The principal and alternating sums of a function defined over a strip in the complex plane

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1. Introduction and summary

remove sum [illegible] of polynomials of

[illegible] with Nörlund def's 'introduce' notation $b_x^{(n)}$

add use of Dirichlet sums

add comment on their [illegible]; Nörlund's [illegible] the [illegible] Borel sum [illegible]

Nörlund sum & rep. of B as [illegible]

give main theory

2 Notation and preliminaries

add $S(\sigma, \sigma')$ Λ $f \in \Lambda(D)$ ~~[illegible]~~ $f(z,w) \in \Lambda_0(D)$ $f(z,w) \in \Lambda^{\zeta}_{z,w}$ [illegible]

definition of Stieltjes integral by integration by parts

n always a fixed integer M, m non-neg integers

2.1 Periodic ~~[illegible]~~ Bernoulli and Euler polynomials

Give properties not already quoted in introduction

Lobatto $B^{(n)}_{z} = \int t \ldots$

2.2. Series whose coefficients are ~~moments~~ generalized moments

Consider series whose coefficients [illegible] may be represented as products of binomial coeffts and Stieltjes or Lebesgue moments.

Series ( ) ... obtained by dropping term when $0 < u < \pi$

Description $\overline{B}_{,1}$ + $\overline{B}$ summability. || it differs only from the series when $n=0$ in the term $b_n^{(n)}(u)$ (since $\overline{b}_n^{(n)}(0) = 0$ but $b_n^{(n)}(0) \neq 0$) (??)

$\frac{\Gamma(m,k)}{\Gamma(m+1)\Gamma(k)}$

clear up a) clause ib) (a fn. of second variable x etc)

b) ... symmetric

(necessary to refer to proof of 1st Euler-Maclaurin theorem)

consider above but $\frac{1}{(1+zt)^k}$ by $\frac{1}{1+z u(t)}$ $u(t)$ remain in $-\delta^* \leq Re(u) \leq \delta'$

2.3 A class of functions

Define $M(M; x^-, x^+)$ as class

take out end of ... theorem

add clause stating that $M(M+r \ldots) \subset M(M$

$a \in \notin M($
$f \in M \Rightarrow af \in M$ ... $\subset M$ ...

Not supposed that $\int |f(x+ 2u)| \ldots < \infty$ ... satisfied ...

(it may be of $M(\ldots x^-, x^+)$
In two intervals (x^-, x^+)
e.g. $1/x$)

... the ... follows ... subclass of $M($ ...) finite number of series

3. The Euler-Maclaurin series and the principal sum

e.g. ... if going be $\mathcal{L}(\ldots)$ ... natural ...

With $[x_1, \ldots] \subseteq [-\infty, \infty]$

Definition $\mathfrak{B}(n; \infty, \rho)$ class of complex valued functions defined on $[\ldots]$

for which $\int_\alpha^\beta |y^{n-1} \mathfrak{B}_\rho(u)| du < \infty$ / $\hat{\mathfrak{B}}(n; \infty, \rho)$ is that subclass of $\mathfrak{B}$ whose members ρ satisfy ... $\hat{\tilde{\mathfrak{B}}}$

then define $\overline{f}_\Delta(n; \rho, \infty, \rho)$

series $\overline{\mathcal{U}}$, $\overline{\mathcal{U}}_e$, $\overline{\mathcal{U}}_o$

partial sum $\overline{F}$ $\overline{F}_e$ $\overline{F}_o$

Remark that series $\ldots (\overline{b}_n^{(n)}(u))$ ... special case ... obtained by taking $\ldots \frac{1}{2} \operatorname{sign}(u - u') \times (\mathfrak{B}, \ldots)$ ... small interval containing u as interior point

~~In the first theorem~~ It is first established that the coefficients ( ) are, under certain conditions, (,) and which both ~~functions~~ occur as remainder terms of these series generalised moments, functions ~~representing~~ and generating these series ( ) are and are asymptotically represented by these series are determined ~~derived~~ determined

(refer to Boole series th. for clearer exposition)

clear up condition on l which the ~ are symmetric

Definition . The functions $F(n, m, \rho, \cdot, \cdot, \psi, z \mid \omega)$ are henceforth ~~the~~ as defined by formulae .

~~It is now established~~ that the series (1) are Borel summable to their generating functions

(again refer to Boole series th.)

Describe Hardy's result.

~~It is now established that,~~ subject to ~~under~~ certain ~~further~~ conditions upon ϕ, the series (1*) are semi convergent; and their oscillating ~~or~~ monotone character is described

(again refer to Boole series th.)

Describe Hardy's result.

Subject to certain conditions upon ϕ, the series (1) is convergent ~~are~~ converge

(again ref to Boole)

It can occur that the imaginary axis in the ω-plane is a natural barrier of the functions F^{*}; alternatively it is possible that these functions can be continued analytically across this axis, and that (1) represents these functions asymptotically over concave sectors

Although the method of Borel summation is a powerful analytical device, it only ~~provides an analytical~~ defines, by means of an integral, expression, for a sum of the series being treated; the method is certainly difficult to implement in computational practice; ~~the method does not~~ it is non-approximative in the sense that it does not entail the recursive construction of a sequence of functions converging to the ~~value~~ sum function, one member of which may be accepted as a suitable approximation to the limit. The use of approximating fractions is an approximative method possessing the above property which Borel's method does not.

Def of approximating fraction

It is possible to show that, for the special case ~~in which~~ $n = 1$, the use of approximating fractions and Borel's method produce the same results

Th...

It is ~~greatly~~ to be regretted that a result similar to that of the above theorem cannot at present be stated for general values of n. The convergence result for approximating fractions upon which the above results are based itself derives from the Stieltjes, Hamburger & R. Nevanlinna theory of continued fractions associated with ~~sequen~~ series whose coefficients are simple moments of the form $c(k)$, $k = 1, 2, \ldots$ Until ~~more general~~ a convergence theory concerning series whose coefficients are generalized moments has been established of a desired extension of the above theorem must remain in abeyance.

The Borel sum of the series $\mathcal{F}$ is obtained from, in particular, the values of the derivatives of ψ; knowledge of a decomposition which ψ may have is not required. Nevertheless ψ has such a decomposition, and the ~~fo~~ same is true of the functions $\bar{F}^{\pm}$.

The

Subject to auxiliary conditions ~~upon~~ ψ, the functions $F^{\pm}$ may be continued analytically across the imaginary axis in the ω-plane, and are ~~is~~ analytic over concave sectors (c.f. clause ( ) of Theorem). The required conditions are satisfied by ~~all~~ functions of a well-defined subclass of $M(M; x^-, x^+)$, and for the ~~fo values~~ functions $F^{\pm}$ derived from the functions ψ of this subclass, completion results may be given.

The principal sum

The functions $\bar{F}^{\pm}$ ~~[illegible]~~ generating ~~[illegible]~~ series $\mathcal{F}$ whose coefficients are ~~[illegible]~~ derived from periodic Bernoulli polynomials are, analytic over a strip in the ~~z-plane~~ domain of definition of one argument ~~(z etc)~~ and over a half plane in that of the other (z and ω respectively above). In this section functions $F^{\pm}$ which generate series $\mathcal{F}$ whose coefficients are derived from direct Bernoulli polynomials are considered.

~~the second~~ Functions of the second type ~~[illegible]~~ represent a solution to the difference equation considered in the introduction; they may, by means of a decomposition, be expressed as a linear combination of functions of the first type; the are, in general, analytic over strips in the domains of definition of both arguments.

Definition (compare Bate's series [illegible])

The domains over which the above functions $F^{\pm}$ are analytic are now established, completion ~~theorems~~ relationships for these functions are obtained, and decompositions are given.

Th. // comment upon domain $\Delta^{\pm}$: envelope as x ranges over $\mathcal{S}(x^-, x^+)$

It is now shown that the series $\mathfrak{F}$ represents the function $F^{\pm}$ asymptotically over certain sectors, that ~~under certain conditions the series converges~~, that this series is Borel summable to the functions of $F^{\pm}$ and that under certain conditions ~~the series~~ it converges

(change round [illegible] as stated?)

Th.

Having determined the domains over which the functions $F^{\pm}$ are analytic, and having established their connections with the series $\mathfrak{F}$, it is now shown that they satisfy a system of difference equations ~~which~~

equivalent to the single difference equation ( ), and, conversely, that these functions can be constructed recursively by a summation process

Th. // Add clause stating that although F . only defined directly by formulae of form ( ) over strip S(. .), can nevertheless be continued analytically throughout S(. .) by use of difference equation

Having shown that the functions $F^{\pm}$ are solutions of the difference equation ( ) (with ϕ of that equation having the form displayed upon the right hand side of formula ( ))

As has been mentioned in the introduction, the function F' of formula ( ) is, when $\nu(u) = \frac{1}{2}\operatorname{sgn}(u)$ and $[\omega_1, \Omega]$ is an arbitrarily small interval containing the origin, Nörlund's principal n^{th} order sum of the function ψ^+. For the case in which $n=1$, and ψ is a meromorphic function of order . . at infinity whose only singularities are simple poles at the points β_ν with corresponding residues γ_ν $(\nu=1,\ldots,N)$, Nörlund gives the completion result

$F(x|\omega) =$

where

$\Pi($ $)$

($F(x-\omega|-\omega)$ in this notation is our F^-). It will be noticed that if the function B of formula ( ) ~~has poles~~ has zeros ~~at the~~

of magnitude l_ν at the point $x+j\omega$ in the range $-\infty < j\omega < x^-$
$(1 \le \nu \le N)$
~~$(\nu=1,\dots,N)$~~ ~~form~~ and is constant over (x^-, ∞) (so that ψ

is simply the single function ψ^+ in terms of the decomposition $\mathcal{F}(\ )$)

the results of formulae $(\ldots)$ are equivalent. On the one hand, the

completion result (b) of clause ( ) of the above theorem deals with functions

that are possibly not meromorphic, and also concerns completion ~~results~~ terms

for sums of general order; on the other hand Nörlund's special

result concerns functions whose singularities are not confined to

the real axis.

(Discussion smoothness of behaviour of Π as fn of ω; also variation of Π

as x transfers to another strip)

Boole's series and the alternating sum

Replace stated theorems by general account

The functions of dealt with ~~very~~ above, may, on the one hand, be entire (for example $\psi(z) = \frac{1}{2}(e^{z}+e^{-z})$ is a member of $M(0;-\infty,\infty)$); on the other, they may have the lines $Re(z) = x^-, x^+$ as natural barriers: for example, the function

$$\psi(z) = \int_{-\infty}^{\infty}\left[\int_{0}^{\infty} e^{-(z-x^-)v} + \int_{-\infty}^{0} e^{-(z-x^+)v}\right]\sin(\lambda v)\,dv\,d\Lambda(\lambda)$$

$$= \int_{-\infty}^{\infty}\left[\frac{\lambda}{\lambda^2+(z-x^-)^2} - \frac{\lambda}{\lambda^2+(z-x^+)^2}\right]d\Lambda(\lambda)$$

where Λ is bounded ~~and~~ nondecreasing and nonanalytic over $(-\infty,\infty)$, is such a function. It may also occur that a function ψ is a member of two classes $M(M;x^-,x^+)$ with disjoint intervals (x^-,x^+): for example $\psi(z) = \frac{1}{z}$ is a member of both $M(0;0,\infty)$ and $M(0;-\infty,0)$.

It is, furthermore, not assumed that ~~the integrals~~ $\int_{-\infty}^{\infty}|\psi^{(M)}(x+iv)|dv \leq A < \infty$
$\in A\{\cup(x^-,x^+)\}$

~~although~~ although functions ψ^{V} for which this is true and for which uniformly for ~~x in~~ every closed subinterval of $x^- < x < x^+$, $\lim \psi^{(M)}(x+iy) = 0$ as $|y|$ tends to infinity do have a representation
of the form ( ) [ ] p. 416??; and are accordingly members of $M(M;x^-,x^+)$,
– Hamburger Math Zeit 10 (1921) 240–254

The problem of determining a function $F^{(n)}$ which satisfies the equation

$$\Delta_\omega^{n} F^{(n)}(x|\omega) = \phi(x)$$

(Δ_ω^n is the [illegible] n is a positive integer; ~~and~~ Δ_ω is the forward difference operator with respect to the indicated argument: $\Delta_\omega F(z) = \frac{F(z+\omega)-F(z)}{\omega}$) is, when ϕ is a polynomial, easily solved with the help of the Bernoulli polynomials $B_\nu^{(n)}$. The latter polynomials are defined by means of the expansion

$$\frac{e^{wt}t^{n}}{(e^{t}-1)^{n}} = \sum \frac{B_\nu^{(n)}(w)\,t^\nu}{\nu!}$$

(summation is with respect to ν which runs, when no indices are attached to the summation sign, from 0 to ∞), and satisfy the relationships

$$D B_\nu^{(r)}(w) = \nu B_{\nu-1}^{(r)}(w) \qquad \Delta B_\nu^{(r)}(w) = \nu B_{\nu-1}^{(r-1)}(w)$$

~~$\Delta B_\nu^{(r)}(w) =$~~

D is the differential operator with respect to the indicated argument (D is d/dw and Δ is Δ_1). ~~Setting~~ It is easily verified by differentiation with respect to w and use of the first of relationships ( ), that the value of the sum

$$F(x+h\omega) \text{ is } \sum_{r} \frac{B_\nu^{(r)}(w-h)}{\nu!} D^{\nu-r}\phi(x+h\omega)\,\omega^\nu.$$

($D^{-r}\phi(x)$ ~~for $r=1,2,\ldots$~~ is the repeated integral $I^{(r)}$ which may be expressed as

$$I^{(r)}(x) = \int_a^x \frac{(x-\tau)^{r-1}}{(r-1)!}\phi(\tau)\,d\tau$$

~~$F^{(r)}$ is a reper~~ (a being some fixed finite complex number) and use of the first of relationships ( )

for $r=1,2,\ldots$) ~~it is easily verified, by differentiation with respect to~~ h,

~~h, that~~ $\hat{F}$ is independent of h, so that if

$$F^{(r)}(z|\omega) = \sum_{\nu}{}' \frac{B_{\nu}^{(n)}(a)}{\nu!} D^{\nu-r}\phi(z)\omega^{\nu}$$

then

$$F^{(r)}(z+h\omega|\omega) = \sum_{\nu}{}' \frac{B_{\nu}^{(r)}(h)}{\nu!} \omega D^{\nu-r}\phi(z)\omega^{\nu}$$

Using the second of relationships ( ), it follows that

$$\underset{\omega}{\Delta} F^{(r)}(z|\omega) = F^{(r-1)}(z|\omega)$$

for $n = n, n-1, \ldots, 1$, ~~and~~ Since $F^{(0)}(z|\omega) = \phi(z)$, ~~it follows that~~ the function $F^{(n)}$ defined by (the terms of the Euler-Maclaurin series) ~~formula~~ ( ) is a solution of equation ( ).

The problem of obtaining ~~a~~ solution of ~~the~~ equation ( ) when ϕ is not a polynomial is a classic problem of the calculus of finite differences (~~for a syste~~ Milne-Thomson [ ] gives us a systematic review of the various methods relating to the case in which $n=1$). Of all the proposed methods of solution, the most principle ~~far reaching~~ is that due to Nörlund [ ]. It is evident that, subject to convergence of both integral and sum, the function

$$F^{(1)}(x|\omega) = \int_a^{\infty} \phi(t)\,dt - \omega \sum_{\nu}{}' \phi(x+\nu\omega)$$

satisfies equation ( ) when $n=1$. When $\phi(x) = 1/x$, for example, this direct definition fails; nevertheless, if the upper limits of integration

and summation are replaced by $\arg(\omega)r$ and r respectively, ~~at~~ r being a positive integer, the limit of the resulting difference as r tends to infinity still defines a function which satisfies equation ( ) when $n=1$. The
When ϕ is a more general function, it can occur that
artifice just described fails in its intention. ~~when ϕ is a~~ Nörlund therefore introduces a converging factor into the integral and sum of formula ( ) and defines the principal sum (Hauptsumme) of ϕ to be

$$F^{(1)}(x|\omega) = \lim_{\eta \to 0} \left[\int_a^{\arg(\omega)\infty} e^{-\eta t^p \{\ln(t)\}^q} \phi(t)\,dt - \omega \sum e^{-\eta \{\ln(\omega)\}^q} \phi(x+\omega\nu) \right]$$

$p, q \geq 0$ being suitably chosen. In many cases that which he considers, the principal sum ~~the~~ may be exhibited by the use of an extension of Poisson's remainder formula for the Euler-Maclaurin series. The n^{th} order periodic Bernoulli functions $\bar{B}_\nu^{(n)}$ (~~the~~ Nörlund's notation is adhered to; the bar does not denote a complex conjugate) are defined by Nörlund ~~~~ by setting $\bar{B}_\nu^{(n)}(w) = B_\nu^{(n)}(w)$ for $0 \leq w < n$ and stipulating that $\Delta^n \bar{B}_\nu^{(n)}(w) = 0$ $(\nu = 0, 1, ..)$ for all finite real w. ~~$\bar{B}_\nu^{(n)}(w) = B_\nu^{(n)}$~~ (when $n=1$, this definition is that of simple periodicity: $\bar{B}_\nu^{(1)}(w+1) = \bar{B}_\nu^{(1)}(w)$; for higher values of n the definition is that of periodicity in a more general sense). Subject to suitable conditions, the function ~~F~~ of formula ( ) may be expressed in the form

$$F^{(n)}(x|\omega) = \sum_{\nu}^{m+1} \frac{B_\nu^{(n)}(\omega)}{\nu!} \nabla^{\nu-n} \phi(x)\omega^\nu + \int_0^\infty \frac{\bar{B}_m^{(n)}(-t)}{m!} \nabla^{\nu} \phi(x+\omega t)\,dt$$

where $m \leq \frac{1}{2}$ is a positive integer.

It is evident that, ~~subject~~ for $\overset{n=1 \text{ or } 2}{\text{appropriate}}$ ϕ, ~~and~~, equation ( ) has a second solution

$$F^{(1)}(z-\omega|\omega) = \int_{a}^{\arg(-\omega)\infty} \phi(t)dt + \omega \sum' \phi(z-\omega-\nu\omega)$$

This definition of a solution ~~...~~ may also be extended by the introduction of a suitable convergence factor. For certain ϕ, Nörlund characterises, by means of a completion theorem (Ergänzungssatz), the function $\Pi^{(1)}(z|\omega) = F^{(1)}(z|\omega) - F^{(1)}(z-\omega|-\omega)$.

The ~~solution of the~~ equation

$$\overset{n}{\underset{\omega}{\nabla}} G^{(n)}(z|\omega) = \phi(z)$$

($\underset{\omega}{\nabla}$ is the averaging operation: $\underset{\omega}{\nabla} G(z) = \frac{1}{2}(G(z+\omega)+G(z))$) may be attacked, ~~using Euler polynomials, and periodic Euler functions,~~ ~~in a manner~~ by (application of theorems) methods similar ~~to~~ to those described above relating to equation ( ). ~~If when ϕ is a polynomial, the~~ a solution to equation ( ) may be expressed ~~exhibited~~ ~~as~~ the sum of a terminating Boole's series. Nörlund also introduces a convergence factor into ~~the sum~~

$$G^{(1)}(z|\omega) = \frac{1}{2} \sum' (-1)^{\nu} \phi(z+\nu\omega)$$

and ~~such expression~~, for certain ϕ, ~~exhibits~~ ~~the~~ a solution to equation ( ) ~~in the form of a ... by use~~ in a form ~~involving~~ in which an integral involving a periodic Euler function occurs; he also gives a completion theorem for two functions $G^{(1)}(z|\omega)$ and $G^{(1)}(z-\omega|-\omega)$.

Subject to convergence, the sum

$$S(z) = \sum_i \phi(z+\nu)$$

satisfies the difference equation $S(z+1)-S(z) = -\phi(z)$. Thus, referring to formula ( ), the expression

$$-F^{(1)}(0|1) = -\int_\infty^0 \phi(t)dt - \sum_0^m \frac{B^{(1)}_{\nu+1}(0)}{(\nu+1)!} D^\nu\phi(0) - \int_0^\infty \frac{\bar{B}^{(1)}_{m+1}(-t)}{(m+1)!} D^{m+1}\phi(t)dt$$

may be interpreted as being a sum of the series $\sum_i \phi(\nu)$.

Such expressions occur in ~~the work of~~ Ramanujan's ~~papers~~ writings on divergent series. According to Hardy, who placed much of Ramanujan's ~~work~~ work theory upon a sound basis: ~~the series $\sum_i \phi(\nu)$ is said to summable R(a) to S~~ if, for some positive integer m, the delayed Euler-Maclaurin series

$\sum_i \frac{B^{(1)}_{m+\nu+1}(0)}{(m+\nu+1)!} D^{m+\nu}\phi(0)$ is $(B',1)$ summable to S', then the series $\sum_0 \phi(\nu)$ is (R,a) summable to S, where

$$S = -\int_a \phi(t)dt - \sum_0^m \frac{B^{(1)}_{\nu+1}(0)}{(\nu+1)!} D^\nu\phi(0) - S'$$

~~is the (R,a) sum of the series~~

(the series $\sum_i f_\nu$ is (B',k) summable upon divergent series. ~~According to Hardy ([ ]), if~~

~~First~~ Hardy ([ ]) defines Ramanujan summability as follows: if for some a and positive integer m, ϕ is such that the integral expressions in formula ( ) exist, then $\sum \phi(\nu)$ is (R', a) summable to S (where $S = -F^{(1)}(a,1)$). Hardy ([ ]) also showed that for certain ϕ the delayed Euler-Maclaurin series

$$\sum \frac{B^{(1)}_{m+\nu+1}(a)\Delta^{m+\nu}\phi(a)}{(m+\nu+1)!}$$

is ~~$(B', 1)$ summable to ~~ ~~$\frac{\text{by the }(B',1)\text{ method of Borel}}{}$~~ ~~Sum~~ the last integral in formula ( ) [if for a fixed $\chi \in (0,\infty)$ a) the series $\sum f_\nu u^\nu/(\chi\nu)!$ converges for small values of u, b) the function $f_\chi(u)$ so defined is regular for $u \in (0,\infty)$ and c) the integral

$$S_\chi = \chi^{-1}\int_0^\infty \exp(-u^{1/\chi})\, u^{(1/\chi)-1} f_\chi(u)\,du$$

exists, then $\sum f_\nu$ is said to be (B', χ) summable to S_χ]:
this general method of summation was introduced by Le Roy [ ]; the case in which $\chi = 1$ was studied in detail by Borel)

~~Evidently a Hardy-Ramanujan sum may be proposed: if the first integral upon the right hand of form $\overline{I}^{(1)}(a)$ exists~~

the first integral upon the right hand side of formula ( ) exists, and for some positive integer m the series ( ) is $(B', 1)$ summable to S', then $\sum \phi(\nu)$ is (H, R, a) summable to

$$S = -\int_a^0 \phi(t) - \sum_0^m \frac{B^{(1)}_{\nu+1}(a)\Delta^\nu \phi(a)}{(\nu+1)!} ~~- S'~~$$

Equally clearly, use of $(B',1)$ summation to sum the ~~extended version~~ an series ~~of series ( ) in which the argument of ϕ is~~ the series $\{\ \}$ $\phi(x)$ is replaced $\overset{m+2}{}$ of $\phi(x)$ may be used to ~~define~~ derive a solution of equation ( ) when $n=1$, ~~In many cases the Nörlund~~ and the same artifice may be used to construct a solution of equation ( ).

Equations (,) ~~describe~~ concern the local behaviour of the functions $F^{(n)}$, $G^{(n)}$ and ϕ. ~~It is conceivable that $F^{(n)}$ exists when~~ If ϕ is a function defined over a strip of the form $x^- \le Re(z) < x^+$, and having the two lines $Re(x) = x^{\pm}$ as natural barriers, ~~In such a case~~ the Nörlund and Hardy–Ramanujan definitions of a solutions to equations (,) break down, since, when ω lies in one of the half planes $Re(\omega) \lessgtr 0$, expressions of the form (,) are undefined. In this paper a solution of equations (,) when ϕ is defined over a strip ~~are~~ is ~~obtain~~ constructed. As a preliminary, a new formula exhibiting the periodic Bernoulli functions as ~~compound~~ composite moments is obtained:

$$\frac{\overline{\mathbb{B}}^{(n)}_{m+\tau}(u)}{(m+\tau)!} = i^{-\tau} \sum_{k=1}^{n} \binom{-k}{\tau} \int_{-1/(2\pi)}^{1/(2\pi)} v^{\tau}\, d\sigma(n,m,k;u|v)$$

~~where~~ for $m = n+1, n+2, \ldots$, where

$$\sigma(n,m,k;u|v) = \frac{(-1)}{2\pi i} \sum_{j=0}^{n-k} \binom{m-j-k}{n-j-k} (-1)^j \frac{B^{(n)}_j(n)}{j!} \int_{-\infty}^{\infty} \frac{\overline{B}_{m-j}(u+vy)}{(m-j)!} \frac{e^{-iy}-1}{y}\, dy$$

When $n=1$, formulae (,) ~~assumes the simple form~~ reduces to

$$\frac{\overline{\mathbb{B}}^{(1)}_{m+\tau}(u)}{(m+\tau)!} = i^{-\tau} \int_{-1/2\pi}^{1/2\pi} u^{\tau} d\left[\int_{-\infty}^{\infty} \frac{\overline{B}_m(u+vy)}{m!} \frac{e^{-iy}-1}{y}\, dy \right]$$

for $z = \cdots$ where $m = 2, 3, \ldots; z = 0, 1, \ldots$

~~The~~ the ~~Ber~~ periodic Bernoulli functions are (apart from the factor $(m+z)!\, i^z$) simple power moments in the commonly accepted sense.

Use of this ~~result~~ formula leads to ~~the result~~ a new form of the remainder term of the Euler-Maclaurin series, namely, when $Re(\omega) > 0$

$$F^+(n, m; z|\omega) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} \frac{d\sigma(n, m, k; z; t)}{(1 + i\omega t)^k}$$

where

$$\sigma(n, m, k; z; t) = \frac{(-1)^{n-1}}{2\pi i} \sum_{j=0}^{n-k} (-1)^j \binom{m-j-k}{m-n} \frac{B_j^{(n)}(0)}{j!} \int_{-\infty} \frac{\bar{B}_{m-j}(ty)}{(m-j)!} \Delta^{2} \frac{\{\phi(z+iy) - \phi(z)\}}{y} dy$$

for $m = n+1, n+2, \ldots; k = 1, \ldots n$: ~~the difference between~~ subject to suitable conditions ~~upon ϕ and x~~

$$E^+(n, m; x|\omega) - E^+(n, m+1; x|\omega) = \frac{B_m^{(n)}(0)}{m!} \Delta^m \phi(z) \omega^m$$

~~for m~~ and the function

$$F^+(n; 0; x|\omega) = \sum_{\nu=0}^{m-1} \frac{B_\nu^{(n)}(0)}{\nu!} \Delta^{\nu} \phi(z) \omega^\nu + E^+(n, m; z|\omega)$$

satisfies equation ( ). One difference between the Poisson-Nörlund form of the remainder term of the Euler-Maclaurin series ~~form~~, exhibited by formula ( ), and the new form displayed by formulae (,) is clearly visible ~~immediately apparent~~: the latter makes appeal to the behaviour of ϕ over a strip alone, ~~but serves nevertheless to~~ ~~define~~ It is established that the series ( ) represents $F(n, 0; x|\omega)$

asymptotically as ω tends to zero in the sector $-\frac{1}{2}\pi < \arg(\omega) < \frac{1}{2}\pi$ and that this series is $(B', 1)$ summable to the value of this function over the half-plane $Re(\omega) > 0$. A formula similar to ( ) defines a function $\bar{E}(n, m, x|\omega)$ over the half-plane $Re(\omega) < 0$ which has properties similar to those of its counterpart. It can occur that the imaginary axis is a natural barrier for both functions. Alternatively it can occur that the series ( ) convergent over a disc $|\omega| < \gamma$ $(0 < \gamma < \infty)$ ~~for sufficiently small ω~~, in which case the two functions are effectively the same and may ~~be obtained~~ alternatively be defined by analytic continuation over the complex plane cut along the two segments of the imaginary axis $\pm i[\gamma, \infty]$. Between these two extremes it can occur that both functions may be continued across the imaginary axis; a completion theorem relating the two functions for values of the argument for which they are both defined is given.

Boole's series and the alternating sum are ~~defined in~~ treated in the same way. In particular, the periodic Euler functions may be expressed in the form

$$\frac{\bar{E}^{(n)}_{m+\tau}(u)}{(m+\tau)!} = i^{\tau} \sum_{k=1}^{n}{}' \binom{-k}{\tau} \int_{-1/2\pi}^{1/2\pi} v^{\tau} d\sigma(n, m, k; u|v)$$

~~where~~ for $m = n+1, n+2, \ldots$; $\tau = 0, 1, \ldots$, where

$$\sigma(n, m, k; u|v) = \frac{(-1)^{n+k} 2^{n-1}}{2\pi i} \sum_{j=k-n}^{n-1}{}' \binom{m+j+1-k}{m} \frac{B^{(n)}_{n-j-1}(u)}{(n-j-1)!} \int_{-\infty}^{\infty} \frac{\bar{E}_{m+j}(u+vy)}{(m+j)!} \frac{e^{-iy}-1}{y}\, dy$$

~~and~~ The ~~exact~~ form corresponding to ( ) for the remainder term of the Boole's series is

$$\overset{*}{G}{}^{+}(n,m;z|\omega) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} \frac{d\sigma(n,m,k;z|t)}{(1+in\omega t)^{k}}$$

where

$$\sigma(n,m,k;z|t) = \frac{(-1)^{n+k}\, 2^{n-1}}{2\pi i} \sum_{j=k-1}^{n-1} \binom{m+j+1-k}{m} \frac{B^{(n)}_{n-j-1}(0)}{(n-j-1)!} \int_{-\infty}^{1} \frac{\bar{E}_{m+j}(u+vy)}{(m+j)!} \frac{\partial^{m}\{\psi(z+iy)-\psi(z)\}}{y}\, dy$$

for $m = n+1, n+2, \ldots, k = 1, \ldots, n$.

The treatment given in this paper is more general than that sketched above. The series considered have, in one case for example, the form

$$\sum_{\nu=0}^{q} \int_{\alpha}^{\beta} \frac{B^{(n)}_{\nu}(w)}{\nu!}\, d\rho(w)\, \partial^{\nu-n} \psi(x)\, \omega^{\nu}.$$

~~and the functions governed~~ Its associated function satisfies an equation of the form ( ) in which ϕ is the integral transform

$$\phi(z) = \int_{\alpha}^{\beta} \psi(z+w\omega)\, d\rho(w).$$

The above simplified theory is obtained by taking $\rho(w)$ to be the function $\frac{1}{2}\operatorname{sign}(w)$.

Notation and preliminaries

With θ_1, θ_2 prescribed real numbers ($\theta_1 < \theta_2$), $\Delta(\theta_1, \theta_2)$ is the finite open sector containing the points of the set $\{z: \theta_1 < \arg(z) < \theta_2, 0 < |z| < \infty\}$; with $\theta_1 \leq \theta_2$, $\bar{\Delta}(\theta_1, \theta_2)$ is the finite closed sector (or ray) containing the points of the set $\{z: \theta_1 \leq \arg(z) \leq \theta_2, 0 < |z| < \infty\}$. $i[-\infty,\infty]$ is the set of points belonging to the imaginary axis in the complex plane. $BE(i[-\infty,\infty])$ ~~is the~~ ~~...~~ denotes a domain in the complex plane containing the points z for which ~~$|z-t| \geq \delta$~~ $|Re(z)| \geq \delta$, $|z| < T$ where $\delta \in (0,\infty)$ arbitrarily small and $T \in (0,\infty)$ arbitrarily large are fixed. [?]

The index of single summation is always ν; if the upper limit is infinity it is omitted from the summation sign; if the lower limit is also zero, it too is omitted: $\sum^{n} a_\nu$, $\sum_1 a_\nu$ and $\sum a_\nu$ denote $\sum_{\nu=1}^{n} a_\nu$, $\sum_{\nu=1}^{\infty} a_\nu$ and $\sum_{\nu=0}^{\infty} a_\nu$ respectively. Order relationships (e.g. $f_\nu = O(\nu!)$) are tacitly assumed to hold for values of the argument tending to infinity; furthermore, use of simple order relationships such as $f_\nu = O(\nu!\, \xi^\nu)$, $f_\nu = O(\xi'^\nu)$, ... implies that ξ, ξ' are fixed finite positive real numbers.

Simple integral expressions and Stieltjes integral expressions

[1] An expression such as z^α, where z is complex and $\alpha \in (-\infty, \infty)$ refers to that branch of this function which assumes positive real values for positive real z. $x!$ denotes $\Gamma(x+1)$ for general real values of x.

The periodic Bernoulli polynomials $\mathcal{B}_{\nu}(x)$ ($\nu = 1, 2, \ldots$) are defined for $x \in [0, 1)$ as the coefficients of $t^\nu/(\nu+1)!$ when $|t| < 2\pi$ in the expansion

$$\frac{te^{xt}}{e^t - 1} - \frac{1}{t} = \sum_{1}^{\infty} \frac{\mathcal{B}_{\nu+1}(x)}{(\nu+1)!} t^\nu$$

and by use of the relationship $\mathcal{B}_{\nu}(x+1) = \mathcal{B}_{\nu}(x)$ ($\nu = 1, 2, \ldots$; $0 \le x < \infty$) over the finite nonnegative real axis. They are alternatively defined by the formulae $\mathcal{B}_1(x) = x - \frac{1}{2} - [x]$ and, for $\tau = 1, 2, \ldots$

$$(\ ) \quad \frac{\mathcal{B}_{2\tau}(x)}{(2\tau)!} = \frac{(-1)^{\tau-1}}{2^{2\tau-1}\pi^{2\tau}} \sum_{1}^{\infty} \frac{\cos(2\nu\pi x)}{\nu^{2\tau}}$$

$$(\ ) \quad \frac{\mathcal{B}_{2\tau+1}(x)}{(2\tau+1)!} = \frac{(-1)^{\tau-1}}{2^{2\tau}\pi^{2\tau+1}} \sum_{1}^{\infty} \frac{\sin(2\nu\pi x)}{\nu^{2\tau+1}}$$

(formula ( ) also defines $\mathcal{B}_1(x)$ for non-integer values of x in $[0,\infty)$)

$B_{\nu+1} = \mathcal{B}_{\nu+1}(0)$ ($\nu = 1, 2, \ldots$) are the Bernoulli numbers; $B_{2\tau+1} = 0$ ($\tau = 1, 2, \ldots$).

denote Riemann integrals and Riemann–Stieltjes integrals respectively.

Le Roy's method ([ ], see [ ] §4.16) for transforming the power series $\mathcal{G} = \sum i\, b_\nu z^\nu$ with complex number coefficients over a sector $\bar{\Delta}(\theta_1, \theta_2)$ functions as follows: it is assumed that for $z \in \bar{\Delta}(\theta_1, \theta_2)$ a) the series $\sum i\, b_\nu (zu)^\nu / (\gamma\nu)!$ converges for small values of u, b) the function $f_\gamma(zu)$ defined by analytic continuation of the sum of this series is regular in u for all $u \in (0, \infty)$ and c) the integral

$$S_\gamma(z) = \gamma^{-1} \int_0^\infty \exp(-u^{1/\gamma})\, u^{(1/\gamma)-1} f_\gamma(zu)\, du$$

exists; $\mathcal{G}$ is then said to be summable $(\mathcal{B}', \gamma)$ to S_γ over $\bar{\Delta}(\theta_1, \theta_2)$. The case in which $\gamma = 1$ was studied in detail by Borel ([ ] Ch. 3)

~~The method of $\mathcal{B}^2$-summation introduced by Hardy ([ ] §1§3.16) functions, in an implementation due to the author [ ], as follows: it is assumed~~

The series ... may also be transformed over the sector $\bar{\Delta}(\theta_1, \theta_2)$ by the method of $\mathcal{B}^2$-summation introduced by Hardy ([ ] §13.16), which, in an implementation due to the author, functions as follows: it is assumed that for $z \in \bar{\Delta}(\theta_1, \theta_2)$ a) the series $\sum i\, b_\nu (zu)^\nu / (\nu!)^2$ converges for small values of u, b) the function $\hat{f}(zu)$ defined by analytic continuation of the sum of this series is regular for all $u \in (0, \infty)$ and c) the integral

$$\hat{\upsilon}(z) = 2\int_0^\infty K_0(2u^{\frac{1}{2}})\,\hat{f}(zu)\,du$$

exists (K_0 being a modified Bessel function of the second kind); the series φ is then said to be summable $(\mathcal{E}^2)$ to $\hat{S}$ over $\bar{\Delta}(\theta_1, \theta_2)$.

