ $(\nu = \bar{I}_0^s)$, $F^{[s]}(\zeta, x)$ may be approximated by a finite linear combination of values of $G(\zeta t, x)$ for various values of t, possibly together with values of the derivatives of this function.

Theorem . Let $h^{[s]}(z) = h(z : a^{[s]}, v^{[s]}; J(s); u^{[s]}, \phi^{[s]})$,

$h^{[s]} \leftarrow h^{[s]}$ and $\zeta \in Z$.

(i) Let $\hat{G}(z = N\{0\}, m) \in A_{\frac{\epsilon}{z}}$, $\hat{G} \leftarrow \mathcal{G}(m)$. Set $F^{[s]}(\zeta, m) = F(x; \mathbb{C}; G^{[s]}, h^{[s]})$, where $G^{[s]}$ denotes $G^{[s]}(\zeta; z, x)$ and $\mathbb{C}$ is a circle enclosing the origin lying in $N\{0\}$, and is described in an anticlockwise direction.

a) $\sum h_\nu^{[s]} G_\nu^{[s]}(\zeta, m) \to F^{[s]}(\zeta, m)$ $(s = \bar{\mathbb{I}}_0^{\tau})$

b) $\mathbb{F}^{[s]}(m) \to \mathbf{F}^{[s]}(\zeta, m)$ $(s = \bar{\mathbb{I}}_0^{\tau})$

c) $\sum h_\nu^{[s;i,j]} G^{[s]}(\zeta, m) \to F_{i,j}^{[s]}(\zeta, m)$ $(s = \bar{\mathbb{I}}_1^{\tau}; i, j = \mathbb{I})$

d) $G^{[s]}(\zeta; z, m)$ is an entire function of z, and

$\sum h_\nu^{[s]} G_\nu^{[s]}(\zeta; Z, m) \to G^{[s]}(\zeta; Z, m)$ $(s = \bar{\mathbb{I}}_1^{\tau})$.

(ii) Let $\hat{g}(N\{0\}) \in A$, $g \leftarrow g$. Set $f^{[s]}(x) = f(x : \mathbb{C}; g^{[s]}, h^{[s]})$ where $\mathbb{C}$ is as in (i)

a) $f^{[s]}$ is an entire function, and $f^{[s]} \leftarrow h^{[s]} g^{[s]}$ $(s = \bar{\mathbb{I}}_1^{\tau})$

b) $f^{[s]}(Z) \to f^{[s]}(Z)$. $(s = \mathbb{I}_0^{\tau})$

c) $f^{[s]}_{i,j}$ is an entire function, and $f^{[s]}_{i,j} \leftarrow h^{[s]}_{i,j}\, g^{[s]}_{i,j}$ $(s = \overline{\mathbb{I}}^{\tau}_{1};$

$i,j = \mathbb{I})$

d) $\underset{\circ}{g}^{[s]}$ is an entire function, and $\underset{\circ}{g}^{[s]} \leftarrow g^{[s]}$ $(s = \overline{\mathbb{I}}^{\tau}_{1})$

B. Let $h^{[s]}(z) = h(z : a^{[s]}, v^{[s]}; I[s], J[s]; u^{[s]}, t^{[s]}; p^{[s]})$

$|t^{[s]}_{\nu}| < 1$ $(s = \overline{\mathbb{I}}^{\tau}_{0},\ \nu = \mathbb{I}^{I[s]}_{1})$, $h^{[s]} \leftarrow h^{[s]}$ and $\zeta \in \mathbb{Z}$.

(i) Let $G(z = \mathbb{D}_{\gamma}, M) \in A_z$, where $\gamma > |\zeta|$. Define $F^{[s]}(\zeta, M)$

as in A(i) where now $\mathbb{C}$ encloses $\mathbb{D}_1$ and lies in $\mathbb{D}_{\gamma/|\zeta|}$

a) $\sum h^{[s]}_{\nu} G^{[s]}_{\nu}(\zeta, M) \Rightarrow F^{[s]}(\zeta, M)$

b) $\mathbb{F}^{[s]}(M) \Rightarrow F^{[s]}(\zeta, M)$

? c) $\sum h^{[s;i,j]}_{\nu} G^{[s]}_{\nu}(\zeta, M) \to F^{[s]}_{i,j}(M)$ $(s = \overline{\mathbb{I}}_{0};\ F^{[s]}_{i,j} = \pi\{F^{[s]}\}$

d) $G^{[s]}(\zeta; z = \mathbb{D}_1, M) \in A_z$ and $\sum G^{[s]}_{\nu}(\zeta, M)(\mathbb{D}_1)^{\nu} \Rightarrow$

$G^{[s]}(\zeta; \mathbb{D}_1, M)$

(ii) Let $g(\overline{\mathbb{D}}_{\gamma}) \in A$ and $\underset{\circ}{g} \leftarrow g$. Define $f^{[s]}(x)$ as in A(ii)

where now $\mathbb{C}$ encloses $\mathbb{D}_1$ and lies in $\mathbb{D}_{1+\delta}$.

a) $f^{[s]}(\mathbb{D}_\gamma) \in A$ and $f^{[s]} \leftarrow h^{[s]} g^{[s]}$ $(s = \overline{\mathbb{I}}_0^{\tau})$

b) $\mathfrak{f}^{[s]}(\mathbb{D}_\gamma) \rightarrow f^{[s]}(\mathbb{D}_\gamma)$ $(s = \overline{\mathbb{I}}_0^{\tau})$

c) $f_{i,j}^{[s]}(\mathbb{D}_\gamma) \in A$ and $f_{i,j}^{[s]} \leftarrow h_{i,j}^{[s]} g^{[s]}$ $(f_{i,j}^{[s]} = \Pi\{\mathfrak{f}^{[s]})$

d) $g^{[s]}(\overline{\mathbb{D}}_\gamma) \in A$ and $g^{[s]} \leftarrow g^{[s]}$.

Proof A(i). Using the theory given in the proof of ... A(i), and the formulae

$$F_{i,j}^{[s]}(\zeta, x) = (2\pi i)^{-s-1} \int_C \cdots \int_C \left\{ \Pi_0^{s-1} z_\nu^{-1} h^{[\nu]}(z_\nu^{-1}) \right\} z_s^{-1} P_{i,j}^{[s]}(z_s^{-1})$$
$$G(\zeta \Pi_0^s z_\nu ; x) \Pi_0^s dz_\nu$$

and

$$F^{[s]}(\zeta, x) = (2\pi i)^{-s-1} \int_C \cdots \int_C \left\{ \Pi_0^s z_\nu^{-1} h^{(\nu)}(z_\nu^{-1}) \right\} G(\zeta \Pi_0^s z_\nu ; x) \Pi_0^s dz_\nu$$

we show by induction that the result of b) holds. The result of a) follows from formula ( ), that of c) from ( ), and d) from the relationship $G^{[s+1]}(\zeta; z, x) = F^{[s]}(\zeta z, x)$ $(s = \overline{\mathbb{I}}_0^{\tau-1})$ and use of formula ( )

(ii) Now formulae ( ) and ( ) are to be replaced by

$$f_{i,j}^{[s]}(x) = (2\pi i)^{-s-1} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \{\Pi_0^{s-1} z_\nu^{-1} h^{[\nu]}(z_\nu^{-1})\} z_s^{-1} P_{i,j}^{[s]}(z_s^{-1}) \, g(x \Pi_0^s z_\nu) \Pi_0^s dz_\nu$$

and

$$f^{[s]}(x) = (2\pi i)^{-s-1} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} \{\Pi_0^s z_\nu^{-1} h^{[\nu]}(z_\nu^{-1})\} g(x \Pi_0^s z_\nu) \Pi_0^s dz_\nu$$

B. With the new contour $\mathbb{C}$, the above formulae and methods of proof are valid.

The structural results of ... A(i)..(ii).. B.(i)..(ii) apply, of course, to the arrays of functions $F^{[s]}$ and $f^{[s]}$ considered in the above theorem

By taking $g(x) = e^x$, we may, by taking $h_\nu^{(s)} = 1/\nu!$ $(s = \mathbb{I}_0^{\tau})$ obtain an array of approximations, each having the form of an exponential cum polynomial sum, to the sum $f^{[\tau]}(x)$ of the series $\sum x^\nu/(\nu!)^{\tau+2}$.

