Variants of the remainder terms of the Euler-Maclaurin and Boole series

From $\underset{\omega}{\Delta^n} F(z|\omega) = \phi(z)$ derive Nörlund form from Darboux sum

Plan of paper on variants...
developed from notes on The principal and alternating sums of a function defined over a strip

p156 of notes

backward form p158

brief description of Nörlund's derivation

series
integral expression for $\bar{b}_z^{(n)}$

solution of $\underset{\omega}{\nabla} G(z|\omega)$ same way

p173
Hardy's classification of ~~remain~~ forms of remainder term (add Hermite form)

Purpose of this note to extend these forms to general order n and
treat Boole series in same way

as a preliminary give some results for periodic Bernoulli and Euler
polynomials.

Periodic Bernoulli and Euler functions

Define $\bar{b}_z^{(n)\pm}(n)$ $\mathrm{Im}(u) \gtrless 0$ by integral expression p136 137

$\bar{b}_z^{(n)}(u)$ by sum

alternatively: define in terms of $b_n^{(1)+}(u)$
and recursion

special form for $n=1$

probably for easier

prove satisfy recursion on p136

$\rho\ \Delta\ \nabla$

also successive orders

define $\bar{e}_z^{(n)\pm}$

as above

bs from p138
recursion on p138

Nörlund form

General on p.156
backward form 158, 159 } ~~Bernoulli~~ Euler Maclaurin formulae

p173 for Euler
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backward form p175

Plana form

p165 final version p167 forward form backward form.

177 for Boole's series

Poisson form

p169

p177 for Boole's series

Ramanujan form

171 & 172

Boole's series variant not given

Mention further forms due to author.

Widder p 273

$$f(x) = \int_{-\infty}^{\infty} e^{-xt} d\alpha(t) \quad a < x < b \quad \alpha \text{ nondecreasing}$$

iff $f(x+y)$ of positive type in $a < 2x < b$ $a < 2y < b$

— . —

$k(x,y)$ continuous in $a<x<b$, $a<y<b$ is of positive type there if it is of positive type in every closed sq. interior to open sq.

— " —

fn. $k(x,y)$ which is continuous in $(a \le x \le b, a \le y \le b)$ is of +ve type there if for every real fn $\phi(x)$ continuous in $(a \le x \le b)$

$$J(\phi) = \int_a^b \int_a^b k(x,y)\phi(x)\phi(y)\,dx\,dy \geq 0$$

— . —

Def. $\psi(z) = \sum_{z=1}^{4} i^{z}\psi_z(z)$ where, fn $z=1,\dots,4$

$\psi \in M(M; x^-, x^+)$ means that

a) $\psi_z \in A\{S(x^-,x^+)\}$ and for all ~~arbitrarily small~~ $\delta \in (0, \frac{1}{2}(x^+ - x^-))$ ~~not~~ at $x' = \frac{1}{2}x^+ + \delta, \frac{1}{2}x'' - \delta$

b)
$$\int_{\frac{1}{2}x'}^{\frac{1}{2}x''}\int_{\frac{1}{2}x'}^{\frac{1}{2}x''} \partial^{M}\psi_z(x+y) \quad \phi(x)\phi(y)\,dx\,dy \geq 0$$

for every function ϕ which is real and continuous in $(\frac{1}{2}x', x'')$

~~where $x' = \frac{1}{2}x^+ + \delta$, $x'' = \frac{1}{2}x^+ - \delta$,~~

Th. $\psi \in M(M; x^-, x^+)$ if and only if $\psi \in A\{S(x^-,x^+)\}$ and ~~$\psi(z)$ has $= \psi^+(z) + \psi^-(z)$ where for~~

~~$z=3,4$ of the functions ψ_z $\psi^+(z) = \sum i^z \psi_z(z)$ with ψ_z satisfying~~

~~a) $\psi_z \in A\{$ relationship ( ) at each and of Definition~~

~~$$\partial^{M}\psi_z(z) = \int_{-\infty}^{\infty} e^{-zv}\,d\sigma_z(v)$$~~

~~where~~

with

$$\sigma(v) = \sum_{r=1}^{4} i^{r} \sigma_r(v)$$

and σ_r being ~~and~~ ~~nondec~~ real valued and nondecreasing over $(-\infty,\infty)$ $(r=1,\dots,4)$,

$$\mathfrak{D}^M\psi(z) = \int_{-\infty}^{\infty} e^{-zv}\, d\sigma(v)$$

the integral converging absolutely for ~~$x^- < x < x^+$~~, $z \in S(x^-, x^+)$

(ii) ~~The function~~ $\psi \in M(M; x^-, x^+)$ if and only if ψ has the ~~has a~~ decomposition $\psi(z) = \psi^+(z) + \psi^-(z)$

a) where $\psi^+ \in M(M; x^-,\infty)$ ~~$A\{S(x^-,\infty)\}$~~ and $\psi^- \in M(M;-\infty,x^+)$ ~~$A\{S(-\infty,x^+)\}$~~ and

b)

$$\psi^\pm(z) = \sum_{r=1}^{4} i^r \psi_r^\pm(z),$$

where

$$(-1)^\nu \mathfrak{D}^{M+\nu}\psi_r^+(x) \ge 0 \qquad \mathfrak{D}^{M+\nu}\psi_r^-(x) \ge 0$$