The author has introduced extensions of the above two methods which permit certain series to be transformed over ~~most~~ larger sectors than those for which the above methods are effective. The first extension functions as follows: it is assumed ~~that~~ for that the series $\sum f_\nu x^\nu/(\gamma\nu)!$ converges in the neighbourhood of the origin, and that the function $\overset{(\cdot)}{f}_\gamma$ obtained by analytic continuation of the sum of this series is uniformly bounded over $\bar{\Delta}(\phi_1, \phi_2)$; set

$$R_\gamma(z) = \gamma^{-1}\int_0^{\infty\exp\{i\phi(\gamma,z)\}} \exp(-u^{1/\gamma})\, u^{(1/\gamma)-1}\, \tilde{f}_\gamma(zu)\,du$$

where

$$\phi(\gamma, z) = \frac{\gamma\pi\{\frac{1}{2}(\phi_1 + \phi_2) - \arg(z)\}}{\gamma\pi + \phi_2 - \phi_1};$$

the series φ is then said to be summable $(\mathcal{E}, \gamma)$ to R_γ over $\Delta(\phi_1 - \frac{1}{2}\gamma\pi, \phi_2 + \frac{1}{2}\gamma\pi)$. For the second extension it is assumed that the series $\sum f_\nu x^\nu/(\nu!)^2$ converges in the neighbourhood of the origin and that the function $\hat{f}$ obtained by analytic continuation of the sum of this series is uniformly bounded over

$\bar{\Delta}(\phi_1, \phi_2)$; set

$$\hat{R}_r(z) = 2\int_0^{\infty \exp\{i\phi(z,z)\}} K_0(2u^{1/2})\hat{f}(zu)\,du;$$

the series $^0\mathcal{F}$ is then said to be summable $(\bar{B}^2)$ to $\hat{R}$ over $\Delta(\phi_1-\pi, \phi_2+\pi)$. (That the above functions $R_\gamma(z)$ and $\hat{R}(z)$ are well defined for values of z lying in the associated summability sectors is easily verified).

The approximating fraction (Näherungsbruch [,]) or Padé quotient ([ ], [ ] Ch 5, [ ] Ch. 20) $P_{i,j}$ ($i, j \geq 0$, being fixed finite integers) derived from the series $\phi^0\mathcal{F}$ with $f_0 \neq 0$, is that irreducible rational function whose numerator polynomial is of degree $\leq j$ and whose denominator polynomial $D_{i,j}$ is of degree $\leq i$ with $D_{i,j}(0) = 1$, whose series expansion in ascending powers of z agrees with $^0\mathcal{F}$ for the greatest number of initial terms. The quotients $\{P_{i,j}\}$ may be placed in a two dimensional array, the Padé table, in which i and j correspond to row and column numbers respectively. For convenience in exposition we append the quotient $P_{0,-1}(z) = 0$ to the Padé table. For a fixed finite integer $m \geq 0$, the quotients $P_{i,i+m-1}$ $(i = 0, 1, \ldots)$ and $P_{i+m,i}$ $(i = 0, 1, \ldots)$ lie on forward diagonals in the Padé table.

The Bernoulli polynomials ~~of order n~~ $B_\nu^{(n)}(y)$ of order n and degree ν $(n,\nu=0,1,\dots)$ may be defined by the relationship

$$\left\{\frac{z}{e^z-1}\right\}^n e^{yz} = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!} B_\nu^{(n)}(y)$$

holding for $|z|<2\pi$. They satisfy the relationships

$$B_\nu^{(n)}(y+1) - B_\nu^{(n)}(y) = \nu B_{\nu-1}^{(n-1)}(y)$$

~~for $n=1,2,\dots$ and $\nu=0,1,\dots$ and~~ $B_\nu^{(n)}(n-y) = (-1)^\nu B_\nu^{(n)}(y)$

$$B_\nu^{(n+1)}(y) = \left(1-\frac{\nu}{n}\right)B_\nu^{(n)}(y) + \nu\left(\frac{y}{n}-1\right)B_{\nu-1}^{(n)}(y)$$

~~for $n,\nu=1,2,\dots$ and $B_\nu^{(n)}(n-y)$ for $n=0,1,\dots$~~ for $n,\nu=1,2,\dots$. The periodic Bernoulli polynomials $\tilde{B}_\nu$ of order one and degree ν $(\nu=0,1,\dots)$ are defined over $(-\infty,\infty)$ by setting $\tilde{B}_\nu(y) = B_\nu^{(1)}(y)$ $(y\in[0,1))$ and imposing the condition $\tilde{B}_\nu(y+1) = \tilde{B}_\nu(y)$ $(y\in(-\infty,\infty); \nu=0,1,\dots)$. For $\tau=1,2,\dots$

$$\frac{\tilde{B}_{2\tau}(y)}{(2\tau)!} = \frac{(-1)^{\tau-1}}{2^{2\tau-1}\pi^{2\tau}} \sum_{\nu=1}^{\infty} \frac{\cos(2\nu\pi y)}{\nu^{2\tau}}$$

$$\frac{\tilde{B}_{2\tau+1}(y)}{(2\tau+1)!} = \frac{(-1)^{\tau-1}}{2^{2\tau}\pi^{2\tau+1}} \sum_{\nu=1}^{\infty} \frac{\sin(2\nu\pi y)}{\nu^{2\tau+1}}$$

(formula ( ) also defines $\tilde{B}_1(y)$ for non integer values of y in $(-\infty,\infty)$; since $B_1(y) = y-\frac{1}{2}$, $\tilde{B}_1(r) = -\frac{1}{2}$ $(r=\dots,-1,0,1,\dots)$, but the sum of the series in formula ( ) to x<0 for these values of y).

The Euler polynomials $E_\nu^{(n)}$ of order n and degree ν $(n,\nu=0,1,\dots)$ may be defined by the relationship

$$\left\{\frac{2}{e^z+1}\right\}^n e^{yz}=\sum{}' \frac{z^\nu}{\nu!}E_\nu^{(n)}(y)$$

holding for $|z|<2\pi$. They satisfy the relationships

$$E_\nu^{(n)}(y+1)+E_\nu^{(n)}(y)=2E_\nu^{(n-1)}(y)$$

$$E_\nu^{(n)}(n-y)=(-1)^\nu E_\nu^{(n)}(y)$$

$$E_\nu^{(n+1)}(y)=\frac{2}{n}E_{\nu+1}^{(n)}(y)+\frac{2}{n}(n-y)E_\nu^{(n)}(y)$$

holding for $n=1,2,\dots$, $\nu=0,1,\dots$. The periodic Euler polynomials $\tilde{E}_\nu$ of order one and degree ν $(\nu=0,1,\dots)$ are defined over $(-\infty,\infty)$ by setting $\tilde{E}_\nu(y)=E_\nu^{(1)}(y)$ $(y\in[0,1))$ and imposing the condition $\tilde{E}_\nu(y+1)+\tilde{E}_\nu(y)=0$ $(y\in(-\infty,\infty)$; $\nu=0,1,\dots)$. For $\tau=1,2,\dots$

$$\frac{\tilde{E}_{2\tau-1}(y)}{(2\tau-1)!}=\frac{4(-1)^\tau}{\pi^{2\tau}}\sum \frac{\cos\{(2\nu+1)\pi y\}}{(2\nu+1)^{2\tau}}$$

$$\frac{\tilde{E}_{2\tau}(y)}{(2\tau)!}=\frac{4(-1)^\tau}{\pi^{2\tau+1}}\sum{}' \frac{\sin\{(2\nu+1)\pi y\}}{(2\nu+1)^{2\tau+1}}$$

(formula ( ) also defines $\tilde{E}_0(y)$ for ~~nonin~~ integer values of y in $(-\infty,\infty)$; ~~since $\tilde{E}_0^{(1)}(y)=1$, $E_\nu^{(1)}$~~ but not at integer values)

Theorem A. i) For $\tau = n+1, n+2, \ldots$ the functions set

$$\overline{B}_\tau^{(n)+}(u) = n \binom{\tau}{n} i^{n-\tau+1} \int_0^\infty \frac{(u+iv)_{n-1} v^{\tau-n}}{e^{2\pi(v-iu)}-1} dv$$

For $\tau = n+1, n+2, \ldots$:

a) ~~are analytic over~~ the ~~For $\tau = n+1, n+2, \ldots$~~ $\overline{B}_\tau^{(n)+} \in A\{H_i^+\}$ and

$B_\tau^{(n)+}$ is differentiable ~~for~~ over $(-\infty,\infty)$ ~~($\tau = n+1, n+2, \ldots$)~~, functions

$\overline{B}_\tau^{(n)-}$ defined by reversing the sign of i above ~~behave in a similar way~~ have similar properties over H_i^- and $(-\infty,\infty)$.

b). Individually the functions $\overline{B}_\tau^{(n)\pm}$ satisfy relationships of the form ( );

b) ~~over~~ for $u \in (-\infty,\infty)$

$$\overline{B}_\tau^{(n)}(u) = \overline{B}_\tau^{(n)-}(u) + B_\tau^{(n)+}(u)$$

ii) For $\tau = n+1, n+2, \ldots$ and $u \in (-\infty,\infty)$

$$\overline{B}_\tau^{(n)}(u) = \binom{\tau}{n} i^{n-\tau+1} \int_{-\infty}^{\infty} t^\tau w(t) dt$$

where $w(t) = n t^{-n} (\overset{(n)}{u} + it)_{n-1} \hat{w}^{\pm}(t)$ with $\hat{w}^+(t) = \{e^{2\pi(t-iu)} - 1\}^{-1}$ ~~over~~ for $0 \le t < \infty$ and $w^-(t) = \{1 - e^{2\pi(iu-t)}\}^{-1}$ for $-\infty < t \le 0$.

(iii) For $m = n+1, m+2, \ldots, \tau = 0, 1, \ldots$ and $u \in (-\infty,\infty)$,

$$\overline{b}_{m+\tau}^{(n)}(u) = i^{-\tau} \sum_{k=1}^{n} \binom{-k}{\tau} \int_{-1/(2\pi)}^{1/(2\pi)} v^\tau b(k, m, n; u/v)$$

where

$$\sigma(k,m,n;u|v)=\frac{(-1)^{n-1}}{2\pi i}\sum_{j=0}^{n-k}\binom{m-j-k}{m-n}(-1)^j b_j^{(n)}(u)\int_{-\infty}^{\infty}\bar{b}_{m-j}^{(1)}(u+vy)\,\frac{e^{-iy}-1}{y}\,dy$$

$\sigma(k,m,n;u)$ being of bounded variation over $(-(2\pi)^{-1},(2\pi)^{-1})$.

(iv) For $m=n+1,n+2,\ldots$ and $\eta\in H^{\pm}$

$$\int_0^{\infty}\bar{b}_{m-1}^{(n)}(u-t)e^{\eta t}dt=\sum_{k=1}^{n}\int_{-1/(2\pi)}^{1/(2\pi)}\frac{d\sigma(k,m,n;u|v)}{(1+i\eta v)^k}$$

B. i) Set

$$\bar{E}_z^{(n)+}(u)=\frac{(-2)^{n-1}}{(n-1)!}\,i^{-z}\int_0^{\infty}v^z(u+iv)_{n-1}\,\mathrm{cosec}\{\pi(u+iv)\}dv$$

For $z=n+1,n+2,\ldots$

a) ~~$\bar{E}_z^{(n)+}\in A\{H_i^+\}$ and $E_z^{(n)+}$ is differentiable over $(-\infty,\infty)$;~~

a) the results of clause A(i a,b) hold for the above functions $\bar{E}_z^{(n)+}$ and similarly defined functions $\bar{E}_z^{(n)-}$

b) individually the functions $\bar{E}_z^{(n)\pm}$ satisfy relationships of the form ( ).

(ii) For $z=n+1,n+2,\ldots$ and $u\in(-\infty,\infty)$

$$\bar{E}_z^{(n)}(u)=i^{-z}\int_{-\infty}^{\infty}t^z w(t)dt$$

where $w(t)=\frac{(-2)^{n-1}}{(n-1)!}(u+it)_{n-1}\,\mathrm{cosec}\{\pi(u+it)\}$

(iii) For $m = n+1, n+2, \ldots$ $\tau = 0$ and $u \in (-\infty, \infty)$

$$\bar{e}^{(n)}_{m+\tau}(u) = i^{\tau} \sum_{k=1}^{n} \binom{-k}{\tau} \int_{-1/(2\pi)}^{1/(2\pi)} v^{\tau} d\sigma(k,m,n;u|v)$$

where

$$\sigma(k,m,n;u|v) = \frac{(-1)^{n+k}\, 2^{n-1}}{2\pi i} \sum_{j=k-1}^{n-1} \binom{m+j+1-k}{m} b^{(n)}_{n-j-1}(u) \int_{-\infty}^{\infty} \bar{e}_{m+j}(u+vy) \frac{e^{-iy}}{y} dy$$

$\sigma(k,m,n;u)$ being of bounded variation over $(-(2\pi)^{-1}, (2\pi)^{-1})$.

(iv) For $m = n+1, n+2, \ldots$ and $\eta \in H^{\pm}$

$$\pm \int_0^{\infty} \bar{e}^{(n)}_{m-1}(u-t) e^{\mp \eta t} dt = \sum_{k=1}^{n} \int_{-1/(2\pi)}^{1/(2\pi)} \frac{d\sigma(k,m,n;u|v)}{(1+i\eta v)^k}$$

Proof. The relationship

$$\bar{b}^{(n)}_{\tau}(u) = (-1)^{n-1} \sum_{0}^{n-1} \binom{n-\nu-1}{\tau-n} (-1)^{\nu} b^{(n)}_{\nu}(u)\, \bar{b}_{\tau-\nu}(u)$$

~~is easi~~ holding for $\tau = n, n+1, \ldots$ is easily verified. It is evidently correct when $n=1$. Assuming it to hold for some $n \geq 1$, use is made of the recursion

$$b^{(n+1)}_{\nu+1}(u) = \left(\frac{n-\nu-1}{n}\right) b^{(n)}_{\nu+1}(u) + \left(\frac{u}{n} - 1\right) b^{(n)}_{\nu}(u).$$

~~to shows that the function represented on the right hand side~~

~~of formula ( ) with n replaced by n+1 is~~

Denote the ~~same~~ expression upon the right hand side of formula ( )

by $\phi_r^{(n)}(u)$ with $0<u<n$. Form the combination $(1-\frac{r}{n})\phi_r^{(n)}(u)+(\frac{u}{n}-1)\phi_{r-1}^{(n)}(u)$ and in the second component replace all terms of the form $(\frac{u}{n}-1)b_{\nu}^{(n)}(u)$ by their equivalent expressions derived from formula ( ); the combination is shown to be $\phi_r^{(n+1)}(u)$. The similar combination of functions $\bar{b}_r^{(n)}(u)$ (i.e. $b_r^{(n)}(u)$ if $0<u<n$) is equivalent to $\bar{b}_r^{(n+1)}(u)$: relationship ( ) holds with n replaced by $n+1$ and $0<u<n$. With $n-1\leq u<n$, form the combination $\phi_r^{(n+1)}(u)+\phi_{r-1}^{(n)}(u)$; the result is shown to be $\phi_r^{(n+1)}(u+1)$. But $\bar{b}_r^{(n+1)}(u)+\bar{b}_{r-1}^{(n)}(u)=\bar{b}_r^{(n+1)}(u+1)$ also: the range of validity thus has been extended from $0<u<n$ to $0<u<n+1$. ~~Continue this process, and modify it, to show that relationship ( ) holds with n replaced by $n+1$ and all finite real u. (Relationship ( ) is given in slightly concealed form by Nörlund ([ ] p. 155 eqn. (101)) i.e. $\phi_r^{(n+1)}(u)$ represents $b_r^{(n+1)}(u)$ for $0<u<n+1$.~~

Evidently $\Delta^{n+1}\phi_r^{(n+1)}(u)=0$ ~~for all real u~~ $(-\infty<u<\infty)$: $\phi_r^{(n+1)}(u)$ represents $\bar{b}_r^{(n+1)}(u)$ for all real u. (Relationship ( ) is given in slightly concealed form by Nörlund ([ ] p. 155 eqn (101)).

For $r=1,2,\ldots$ and $-\infty<u<\infty$

$$\bar{b}_r(u)=-\sum{}'(2i\nu\pi)^{-r}e^{2i\nu\pi u};$$

~~or $\bar{b}_r(u)=\bar{b}_r^{+}(u)+\bar{b}_r^{-}(u)$, where~~

$$\bar{b}_r^{+}(u)=-i^{-r}\sum_1(2\nu\pi)^{-r}e^{2i\nu\pi u}$$

~~and $\bar b_r(u) = (-1)^r b$~~

~~Setting~~ For $r = 1, 2, \dots$, set

$$\bar b_r^+(u) = -i^{-r}\sum_1 (2\nu\pi)^{-r} e^{2\nu\pi i u}, \quad \bar b_r^-(u) = -i^{r}\sum_1 (2\nu\pi)^{-r} e^{-2\nu\pi i u}.$$

$\bar b_r^+(u)$ and $\bar b_r^-(u)$ are ~~defined~~ analytic for ~~R~~ $\mathrm{Im}(u) \geq 0$ and $\mathrm{Im}(u) < 0$ respectively ~~and~~ are continuous for ~~[illegible]~~ real u ~~and~~, when $r > 1$ are differentiable for real u; ~~$(r = 1, 2, \dots)$~~. ~~and~~ also $b_r(u) = b_r^+(u) + b_r^-(u)$ for real values of u $(r = 1, 2, \dots)$.

Define functions $\bar b_r^{(n)\pm}(u)$ by means of formulae similar to ( ) in which $\bar b_{r-\nu}$ is replaced by $\bar b^{\pm}_{r-\nu}$: again $\bar b_r^{(n)}(u) = \bar b_r^{(n)+}(u) + \bar b_r^{(n)-}(u)$ ~~$(r = n, n+1, \dots)$~~ and, ~~as~~ the similar relationship holds for ~~the~~ functions $\bar B_r^{(n)}$ and $\bar B_r^{(n)\pm}$ $(r = n, n+1, \dots)$. Using Euler's ~~expression~~ formula for the ~~together~~ ~~of Γ-function, $\bar B_r^{(n)}$ has the representation ( ). The results of~~ $\leftarrow 0$

clauses ~~(i)~~ Ai b) follow from the above remarks; that of clause Aic) may ~~be~~ easily be verified. The result of clause Aii) follows from formula ( ) and its companion concerning the function $\bar B_r^{(n)-}$.

From formula ( )

$$\bar b_{j+r+1}(u) = i^{-r}\int_{-1/(2\pi)}^{1/(2\pi)} v^r d\sigma(j, 1, 1; u \mid v)$$

$m+r+1$?

for $j = 1, 2, \dots$, $r = 0, 1, \dots$ where $\sigma(j, 1, 1; u|v)$ has a saltus of magnitude $-i^{-j-1}\nu^j e^{iu/v}$ at the point $v = (2\nu\pi)^{-1}$ $(\nu = \dots, -1, 0, 1, \dots)$ and no other point of variation in the range $-(2\pi)^{-1} \leq v \leq (2\pi)^{-1}$. ~~(it is, of course, of bounded variation in this range)~~ Again by use of formula ( ),

$$\sigma(j, 1, 1; u \mid v) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \bar b_{j+1}(u + vy)\frac{e^{-iy}}{y}dy$$

for $j=1,2,\ldots$, $u\in(-\infty,\infty)$ and $v\in[-(2\pi)^{-1},(2\pi)^{-1}]$. Combining formulae (, ,) and using an elementary decomposition of $\binom{\tau-\nu-1}{\tau-n}$ in terms of coefficients of the form $\binom{-k}{\tau}$, formulae (,) are derived. Since the relevant functions $\varsigma(j,1,,1,u|v)$ are of bounded variation ~~in~~ over the range $-(2\pi)^{-1}\leq v\leq(2\pi)^{-1}$, the same is true of the functions defined by formula ( ).

Nörlund has shown that for $m=n+1,n+2,\ldots$ and $\eta\in H^{+}\cap D_{2\pi}$

$$\int_0^\infty \overline{b}_{m-n}^{(n)}(u-t)e^{-\eta t}=\sum \overline{b}_{m+\nu}^{(n)}(u)(-\eta)^{\nu}$$

The above series ~~on the right hand side of th~~ is, from the results of the preceding clause, also generated by the expression on the right hand side of relationship ( ) when $\eta\in D_{2\pi}$. This relationship (with $e^{-\eta t}$ for $e^{\mp\eta t}$) persists for values of η for which both constituents are defined, namely over H^{+}. The further result of clause is demonstrated in the same way (the expression upon the right hand side of relationship ( ) ~~is of conv~~ represents a function analytic over the cut plane $\mathbb{C}\setminus\pm i[2\pi,\infty]$).

Part B of the theorem is demonstrated in the same way. The counterparts to formulae (, , , ,) are in succession

$$\bar{E}^{(n)}_{z}(x) = 2^{n-1}\sum_{\nu=0}^{n-1}\frac{(-1)^{\nu}}{(n-\nu-1)!}\, b^{(n)}_{\nu}(x)\,\bar{E}_{z+n-\nu-1}(x)$$

$$E^{(n+1)}_{\nu}(x) = \frac{2}{n}\left\{E^{(n)}_{\nu+1}(x) + (n-x)\,E^{(n)}_{\nu}(x)\right\}$$

$$\bar{e}_{z}(x) = 2\sum{}''\left\{(2\nu-1)\pi i\right\}^{-z-1} e^{(2\nu-1)x\pi i}$$

and relationships (,) with b replaced by e. (rel^ given by Nörlund, p153 eq (97))

Lemma 1 Let the function $\sigma \in BV(-\infty,\infty)$ be such that the moment

$$f_\nu = \int_{-\infty}^{\infty} t^\nu \, d\sigma(t)$$

exists for arbitrarily large positive integer ν. Let l be a positive integer. For $m = 0,1,\dots$, let $f^{(m)+}$ be the function defined over H_l^+ by means of the formula

$$f^{(m)+}(z) = \frac{\Gamma(m+l)}{\Gamma(m+1)\Gamma(l)} \int_{-\infty}^{\infty} t^m \, {}_2F_1(m+l, 1; m+1; zt)\, d\sigma(t)$$

and define functions $f^{(m)-}$ over H_l^- similarly. ~~Set~~ For $m, \tau = 0,1,\dots$, set

$$f^{(m,\tau)}(z) = \sum_0^{\tau-1} (-1)^{m+\nu} \binom{-l}{m+\nu} f_{m+\nu} z^\nu \longleftarrow \hat{f}^{(2m,\tau)} = \sum_0^{\tau-1} \binom{-l}{2m+\nu} f_{2m+\nu} z^\nu$$

~~Let f~~ For $m = 0,1,\dots$ let $f^{(m)}$ be the series

$$\sum_0^\infty (-1)^{m+\nu} \binom{-l}{m+\nu} f_{m+\nu} z^\nu$$

and let $\hat{f}^{(2m)}$ be the series

$$\sum_0^\infty \binom{-l}{2m+\nu} f_{2m+\nu} z^\nu .$$

The order relationships involving κ or ξ given below are assumed to hold for fixed $\kappa \in [0,\infty)$, $\xi \in (0,\infty)$.

~~(i) a) $f^{(m)}$ is analytic over H_l^+ (m=0,1,...)~~

(i) a) $f^{(m)+}$ and $f^{(m)-}$ are analytic over H_l^+ and H_l^- respectively $(m = 0,1,\dots)$

b) Let σ be nonanalytic at every point of $[\alpha, \beta] \subseteq [-\infty,\infty]$. Then the segments ~~the segment~~ of the real axis made up of the points of the $[\alpha,\beta]^{-1}$ are natural barriers of the functions $f^{(m)\pm}$ $(m = 0,1,\dots)$; they cannot be continued analytically across these segments.

c) Let σ be analytic over an open domain $\mathfrak{D}$ in the complex plane containing all points of the segment $(\alpha,\beta) \subseteq (-\infty,\infty)$. Then $f^{(m)+}$ as defined over H_i^+ by formula ( ) can be continued analytically across the segments $(\alpha,\beta)^{-1}$ of the real axis, a corresponding result also holds for $f^{(m)-}$ and, denoting the two functions directly defined and obtained by analytic continuation by $f^{(m)\pm}$ respectively,

$$\cancel{f^{(r,l)}(z) = f^{-}}$$

$$f^{(m)+}(z) = f^{(m)-}(z) - \frac{2\pi i(-1)^{\ell}}{\Gamma(\ell)}\mathfrak{D}^{\ell}\sigma(\lambda)$$

where $\lambda = z^{-1}$ and $\mathfrak{D}$ is $d/d\lambda$, for $\lambda \in \mathfrak{D}$ and $m = 0,1,\ldots$ If σ as defined over $\mathfrak{D}$ is a polynomial of degree less than ℓ, $f^{(m)+}$ and $f^{(m)-}$ are the same function in the sense that $f^{(m)+}$ as continued analytically across $(\alpha,\beta)^{-1}$ into H_i^- is $f^{(m)-}$, and conversely.

(ii) a) For $m, \tau = 0,1,\ldots$ and $z \in H_i^+$

$$f^{(m)+}(z) = f^{(m,\tau)}(z) + z^{\tau} f^{(m+\tau)+}(z)$$

and a similar result holds for the functions $f^{(m)-}$ over H_i^-

b) Let σ be constant over $(-\infty,\alpha)$ $(\alpha \geq 0)$, real valued and nondecreasing over $[\alpha,\infty)$. For $m=0,1,\ldots$ $f^{(m)\pm}$ are obtained from each other by analytic continuation across the negative real axis $(-\infty, ...)$ (and across ~~(α^{-1},∞) if~~ the segment (α^{-1},∞) if $\alpha>0$) they are the same function, $f^{(m)}$ say, and $f^{(m)}$ is semi-convergent over H_- in the sense that in conjunction with formula ( )

$$|f^{(m+r)}(z)| \leq \left|\binom{-\ell}{m+r} f_{m+r}\right|$$

for $r=0,1,\ldots$ and $z\in H^-$.

Furthermore, when $z\in(-\infty,0)$, the partial sums $f^{(m,r)}(z)$ oscillate about the value of $f^{(m)}(z)$:

$$f^{(m,2r)}(z) < f^{(m)}(z) < f^{(m,2r+1)}(z)$$

for $m,r=0,1,\ldots$.

b) ~~$f^{(m)}$ represents~~ For $m=0,1,\ldots$ $f^{(m)}$ represents $f^{(m)+}$ and $f^{(m)-}$ asymptotically ~~analytically~~ as the argument tends to $z=0$ in the sectors $\Delta(0,\pi)$, ~~and~~ and $\Delta(\pi,2\pi)$ respectively.

d) Let σ be constant over $(-\infty,\alpha)$ and (β,∞) where $\max|\alpha|,\beta=\gamma^{-1}$ ~~for~~ $(0,\infty)$. For $m=0,1,\ldots$ $f^{(m)\pm}$ are obtained from each other by analytic continuation across the segments ~~obtained by removing~~ $(-\infty,\infty)\setminus[\alpha,\beta]^{-1}$, they are the same function, $f^{(m)}$ say, and $f^{(m)}$ converges to $f^{(m)}$ over D_γ.

~~If, in addition, $\alpha\geq 0$ and σ is real valued and nondecreasing~~

(iii) Let $f_{2s+1} = 0$ $(s = 0, 1, \ldots)$

a) When $z \in H_1^+$

$$f^{(2m+1)+}(z) = z f^{(2m+2)+}(z)$$

for $m = 0, 1, \ldots$ and

$$f^{(2m)+}(z) = \hat{f}^{(m,2\tau)}(z) + z^{2\tau} f^{(2m+2\tau)+}(z)$$

for $m, \tau = 0, 1, \ldots$ Similar formulae hold for the functions $f^{(2m)-}$.

b) For $m = 0, 1, \ldots$ $\hat{f}^{(m)}$ represents $f^{(2m)+}$ and $f^{(2m)-}$ asymptotically as the argument (z in formula ( )) tends to zero in the sectors $\Delta(0, \pi)$ and $\Delta(\pi, 2\pi)$ respectively

c) Let $f_{2s} = O((2s + \tfrac{l}{2})!\, \xi^{2s})$, let ξ be ~~a~~ ~~nonde~~ real valued and nondecreasing, and let $l = 1$.

For $m = 0, 1, \ldots$ $\hat{f}^{(2m)}$ is semi convergent over $\Delta(\frac{1}{4}\pi, \frac{3}{4}\pi)$ and $\Delta(\frac{5}{4}\pi, \frac{7}{4}\pi)$ in the sense that in conjunction with formula ( )

$$|f^{(2m+2\tau)+}(z)| \leq \left| \binom{-l}{2m+2\tau} f_{2m+2\tau} \right|$$

for $\tau = 0, 1, \ldots$ and $z \in \bar{\Delta}(\frac{1}{4}\pi, \frac{3}{4}\pi)$; a similar result holds for the functions $f^{(2m)-}$ over $\bar{\Delta}(\frac{5}{4}\pi, \frac{7}{4}\pi)$. (Semi convergence over $i(0,\infty)$ and $-i(0,\infty)$ holds for $l = 1, 2, \ldots$ ~~general values of positive~~ integer ~~values of l~~ $-B$).

Furthermore, when $z \in i(0, \infty)$, the partial sums $\hat{f}^{(2m,\tau)}$ oscillate about the value of $f^{(2m)+}(z)$:

$$\hat{f}^{(2m,2r)}(z) < f^{(2m)+}(z) < \hat{f}^{(2m,2r+1)}(z)$$

for $m,r = 0,1,\dots$; similarly, when $z \in i(-\infty,0)$, the partial sums $\hat{f}^{(2m,r)}(z)$ oscillate about the value of $f^{(2m)-}(z)$.

If the $\{f_{2\nu}\}$ are further restricted by the condition $f_{2\nu} = O(\gamma^{-2\nu})$ $(\gamma \in (0,\infty))$ then $f^{(2m)\pm}$ are both obtained by analytic continuation of the function $f^{(2m)}$ to which $\hat{f}^{(2m)}$ converges over $\mathbb{D}_\gamma$; for $z \in i(-\gamma,\gamma)$,

$$\hat{f}^{(m,2r)}(z) \leq \hat{f}^{(2m,2r+2)} \leq f^{(2m)}(z) \leq f^{(2m,2r+3)}(z) \leq f^{(2m,2r+1)}(z)$$

and for $z \in (-\gamma,\gamma)$

$$\hat{f}^{(2m,r)}(z) \leq \hat{f}^{(2m,r+1)}(z) \leq f^{(2m)}(z)$$

both sets of inequalities holding for $m,r = 0,1,\dots$.

(iv) a) Let $f_\nu = O((\nu+\kappa)!\,\beta^{\nu})$. Then $f^{(m)}$ is $(B',1)$ summable to $f^{(m)+}$ over $i(0,\infty)$ and to $f^{(m)-}$ over $-i(0,\infty)$, and is $(B,1)$ summable to $f^{(m)+}$ over H_1^+ and to $f^{(m)-}$ over H_1^- $(m = 0,1,\dots)$.

b) Let σ also be a function of a second variable x, and for all $t \in (-\infty,\infty)$ let σ be analytic over an open domain $\mathbb{D}$ in the complex x-plane with $f_{2\nu+1} = 0$ and $f_{2\nu} = O((2\nu+\kappa)!\,\beta^{\nu})$. At an infinite set of points ~~in~~ with a limit point in D let

~~Let~~ σ be real valued and nondecreasing. ~~...~~ Then $f^{(m)}$ is both $al(\bar{B}^2)$ $(\bar{B}', 2)$ summable to $f^{(m)+}$ over $i(0,\infty)$ and to $f^{(m)-}$ over $-i(0,\infty)$, and is both $al(\bar{B}^2)$ $(\bar{B}, 2)$ summable to $f^{(m)+}$ over $\Delta(\cancel{0,\infty})$ H_i^+ and to $f^{(m)-}$ over $\Delta(\cancel{0,\infty})$ H_i^-

$(m = 0, 1, \ldots)$ (so that ~~f~~ $f^{(m)+}$ and $f^{(m)-}$ are both equivalent to a single function $f^{(m)}$ ~~in the sense of~~ as described in clause (ii)c))

b) Let σ be constant over $[-\infty, 0]$ and let $f_\nu = O((\nu+\nu)!\,\xi^\nu)$

~~for fixed $\nu \in [0,\infty)$~~. Then $f^{(m)}$ is $(\bar{B}', 2)$ summable to $f^{(m)}$ over $(-\infty, 0)$

and is $(\bar{B}, 2)$ summable to $f^{(m)}$ over ~~$H_i^+ \cup (-\infty, 0) \cup H$~~ $C \setminus (0,\infty)$

Proof The above results are stated in a form suitable for later use. Nevertheless their proofs are made a little clearer by a change of notation.

~~Proof. The existence of the single moment ( ) for arbitrarily large~~ ~~ν implies that all moments f_ν ~~exist for~~ $(\nu = 0, 1, \ldots)$ exist.~~

Define the function $F^{(m)\mp}$ over $H_i^{\mp}$ by means of the formula

$$F^{(m)\mp}(\lambda) = \frac{\Gamma(m+l)\lambda^{-l}}{\Gamma(m+1)\Gamma(l)} \int_{-\infty}^{\infty} t^m {}_2F_1(m+l, 1; m+1; \lambda^{-1}t)\, d\sigma(t)$$

so that

$$F^{(m)\mp}(\lambda) = \frac{\Gamma(m+l)\ldots}{\Gamma(m+1)\Gamma(l)} \int_{-\infty}^{\infty} t^m (\lambda - t)^{-1} {}_2F_1(1-l, 1; m+1\ (\lambda-t)^{-1}t)\, d\sigma(t)$$

The hypergeometric function occurring in the ~~above~~ latter integrand is a polynomial of degree $l-1$ in $(\lambda-t)^{-1}t$. Evidently $F^{(m)\mp}$ is analytic over $H_i^{\mp}$, and since // $f^{(m)+}(z) = \lambda F^{(m)\mp}(\lambda)$, ~~...~~ $\lambda = z^{-1}$, // $f^{(m)+}$ is analytic over H_i^+. Functions $F^{(m)+}$ are ~~similarly~~

defined over H_i^{+} by a formula similar to ( ); ~~that~~ that $f^{(m)\mp}$ is analytic over $H_i^{\mp}$ may be demonstrated in the same way.

From formula ( )

$$\lambda^{-l}F^{(0)\mp}(\lambda) = \int_{-\infty}^{\xi} (\lambda - t)^{-l}\, d\sigma(t)$$

~~It is known ( ) that and $F^{(0)-}$ is similarly defined.~~

When $l = 1$ it is known ( ) that if σ is as described in clause (ib) then the segment $[\alpha, \xi]$ of the real axis is a natural barrier of the function $\lambda^{-1}F^{(0)\mp}(\lambda)$ ~~and $\lambda^{-1}F^{(0)-}$~~ (λ). $\lambda^{-l}F^{(0)\mp}(\lambda)$ is evidently a constant multiple of the $(l-1)^{th}$ derivative of $\lambda^{-1}F^{(0)\mp}(\lambda)$: ~~thus~~ the above segment is a natural barrier of the functions $\lambda^{-l}F^{(0)}$ defined by formula ( ) for ~~general~~ $l = 1, 2, \ldots$ Setting

$$F^{(m,\tau)}(\lambda) = \sum_{0}^{\tau-1} (-1)^{m+\nu} \binom{-l}{m+\nu} f_{m+\nu}\, \lambda^{-\nu}$$

for $m, \tau = 0, 1, \ldots$ it follows from the recursion for hypergeometric functions of contiguous orders that

$$\lambda^{l} F^{(m)\mp}(\lambda) = F^{(m,\tau)}(\lambda) + \lambda^{l-\tau} F^{(m+\tau)\mp}(\lambda)$$

for $m, \tau = 0, 1, \ldots$ and $\lambda \in H^{\mp}$; $F^{(m)\mp}(\lambda)$ is a linear function of $F^{(0)\mp}(\lambda)$ with coefficients that are rational functions of λ. The segment $[\alpha, \xi]$ is ~~a~~ a natural barrier of all functions $F^{(m)\mp}$ $(m = 0, 1, \ldots)$. Relationships ( ) may now be used to prove ~~clear~~ the result of clause (ib).

~~Formula ( ) is a contour integral~~

In formula ( ), the contour of integration (i.e. the real axis) may, under the conditions of clause (ic), be deformed in such a way that the segment $(\alpha, 1)$ becomes a path lying in $H_i^{+} \cap D$. The value of $F^{(0)-}(\lambda)$ is, for $\lambda \in H_i^{-}$, thereby unchanged; furthermore, $F^{(0)-}$ is now expressed for values of $\lambda \in H_i^{+}$ lying below the deformed contour: $F^{(0)-}$ may be continued analytically into $H_i^{+} \cap D$. For a fixed $\lambda \in H_i^{+} \cap D$, the deformed contour may be restored to its original position and, if a small circle surrounding the fixed value of λ is left in position, the value of $F^{(0)-}$ for this fixed value is left unchanged. That component of the expression yielding the value of $F^{(0)-}$ which involves an integral of the form ( ) is, since λ is now in H_i^{+}, $F^{(0)+}(\lambda)$. Thus

$$F^{(0)+}(\lambda) = F^{(0)-}(\lambda) + \frac{2\pi i(-1)^{l}}{(l-1)!}\partial^{l}\sigma(\lambda)$$

for all $\lambda \in D$. Using formula ( ), it follows that

$$F^{(m)+}(\lambda) = F^{(0)-}(\lambda) + \frac{2\pi i\, \lambda^{m}(-1)^{l}}{(l-1)!}\partial^{l}\sigma(\lambda)$$

for all $\lambda \in D$; and the result of clause (ic) follows from formula ( ).

The result of clause (iia) is a direct consequence of relationships (,).