Theorem Let $h^{[s]} = m\sigma\{\sigma^{[s]} \in BND[\alpha[s], \beta[s]]\}$.

A. Set $F^{[s]}(\zeta, x) = F(x : \sigma^{[s]} ; G^{[s]}(\zeta); \alpha[s]; \beta[s])$.

(i) Let $[\alpha[s], \beta[s]] \subseteq [0, \infty]$, $\xi \in [0, \infty)$

? a) Let $G \in \tilde{Q}\{M; 0, \infty\}$, $G \sim g(m)$ and $h^{(s)}_{m'} \in DS$. Then

$U^{[s]}_{m'}(m) \to F^{[s]}(\xi, m)$

(ii) Let $[\alpha[s], \beta[s]] \subseteq [1, \infty]$, $\xi \in [0, \infty)$

? a) Let $G \in Q\{M; \xi, \infty\}$, $g(M) \in W$ and $h^{[s]}_{m'} \in DS$ $(s = I^{*}_{0})$.

Then $\mathbb{U}^{[s]}_{m'}(m) \to F^{[s]}(\xi, m)$

: b) Let $G \in R\{M; \xi, \infty\}$ and with $\tilde{h}^{[s]}$ determined from $h^{[s]}$ by a formula similar to ( ), let $\tilde{h}^{[s]}_{m''} \in DS$. Then

$\mathbb{L}^{[s]}_{m''}(m') \to F^{[s]}(\xi, m')$.

(iii) Let $[\alpha[s], \beta[s]] \subseteq [0, 1]$, $\xi \in [0, 1]$

a) Let $G \in Q\{M; 0, 1\}$ and $G \sim g(m)$. Then $\mathbb{U}^{[s]'}(m) \to F(\xi, m)$

b) Let $G \in R\{M; 0, 1\}$ and $G \leftarrow g(m)$. Then $\mathbb{F}^{[s^{*}]'}(m) \to F(\xi, m)$

(iv) Let $[\alpha[s], \beta[s]] \subseteq [-\infty, \infty]$, $\xi \in (-\infty, \infty)$. Let $G \in Q\{M; -\infty, \infty\}$

$G^{\pm} \sim g(m)$ and $h^{[s]}_{2m'} \in DH$ Then $\mathbb{U}^{[s]}_{2m'}(m) \to F^{(s)}(\xi, m)$, and if $\sigma^{[s]} \in Sy$ $(s \equiv \mathbb{I}_0^{\pi})$ with $G(t,m) = G(-t,m)$ $(t = [0,\infty])$, then

$\overline{\mathbb{D}}_{2m'}(m) \Rightarrow F^{[s]}(\xi, m)$.

(v) Let $[\alpha[s], \beta[s]] \subseteq [-1,1]$, $\xi \in [-1,1]$

a) Let $G \in Q\{M; -1,1\}$, $G \pm \sim \mathcal{G}(m)$. Then $\mathbb{D}^{[s]'}(m) \Rightarrow F^{[s]}(\xi, m)$.

and if $\sigma^{[s]} \in Sy$ $(s \equiv \mathbb{I}_0^{\pi})$ with $G(t,m) = G(-t,m)$ $(t = [0,\infty])$

then $\overline{\mathbb{D}}^{[s]'}(m) \Rightarrow F^{[s]}(\xi, m)$

b) Let $G \in \overline{Q}\{M; -1,1\}$, $G \pm \sim \mathcal{G}(m)$. Then $\mathbb{U}^{[s]'}(m) \Rightarrow F^{[s]}(\xi, m)$

c) Let $G \in R\{M; -1,1\}$. Then $\mathbb{L}^{[s]}(m) \Rightarrow F^{[s]}(m)$

d) Let $G \in R\{M; -1,1\}$ with $G \leftarrow \mathcal{G}(m)$. Then $F^{[s]'}(m) \Rightarrow F^{[s]}(\xi, m)$

B. Let $f^{[s]}(x) = f[x : \sigma^{[s]}; g^{[s]}; \alpha[s], \beta[s]]$

(i) Let $[\alpha[s], \beta[s]] \subseteq [0,\infty]$, $\xi \in (0,\infty)$. Set $\tilde{\Delta} = \Delta[0,\infty]_{\phi}^{\psi}$

$\Delta = \Delta[0,\infty]_0^{\psi}$.

a) Let $g(\tilde{\Delta}) \sim g$, $g(\Delta_{\phi}^{\psi}) \sim g$ and $h_{m'}^{[s]} \in \mathbb{D}S$. Then

$\mathfrak{u}_{m'}^{[s]}(\mathcal{B}\{\Delta\}) \to f(\mathcal{B}\{\Delta\})$.

(ii) Let $[\alpha[s],\beta[s]] \le [1,\infty]$, $\overset{\infty}{\Delta}\xi = \Delta[\rho,\infty]_{\phi}^{\psi}$, $\Delta = \Delta[\rho\kappa,\infty]_{\phi}^{\psi}$.

a) Let $g(\overset{\infty}{\Delta}) \in q$ and $h_{m'}^{[s]} \in \mathbb{D}\delta$. Then $u_{m'}^{[s]}(B\{\Delta\}) \to f(B\{\Delta\})$

b) Let $g(N\{\overset{\infty}{\Delta}\}) \in A$ and for the sequences $\tilde{h}^{[s]}$ of (i)b), let $\tilde{h}_{m}^{[s]} \in \mathbb{D}\delta$. Then $\mathbb{V}_{m'}^{[s]}(\Delta) \to f^{[s]}(\Delta)$

(iii) Let $[\alpha[s],\beta[s]] \le [0,1]$, $\tilde{\Delta} = \Delta[0,\omega]_{\phi}^{\psi}$, $\Delta = \Delta[0,\omega/\beta]_{\phi}^{\psi}$

a) Let $g(\overset{\infty}{\Delta}) \in q$ and $g(\Delta_{\phi}^{\psi}) \sim g$. Then $u^{[s]'}(B\{\Delta\}) \to f(B\{\Delta\})$

b) Let $g(N\{\overset{\infty}{\Delta}\}) \in A$, then $\mathbb{V}^{[s]'}(\Delta) \to f^{[s]}(\Delta)$ and, if $g \leftarrow g$, $f^{[s]'}(B\{\Delta\}) \to f^{[s]}(B\{\Delta\})$.

(iv) Let $[\alpha[s],\beta[s]] \le [-\infty,\infty]$. Let $\Delta = \pm\Delta[0,\infty]_{\phi}^{\psi}$

a) Let $g(\Delta) \in q$, $g \in (\pm\Delta_{\phi}^{\psi}) \sim g$ and $h_{2m'}^{[s]} \in \mathbb{D}H$. Then $u_{2m'}(B\{\Delta\}) \to f^{[s]}(\Delta)$ and if, in addition, $\sigma \in Sy$ and $g(x) = g(-x)$ $(x = \Delta)$, then $\overline{\overline{u}}_{2m'}^{[s]}(B\{\Delta\}) \to f^{[s]}(B\{\Delta\})$.