From formula ( )

$$F^{(m)\pm}(\lambda) = \frac{\Gamma(l+m)}{\Gamma(l)\Gamma(m)} \int_{-\infty}^{\infty} t^{m} \left(1-\frac{t}{\lambda}\right)^{-l} \int_0^1 u^{m-1}\left(1-\frac{ut}{\lambda}\right)^{l-1} du\, d\sigma(t)$$

$$= \frac{\lambda\,\Gamma(l+m)}{\Gamma(l)\Gamma(m)} \int_{-\infty}^{\infty} \int_0^t u^{m-1} \left\{\frac{\lambda-u}{\lambda-t}\right\}^{l} (\lambda-u)^{-1} du\, d\sigma(t)$$

for $m=1,2,\ldots$, when $\lambda = \xi - i\eta$ $(\eta \in (0,\infty))$. For all such λ and $u \in (0,\infty)$, $t \in (-\infty,\infty)$

$$\left|\frac{\lambda}{\lambda-u}\right| \leq 1+\frac{|u|}{\eta} \quad \left|\frac{\lambda-u}{\lambda-t}\right| \leq 1+\frac{|t|}{\eta}.$$

Hence

~~$\lim \lambda^{\tau}\{F^{(m)}(\lambda) - \sum_{0}^{\tau-1} (-1)^{m+i} \binom{-l}{m+i}$~~

Hence $\lim \lambda^{\tau}\{F^{(m)}(\lambda) - F^{(m,\tau)}(\lambda)\} = 0$ as λ tends to infinity

~~in the half-plane $\mathrm{Im}(\lambda) \leq \eta$. The results of clause (iia) follows immediately~~

in the half-plane $\mathrm{Im}(\lambda) \leq \eta$ for $m=1,2,\ldots$ and that this also occurs when $m=0$ is a consequence of relationship ( ). The results of clause (iia) follow immediately

~~That $F^{(m)\pm}$ are the same function, $F^{(m)}$ say, which has a~~ representation of the form ( ) over $\mathbb{C}\setminus[0,\infty]$ when σ is as described in clause (iic). follows from clause (ic). A formula similar to ( ) holds for $F^{(m)}$, and in this formula the lower limit $t=-\infty$ may be replaced by $t=\alpha$. When $\lambda \in H^-$,

$$\left|\frac{\lambda}{\lambda-u}\right|, \left|\frac{\lambda-u}{\lambda-t}\right| \leq 1$$

for all ~~u, t~~ $u \in [0,\Gamma]$, $t \in [\alpha, \infty]$. Hence, when $\lambda \in H^-$, $|F^{(m)}(\lambda)| \leq \frac{c}{m!\eta\delta}$ $(m=1,2,\ldots)$ and that this relationship also holds when $m=0$ may be verified directly, as may also the first result of clause (iic). The second result also follows from relationship ( ).

Under the conditions of clause (iid), $f_\nu = O(\gamma^{-\nu})$ and f converges to $f^{(m)}$ over D_γ. ~~The additional results concerning this convergent series may be ~~deduced~~ demonstrated by the use of formula ( ).~~

The results of clauses (iii a) follow from formula ( ), and that of clause (iiib) is a consequence of clause (iib).

When $f_{2\nu+1}=0$ $(\nu=0,1,\ldots)$, σ and σ_1, where $\sigma_1(t)=\frac{1}{2}\{\sigma(t)-\sigma(-t)\}$ generate the same moments: ~~it the ... that ... may be replaced by σ_1 in~~ formula ( ) still holds for $\nu=0,1,\ldots$ when σ is replaced by σ_1. Since σ is non-decreasing the $\{f_\nu\}$ are ~~moments of~~ Hamburger moments ~~problem~~ and, since $f_\nu = O(\nu!\,\xi'^{\nu})$ for some $\xi' \in (0,\infty)$, the associated Hamburger moment problem is determinate: ~~there is only~~

~~one bounded form~~ σ is the only bounded nondecreasing function, normalised by a condition such as $\sigma(0)=0$ to generate the moments Since σ_1 is bounded and nondecreasing, $\{b\}$. ~~Hence~~ ~~[illegible]~~ $\sigma_1=\sigma$ and $\sigma(-t)=-\sigma(t)$ for all $t\in(-\infty,\infty)$.
Formula ( ) now yields ~~[illegible]~~
~~The results of clause (iiic) may be verified as above.~~

Under the conditions of clause (iva), the function f_1 ~~produced~~ (see formula ( ) and the accompanying text) produced by the transformation of the series $f^{(m)}$ is, when $z\in i(0,\infty)$ and $|u|<1/\frac{1}{3}|z|$,

$$f_1(zu)=\frac{\Gamma(m+l)}{\Gamma(m+1)\Gamma(l)}\int_{-\infty}^{\infty}t^m{}_1F_1(m+l;m+1;uzt)\,d\sigma(t)$$

$$=\frac{\Gamma(m+l)}{\Gamma(m+1)\Gamma(l)}\int_{-\infty}^{\infty}t^m e^{uzt}{}_1F_1(1-l;m+1;uzt)\,d\sigma(t)$$

(the confluent hypergeometric function in the latter integrand is, of course, a polynomial of degree $l-1$ in uzt.
~~$f_1(zu)$ is ~~ regular for $|u|<1/\frac{1}{3}|z|$
of ~~when $z\in i(0,\infty)$,~~ $f_1(zu)$ is ~~an analytic function of u for $|u|<1/\frac{1}{3}|z|$,~~ ~~and is~~ also regular for $u\in[0,\infty)$. The function S_1 of formula ( ) yields the function $f^{(m)+}$ of formula ( ). The further results of clause (iva) are demonstrated in the same way.

Denoting the function σ considered in clause (ivb) by $\sigma(x,t)$, it may be shown, as in the proof of clause (iiic), that $\sigma(x,-t)=-\sigma(x,t)$ $(-\infty<t<\infty)$ ~~for every~~ at every point x of the infinite set ~~with limit~~ mentioned. Expanding $\sigma(x,t)$ as a power series about the limit point

$x = x'$, it is easily demonstrated that $\sigma(x, -t) = -\sigma(x, t)$ $(-\infty < t < \infty)$ for all x sufficiently close to x' and indeed, throughout

III. The function f_x (see formula ( ) and the accompanying text) produced by the transformation of the series $f^{(2m)}$ is, ~~when $z \in (0,\infty)$ and $|u| \leq 1$~~ ~~obtained by~~ $f_1(zu^{\frac{1}{2}})$, where f_1 is ~~given by~~ obtained by replacing m by $2m$ in formula ( ). Again, the function S_w of formula ( ) yields, when $z \in (0,\infty)$, the function $f^{(2m)+}$ for of formula ( ), and the further results of clause (ivb) are demonstrated in the same way.

A number of results further to those of Lemma 1 are easily derived: if σ is constant over $[-\infty, x)$ $(x \geq 0)$ and real valued and nondecreasing ~~there~~ (so that $f^{(m)\pm}$ may be regarded as the same function $f^{(m)}$, defined over $\mathbb{C} \setminus [\xi, \infty]$ by of formula ( )) $f^{(m)}$ ~~is semi-convergent~~ over H_- in the sense that in conjunction with formula ( )

$$|f^{(m,\tau)}(z)| \leq |\binom{-1}{m+\tau} d_{m+\tau}|$$

for ~~each~~ $m, \tau = 0, 1, \ldots$ and $z \in H^-$. Furthermore, when $z \in (-\infty, 0)$, the partial sums $f^{(m,\tau)}(z)$ oscillate about the value of $f^{(m)}(z)$: $f^{(m,2\tau)}(z) < f^{(m)}(z) < f^{(m,2\tau+1)}(z)$ for $m, \tau = 0, 1, \ldots$ If, in addition, σ is also constant over $[\xi, \infty]$, then for $z \in (-\xi^{-1}, 0)$,

$$f^{(m,2\tau)}(z) \leq f^{(m,2\tau+2)}(z) \leq f^{(m)}(z) \leq f^{(m,2\tau+3)}(z) \leq f^{(m,2\tau+1)}(z),$$

and for $z \in (0, \beta^{-1})$ $f^{(m,2r)}(z) \leq f^{(m,2r+1)}(z) \leq f^{(m)}(z)$ with $m, r = 0, 1, \ldots$ in both cases. ~~If only the~~ Again, if only the moments $f_\nu = 0, 1, \ldots, l$ exist, then the functions $F^{(n)\pm}(\lambda)$ are uniformly bounded over the half-planes $Im(\lambda) \geq \delta \in (0, \infty)$, $Im(\lambda) \leq \delta$ respectively.

$\boxed{\;}$ ~~relationship ( )~~ an analogue of relationship ( ) holds over $\mathbb{C} \setminus [0, \infty^{-1}]$, $G^{(m)}$ represents $f^{(m)}$ asymptotically as z tends to zero in the sector $\Delta(0, 2\pi)$, and is semi-convergent

$\boxed{\;}$ ~~If, alternatively, $b \neq 1$ and~~

If $\sigma\, t_\nu \in BV(\alpha, \infty)$ $(\alpha \geq 0)$, and ~~Ka~~ $f_\nu = O((\nu + \kappa)! \frac{1}{3}^{\nu})$ (a weaker condition that that imposed in clause (va)) and $l = 1$, then $f^{(m)}$ is both $(B', 2)$ and (B^2) summable to $f^{(m)}$ over $(-\infty, 0)$, and both $(B, 2)$ and (B^2) summable to $f^{(m)}$ over $\mathbb{C} \setminus [0, \infty]$ $(m = 0, 1, \ldots)$.

Definition $\psi \in M(M; x^-, x^+)$ means that is the class of functions ψ for which

$$\psi(z) = \sum_{\nu=1}^{4} i^{\nu} \psi_\nu(z)$$

where, for $\nu = 1, \ldots, 4$

a) $\psi_\nu \in A\{S(x^-, x^+)\}$ and

b) for all $\delta \in (0, \frac{1}{4}(x^+ - x^-))$ and $x' = \frac{1}{2}x^- + \delta$, $\frac{1}{2}x^+ - \delta$

$$\int_{x'}^{x''} \int_{x'}^{x''} \Delta^{III} \psi_\nu(x+y)\phi(x)\phi(y)\,dx\,dy \geq 0$$

for every function ϕ which is real and continuous in $[x', x'']$

Theorem (i) $\psi \in M(M; x^-, x^+)$ if and only if $\psi \in A\{S(x^-, x^+)\}$

and with

$$\sigma(v) = \sum_{\nu=1}^{4} i^{\nu} \sigma_\nu(v)$$

σ_ν being real valued and nondecreasing over $(0, \infty)$ $(\nu = 1, \ldots, 4)$,

$$\Delta^{III}\psi(z) = \int_{-\infty}^{\infty} e^{-zv}\,d\sigma(v)$$

the integral converging absolutely for $z \in S(x^-, x^+)$.

(ii) $\psi \in M(M; x^-, x^+)$ if and only if ~~ψ has the decomposition~~

$\psi(z) = \psi^+(z) + \psi^-(z)$ where $\psi^+ \in M(M; x^-, \infty)$, $\psi^- \in M(M; -\infty, x^+)$;

$\Delta^M \psi^\pm$ then have the representations

$$\Delta^M \psi^+(z) = \int_0^\infty e^{-zv} d\sigma^+(v), \quad \Delta^M \psi^-(z) = \int_{-\infty}^0 e^{-zv} d\sigma^-(v)$$

where, in terms of formula ( ), $\sigma(v) = \sigma^\pm(v)$ for $v \in \pm[0, \infty)$

(iii) $\psi \in M(M; x^-, x^+)$ if and only if $\psi(z) = \psi^+(z) + \psi^-(z)$ where

$$\psi^\pm(z) = \sum_1^4 i^\nu \psi_\nu^\pm(z)$$

and, with $\psi_\nu^+ \in A\{S(x^-, \infty)\}$, $\psi_\nu^- \in A\{S(-\infty, x^+)\}$

$$(-1)^r \Delta^{M+r} \psi_\nu^+(x) \geq 0 \qquad \Delta^{M+r} \psi_\nu^-(x) \geq 0$$

for $\nu = 1, \ldots, 4$, $r = 0, 1, \ldots$ and all $x \in (x^-, \infty)$ and $x \in (-\infty, x^+)$ respectively; $\psi^\pm$ are then the same functions as those of clause (ii) and

$$\Delta^M \psi_\nu^+(z) = \int_0^\infty e^{-zv} d\sigma_\nu^+(v), \quad \Delta^M \psi_\nu^-(z) = \int_{-\infty}^0 e^{-zv} d\sigma_\nu^-(v)$$

where $\sigma^\pm(v) = \sum_1^4 i^\nu \sigma_\nu^\pm(v)$ for $v \in \pm[0, \infty)$.

[From a theorem due to Mercer [: Mathias Bochner!]]

Proof. ~~The function~~ $\cancel{\psi_\omega}$ $\mathfrak{D}^{M}\psi_{\omega} \in A\{S(x^{-}, x^{+})\}$ and condition b) of Definition .. is satisfied if and only if $\mathfrak{D}^{M}\psi_{\omega}$ has a representation of the form ( ) in which the appropriate function σ_ω is real valued and nondecreasing over $(0, \infty)$, the integral converging for $z \in S(x^{-}, x^{+})$; since σ_ω is nondecreasing, the abscissae of convergence and absolute convergence are the same. The result of clause (i) is now a consequence of the defining relationship ( ). That of clause (ii) follows from the decomposition of ψ into two functions of the form ( ).

~~The functions $\mathfrak{D}^{M}\psi_{\omega}^{\pm}$ sati~~

From a theorem of Bernstein and Widder, the functions $\mathfrak{D}^{M}\psi_{\omega}^{\pm}$ satisfy the appropriate conditions of the form ( ) if and only if they have representations of the form ( ) as described. The result of clause (iii) now follows from that of clause (ii).

The Euler Maclaurin series

In this section an extension of the Euler Maclaurin series is introduced and its properties are described.

Notation. ρ and ψ being suitable functions, ~~set~~ ~~$e_\nu^{(\rho,n)}$ be the~~ ~~coefficients~~

$$e_\nu^{(\rho,n)} = \int_0^n \frac{\mathcal{B}^{(n)}_{n+\nu}(y)}{(n+\nu)!}\, d\rho(y)$$

for $n=1,2,\ldots;\ \nu=0,1,\ldots$. For $n=1,2,\ldots;\ j=0,1,\ldots$ let $E^{(n,j)}_{\rho;\psi,x}$ and $\hat{E}^{(n,j)}_{\rho;\psi,x}$ be the series

$$\sum{}' e^{(\rho,n)}_{j+\nu}\, \mathcal{D}^{j+\nu}\psi(x)\,\omega^{\nu}$$

and

$$\sum{}' e^{(\rho,n)}_{j+2\nu}\, \mathcal{D}^{j+2\nu}\psi(x)\,\omega^{2\nu}$$

respectively, and for $n=1,2,\ldots;\ j,\tau=0,1,\ldots$ let

$$E^{(n,j,\tau)}_{\rho;\psi,x}(\omega) = \sum_0^{\tau-1} e^{(\rho,n)}_{j+\nu}\, \mathcal{D}^{j+\nu}\psi(x)\,\omega^{\nu}$$

$$\hat{E}^{(n,j,\tau)}_{\rho;\psi,x}(\omega) = \sum_0^{\tau-1} e^{(\rho,n)}_{j+2\nu}\, \mathcal{D}^{j+2\nu}\psi(x)\,\omega^{2\nu}$$

Theorem. Let $\rho \in BV(0,1)$, and let ~~$[\hat{s},\check{s}]$ be a fixed~~ the interval $[\hat{s},\check{s}] \subseteq [-\infty,\infty]$ be fixed. Let ψ_τ $(\tau=1,\ldots,8)$ be eight functions such that, the integer $J\geq 0$ being fixed,

$$(-\mathcal{D})^{\nu}\psi_\tau(x') \geq 0$$

for $\tau=1,4,\ldots;\ \nu=J,J+1,\ldots$ and all $x'\in(\hat{s},\infty)$, and

$\omega) \partial^{\nu} \psi_{\tau}(x') \geqslant 0$

for $\tau = 5, \ldots, 8$; $\nu = 0, 1$, and all $x' \in (-\infty, \tilde{s})$. Let ψ be the function defined over $(\hat{s}, \tilde{s})$ by means of the formula

$$\psi(x) = \sum_{\tau=1}^{8} i^{\tau} \psi_{\tau}(x)$$

and over the strip $\hat{s} < Re(x) < \tilde{s}$ by analytic continuation of this function.

(i) a) For $r = J, J+1, \ldots$ $(r > 0)$, $\tau = 0, 1, \ldots$ and $\hat{s} < Re(x) < \tilde{s}$

$$e^{(\rho, n)}_{r+\tau} \partial^{r+\tau} \psi(x) = i^{\tau} \sum_{l=1}^{n} \binom{-l}{\tau} \int_{-\infty}^{\infty} t^{\tau} d\sigma(\rho; \omega\psi, x; n, \overset{r}{l}; t)$$

where $\sigma(\rho; \psi, x; n, \overset{r}{l}) \in BV(-\infty, \infty)$ is given ~~by the formula~~ by the formula

$$\cancel{\sigma(\rho; \psi, x; n, \overset{r}{l}; t) = \frac{(-1)}{2\pi i} \sum_{j=0}^{n-1} \sum^{n-l} (-1)^{j} \binom{r-j-l}{r-n} \int_{-\infty}^{\infty} \int_{0}^{n} \frac{B^{(n)}_{j}(y) \tilde{B}_{r-j+1}(y+tw)}{j!\,(r-j+1)!}}$$

$$= \frac{(-1)}{2\pi i} \sum_{j=0}^{n-1} \sum^{n-l} (-1)^{j} \binom{r-j-l}{r-n} \int_{-\infty}^{\infty} \int_{0}^{n} \frac{B^{(n)}_{j}(y) \tilde{B}_{r-j+1}(y+tw)}{j!\,(r-j+1)!} \frac{\omega) \partial^{r} \psi(x+iw) - \partial^{r} \psi(x)}{w} d_{j}(y)\, dw$$

(where $\partial = d/dx$)

for $t (\neq 0) \in (-\infty, \infty)$, and, $\sigma(\rho; \psi, x; n, \overset{r}{l}, 0) = 0$,

$$\sigma(\rho; \psi, x; n, l; \pm\infty) = \frac{(-1)}{2\pi i} \sum_{j=0}^{n-1} \sum^{n-l} (-1)^{j} \binom{r-j-l}{r-n} \int_{-\infty}^{\infty} \int \frac{B^{(n)}_{j}(y) \tilde{B}_{r-j+1}(y \pm (2\pi)^{-1} w)}{j!\,(r-j+1)!} \frac{e^{-iw} - 1}{w} dw\, d_{j}(y) * \omega) \partial^{r} \psi(x)$$

$$\varpi(\rho;\psi,x;n,l;\pm\infty)=\frac{(-1)^{n-1}}{2\pi i}\sum_{j=0}^{n-l}(-1)^{j}\binom{r-j-l}{r-n}\int_{-\infty}^{\infty}\frac{B_j^{(n)}(y)\tilde{B}_{r-j+1}(y\pm(2\pi)^{-1}w)}{j!\,(r-j+1)!}\,\frac{e^{-iw}-1}{w}\,d\rho(y)\,x^{r}\,\mathcal{D}^{r}\psi(x)$$

b) If the above conditions imposed upon the ψ_τ hold with $J=0$, and

(1) $\sum_\nu \left| \int_0^n e^{2i\nu\pi y}\, d\rho(y)\right| / \nu < \infty$, then the representation ( ) of formulae (,) also holds for $r=0$, $\tau=0,1,\ldots$.

~~(ii) Let $E^{(n,j)+}_{\rho;\psi,x}$~~

(ii) For $j=K,K+1,\ldots$ ($K=\max(J,1)$) let $E^{(n,j)+}_{\rho;\psi,x}$ be the function defined over the strip $\hat{S}<x<\tilde{S}$ and the half-plane $\omega\in H^{+}$ by means of the formula

$$E^{(n,j)+}_{\rho;\psi,x}(\omega)=\sum_{l=1}^{n}\int_{-\infty}^{\infty}\frac{d\varpi(\rho;\psi,x;n,l;t)}{(1+i\omega t)^{l}}$$

~~and define functions $E^{(n,j)-}_{\rho;\psi,x}$ over the strip for $\hat{S}<x<\tilde{S}$ and $\omega\in H^{-}$ similarly.~~

Define the functions $E^{(n,j)+}_{\rho;\psi;x}$ ($j=0,1,\ldots,K-1$ ~~and $j=0$ if $J=0$ in the above~~) by means of the formulae

$$E^{(n,K-\tau)+}_{\rho;\psi,x}(\omega)=e^{(\rho,n)}_{J-\tau}\,\mathcal{D}^{K-\tau}\psi(x)+i\omega\,E^{(n,K-\tau+1)+}_{\rho;\psi,x}(\omega)$$

for $\tau=1,\ldots,K$, ~~and the functions $E^{(n,j)-}_{\rho;\psi,x}$ ($j=0,1,\ldots,K-1$) similarly~~

a) For ~~$\omega\in H^{+}$~~ fixed $\omega\in H^{+}$, the functions $E^{(n,j)\pm}_{\rho;\psi,x}(\omega)$ ($j=K,K+1,\ldots$) have a representation of the form (,) as described (with different η_τ)

above with $\tau_j = 0$; ~~the functions~~ for $j = 0, 1, \ldots, k-1$ they have a representation as described with $\tau_j = K - j$. ~~A similar result holds for the functions $E^{(n,j)-}_{\rho,\psi,x}$.~~

b) For $j, \tau = 0, 1, \ldots$, $\hat{s} < Re(x) < \tilde{s}$ and $\omega \in H^+$

$$E^{(n,j)+}_{\rho;\psi,x}(\omega) = E^{(n,j;\tau)}_{\rho;\psi,x}(\omega) + \omega^{\tau} E^{(n,j+\tau)+}_{\rho;\psi,x}(\omega)$$

~~and a similar relationship holds for the functions $E^{(n,j)-}_{\rho;\psi,x}$~~

c) For $j = 0, 1, \ldots$ and $\hat{s} < Re(x) < \tilde{s}$

$$E^{(n,j)+}_{\rho,\psi,x}(\omega) \sim \mathcal{E}^{(n,j)}_{\rho,\psi,x}(\omega)$$

as ω tends to zero in $\Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$; $\mathcal{E}^{(n,j)}_{\rho,\psi,x}$ ~~also represents $E^{(n,j)-}_{\rho,\psi,x}$ asymptotically in a similar sense over $\Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$~~.

d) If the conditions of clause (ib) hold, $E^{(n,0)\pm}_{\rho,\psi,x}(\omega)$ have ~~by~~ ~~of the form~~ direct representations of the form ( ), and ~~the~~ use of formulae ( ) ~~such of the form to define this function~~ of the form ( ) to define these functions may be dispensed with.

~~(iii) For $0 \le x \le n$ either~~

~~A: let $\rho(n-y) = \rho(y)$ or~~

~~B: let $\rho(n-y) = 2\rho(\frac{n}{2}) - \rho(y)$~~

~~a) $e^{(\rho,n)}_{2\nu+1} = 0$ $(\nu = 0, 1, \ldots)$ in case A, and $e^{(\rho,n)}_{2\nu+2} = 0$ in case B. $(\nu = 0, 1, \ldots)$~~

(iii) Let $n' = n \bmod (2)$. For $0 \le x \le n$, ~~[illegible]~~ either

A: let $\rho(n-y) = \rho(y)$ or

B: let $\rho(n-y) = 2\rho(\frac{n}{2}) - \rho(y)$

a) $\varepsilon^{(\rho,n)}_{2\nu+n'} = 0$ in case A and $\varepsilon^{(\rho,n)}_{2\nu+n'+1} = 0$ in case B $(\nu = 0, 1, \ldots)$

b) In equation ( )

$$(\ast) \qquad \varpi(\rho;\psi,x;n,m+l;-t) = -\varpi(\rho;\psi,x;n,m,l;t)$$

for $l = 1, \ldots, n$, $t \in (-\infty, \infty)$ and all $m \ge 0$ for which $m-n$ is odd in case A and $m-n$ is even in case B

c) Let the functions $E^{(n,j)+}_{\rho;\psi,x}$ be as defined in clause (ii), and let $x \in R(\xi, \tilde{\xi})$

In case A:

α) For $j = 0, 1, \ldots$

$$E^{(n,2j+n')+}_{\rho;\psi,x}(\omega) = \omega\, E^{(n,2j+n'+1)}_{\rho;\psi,x}$$

β) For ~~$j, \tau = 0, 1, \ldots$~~ ~~$(2j(2j+n'-1) \ge 0)$~~ $j = 1-n', 2-n', \ldots, \tau = 0, 1, \ldots$

$$E^{(n,2j+n'\pm 1)+}_{\rho;\psi,x}(\omega) = \hat{E}^{(n,2j+n'\pm 1;\tau)}_{\rho;\psi,x}(\omega) + \omega^{2\tau} E^{(n,2j+2\tau+n'\pm 1)}_{\rho;\psi,x}(\omega)$$

γ) For ~~$j = 0, 1, \ldots$~~ $j = 1-n', 2-n', \ldots$

$$E^{(n,2j+n'\pm 1)+}_{\rho;\psi,x}(\omega) \sim \hat{E}^{(n,2j+n'+1)\#}_{\rho;\psi,x}(\omega)$$

as ω tends to zero in $\Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$

In case B the above results $\alpha-\gamma$ ~~(α)~~ hold with $2j$ replaced, consistently, by $2j+1$, ~~and the results β, γ hold with $2j+1$ replaced by $2j$~~

d) If the conditions ( ) hold, relationship (+) holds for all $m \geq 0$ for which $m-n$ is odd in case A and $m-n$ is odd? in case B.

b) Let the functions $E^{(n,j)\pm}_{\rho;\nu,x}$ be as defined in clause (ii), and let $\hat{s} < Re(x) < \tilde{s}$.

In case A

α) ~~*~~ For $j = 0, 1, \ldots$

$$E^{(n,2j+1)+}_{\rho;\nu,x}(\omega) = \omega E^{(n,2j+2)+}_{\rho;\nu,x}(\omega)$$

β) For $j, \tau = 0, 1, \ldots$

$$E^{(n,2j)+}_{\rho;\nu,x}(\omega) = \hat{E}^{(n,2j;\tau)}_{\rho;\nu,x}(\omega) + \omega^{2\tau} E^{(n,2j+2\tau)+}(\omega)$$

γ) For $j = 0, 1, \ldots$

$$E^{(n,2j)+}_{\rho;\nu,x}(\omega) \sim \hat{\varepsilon}^{(n,2j)}_{\rho;\nu,x}(\omega)$$

as ω tends to zero in $\Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$

In case B the above results α–γ) hold with $2j$ replaced consistently by $2j+1$.

c) If, in case B, $d_0(0) = 0$, then $e^{(\rho,n)}_0 = 0$ also, and $E^{(n,0)+}_{\rho;\nu,x}(\omega) = \omega E^{(n,1)}_{\rho;\nu,x}(\omega)$; furthermore, condition (?) automatically holds??

(iv) Functions $E^{(n,j)-}_{\rho;\nu,x}$ may be defined over H^- by a formula similar to ( ), and to all of the above results stated for $E^{(n,j)+}_{\rho;\nu,x}$ over H^+ ~~also~~ there corresponds a result holding for $E^{(n,j)-}_{\rho;\nu,x}$ holding over H^-; the asymptotic results corresponding to (,) hold over the sector $\Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$. $E^+(\omega) = E^-(-\omega)$.

Proof. It follows from a result of Hausdorff [.], S. Bernstein [ ]

and Widder that the conditions ( ) and ( ) imply that

$$\mathcal{D}^J \psi^{(\tau)}(x) = (-1)^J \int_0^\infty e^{-xv} d\sigma^{(\tau)}(v)$$

for $\tau = 1, \dots, 4$ and $\hat{S} < x < \infty$, and

$$\mathcal{D}^J \psi^{(\tau)}(x) = \int_0^\infty e^{xv} d\sigma^{(\tau)}(v)$$

for $\tau = 5, \dots, 8$ and $-\infty < x < \infty$. ~~The function represented~~ where

the $\sigma^{(\tau)}$ are real valued and nondecreasing over $[0, \infty]$ and such

that the indicated integrals exist for the stated ~~indicated~~ values of x.

The functions represented by formulae (,) are analytic for

$\hat{S} < Re(x) < \infty$ and $-\infty < Re(x) < \check{S}$ respectively, and their successive ~~derivatives~~

~~are represented by absolutely convergent integral expressions~~

obtained by differentiation of the exponential functions involved.

Changing the variable of integration in formula ( ), and using

formula ( ), it follows that for $\hat{S} < x < \check{S}$

$$\mathcal{D}^J \psi(x) = \int_{-\infty}^{\infty} e^{-xv} d\sigma(v)$$

where

$$\sigma(v) = (-1)^J \sum_1^4 i^\nu \sigma^{(\nu)}(v) + \sum_5^8 i^\nu \sigma^{(\nu)}(v)$$

and hence that for $j = J, J+1, \dots$, $\nu = 0, 1, \dots$

$$\mathcal{D}^{j+\nu} \psi(x) = (-1)^\nu \int_{-\infty}^{\infty} v^\nu d\sigma(\psi, x; j, \nu)$$

where " $d\sigma(\psi,x;j,v) = (-1)^{j-J} v^j e^{-xv} d(v)$ " $(-\infty<v<\infty)$, and $\sigma(\psi,j)\in$

~~$d\sigma(\psi,j;v) = (-1)^{j-J} v^j e^{-xv} d\sigma(v)$~~

$BV(-\infty,\infty)$.

It follows from the result of Lemma 2, clause (ii) ~~that~~

~~$$e^{(p,n)}_{m+\nu} = i^{-\nu}\sum_{l=1}^{n}\binom{-l}{\nu}\int_{-1/(2\pi)}^{1/2\pi} u^{\nu}\, d\sigma(p;n,l,m;u)$$~~

The function

$$\sigma(p;n,l,m;u) = \frac{(-1)^{n-1}}{2\pi i}\sum_{j=0}^{n-l}\binom{n+m-j-l-1}{m-1}(-1)^{j}\int_{-\infty}^{\infty}\int_{0}^{n}\frac{B_j^{(n)}(y)\tilde{B}_{n+m-j}(y+un)}{j!\,(n+m-j)!}\frac{e^{-iw}-1}{w}\,d(p(y))\,dw$$

has a saltus of magnitude

$$(-1)^{n-1} i^{-n-m} u^{n+m-1}\sum_{j=0}^{n-l}\binom{n+m-j-l-1}{m-1}\int_{0}^{n}\frac{B_j^{(n)}(y)e^{iy/u}}{j!}\,d(p(y))\,(iu)^{-j}$$

at the point $u=(\nu\pi)^{-1}$ $(\nu=\ldots-1,1,\ldots)$ and no other point of variation in the range $-(2\pi)^{-1}\le u\le 2\pi$ and hence $\sigma(p;n,l,m)\in$ $BV(-(2\pi)^{-1},(2\pi)^{-1})$ $(n=1,2,\ldots\ m=1,2,\ldots\ l=1,\ldots,n)$ (see formulae (,)).

It follows from the result of clause (ii) of Lemma 2 that for $n,m=1,2,\ldots$ $\nu=0,1,\ldots$

$$(+)\quad e^{(p,n)}_{m+\nu} = i^{-\nu}\sum_{l=1}^{n}\binom{-l}{\nu}\int_{-1/(2\pi)}^{1/(2\pi)} u^{\nu}\, d\sigma(p;n,l,m;u).$$

~~$n,m=1,2,\ldots;\ \nu=0,1,\ldots$~~

Hence

$$e^{(\rho,n)}_{m,\nu}(\cdot)^{m+\nu}\psi(u) = i^{\nu}\sum_{l=1}^{n}\binom{-l}{\nu}\int_{-1/(2\pi)}^{1/(2\pi)} u^{\nu} d\sigma(\rho;n,l,m;u)\int_{-\infty}^{\infty} v^{\nu} d\sigma(\psi,x;m;v)$$

or, setting $t = uv$, formula ( ) holds as described, with

$$\sigma(\rho;\psi,x;n,l;t) = \int_{-\infty}^{\infty} \sigma(\rho;n,l,m;\tfrac{t}{v})\,d\sigma(\psi,x;m,v)$$

~~Replacing $\sigma(\rho;n,l,m;\frac{t}{v})$ and $\sigma(\psi,x;m,v)$ by~~

Inserting the expressions (,) derived for $\sigma(\rho;n,l,m;\frac{t}{v})$ and $\sigma(\psi,x;m,v)$ in the above formula, after having replaced ~~and replacing~~ the variable of integration v by vw in formula ( ), formula ( ) is derived.

If the auxiliary condition of clause (ib) also holds, the sum of the absolute values of the saltus ( ) over the range $-(2\pi)^{-1} \le u \le u \le (2\pi)^{-1}$ is bounded when $m=0$, and $\sigma(\rho;n,l,0) \in BV(-(2\pi)^{-1},(2\pi)^{-1})$. $(n=1,2,\ldots;\ l=1,\ldots,n)$
If $J=0$ in the initial conditions of the theorem, formulae (;) also hold when $m=0$ for these values of n and l, and the result of clause (ib) follows.

~~Setting $\hat{r}=0$ in formula~~

Taking $\sigma(t)$ in the result of clause (i) of Lemma 1_z to be $\sigma(\rho;\psi,x;n,l,t)$ and setting $\hat{r}=1$, $z=-iw$ in that result, it follows that for a fixed $x \in \mathfrak{S}(\hat{S},\check{S})$, the function $E^{(n;j)}_{\rho;\psi,x}$ of formula ( ) is analytic over H^+ ($n = 1, 2, \ldots, j = 1, 2, \ldots$). Furthermore, formula ( ) may be rewritten as

$$E^{(n,j)+}_{p;\vartheta,x}(\omega) = \int_{-\infty}^{\infty} e^{-xv} d\tilde{\sigma}(v)$$

where

$$d\tilde{\sigma}(v) = (-1)^{m}\left[\omega^{m}\sum_{l=1}^{n}\int_{-\infty}^{\infty}\frac{d\sigma(\rho; n, l, m; t)}{(1+i\omega v t)^{l}}\right]d\sigma(v)$$

Since $(1+i\omega v t)^{-l}$ is uniformly bounded for fixed $\omega \in H^+$ and $v, t \in (-\infty,\infty)$, and σ have the decomposition ~~established in the proof~~ ( ), ~~it follows~~ where the $\sigma^{(\tau)}$ are such that ~~formulae of the form ( )~~ absolutely convergent integral representations of the form ( ) hold when $\tau = 1, .., 4$ and corresponding representations hold when $\tau = 5, .., 8$, with $x \in S(\hat{s}, \check{s})$ in both cases, it follows that $\tilde{\sigma}$ also has a decomposition of the form ( ), where the appropriate $\sigma^{(\tau)}$ also have properties similar to those implied by the representation ( ). The first result of clause (iia) follows immediately. The second, by τ-fold differentiation of relationship ( ).

The result of clause (iib) follows partly from the way in which the functions $E^{(n,j)+}_{p;\vartheta,x}$ have been defined when $j < k$ (namely by use of relationships ( )) and partly from our extension of the result of clause ( ) of Lemma 1.

When $j \geq k$,) ~~The result~~ of clause (iic) follows by considering linear combinations of functions with which ~~the~~ the result of clause (iib) of Lemma 1 is associated. The further results, concerning the case in which $j < k$,

~~Again, the result of clause (iid) is a consequence~~

follows from the remark that if the series $\sum i f_{\nu,1} z^{\nu}$ represents the function $f(z)$ asymptotically in a certain sector, the series $\sum i f_{\nu} z^{\nu}$ represents the function $f_0 + z f(z)$ in the same way.

~~Again,~~ the result of clause (iid) is a consequence of that of clause (iib).

~~The spe~~ That the ~~spec~~ coefficients $e_{\nu}^{(\rho,n)}$ have the special forms described in clause (iii)(a) follows from formula ( ). ~~The results of clauses (iiib and c)~~ successive results of stated in clause ~~(~~ (iiib) follow ~~either~~ from ~~a~~ those of clauses (iiia, b) of Lemma 1.

~~The periodic 1~~ Although the periodic Bernoulli functions $\bar{B}_n$ satisfy relationship ( ) ~~for ρ~~ (with $\rho = n$) for $0 < x < n$, ~~they have a saltus at the point~~ this relationship is false when $x = \nu n$ ($\nu = \ldots, -1, 0, 1, \ldots$) However, by imposing the condition $\phi(0)$ (which implies that $\phi(\nu n) = 0$ ($\nu = -1, 0, 1, \ldots$), it follows that $e_{2\nu}^{(\rho,n)} = 0$ ~~when $\nu = 0$ also.~~ not only when $\nu > 0$ but also when $\nu = 0$.

Functions $E_{\rho;\nu,x}^{(n;j)-}$ are equally well defined over H^{-} by a formula analogous to ( ), and the results stated in clause (iv) follow immediately.

~~In case A~~

It follows from formulae (.) that in case A ~~in case A~~ ~~for all t∈~~

(for $m = 2, \ldots, n+1$, $l = 1, \ldots, n+1$)

~~σ~~ $\sigma(\rho;\psi,x;n,l;t)$ is expressible ~~for~~ all $t \in (-\infty,\infty)$ in the form

$$\cancel{2(-1)^{n-1} i^{n-m+1} \sum_1' \left[\sum_{j^*=0} \binom{n+m-2j^*-l-1}{m-1} (-1)^{j^*} (2\nu\pi)^{-n-m+2j^*+1} \int_0^{\frac{1}{2}n} \frac{B_{2j^*}^{(n)}(y)}{(2j^*)!} \sin(2\nu\pi y)\, d\rho(y) \right.}$$

$$\cancel{\left. + \sum_{j^*=0} \binom{n+m-2j-l-2}{m-1} (-1)^{j} (2\nu\pi)^{-n-m+2j+2} \int_0^{\frac{1}{2}n} \frac{B_{2j+1}^{(n)}(y)}{(2j+1)!} \cos(2\nu\pi y)\, d\rho(y) \right] \{\sigma}$$

$$2(-1)^{n-1} i^{n-m+1} \sum_1' \left[\sum_{j=0}^{\left[\frac{1}{2}(n-l)\right]} \binom{n+m-2j-l-1}{m-1} (-1)^{j} (2\nu\pi)^{-n-m+2j+1} t_j'(\nu) \right.$$

$$\left. + \sum_{j=0}^{\left[\frac{1}{2}(n-l-1)\right]} \binom{n+m-2j-l-2}{m-1} (-1)^{j} (2\nu\pi)^{-n-m+2j+2} t_j''(\nu) \right] \left\{ \sigma(\psi,x;m;2\nu\pi t) + (-1)^{n-m} \sigma(\psi,x;m;-2\nu\pi t) \right\}$$

~~where $t(i) = \frac{1}{2}$~~ where

$$t_j'(\nu) = \int_0^{\frac{1}{2}n} \frac{B_{2j}^{(n)}(y)}{(2j)!} \sin(2\nu\pi y)\, d\rho(y), \quad t_j''(\nu) = \int_0^{\frac{1}{2}n} \frac{B_{2j+1}^{(n)}(y)}{(2j+1)!} \cos(2\nu\pi y)\, d\rho(y)$$

In case B, ~~a~~ similar formula holds: i^{n-m+1} and $(-1)^{n-m}$ in expression ( ) must be replaced by i^{n-m} and $(-1)^{n-m+1}$, and $\sin(2\nu\pi y)$ and $\cos(2\nu\pi y)$ in formulae ( ) must be replaced by $\cos(2\nu\pi y)$ and $-\sin(2\nu\pi y)$ respectively. The results of clause (iiib) are evidently correct. The successive results stated in clause (iiic) follow from those of clauses (iiia, b) of Lemma 1.

Theorem . Let n, ρ, ψ, x and the functions $\{E^{(n,j)\pm}_{\rho;\psi,x}\}$ be as described in Theorem and let $x \in S(\hat{s}, \check{s})$

(i) For $j = 0, 1, \ldots$ $\mathcal{E}^{(n,j)}_{\rho;\psi,x}$ is a) summable $(B', 1)$ to $E^{(n,j)+}_{\rho;\psi,x}$ over $[0,\infty)$ and ~~to $E^{(n,j)-}_{\rho;\psi,x}$ over $(-\infty, 0]$ and~~

b) summable $(\bar{B}, 1)$ to $E^{(n,j)+}_{\rho;\psi,x}$ over H^+, ~~and to $E^{(n,j)-}_{\rho;\psi,x}$ over H^-~~

ii) Let $n' = n \bmod (2)$ and let the conditions of clause ( ) of Theorem ~~with $n' = n \bmod (2)$~~ hold. In case A of that clause and for $j = \ldots 1 - n', 2 - n', \ldots$, $\mathcal{E}^{(n, 2j+n'-1)}_{\rho;\psi,x}$ is

a) summable $(B', 2)$ and summable (B^2) to $E^{(n,2j+n'-1)+}_{\rho;\psi,x}$ over $[0,\infty)$ and ~~to $E^{(n,2j+n'-1)-}_{\rho;\psi,x}$ over $(-\infty, 0]$ and~~

b) summable $(\bar{B}, 2)$ and summable $(\bar{B}^2)$ to $E^{(n,2j+n'-1)+}_{\rho;\psi,x}$ over H^+. ~~and to $E^{(n,2j+n'-1)-}_{\rho;\psi,x}$ over H^-~~

In case B, the above results hold with $2j$ replaced consistently by $2j+1$

(iii) The Results corresponding to those of clause (i) may be formulated for the functions $E^{(n,j)-}_{\rho;\psi,x}$ by replacing $[0,\infty)$ and H^+ by $(-\infty, 0]$ and H^-; results corresponding to those of clause (ii) for these functions may be obtained in a similar manner.

Proof. For formula (1) $e^{(p,n)\,(m\geq 1)}_{(m+2)}$ is expressible as the sum of ~~k-consor~~ constituents $e^{(l)}_{\nu}$ $(l=1,\dots,n)$ for each of which an order

~~Proof. Setting~~

~~$e^{(l)}_{\nu} = i^{-\nu}(-\zeta_{\nu}^{\iota})$~~

relationship of the form $e^{(l)}_{\nu} = O((2\pi)^{-\nu})$ holds. Since f is analytic over $S(\hat{s},\tilde{s})$, $\mathcal{D}^{\nu} f(x) = O(\nu!\,\delta^{-\nu})$, where $\delta \in (0, \min\{Re(x)-\hat{s}, \tilde{s}-Re(x)\})$. Hence for $m \geq \max(1, J)$, $\mathcal{E}^{(n,m)}_{\rho,\omega,x}$ may be expressed as a linear combination of n series of the type considered in clause ( ) (with $z = i\omega$) of Lemma ..., and is $(B',1)$ summable as described in clause (ia) of the present theorem. If the series $\sum f_{\nu+1}$ is summable $(B',1)$ to S, the series $\sum f_{\nu}$ is $(B',1)$ summable to $f_0 + S$ ([ ] of). Use of this ~~result~~ property of $(B',1)$ summation extends the result ~~proved so~~ just proved to ~~values of m~~ nonnegative values of $m < \max(1, J)$: ~~and~~ clause (ia) has been dealt with. The remaining results of the theorem are demonstrated in the same way.

signs (ii) the partial sums of $\varepsilon^{(J)}_{\psi,\mu}$ oscillate about $R^{(2J,0)}_{\tilde{\rho},\psi,0}$ and (iii): $\hat{\varepsilon}^{(J)}_{\psi,\mu}$ is semi-convergent ~~$|R^{(2J,2j)}_{\tilde{\rho},\psi,0}(1)| <$~~ in the sense that ~~(or some notation)~~ H

$$|R^{(2J,2j)}_{\tilde{\rho},\psi,0}(1)| < \left| \frac{B_{2J+2j+2}}{(2J+2j+2)!} \mathcal{D}^{2J+2j+1}\psi(\mu) \right| \qquad (j=0,1,\dots)$$

Hardy ( ) has also shown that without the above restrictions a–d) upon ψ, but with ψ assumed analytic in the finite part of the half-plane $Re(\mu) > -\delta$ ($\delta\in(0,\infty)$) and $\psi(\mu)=O(\mu^s)$ for large μ in this region, the intercalated version of $\varepsilon^{(K)}_{\psi,\mu}$

$$\frac{B_{2J+2}}{(2J+2)!}\mathcal{D}^{2J+1}\psi(\mu) + 0 + \frac{B_{2J+4}}{(2J+4)!}\mathcal{D}^{2J+3}\psi(\mu) + 0 + \cdots$$

(This is simply $\varepsilon^{(2J)}_{\tilde{\rho},\psi,\mu}$ with the zero terms reinserted and the argument set equal to unity) is, when $\mu=0$, $(B',1)$ summable to $R^{(2J,0)}_{\tilde{\rho},\psi,0}$ at the point ~~$\mu=$~~ 1 (in a sense readily to be deduced ~~understood~~ from the theory ~~description~~ of $(B',1)$ summability given in §2, ~~and referring to a~~ which refers to ~~a sector of~~ $(B',1)$ summability over a sector, rather than at a single point). He remarks that $\hat{\varepsilon}^{(J)}_{\psi,\mu}$ itself is ~~(B^2) summable~~, when $\mu=0$, (B^2) summable (again in a sense ~~slightly diff.~~ differing slightly from that of §2) to

Theorem . Let $[\hat{s}, \tilde{s}] \subseteq [-\infty, \infty]$ ~~and~~ $\cancel{x \in (\hat{s}, \tilde{s})}$. Let ψ_1, ψ_2 be two functions, such that, the integer $J \geq 0$ being fixed, ψ_1 satisfies ~~relat~~ conditions of the form ( ) over $(\hat{s}, \infty)$ and ψ_2 conditions of the form ( ) over $(-\infty, \tilde{s})$. With $m \geq \max(1, J)$ and $\phi' \in [0, 2\pi)$, let $\psi(x) = e^{i\phi'}\{\psi_1(x) + (-1)^{m-J}\psi_2(x)\}$. With ρ to be further specified below, and $x \in (\hat{s}, \tilde{s})$ let the functions $F^{(n,m)}_{\rho;\psi,x}$. B_j be as defined in ~~Theorem~~ clause ( ) of Theorem . Let $\phi'' \in [0, 2\pi)$ and $n' = n \bmod (2)$.

(i) Let $\rho(n-y) = \rho(y)$ for $0 \leq y \leq n$, and

$$e^{-i\phi''}(-1)^j \int_0^{\frac{1}{2}n} B^{(n)}_{2j}(y)\sin(2\pi\omega y)\,d\rho(y), \quad e^{-i\phi''}(-1)^{j} \int_0^{\frac{1}{2}n} B^{(n)}_{2j+1}(y)\cos(2\pi\omega y)\,d\rho(y)$$

~~for n = 1, 2, ... and~~

be real and nonnegative for $n = 1, 2, \ldots$ and ~~$0 \leq 2j \leq$~~ $j = 0, \ldots, [\frac{1}{2}n]$ in the first integral and $j = 0, \ldots, [\frac{1}{2}n - 1]$ in the second. Let $m = 2j + n' - 1$ $(j \geq 1 - n')$

a) The series $\mathcal{E}^{(n,m)}_{\rho;\psi,x}$ is semi convergent over $\bar{\Delta}(-\frac{1}{4}\pi, \frac{1}{4}\pi)$ in the sense that ~~for j ≥ max~~ relationship ( ) holds and

$$|\omega^{2\tau} E^{(n,m+2\tau)+}_{\rho;\psi,x}(\omega)| \leq |e^{(n,\rho)}_{m+2\tau} \Delta^{m+2\tau}\psi(x)|$$

when $\omega \in \bar{\Delta}(-\frac{1}{4}\pi, \frac{1}{4}\pi)$, $(\tau = 0, 1, \ldots)$

b) When $\omega \in [0, \infty)$, the partial sums of $\mathcal{E}^{(n,m)}_{\rho;\psi,x}$ oscillate about the value of $E^{(n,m)+}_{\rho;\psi,x}(\omega)$: setting $c = \exp[i\{\frac{\pi}{2}(n+m+1) - \phi' - \phi''\}]$

~~$c = \exp\left[i\left\{\tfrac{1}{2}\pi(n+m+2)\right\} - \phi' - \phi''\right]$~~

$$cE^{(n,m;2\tau-1)}_{\rho;\psi,x}(\omega) < cE^{(n,m)+}_{\rho;\psi,x}(\omega) < cE^{(n,m;2\tau)}_{\rho;\psi,x}(\omega)$$

~~for $\tau = 0,1,\ldots$, when we~~ for $\tau = 0,1,\ldots$

c) If the conditions of clause ( ) of Theorem I hold, ~~if~~ $J=0$ and ~~$m=0$~~ n is odd, the results of a, b) above also hold with $m=0$.

(ii) Let $\rho(n-y) = 2\rho(\tfrac{1}{2}n) - \rho(y)$ for $0 \le y \le n$, and let the values

~~$e^{-i\phi''}(-1)^j \int_0^{\frac{1}{2}n} \mathfrak{Z}^{(n)}_{j}(y)\cos(2\pi m y)\,d\rho(y),\ e^{-i\phi}\ldots$~~

of the integrals ( ), with ~~cos~~ $\sin(2\pi y)$ and $\cos(2\pi y)$ interchanged and $(-1)^j$ replaced by $(-1)^{j+1}$ in the second, be nonnegative as described in connection with expressions ( ). Let $m = 2j + n'$ $(j \ge 1-n')$. For the newly ~~defined~~ value of m, the results a) ~~and b)~~ a,-c) (with n even in the case (c), m odd ...) above hold, ~~and with $c = \exp[i\{\tfrac{1}{2}\pi(n+m+2\tau)\} - \phi' - \phi'']$~~

(iii) Results similar to those stated above also hold with regard to the functions $E^{(n,m;\ldots)-}_{\rho;\psi,x}$ ~~etc~~ over $\bar{\Delta}(\tfrac{3}{4}\pi, \tfrac{5}{4}\pi)$ and $(-\infty, 0]$.

~~Proof. The function $e^{i\phi'}\mathfrak{D}^m\psi(x)$ has a representation of the form ( ) in~~

Proof. The function $(-1)^{m+J} e^{-i\phi'} \mathfrak{D}^m \psi(x)$ has a representation of the form ( ) in which σ is nondecreasing over $(-\infty, \infty)$. Subject to the conditions stated in clause (i), it follows from formulae (,) that ~~the for the functions~~ in formula ( ), $c\mathfrak{S}(\rho;\psi,x;n,m,l,t)$

is non-decreasing for $0 \leq t < \infty$, for $l = 1, \ldots, n$. The methods of proof used to establish the results of clauses ( ) of Lemma 1 are easily ~~can be~~ extended so as to concern linear combinations ~~of series of the type ...~~, with positive real coefficients, of the series of the type considered, and the results of clause (i) of the present theorem follows immediately.

When $\overset{\sim}{\underset{1}{\mathfrak{I}}}(0)=\overset{\sim}{\underset{1}{\mathfrak{I}}}(1)=0$, $\overset{\sim}{\underset{1}{\mathfrak{I}}}(x)=-\frac{1}{2}$ $(0<x<1)$, the coefficients of even powers of the argument in the $\overset{\wedge}{\underset{\sim}{\mathfrak{S}}}{}^{(1)}_{\mu,\psi,\mu}$ are identically zero, and when $k=1$ this series becomes the restricted Euler-Maclaurin series

$$\overset{\wedge}{\mathfrak{S}}{}^{(1)}_{\psi,\mu} = \sum_{j=1}^{\infty} \frac{B_{2j+2}}{(2j+2)!} \mathfrak{D}^{2j+1}\psi(\mu)$$

This series, and its delayed form,

$$\overset{\wedge}{\mathfrak{S}}{}^{(k)}_{\psi,\mu} = \sum_{j=1}^{\infty} \frac{B_{2k+2j+2}}{(2k+2j+2)!} \mathfrak{D}^{2k+2j+1}\psi(\mu)$$

$(1\leq k<\infty)$ ~~has~~ have been studied in detail by Hardy ([ ]).

Using properties of the periodic Bernoulli polynomial, he showed that if a) $\psi(\mu)$ is real for $0\leq\mu<\infty$ and, b) for all for $j=J, J+1, \ldots$, b) $\mathfrak{D}^{2j+1}\psi(\mu)$ tends to zero as μ tends to infinity, c) $\int_0^\infty |\mathfrak{D}^{2j+2}\psi(\mu)|\,d\mu<$, and d) $\mathfrak{D}^{2j+2}\psi(\mu)$ is of fixed sign for all $\mu\in[0,\infty)$ (this condition implies (see the footnote to p327 of [ ]) that, depending upon the sign of $\mathfrak{D}^{2J+2}\psi(\mu)$, either $(-\mathfrak{D})^{j}\psi(\mu)\geq 0$ for all $\mu\in[0,\infty)$ and $j=2J+1, 2J+2,\ldots$ or $(-\mathfrak{D})^{j}\psi(\mu)\leq 0$ for these μ and j) the following results hold: ~~(i)~~ when $\mu=0$ and $k=1$ (i) the terms of $\overset{\wedge}{\mathfrak{S}}{}^{(J)}_{\psi,\mu}$ alternate in

Let ρ be as described in Theorem 1.

Theorem . With $\hat{s} = -\infty$ and $\tilde{s} = \infty$, let the functions ψ_τ and ψ be as described in Theorem , and let an interval $[\alpha, \beta] \subset (-\infty, \infty)$ exist such that $\psi_\tau(-x) = O(e^{\alpha x})$, $\psi_\tau(x) = O(e^{-\alpha x})$ for large ~~real~~ positive real x $(\tau = 1, \ldots, \tau)$. Let $\gamma = \max\{|\alpha|, \beta\}/2\pi$. For the resulting function ψ of formula ( ), let $E^{(n,m)\pm}_{\rho;\psi,x}$ be as defined in Theorem ( ) $(l = 1, \ldots, n)$ in formula ( )

(i) The functions $\sigma(\rho;\psi,x;n,m,l)$ are constant over $[-\infty, -\gamma)$ and $(\gamma, \infty]$ $(m \geq 1, J$; $m = J, J+1, \ldots$ $(m > 0))$, and if $J = 0$ and conditions ( ) also hold, this is also true of the functions $\sigma(\rho;\psi,x;n,0,l)$.

(ii) For $m \geq 1, J$, $m \neq 0, 1$, the functions $E^{(n,m)\pm}_{\rho;\psi,x}$ are the same function in the sense that $E^{(n,m)-}_{\rho;\psi,x}$ is obtained from $E^{(n,m)+}_{\rho;\psi,x}$ by analytic continuation across the segment $i(-\gamma', \gamma')$ of the imaginary axis, and for $m \geq 1, J$, this function has the representation

$$E^{(n,m)}_{\rho;\psi,x}(\omega) = \sum_{l=1}^{n} \int_{-\gamma}^{\gamma} \frac{d\sigma(\rho;\psi,x;n,m,l)}{(1 - i\omega t)^l}$$

(iii) a) $\mathcal{E}^{(n,m)}_{\rho;\psi,x}$ converges to $E^{(n,m)}_{\rho;\psi,x}$ over $\mathbb{D}_{1/\gamma}$ $(m = 0, 1, \ldots)$

(iv) ~~Let $n = n' \bmod(2)$ and~~ Let ψ be as described in Theorem ..., and let $n = n' \bmod(2)$ ~~and m~~

a) Let $\rho(n - y) = \rho(y)$ for $0 \leq y \leq n$. For $m = \cancel{J, J+1,}$ ~~odd~~ $m - 2j + n' - 1$ $(j \geq 1 - n')$ and c be as defined in clause ( ) of Theorem , then for $\omega \in (-\gamma, \gamma)$, the ~~partials~~ even and odd order partial sums of $\hat{\mathcal{E}}^{(n,m)}_{\rho;\psi,x}$ are members of monotonically increasing and decreasing sequences respectively in the sense that, ~~for $\tau = 0, 1, \ldots$~~

b) Let $\rho(n-y)=\rho(y)$ for $0\le y\le n$. ~~Then~~ For $j=1-n', 2-n', \ldots$ $\hat{E}^{(n,2j+n'-1)}_{\rho>4,x}$ converges to $E^{(n,2j+n'-1)}_{\rho>4,x}$ over $D_{1/\delta}$.

c) Let $\rho(n-y) = 2\rho(\tfrac{1}{2}n) - \rho(y)$. The preceding result with $2j$ replaced by $2j+1$ holds.

(iv) Let the conditions of clause (i) of Theorem ... be satisfied, with m and c as in that clause

a) When $\omega\in(-\delta,\delta)$ the even and odd order partial sums of $\hat{E}^{(n,m)}_{\rho>4,x}$ are members of monotonic~~ally~~ sequences of ~~contrary senses~~ opposing type ~~increasing and decreasing sequences~~ respectively, in the sense that

$$cE^{(n,m;2\tau-1)}_{\rho>4,x}(\omega) < cE^{(n,m;2\tau+1)}_{\rho>4,x}(\omega) < cE^{(n,m)}_{\rho>4,x}(\omega) < cE^{(n,m,2\tau+2)}_{\rho>4,x}(\omega) < cE^{(n,m;2\tau)}_{\rho>4,x}(\omega)$$

for $\tau = 0,1,\ldots$

b) When $\omega\in i(\epsilon\delta, 4\delta)$, the partial sums of $\hat{E}^{(n,m)}_{\rho>4,x}$ form a monotonic sequence in the sense that

$$cE^{(n,m,\tau-1)}_{\rho>4,x}(\omega) < cE^{(n,m,\tau)}_{\rho>4,x}(\omega) < cE^{(n,m)}_{\rho>4,x}(\omega)$$

for $\tau=0,1,\ldots$

(v) Let the conditions of clause (ii) of Theorem — be satisfied, with m and c as in that clause. Then ~~the~~, with these modifications, the results of iv a–c (with n even in the case of c) above hold.

(c) If the conditions ( ) of Theorem - also hold, $J=0$ and ~~m=0~~ $m=0$ the results of a, b) above also hold with $m=0$

Proof. Under the conditions imposed upon the ψ_γ, formula ( ) ~~may be replaced by~~

$$\Sigma_{\nu}^{J}(\tau) = \int_{-\infty}^{-\beta} e^{-\tau\nu} d\sigma(\nu).$$

the function σ occurring in formula ( ) is constant over $(-\infty, \alpha)$ and (β, ∞). The result of clause (i) now follows directly from formula ( ). That the ~~functions~~ $E^{(n,m)\pm}_{\rho,\gamma,x}$ $(m \geq 1, J)$ both have a representation of the form ( ) now follows from formula ( ). The function $E^{(n,m)}_{\rho,\gamma,x}$ so defined is analytic over $\mathbb{D}_{1/\gamma}$ $\mathbb{D}(\pi\gamma^{-1})$ and analytic continuation ~~...~~ of this function across the segment $i(-\gamma^{-1}, \gamma^{-1})$ is of course possible. For ~~$m < 1, J$~~, $m < \min(1, J)$, the ~~function~~ $E^{(n,m)\pm}_{\rho,\gamma,x}$ is a linear function of $E^{(n,k)\pm}_{\rho,\gamma,x}$ with ~~coefficients~~ polynomials in x as coefficients; ~~and~~ $E^{(n,m)\pm}_{\rho,\gamma,x}$ reduce to a single function for all nonnegative m. $E^{(n,m)}_{\rho,\gamma,x}$ now reduces to the Taylor expansion of $E^{(n,m)}_{\rho,\gamma,x}$ and ~~naturally converges to the value of this function over $\mathbb{D}(\gamma^{-1})$.~~ the result of clause (iii a) follows directly, as do ~~The~~ the results of clauses (iii b, c). The results of clauses (iv) are refinements, ~~of those of~~ (made possible by the ~~same~~ restrictions imposed upon ψ) of those of clauses (,) of Theorem , and are derived in the same way.

Theorem . Let ρ, σ, $[\hat{s}, \tilde{s}]$, $\{\psi_\tau\}$ and ψ be as defined in theorem , and $x \in \mathfrak{S}(\hat{s}, \tilde{s})$.

i) Let σ_τ be nonanalytic over an interval of the form $[v, \infty)$ $(v \in [0, \infty))$ in any one of the integral representations for $\mathfrak{L}^{\sigma}\psi_\tau(x)$ of formulae (.). The imaginary axis is a natural barrier for all functions $E^{(n,m)\pm}_{\rho,\psi,x}$ defined in clause ( ) of theorem : analytic continuation of these functions across this boundary is impossible.

(ii) In all of the representations for $\mathfrak{L}^{\sigma}\psi_\tau(x)$ of formulae (,) let σ_τ be analytic over ~~ ~~ the sector $\Delta(-\phi, \phi)$, with $\lim \{v^{y}\sigma_\tau(v)\}$ ~~$=0$,~~ ~~v being arbitrarily large~~ ~~$y \in (0,\infty)$ being arbitrarily~~ $= 0$, with $y \in (0,\infty)$ arbitrarily large, ~~for all $y \in (0,\infty)$~~ as v tends to infinity in this sector.

a) The functions $E^{(n,m)}_{\rho,\psi,x}$ ~~may be~~ defined in clause ( ) of Theorem may be continued analytically across both the positive and negative parts of the imaginary axis $(m = 0, 1, \ldots)$.

b) Denoting by $E^{(n,m)+}_{\rho,\psi,x}$ the functions directly defined over H^+ in clause ( ) of Theorem and also obtained by analytic continuation as above, $E^{(n,m)+}_{\rho,\psi,x}$ is analytic over $\Delta(-\frac{1}{2}\pi - \phi, \frac{1}{2}\pi + \phi)$, and

$\mathcal{E}^{(n,m)}_{p+4,\pi}$ represents this function asymptotically as its argument tends to zero in this sector $(m=0,1,\ldots)$.

c) With $E^{(n,m)-}_{p+4,x}$ similarly defined, this function is analytic over $\Delta(\frac{1}{2}\pi-\phi, \frac{3}{2}\pi+\phi)$, and $\mathcal{E}^{(n,m)}_{p+4,x}$ also represents this function as above over this sector $(m=0,1,\ldots)$

d) Let $n'=n \bmod(2)$; $n'=res(n,2)$; If $\rho(n-y)=\rho(y)$ $(0\leq y\leq n)$, $\hat{\mathcal{E}}^{(n,2j+n'-1)}_{p+4,x}$ $(j=1-n', 2-n',\ldots)$ represents $E^{(n,2j+n'-1)\pm}_{p+4,x}$ as described above ~~...~~, and if $\rho(n-y)= 2\rho(\frac{1}{2}n)-\rho(y)$ $(0\leq y\leq n)$ this result with $2j-1$ replaced by $2j$ holds

Proof. Under the conditions of clause (i), the function σ in formula ( ) is nonanalytic (either over) $[\nu,\infty)$ or $(-\infty,\nu]$; the same is true of the function $\sigma(\gamma,x)$ of formula ( ). Hence all functions $\sigma(\rho;n,m,l)$ occurring in formulae (.) are nonanalytic over $(-\infty,\infty)$, and, by a simple extension of the result of clause ( ) of Lemma 1, the result of clause (i) follows.

The conditions imposed upon $\sigma_x(\nu)$ in clause (ii) are also satisfied by the derivatives $\mathcal{D}^{k}\sigma_x(\nu)$ $(k=1,\ldots,n)$. Thus ~~$\lim\{t^{\nu},\sigma($~~ $\lim\{t^{\nu}\mathcal{D}^{l}\sigma(\rho,n,m,l;t)\}=0$. $\sigma(\rho,n,m,l)$ is analytic over the sectors $\Delta(-\phi,\phi)$ and $\Delta(\pi-\phi,\pi+\phi)$ $(m\geq \max(1,j), l=1,\ldots,n)$

and $\overline{\lim}\, t^{y} \mathcal{D}^{l} \sigma(\rho, n, m; l; t) = 0$ for these values of m and l as t

tends to infinity ~~over~~ in these ~~two~~ two sectors. The result of clause

(iia) follows simply from clause ( ) of Lemma 1 and those of clauses (iibc)

~~and clause (iid)~~

~~from clause ( ) of that lemma~~

and clause (iid) from clauses ( ) and ( ) respectively of that lemma

Theorem . Let ρ be as described in Theorem 1 with $r=1$, let $[\hat{S}, \hat{\hat{S}}] \subseteq [-\infty, \infty]$ and let j, ψ_1 and ψ_2 be as described at the commencement of Theorem . Let $\phi'' \in [0, 2\pi)$. Subject to conditions given below which ensure their existence, denote the approximating functions derived from the series $\mathcal{E}^{(1,m)}_{\rho,\psi,x}$ by $\{P^{(m)}_{i,j}\}$ and those derived from the series $\hat{\mathcal{E}}^{(1,m)}_{\rho,\psi,x}$ by $\{\hat{P}^{(m)}_{i,j}\}$. ~~Let $\phi'' \in [0, 2\pi)$.~~ [illegible]

(i) Let $m \geq \max(j, 1)$, $2^J \psi(x) = \psi_1(x) - (-1)^{m+J} \psi_2(x)$ and $x \in (\hat{S}, \hat{\hat{S}})$. Let

$$e^{i\phi''} \int_0^1 e^{2i\nu\pi} d\rho(y) \, , \quad (-1)^m e^{i\phi''} \int_0^1 e^{-2i\nu\pi} d\rho(y)$$

be real and nonnegative for $\nu = 0, 1, \ldots$.

a) All diagonal sequences of the form $\{P^{(m)}_{i,k+i}\}$, $\{P^{(m)}_{k+i+a,i}\}$ with increasing i and $k = 0, 1, \ldots$ fixed converge uniformly to $E^{(1,m)+}_{\rho,\psi,x}$ over any bounded domain lying in H^+ and to $E^{(1,m)-}_{\rho,\psi,x}$ over any bounded domain lying in H^-

b) Let ψ_1, ψ_2 also satisfy the conditions given at the commencement of Theorem , with γ as described, so that $E^{(1,m)\pm}_{\rho,\psi,x}$ are the same function $E^{(1,m)}_{\rho,\psi,x}$. The diagonal sequences of the form $\{P^{(m)}_{i,2k+i+1}\}$, $\{P^{(m)}_{2k+i+1,i}\}$ with increasing i and $k = 0, 1, \ldots$ fixed now converge

uniformly to $E^{(1,m)}_{p\nu 4,x}$ over any bounded domain lying in $\mathbb{C}\setminus\{i[\tau^{-1},\infty],$
$i[-\infty,-\tau^{-1}]\}$.

c) Let $j=0$, $\sum' \left| \int_0^1 e^{2i\nu\pi y} d\rho(y)/\nu \right| < \infty$ and $\psi(x) = \psi_1(x) - \psi_2(x)$.
The results of a, b) above now hold with regard to the functions
$E^{(1,0)\pm}_{p\nu 4,x}$ and $E^{(1,0)}_{p\nu 4,x}$ respectively.

(ii) Let ${}_{\nu}\!\psi^{j}(x) = \psi_1(x) + (-1)^j \psi_2(x)$, let $\rho(1-y) = \rho(y)$ $(0 \leq y \leq 1)$, let

$$e^{i\psi''} \int_0^{\frac{1}{2}} \sin(2\nu\pi y)\, d\rho(y)$$

be real and nonnegative for $\nu = 0,1,\ldots$, and let $2m+1 \geq \max(j,1)$

a) ~~For $2m+1 \geq \max(j,1)$~~, $P^{(2m+1)}_{2i+\tau,2j+\tau'}(\omega) = \hat{P}^{(2m+1)}_{i,j}(\omega^2)$ $(i,j = 0,1,\ldots;\tau,\tau' = 0,1)$

b) All diagonal sequences of the form $\{\hat{P}^{(2m+1)}_{i,k+i}\}$, $\{\hat{P}^{(2m+1)}_{k+i+1,i}\}$ with increasing i and $k = 0,1,\ldots$ fixed converge uniformly to $E^{(1,2m+1)+}_{p\nu 4,x}$ over any bounded lying in H^+ and to $E^{(1,2m+1)-}_{p\nu 4,x}$ over any bounded domain lying in H^-.

c) Let ψ_1, ψ_2, γ and $E^{(1,m)}_{p\nu 4,x}$ be as defined in clause (ib). All diagonal sequences of the form $\{P^{(2m+1)}_{i,k+i}\}$, $\{P^{(2m+1)}_{k+i+1,i}\}$, $\{\hat{P}^{(2m+1)}_{i,k+i}\}$, $\{\hat{P}^{(2m+1)}_{k+i+1,i}\}$ with increasing i and $k = 0,1,\ldots$ fixed now converge uniformly to $E^{(1,2m+1)}_{p\nu 4,x}$ over any bounded domain lying in $\mathbb{C}\setminus\{i[\tau^{-1},\infty], i[-\infty,-\tau^{-1}]\}$.

(iii) Let $\Delta^{J}\psi(x) = \psi_1(x) - (-1)^{J}\psi_2(x)$, let $\rho(1-y) = 2\rho(\frac{1}{2}) - \rho(y)$ $(0<y<1)$,

let

$$e^{i\phi''}\int_0^{\frac{1}{2}} \cos(2\nu\pi y)\,d\rho(y)$$

be real and nonnegative for $\nu = 0, 1, \ldots$, and let $2m \geq \max(J, 1)$.

a) With $2m+1$ consistently replaced by $2m$, all results of clause (iia-c) hold.

b) Under the additional assumptions of clause (ic), they also hold with $2m+1$ replaced by zero.

Proof. As in the proof of Theorem —, it may be shown that under the conditions of clause (i), $\epsilon^{(\rho,1)}_{m,\nu}\, c^{m+J}\Delta\psi(x) = c f_\nu$ $(\nu = 0, 1, \ldots)$ where c is a nonzero constant, ~~which need no~~ and f_ν has the form ( ) in which σ is a bounded nondecreasing function. As was demonstrated in the proof of Theorem —, the $\{f_\nu\}$ in this case satisfy the relationship $f_\nu = O(\nu!\,\xi^{\nu})$ where $\xi \in (0, \infty)$. The author has shown, concerning the approximating functions $\{P_{i,j}\}$ derived from ~~this series, that under these conditions all diagonal sequences of~~ a series whose coefficients $\{f_\nu\}$ are as just described, all diagonal sequences of the form $\{P_{i,k+i}\}$, $\{P_{k+i+1,i}\}$ with increasing i and $k = 0, 1, \ldots$ fixed, converge uniformly, to the function defined by the integral expression (??) over H^{i+}, (over any bounded domain in H^{i+}) and behave over H^{i-} in the same way. The approximating fractions derived from the series with coefficients $\{c f_\nu\}$ ~~(…)~~ are of course $\{c P_{i,j}\}$. The result of clause (ia) follows immediately. If, in the preceding discussion, $f_\nu = O(\gamma^{-\nu})$ for ~~some~~ $\gamma \in (0, \infty)$, the diagonal sequences of the forms $\{P_{i,2k+i+\kappa}\}$, $\{P_{2k+i+\kappa,i}\}$ with increasing i and $k = 0, 1, \kappa$ fixed converge uniformly over any bounded domain in $\mathbb{C} \setminus \{[-\infty, -\gamma], [\gamma, \infty]\}$ to ~~the function defined by the integrand~~ f, and the result of clause (ib) follows. The conditions imposed ~~upon~~ in clause (ic) ~~are special~~ constitute a special case of those given in clause ( ) of Theorem ; now $c_\nu^{(\beta,1)} \{\ldots\} \psi(x) = c f_\nu$ $(\nu = 0, 1, \ldots)$ where the $\{f_\nu\}$ are as described above, and the additional result of clause (ic) follows. If the approximating functions derived from the series ~~$\sum f_\nu z^\nu$ are $P_{i,j}$, $f_{2\nu+1} = 0$ $(\nu = 0, 1, \ldots)$~~, those from the series $\sum f_\nu z^\nu$ and $\sum f_{2\nu} z^\nu$ are $P_{i,j}$ and $\hat{P}_{i,j}$ respectively, and $f_{2\nu+1} = 0$ $(\nu = 0, 1, \ldots)$, then $P_{2i+\tau, 2j+\sigma}(z) = \hat{P}_{i,j}(z^2)$ $(i, j = 0, 1, \ldots; \tau, \sigma = 0, 1)$. Using the theory developed in the proof of Theorem —, the remaining results of the present theorem follow from this remark.

~~Theorem . Let ρ satisfy~~ the conditions imposed in Theorem 1, —

Theorem . Let ρ, ψ, f and ψ be as defined in Theorem . ~~With ψ to~~

~~be~~

(i) Subject to conditions to be imposed upon ψ, and with $m \geq \ldots + n - 1$, set

$$R^{(n,m)'}_{\rho;\psi,x}(\omega) = \int_0^\infty \int_0^n \frac{\tilde{B}^{(n)}_{n+m-1}(y-t)}{(n+m-1)!} \Delta^m_\omega \psi(x+\omega t)\, d\rho(y)\, dt$$

$$R^{(n,m)''}_{\rho;\psi,x}(\omega) = -\int_{-\infty}^0 \int_0^m \frac{\tilde{B}^{(n)}_{n+m-1}(y-t)}{(n+m-1)!} \Delta^m_\omega \psi(x+\omega t)\, d\rho(y)\, dt$$

a) In the representation of formula ( ), let $\psi_z(x) = 0$ $(z \geq 5)$. Then for $\omega \in H^+$, $E^{(n,m)+}_{\rho;\psi,x}(\omega) = R^{(n,m)'}_{\rho;\psi,x}(\omega)$, and, for $\omega \in H^-$, $E^{(n,m)-}_{\rho;\psi;x}(\omega) = R^{(n,m)''}_{\rho;\psi,x}(\omega)$.

b) In the representation of formula ( ), let $\psi_z(x) = 0$ $(z \leq 4)$. The results of a) with $R^{(n,m)'}_{\rho;\psi;x}$ and $R^{(n,m)''}_{\rho;\psi;x}$ interchanged hold.

With $J = 0$ and subject to further

(ii) ~~Subject to~~ conditions to be imposed upon ψ, and with coefficients $e^{(n)}_{\nu-n}$ defined by a formula analogous to ( ), set

$$\cancel{S^{(n)'}_{\rho;\psi,x}(\omega) = (-1)^{n-1} \sum_{\nu=0}^{n-1} (-1)^\nu e^{(n)}_{\nu-n} I_{n-\nu}}$$

$$S^{(n)'}_{\rho;\psi,x}(\omega) = (-1)^n \sum{}' \binom{n\nu-1}{n-1} \int_0^n \psi(x+y\omega+\nu\omega)\, d\rho(y).$$

$$- \sum_1^n (-1)^\nu e^{(n)}_{-\nu} I'_\nu(x,\omega)$$

$$S^{(n)''}_{\rho;\psi,x}(\omega) = \sum{}' \binom{n\nu-1}{n-1} \int_0^n \psi(x+y\omega-n\omega-\nu\omega)\, d\rho(y)$$

$$- \sum_1^n e^{(n)}_{-\nu} I''_\nu(x,\omega)$$

where

$$I'_\nu(x,\omega) = \int_0^\infty \int_{t_{\nu-1}}^\infty \cdots \int_{t_1}^\infty \psi(x+\omega t)\,dt\,dt_1 \ldots dt_{\nu-1}$$

$$I''_\nu(x,\omega) = \int_{-\infty}^0 \int_{-\infty}^{t_{\nu-1}} \cdots \int_{-\infty}^{t_1} \psi(x+\omega t)\,dt\,dt_1 \ldots dt_{\nu-1}$$

§ a) In the representation of formula ( ), let $\psi_\tau(x) = O$ $(\tau \geq 5)$, and in formula ( ), let $\sigma_\tau(v) = O(v^{n+\delta})$ for some $\delta \in (0,\infty)$ as v tends to ∞ $(\tau \leq 4)$. Then for $\omega \in H^+$,

$E^{(n,0)+}_{p;4,x}(\omega) = S^{(n)'}_{p;4,x}(\omega)$ and, for $\omega \in H^-$, $E^{(n,0)-}_{p;4,x}(\omega) = S^{(n)''}_{p;4,x}(\omega)$.