(v) Let $[\alpha[s],\beta[s]] \le [-1,1]$, $\Delta = \pm\Delta[0,\omega]_{\phi}^{\psi}$.

a) Let $g(\pm\Delta_\phi^\psi) \sim g$. If $g(\Delta) \in q$, then $\omega^{[s]'}(\mathfrak{B}\{\Delta\}) \to f(\mathfrak{B}\{\Delta\})$, and if $e \in \delta_q$ and $g(x) = g(-x)$ $(x \equiv \Delta)$ also, then $\bar{u}^{[s]'}(\mathfrak{B}\{\Delta\}) \to f^{[s]}(\mathfrak{B}\{\Delta\})$; if $g(\Delta) \in \bar{q}$, then $u^{[s]'}(\mathfrak{B}\{\Delta\}) \to f^{[s]}(\mathfrak{B}\{\Delta\})$.

b) Let $g(\mathfrak{N}\{\Delta\}) \in A$. Then $\eta^{[s]'}(\Delta) \to f^{[s]}(\Delta)$ and, if $g \leftarrow g$, $f^{[s]'}(\mathfrak{B}\{\Delta\}) \to f(\mathfrak{B}\{\Delta\})$.

C. In any of the cases considered in A., $F^{[s]}(\zeta, x)$ has a representation of the form $F^{[s]}(\zeta, x) = F[x : G(\zeta); \hat{\sigma}^{[s]}, \alpha^{[s]}, \beta^{[s]})$ where $\hat{\sigma}^{[s]} \in BND[\hat{\alpha}[s], \hat{\beta}[s]]$, $[\hat{\alpha}[s], \hat{\beta}[s]] = \prod_0^s [\alpha[s], \beta[s]]$, $\hat{\sigma}^{[0]}(t) = \sigma^{[0]}(t)$ $(t \equiv [\alpha[0], \beta[0]]$, $d\hat{\sigma}^{[s+1]}[0] = d\sigma^{[s+1]}(0)\{\hat{\sigma}^{[s]}(\hat{\beta}[s]) - \hat{\sigma}^{[s]}(\hat{\alpha}[s])\}$ if $0 \in (\hat{\alpha}[s], \hat{\beta}[s])$ and $d\hat{\sigma}^{[s+1]}(t) = \int_{\hat{\alpha}[s]}^{\hat{\beta}[s]} d\sigma^{[s+1]}(t/t')\, d\hat{\sigma}^{[s]}(t')$ $\{t (\neq 0) \equiv [\hat{\alpha}[s], \hat{\beta}[s]]$ $(s \equiv \bar{I}_0^{n-1})$. A similar result obtained by replacing $F^{[s]}(\zeta, x)$, $G(\zeta t, x)$ by $f^{[s]}(x)$, $g(tx)$ holds for the functions considered in B.

Proof A(i)a) When $\zeta = 0$, $F_{i,j}^{[s]}(x) = h_0^{[s]} \cdots h_0^{[s]} G_0(x)$ $(s = \overline{\overline{I}}_0^{\tau};\ i, j = \overline{I})$ and the results of the clause are trivially true; we discount this case. When $G(t,x)$ is simply a polynomial ~~in~~ $\sum_0^{\mu-1} G_\nu(x) t^\nu$, $F_{i,j}^{[s]}(\zeta, x) = F^{[s]}(\zeta, x)$ $(s = \overline{\overline{I}}_0^{\tau},\ i+j+1 \geq \mu)$ and again the stated results clearly hold. We discard the polynomial part of $G(t,x)$ and consider the functions constructed from $\hat{G}(t,x)$ in isolation.

That $W_{m'}^{[0]}(M) \to F(\zeta, M)$ under the stated conditions follows immediately from .. A(i)a). We now have

$$G^{[1]}(\zeta; t, x) = F^{[0]}(\zeta t, x) = \int_{\alpha[0]}^{\beta[0]} G(\zeta t t_0, x)\, d\sigma^{[0]}(t_0)$$

with $\zeta \in [0, \infty)$. It is evident that $G^{[1]}(\zeta; t, x)$ is continuous over every bounded subinterval of the range $t \in [0, \infty]$ and that, since $G(t,x)$ tends to a finite limit as t tends to infinity, $G^{[1]}(\zeta; t, x)$ does the same.

Furthermore, if $\zeta \in (0,\infty)$ and $\varkappa\{\zeta(x)\} = \mu_{\nu}$, ~~$\{G_{\mu(\nu)}(\zeta x) t$~~

$\{G_{\mu(\nu)}(\zeta t)^{\mu(\nu)}\}$ is an asymptotic sequence as $t(\in(0,\infty)) \to 0$.

Replacing $G_{\nu}(x)$ by $G_{\nu}(x)\zeta^{\nu}$ and incorporating t as one of the components of x in ..A(i).., we find that $\sum h_{\nu}^{[0]}\zeta^{\nu}G_{\nu}(x)t^{\nu}$ represents $F^{[0]}(\zeta t, x) = G^{[1]}(\zeta; t, x)$ asymptotically as $t(\in(0,\infty)) \to 0$. Hence, adopting the notation used in defining $F^{[s]}(\zeta, x)$, $G^{[1]}(\zeta) \in Q\{M; 0, \infty\}$. The result of the clause is now proved by induction. In place of ~~the~~ formulae of the form (,) we now have

$$F^{[s]}_{r,m+r-1}(\zeta, x) = \sum_0^{m-1} h_{\nu}^{[0]} \dots h_{\nu}^{[s]} G_{\nu}(x)\zeta^{\nu} + \int_{\alpha[s]}^{\beta[s]} \dots \int_{\alpha[0]}^{\beta[0]} G_m(\zeta \Pi_0^s t_{\nu}, x)\, d\sigma_m(t_0) \dots d\sigma_m(t_s)$$

?

and

$$F^{[s]}(\zeta, x) = \int_{\alpha[s]}^{\beta[s]} \dots \int_{\alpha[0]}^{\beta[0]} G(\zeta \Pi_0^s t_{\nu}, x)\, d\sigma(t_0) \dots d\sigma(t_s)$$

where $G_m(t, x) = \{G(t, x) \dot{-} \sum_0^{m-1} G_{\nu}(x) t^{\nu}\}/t^m \quad (t \in [0, \infty])$

(ii)(a) The proof of this result is analogous to that of (i).

b) The result of this clause is proved by induction using an argument similar to that exploited in the proof of ... A.(i)b) with formulae of the form (,) where now $\mathbb{C}$ surrounds $[1,\infty]$ and lies in $N\{[1,\infty]\}$.

The results of (ii)a), (iv), (v)a) and (iii)b), (v)b) are derived by methods similar to those used in the proof of (i) and (ii)b) respectively. The results of part B. are special cases of those of part A., setting $\zeta=1$ and adopting appropriate substitutions for $G_D(x)$ and $G(t,x)$.

C. $F^{[D]}(\zeta,x)$ has a representation of the form stated. Assuming $F^{[s]}(\zeta,x)$ to have one, and using the relationship $G^{[s+1]}(\zeta;t,x)=F^{[s]}(\zeta t,x)$, we have

$$F^{[s+1]}(\zeta,x)=\int_{\alpha[s+1]}^{\beta[s+1]}\int_{\hat{\alpha}[s]}^{\hat{\beta}[s]} G(\zeta t't'',x)\,d\hat{\sigma}^{[s]}(t'')\,d\sigma^{[s+1]}(t')$$

$$=\int_{\hat{\alpha}[s+1]}^{\hat{\beta}[s+1]} G(\zeta t,x)\,d\hat{\sigma}^{[s+1]}(t)$$

where $\hat{\sigma}^{[s+1]}(t)$ is as defined in the theorem.

The structural results of Theorem may, of course, be applied to the arrays of functions $F^{[s]}$ and $f^{[s]}$ considered in the above theorem.