~~b) In the representation of formulas~~

b) ~~Interchange~~ The result of the preceding clause ~~are~~ also hold if the ~~past~~ bracketed conditions $(\tau \geq 5)$ and $(\tau \leq 4)$ are interchanged and the same is done with ~~and also~~ the functions $S^{(n)'}_{p;4,x}$ and $S^{(n)''}_{p;4,x}$. ~~in the preceding clause~~

Proof. For large $y \in (-\infty,\infty)$, $\tilde{B}^{(n)}_\nu(y) = O(|y|^{n-1})$. Subject to the conditions of clause (a), $\psi(t)$ is regular for $Re(t) \geq Re(x)$, and for large $t \in \Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, ~~$\psi(x+\omega t) = O($~~ $D^m \psi(t) = O(|t|^{-m+J})$ $(m \geq J)$. Thus when $m \geq J+n-1$, $R^{(n,m)'}_{p;4,x}$ is well defined over H^+.

[ ]

Nörlund has shown that for $\eta \in H^+ \cap D$ and $r = n+1, n+2, \ldots$

$$\int_0^\infty \frac{\tilde{B}^{(n)}_r(y-t)}{r!} e^{-\eta t}\,dt = \sum \frac{B^{(n)}_{r+\nu+1}(y)}{(r+\nu+1)!}(-\eta)^\nu$$

~~whence~~ Thus, using the moment representation ( ), for the same η and r

$$\int_0^\infty \frac{B_r^{N(n)}(y-t)}{r!} e^{-\eta t} dt = \sum_{l=1}^{n}{}' \int_{-1/(2\pi)}^{1/(2\pi)} \frac{d\sigma(n,r,l;y;u)}{(1+i\eta u)^l}$$

Nörlund's integral representation of formula ( ) provides the analytic continuation ~~of the function of~~ over H^+ of the function defined by the sum of the series ~~over~~ ( ) over $D_{2\pi}$. The integral expression ( ) offers analytic continuation throughout the complex plane except at the points $\eta = \pm 2\nu\pi i$ ($\nu = 1, 2, \ldots$) where, of course, the function has n^{th} order poles. The limit as η tends to zero in H^+ of the value of the left hand integral in relationship ( ) exists. ~~Changing~~ Substituting $\eta = v\omega$ ($v \in [0,\infty)$, $\omega \in H^+$), multiplying relationship ( ) throughout by $(-1)^{m-J} v^{m-J} e^{-xv} d\sigma(v) d\rho(y)$ and integrating over the intervals $[0, n]$ and $[0, \infty]$ with respect to y and v respectively, it is found that

$$R^{(n;m)'}_{\rho;4,x}(\omega) = (-1)^{m-J} \sum_{l=1}^{n}{}' \int_0^\infty \int_{-1/(2\pi)}^{1/(2\pi)} \frac{v^{m-J} e^{-xv}}{(1+i\omega v u)^l} d\sigma(\rho; n, mn-1, l; u) d\sigma(v)$$

where the function $\sigma(\rho; n, mn-1, l)$ is defined by formula ( ). The first result of clause (ia) is now derived as in the proof of clause ( ) of Theorem . The second result of this clause and those of its successors, are derived in the same way.

Under the conditions imposed upon the ψ_i in clause (iii), $\psi(t) = O(t^{-n-\delta})$ for large $t \in \Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$; the sum and integrals in formula ( ) for $S^{(n)'}_{\rho;\psi,x}(\omega)$ exist when $\omega \in H^+$. Since ψ satisfies the conditions imposed in Theorem 1 with $\eta = 0$, the function $E^{(n,1)+}_{\rho;\psi,x}$ of formula ( ) is well defined over H^+, and $E^{(n,0)+}_{\rho;\psi,x}$ may be defined in terms of it by use of formula ( ), ~~Relationship~~ the first result of clause ( ) may be verified directly, and the remaining results of the theorem are similarly obtained. ~~in the same way~~.

Theorem Let B' and s' be such that with B' constant over

(x^-, x^+)

$$\int_{-\infty}^{\infty} \left| \frac{B'(x)}{(z-x)^{s'(p)}} \right| dx < \infty$$

for ~~all~~ $p = 1, 2, \ldots$ and $z \in \mathfrak{S}(x^-, x^+)$, and let

$$(\ldots)\psi^{M'}(z) = \int_{-\infty}^{\infty} \frac{dB'(x)}{(z-x)^{s'}}$$

for $z \in \mathfrak{S}(x^-, x^+)$. Let $h = \max(0, 1 + [-\operatorname{Re}(s')])$ and set $M = M' + h$, $s = s' + h$, $B(x) = (-1)^h \prod_{l=0}^{h-1} (s'+l) B'(x)$ ~~for $z \in \mathfrak{S}(x^-, x^+)$~~. For $m = M+n, M+n+1, \ldots$, $k = 1, \ldots, n$ define

functions $w^{\pm}(k, m, n; \rho, \varkappa, \varsigma; \psi, z)$ over $\Delta(\pi \mp \frac{\pi}{2} - \arg(z - x^+),$

$\pi \mp \frac{\pi}{2} - \arg(z - x^-))$ when $\operatorname{Im}(z) \geq 0$ and over $\Delta(\mp \pi - \arg(z - x^-),$

$\mp \frac{\pi}{2} - \arg(z - x^+))$ when $\operatorname{Im}(z) \leq 0$ (the real ~~segment~~ $\pm(0, \infty)$

in particular belongs to the ~~segment~~ over which $w^{\pm}$ is ~~are~~ defined)

by

$$w^{\pm}(k, m, n; \rho, \varkappa, \varsigma; \psi, z \mid t) =$$

$$\frac{1}{2\pi i} i^{m-n-M-s} t^{m-n} \sum_{j=k}^{n} \binom{m+j-n-k}{m-n} \frac{t^j}{\prod_{l=1}^{M+j} {}^{(s)}(s-l)} T^{\pm}(j, n; M, s; \rho, \varkappa, \varsigma; \psi, z \mid t)$$

where, in the product the term corresponding to $l = s$, if it occurs, is to be omitted, and

a) if $M+j-s$ is not a nonnegative integer

$T^{\pm}(j,n;M,s;\rho,\alpha,\beta;\psi,z|t) =$

$$\frac{1}{\pi t}\left[\left[\int_{-\infty}^{x^-} - \int_{x^+}^{\infty}\right]\int_{\alpha}^{\beta} i\cot\left[\pi\{u + i(z-x)t\}\right]d\rho(u)\,dB(x) \mp \{B(\infty)-B(-\infty)\}\{\rho(\beta)-\rho(\alpha)\}\right]$$

Corollary

i) The functions $w^{\pm}(k,m,n;\rho,\alpha,\beta;\psi,z)$ are analytic over their domains of definition

ii) Setting $w(k,m,n;\rho,\alpha,\beta;\psi,z|t) = w^{\pm}(k,m,n;\rho,\alpha,\beta;\psi,z|t)$ for $t \in \pm(0,\infty)$,

$$\bar{f}_{n+z}(n;\rho,\alpha,\beta)\, {}^{0}\!\Delta^{m-n+z}\psi(z) = i^{z}\sum_{k=1}^{n}\binom{-k}{z}(-1)^{k}\int_{-\infty}^{\infty} t^{z}\, w(k,m,n;\rho,\alpha,\beta;\psi,z|t)\,dt$$

for $m = M+n, M+n+1, \ldots (m > n)$ and $z = 0, 1 \ldots$

(iii) For $m = M+n, M+n+1, \ldots (m > n)$

$$\bar{F}^{\pm}(m,n;\rho,\alpha,\beta;\psi,z|\omega) = \sum_{k=1}^{n}\int_{-\infty}^{\infty}\frac{w(k,m,n;\rho,\alpha,\beta;\psi,z|t)\,dt}{(1+i\omega t)^{k}}$$

for $\omega \in H^{\pm}$ respectively.

$$T^{\pm}(j,n;M,s;\rho,\alpha,\beta;\psi,z|t) = -\left[\int_{x^{+}-i\infty}^{\infty}\int^{(0+)} - \int_{-\infty}^{x^{-}-i\infty}\int_{(0+)}^{0}\right]\int_{\alpha}^{\beta} b_{n-j}^{(n)}(x)\, y^{M+j-s-n} \operatorname{cosec}^2\left[\pi\{w+i(z-x-iy)t\}\right] dy\, dB(x)$$

~~b) if it is true that M+j>n and~~

the symbol $\int_{(0+)}^{i\infty}$ denoting integration over a contour of Hankel type, commencing at $i\infty$, encircling the origin in an anti-clockwise direction, and returning to $i\infty$, the symbol $\int_{-i\infty}^{(0+)}$ having a similar meaning, while

b) if $M+j-s$ is a nonnegative integer

$$T^{\pm}(j,n;M,s;\rho,\alpha,\beta;\psi,z|t) = \frac{i}{s}\left[\int_{x^{+}-i\infty}^{\infty}\int^{i\infty} - \int_{-\infty}^{x^{-}-i\infty}\int_{0}^{0}\right]\int_{\alpha}^{\beta} b_{n-j}^{(n)}(x)\, y^{M+j-s-n-2} \operatorname{cosec}^2\left[\pi\{w+i(z-x-iy)t\}\right] dy\, dB(x)$$

c) and if $M+j=s$

(iv) For $m = 0, 1, \ldots$ The functions ~~defined in the preceding clause~~ may be
$\overline{F}^{\pm}(m, n; \rho, \alpha, \beta; \psi, z \mid \omega)$
continued analytically across the two open segments $\pm i(0, \infty)$

of the imaginary axis in the ω-plane, and ~~setting~~ defining
the concave open domains $\Delta^{\pm}$ by

$$\Delta^{+}(z) = \Delta\left(-\arg(z - x^{+}), \pi - \arg(z - x^{-})\right)$$

$$\Delta^{-}(z) = \Delta\left(\pi - \arg(z - x^{+}), 2\pi - \arg(z - x^{-})\right)$$

when $\operatorname{Im}(z) \geq 0$ and

$$\Delta^{+}(z) = \Delta\left(-\pi - \arg(z - x^{-}), -\arg(z - x^{+})\right)$$

$$\Delta^{-}(z) = \Delta\left(-\arg(z - x^{-}), \pi - \arg(z - x^{+})\right)$$

when $\operatorname{Im}(z) \leq 0$, $\overline{F}^{\pm}(m, n; \rho, \alpha, \beta; \psi, z \mid \omega) \in A_{\omega}(\Delta^{\pm}(z))$. ~~$(m = M, M+1, \ldots)$~~.

(v) For values of $\omega \in \{\Delta^{+}(z) \cap \Delta^{-}(z)\} \cap H_{i}^{\pm}$

$$\overline{F}^{+}(m, n; \rho, \alpha, \beta; \psi, z \mid \omega) = \overline{F}^{-}(m, n; \rho, \alpha, \beta; \psi, z \mid \omega) + \omega^{-m}\,\overline{\Pi}(M, n; \rho, \alpha, \beta; \psi, z \mid \omega)$$

where

$$\overline{\Pi}(M, n; \rho, \alpha, \beta; \psi, z \mid \omega) = 2\pi\omega^{M+n-1} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(k-1)!}\, \partial^{k-1}\, W_{i}^{\pm}(k, M+n; \rho, \alpha, \beta; \psi, z/\lambda)$$

∂ being $d/d\lambda$, where $\lambda = i/\omega$ $(m = 0, 1, \ldots)$.
~~where $\partial = d/d\lambda$ and $\lambda = i/\omega$ $(m = M, M+1, \ldots)$~~

(vi) a) $\overline{\mathcal{F}}^{\pm}(m, n; \rho, \alpha, \beta; \psi, z \mid \omega)$ represents $\overline{F}^{\pm}(m, n; \rho, \alpha, \beta; \psi, z \mid \omega)$
asymptotically as ω tends to zero in ~~the~~ $\Delta^{\pm}(z)$

b) If $\rho \in \hat{B}(n; \alpha, \beta)$ (or $\rho \in \tilde{B}(n; \alpha, \beta)$) $\overline{\mathcal{F}}^{\pm}$ in the preceding clause may be
replaced by $\hat{\mathcal{F}}$ (or $\tilde{\mathcal{F}}$).

Proof. The immovable singularities in Π^+ (if there are any) of the term T^+ defined by integral expressions of the form (, ,) are confined to those values of t for which $u + i(z-x)~~\ldots~~ = \nu$ $(\nu = \ldots, -1, 0, 1, \ldots)$ where u and x range over all values at which ρ and B have points of increase. These singularities are, ~~when $\operatorname{Im}(z) \geq 0$,~~ confined to the complement in H^+ of the sector $\Delta(\frac{\pi}{2} - \arg(z - x^+), \frac{\pi}{2} - \arg(z - x^-))$, ~~and~~ when $\operatorname{Im}(z) \geq 0$ and of $\Delta(-\pi - \arg(z-x^-), -\frac{\pi}{2} - \arg(z - x^+))$ when $\operatorname{Im}(z) \leq 0$; the ~~function~~ term is analytic over the sector itself. Thus w^+ is analytic over its domain of definition; ~~and~~ the same also follows with regard to w^-. (~~It may~~ ~~be~~ If ρ and B are both analytic over intervals in $[x^-, x^+]$ and $[-\infty, x^-]$, $[x^+, \infty]$, $w^{\pm}$ may, of course, be shown to be analytic over more extensive regions.);

Subject to the conditions imposed upon B and ρ

$$\mathcal{D}^{m-n+\tau}\psi(z) = \frac{(-1)^{m-n+\tau-M}}{\Gamma(s)}\left[\int_{-\infty}^{x^-}\int_{0}^{\infty} - \int_{x^+}^{\infty}\int_{-\infty}^{0}\right] v^{m+\tau-M-n+s-1} e^{-(z-x)v}\, dv\, d\ell(x)$$

when $m \geq M+n$ ~~($m \geq n$)~~ and $\tau = 0, 1, \ldots$ for $z \in \mathcal{S}(x^-, x^+)$. Also

$$\bar{b}^{(n)}_{m+\tau}(u) = (-1)^{\tau} \sum_{k=1}^{n}{}' (-1)^{k-1} \binom{-k}{\tau} \sum_{j=k}^{n} \binom{m+j-n-k}{m-n} b^{(n)}_{n-j}(u)\, \bar{b}_{m-n+j+\tau}(u)$$

and

$$\bar{b}^{(1)}_{m-n+j+z}(u) = -\Sigma_1 (2i\nu\pi)^{-m+n-j-z} e^{2i\nu\pi u} - (-1)^{m-n+j+z} \Sigma_1 (2\nu\pi)^{-m+n-j-z} e^{-2\nu\pi u}$$

Hence, when $m > n$,

$$\bar{f}_{m+z}(n; \rho, \alpha, \beta) \mathfrak{D}^{m-n+z} \psi(z) =$$

$$(-1)^M \sum_{k=1}^{n} (-1)^k \binom{-k}{r} \sum_{j=k}^{n} \binom{m+j-n-k}{m-n} i^{+m-n-j+z} (t_1+t_2+t_3+t_4)$$

where

$$t_1 = \frac{1}{\Gamma(s)} \Sigma_1 \int_0^\infty \int_{-\infty}^{x^-} \int_\alpha^\beta b^{(n)}_{n-j}(u) v^{m+z-M-n+s-1} e^{-(z-x)v} (2\nu\pi)^{-m+n-j-z} e^{2i\nu\pi u}$$

$$d\rho(u)\,dx\,dv$$

t_2 is derived from t_1 by replacing the symbols $\int_0^\infty \int_{-\infty}^{x^-}$ by $-\int_{-\infty}^{0} \int_{x^+}^{\infty}$, t_3 is derived from t_1 by multiplication by $(-1)^{m-n+j+z}$ and changing the sign of ν in the exponential term, and t_4 is obtained from t_3 in the same way that t_2 was derived from t_1. Setting $v = 2\nu\pi t$

$$t_1 = \frac{1}{\Gamma(s)} \int_0^\infty \int_{-\infty}^{x^-} \int_\alpha^\beta b^{(n)}_{n-j}(u)\, t^{m+z-M-n+s-1} \Sigma_1 (2\nu\pi)^{s-M-j} e^{2\nu\pi i\{iu-(z-x)t\}}$$

$$d\rho(u)\,dx\,dt$$

When $M+j-s$ is not a ~~nonnegative~~ ~~positive~~ nonnegative integer

$$(2\nu\pi)^{s-M-j-1} = \frac{i\Gamma(s-M-j)}{2\pi} \int_{(0+)}^{\infty} (-h)^{M+j-s} e^{-2\nu\pi h} dh$$

the contour of integration being of Hankel type. Also $\Sigma_1 (2\pi\nu) e^{2\nu\pi\{iu-(z-x)t\}} =$

$-\frac{\pi}{2}\operatorname{cosec}^2[\pi\{u+i(z-x)t\}]$. Consequently

$$t_1=\frac{-\pi i\,\Gamma(s-M-j)}{-2\Gamma(s)}\int_0^\infty\int_{-\infty}^{x^-}\int_{(0+)}^{\infty}\int_\alpha^\beta b^{(n)}_{n-j}(u)\,t^{m+z-M-n+s-1}(-h)^{M+j-s}\operatorname{cosec}^2[\pi\{u+i\{(z-x)t+h\}\}]\,d\rho(u)\,dh\,dx\,dt$$

or changing the variable of integration ~~to $y=iw/t$~~ in the third integral to $y=ih/t$

$$t_1=\frac{\pi i^{M+j-s}}{2\prod_{l=1}^{M+j}(s-l)}\int_0^\infty\int_{-\infty}^{x^-}\int_{(0+)}^{(i\infty)}\int_\alpha^\beta b^{(n)}_{n-j}(u)\,t^{m-n+j+z}\,y^{M+j-s}\operatorname{cosec}^2[\pi\{u+i(z-x-iy)t\}]\,d\rho(u)\,dy\,dx\,dt$$

If $M+j-s$ is a nonnegative integer

$$(2\pi)^{s-M-j-1}=\frac{1}{\Gamma(M+j-s+1)}\int_0^\infty h^{M+j-s}e^{-2\pi h}\,dh$$

and formula ( ) has the alternative formulation

$$t_1=-\frac{\pi}{2(s-1)!\,(M+j-s)!}\int_0^\infty\int_{-\infty}^{x^-}\int_0^\infty\int_\alpha^\beta b^{(n)}_{n-j}(u)\,t^{m+z-M-n+s-1}\,h^{M+j-s}\operatorname{cosec}^2[\pi\{u+i\{(z-x)t+h\}\}]\,d\rho(u)\,dh\,dx\,dt$$

or, again

$$t_1=\frac{\pi i^{M+j-s-1}}{2s\prod_{l=1}^{M+j}(s-l)}\int_0^\infty\int_{-\infty}^{x^-}\int_{0}^{\infty}\int_\alpha^\beta b^{(n)}_{n-j}(u)\,t^{m-n+j+z}\,y^{M+j-s}\operatorname{cosec}^2[\pi\{u+i(z-x-iy)t\}]\,d\rho(u)\,dy\,dx$$

If $M+j=s$, then, since $\sum_1' e^{2\pi\{iu-(z-x)t\}}=\frac{1}{2}[i\cot[\pi\{u+i(z-x)t\}]-$

t_1 may be evaluated without use of Γ-function integrals

$$t_1 = \frac{1}{2\prod\limits_{l=1}^{M+j}{}^{(s)}(s-z)} \int_0^\infty \int_{-\infty}^{x^-} \int_\alpha^\beta b_{n-j}^{(n)}(u)\, t^{m+z+j-n-1} \left[i \cot[\pi_{11}\{u+i(z-x)t\}] - 1 \right] d\rho(u)\, dx\, dt$$

The terms t_2, t_3 and t_4 may be treated in the same way, and the result of clause (ii) follows.

The derivative $\partial x^M \psi$ considered in the theorem has a representation of the form ( ) ~~save~~ if $\psi \in M(M; x^-, x^+)$. As is easily verified, the functions $\varpi(k, m, n; \rho, \alpha, \beta; \psi, z)$ ~~of~~ of clause ( ) of Theorem — in the present case are differentiable ~~when~~ ~~m>n~~ and their derivatives are the corresponding w-functions. ~~of~~ ~~the~~ [illegible] The result of clause (iii) follows from clause ( ) of Theorem .

~~The results of clauses (iv — vi) are consequence of clause ( )~~

~~of Theorem~~.

The result of clause (iv) follows directly from clause ( ) of Theorem — ; that of clause (v) ~~with m = M n~~ from clause ( ) of Theorem in conjunction with the recursion ( ), and those of clause (vi) from clause ( ) of Theorem. (It should perhaps be pointed out that the functions $w^{\pm}(M; m, n; \rho, \alpha, \beta; \psi, z/t)$ are analytic over $\pm(0, \infty)$; they may, however, possess a nonnegligible singularity at the origin.)

Definition. With $n \geq 1$, ~~and~~ $[\alpha, \beta] \subseteq [-\infty, \infty]$ and $\rho \in \mathbb{B}_\infty(\alpha, \beta)$ functions $\tilde{\rho}$ and $\hat{\rho}(r)$ $(r = 1, \dots, n)$ are defined as ~~[illegible]~~ follows; if a) $\alpha < 0$, $\beta > n$ ~~$[0, n] \subset (\alpha, \beta)$~~ then, with $n' = -[-\beta]$, $m'' = -[\alpha]$ and ρ extended over the range $[-m'', \alpha)$ if $\alpha > -m''$ by setting $\rho(y) = \rho(\alpha)$ over ~~the~~ the first interval, and over the range $(\beta, n']$ (if $\beta < n'$) ~~by setting~~ $\rho(y) = \rho(\beta)$ over ~~the interval~~ the second, set $\rho(y)$

(i) $\tilde{\rho}$ is defined over $[-m'', n' - n]$ by defining the function ~~is given~~ over the interval $i \leq y < i+1$ by means of the formula

$$\tilde{\rho}(y) = (-1)^n \sum_{\nu=0}^{i+m''} \binom{n+\nu-1}{\nu} \rho(y-\nu)$$

for $i = -m'', -m''+1, \dots, -1$, and by means of the formula

$$\tilde{\rho}(y) = \sum_{\nu=0}^{n'-n-i-1} \binom{n+\nu-1}{\nu} \rho(y+n+\nu) + (-1)^n \sum_{\nu=0}^{m''-1} \binom{n+\nu-1}{\nu} \rho(-\nu)$$

for $i = 0, 1, \dots, n'-n-1$ and setting

$$\tilde{\rho}(n'-n) = \rho(\beta) + (-1)^n \sum_{\nu=0}^{m''-1} \binom{n+\nu-1}{\nu} \rho(-\nu).$$

(ii) $\hat{\rho}(r)$ is defined over $[0, r]$ by means of the formulae

$$\hat{\rho}(r/y) = \rho'(r/y) + \rho''(r/y)$$

for $r = 1, \dots, n-1$ and

$$\hat{\rho}(n/y) = \rho(y) + \rho'(n/y) + \rho''(n/y)$$

where

$$\rho'(r/y) = (-1)^{n+r+1} \sum_{\nu=0}^{m''-1} \binom{n-r+\nu}{\nu} \rho(y-\nu-1)$$

when $0 \leq y \leq 1$ and $\rho'(r/y) = \rho'(r/1)$ when $1 < y \leq r$, and

$$\rho''(r/y) = \sum_{\nu=0}^{n'-n-1} \binom{n-r-\nu}{\nu} \rho(y+n-r+\nu-1)$$

when $r-1 \leq y \leq r$ and $\rho''(r|y) = \rho''(r|r-1)$ when $0 \leq y < r-1$, ~~all for $r=1,\dots,n$.~~

b) ~~a~~ if $\alpha \geq 0, \beta > n$, ρ is extended over the range $(\beta, n']$ as in a) above

(i) $\tilde{\rho}$ is defined over $[0, n'-n]$ by means of formulae (,) alone and

(ii) ~~$\tilde{\rho}$ &~~ the functions $\hat{\rho}(r|y)$ are defined over $[0, r]$ $(r=1,\dots,n)$ by deleting ~~ref~~ all references to the functions $\rho'(r)$.

c) if $\alpha < 0, \beta \leq n$, ρ is extended over the range $[-m'', \alpha)$ as in a) above

(i) $\tilde{\rho}$ is defined over ~~$[-0, m]$~~ $[-m'', 0]$ by means of formula ( ) alone and

(ii) the functions $\hat{\rho}(r|y)$ are defined over $[0, r]$ $(r=1,\dots,n)$ by deleting all references to the functions $\rho''(r)$

d) if $\alpha \geq 0, \beta \leq n$ the functions $\tilde{\rho}$ and $\hat{\rho}(r)$ $(r=1,\dots,n-1)$ are undefined and $\hat{\rho}(n)$ is simply ρ.

Coefficients f_ν are defined by setting

$$\cancel{f_\nu = \int_\alpha^\beta \frac{B_\nu^{(n)}(\eta)}{\nu!}}$$

$$f_\nu(n; \rho, \alpha, \beta) = \int_\alpha^\beta \frac{B_\nu^{(n)}(\eta)}{\nu!} d\rho(\eta)$$

for $\nu = 0, 1, \dots$.

Derivatives of nonnegative order are defined by setting

$$\mathfrak{D}^\tau \psi(x) = d^\tau \psi(x) / dx^\tau$$

for $\tau = 0, 1, \dots$. With ψ assumed analytic in a strip of the form $S(x^-, x^+)$ and a a fixed point in this strip, derivatives of negative order are defined by setting

$$\mathfrak{D}^{-r}\psi(x) = \int_a^x \frac{(x-z)^{r-1}}{(r-1)!}\psi(z)dz$$

for $r = 1, 2, \ldots$, where the path of integration is the straight line joining a and x.

~~The functions $F(n;\rho,\alpha$ F is defined by mean~~

With $m \geq 0$ fixed, In case a) above, the function F^+ is defined by means of the formula

$$F^+(n;\rho,\alpha,\beta;\psi,x/\omega) = \sum_0^{n+m-1} f_\nu(\rho;\alpha,\beta)\mathfrak{D}^{\nu-n}\psi(x)\omega^\nu$$

$$+\omega^{m+n}\left[\sum_{r=1}^{n} E^+(r,m;\hat{\rho}(r),0,r;\psi,x/\omega) + E(0,m;\tilde{\rho},\alpha,\beta-n;\psi,x/\omega)\right]$$

where

$$E(0,m;\tilde{\rho},\alpha,\beta-n;\psi,x/\omega) =$$

$$\omega^{-m}\left[\int_\alpha^{\beta-n}\psi(x+y\omega)d\tilde{\rho}(y) - \sum_{\nu=0}^{m-1}\int_\alpha^{\beta-n}\frac{y^\nu}{\nu!}d\tilde{\rho}(y)\mathfrak{D}^\nu\psi(x)\omega^\nu\right];$$

In case b), ~~α in~~ the parameter α occurring in ~~relating to~~ the function E is replaced by zero consistently; in case c), ~~$\beta-n$~~ the parameter $\beta-n$ is treated in the same way; in case d) the expression in square brackets in formula ( ) is replaced by $E^+(n,m;\rho;\alpha,\beta;\psi;x/\omega)$ alone. Functions F^- are defined in terms of functions E^- in the same way.

The series $\mathfrak{F}(n;\rho,\alpha,\beta,\psi,x)$ in ascending powers of ω is ~~are~~ defined by the formula

$$\mathfrak{F}(n;\rho,\alpha,\beta;\psi,x/\omega) = \sum f_\nu(n;\rho,\alpha,\beta)\mathfrak{D}^{\nu-n}\psi(x)\omega^\nu$$

Similarly

$$\tilde{\mathfrak{F}}(n;\rho,\alpha,\beta;\psi,x\,|\,\omega) = \sum_{\nu} f_{\nu}(n;\rho,\alpha,\beta)\,\partial^{\nu-n}\psi(x)\,\omega^{\nu}$$

$$\hat{\mathfrak{F}}(n;\rho,\alpha,\beta;\psi,x\,|\,\omega) = \sum_{\nu} f_{\nu+1}(n;\rho,\alpha,\beta)\,\partial^{\nu-n+1}\psi(x)\,\omega^{\nu+1}$$

Theorem. Where relevant, the following abbreviations ~~are used in the clauses~~ $\alpha' = \min(0,\alpha)$, $\beta' = \max(0,\beta - n)$, $x' = x^- - Re(x)$, $x'' = x^+ - Re(x)$ are used in the following clauses

A. Let $[\alpha,\beta] \subset (-\infty,\infty)$, $\rho \in P_n(\alpha,\beta)$ where and let $\psi \in M(M; x^-, x^+)$.

(i) Let $x \in S(x^-, x^+)$ be fixed, and set $\omega^+(n;\alpha,\beta;x) = \min(x'/\alpha', x''/\beta')$ $\omega^-(n;\alpha,\beta;x) = \max(x''/\alpha', x'/\beta')$. The functions $F^{\pm}(n;\rho,\alpha,\beta;\psi,x|\omega)$ are analytic over the strips $S(0,\omega^+(n;\alpha,\beta;x))$ and $S(\omega^-(n;\alpha,\beta;x),0)$ in the ω-plane respectively

(ii) Let $\omega^+ = (x^+ - x^-)/(\beta' - \alpha')$, $\hat{x}^-(\omega) = x^- - \alpha' Re(\omega)$, $\hat{x}^+(\omega) = x^+ - \beta' Re(\omega)$, $\tilde{x}^-(\omega) = x^- - \beta' Re(\omega)$, $\tilde{x}^+(\omega) = x^+ - \alpha' Re(\omega)$. With $\omega \in S(0,\omega^+)$ fixed, $F^+(n;\rho,\alpha,\beta;\psi|\omega)$ is analytic over the strip $S(\hat{x}^-(\omega), \hat{x}^+(\omega))$ in the x-plane, and with $\omega \in S(-\omega^+,0)$ fixed, $F^-(n;\rho,\alpha,\beta;\psi,|\omega)$ is analytic over the strip $S(\tilde{x}^-(\omega), \tilde{x}^+(\omega))$.

B. Over the half-planes over which ψ is defined below let $\psi(x) = O(e^{\gamma|Re(x)|})$ as $|Re(x)| \to \infty$ in that half-plane; over the semi-infinite interval over which ρ is defined below let $\rho(y) = O(e^{-\gamma'|y|})$ as $|y| \to \infty$ in that interval; set $\Gamma = \gamma'/\gamma$.

(i) Let $\psi \in M(M; x^-, \infty)$ ~~and let~~ $\cancel{x_1}$

a) Let $\alpha > -\infty$, $\beta = \infty$

~~α) With~~ $\cancel{\omega \in S(0, \min(\Gamma, x'/\alpha'))}$ ~~fixed,~~ $\cancel{F^+(n; \rho, \alpha, \beta; \psi, x)}$

α) With $x \in S(x^-, x^+)$ fixed, $F^+(n; \rho, \alpha, \beta; \psi, x)$ is analytic over the strip $S(0, \min(\Gamma, x'/\alpha'))$ in the ω-plane

β) With $\omega \in S(0, \Gamma)$ fixed, $\cancel{F^+(n; \rho, \alpha, \beta, \psi(\omega)}$ ~~is analytic~~ $F^+ \in A_x\{S(x^- - \alpha' Re(\omega), \infty)\}$

~~over the strip~~ $\cancel{S(x^- - \alpha' Re}$ ~~in the x-plane~~

b) Let $\alpha = -\infty$, $\beta < \infty$

α) With $x \in S(x^-, x^+)$ fixed, $\cancel{F^-(n \quad) \text{ is}}$

$F^- \in A_\omega\{S(\max(-\Gamma, x'/\beta'), 0)\}$

β) With $\omega \in S(-\Gamma, 0)$ fixed, $F^- \in A_x\{S(x^- - \beta' Re(\omega), \infty)\}$

(ii) Let $\psi \in M(M; -\infty, x^+)$

a) Let $\alpha > -\infty$, $\beta = \infty$

α) With $x \in S(x^-, x^+)$ fixed, $F^- \in A_\omega\{S(\max(-\Gamma, x''/\alpha'), 0)\}$

β) With $\omega \in S(-\Gamma, 0)$ fixed, $F^- \in A_x\{S(-\infty, x^+ - \alpha' Re(\omega))\}$

b) Let $\alpha = -\infty$, $\beta < \infty$

α) With $x \in S(x^-, x^+)$ fixed, $F^+ \in A_\omega\{S(0, \min(\Gamma, x''/\beta'))\}$

β) With $\omega \in S(0, \Gamma)$ fixed, $F^+ \in A_x\{S(-\infty, x^+ - \beta' Re(\omega))\}$

C Let $\psi \in M(M; -\infty, \infty)$ with $\psi(x) = O(e^{\Gamma' Re(x)})$ as $Re(x) \to \infty$ $(0 \le \Gamma', \Gamma'' < \infty)$ and $\psi(x) = O(e^{-\Gamma'' Re(x)})$ as $x \to -\infty$. Let $\alpha = -\infty, \beta = \infty$ with $(0 < \gamma', \gamma'' < \infty)$ $\rho(y) = O(e^{-\gamma' y})$ as $y \to \infty$ and $\rho(y) = O(e^{\gamma'' y})$ as $y \to -\infty$. Set $\omega^+ = \min\left\{\frac{\gamma'}{\Gamma'}, \frac{\gamma''}{\Gamma''}\right\}$ and $\omega^- = \max\left\{-\frac{\gamma'}{\Gamma''}, -\frac{\gamma''}{\Gamma'}\right\}$. For all $x \in \mathbb{C}$, ~~the~~ ~~functions~~ $F^+ \in A_\omega\{S(0, \omega^+)\}$ and $F^- \in A_\omega\{S(\omega^-, 0)\}$. For ~~any~~ fixed $\omega \in S(0, \omega^+)$ ~~$\omega \in S(0,\omega^+)$ fix, F^+ is an e~~ and $\omega \in S(\omega^-, 0)$ respectively, F^+ and F^- are entire functions of x.

D. If in any of the above results in which it has been shown that for a fixed x, F^+ is analytic over a strip of the form $S(0, \omega^+)$ in the ω-plane, σ in the representation ( ) is non-analytic over some interval of the form (V, ∞) or $(-\infty, -V)$ ~~for that x~~ $(0 \le V < \infty)$, then the imaginary axis is a natural barrier of F^+ in the ω-plane; a similar remark ~~concerns~~ holds for the function F^-

E. Let $\omega \Delta^m \psi$ have the representation ( ) of Theorem , and let $-\infty < \alpha$ ~~$\le \beta$~~, $\beta < \infty$.

(i) For $x \in S(x^-, x^+)$ fixed the results of clause $A i)$ the hold, but the function $F^{\pm}$ can be continued analytically across the imaginary arcis

(ii) Denote by $F^{\pm}$ the functions directly defined by formulae such as ( ) and ( ), and obtained by analytic continuation. Let $\Delta^{\pm}(x)$ be the regions defined in clause ( ) of Theorem . Let $d_1^{(n)} = \mathrm{Im}(x)/\beta'$, $d_2^{(n)} = \mathrm{Im}(x)/\alpha'$. Let $\mathbb{D}(n,x)$ be the regions defined by

$$\mathbb{D}(n,x) = \{\omega : \omega \in \Delta(\arg(x^{+}-x), 0) \cup \Delta(-\pi, \arg(x-x^{-})-\pi) : \mathrm{Im}(\omega) \geq d_1^{(n)}\}$$
$$\cup \{\omega : \omega \in \Delta(0, \arg(x-x^{+})) \cup \Delta(\pi+\arg(x^{-}-x), \pi) ; \mathrm{Im}(\omega) \leq d_2^{(n)}\}$$

if $\mathrm{Im}(x) > 0$, and by

$$\mathbb{D}(n,x) = \{\omega : \omega \in \Delta(0, \arg(x^{+}-x)) \cup \Delta(\pi+\arg(x-x^{-}), \pi) ; \mathrm{Im}(\omega) \leq d_1^{(n)}\}$$
$$\cup \{\omega : \omega \in \Delta(\arg(x-x^{+}), 0) \cup \Delta(-\pi, \arg(x^{-}-x)-\pi) ; \mathrm{Im}(\omega) \geq d_2^{(n)}\}$$

if $\mathrm{Im}(x) < 0$. Then ~~functions $F^{\pm}$~~ $F^{\pm} \in A_{\omega}\{\Delta^{\pm}(x) \setminus \mathbb{D}(n,x)\}$.

(iii) With $x \in (x^{-}, x^{+})$ fixed, set $\omega^{+} = \min(x''/\beta', x'_{0}/\alpha')$, $\omega^{-} = \max(x'/\beta', x''/\alpha')$. Then $F^{+} \in A_{\omega}\{\mathbb{C} \setminus \{[-\infty, 0] \cup [\omega^{+}, \infty]\}\}$ and $F^{-} \in A_{\omega}\{\mathbb{C} \setminus \{[-\infty, \omega^{-}] \cup [0, \infty]\}\}$

(iv) Set

$$\Pi(n; \rho, \alpha, \rho; \vartheta, x|\omega) = 2\pi\omega^{M+n-1} \sum_{r=1}^{n} \sum_{l=1}^{r} \frac{(-1)^{l}}{(l-1)!} \omega \, \partial^{l-1} W(n, r, l; \rho, \vartheta, x|\lambda)$$

where $\lambda = i/\omega$, $\partial = d/d\lambda$ and

$$W(n,r,l;\rho,\alpha,\beta;\psi,x|z) =$$

$$\frac{(-1)^{M-1}\pi}{2}\sum_{j=l}^{r}\binom{l-j-1}{M}\left[\int_{-\infty}^{x^-}\int_{0}^{i\infty}+\int_{x^+}^{\infty}\int_{-i\infty}^{0}\right]\int_{\alpha}^{\beta}\frac{\zeta^{(n)}_{r-j}(y)\,z^{M+j-1}}{(r-j)!\,(M+j-1)!}\operatorname{cosec}^2\Big[$$

$$\pi\{y+z+(x-s)iz\}\Big]\,d\rho(r|y)\,dz\,dB(s)$$

Over regions in the ω-plane in which the functions $F^{\pm}$ are, according to the results of the preceding two clauses, both analytic, the relationship

$$F^+(n;\rho,\alpha,\beta;\psi,x|\omega) = F^-(n;\rho,\alpha,\beta;\psi,x|\omega) + \Pi(n;\rho,\alpha,\beta;\psi,x|\omega)$$

prevails.

When $n=1$ and $M=0$, Π has the simpler representation

$$\Pi(1;\rho,\alpha,\beta;\psi,x|\omega) = \pi i\{\rho(\beta)-\rho(\alpha)\}\{B(\infty)-B(-\infty)\}$$

$$+\pi\left[\int_{x^+}^{\infty}-\int_{-\infty}^{x^-}\right]\int_{\alpha}^{\beta}\cot\left[\pi\left\{\frac{x-s}{\omega}+y\right\}\right]d\rho(y)\,dB(s)$$

F. Let $\psi\in M(M;-\infty,\infty)$, with $\psi(x)=O(e^{\gamma|Re(x)|})$ as $|Re(x)|\to\infty$, and let $[\alpha,\beta]\subset(-\infty,\infty)$.

(i) For $x\in\mathbb{C}$ fixed, the functions $F^{\pm}$ directly defined by formulae such as ( ) and ( ) represent the same function, F say, ~~which~~

~~is analytic~~ and $F \in A_0\{\mathbb{C}\setminus\{\pm i[2\pi/\gamma,\infty]\}\}$.

(ii) For any fixed $\omega \in \mathbb{C}\setminus\{\pm i[2\pi/\gamma,\infty]\}$, F is an entire function of x.

G Let $\psi \in M(M;x^-,x^+)$ have the decomposition $\psi = \psi^+ + \psi^+$ of clause of Lemma , and let R' ~~and~~ R'' be the functions defined ~~by~~ in ~~of~~ clause ( ) of Theorem . Set

G(n+1,n)

$$F'(n;\rho,\alpha,\beta;\psi',x|\omega) = \sum_0^{n+M-1} w(\rho;\alpha,\beta)\,\partial^{\nu-n}\psi'(x)\,\omega^{\nu}$$
$$+\,\omega^{M+n}\left[\sum_{r=1}^{m} R'(n;\rho(r),0,r;\psi',x|\omega) + E(0,m;\tilde{\rho},\alpha,\beta-n;\psi',x|\omega)\right]$$

where $\{\rho(r)\}$, $\tilde{\rho}$ and E are to be interpreted, according to the values of α and β, as in Definition – ; ~~and~~ define F'' in terms of R'' in a similar way.

Under any of the conditions ~~under which Γ^{+}~~, given in the above clauses, under which F^+ ~~has to~~ exists, this function has the decomposition

$$F^+(n;\rho,\alpha,\beta;\psi,x|\omega) = F'(n;\rho,\alpha,\beta;\psi^+,x|\omega) + F''(n;\rho,\alpha,\beta;\psi,x|\omega).$$

F^- has a similar decomposition in which ψ^+ and ψ^- are ~~int~~ interchanged.

Proof. ~~The functions~~ In formula ( ), the ~~functions occurring in the~~ terms involved in the first partial sum are analytic in x for all $x \in \mathfrak{S}'(x^-, x^+)$ and in ω for all $\omega \in \mathbb{C}$. The functions E^+ are analytic for all $x \in \mathfrak{S}(x^-, x^+)$ and $\omega \in H^+$. When $\alpha < 0$, $\beta > n$, the last term in formula ( ) is analytic in x and ω for all x, ω such that $x + \alpha\omega$, $x + (\beta - n)\omega \in \mathfrak{S}(x^-, x^+)$; similar statements ~~may be m~~ concerning this function may be made when either $\alpha > 0$ or $\beta < n$; and when both latter conditions hold, the last term may be disregarded. The results of part A follows from these considerations. ~~(if it should be taken into account)~~

Under the conditions of any of the clauses of part B, the integral in formula ( ) extends over a semi-infinite range, and the stated conditions suffice to ensure that this integral exists; under the conditions of part C the integral concerned extends ~~over a form~~ over $(-\infty, \infty)$.

The result of part D is a consequence of part I of Theorem 5. Under the conditions of ~~part~~ clauses Di–iii), the functions E^+ are, for $x \in \mathfrak{S}(x^-, x^+)$ fixed, analytic over the domain $\Delta'(x)$ in the ω-plane; when $\alpha < 0$ or $\beta < n$, the integral in formula ( ) represents a function which is analytic over $\mathbb{C} \setminus \mathbb{D}(n, x)$. The results of clauses Di–iii) follows immediately. The results of clause Div) follows from those of part of Theorem .

Under the conditions of part F, the functions E^+ in formula ( ) ~~are~~, may, for all $x \in \mathbb{C}$, ~~for all $x \in \mathbb{C}$, analytic in $\mathbb{D}_{2\pi/\gamma} \cap H^+$; they may~~ be continued analytically ~~across that diameter part of the imaginary~~ across the segment $i(-2\pi/\gamma, 2\pi/\gamma)$ of the imaginary axis and yield thereby the similarly defined functions E^-. The function defined by formula ( ) is analytic for all $x \in \mathbb{C}$ and $\omega \in \mathbb{C}$. The functions $F^{\pm}$ are the same (F say) and, since the functions $E^{\pm}$ are analytic in ω for all $\omega \in D_{\pi/8}$, this is also true of F. The further results of part F follow immediately.

The decomposition described in part G is a consequence of that described in part — of Theorem — .

It is to be remarked that the function $F^{+}(n;\rho,\alpha,\beta;\psi,x|\omega)$ may be well-defined where $F^{+}(n-1;\rho,\alpha,\beta;\psi,x|\omega)$ is not; to cite an extreme example, when $\alpha=0$, $n-1 \leq \rho \leq n$, $\alpha|\psi \in M(M;x^-,x^+)$ the former function is defined, when $x \in \mathfrak{I}(x^-,x^+)$, for all $\omega \in H^{+}$ whereas (vide clause Ai)) the latter may (if $x^+ < \infty$) be defined only over a strip in H^{+}. Similar remarks may be made concerning the functions F^-. Again, under the conditions of a clause such as Biii), $F^{+}(n;\rho,\alpha,\beta;\psi,x|\omega)$ may be well defined when its counterpart F^- is not.

Theorem A (i) Where, in Theorem --, $F^{\pm}$ is shown for a fixed x to be analytic over a strip of the form $\mathcal{S}(0, \omega^+)$, $F^{\pm}(\omega) \sim \mathcal{F}(\omega)$ as ω tends to zero in the sectors $\Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ ~~and $\Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$ respectively~~
a similar result holds with regard to the function F^- over the sector $\Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$

(ii) If ψ and $[\alpha, \beta]$ have the special form described in ~~clause ( )~~ part E of Theorem --, ~~$F^{+}(\omega) \sim \mathcal{F}(\omega)$ as ω~~ with $\Delta^{\pm}(x)$, the regions specified defined in clause ( ) of Theorem --, then $F^{\pm}(\omega) \sim \mathcal{F}(\omega)$ as ω tends to zero in $\Delta^{\pm}(x)$.

(iii) If, in addition, $\rho \in \hat{\mathcal{B}}_{\infty}(\alpha, \beta)$ or $\rho \in \tilde{\mathcal{B}}_{\infty}(\alpha, \beta)$, $\mathcal{F}$ in the preceding two clauses may be replaced by $\hat{\mathcal{F}}$ or $\tilde{\mathcal{F}}$ respectively

~~(iv) If, alternatively, $\rho \in \tilde{\mathcal{B}}_{\infty}(\alpha, \beta)$, $\mathcal{F}$ in clause (i, ii) may be replaced by $\tilde{\mathcal{F}}$.~~

B. (i) If ψ_γ and $[\alpha, \beta]$ are as described in part F of Theorem --, so that $F^{\pm}$ are the same function F, $\mathcal{F}$ converges to F over $\mathbb{D}_{2\pi/\gamma}$

(ii) If, in addition, $\rho \in \hat{\mathcal{B}}_{\infty}(\alpha, \beta)$ or $\rho \in \tilde{\mathcal{B}}_{\infty}(\alpha, \beta)$ in the preceding clause may be replaced by $\hat{\mathcal{F}}$ or $\tilde{\mathcal{F}}$ respectively.

C. (i) Where in Theorem --, F^+ is shown for a fixed x to be analytic over a strip of the form $\mathcal{S}(0, \omega^+)$, $\mathcal{F}$ is ~~$(B, 1)$~~

a) $(B', 1)$ summable to F^+ over $(0, \omega^+)$ and

b) $(\bar{B}, 1)$ summable to F^+ over $\mathcal{S}(0, \omega^+)$;

~~or~~ similar results hold with regard to ~~the~~ functions F^{*-} defined over strips of the form $\mathcal{S}(\omega^-,0)$.

(ii) If, in addition, $\rho \in \hat{B}_\infty(\alpha,\beta)$ or $\rho \in \tilde{B}_\infty(\alpha,\beta)$, then $\hat{f}$ or $\tilde{f}$ is

a) both $(B',2)$ and B^2-summable to F^+ over $(0,\omega^+)$ and

b) ~~$(\bar{B},2)$~~ both $(\bar{B},2)$ and $\bar{B}^2$- summable to F^+ over $\mathcal{S}(0,\omega^+)$;

similar results hold with regard to ~~the~~ functions F^- defined over strips of the form $\mathcal{S}(\omega^-,0)$.

Proof. Supposing, under the condition of part A(i), that $\alpha<0,\ \beta>n$, and using relationship ( ), it follows from n-fold use of relationship ( ) that when $r \geq n$,

$$\int_n^{n'} \frac{B_r^{(n)}(y)}{r!}d\rho(y) = \sum_{i=n}^{n'-1}\int_i^{i+1}\frac{B_r^{(n)}(y)}{r!}d\rho(y)$$

$$= \sum_{i=n}^{n'-1}\int_i^{i+1}\left\{\frac{B_r^{(n)}(y-i+n-1)}{r!}+\sum_{\nu=1}^{i-n+1}\frac{B_{r-1}^{(n-1)}(y-\nu)}{(r-1)!}\right\}d\rho(y)$$

$$= \sum_{i=n}^{n'-1}\int_{n-1}^{n}\frac{B_r^{(n)}(y)}{r!}d\rho(y+i-n+1)+\sum_{i=n}^{n'-1}\sum_{\nu=1}^{i-n+1}\int_{i-\nu}^{i-\nu+1}\frac{B_{r-1}^{(n-1)}(y)}{(r-1)!}d\rho(y+\nu)$$

and

$$\int_{-m''}^{0}\frac{B_r^{(n)}(y)}{r!}d\rho(y)=\sum_{i=0}^{m''-1}\int_0^1\frac{B_r^{(n)}(y)}{r!}d\rho(y-i-1)-\sum_{i=0}^{m''-1}\sum_{\nu=0}^{i}\int_{-i+\nu-1}^{-i+\nu}\frac{B_{r-1}^{(n-1)}(y)}{(r-1)!}d\rho(y-\nu).$$

Thus

$$\int_\alpha^\beta\frac{B_r^{(n)}(y)}{r!}d\rho(y)=\int_0^n\frac{B_r^{(n)}(y)}{r!}d\rho(y)+\int_0^1\frac{B_r^{(n)}(y)}{r!}d\rho'(n|y)+\int_{n-1}^{n}\frac{B_r^{(n)}(y)}{r!}d\rho''(n|y)$$

$$+\int_{-m''}^{0}\frac{B_{r-1}^{(n-1)}(y)}{(r-1)!}d\rho'_{n-1}(y)+\int_{n-1}^{n'-1}\frac{B_{r-1}^{(n-1)}(y)}{(r-1)!}d\rho''_{n-1}(y)$$

where

$$\rho'(n|y)=\sum_{\nu=0}^{m''-1}\rho(y-\nu-1),\qquad \rho''(n|y)=\sum_{\nu=0}^{n'-n-1}\rho(y+\nu+1)$$

$$\rho'_{n-1}(y)=-\sum_{\nu=0}^{m''+j}\rho(y-\nu)\quad (j\leq y<j+1;\ j=-m'',-m''+1,\dots,-1)$$

$$\rho''_{n-1}(y)=\sum_{\nu=1}^{n'-j-1}\rho(y+\nu)\quad (j\leq y<j+1;\ j=n-1,n,\dots,n'-2)$$

The coefficient $f_r(n;\rho,\alpha,\beta)$ has thus been expressed in terms of a coefficient of the form $f_r(n;\rho(n),0,n)$ and an integral involving the polynomial $B^{(n-1)}_{z-1}$ over the range $[-m'', n'-1]$. The latter integral may be reduced to a form involving a coefficient $f_{z-1}(n-1; \rho(n-1), 0, n-1)$ and an integral involving the polynomial $B^{(n-2)}_{z-2}$ over the range $[-m'', n'-2]$. Continuing this reduction

$$f_z(n; \rho, \alpha, \beta) = \sum_{r=1}^{n} f_{z-n+r}(r; \hat{\rho}(r), 0, r) + f_{z-n}(0; \tilde{\rho}, \alpha, \beta-n)$$

where

$$f_{z-n}(0; \tilde{\rho}, \alpha, \beta-n) = \int_{\alpha}^{\beta-n} \frac{y^{z-n}}{(z-n)!} d\tilde{\rho}(y)$$

the functions $\hat{\rho}(r)$ and $\tilde{\rho}$ being as described in Definition ... . Over the range $[0, r]$ the functions $B^{(r)}_z$ and $\bar{B}^{(r)}_z$ are the same $(r = 1, \ldots, n)$. ~~The~~ ~~series~~ When $x \in \mathfrak{D}(x^-, x^+)$, the series

$$\sum f_{r+m+\nu}(r; \hat{\rho}(r), 0, r)\, \mathfrak{D}^{r+m+\nu}\psi(x)\omega^{\nu}$$

represents $E^+(r, m; \hat{\rho}(r), 0, r; \psi, x|\omega)$ asymptotically ~~tends to~~ as ω tends to zero in the sector $\Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, $(r = 1, \ldots, n)$. As is easily verified, the series

$$\sum f_{m+\nu}(0; \tilde{\rho}, \alpha, \beta-n)\, \mathfrak{D}^{m+\nu}\psi(x)\omega^{\nu}$$

represents $E(0, m; \tilde{\rho}, \alpha, \beta-n; \psi, x|\omega)$ in the same way. Combining these results, it follows that $\mathfrak{J}(n; \rho, \alpha, \beta; \psi, x|\omega)$ represents $F^+(n; \rho, \alpha, \beta; \psi, x)$ asymptotically in the sector $\Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. If $\alpha \geq 0$ or $\beta \leq n$, the reductions described above are a little simpler; when both conditions hold, no reduction is necessary.

The further result of clause Aiii), consequent upon the special form assumed for ψ, follows from part — of Theorem —.

The above reductions are symmetry preserving in the sense that if $\rho \in \hat{B}^{n;}_{\infty}(\alpha,\beta)$, then ~~$\hat{\rho}(n) \in \hat{B}_{\infty}(\alpha,\beta)$, $\hat{\rho}(n-1) \in \tilde{B}_{\infty}(\alpha,\beta)$,~~ $\hat{\rho}(n-2i) \in \hat{B}^{n-2i;}_{\infty}(\alpha,\beta)$, $\hat{\rho}(n-2i-1) \in \tilde{B}^{n-2i-1;}_{\infty}(\alpha,\beta)$ for all i such that these functions are defined and $\tilde{\rho} \in \hat{B}(0;\alpha,\beta)$ if n is even and $\tilde{\rho} \in \tilde{B}(0;\alpha,\beta)$ if n is odd; ~~interchanging the hat and tilde consistently, a similar~~ the corresponding result derived by interchanging the hat and tilde consistently also ~~holds. The result of clause Aiii) now follows from the connect~~ holds. The series $\vartheta(n;\rho,\alpha,\beta;\psi,x)$ may, under the assumptions of clause Aiii) may be decomposed into a finite sum ~~involving~~ of series involving only even or odd powers of ω and each of which, by use of clause — of Theorem — and an independent proof concerning the function $E(0,m;\tilde{\rho},\alpha,\beta,n;\psi x)$, an a subsidiary result of the same form as that stated in clause Aiii) holds. These subsidiary results may be combined to obtain the general result given.

The results of parts B and C, concerning convergence and summability, are derived in the same way.

Theorem . A) With $\rho \in B_{\infty}(\alpha,\beta)$ and $h \in (-\infty,\infty)$, define ρ_h over $[\alpha+h,\beta+h]$ by means of the formula $\rho_h(y) = \rho(y-h)$ $(\alpha+h \le y \le \beta+h)$. ~~For certain x and ω, let $F(n;\rho,\alpha,\beta;\psi,x|\omega)$ satisfy one of the conditions~~

~~certain~~

Let ψ and ρ be such that (as a consequence of one of the results of theorem ......,) $F^{+}(n;\rho,\alpha,\beta;\psi,x|\omega)$ is analytic in the neighbourhood of x_0 for a certain ω! Then ~~for sufficiently small real h, the~~ ~~for which, when h is~~ such that for all $h \in (h^-,h^+)$ an interval (h^-,h^+) containing the origin exists the two functions $F^{+}(n;\rho,\alpha,\beta;\psi,x+h\omega|\omega)$ and $F^{+}(n;\rho_h,\alpha+h,\beta+h;\psi,x|\omega)$ are, for the ω in question, ~~both analytic in the neighbourhood of~~ ~~identical~~ x and are identical. A similar result holds with regard to the function F^{-}.

B) i) Let $\rho \in B_{\infty}(\alpha,\beta)$ where $[\alpha,\beta] \subset (-\infty,\infty)$, and let $\psi \in M(M,x^-,x^+)$. Let $\alpha' = \min(0,\alpha)$, $\beta' = \max(0,\beta-1)$, $\omega^{+} = (x^{+}-x^{-})/(\beta'-\alpha'+1)$.

a) Set $\hat{x}^{-}(\omega) = x^{-} - \alpha' \operatorname{Re}(\omega)$, $\hat{x}^{+}(\omega) = x^{+} - (\beta'+1)\operatorname{Re}(\omega)$. With $\omega \in S(0,\omega^{+})$ fixed, ~~the functions $F^{+}(r;\rho,\alpha,$~~ the functions $F^{+}(r;\rho,\alpha,\beta;\psi,x|\omega)$ ~~are analytic over~~ $(r=1,2,\dots)$ are analytic over the strip $S(\hat{x}^{-}(\omega),\hat{x}^{+}(\omega))$ in the x-plane, and satisfy the system of difference equations

$$\underset{\omega}{\Delta} F^{+}(1;\rho,\alpha,\beta;\psi,x|\omega) = \int_{\alpha}^{\beta} \psi(x+y\omega)\,d\rho(y)$$

$$\underset{\omega}{\Delta} F^{+}(r+1;\rho,\alpha,\beta;\psi,x|\omega) = F^{+}(r;\rho,\alpha,\beta;\psi,x|\omega)$$

for $r=1,2,\ldots$, where $\underset{\omega}{\Delta} g(x) = \{g(x+\omega)-g(x)\}/\omega$.

b) Set $\tilde{x}^{-}(\omega) = x^{-}-(\alpha'-1)\,\mathrm{Re}(\omega)$, $\tilde{x}^{+}(\omega) = x^{+}-\beta'\,\mathrm{Re}(\omega)$. With $\omega\in(-\omega',\omega)$ fixed, the functions $F^{-}(r;\rho,\alpha,\beta;\psi,x|\omega)$ $(r=1,2,\ldots)$ are analytic over the strip $S(\tilde{x}^{-}(\omega),\tilde{x}^{+}(\omega))$ in the x-plane, and satisfy ~~a~~ ~~s~~ system of difference equations analogous to those given above.

(ii) Under the conditions of clause Bia(β) of theorem —, the functions F^{+} concerned satisfy the above difference equations for $x\in S(x^{-}-\alpha'\,\mathrm{Re}(\omega),\infty)$; under those of clause Biib(β), the functions F^{-} concerned satisfy them for $x\in S(-\infty,x^{+}-\alpha'\,\mathrm{Re}(\omega))$; under those of clause Bib(β), the functions F^{-} concerned satisfy them for $x\in S(x'-\beta''\,\mathrm{Re}(\omega),\infty)$ where $\beta''=\max(0,\beta-1)+1$, and under those of clause Biib(β), the functions F^{+} concerned satisfy them for $x\in S(-\infty,x^{+}-\beta''\,\mathrm{Re}(\omega))$.

(iii) Where, as in ~~clauses~~ part C and clause Fii of theorem - it has been shown that for a certain ω, $F^{\pm}$ ~~or F~~ are entire functions of x, the functions concerned satisfy the above difference equations for all finite complex x.

C. With $\rho^{(1)}=\rho\in B_{00}(\alpha,\beta)$ define further functions $\rho^{(r)}$ $(r=2,\dots,n)$ on $[r\alpha, r\beta]$ by setting $\rho^{(r)}(y)=\int_{\alpha}^{\beta}\rho^{(r-1)}(y-y')\,d\rho(y')$ when $r<n$, and extending $\rho^{(r)}$ over $(r\beta,(r+1)\beta-\alpha]$ by setting $\rho_r(y)=\rho_r(r\beta)$ ~~at~~ in this interval and over $[(r+1)\alpha-\beta, r\alpha)$ by setting $\rho_r(y)=\rho_r(r\alpha)$ there. Set $F_0^{\pm}(x)=\psi(x)$ and $F_r^{\pm}(x)=F^{\pm}(r;\rho_r,r\alpha,r\beta;\psi,x|\omega)$ $(r=1,\dots,n)$ and $\alpha'(n)=\min(0,n\alpha)$, $\beta'(n)=\max(0,n(\beta-1))$.

(i) Let $[\alpha,\beta]\subset(-\infty,\infty)$ and $\psi\in M(M;x^-,x^+)$ and set $\omega^+=(x^+-x^-)/(\beta'(n)-\alpha'(n))$.

a) Set $\hat{x}^-(\omega)=x^- -\alpha'(n)Re(\omega)$, $\hat{x}^+(\omega)=x^+-\beta'(n)Re(\omega)$. With $\omega\in S(0,\omega^+)$ fixed, ~~the functions $F_r^{\pm}(x)$ $(r=0,1,\dots,n)$ are anal~~ ~~$F_r^+\in A_x\{S\{\hat{x}^-(\omega),\hat{x}^+(\omega)\}\}$ $(r=0,1,\dots,n)$, and these functions satisfy~~ ~~the may alternatively be constructed by use of the relationships~~

$$F^+(m;\rho_m,m\alpha,m\beta;F_r,x|\omega)=F^+_{m+r}(x)$$

for ~~$r=0,1,$~~ $m=1,\dots,n$, $r=0,1,\dots,n-m$ and all $x\in S\{\hat{x}^-(\omega),\hat{x}^{+}(\omega)\}$.

b) Set $\tilde{x}^-(\omega)=x^- -\beta'(n)Re(\omega)$; $\tilde{x}^+(\omega)=x^+-\alpha'(n)Re(\omega)$. With $\omega\in S(-\omega^+,0)$ fixed, ~~$F_r^-\in A_x\{S\{\tilde{x}^-(\omega),\tilde{x}^+(\omega)\}\}$ $(r=0,\dots,n)$ and these functions satisfy~~ relationships similar to ( ) involving functions F^- hold over $S\{\tilde{x}^-(\omega),\tilde{x}^+(\omega)\}$

(ii) Under the conditions of clause Biaβ) of Theorem ..., relationships of the form ( ) hold over $S(x^- - \alpha'(n)Re(\omega),\infty)$; under those of clause

~~Biib/3) similar relationships for~~

Biib/3) they hold over $S(-\infty, x^+ - \beta'(n) Re(\omega))$; under those of clause

Biib/3) similar relationships involving functions F^- hold over $S(x^- - \beta'(n) Re(\omega), \infty)$ and under those of clause Biiia/3), they hold over $S\cancel{(-\infty, x^+ - \beta'(n) Re(\omega))}$ $S(-\infty, x^+ - \alpha'(n) Re(\omega))$

(iii) Where, as in part C and clause Fii) of Theorem – , it has been shown that for a certain ω, $F^\pm$ are entire functions of x, the functions concerned satisfy relationships of the form ( ) for ~~all finite com~~ $m = 1, 2, \ldots, r = 0, 1, \ldots$ and all finite complex x.

~~Proof. It has been shown that~~

Proof. To any of the results of theorem in which it has been shown that $F^+(n;\rho,\alpha,\beta;\psi,x|\omega)$ is for a certain ω analytic in the neighbourhood of x, there corresponds a result to be derived from part C of Theorem stating that ${}^{\omega}J(n;\rho,\alpha,\beta;x)$ is summable $(B',1)$ to the value of this function. It is clear that for the above fixed x and ω and sufficiently small real h $F^+(n;\rho_h,\alpha+h,\beta+h;\psi,x|\omega)$ is analytic in the neighbourhood of x. In short, ~~for~~ over an ~~proper interval~~ open interval containing the origin, the series

$$\sum\nolimits_1' \int_{\alpha}^{\beta} \frac{B_{\nu}^{(n)}(y+h)}{\nu!} d\rho(y) \Delta^{\nu-n} \psi(x-h\omega)\omega^{\nu}$$

is summable to a function, $S(h;x,\omega)$ say. The derivative with respect to h of $S(h,x,\omega)$ is the difference between the $(B',1)$ sums of the series

$$\sum\nolimits_1' \int_{\alpha}^{\beta} \frac{B_{\nu-1}^{(n)}(y+h)}{(\nu-1)!} d\rho(y) \Delta^{\nu-n} \psi(x-h\omega)\omega^{\nu}$$

and

$$\sum\nolimits_1' \int_{\alpha}^{\beta} \frac{B_{\nu}^{(n)}(y+h)}{\nu!} d\rho(y) \Delta^{\nu-n+1} \psi(x-h\omega)\omega^{\nu+1}$$

This difference is easily verified to be zero. In short $S(h,x,\omega)=F^+(n;\rho,\alpha,\beta;\psi,x|\omega)$ for all h in the stated interval. ~~and~~ Replacing x by $x+h\omega$, ~~$S(h,x+h\omega,\omega)=F^+(n;\rho,\alpha,\beta;\psi,x+h\omega|\omega)$~~

$F^+(n;\rho_h,\alpha+h,\beta+h;\psi,x|\omega)=F^+(n;\rho,\alpha,\beta;x+h\omega|\omega)$ as required

That, under the conditions of clause B i), the functions $F^{+}(r;\rho,\alpha,\beta;\psi,c|\omega)$ are analytic as described, follows from clause ( ) of Theorem . Furthermore of when $h = 0, 1$,

$F^{+}(r;\rho,\alpha,\beta;\psi+h\omega|\omega)$ is the $(B',1)$ sum of the series

$$\sum \int_{\alpha}^{\beta} \frac{B_{\nu}^{(r)}(y+h)}{\nu!} d\rho(y)\, \mathcal{D}^{\nu-r}\psi(x)\,\omega^{\nu}.$$

The difference between the values of these two functions is the $(B',1)$ sum of the difference series whose terms involve polynomials of the form $B_{\nu}^{(r)}(y+1) - B_{\nu}^{(r)}(y) = \nu B_{\nu-1}^{(r-1)}(y)$. Replacing r by $r+1$, relationship ( ) is obtained. The special result ( ) is easily verified. The further results of part B are derived in the same way.

~~Under the stated conditions, relationship ( ) holds when $m+r=1$~~ ~~evidently holds when $m+r=1$~~. Assume that it holds for all appropriate $m+r \leq i$, ~~where~~ for some $i < m$. This is easily verified by use of series of the form ( )

$$\underset{\omega}{\Delta} F^{+}_{m+r}(x) = \int_{\alpha}^{\beta} F_{m+r-1}(x+y\omega)\,d\rho(y)$$

$$\underset{\omega}{\Delta} F^{+}(m;\rho_m, m\alpha, m\beta; F_r, x|\omega) = \int_{\alpha}^{\beta} F^{+}(m-1;\rho_{m-1},(m-1)\alpha,(m-1)\beta;F_r, x+y\omega)\,d\rho(y)$$

$$= \phi(m,\overset{+}{r},x)$$

Set $F^{+}_{m+r}(x) - F^{+}(m;\rho_m, m\alpha, m\beta; F_r, x|\omega)$. From the above two relationships, $\underset{\omega}{\Delta}\phi(m,r,x) = 0$, so that $\phi(m,\overset{+}{r},x)$ is periodic in x with period ω. Decompose ψ in the form $\psi^{+}+\psi^{-}$ given by relationship ( ). With ψ replaced by ψ^{+}, $F^{+}_{m+r}(x)$ is analytic in

x for arbitrarily large positive $\mathrm{Re}(x)$, and as $\mathrm{Re}(x)$ tends to infinity ~~in the direction $\arg(x) = \arg(\omega)$~~, $\Delta^{m+r+M+1} F^{+}_{mr}(x)$ tends to zero. The same is true of the other constituent ~~ϕ~~ of $\phi(m, r, \psi^{+}, x)$. Hence $\Delta^{m+r+M+1} \phi(m, r, \psi^{+}, x)$ ~~$= 0$ identically, and~~ tends to zero as x tends to infinity in the ~~manner des~~ in the direction $\arg(x) = \arg(\omega)$ and since ϕ is periodic, $\phi(m, r, \psi^{+}, x) = 0$ identically. ~~Similarly~~ Considering the behaviour of $\phi(m, r, \psi^{-}, x)$ for large negative $\mathrm{Re}(x)$, it is finally shown that $\phi(m, r, \psi, x) = 0$ identically. Relationship ( ) is true as stated. The further results of part C are derived in the same way.

~~Under the~~

It is clear that under the conditions of clause B(a), the functions $F^{+}_{mr}(x)$ upon the left hand side of relationship ( ) are well-defined for the stated values of x and ω. Furthermore (~~from clause~~ by use of clause ( ) of Theorem), ~~$\Delta^{r+M} F_r(x)$~~ for fixed $\omega \in S(0, \omega^{+})$, $1 \le r \le n$, $\Delta^{r+M} F_r \in M(0; x^{-}(\omega), x^{+}(\omega))$: the functions upon the right hand side of relationship ( ) also exist for the stated values of m and r. Relationship ( ) itself evidently holds when $m+r = 1$.

Notation With ${}_1\alpha \in \mathcal{B}_{21}(\omega, {}_1s)$ set

$$\bar{g}_{\nu}(n; \rho, \alpha, {}_1 s) = \int_{\alpha_0}^{\rho_0} \frac{\bar{E}_{\nu}^{(n)}(u)}{\nu!} \, d\, {}_1\rho(u)$$

for $n = 1, 2, \ldots$; $\nu = 0, 1, \ldots$. For suitably defined ψ, and $n = 1, 2, \ldots$

$j = 0, 1, \ldots$

and $\bar{\mathcal{Y}}_e(n, j; \rho, \alpha, {}_1 s; \psi, z \mid \omega)$ be the series

let $\bar{\mathcal{Y}}(n, j; \rho, \alpha, {}_1 s; \psi, z \mid \omega)$ be the series

$$\sum \bar{g}_{j\nu}(n; \rho, \alpha, {}_1 s) \, \omega^{j\nu} \psi(z) \omega^{\nu}$$

for $j = 0, 1, \ldots$, ~~let~~ and let $\bar{\mathcal{Y}}_e(n, j; \rho, \alpha, {}_1 s; \psi, z \mid \omega)$ be the series

$$\sum \bar{g}_{j\nu}(n; \rho, \alpha, {}_1 s) \, \omega^{j\nu} \psi(z) \omega^{\nu}$$

respectively

Notation With $\rho \in \mathcal{B}_n(\alpha,\beta)$, set

$$b_\nu(n;\rho,\alpha,\beta) = \int_\alpha^\beta \frac{\overset{(n)}{E}(y)}{(2\nu)!}\, d\rho(y)$$

for $n = 1,2,\ldots, \nu = 0,1,\ldots$. For ~~$n=1,2,\ldots; j=0,1,\ldots$ let $\mathcal{B}$ suitably defined~~ ψ and $n = 1,2,\ldots; j = 0,1,\ldots$ let $\mathcal{B}(n,j;\rho,\alpha,\beta;\psi,x|w)$ ~~be and~~

~~$\hat{\mathcal{B}}(n,j;\rho,\alpha,\beta;\psi,x|w)$ and $\tilde{\mathcal{B}}(n,j;\rho,\alpha,\beta;\psi,x|w)$~~ be the series

$$\sum b_\nu(n;\rho,\alpha,\beta)\, \Delta^{j\nu}\psi(x)\omega^\nu$$

for $j = 0,1,\ldots$ let $\mathcal{A}(n,j;\rho,\alpha,\beta;\psi,x|w)^{j+\nu}$ be

$$\sum_{\nu = j - p(n) + \nu - 1} b_{p(n)+2\nu}(n;\rho,\alpha,\beta)\, \Delta^{j+p(n)+2\nu}\psi(x)\omega^{p(n)+2\nu}$$

for $j = 1,2,\ldots$, and let $\mathcal{E}(n,j;\rho,\alpha,\beta;\psi,x|w)$ be the series

$$\sum_{\nu = j - p(n) + 2\nu} b_{p(n)+2\nu+1}(n;\rho,\alpha,\beta)\, \Delta^{2j+p(n)+2\nu}\psi(x)\omega^{p(n)+2\nu+1}$$

for $j = p(n), p(n)+1,\ldots$ respectively, where ~~$p(n)=0$ if n is even and~~ $p(n) = n - 2\lfloor n/2\rfloor$.

Theorem. A Let $\psi \in M(M; x^-, x^+)$

(i) a) For $r = M, M+1,\ldots$ $(r > \nu)$, $\tau = 0,1,\ldots$ and $x \in \mathfrak{S}(x^-,x^+)$

~~$C_\nu(n;\rho,\alpha,\beta$~~

$$b_{r+\tau}(n;\rho,\alpha,\beta)\, \Delta^{r+\tau}\psi(x) = i^\tau \sum_{l=1}^{n} \binom{-l}{\tau} \int_{-\infty}^{\infty} t^\tau\, d\sigma(n;r,l;\rho,\alpha,\beta;\psi,x|t)$$

where

$$\sigma(n,r,l;\rho,\alpha,\beta;\psi,x|t) = \frac{(-1)^{n+l}2^{n-1}}{2\pi i} \sum_{j=l-1}^{n-1} \binom{r+j-l+1}{r} \int_{-\infty}^{t}\int_\alpha^\beta \frac{D_{n-j-1}(y)\overset{(n)}{\tilde{E}}_{l,j}(y+lw)}{(n-j-1)!\,(r+j)!}\, \frac{\Delta^r\psi(x+lw) - \Delta^r\psi(x)}{w}\, d\rho(y)\,dw$$

(where $\mathfrak{D} = d/dx$) for $t(\neq 0) \in (-\infty, \infty)$, and in terms of formula ( )

$$\sigma(n, r, k; \rho, \alpha, \beta; \psi, x \mid \pm\infty) = \tilde{\sigma}(n, r, l; \rho, \alpha, \beta; \pm 1/2\pi)\, {}^{\omega}\mathfrak{D}^{r}\psi(x)$$

b) If $\psi \in M(0; \infty, \cdot)$ and

$$\sum_{j}{}'' \left| \int_{\alpha}^{\beta} e^{(2\lambda-1)i\pi j} \right| / (2\lambda - 1) \,|\, < \infty$$

then the representations of formulae (,) also hold for $r = 0, 1, \ldots; \tau = 0, 1, \ldots$

ii) For $m = K, K+1, \ldots$ $(K = \max(M, 1))$ let $G^{+}(n, m; \rho, \alpha, \beta; \psi, x \mid \omega)$ be the function defined over the strip $x \in \mathfrak{S}(x^{-}, x^{+})$ and the half-plane $\omega \in H^{+}$ by means of the formula

$$B^{+}(n, m; \rho, \alpha, \beta; \psi, x \mid \omega) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} \frac{d\sigma(n, m, l; \rho, \alpha, \beta; \psi, x \mid t)}{(1 + i\omega t)^{l}}$$

Define the functions $B^{+}(n, m; \rho, \alpha, \beta; \psi, x \mid \omega)$ $(m = 0, 1, \ldots, K-1)$ by means of the formulae

$$B^{+}(n, K-z; \rho, \alpha, \beta; \psi, x \mid \omega) = c_{K-z}(n; \rho, \alpha, \beta)\, {}^{\omega}\mathfrak{D}^{K-z}\psi(x) + \omega B^{+}(n, K-z+1; \rho, \alpha, \beta; \psi, x \mid \omega)$$

for $z = 1, \ldots, K$.

a) For fixed $\omega \in H^{+}$, $B(n; m; \rho, \alpha, \beta; \psi \mid \omega) \in M_{\infty}^{m(K)}(j(m); x^{-}, x^{+})$ and $\mathfrak{D}^{j} \psi$ $(j = 0, 1, \ldots, m(K))$, where $j(m)$ ~~if $m \geq K$ and $j(m) = K - m$ if $m < K$~~ $j(m) = \max(0, K - m)$

b) For $m, \tau = 0, 1, \ldots$, $x \in \mathfrak{S}(x^-, x^+)$ and $\omega \in H^+$

$$B^+(n, m; \rho, \alpha_1, \iota; \psi, x|\omega) = B(n, m; \tau; \rho, \alpha_1, \iota; \psi, x|\omega) + \omega^\tau B(n, m+\tau; \rho, \alpha_1, \iota; \psi, x|\omega)$$

c) For $m = 0, 1, \ldots$ and fixed $x \in \mathfrak{S}(x^-, x^+)$

$$B^+(n, m; \rho, \alpha_1, \iota; \psi, x|\omega) \sim B(n, m; \rho, \alpha_1, \iota; \psi, x|\omega)$$

as ω tends to zero in $\Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$

d) If the conditions of clause ib) hold, the function $G^+(n, 0; \rho, \alpha_1, \iota; \psi, x|$ $\omega)$ have a direct representation of the form ( ), and use of formulae such as ( ) to define these functions may be dispensed with.