By taking $g(x) = 1/(1+x)$, $[\alpha[s], \beta[s]] = [0, \infty]$, $\sigma^{[s]} = 1 - e^{-t}$ $(s \equiv \overline{1}_0^{\tau})$ we may obtain a sequence of rational approximations to the integral $\int_0^{\infty} \{e^{-t}/(1+xt^{\tau})\}dt$, which is asymptotically represented by the series $\Sigma(-1)^{\nu}(\nu!)^{\tau}x^{\nu}$ for $x \in \Delta_{-\pi}^{\pi}$.

It appears that the results of ... A(i)b), ( )b) and B(i)b).. ( )b) cannot be applied recursively. If $G(z,x)$ has a singularity at $z = z'$ and $\alpha[0] = 0$, $\beta[0] = \infty$ with $d\sigma^{[0]}(t) > 0$ $(t = [0,\infty])$ and $b(\neq 0) \in \mathbb{Z}$, then $G^{[1]}(b; z, x)$ has singularities distributed uniformly along the line $z = \arg(z') - \arg(\zeta)$. In this case $G^{[1]}(\zeta; z, x)$ is not analytic in $N\{0\}$ and hence not in $N\{[0,\infty]\}$.

Theorem Let $m \in I$ and $h^{(s)} = ms\{\sigma^{[s]} \in BND[\alpha[s], \beta[s]]\}$.

A. Let $\zeta(m) = ms\{\Lambda(m); \alpha', \beta'\}$, $G(t, m) = st\{t : \Lambda(m) : \alpha', \beta'\}$ and set $F^{[s]}(\zeta; x) = F[x : \sigma^{[s]}; G^{[s]}(\zeta); \alpha[s], \beta[s]]$.

(i) Let $\Delta(y, M) \in BND_y[\alpha', \beta']$

a) Let $\zeta \in (0,\infty)$, $[\alpha[s], \beta[s]] \subseteq [0,\infty]$ and $[\alpha', \beta'] \subseteq [-\infty, 0]$.

(1) $0 < F^{[s]}(\zeta, m) < \infty$. (2) $F^{[s]}_{i,j}(\zeta, m) < F^{[s]}(\zeta, m)$ $(F_{i,j} = O^{[s]})$ $F^{[s]}_{r,2m+r-1}(\zeta, m)\big|_0^\infty \in MI\{F^{[s]}(\zeta, m)\}$, $F^{[s]}_{r,2m-r+1}(\zeta, m)\big|_0^{m+1} \in MI\{F^{[s]}_{m+1,m}(\zeta, m)\}$, (2) $F^{[s]}_{i,j}(\zeta, m) > F^{[s]}(\zeta, m)$ $(F_{i,j} = E^{(s)})$ $F^{[s]}_{r,2m+r}(\zeta, m)\big|_0^\infty \in MD\{F^{[s]}(\zeta, m)\}$, $F^{[s]}_{r,2m-r}(\zeta, m)\big|_0^m \in MD\{F^{[s]}_{m,m}(\zeta, m)\}$.

b) Let $\zeta \in (0,\infty)$, $[\alpha[s], \beta[s]] \subseteq [0,1]$ and $[\alpha', \beta'] \subset [0, 1/\zeta]$. Then $0 < F^{[s]}_{i,j}(\zeta, m) < F^{[s]}(\zeta, m) < \infty$ $(F^{[s]}_{i,j} = U^{[s]})$.

$F^{[s]}_{r,m+r}(\zeta,m)\big|_0^{\infty} \in MI\{F^{[s]}(\zeta,m)\}$, $F^{[s]}_{r,2m-r}(\zeta,m)\big|_0^{m} \in MI\{F^{[s]}_{m,m}(\zeta,m)\}$,

$F^{[s]}_{r,2m-r+1}(\zeta,m)\big|_0^{m+1} \in MI\{F^{[s]}_{m+1,m}(\zeta,m)\}$

c) Let $\zeta(\neq 0) \in (-\infty,\infty)$, $[\alpha[s],\beta[s]] \subseteq [-1,1]$ and $[\alpha',\beta']$ $\subset (-1/\zeta, 1/\zeta)$. Then $0 < F^{[s]}_{i,j}(\zeta,m) < F^{[s]}(\zeta,m)$ $(F_{i,j} \equiv \mathbb{O}^{[s]})$ $F^{[s]}_{r,2m+r-1}(\zeta,m)\big|_0^{\infty} \in MI\{F^{[s]}(\zeta,m)\}$, $F^{[s]}_{r,2m-r+1}(\zeta,m) \in MI$ $\{F^{[s]}_{m+1,m}(\zeta,m)\}$.

(ii) Let $\Delta(y,m) \in \mathcal{B}M_y[\alpha',\beta']$.

a) The results of (i)a)(2) and b,c) hold with $F^{[s]}(\zeta,m)$ and the relevant $F^{[s]}_{i,j}(\zeta,m)$ replaced by $|\operatorname{Re} F^{[s]}(\zeta,m)|$, $|\operatorname{Re} F^{[s]}_{i,j}(\zeta,m)|$, by $|\operatorname{Im} F^{[s]}(\zeta,m)|$ and $|\operatorname{Im} F^{[s]}_{i,j}(\zeta,m)|$, and by $|F^{[s]}(\zeta,m)|$, $|F^{[s]}_{i,j}(\zeta,m)|$ respectively.

b) When, as in (i)a)(1), $[\alpha',\beta'] \subset [-\infty,0]$, $[\alpha[s],\beta[s]] \subseteq$ $[0,\infty]$, we have $|E^{[s]}_{r,2m+r-1}(\zeta,m)|\big|_0^{\infty} \in MD\{0\}$, $|E^{[s]}_{m+1,m,r,2m-r+1}(\zeta,m)|\big|_0^{m+1} \in MD\{0\}$.

B. Let $g \in \mathrm{ms}\{D; \hat{\alpha}, \hat{\beta}\}$, $g(x) = \mathrm{st}\{x : D; \hat{\alpha}, \hat{\beta}\}$ and set

$f^{[s]}(x) = f[x : e^{[s]}, g^{[s]}; \alpha[s], \beta[s]]$, $[\hat{\alpha}, \hat{\beta}] \times [\tilde{\alpha}, \tilde{\beta}] = [\check{\alpha}, \check{\beta}]$

(i) Let $D \in \mathrm{BND}[\alpha, \beta]$. With F, $\mathbb{m}$, $\mathbb{F}$, $\mathbb{U}$, $\mathbb{D}$ and $\mathbb{E}$ replaced by f, $[\tilde{\alpha}, \tilde{\beta}]$, $\mathfrak{f}$, u, $\mathfrak{o}$ and e, and $\zeta = 1$ in each case, the results of A(i) hold.

(ii) Let $D \in \mathrm{BM}[\hat{\alpha}, \hat{\beta}]$. With modifications similar to those just given, the results of A.(ii) hold.

Proof. A.(i)a) Set $\Delta^{[0]}(y, x) = \Delta(y/\zeta, x)$. We have $\mathfrak{f}^{[0]}(\zeta, m)$
$= \mathrm{ms}\{\Delta^{[0]}(m) : \zeta\alpha', \zeta\beta'\}$ where $\Delta^{[0]}(y, m) \in \mathrm{BND}_y[\zeta\alpha', \zeta\beta']$.
Set $[\alpha'[0], \beta'[0]] = [\zeta\alpha', \zeta\beta']$ and $[\alpha'[s+1], \beta'[s+1]] =$
$[\alpha[s], \beta[s]] \times [\alpha'[s], \beta'[s]]$ $(s \equiv \overline{\mathbb{I}}_0^{\tau-1})$. Then $\mathfrak{f}^{[s]}(\zeta, m) =$
$\mathrm{ms}\{\Delta^{[s]}(m); \alpha'[s], \beta'[s]\}$ where in each case $\Delta^{[s]}(y, m) \in$
$\mathrm{BND}_y[\alpha'[s], \beta'[s]]$, $[\alpha'[s], \beta'[s]] \subseteq [-\infty, 0]$ $(s = \overline{\mathbb{I}}_0^{\tau})$. The result of ... A ... may now be applied.