(iii) a) If $\rho \in \hat{B}_n(\alpha_1 \iota)$ then $b_{2\nu}(\rho; \alpha_1, \iota) = 0$ and if $\rho \in \tilde{B}_n(\alpha_1 \iota)$, $b_{2\nu+1}(\rho; \alpha_1, \iota) = 0$ ~~($\nu = 0, 1, \ldots$)~~ for $\nu = 0, 1, \ldots$ in both cases

b) In equation ( )

$$\sigma(n, m, l; \rho, \alpha_1, \iota; \psi, x|-t) = \sigma(n, m, l; \rho, \alpha_1, \iota; \psi, x|t)$$

for $l = 1, \ldots, n$, $t \in (-\infty, \infty)$ and all $m \geq K$ for which $m-n$ is odd when $\rho \in \hat{B}_n(\alpha_1 \iota)$ and $m-n$ is even when $\rho \in \tilde{B}_n(\alpha_1 \iota)$

c) Let the functions $B^+(n, m; \rho, \alpha_1, \iota; \psi, x|\omega)$ be as defined in clause (ii), and let $x \in \mathfrak{S}(x^-, x^+)$ be fixed

If $\rho \in \hat{B}(\alpha_1 \iota)$:

α) for $j = 0, 1, \ldots$

$$\mathcal{B}^{+}(n, p(n)+2j ; \rho, \alpha, \beta; \psi, x|\omega) = \omega \mathcal{B}^{+}(n, p(n)+2j+1 ; \rho, \alpha, \beta; \psi, x|\omega)$$

β) For $j = 1-p(n), 2-p(n), \ldots, \tau = 0, 1, \ldots$

$$\overset{2j-1}{\mathcal{B}^{+}}(n, p(n)+2j-1 ; \rho, \alpha, \beta; \psi, x|\omega) =$$

$$\overset{2j+1,\tau}{\hat{\mathcal{B}}_e}(n; p(n)+2j-1+\tau ; \rho, \alpha, \beta; \psi, x|\omega)$$
$$+ \omega^{2\tau} \overset{2j-1+2\tau}{\mathcal{B}^{+}}(n, p(n)+2j+2\tau-1 ; \rho, \alpha, \beta; \psi, x|\omega)$$

γ) For $j = 1-p(n), 2-p(n), \ldots$

$$\overset{2j+1}{\mathcal{B}^{+}}(n, p(n)+2j-1 ; \rho, \alpha, \beta; \psi, x|\omega) \sim \overset{2j+1}{\mathcal{B}_e}(n, p(n)+2j-1 ; \rho, \alpha, \beta; \psi, x|\omega)$$

as ω tends to zero in $\Delta(-\frac{1}{2}\pi, \frac{1}{2}\pi)$

If $\rho \in \tilde{\mathcal{B}}_n(\alpha, \beta)$, the above results α)–γ) hold with $\overset{2j+1}{2j}$ replaced consistently by $\overset{2j}{2j-1}$, j running from and $j =$

δ) If condition ( ) holds, relationship ( ) holds for all $m \geq M$ for which $m-n$ is odd when $\rho \in \hat{\mathcal{B}}_n(\alpha, \beta)$ and $m-n$ is even when $\rho \in \tilde{\mathcal{B}}_n(\alpha, \beta)$

(iv) Functions $\mathcal{B}^{-}(n, m; \rho, \alpha, \beta; \psi, x|\omega)$ may be defined over H^{-} by formulae similar to (,) and to each of the above results stated for $\mathcal{B}^{+}(n, m; \rho, \alpha, \beta; \psi, x|\omega)$ over H^{+} there corresponds a result for $\mathcal{B}^{-}(n, m; \rho, \alpha, \beta; \psi, x|\omega)$ over H^{-}; the asymptotic results corresponding to (,) hold over the sector $\Delta(\frac{1}{2}\pi, \frac{3}{2}\pi)$

Theorem . With $\rho \in \mathfrak{B}_n(\alpha,\varepsilon)$, $\psi \in \mathfrak{M}(M; x^-, x^+)$ and $x \in \mathfrak{Z}(x^-, x^+)$, let the functions $\mathfrak{B}^{\pm}(n, m; \rho, \alpha, \beta; \psi, x|w)$ be as described in Theorem .

(i) For $j = 0, 1, \ldots$, $\mathfrak{B}(n, m; \rho, \alpha; \psi, x)$ is

a) summable $(B', 1)$ to $\mathfrak{B}^+(n, m; \rho, \alpha, \beta; \psi, x)$ over $[0, \infty)$ and

b) summable $(\bar{B}', 1)$ to this function over H^+

(ii) Let $\rho \in \hat{\mathfrak{B}}_n(\alpha, \varepsilon)$. For $j = 1 - p(n), 2 - p(n), \ldots$ $\hat{\mathfrak{B}}_2^{2j+1}(n, p(n) + 2j - 1; \rho, \alpha, \beta; \psi, x)$

is and summable(B^+) $2j+1$

a) summable $(B', 2)$ to $\mathfrak{B}^+(n, p(n) + 2j - 1; \rho, \alpha, \beta; \psi, x)$ ~~and~~ over $[0, \infty)$

and

b) summable $(\bar{B}, 2)$ and summable $(\bar{B}^2)$ to this function over H^+

If $\rho \in \tilde{\mathfrak{B}}_n(\alpha, \varepsilon)$, the above results hold with $2j$ replaced consistently by $2j+1$.

(iii) Results corresponding to those of clause (i) may be formulated for the functions $\mathfrak{B}^-$ by replacing $[0, \infty)$ and H^+ by $(-\infty, 0]$ and H^-; results corresponding to those of clause (ii) for these functions may be obtained in a similar manner.

Theorem . ~~Let $\psi_1 \in M(M, x^-, x^+)$.~~ Set $n' = [n/2]$ ~~and in the following~~

~~Let $z \in (x^-, x^+)$.~~

$\phi', \phi'' \in [0, 2\pi)$ and in the following let $z \in (x^-, x^+)$

~~(i)~~ Let $\psi(z) = e^{i\phi'}\psi_\phi(z)$ where $\phi \in [0, 2\pi)$ ~~ψ_1 than~~ $\psi_\phi \in M(M, x^-, x^+)$ and in

(i) In

the representation ( ) for ψ' let ~~σ is nondecreasing~~ $v^{M-1} d\sigma(v) \geq 0$ for all real v. Let $p \in \hat{E}_1(x_1, z)$ and let

$$4e^{-i\phi''}(-1)^{j+1}\int_{\alpha}^{p} B^{(n)}_{n-2j-1}(u)\cos\{(2\nu+1)\pi u\}, \quad e^{-i\phi''}(-1)^{j}\int_{\alpha}^{p} B^{(n)}_{n-2j-2}(u)\sin\{(2\nu+1)\pi u\}$$

be real and nonnegative for $\nu = 0, 1, \ldots$ and $j = 0, 1, \ldots, n' + p(n) - 1$ in the first integral and $j = 0, 1, \ldots n' - 1$ in the second. Let ~~$2\mu+1$~~ $2m+1 \geq \max(M, n+1)$.

a) The series $\bar{\mathcal{G}}_e(n, 2m+1; \nu, \alpha, \beta; \psi, z \mid \omega)$ is semi-convergent over $\bar{\Delta}(-\frac{1}{4}\pi, \frac{1}{4}\pi)$ in the sense that relationship ( ) holds and

$$|\cancel{\omega}\, G(n, 2m+2\tau+1; \nu, \alpha, \beta; \psi, z \mid \omega)| \leq$$

$$-\mathcal{g}^{(r)} \big| \mathcal{S}_{2m+2\tau+1}(n; \nu, \alpha, \beta)\, \Delta^{2m+2\tau+1} \psi(z)\, \omega^{2\tau} \big|$$

when $\omega \in \bar{\Delta}(-\frac{1}{4}\pi, \frac{1}{4}\pi)$. $(\tau = 0, 1, \ldots)$

b) When $\omega \in [\ldots)$ the partial sums of $\bar{\mathcal{G}}_e(n, 2m+1; \nu, \alpha, \beta; \psi, z \mid \omega)$ oscillate about the value of $\bar{G}^{+}(n, 2m+1; \nu, \alpha, \beta; \psi, z \mid \omega)$

setting $c = (-1)^{n'} e^{\mp\phi - i(\phi' + \phi'')}$

$$-c\bar{G}(n, 2m+2r+1; \rho, \alpha, \beta; \psi, z|\omega) < c\bar{G}(n, 2m+1; \rho \ldots$$

$$\leq c\bar{G}(n, 2m+1, 2r+1; \rho, \alpha, \beta; \psi, z|\omega) < c\bar{G}^{+}(n, 2m+1; \rho, \alpha, \beta; \psi, z|\omega)$$

$$< c\bar{G}(n, 2m+1, 2r; \rho, \alpha, \beta; \psi, z|\omega)$$

for $r = 0, 1, \ldots$

(ii) In the representation ( ) for ψ' let $v^{M} d\sigma(v)$ for all real v. Let $\rho \in \tilde{\mathfrak{B}}(\alpha, \beta)$, and let the values of the integrals ( ) with $\cos\{(2r+1)\pi u\}$ and $\sin\{(2r+1)\pi u\}$ interchanged and $(-1)^{s\psi}$ replaced by $(-1)^{s\psi}$ in the ~~first~~ second. Let $2m \geq \max(M, n)$. For the newly restricted value of m, the above results with $2m+1$ replaced by $2m$ hold

(iii) Results similar to those stated above also hold with regard to the functions $\bar{G}^{-}$ ~~defined~~ over $\bar{\Delta}(\frac{3}{4}\pi, \frac{5}{4}\pi]$ and $(-\infty, 0]$.

Theorem Let $\psi \in \mathfrak{M}(M, \omega, \infty)$ with $\psi(z) = O(e^{\gamma_{\pi}/\beta' \operatorname{Re}(z)})$ as $|\operatorname{Re}(z)|$ tends to infinity; set $\delta = 2\pi/\delta'$. Let $\rho \in \mathcal{P}_n(\alpha, \iota)$

(i) The functions $s^{k}(n, m; \rho, \alpha, \iota; \psi, z/t)$ $(k = 1, \ldots, n)$ in formula ( ) are constant over $[-\infty, -\delta)$ and $(\delta, \infty]$.

(ii) The functions $\bar{G}^{\pm}(n, m; \rho, \alpha, \rho; \psi, z/\omega)$ are the same function $G(n, m; \rho, \alpha, \iota; \psi, z/\omega)$ in the sense that they may be obtained from each other by analytic continuation across the segment $i(-\delta^{-1}, \delta^{-1})$ of the imaginary axis, and this function has the representation

a ... the form ... the limits

$$G(n, m; \rho, \alpha, \iota; \psi, z/\omega) = \sum_{k=1}^{n} \int_{-\delta}^{\delta} \frac{ds(n, m, k; \rho, \alpha, \iota; \psi, z/t)}{(1 + i\omega t)^k}$$

(iii) a) $\bar{J}(n, m; \rho, \alpha, \iota; \psi, z)$ converges to $G(n, m; \rho, \alpha, \iota; \psi, z/\omega)$ over $D_{1/\delta}$ $(m = 0, 1, \ldots)$

b) Let $\rho \in \hat{\mathcal{B}}_n(\alpha, \iota)$. For $m = 0, 1, \ldots$ $G_e(n, 2m+1; \rho, \alpha, \iota; \psi, z)$ converges to $G(n, 2m+1, \rho, \alpha, \iota; \psi, z)$ over $D_{1/\delta}$

c) Let $\rho \in \overset{\infty}{\mathcal{P}}_n(\alpha, \iota)$. The preceding result with $2m+1$ replaced by $2m$ holds.

(iv) Let the conditions of clause (i) of Theorem — hold, with $m \cdot 1/z$ restricted and c defined as in that clause.

a) When $\omega \in (-\gamma^{-1}, \gamma^{-1})$ the even and odd order partial sums of

the $\bar{\mathcal{T}}_{e}(n, 2m+1; \nu, \alpha_1, \iota; 4, z|\omega)$ are members of monotonic sequences

of opposing types respectively, in the sense that

$ck(2\tau-1, 2m+1|\omega) < ck(2\tau+1, 2m \ldots ck(2m+1|\omega)$

$$c\bar{G}_e(n, 2m+1; 2\tau-1; \nu, \alpha_1, \iota; 4, z|\omega) < c\bar{G}_e(n, 2m+1; 2\tau+1; \nu, \alpha_1, \iota; 4, z|\omega)$$

$$< c\bar{G}_e(n, 2m+1; \nu, \alpha_1, \iota; 4, z|\omega) < c\bar{G}_e(n, 2m+1; 2\tau; \nu, \alpha_1, \iota; 4, z|\omega)$$

$$< c\bar{G}_e(n, 2m+1, 2\tau; \nu, \alpha_1, \iota; 4, z|\omega)$$

for $\tau = 0, 1, \ldots$

b) When $\omega \in i(-\gamma^{-1}, \gamma^{-1})$, the partial sums of $\bar{\mathcal{T}}_{e}(n, 2m+1; \nu, \alpha_1, \iota; 4, z|\omega)$

form a monotonic sequence in the sense that

$$c\bar{G}(n, 2m+1, \tau-1; \qquad\qquad) < c\bar{G}(n, 2m+1; \tau;$$

$$< c\bar{G}(n, 2m+1;$$

$$c\bar{G}_e(\tau-1, 2m+1|\omega) < c\bar{G}_e(\tau, 2m+1|\omega) < c\bar{G}_e(2m+1|\omega)$$

for $\tau = 0, 1, \ldots$

(v) Let the conditions of clause (ii) of Theorem   hold, with m restricted and c defined as in that clause, the results of the preceding clause with $2m+1$ replaced consistently by $2m$ hold.

Theorem . Let $\rho \in \mathcal{B}_n(\alpha, \iota)$, and $\psi \in M(M; x^-, x^+)$ and $z \in S(x^-, x^+)$

i) Let $V \in (0, \infty)$ and, in the representation ( ), let σ be nonanalytic over either (V, ∞) or $(-\infty, -V)$. The imaginary axis in the ω-plane is a natural barrier ~~of~~ for both functions $G^{\pm}(m, n; \rho, \alpha, \iota; \psi, x | \omega)$

ii) Let $\phi \in (0, \frac{1}{2}\pi)$ and, in the representation ( ), let σ be analytic in the two sectors $\Delta(-\phi, \phi)$, $\Delta(\pi - \phi, \pi + \phi)$ ~~(…)~~ with $\lim \{ v^y \sigma(v) \} = 0$ ($y \in (0, \infty)$ being arbitrarily large) as v tends to infinity in these sectors

a) The functions $G^{\pm}(m, n; \rho, \alpha, \iota; \psi, x | \omega)$ may be continued analytically across both positive and negative parts of the imaginary axis $(m = 0, 1, \dots)$

b) Denoting by $\bar{G}^{\pm}(m, n; \rho, \alpha, \iota; \psi, x | \omega)$ the functions directly defined as in clause ( ) of Theorem and also obtained by analytic continuation, the functions $G^{\pm}(m, n; \rho, \alpha, \iota; \psi, x | \omega)$ are analytic over $\Delta(\mp \frac{1}{2}\pi - \phi, \pi \mp \frac{1}{2}\pi + \phi)$ ~~and $\mathcal{G}(m, n$ and~~ and ~~$\Delta(\frac{1}{2}\pi - \phi, \frac{3}{2}\pi + \phi)$~~ and the series $\mathcal{G}(m, n; \rho, \alpha, \iota; \psi, x | \omega)$ represents the functions asymptotically as ω tends to zero in these sectors $(m = 0, 1, \dots)$

c) ~~Let $\rho \in \hat{\mathcal{B}}_n(\alpha, \iota)$~~ If, in addition, $\rho \in \hat{\mathcal{B}}_n(\alpha, \iota)$, then the series $\mathcal{G}_e(2m+1, n; \rho, \alpha, \iota; \psi, z | \omega)$ represents $\bar{G}^{\mp}(2m+1, n; \rho, \alpha, \iota; \psi, z | \omega)$ as ω tends to zero in the above sectors; while if $\rho \in \tilde{\mathcal{B}}_n(\alpha, \iota)$ the corresponding result with $2m+1$ replaced by $2m$ holds $(m = 0, 1, \dots)$

Theorem Let $\rho \in \mathfrak{S}_1(\alpha, \beta)$, and $\psi \in \mathfrak{M}(M; \alpha, \beta)$ and $z \in (x^-, x^+)$, and ϕ'' $\in [0, 2\pi)$.

(and let σ' be the function occurring in the representation ( ) for ψ'; $\phi' \in [0, 2\pi)$),

Theorem Let $\rho \in \mathfrak{S}_1(\alpha, \beta)$ and $\psi(z) = e^{i\phi'} \psi'(z)$ where $\psi' \in \mathfrak{M}(M, x^-, x^+)$ with r fixed, $\phi'' \in [0, 2\pi)$,

~~and~~ in the following let ~~ϕ~~ $z \in (x^-, x^+)$. Denote by $\mathcal{A}^{(m,r)}_{\psi}$ the diagonal sequence of approximating functions $P^{(m)}_{j, j+r}(\omega)$ $(j = 0, 1, \ldots)$ when $r \geq 0$ and $P^{(m)}_{j-r, j}(\omega)$ $(j = 0, 1, \ldots)$ when ~~$r = 1, 2, \ldots$~~ $r < 0$ derived from the series $\overline{\mathcal{G}}(m, 1; \rho, \alpha, \beta; \psi, z | \omega)$; ~~similarly,~~ denote by $\hat{\mathcal{A}}^{(m,r)}_{e}$ the similar sequence of approximating functions $\hat{P}^{(m)}_{j, j+r}(\omega^2)$ derived from the series $\overline{\mathcal{G}}_e(m, 1; \rho, \alpha, \beta; \psi, z | \omega)$ (viewed as a series in ascending powers of ω^2).

i) Let $m \geq \max(M, 2)$, and ~~in the representation ( ) for ψ' let~~

$v^{m+M} d\sigma'(v) \geq 0$ for all real v. Let

$$e^{i\phi''} \int_{\alpha}^{\beta} e^{(2\nu+1)\pi i u} \, d\sigma(v), \quad (-1)^m e^{i\phi''} \int_{\alpha}^{\beta} e^{-(2\nu+1)\pi i u} \, d\sigma(v)$$

be real and nonnegative for $\nu = 0, 1, \ldots$

a) $\mathcal{A}^{(m,r)}$ converges uniformly to $E^{\pm}(m, 1; \rho, \alpha, \beta; \psi, x)$ over any bounded domain lying in $H^{\pm}$, ~~and to $E(m, 1$~~ $(r = \ldots -1, 0, 1, \ldots)$ ~~$(r = 0, 1, \ldots)$~~

b) Let $\psi(z) = O(e^{\gamma z})$ as $|Re(z)| \to \infty$ for some $\gamma \in (0, \infty)$. (so that $E^{\pm}$ are the same function E; vide clause ( ) of Theorem)

In addition to the general result of the preceding subclause $d^{(m,2r+1)}$ converges uniformly to $E(m,1;\rho,\alpha_1,\beta;\psi,z)$ over any bounded domain lying in $\mathbb{C}\setminus\{\pm i[2\pi/\gamma,\infty]\}$ $(r=0,1,\ldots)$.

(ii) Let $\rho\in\hat{B}_1(\alpha_1,\beta)$, $2m+1>\max(M,2)$, and $v^{M+1}d\sigma(v)\geq 0$ for all real v. Let

$$e^{i\phi'}\int_{-\infty}^{\infty}\sin\{(2\nu+1)\pi u\}d\rho_1(u)$$

be real and nonnegative for $\nu=0,1,\ldots$.

a) $P^{(2m+1)}_{2j+1,2k+1}(w)=\hat{P}^{(2m+1)}_{j,k}(w^2)$ $(j,k=0,1,\ldots;\ \nu=0,1,\ldots)$

b) $d_e^{(2m+1,r)}$ converges uniformly to $E^{\pm}(m,2m+1,m;\rho,\alpha_1,\beta;\psi,z)$ over any bounded domain lying in $H^{\pm}$ $(r=\ldots-1,0,1,\ldots)$

c) If the additional restriction of clause i b) above holds, ~~convergence described in the preceding clause holds over any bounded domain~~ both $d_o^{(2m+1,r)}$ and $d_e^{(2m+1,r)}$ converge uniformly to $E(2m+1,1;\rho,\alpha_1,\beta;\psi,z)$ over any bounded domain lying in $\mathbb{C}\setminus\{\pm[2\pi/\gamma,\infty]\}$ $(r=\ldots-1,0,1,\ldots)$

(iii) Let $\rho\in\tilde{B}_1(\alpha_1,\beta)$, $2m>\max(M,2)$ and $v^{M}d\sigma(v)\geq 0$ for all real v. Let

$$e^{i\phi''}\int_{-\infty}^{\infty}\cos\{(2\nu+1)\pi u\}d\rho_1(u)$$

be real and nonnegative for $\nu=0,1,\ldots$. The results of clause (ii) with $2m+1$ consistently replaced by $2m$ hold.

Theorem Let $\rho \in \mathcal{E}_n(\alpha, \beta)$ and $\psi \in \mathcal{M}(M; x^-, x^+)$ with ψ^+ and ψ^- being the two functions occurring in the decomposition ( ).

(i) Set

$$R'(m,n;\rho,\alpha,\beta;\psi',z|\omega) = \int_0^\infty \int_\alpha^\beta \frac{\bar{E}^{(n)}_{m+1}(y-1)}{(m-1)!} \Delta \psi(z+\omega t)\, d\rho(y)\, dt$$

$$R''(m,n;\rho,\alpha,\beta;\psi',z|\omega) = -\int_{-\infty}^0 \int_\alpha^\beta \frac{\bar{E}^{(n)}_m(y-1)}{m!} \Delta^{m+1} \psi(z+\omega t)\, d\rho(u)\, dt$$

Then for ~~m ≥ M, m = M, M+1, ... (m ≥ n) and~~ $\omega \in H^+$, $\bar{G}^+$ has the decomposition

$$\bar{G}^+(m,n;\rho,\alpha,\beta;\psi,z|\omega) = R'(m,n;\rho,\alpha,\beta;\psi^+,z|\omega) + R''(m,n;\rho,\alpha,\beta;\psi^-,z|\omega)$$

~~and for $\omega \in H^-$~~ and G^- is ~~ref~~ has a similar representation

~~$\bar{G}^-(m,n;\rho,\alpha,\beta;\psi$~~ over H^- derived by interchanging ψ^+ and ψ^-

~~$m \geq M+n$~~, $(m = M+n, M+n+1, \dots)$

(ii) ~~Set~~ Let $M=0$ and in ~~the formula~~ expression ( ) for $\Delta^m \psi$, let $\varepsilon(v) = O(v^{n+\delta})$ as v tends to zero for some $\delta > 0$. Set

$S'(m,n;\rho,\alpha,\beta;\psi',z|\omega) =$

$$2^{-n} \sum (-1)^{\nu} \binom{n+\nu-1}{n-1} \int_\alpha^\beta \psi(z+u\omega+\nu\omega)\, d\rho(u)$$

$$S''(m,n;\rho,\alpha,\beta;\psi',z|\omega) = -2^{-n} \sum (-1)^{\nu} \binom{n+\nu-1}{n-1} \int_\alpha^\beta \psi(z-n\omega+u\omega-\nu\omega)\, d\rho(u)$$

Then for $\omega \in H^+$ and $m = 0$, $\bar{G}^+$ has ~~the~~ a decomposition similar to $\mathcal{G}(\ )$ with $m(\omega)$ set equal to zero and R', R'' replaced by S', S''.

$\bar{G}^-$ has a similar decomposition over H^- derived as in clause (i).

Definition With $n \geq 1$, $[\alpha, \zeta] \subseteq [-\infty, \infty]$ and $\rho \in B_n[\alpha, \zeta]$ functions $\tilde{\rho}$ and $\hat{\rho}(r)$ $(r = 1, \ldots, n)$ are defined as follows:

a) if $\alpha < 0$, $\zeta > n$ then with $n' = -[-\zeta]$, $m'' = -[\alpha]$ and ρ extended over the range $[-m'', \alpha)$ if $\alpha > -m''$ and over the range $(\zeta, n']$ if ~~$n' \neq$~~ $n' > \zeta$ by setting $\rho(y) = \rho(\alpha)$ over the first interval and $\rho(y) = \rho(\zeta)$ over the second.

(i) $\tilde{\rho}$ is defined over $[-m'', n' - n]$ by setting, for $j \leq y < j+1$

$$\tilde{\rho}(y) = 2^n \sum_{\nu=0}^{m''+j} (-1)^\nu \binom{n+\nu-1}{\nu} \rho(u-\nu)$$

for $j = -m'', -m''+1, \ldots, -1$,

$$\tilde{\rho}(y) = 2^n \left[\sum_{0}^{n'-j-n-1} (-1)^\nu \binom{n+\nu-1}{\nu} \rho(u+n+\nu) + \sum_{0}^{m''-1} (-1)^\nu \binom{n+\nu-1}{\nu} \rho(-\nu) \right]$$

for $j = 0, 1, \ldots, n'-n-1$, and setting

$$\tilde{\rho}(n'-n) = 2^n \left\{ \rho(\zeta) + \sum_{0}^{m''-1} (-1)^\nu \binom{n+\nu-1}{\nu} \rho(-\nu) \right\}$$

(ii) $\hat{\rho}(r)$ is defined over $[0, r]$ by setting

$$\hat{\rho}(r|u) = \rho'(r|u) + \rho''(r|u)$$

for $r = 1, \ldots, n-1$ and

$$\hat{\rho}(n|u) = \rho(u) + \rho'(n|u) + \rho''(n|u)$$

where

$$\rho'(r/u) = -2^{n-r}\sum_{0}^{n''-1}(-1)^{\nu}\binom{n-r+\nu}{\nu}\rho(u-\nu-1)$$

for $0 \le u \le 1$, $\rho'(r/u) = \rho'(r/1)$ when $1 < u \le r$ and

$$\rho''(r/u) = -2^{n-r}\sum_{0}^{n'-n-1}(-1)^{\nu}\binom{n-r+\nu}{\nu}\rho(u+n-r+\nu+1)$$

for $r-1 \le u \le r$, ~~and~~ $\rho''(r/u) = \rho''(r/r-1)$ when $0 \le u < r-1$ ~~the~~

~~additional conditions being~~ ~~on~~

~~intervals over which~~ $\rho'(r)$ ~~an~~, $\rho''(r)$ are defined ~~to be constant~~

~~being omitted above~~

Coefficients g_ν are defined by ~~omitting~~ deleting the terms from formula ( ). The functions $G^{\pm}_{\nu}(n;\rho,\alpha,\beta;\psi,z/w)$ are defined by ~~a~~ formulae similar to (,) ~~with in~~ with $F, f, 2^{\nu-n}$ replaced by $G, g, 2^{\nu}$. The Series $\mathfrak{G}^{\pm}(n,\rho,\alpha,\beta;\psi,z/w)$ ~~are de~~ is defined by a formula similar to ( ), with $\mathfrak{F}, f$ replaced by $\mathfrak{G}, g$, and series $\mathfrak{G}_e$ are defined in the same way

Theorem. Let B' and s' be such that with B' constant over (x^-, x^+)

$$\int_{-\infty}^{\infty} \left| \frac{B'(x)}{(z-x)^{s'+\nu}} \right| dx < \infty$$

for $\nu = 1, 2, \ldots$ and all $z \in S(x^-, x^+)$ and let

$$\Delta^{M'} f(z) = \int_{-\infty}^{\infty} \frac{dB'(x)}{(z-x)^{s'}}$$

for $z \in S'(x^-, x^+)$. Let $h = \max(0, 1 + [-\mathrm{Re}(s')])$ and set $M = M' + h$,
$s = s' + h$, $B(x) = (-1)^h \prod_{l=0}^{h-1} (s' + l) B'(x)$

For $m = M, M+1, \ldots$ $k = 1, 2, \ldots, n$ define functions $w^{\pm}(k, m, n;$
$\rho, \alpha, \zeta, \phi, z)$ over $\Delta(\pi \mp \frac{\pi}{2} - \arg(z - x^+), \pi \mp \frac{\pi}{2} - \arg(z - x^-))$ when
$\mathrm{Im}(z) \geq 0$ and over $\Delta(\mp \pi - \arg(z - x^-), \pm \frac{\pi}{2} - \arg(z - x^+))$ when
$\mathrm{Im}(z) \leq 0$ (the real segment $\pm(0, \infty)$ in particular belongs to the
segment over which $w^{\pm}$ is defined) by

$$w^{\pm}(k, m, n; \rho, \alpha, \zeta, \phi, z | t) =$$
$$(-2)^{n-1} i^{M-m-s-1} \imath^{m-1} \sum_{j=k}^{n} \binom{m+j-k}{m} \frac{t^j}{\prod_{l=1}^{M+j-1 (s)} (s-l)} T^{M}(j, n; M, s; \rho, \alpha, \zeta, \phi, z | t)$$

where, in the product, the term corresponding to $l = s$, if it occurs, is
to be omitted, and

a) if $M+j-s$ is not a positive integer

$$T^{\mp}(j,n;M,s;\rho,\alpha,\beta;\psi,z|t)=$$

$$-\frac{1}{2\pi i}\left[\int_{x^+}^{\infty}\int_{-i\infty}^{(0+)}+\int_{-\infty}^{x^-}\int_{(0+)}^{i\infty}\right]\int_{\alpha}^{\beta} b_{n-j}^{(n)}(u)\,y^{M+j-s-1}\operatorname{cosec}\left[\pi\{u+i(z-x-iy)t\}\right]d\rho(u)\,dy\,dB(x)$$

while

b) if $M+j-s$ is a positive integer

$$T(j,n;M,s;\rho,\alpha,\beta;\psi,z|t)=$$

$$\left[\int_{x^+}^{\infty}\int_{-i\infty}^{0}+\int_{-\infty}^{x^-}\int_{0}^{\infty}\right]\int_{\alpha}^{\beta} b_{n-j}^{(n)}(u)\,y^{M+j-s-1}\operatorname{cosec}\left[\pi\{u+i(z-x-iy)t\}\right]d\rho(u)\,dy\,dB(x)$$

c) and if $M+j=s$

$$T(j,n;M,s;\rho,\alpha,\beta;\psi,z|t)=$$

$$\left[\int_{-\infty}^{x^-}-\int_{x^+}^{\infty}\right]\int_{\alpha}^{\beta} b_{n-j}^{(n)}(u)\operatorname{cosec}\left[\pi\{u+i(z-x)t\}\right]d\rho(u)\,dB(x)$$

i) The functions $w^{\mp}(k,m,n;\rho,\alpha,\beta;\psi,z)$ are analytic over their domains of definition.

ii) Setting $w(k,m,n;\rho,\alpha,\beta;\psi,z|t)=w^{\mp}(k,m,m,n;\rho,\alpha,\beta;\psi,z|t)$ for $t\in\pm(0,\infty)$

$$\bar{\mathfrak{S}}_{m+z}(n;\rho,\alpha,\beta)\,\Delta^{m+z}\psi(z)=$$

$$i^{z}\sum_{k=1}^{n}\binom{-k}{z}(-1)^{k}\int_{-\infty}^{\infty} t^{z} w(k,m,n;\rho,\alpha,\beta;\psi,z\,|\,t)\,dt$$

for $m = M, M+1, \ldots$ $(m>n)$ and $z = 0, 1, \ldots$

(iii) For $m = M, M+1, \ldots$ $(m>n)$

$$\overline{G}^{\pm}(m,n;\rho,\alpha,\beta;\psi,z\,|\,\omega) = \sum_{k=1}^{n}\int_{-\infty}^{\infty}\frac{w(k,m,n;\rho,\alpha,\beta;\psi,z\,|\,t)}{(1+i\omega t)^{k}}\,dt$$

for $\omega \in H^{\pm}$ respectively

(iv) For $m = 0, 1, \ldots$ the functions $\overline{G}^{\pm}(m,n;\rho,\alpha,\beta;\psi,z\,|\,\omega)$ may be continued analytically across the two open segments $\pm i(0,\infty)$ of the imaginary axis in the ω-plane, and defining the concave open domains $\Delta^{\pm}$ by

$$\Delta^{+}(z) = \Delta(-\arg(z-x^{+}),\ \pi-\arg(z-x^{-}))$$

$\frac{\pi}{2} + \frac{\pi}{2} \ldots \frac{3\pi}{2} \mp$

$$\Delta^{-}(z) = \Delta(\pi-\arg(z-x^{+}),\ 2\pi-\arg(z-x^{-}))$$

when $\operatorname{Im}(z) \geq 0$ and

$$\Delta^{+}(z) = \Delta(-\pi-\arg(z-x^{-}),\ -\arg(z-x^{+}))$$

$$\Delta^{-}(z) = \Delta(-\arg(z-x^{-}),\ \pi-\arg(z-x^{+}))$$

when $\operatorname{Im}(z) \leq 0$. $\overline{G}^{\pm}(m,n;\rho,\alpha,\beta;\psi,z\,|\,\omega) \in A_{0}(\Delta^{\pm}(z))$

(v) For values of $\omega \in \{\Delta^+(z) \cap \Delta^-(z)\} \cap H_i^{\pm}$

$$\overline{G}^+(m,n;\rho,\alpha,\varsigma;\psi,z|\omega) = \overline{G}^-(m,n;\rho,\alpha,\varsigma;\psi,z|\omega) + \omega^{-m}\,\overline{B}(M,n;\rho,\alpha,\varsigma;\psi,z|\omega)$$

where

$$B(M,n;\rho,\alpha,\varsigma;\psi,z|\omega) = 2\pi\omega^{M-1}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(k-1)!}\,\mathcal{D}^{k-1}\,\kappa^{\pm}(k,M;\rho,\alpha,\varsigma;\psi,z|\lambda)$$

$\mathcal{D}$ being $d/d\lambda$, where $\lambda = i/\omega$ $(m = 0,1,\ldots)$

(vi) a) $\overline{\mathcal{G}}(m,n;\rho,\alpha,\varsigma;\psi,z|\omega)$ represents $\overline{G}^{\pm}(m,n;\rho,\alpha,\varsigma;\psi,z|\omega)$ asymptotically as ω tends to zero, in $\Delta^{\pm}(z)$

b) If $\rho \in \hat{B}(n;\alpha,\varsigma)$ (or $\rho \in \tilde{B}(n;\alpha,\varsigma)$), $\overline{\mathcal{G}}$ in the preceding clause may be replaced by $\hat{\overline{\mathcal{G}}}$ (or $\tilde{\overline{\mathcal{G}}}$)

Theorem With $B \in BV(-\infty,\infty)$ constant over the interval $(\hat{s},\tilde{s}) \subset (-\infty,\infty)$ and

$J \geq 0$, let

$$\mathcal{L}_J^{0}(x) = \int_{-\infty}^{\infty} \frac{dB(s)}{x-s}$$

i) a) For $J>0$ and $l=1,\dots,n$, $j=l,\dots,n$, $m=J, J+1,\dots$ ~~($\tilde{s}$)~~ the functions

$W(\rho;\psi,x;n,m,l,t) =$

$$\frac{(-1)^{J-1}(it)^{m-J}}{2}\frac{\pi}{?}\sum_{j=l}^{n}\binom{l-j-1}{m}\left[\int_{-\infty}^{\hat{s}}\int_{0}^{i\infty}+\int_{\tilde{s}}^{\infty}\int_{-i\infty}^{0}\right]\int\int_{1}^{n}\frac{\mathcal{L}^{(n)}_{n-j}(y)z^{J+j-1}}{(n-j)!(J+j-1)!}\operatorname{cosec}^2\{\pi\{y+z+(x-s)it\}\}$$

$d\rho(y)\,dz\,dB(s)$

are analytic for $x \in S(\hat{s},\tilde{s})$ and, when $Im(x) \geq 0$ over the first of the

sectors

$$\Delta\left(-\frac{\pi}{2}-\arg(\tilde{s}-x), \frac{\pi}{2}-\arg(x-\hat{s})\right), \Delta\left(-\frac{\pi}{2}-\arg(x-\hat{s}), \frac{\pi}{2}-\arg(\tilde{s}-x)\right)$$

and when $Im(x) \leq 0$ over the second; furthermore the functions $W(\rho;\psi,x;{}^{n,m,l}_{t})$

~~defined~~ obtained from the above formula with $(-1)^{J-1}$ replaced by $(-1)^{J}$

and the symbols $\int_0^{i\infty}$ and $\int_{-i\infty}^{0}$ interchanged are also analytic for $x \in S(\hat{s},\tilde{s})$

and for t belonging to the first or second of the sectors obtained from ( ) by adding $\frac{\pi}{2}$

to all limits according as to whether $Im(x) \geq 0$ or ≤ 0

~~With $W(\rho;\psi,x;n,m,l,t)$ as just defined over in particular~~

the ~~sectors~~ ~~segments~~ intervals $(0,\infty)$ and $(-\infty)$

$$e^{(\rho,n)}_{m+\tau} = i^{\tau}\sum_{l=1}^{n}\binom{-l}{?}\int_{-\infty}^{\infty} t^{\tau} W(\rho;\psi,x;n,m,l;t) \quad \text{for } m=J, J+1,\dots, \tau=0,1,\dots$$

Formula ( ) also holds ~~when~~ as described ~~above~~ ~~J=~~ if $J=0$, now with $m=1,2,\ldots$ if the term corresponding ~~(such [illegible])~~ to $j=1$ in the sum ( ) for the function $w(\rho;\psi,x,n,m,1;t)$ is replaced by

$$\frac{1}{2}(it)^{m}\left(\frac{-1}{m}\right)\left[\left[\int_{\tilde{s}}^{\infty}-\int_{-\infty}^{\hat{s}}\right]\int_{0}^{n}\frac{B_{n-1}(y)}{(n-1)!}\cot\{\pi\{y+(x-s)ti\}\}d\rho(y)\,d\ell(s)\right.$$

$$\left.-i\{\ell(\infty)-\ell(-\infty)\}\int_{0}^{n}\frac{B_{n-1}(y)}{(n-1)!}d\rho(y)\right]$$

~~When~~ ~~the~~

With $w(\rho;\psi,x,n,m,\ell/t)$ as just ~~described~~ defined over in particular the intervals $(-\infty,0)$ and $(0,\infty)$

$$e_{m+\tau}^{(m,n)}\partial^{m+\tau}\psi(x)=i^{\tau}\sum_{\ell=1}^{n}\binom{-\ell}{\tau}\int_{-\infty}^{\infty}t^{\tau}w(\rho;\psi,x;n,m,\ell/t)\,dt$$

for $m=J,J+1,\ldots$ $(m>0)$, $\tau=0,1,\ldots$

Theorem . With $B \in \hat{N}(-\infty,\infty)$ constant over the interval $(\hat{S},\tilde{S}) \subset (-\infty,\infty)$

~~let and $J \geq 0$ let~~

$$\cancel{\mathcal{A}^{\hat{S}}\psi(x) = \int_{-\infty}^{\infty} \frac{dB(s)}{x-s}}$$

i) a) For $l = 1,\dots,n$, $j = l,\dots,n$, $m = J, J+1,\dots$ $(J>0)$, the functions

$$W(\rho;\psi,x,n;m,l;t) =$$

$$(it)^{m-j}\sum_{j=l}^{n}\binom{m+j-l}{m}\left[\int_{\tilde{S}}^{\infty}\int_{0}^{i\infty}+\int_{-\infty}^{\hat{S}}\int_{-i\infty}^{0}\right]\int_{0}^{\infty}\frac{D_{n-j}^{(n)}(y)\,z^{J+j-1}}{(n-j)!\,(J+j-1)!}\operatorname{cosec}^{2}\left[\pi\{(s-x)i+y+z\}\right]dz\,dB(s)\,\rho(y)$$

are analytic for all $x \in S(\hat{S},\tilde{S})$ and at all points $t \in (-\infty,0) \cup (0,\infty)$

b) For $m = J, J+1,\dots$ $(J>0)$, the function $E_{\rho;\psi,x}^{(n,m)+}$ of formula ( ) is now given ~~by~~ over H^{+} by

$$E_{\rho;\psi,x}^{(n,m)+}(\omega) = \sum_{k=1}^{n}\int_{-\infty}^{\infty}\frac{W(\rho;\psi,x;n;m,l;t)\,dt}{(1+i\omega t)^{l}}$$

and for these values of m $E_{\rho;\psi,x}^{(n,m)-}$ has a similar alternative representation over H^{-}

c) ~~For $m = 0,1,\dots$. The~~ functions $E_{\rho;\psi,x}^{(n,m)\pm}$ directly defined ~~by~~ by may be continued analytically across the segments $\pm i(0,\infty)$ of the imaginary axis and, denoting the functions as directly defined by formulae of the type ( ) and ~~by~~

~~obtained by analytic continuation by the symbols $E^{(n,m)\pm}$~~

c) For $m=0,1,\ldots$ (?), the functions $E^{(n,m)\pm}_{p;4,x}$ directly defined by formulae of the type ( ) and ~~given~~ obtained ~~for $m<\max(J,1)$~~ by use of relationship when $m<\max(J,1)$ ~~are~~ may be continued analytically across the segments of the imaginary axis

~~d) Denoting by the same symbol the functions directly defined and~~

d) Denoting by $E^{(n,m)\pm}_{p;4,x}$ the functions directly defined and obtained by analytic continuation, $E^{(n,m)+}_{p;4,x}$ and $E^{(n,m)-}_{p;4,x}$ are ~~to~~, when $\mathrm{Im}(x)\geq 0$, analytic over the sectors $\Delta(-\pi-\arg(\tilde{s}-x),\pi-\arg(x-\hat{s}))$, ~~and~~ $\Delta(-\arg(\tilde{s}-x),2\pi-\arg(x-\hat{s}))$ respectively and at points common to ~~over the union of~~ these two open sectors, the relationship

$$E^{(n,m+\tau)+}_{p;4,x}(w)=E^{(n,m+\tau)-}_{p;4,x}(w)-2\pi i\lambda\sum_{l=1}^{n}\frac{(-1)^{l}}{(l-1)!}\partial_{\lambda}^{\,l-1}N(p;4,x;n,m+\tau,l;\lambda)$$

where $\lambda=i/w$, holds for all $m,\tau=0,1,\ldots$ such that ~~m~~ $m+\tau\geq\max(J,1)$.