The remaining results may be derived from those of Theorem in the same way.

Theorem. Let $m \in \mathbb{I}$ and $\mathcal{L}_k(\alpha,\beta)$, $\overline{\mathcal{D}}_k(\alpha,\beta)$, K_k and L_k be as defined in ..

A. Let $F^{[s]}(h,x) = F(x : \sigma^{[s]}, G^{[s]}(h); \alpha[s], \beta[s])$ $(x \in \mathbb{Z})$

(i) Set $M_{k,m}^{[s;\alpha,\beta]} = \max |G^{[s]}(h;z,x) - \sum_0^{m-1} G^{[s]}(h,x) z^j| / |z^m|$.

$(z = \mathcal{L}_k(\alpha,\beta))$

a) Let $[\alpha[s],\beta[s]] \subseteq [0,1]$, $h^{[s]} = ms\{\sigma^{[s]} \in BND[\alpha[s],\beta[s]]\}$,

$h \in (0,\infty)$, $G(z = \overline{\mathcal{D}}_k(0,h), x) \in A_z$

(1) Let $G \leftarrow h(x)$. Then $|E_{r,m+r-1}^{[s]}(h,x)| \leq K_k L_k M_{k,m}^{[s;0,1]} |h_m^{[s]}| / z^{2r+1}$ $(r \in \mathbb{I})$

(2) $|E_{m+r+1,r}^{[s]}(h,x)| = O(k^{2r})$ as $r \to \infty$

b) Let $[\alpha[s],\beta[s]] \subseteq [-1,1]$, $h^{[s]} = ms\{\sigma^{[s]} \in BND[\alpha^{[s]},\beta^{[s]}]\}$,

$h \in (-\infty,\infty)$, $G(z = \overline{\mathcal{D}}_k(-h,h), x) \in A_z$. (1) Let $G \leftarrow h(x)$. Then

$|E_{r,2m+r-1}^{[s]}(h,x)| \leq K_k L_k M_{k,2m}^{[s;-1,1]} |h_{2m}^{[s]}| / k^{2r+1}$ $(r \in \mathbb{I})$. (2) $|E_{2m+r+1,r}^{[s]}(h,x)|$

$= O(k^{2r})$ as $r \to \infty$

(ii) Let $[\alpha[s], \beta[s]] \subset [0,\infty) \langle (-\infty,\infty) \rangle$, $h^{[s]} = $ mo$\{\sigma^{[s]} \in BND[\alpha[s], \beta[s]]\}$, $b(\neq 0) \in (-\infty,\infty)$ and $G(z,x)$ be an entire function of z. Then $E^{[s]}_{m+r+1,r}(b,x) \langle E^{[s]}_{2m+r+1,r}(b,x) \rangle$ tends to zero as $r \to \infty$ faster than the terms of any geometric progression. If in addition, $G \in \mathcal{G}(x)$, this result holds for the functions $E^{[s]}_{r,m+r-1}(b,x) \langle E^{[s]}_{r,2m+r-1}(b,x) \rangle$.

B. Let $f^{[s]}[x] = f[x : \sigma^{[s]}; g^{[s]}; \alpha[s], \beta[s]]$ $(x \in \mathbb{Z})$.

(i) Set $\hat{M}^{[s;\alpha,\beta]}_{k,m} = \max |g^{[s]}(z) - \sum_0^{m-1} g^{[s]}_\nu z^\nu| / |z^m|$ $(z = \mathcal{L}_k(\alpha,\beta))$

a) Let $[\alpha[s], \beta[s]] \subseteq [0,1]$, $h^{[s]} = $ mo$\{\sigma^{[s]} \in BND[\alpha[s], \beta[s]]\}$, $g(\mathcal{D}_k(0,1)) \in A$. (1) Let $g \in g$. Then $|e^{[s]}_{r,m+r-1}(x)| \leq K_k L_k \hat{M}^{[s;0,1]}_{k,m} |x^m| k^{2r+1}$ $(r \geq \mathrm{I})$. (2) $|e^{[s]}_{m+r+1,r}(x)| = O(k^{2r})$ as $r \to \infty$

b) Let $[\alpha[s], \beta[s]] \subseteq [-1,1]$, $h^{[s]} = $ mo$\{\sigma^{[s]} \in BND[\alpha[s], \beta[s]]\}$, $g(\mathcal{L}_k(-1,1)) \in A$. (1) Let $g \in g$. Then $|e^{[s]}_{r,2m+r-1}(x)| \leq K_k L_k \hat{M}^{[s;-1,1]}_{k,2m} |x|^{2m} k^{2r+1}$ $(r \geq \mathrm{II})$. (2) $|k e^{[s]}_{2m+r+1,r}(x)| = O(k^{2r})$ as $r \to \infty$.

(ii) Let $[\alpha[s],\beta[s]] \subset [0,\infty) \langle(-\infty,\infty)\rangle$, $h^{[s]} = \mathrm{mo}\{\sigma^{[s]} \in BND[\alpha[s],\beta[s]]\}$ and g be an entire function. Then $e^{[s]}_{m+r+1,r}(x) \langle e^{[s]}_{2m+r+1,r}(x)\rangle$ tends to zero as $r \to \infty$ faster than the terms of any geometric progression. If, in addition, $g \prec g$, this result holds for the functions $e^{[s]}_{r,m+r-1}(x) \langle e^{[s]}_{r,2m+r+1}(x)\rangle$.

Proof. A.(i)a)(1) If $G(z = \mathbb{D}_k(0,\xi), x) \in A_z$, then $G^{[s]}(\xi; z = \mathbb{D}_k(0,1), x) \in A_z$. It follows from .. A .. that $F^{[s]}(\xi, x) \to F^{[s]}(\xi, x)$ and that each function $G^{[s+1]}(\xi; z, x) = F^{[s]}(\xi z, x)$ is given by formula ( ), where $\mathbb{C}$ surrounds the real ~~axis~~ segment $[0,1]$ and lies in $N\{[0,1]\}$. We may extend the domain of definition of each $\sigma^{[s]}$ where necessary, by setting $d\sigma^{[s]}(t) = 0$ $(t \in [0,\alpha[s]])$ $(\alpha[s] > 0)$ and $d\sigma^{[s]}(t) = 0$ $(t \in (\beta[s],1])$ $(\beta[s] < 1)$.

The result of .. A.(i).. may be applied with $[\alpha,\beta] = [0,1]$ to each of the sequences of functions $F^{[s]}_{r,m+r+1}(\xi, x)$ $(r \in \mathbb{I})$ produced

The remaining results of the theorem may also be deduced from those of Theorem .

By restricting the intervals $[\alpha[s], \beta[s]]$ in A(i)a)(1) to $[0,1]$, we ensure that all functions $G^{[s]}(\xi; z, x)$ are analytic within and upon the same ellipse $\mathcal{L}_k(0,1)$, and derive the same lower bound for the rate of convergence of each sequence $F^{[s]}_{r, m+r-1}(\xi, m)$ considered. We could, of course, treat the case in which $[\alpha[s], \beta[s]]$ are simply confined to $[0,\infty)$, when each function $G^{[s]}(\xi; z, x)$ becomes analytic within and upon its own ellipse $\mathcal{L}_{k[s]}(\alpha[s], \beta[s])$ and the rate of convergence differs for each sequence considered. But the result deduced in this way is then no more than that directly to be deduced from — A.(i).. . In the case of A(ii), no simplification is to be obtained by restricting the intervals to $[0,1]$ or $[-1,1]$. Similar considerations relate to the other clauses of the Theorem.

Again (see the remarks following Theorem) it appears that the results of A.(i)..., B.(i)... cannot be applied recursively. If $d\sigma^{[s]}(t) > 0$ $(t = [0,\infty])$, $G^{[1]}(\xi; z, x)$ is not analytic in a domain of the form $\mathbb{D}(\eta; \phi_1, \phi_2)$ as described in Theorem . Hence inequalities such as those given in ... cannot be derived when $s \geq 1$ for $F^{[s]}(\xi, x)$ when the functions $h^{[s]} = m_s\{BND[\alpha[s], \beta[s]]\}$ and $[\alpha[s], \beta[s]] \subseteq [0,\infty] \langle [-\infty, \infty] \rangle$.

We conclude this section by giving an example in which iterated application of our algorithm, as described in this section, results in the determination of a unique sum function of the formal power series $\sum h_\nu z^\nu$ even when the sequence h is associated with an indeterminate moment problem. It was remarked by Stieltjes that when $C \in (0,\infty)$, $c \in (0, \frac{1}{2})$

then $\int_0^\infty t^\nu \exp(-Ct^c)\sin(Ct^c\tan(c\pi)) = 0$ $(\nu \equiv \mathbb{I})$, i.e. the Stieltjes moment problem associated with the sequence $h_\nu(C, 1/c) = \int_0^\infty t^\nu \exp(-Ct^c)dt$ $(\nu \equiv \mathbb{I})$ has a continuum of solutions $\sigma(t; C, c, C')$ $= \int_0^t \exp(-Ct^c)\{1 + C'\sin(Ct^c\tan(c\pi))\}dt$ $(C' \equiv (-1,1))$. We set $h_\nu^{[s]} = A\Gamma(\nu + 1 + \frac{s}{r}) = A\int_0^\infty t^\nu d\sigma^{[s]}(t)$ where $\sigma^{[s]}(t) = \int_0^t e^{-t}t^{s/r}dt$ $(s \equiv \mathbb{I}_0^{r-1})$ and $A = (2\pi r^3)^{1/2r}(2\pi)^{-1/2}$ $(r \in \mathbb{I}_3)$, $g^{[0]}(x) = (1-x)^{-1}$. The last series to be transformed has coefficients $\prod_0^{r-1} h_{\nu'}^{[s]}$ $= \Gamma((\nu'+1)r) = h_{\nu'}(r, 1/r)$ $(\nu' \equiv \mathbb{I})$. Each moment sequence $h^{[s]}$ $(s \equiv \mathbb{I}_0^{r-1})$ is associated with a determinate Stieltjes moment problem and in particular, from ( ), $f^{[r-1]}(\Delta) \to f^{[r-1]}(\Delta)$ where $\Delta = \mathbb{Z} - N\{[0,\infty]\}$ and

$$f^{[r-1]}(x) = r^{3/2}(2\pi)^{(1-r)/2}\int_0^\infty e^{-t_{r-1}} t_{r-1}^{(r-1)/r}\int_0^\infty e^{-t_{r-2}} t_{r-2}^{(r-2)/r} \cdots \int_0^\infty \frac{e^{-t_0}}{1 - t_0 \cdots t_{r-1}x} dt_0 \cdots dt_{r-2} dt_{r-1}$$

From .. ϵ ..., $f^{[r-1]}(x)$ has a representation of the form $f^{[r-1]}(x) = \int_0^\infty (1-xt)^{-1}\Omega_{r-1}^{[r-1]}(t)$ with $\Omega^{[r-1]}(t) \geq 0$ $(t \equiv [0,\infty])$. The functions

$\Omega_s^{[r-1]}(t)$ satisfy the recursion $\Omega_0^{[r-1]}(t) = Ae^{-t}$, $\Omega_{s+1}^{[r-1]}(t) = \int_0^\infty e^{-t/t'}(t/t')^{s/r}\Omega_s^{[r-1]}(t')dt'$ $(s \in \mathbb{I}_0^{r-2}, t = [0, \infty])$, and it is easily verified that $\Omega_1^{[r-1]}(t) = 2At^{\frac{1}{2}(1+1/r)}K_{1-1/r}(2t^{\frac{1}{2}})$ where $K_\nu(z)$ is the Bessel function of the second kind. What the remaining $\Omega_s^{[r-1]}(t)$ are, and whether $d\Omega_{r-1}^{[r-1]}(t)/dt = \varpi(t: r, 1/r\}; C')$ for some $C' \in (-1, 1)$, remains to be demonstrated.

Convergence of row and column sequences

Notation. $\mathbb{R}\mathbb{F}_i$ $(i \in \mathbb{I})$ <$\mathbb{C}\mathbb{F}_j$ $(j \in \mathbb{I})$> is the row <column> sequence of functions $F_{i,j}$ $(j \langle i \rangle \equiv \mathbb{I})$. The symbols $\mathbb{r}f_i$ and $\mathbb{c}f_j$ have similar meanings.

Theorem. Let $h \leftarrow h$

A. Set $F(x) = F(x : \mathbb{C}; h, G)$

(i) Let $h(z)$ possess a pole of order $N(\nu,\nu')$ at $z = t_{\nu,\nu'}^{-1}$ $(\nu \equiv \mathbb{I}_1^r, \nu' \equiv \mathbb{I}_1^{r(\nu)})$, where $|t_{\nu,\nu'}| = t(\nu) \in (0,\infty)$ $(\nu \equiv \mathbb{I}_1^r, \nu' \equiv \mathbb{I}_1^{r(\nu)})$ and $t(\nu+1) < t(\nu)$ $(\nu \equiv \mathbb{I}_1^{r-1})$ and let $h(z)$ be otherwise regular for $z \in \mathbb{D}_{1/t(r+1)}$ where $t(r+1) < t(r)$. Set $N(\tau) = \sum_1^{\tau} \sum_1^{r(\nu)} N(\nu,\nu')$ $(\tau \equiv \mathbb{I}_0^r)$

a) Let $r' \in \mathbb{I}_1^r$, $\gamma' > t(r')$. Let $G(z \equiv \mathbb{D}, \hat{M}(\gamma')) \in A_z$, where $\mathbb{D}$ is the aggregate of $\bar{\mathbb{D}}_{\gamma'}$ and (if $r' > 1$) the $N(r'-1)$ neighborhoods $\mathbb{N}(t_{\nu,\nu'})$ $(\nu \equiv \mathbb{I}_1^{r'-1}, \nu' \equiv \mathbb{I}_1^{r(\nu)})$. Let $G \in \mathcal{G}(\hat{M}(\gamma'))$.

Let $\mathbb{C}$ be the perimeter of $\mathbb{D}_{t(r+1)}$ together with $N(r)$ circles lying in $N\{t_{\nu,\nu'}\}$ and surrounding $t_{\nu,\nu'}$ ($\nu = \mathrm{I}_1^r$, $\nu' = \mathrm{I}_1^{r(\nu)}$).

Then $\mathbb{RF}_{N(\tau)}(M(\gamma')) \to \mathbf{F}(M(\gamma'))$ ($\tau = \mathrm{I\!I}_{r'}^r$), and

$|E_{N(\tau),j}(x)| = o\{(t(\tau+1)/\gamma')^j\}$ as $j \to \infty$ ($\tau = \mathrm{I\!I}_{r'-1}^{r-1}$, $x = \hat{M}(\gamma')$).