When $\mathrm{Im}(x)\leq 0$, the sector pair ( ) must be replaced by ~~$\tilde{s}-x$ and $x-\hat{s}$ must be interchanged in the above results,~~ $\Delta(-\pi-\arg(x-\hat{s}),\pi-\arg(\tilde{s}-x)),\ \Delta(-\arg(x-\hat{s}),2\pi-\arg(\tilde{s}-x))$

e) For $m=0,1,\ldots$, $\mathcal{E}^{(n,m)}_{p;4,x}(w)$ represents ~~$E^{(n,m)\pm}$~~ the functions $E^{(n,m)\pm}_{p;4,x}(w)$ asymptotically as w tends to zero ~~over~~ in the open sectors over which these functions are analytic as described in clause d). If p is subjected to the conditions A or B of clause ( ) of Theorem ..., the asymptotic result of formula ( ), or of its counterpart as described, hold for the series $\hat{\mathcal{E}}^{(2m+1, 2m)}_{p;4,x}$ or $\hat{\mathcal{E}}^{(n, 2m+1)}_{p;4,x}$ over an extended sector, as just described.

---

f) If $\zeta = 0$ and ρ satisfies the condition imposed in clause ( ) of Theorem ..., the results of clauses a, b) hold with $m = 0$ and that of clause d) holds with $m = t = 0$. ~~Furthermore~~ Furthermore, the functions $E^{(1,0)\pm}_{p;4,x}$ has the special representation ~~$E(\rho; 4; x; \pi; m, t; \omega)$~~

$$E^{(1,0)+}_{p;4,x}(\omega) = \frac{1}{\omega}\left[\int_{-\infty}^{\hat{S}}\int_0^1 \Psi\left(y+\frac{x-s}{\omega}\right)d\rho(y)\,dB(s) - \{\rho(1)-\rho(0)\}\int_{-\infty}^{\hat{S}} m\left(\frac{x-s}{\omega}\right)dB(s)\right.$$

$$\left. + \int_{\hat{S}}^{\infty}\int_0^1 \Psi\left(1-y-\frac{x-s}{\omega}\right)d\rho(y)\,dB(s) - \{\rho(1)-\rho(0)\}\int_{\hat{S}}^{\infty} m\left(\frac{s-x}{\omega}\right)dB(s)\right]$$

holding over the first of the sectors ( ) when $Im(x) \geq 0$ and over the first of the pair ( ) if $Im(x) < 0$ the second of the second sector of the pair ~~and E~~ over H^+, ~~and~~; $E^{(1,0)-}_{p;4,x}$ is given over H^- by a similar formula in which the argument pairs $y + \frac{x-s}{\omega}$, $x-s$ and $1-y-\frac{x-s}{\omega}$, $\frac{s-x}{\omega}$ are interchanged. When $Im(x) \geq 0$

$$E^{(1,0)-}_{p;4,x}(\omega) = E^{(1,0)+}_{p;4,x}(\omega) + \frac{\pi i}{\omega}\{\rho(1)-\rho(0)\}\{B(\infty)-B(-\infty)\}$$

$$+\frac{\pi}{\omega}\int_{-\infty}^{\infty}\int_0^1 \cot\left\{\pi\left(\frac{x-s}{\omega}+y\right)\right\}d\rho(y)\,dB(s)$$

at points common to the ~~two sectors ( )~~ sectors of the relevant pairs ( ) or ( ).

f) If $\zeta=0$ and ρ satisfies the condition imposed in clause ( ) of Theorem ——, the results of clauses a,b) hold with $m=0$, and that of clause d) holds with $m+r=0$ ——

Theorem ——, ~~and the term in the sum formed ( )~~ corresponding to $j=1$, for the function $w(\rho;\psi,x,m,m,1,t)$ is replaced by

$$-\frac{1}{2}(it)^{m}\binom{-1}{m}\left[\left\{\int_{\tilde{s}}^{\infty}-\int_{-\infty}^{\hat{s}}\right\}\int_{0}^{n}\frac{B_{n-1}(y)}{(n-1)!}\cot\{\pi\{y+(x-s)t i\}\}\,d\rho(y)\,dB(s)\right.$$
$$\left.-\,i\{B(\infty)-B(-\infty)\}\int_{0}^{n}\frac{B_{n-1}(y)}{(n-1)!}\,d\rho(y)\right]$$

when $t>0$, and reversing the sign of this expression when $t<0$, the results of clauses (a,b) hold with $\overset{m}{\zeta}=0$ and that of clause d) holds with $m+r=0$.

(ii) If $\int_{-\infty}^{\infty}\ln(1+s)\,dB(s)<\infty$, the function $E^{(1,\omega)+}_{\rho,0,1;\psi,x}(\omega)$ has the special representation

$$E^{(1,\omega)}_{\rho,0,1;\psi,x}(\omega)=\frac{1}{\omega}\left[\int_{-\infty}^{\hat{s}}\int_{0}^{1}\Psi\left(y+\frac{x-s}{\omega}\right)d\rho(y)\,dB(s)-\{\rho(1)-\rho(0)\}\int_{-\infty}^{\hat{s}}\ln\left(\frac{x-s}{\omega}\right)dB(s)\right.$$
$$\left.+\int_{\tilde{s}}^{\infty}\int_{0}^{1}\Psi\left(1-y-\frac{x-s}{\omega}\right)d\rho(y)\,dB(s)-\{\rho(1)-\rho(0)\}\int_{\tilde{s}}^{\infty}\ln\left(\frac{s-x}{\omega}\right)dB(s)\right]$$

(Ψ being the logarithmic derivative of the Γ-function) holding over the first of the sectors ( ) when $\operatorname{Im}(x)\geq 0$ and over the first of the sectors ( ) if $\operatorname{Im}(x) \leq 0$; $E^{(1,0)-}_{\rho;\psi;x}$ is given over the second sector of the relevant pair by a similar formula in which (corresponding numbering) the argument pairs $y + \frac{x-s}{\omega}, x-s$ and $1-y-\frac{x-s}{\omega}, \frac{s-x}{\omega}$ are interchanged. ~~When Im(x) ≥ 0~~ Also

$$E^{(1,0)+}_{\rho;0,1;\psi;x}(\omega) = E^{(1,0)-}_{\rho;0,1;\psi;x}(\omega) + \frac{\pi i}{\omega}\left\{\rho_1(1) - \rho_1(0)\right\}\left\{B(\infty) - B(-\infty)\right\}$$

$$\cancel{+\frac{\pi}{\omega}\int_{-\infty}^{\hat{s}}\int_0^1 \cot\{\pi}$$

$$+\frac{\pi}{\omega}\left[\int_{\hat{s}}^{\infty} - \int_{-\infty}^{\hat{s}}\right]\int_0^1 \cot\left\{\pi\left\{\frac{x-s}{\omega} + y\right\}\right\} d\rho_1(y)\, dB(s)$$

at points common to the sectors of the relevant pair ( ) or ( ).

Proof. The immovable singularities in H^+ of the function defined by formula ( ), are those values of $t \in H^+$ for which $y + (x-s)it = \nu$ where ~~y ranges over all~~ y and s range over all values for at which ρ and B have points of increase and $\nu = \ldots, -1, 0, 1, \ldots$ These singularities are confined to the complement in ~~the complex plane~~ H^+ of the first or second of the ~~second~~ sectors ( ) (depending upon whether $Im(x) \gtrless 0$), and the function is analytic over the appropriate sector itself. (This sector contains the real segment $(0, \infty)$). Similar remarks may be made concerning the function defined over the sectors obtained from ( ) by adding π to the limits.

Proof. ~~It is clear that ψ is as described at the outset to commencement of Theorem~~, and further, ~~that~~ For $x \in S(\hat{s}, \tilde{s})$, for $m = J, J+1, \ldots (m \geq 0), \tau = 0, 1, \ldots$

$$(*) \quad \mathcal{D}^{m+\tau}\psi(x) = (-1)^{m+\tau-J}\left[\int_1^{\infty}\int_{-\infty}^{\hat{s}} - \int_{-\infty}^{0}\int_{\tilde{s}}^{\infty}\right] v^{m+\tau-J} e^{-(x-s)v}\, dB(s)\, dv$$

$m = J, J+1, \ldots (m \geq 0), \tau = 0, 1, \ldots$ ~~when $m \geq 0$~~

Using formulae (, ,). it follows that for ~~the above values of m, t ...~~

$$e^{(\rho, m)}_{m+\tau}\mathcal{D}^{m+\tau}\psi(x) = (-1)^{J} i^{\tau-m} \sum_{\substack{\cancel{j=\ell} \\ \ell=1}}^{n}{}' \binom{-\ell}{t} \sum_{j=\ell}^{n}{}' \binom{m+j-\ell}{m} i^{j}(t_1 + t_2 + t_3 + t_4)$$

where

$$t_1 = \int_0^{\infty}\int_{-\infty}^{\hat{s}}\int_0^{n} \frac{B^{(n)}_{n-j}(y)}{(n-j)!} v^{m-J+\tau} e^{-(x-s)v} \sum_{1}^{?} (2\pi\pi)^{-m-j-\tau} e^{2i\pi\pi y}\, d\rho(y)\, dB(s)\, dv$$

t_2 is obtained from t_1 by multiplication by $(-1)^{m+j+\tau}$ and replacement of $e^{2i\pi\pi y}$ by $e^{-2i\pi\pi y}$, t_3 is obtained from t_1 by reversing its sign and replacement of the symbols $\int_0^{\infty}\int_{-\infty}^{\hat{s}}$ by $\int_{-\infty}^{0}\int_{\tilde{s}}^{\infty}$, and t_4 is obtained from t_3 in exac the same way that t_2 is obtained from t_1. Setting ~~$t = 2\pi\pi$~~ $v = 2\pi\pi t$ $(\pi = 1, 2, \ldots)$, t_1 can, ~~when $J+j \geq 1$~~ be expressed in the form

$$t_1 = \int_0^{\infty}\int_{-\infty}^{\hat{s}}\int_0^{n} \frac{B_{n-j}(y)}{(n-j)!} t^{m+\tau-J} \sum_{1}^{?} (2\pi\pi)^{-J-j+1} e^{2\pi\pi\{iy - (x-s)t\}}\, d\rho(y)\, dB(s)\, dt$$

and subsequently as

$$t_1 = -\frac{\pi i^{-J-j}}{2} \int_0^{\infty} t^{m-J+\tau} \int_{-\infty}^{\hat{s}}\int_0^{i\infty}\int_0^{n} \frac{B_{n-j}(y) z^{J+j-1}}{(n-j)!(J+j-1)!} \operatorname{cosec}^2\left[\pi\{y + z + (x-s)it\}\right] d\rho(y)\, dz\, dB(s)\, dt$$

Similar expressions may be derived for t_2, t_3 and t_4, and

The above analysis may still holds when $j = 0$ and $j \geq 1$

formula ( ) is obtained. The special result concerning expression ( ) when $j = 0$ (and $j \geq 1$) is derived ~~as above~~ without ~~the~~ use of formulae of the form ( ), as above (the w_j functions with $t \geq 1$ do not involve ~~de~~ in the sums ( ) with $j = 1$).

This is evident from formula (4.) (of (a.b)) (~~settin~~ replacing m by l, j), ψ

satisfies is as described at the ~~aat~~ commencement of Theorem

~~The function $w(\rho;\psi\ x, n, m$~~

It is easily verified that ~~in-terms of~~ the functions $\varpi(\rho;\psi, x; n, m, l; t)$

of formula are, for the function ψ now being considered, analytic

at all points $t \in (-\infty, 0) \cup (0, \infty)$ and that, with $D = d/dt$ and for the l, j, m as stipulated in clause ia),

$$\frac{d}{dt}\varpi(\rho;\psi, x, n, m, l; t) = W(\rho;\psi, x, n, m, l; t)$$

The result of clause ib) follows immediately, as does that of clause ic) ~~in~~ from clause ( ) of Theorem ... in view of the remark just made concerning the nature of the functions occurring in relationship ( ) over $(-\infty, 0) \cup (0, \infty)$.

It follows from clause ( ) of Theorem that relationship ( ) obtains for all $m, r = 0, 1, \ldots$ with $\{m + r \geq \max(j, 1)\}$ over all regions in the complex w-plane such that the constituent functions in this relationship are analytic, i.e. over the sectors ( ) or ( ) according as to whether $\mathrm{Im}(x) \geq 0$ or ≤ 0. The functions $\varpi(\rho;\psi, x, n, m, l; t)$, for l, j, m as prescribed in clause ia), satisfy the conditions of clause ( ) of or second (depending upon the conditions [illegible]) 1

Theorem — over the first or second (depending upon the conditions [illegible]) of the sectors ( ) and over the first or second of the second of the sectors obtained from them by adding π to the limits; the results of clause (ii) now follow from those of clause ( ) of theorem — .

The series $\mathcal{E}^{(1,0)}_{\rho;\psi,x}$ derived from the function ψ described in clause (ii) is, according to clause ( ) of Theorem , $(\mathcal{B}',1)$ summable over $(0,\infty)$ to a ~~spec~~ function ~~having a special form of that~~ of the ~~index~~ of the form ( ) with ~~m~~ $n=1, m=0$. An alternative expression for this function may be given. The function f_1 of formula ( ) is, in the case being considered, given by

$$f_1(\omega u) = \int_{-\infty}^{\infty}\int_0^1 \left[\frac{e^{-yu/\lambda(x,s,\omega)}}{e^{-u/\lambda(x,s,\omega)}-1} + \frac{\lambda(x,s,\omega)}{u}\right]\frac{1}{x-s}\,d\rho(y)\,dB(s)$$

where $\lambda(x,s,\omega)=(x-s)/\omega$. For $\omega\in(0,\infty)$, $f_1(\omega u)$ is analytic ~~on~~ for $|u|<2\pi\{\min|x-\hat{s}|,|x-\tilde{s}|\}/\omega$ and regular, when $x\in S(\hat{s},\tilde{s})$ is fixed, over the sector $u\in\Delta(-\frac{\pi}{2}+\arg(x-\hat{s}),\frac{\pi}{2}+\arg(\tilde{s}-x))$ when $\operatorname{Im}(x)\geq 0$ and over the sector obtained by interchanging $x-\hat{s}$ and $\tilde{s}-x$ when $\operatorname{Im}(x)\leq 0$, and over $(0,\infty)$ for all $x\in S(\hat{s},\tilde{s})$. The function S_1 of formula ( ) may be expressed as

$$S_1(\omega) = \frac{1}{\omega}\left[\int_0^{\infty}\int_{-\infty}^{\hat{s}}\int_0^1\left\{\frac{e^{-z\{y+\lambda(x,s,\omega)\}}}{e^{-z}-1}+\frac{e^{-z}}{z}+\frac{e^{-z\lambda(x,s,\omega)}-e^{-z}}{z}\right\}d\rho(y)\,dB(s)\,dz\right.$$

$$\left.+\int_0^{\infty}\int_{\tilde{s}}^{\infty}\int_0^1\left\{\frac{e^{-z\{1-y-\lambda(x,s,\omega)\}}}{e^{-z}-1}+\frac{e^{-z}}{z}+\frac{e^{z\lambda(x,s,\omega)}-e^{-z}}{z}\right\}d\rho(y)\,dB(s)\,dz\right]$$

~~and~~ S_1 is the function $E^{(1,0)+}_{\rho;\psi,x}$ of formula ( ). That $\mathcal{E}^{(1,0)}_{\rho;\psi,x}$ is $(\mathcal{B},1)$ summable to $E^{(1,0)+}_{\rho;\psi,x}$ over the sector described in clause (ii) is demonstrated in a similar fashion, as is also the correctness of the ~~[illegible]~~ counterpart to formula ( ) for $E^{(\tau,0)-}_{p;q,x}$. Relationship ( ) follows from a simple property of the function Ψ.

Lemma . A) i) For $n=1,2,\ldots,\tau=n,n+1,\ldots$ and $0\le y\le n$

$$\frac{B_\tau^{(n)}(y)}{\tau!}=(-1)^{n-1}\sum_{\nu=0}^{n-1}\binom{\tau-\nu-1}{n-\nu-1}(-1)^\nu\frac{B_\nu^{(n)}(y)}{\nu!}\frac{\tilde{B}_{\tau-\nu}(y)}{(\tau-\nu)!}$$

ii) For $n=1,2,\ldots,r=n,n+1,\ldots,\tau=0,1,\ldots$ and $0\le y\le n$

$$\frac{B_{r+\tau+1}^{(n)}(y)}{(r+\tau+1)!}=i^{-\tau}\sum_{l=1}^{n}\binom{-l}{\tau}\int_{-1(2\pi)}^{1(2\pi)}u^\tau d\sigma(n,l,r,y;u)$$

where ~~$\sigma(n,l,r,y)\in BV(-\frac{1}{2\pi},-(2\pi)^{-1},(2\pi)^{-1})$ $(n=1,2,\ldots;l=1,\ldots,n;$ $r=n,n+1,\ldots;0\le y\le n)$~~ and

$$\sigma(n,l,r,y;u)=\frac{(-1)^{n-1}}{2\pi i}\sum_{j=0}^{n-l}\binom{r-j-l}{r-n}(-1)^j\frac{B_j^{(n)}(y)}{j!}\int_{-\infty}^{u}\frac{\tilde{B}_{r-j+1}(y+uw)}{(r-j+1)!}\frac{e^{-iw}-1}{w}dw$$

and $\sigma(n,l,r,y)\in BV(-(2\pi)^{-1},(2\pi)^{-1})$ $(n=1,2,\ldots;l=1,\ldots,n;r=n,n+1,\ldots;$ $0\le y\le n)$.

B) i) For $n=1,2,\ldots,\tau=n,n+1,\ldots$ and $0\le y\le n$

$$E_\tau^{(n)}(y)=\frac{2^{n-1}}{(n-1)!}\sum_{\nu=0}^{n-1}\binom{n-1}{\nu}(-1)^\nu B_\nu^{(n)}(y)\tilde{E}_{\tau+n-\nu-1}(y)$$

ii) For $n=1,2,\ldots,r,\tau=0,1,\ldots$ and $0\le y\le n$

$$\frac{E_{r+\tau}^{(n)}(y)}{(r+\tau)!}=i^\tau\sum_{l=1}^{n}\binom{-l}{\tau}\int_{-1(2\pi)}^{1(2\pi)}u^\tau d\hat{\sigma}(n,l,r,y;u)$$

where $\hat{\sigma}(n,l,r,y;u)=(-1)^{n+l}\frac{2^{n-1}}{2\pi i}\sum_{j=l-1}^{n-1}\binom{r+j+1-l}{r}\frac{B_{n-j-1}^{(n)}(y)}{(n-j-1)!}\int_{-\infty}^{u}\frac{\tilde{E}_{r+j}(y+uw)}{(r+j)!}\frac{e^{-iw}-1}{w}dw$

and $\hat{\varepsilon}(n,l,r,y) \in BV(-(2\pi)^{-1}, (2\pi)^{-1})$ $(n=1,2,\ldots; l=1,\ldots,n; r=n,n+1,\ldots;$ $0 \leq y < n)$

Proof. ~~Express~~ Denote the expression upon the right hand side of formula ( ) by $\hat{B}_{\tau}^{(n)}(y)/k!$. Evidently $\hat{B}_{\tau}^{(n)}(y) = B_{\tau}^{(n)}(y)$ ~~/k!~~ as described in clause (ii) when $n=1$. Assume that this relationship ~~is~~ also holds as described for some $n \geq 1$. ~~With $0 \leq y \leq n$, form the linear combination of~~ Let $0 \leq y < n$, and replace $B_{\nu}^{(n)}$ and $B_{\nu-1}^{(n)}$ ~~$(\nu \geq n+1)$ by $\hat{B}_{\nu}^{(n)}$~~ upon the right hand side of formula ( ) by $\hat{B}_{\nu}^{(n)}$ and $B_{\nu-1}^{(n)}$ $(\nu \geq n+1)$; the resulting expression is $\hat{B}_{\nu}^{(n+1)}$, i.e. the stated result holds ~~for~~ with n replaced by $n+1$ and $0 \leq y < n$. ~~Rep. Replace y by $y+1$~~ Using relationship ( ), it is found that $\hat{B}_{\tau}^{(n+1)}(y+1) = \hat{B}_{\tau}^{(n+1)}(y) + \tau \hat{B}_{\tau-1}^{(n)}(y)$ when, in particular, $n-1 \leq y < n$; hence $\hat{B}_{\tau}^{(n+1)}(y) = B_{\tau}^{(n+1)}(y)$ over the complete range $0 \leq y < n+1$.

From formulae ( )

$$\frac{\tilde{B}_{\tau}(y)}{\tau!} = -\sum{}' (2i\nu\pi)^{-\tau} e^{2i\nu\pi y}$$

for $\tau = 2,3,\ldots$ and $y \in (-\infty,\infty)$. Thus

$$\frac{\tilde{B}_{j+\tau+1}(y)}{(j+\tau+1)!} = i^{-\tau}\int_{-1/(2\pi)}^{1/(2\pi)} u^{\tau}\, d\sigma(1,1,j,y;u)$$

~~where~~ for $j=1,2,\ldots, \tau=0,1,\ldots$, where $\sigma(1,1,j,y)$ has a saltus of magnitude $-i^{-j-1}u^{j}e^{iy/u}$ at the point $u=(\nu\pi)^{-1}$ $(\nu=\ldots,-1,1,\ldots)$ and no other point of variation in the range $-(2\pi)^{-1}\le u\le(2\pi)^{-1}$; ~~then~~ the sum of the absolute values of these saltus in this range is bounded for $j\geq 1$: $\sigma(1,1,j;y)\in BV(-(2\pi)^{-1},(2\pi)^{-1})$. Furthermore, using formula ( ) with r replaced by $j+1$,

$$\sigma(1,1,j,y;u)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{B}_{j+1}(y+uw)}{(j+1)!}\,\frac{e^{-iw}-1}{w}\,dw$$

for $j=1,2,\ldots$, $y\in(-\infty,\infty)$ and $u\in[-(2\pi)^{-1},(2\pi)^{-1}]$. The function $\sigma(n,l,r,y;u)$ of formula ( ) has a saltus of magnitude

$$(-1)^{n}i^{-r-1}u^{r}e^{iy/u}\sum_{j=0}^{n-l}\binom{r-j-l}{r-n}\frac{B_{j}^{(n)}(y)}{j!}(iu)^{-j}$$

at the point $u=(\nu\pi)^{-1}$ $(\nu=\ldots,-1,1,\ldots)$ and no other point of variation in the range $-(2\pi)^{-1}\le u\le(2\pi)^{-1}$, and hence $\sigma(n,l,r,y)$ is as characterised in clause (Aii): the moments ~~in formu~~ occurring in formula ( ) exist. From formulae (,)

$$\frac{B_{r+\tau+1}^{(n)}(y)}{(r+\tau+1)!}=\frac{(-1)^{n-1}i^{-\tau}}{2\pi i}\sum_{\nu=0}^{n-1}\binom{r+\tau-\nu}{n-\nu-1}(-1)^{\nu}\frac{B_{\nu}^{(n)}(y)}{\nu!}\int_{-1/(2\pi)}^{1/(2\pi)}u^{\tau}d\left[\int_{-\infty}^{\infty}\frac{\tilde{B}_{r-\nu+1}(y+uw)}{(r-\nu+1)!}\,\frac{e^{-iw}-1}{w}\,dw\right]$$

for $n,r=1,2,\ldots$, $\tau=0,1,\ldots$ and $y\in(-\infty,\infty)$. Formula ( ) is derived by use of elementary identities involving binomial coefficients.

Formula ( ) may be derived by use of relationships (,), as in the proof of formula ( ). ~~It follows~~ From formulae (,)

$$\frac{E_\tau(y)}{\tau!} = 2\sum{}'' \{(\nu-1)\pi i\}^{-\tau-1} e^{(\nu-1)y\pi i}$$

for $\tau = 1, 2, \ldots$. The result of clause (iii) may now be demonstrated by methods analogous to those used to prove clause (ii).

~~Formula ( ) is given by Nörlund ([ ] p. 153 eq. formula (97));~~

~~he does not give formula ( ) but gives instead a formula~~

Formulae (,) are given in slightly ~~modified~~ concealed form by Nörlund ([ ] p 155 eq. (101) and p 153, eq. (97))

$B_\nu^{(n+1)} = \frac{\nu!}{\cdot}$

$\sum_{s=0}$

$\underline{\underline{134'}}$ m, r $\alpha \in seq(\mathbb{K}, \mathbb{K})$ $|\alpha| \geq m+r$ $z \in \mathbb{K} \setminus \alpha[m, m+r]$

$$\frac{1}{z-\alpha_m} = \sum_{\lambda=0}^{r} \frac{\pi(\alpha \| m+\lambda+1, r-\lambda | \alpha_m)}{\pi(\alpha \| m+\lambda, r-\lambda+1 | z)} = \sum_{\lambda=0}^{r} \frac{\pi(\alpha \| m+r-\lambda+1, \lambda | \alpha_m)}{\pi(\alpha \| m+r-\lambda, \lambda+1 | z)}$$

$$\frac{1}{z-\alpha_{m+r}} = \sum_{\lambda=0}^{r} \frac{\pi(\alpha \| m, \lambda | \alpha_{m+r})}{\pi(\alpha \| m, \lambda+1 | z)} = \sum_{\lambda=0}^{r} \frac{\pi(\alpha \| m, r-\lambda | \alpha_{m+r})}{\pi(\alpha \| m, r-\lambda+1 | z)}$$

$\underline{\underline{134}}$

$m, i \in \overline{\mathbb{N}}$, $\lambda \in [i]$ $\alpha \in seq(\mathbb{K})$ $|\alpha| \geq m+i$

$$\pi(\alpha \| m, i; \lambda) = \sum_{\omega=\lambda}^{i} \pi(\alpha \| m+\omega+1, i-\omega | \alpha_{m+\lambda}) \pi(\alpha \| m, \omega)$$

$$= \sum_{\omega=0}^{\lambda} \pi(\alpha \| m, \omega | \alpha_{\cancel{m+i}\, m+\lambda}) \pi(\alpha \| m+\omega+1, i-\omega) \quad \langle \mathbb{K} \rangle$$

$\underline{\underline{135}}$

$m, i \in \overline{\mathbb{N}}$, $\alpha \in seq(\mathbb{K})$, $f \in \overline{seq}(\mathbb{K})$ $m+i \geq |\alpha|, |f|$?

$$L(\alpha, f \| m, i) = \sum_{\omega=0}^{i} \delta(\alpha, f \| m, \omega) \pi(\alpha \| m, \omega) \quad \langle \mathbb{K} \rangle$$

$$= \sum_{\omega=0}^{i} \delta(\alpha, f \| m+\omega, i-\omega) \pi(\alpha \| m+\omega+1, i-\omega) \quad ''$$

$$= \sum_{\omega=0}^{i} \delta(\alpha, f \| m+i-\omega, \omega) \pi(\alpha \| m+i-\omega+1, \omega) \quad ''$$

$\underline{\underline{137}}$

$m, i \in \overline{\mathbb{N}}$ $\alpha \in seq(\mathbb{K})$ $f, g \in \overline{seq}(\mathbb{K})$

i) $D = \cup [\pi(\alpha \| m, \tau | \alpha_{m+\tau})]$ $(\tau, D = [i])$

$n \in [m+i]$, $k = max(m, n)$ $h = min(i, m+n+i)$

a) $|\alpha|, |g| \geq k+i$

$$\mathrm{row}\left[\delta(\alpha,g\,\|\,n,m-n+\nu)\right]^{\nu=[i]}\mathcal{D}=$$

$$\mathrm{row}\left[0^{[n-m-1]}\;\Big|\;\left[\frac{\pi(\alpha\,\|\,m,n-m\,|\,\alpha_{n+\nu})}{\pi(\alpha\,\|\,n,m-n\,|\,\alpha_{m+\nu})}\left\{g_{k+\nu}-\Lambda(\alpha,g\,\|\,n,m-n-1\,|\,\alpha_{m+\nu})\right\}\right]^{\nu=[h]}\right]$$

b)' $|\alpha|,|f|\geq k+i$

$$\mathrm{row}\left[\pi(\alpha\,\|\,m+\nu+1,i-\nu)\Lambda(\alpha;f\,\|\,n,m-n+\nu)\right]^{\nu=[i]}\mathcal{D}=$$

$$\mathrm{row}\left[0^{[n-m-1]}\;\Big|\;\left[\pi(\alpha\,\|\,m,n-m\,|\,\alpha_{n+\nu})\right.\right.$$

$$\left\{\frac{\pi(\alpha\,\|\,n,m-n+i\,;\,k-n+\nu)}{\pi(\alpha\,\|\,n,m-n\,|\,\alpha_{m+\nu})}\left\{f_{k+\nu}-\Lambda(\alpha;f\,\|\,n,m-n-1\,|\,\alpha_{m+\nu})\right\}\right.$$

$$\left.\left.\left.+\pi(\alpha\,\|\,m,i;\nu)\Lambda(\alpha,f\,\|\,n,m-n-1)\right\}\right]^{\nu=[h]}\right]\quad\langle K\rangle$$

c) $|\alpha|\geq m+i$: $\mathcal{D}^{-1}=U\left[\mu(\alpha\,\|\,m,\nu;\tau)\right]\;(\tau,\nu=[i])$

~~ii) $\acute{\mathcal{D}}=U\left[\mu(\alpha\,\|\,m,\nu;\tau)\right]\;(\tau,\nu=[i])$~~

ii) $\acute{\mathcal{D}}=L\left[\pi(\alpha\,\|\,m+\tau+1,i-\tau\,|\,\alpha_{m+\nu})\right]\;(\tau,\nu=[i])$

$j\in\overline{N}$, $h=\min(i,j)$

a) $|\alpha|,|g|\geq m+\max(i,j)$

$$\mathrm{row}\left[\delta(\alpha,g\,\|\,m+\nu,j-\nu)\right]^{\nu=[i]}\acute{\mathcal{D}}=$$

$$\mathrm{row}\left[\left[\frac{\pi(\alpha\,\|\,m+j+1,i-j\,|\,\alpha_{m+\nu})}{\pi(\alpha\,\|\,m+i+1,j-i\,|\,\alpha_{m+\nu})}\left\{g_{m+\nu}-\Lambda(\alpha,g\,\|\,m+i+1,j-i-1\,|\,\alpha_{m+\nu})\right\}^{\nu=[h]}\;\Big|\;0^{[i-j-1]}\right]\right]$$

b) $|\alpha|,|f|\geq m+\max(i,j)$

$$\text{mw}\left[\pi(\alpha\|m,\nu)\,\Delta(\alpha;f\|m+\nu,j-\nu)\right]^{\nu=[i]}\,\acute{D}=$$

$$\text{mw}\Big[\pi(\alpha\|m+j+1,i-j|\alpha_{m+\nu})$$

$$\left\{\frac{\pi(\alpha\|m,j;\nu)\{f_{m+\nu}-\Delta(\alpha;f\|m+j+1,j-i-1|\alpha_{m+\nu})\}}{\pi(\alpha\|m+i+1,j-i|\alpha_{m+\nu})}\right.$$

$$\left.+\pi(\alpha\|m,i;\nu)\,\Delta(\alpha;f\|m+i+1,j-i-1)\right\}\Big]^{\nu=[k]}\Big|0^{[\ell-j-1]}\Big]$$

c) $|\alpha|\geq m+i$ $\acute{D}^{-1}=L\left[\mu(\alpha\|m+\nu,i-\nu;\tau-\nu)\right]\ (\tau,\nu=[i])$

145

$m,i\in\bar{N}$, $\alpha\in seq(K)$ $f\in\overline{seq}(K)$, $m+i\leq|\alpha|,|f|$, $x\in K\setminus\alpha[m,m+i]$

$L(\alpha;\dfrac{\langle f\rangle}{x-\langle\alpha\rangle}\|m,i|z)$ is

$\pi(\alpha\|m;i+1|x)\,\delta(\alpha,\dfrac{\langle f\rangle}{(x-\langle\alpha\rangle)^2}\|m,i)$ when $z=x$

and

$$\frac{1}{z-x}\left\{\frac{\pi(\alpha\|m,i+1|z)}{\pi(\alpha\|m,i+1|x)}\Delta(\alpha,f\|m,i|x)-\Delta(\alpha,f\|m,i|z)\right\}$$

~~when z~~ for $z\in K\setminus x$: (latter expression reduces to $\dfrac{f_{m+\nu}}{x-\alpha_{m+\nu}}$ when

$x=\alpha_{m+\nu}$ $(\nu=[i])$)