(1) If $h(z^{-1})$ possesses singularities in $\overline{\mathbb{D}}_{t(r+1)}$, then this relationship holds for $\tau = r$

(2) If $h(z^{-1})$ possesses no such singularities, $E_{N(r),j}(x)$ tends as $j \to \infty$, to zero faster than the terms of any geometric progression ($x = \hat{M}(\gamma')$)

b) Let $G(z,m)$ be an entire function of z with $G \in \mathfrak{g}(m)$ and let $\mathbb{C}$ be the circle $|z| = \gamma' > |t_1|$. $E_{N(\tau),j}(x)$ then tends, as $j \to \infty$, to zero faster than the terms of any geometric progression.

(ii) or $h(z)$ possess a zero of order $n(\nu,\nu')$ at $z = u^{-1}_{\nu,\nu'}$ $(\nu \equiv \mathbb{I}_1^s, \nu' \equiv \mathbb{I}_1^{s(\nu)})$ where $|u_{\nu,\nu'}| = u(\nu) \in (0,\infty)$ $(\nu \equiv \mathbb{I}_1^s, \nu' \equiv \mathbb{I}_1^{s(\nu)})$ and $u(\nu+1) < u(\nu)$ $(\nu \equiv \mathbb{I}_1^{s-1})$ and let $h(z)^{-1}$ be otherwise regular for $z \equiv \mathbb{D}_{1/u(s+1)}$ where $u(s+1) > u(s)$. Set $n(\tau) = \sum_1^{\tau} \sum_1^{s(\nu)} n(\nu,\nu')$ $(\tau \equiv \mathbb{I}_0^s)$.

a) Let $s' \in \mathbb{I}_1^s$, $\gamma'' > u(s')$. Let $h(z)$ possess poles of arbitrary order at $z = \tilde{t}_\nu^{-1} \in \overline{\overline{\mathbb{D}}}_{1/\gamma''}$ $(\nu \equiv \mathbb{I}_1^{\tilde{r}(\gamma'')})$, being otherwise regular in $\mathbb{D}_{1/\gamma''}$. Let $G(z \equiv \tilde{\tilde{\mathbb{D}}}, \tilde{M}(\gamma'')) \in A_z$, where $\tilde{\tilde{\mathbb{D}}}$ is the aggregate of $\overline{\mathbb{D}}_{\gamma''}$ and (if $\tilde{r}(\gamma'') > 0$) the $\tilde{r}(\gamma'')$ neighborhoods $\mathbb{N}\{\tilde{t}_\nu\}$ $(\nu \equiv \mathbb{I}_1^{\tilde{r}(\gamma'')})$. Let $\mathbb{C}$ be the perimeter of $\mathbb{D}_{\gamma''}$ together with $\tilde{r}(\gamma'')$ circles lying in $\mathbb{N}\{\tilde{t}_\nu\}$ and surrounding $\tilde{t}_\nu$ $(\nu \equiv \mathbb{I}_1^{\tilde{r}(\gamma'')})$. Then $\mathbb{C}\mathbb{F}_{n(\tau)}(\tilde{M}(\gamma'')) \to F(\tilde{M}(\gamma''))$ $(\tau \equiv \mathbb{I}_{s'}^s)$, and $|E_{i,n(\tau)}(x)| = o\{(u(\tau+1)/\gamma'')^i\}$ as $i \to \infty$ $(\tau \equiv \mathbb{I}_{s'}^{s-1}, x \equiv \tilde{M}(\gamma''))$.

(1) If $h(z^{-1})^{-1}$ possesses singularities in $\overline{\mathbb{D}}_{t(s+1)}$, then this relationship holds for $\tau = s$.

(2) If $h(z^{-1})^{-1}$ possesses no such singularities, $E_{i,n(s)}(x)$ tends, as $i \to \infty$, to zero faster than the terms of any geometric progression

B. Set $f(x) = f(x : \mathbb{C}; h; G)$.

(i) Let h be as in A(i).

a) Let $g(\overline{\mathbb{D}}_K) \in A$ with $g \prec g$. Let M contain all singular points of g. Set $\hat{\sigma T}(\tau) = \bigcup_1^{\tau} \bigcup_1^{r(\omega)} t_{\omega,\nu}$ $(\tau = \mathrm{I}_0^r)$. Then $\mathrm{r}f_{N(\tau)}(\hat{\mathbb{D}}(\tau)) \to f(\hat{\mathbb{D}}(\tau))$ where $\hat{\mathbb{D}}(\tau) = \mathbb{D}_{K/t(\tau)} - \mathbb{N}\{\hat{\sigma T}(\tau) * M\}$ $(\tau = \mathrm{I}_0^r)$, and $|e_{N(\tau),j}(x)| = o\{(t(\tau+1)|x|/K)^j\}$ as $j \to \infty$ $(\tau = \mathrm{I}_0^{r-1}, x = \hat{\mathbb{D}}(\tau))$. If h is as in A(i)a)(1), then this relationship also holds for $\tau = r$. If h is as in A(i)a)(2), then $e_{N(r),j}(x)$ tends, as $j \to \infty$, to zero faster than the terms of any geometric progression.

b) Let g be an entire function, with $g \prec g$. Then $\mathscr{f}_{N(\tau)}(\mathbb{Z}) \to f(\mathbb{Z})$ $(\tau = \mathbb{I}_0^r)$ and $e_{N(\tau),j}(x)$ tends, as $j \to \infty$, to zero faster than the terms of any geometric progression $(\tau = \mathbb{I}_0^r,\ x = \mathbb{Z})$.

(ii) Let h_s be as in A(ii)

a) Let $h(z)$ possess poles of arbitrary order at $z = T_\nu^{-1} \in \mathbb{D}_{1/u(s+1)}$ $(\nu = \mathbb{I}_1^{\tilde{r}})$ and be otherwise regular for $z \in \mathbb{D}_{1/u(s+1)}$.

Denote by $\tilde{T}(\tau)$ the set of T_ν^{-1} for which $T_\nu > u_\tau$ $(\tau = \mathbb{I}_1^s)$

Let $\bar{\mathbb{D}}_K$ and M be as in (i)a). Then $\mathscr{f}_{n(\tau)}(\tilde{\mathbb{D}}(\tau)) \to f(\tilde{\mathbb{D}}(\tau))$, where $\tilde{\mathbb{D}}(\tau) = \bar{\mathbb{D}}_{K/u(\tau)} - N\{\tilde{T}(\tau) \times M\}$ $(\tau = \mathbb{I}_0^s)$, and $|e_{i,n(\tau)}(x)| = o\{(u(\tau+1)\,|x|/K)^i\}$ as $i \to \infty$ $(\tau = \mathbb{I}_0^{s-1},\ x = \tilde{\mathbb{D}}(\tau))$.

If h is as in A(ii)a)(1), then this relationship holds for $\tau = s$. If h is as in A(ii)a)(2) then $e_{i,n(s)}(x)$ tends, as $i \to \infty$, to zero faster than the terms of any geometric progression.

b) Let g be an entire function. Then $c f_{n(\tau)}(\mathbb{Z}) \to f(\mathbb{Z})$ $(\tau \equiv \mathbb{I}_0^s)$ and $e_{i,n(\tau)}$ tends as $i \to \infty$, to zero faster than the terms of any geometric progression $(\tau \equiv \mathbb{I}_0^s, x \equiv \mathbb{Z})$.

Proof. A(i)a) Making extensive use of the theory contained in Hadamard's dissertation upon Taylor series [ ], Montessus de Ballore showed that if $h(z)$ satisfies the stated conditions, then each row sequence $P_{N(\tau),j}$ $(j \equiv \mathbb{I})$ converges to $h(z)$ uniformly for $z \equiv \mathbb{D}_{1/t(\tau+1)} - N\{\cup_1^{\tau} \cup_1^{\nu(\omega)} t_{\omega,\omega'}^{-1}\}$ $(\tau \equiv \mathbb{I}_0^{r-1})$ and, furthermore, $|h(z) - P_{N(\tau),j}(z)| = O\{(|z|/t(\tau+1))^j\}$ as $j \to \infty$ for these values of z. If $h(z)$ possesses singularities on or outside the circle $z = t(r+1)$, then this relationship holds with $\tau = r$; if $h(z)$ possesses no such singularities, then $t(r+1)$ may be taken arbitrarily large, i.e. $|h(z) - P_{N(r),j}(z)|$ tends, as $j \to \infty$, to zero

faster than the terms of any geometric progression. This result may, of course, be formulated in terms of the functions $z^{-1}h(z^{-1})$, and $z^{-1}P_{n(\tau),j}(z^{-1})$. Under the stated assumptions concerning G, $\mathbb{C}$ may be deformed to be a circle slightly larger than the perimeter of D_{γ}, and with centre also at the origin, together with $N(r'-1)$ circles lying in $N\{t_{\nu,\nu'}\}$ and surrounding $t_{\nu,\nu'}$ $(\nu = \mathrm{I}_1^{r'-1}; \nu' = \mathrm{I}_1^{r(\nu)})$. It follows from the above result, that for some $j' \in \mathrm{I}$,
$F_{n(\tau),j}(x) = F(x : \mathbb{C}; P_{n(\tau),j}, G)$ $(j = \mathrm{I}_{j'})$. The length of $\mathbb{C}$ is finite. Upon the small circles that go to make up $\mathbb{C}$, $|z| > \gamma'$. A simple argument suffices to obtain the results stated.

b) If $G(z, m)$ is an entire function of z, γ' may be taken arbitrarily large in the above argument.

(ii) It was also shown by Montessus de Ballore [ ] that if $h(z)$ is as described in a), $P^{-1}_{i,n(\tau)}(z)$ $(i = \overline{I})$ converges to $h(z)^{-1}$ uniformly for $z \in \mathbb{D}_{1/u(\tau+1)} - N\{U_1^{\tau} U^{s(\iota)} u^{-1}_{\nu,\nu'}\}$ $(\tau = \overline{I}_0^{s-1})$ with order relationships and further results similar to those described above for the polar case. The remainder of the proof is analogous to that of the preceding clause.

B.(i) Let $\tau \in \overline{I}_0^{r}$, $x \in \hat{\mathbb{D}}(\tau)$. In A(i)a), we set $G(z,x) = g(zx)$ and $\gamma' = \kappa/x$. Since $g(\overline{\mathbb{D}}_\kappa) \in A$, $G(z \in \overline{\mathbb{D}}_{\gamma'}, x) \in A_z$ in this case. Furthermore $|x| < \kappa / t_\tau$ so that $\gamma' > t_\tau$. We now apply the result of A.(i)a) considering the case $\tau = r' - 1$ alone.

The remaining results of B. are proved in the same way.

For the sake of clarity, we remark that the results of A(i)a) for example, concern the sequences $RF_{N(\tau)}$ alone and say nothing concerning other sequences. It may not be assumed that if $\mathbb{R}\mathbb{F}_{N(\tau)}(M) \to F(M)$, then this result also holds with $N(\tau)$ replaced by $N(\tau)+1$. Indeed, with reference to the Padé table, Perron [ ] gives an example in which $\mathbb{P}_{0,j}(\mathbb{B}) \to h(\mathbb{B})$ as $j \to \infty$, but the sequence $\mathbb{P}_{1,j}(z)$ diverges over a point set everywhere dense in $\mathbb{B}$. It is also possible to construct examples in which the conditions of Montessus de Ballore's theorem hold, and an intervening sequence $P_{i,j}(z)$ $(i \neq N(\tau))$ converges, as $j \to \infty$, but not to $h(z)$.

Integral transforms of Padé quotients

It is clear that if, with $\mathbb{P}_S$ prescribed, $\mathbb{P}_S(\overline{\mathbb{D}}') \to h(\overline{\mathbb{D}}')$ and $\Psi(m) = \int_{\mathbb{K}'} h(z)\psi'(z,m)dz$, where $\mathbb{K}'$ is a prescribed rectifiable contour lying in $\overline{\mathbb{D}}'$, and the function ψ' is prescribed, then $\int_{\mathbb{K}} P_{i,j}(z)\psi'(z,m)dz \to \Psi(m)$ where the $P_{i,j}$ are the ordered members of $\mathbb{P}_S$. Alternatively, if the sequence $z^{-1}P_{i,j}(z^{-1})$ $(P_{i,j} \in \mathbb{P}_S)$ converges to $z^{-1}h(z^{-1})$ for $z \in \overline{\mathbb{D}}$ and $\Psi(m) = \int_{\mathbb{K}} z^{-1}h(z^{-1})\psi(z,m)dz$, with $\mathbb{K}$ lying in $\mathbb{D}$, then $\int_{\mathbb{K}} z^{-1}P_{i,j}(z^{-1})\psi(z,m)dz \to \Psi(m)$. The convergence of the sequence of integral transforms of Padé quotients in question can be established for the special series $\sum i h_\nu z^\nu$ considered in §§ using the convergence theory of the Padé table given in those sections. Indeed, taking $\mathbb{K}$ to be $\mathbb{C}$, a contour enclosing the singular points of $h(z^{-1})$ and setting $\psi(z,x) = G(z,x)$, this artifice has already been exploited to prove convergence of the sequence $\mathbb{F}_s(M)$ ($\mathbb{F}_s \Leftrightarrow \mathbb{P}_s$) of mappings which then results.

We wish to point out that even when $\mathbb{K}$ and $\mathbb{C}$ are not identical, our convergence theory nevertheless includes that of the above sequence of integral transforms. Taking, for example, h to be one of the functions h_A or h_B described in Theorem ..., and $\mathbb{C}$ a rectifiable contour as therein described surrounding the singularities of $h(z^{-1})$ but no part of $\mathbb{K}$, we set $G(z,M) = (2\pi i)^{-1} \int_{\mathbb{K}} (z'-z)^{-1} \psi(z,M)\,dz'$. We then find that $F(M)$, the mapping to which convergence is proved in Theorem ... is given by

$$F(M) = (2\pi i)^{-1} \int_{\mathbb{C}} z^{-1} h(z^{-1}) G(z,x)\,dz$$

$$= (2\pi i)^{-1} \int_{\mathbb{C}} z^{-1} h(z^{-1}) \left[(2\pi i)^{-1} \int_{\mathbb{K}} (z'-z)^{-1} \psi(z',M)\,dz'\right] dz$$

$$= (2\pi i)^{-1}\int_{K}\Big[(2\pi i)^{-1}\int_{C}\{z(z'-z)\}^{-1}h(z^{-1})\,dz\Big]\psi(z',M)\,dz'$$

$$= (2\pi i)^{-1}\int_{K} z'^{-1}h(z'^{-1})\psi(z',M)\,dz' = \Psi(M)$$

since C can be moved into a position surrounding the points z' of K. Again, taking $h(z) = st\{\sigma;\alpha,\beta\}$ where σ, α, and β are as described in one of the convergence clauses of Theorem ..., we find that

$$\Psi(M) = \int_{K}\Big[(2\pi i)^{-1}\int_{\alpha}^{\beta}(z-t)^{-1}d\sigma(t)\Big]\psi(z,M)\,dz$$

$$= \int_{\alpha}^{\beta}\Big[(2\pi i)^{-1}\int_{K}(z-t)^{-1}\psi(z,M)\,dz\Big]d\sigma(t)$$

$$= \int_{\alpha}^{\beta}G(t,M)\,d\sigma(t) = F(M)$$

where $F(M)$ is the limit mapping considered in each clause of Theorem .