$(\nu=0,1,\dots)$ and for all $x\in(x^-,\infty)$ and $x\in(-\infty,x^+)$ respectively

$\mathfrak{D}^M\psi^+$ and $\mathfrak{D}^M\psi^-$ then have ~~having~~ the representation

$$\mathfrak{D}^M\psi^+(z) = \int_0^\infty e^{-zv}\, d\sigma^+(v)$$

$$\mathfrak{D}^M\psi^-(z) = \int_{-\infty}^0 e^{-zv}\, d\sigma^-(v)$$

where, in terms of the repr. ( ), $\sigma(v) = \sigma^\pm(v)$ for $v \in \pm[0,\infty)$

~~In terms of the representation ( ),~~

~~$\sigma(v) = \sigma^\pm$~~

(iii) a) The ψ_r ~~of relationship ( ) and~~ σ_r of formulae relationships (,) respectively are connected by the relationship

$$\psi_r(z) = \int_{-\infty}^{\infty} e^{-zv}\, d\sigma_r(v)$$

b) $\psi \in M(M; x^-, x^+)$ if and only if ψ has . . .

where $\psi^\pm(z) = \cdots$ —

and with $\psi^\pm_r \in A\{S(x^-,\infty)\}$ &

$(-1)^\nu \cdots$ —

$\psi^\pm$ are then the same functions as those of clause ii and

$$\psi_r^+(z) = \sum_0^\infty \int_0^\infty e^{-zv} \sigma_r^+(v)$$

$\psi_r^-(z) \cdot$ where

$$\sigma^\pm(v) = \sum i^r \sigma_r^\pm(v)$$

for $v \in \pm[0,\infty)$

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$$\bar b_z^{(n)}(y) = (-1)^{n-1} \sum_0^{n-1} \binom{z-\nu-1}{z-n} (-1)^\nu b_\nu^{(n)}(y)\, \bar b^{(1)}_{z-\nu}(u)$$

$$\bar b^{(n)}_{m+z}(u) = \sum_0^{n-1} \binom{z+m+\nu-n}{\nu} (-1)^\nu b^{(n)}_{n-\nu-1}(u)\, \bar b_{m-n+\nu+z+1}(u)$$

$$= -\sum_{j=1}^{n} \binom{z+m+j-n-1}{j-1} (-1)^j b^{(n)}_{n-j}(u)\, \bar b_{m-n+z+j}(u)$$

$$= (-1)^z \sum_{k=1}^{n} (-1)^{k-1}\binom{-k}{z} \sum_{j=k}^{n} \binom{m+j-n-k}{m-n} b^{(n)}_{n-j}(u)\, \bar b_{m-n+j+z}(u)$$

$$\bar b_{m-n+j+z}(u) = -\sum{}' (2i\nu\pi)^{-m+n-j-z} e^{2i\nu\pi u}$$

$$b_z^{(n)}(u) = (-1)^{n} \sum_0^{n-1}\binom{z-\nu-1}{z-n}(-1)^\nu b^{(n)}_\nu(u) \sum_{j=-\infty}^{\infty}{}' (2ij\pi)^{\nu-z} e^{2ij\pi u}$$

$$(2ij\pi)^{\nu-z} = \frac{1}{\Gamma(z-\nu)}\int_0^\infty h^{z-\nu-1} e^{-2ij\pi h}\, dh \qquad j>0$$

$$b_z^{(n)+}(u) = (-1)^n \sum_0^{n-1} \frac{(z-\nu-1)!\, i^{\nu-z}}{(z-n)!\,(n-\nu-1)!}(-1)^\nu b_\nu^{(n)}(u) \sum_{j=1}^{\infty}\frac{1}{(z-\nu-1)!}\int_0^\infty h^{z-\nu-1} e^{-2ij\pi h}\cdot e^{2ij\pi u}$$

$Re(iu)<0$

$$= (-1)^n i^{-z}\int_0^\infty \sum_0^{n-1}\frac{i^{-\nu}\, B^{(n)}_\nu(u)}{(z-n)!\,(n-\nu-1)!\,\nu!}\, h^{z-\nu-1}\, \frac{e^{-2\pi h+2i\pi u}}{1-e^{-2\pi h+2i\pi u}}\, dh$$

i.e. $Im(u) \geq$

i^{1-n}

$$= \frac{(-1)^n i^{-z}}{(z-n)!}\int_0^\infty \frac{h^{z-n}}{e^{2\pi(h+iu)}-1}\sum_0^{n-1}\frac{B^{(n)}_\nu(u)}{\nu!}\,\frac{(ih)^{n-\nu-1}}{(n-\nu-1)!}\, dh$$

$$\boxed{= \frac{i^{-z+1}\, i^n}{(z-n)!\,(n-1)!}\int_0^\infty \frac{h^{z-n}}{e^{2\pi(h-iu)}-1}\, B^{(n)}_{n-1}(u+ih)\, dh \qquad Im(u) \geq 0}$$

$$\frac{\partial}{\partial u}: \frac{e^{2\pi(h-iu)}\cdot(-2\pi i)}{\{e^{2\pi(h-iu)}-1\}^2}$$

$$b_z^{(n)-}(u) = (-1)^n i^{-z} \int_0^\infty \sum_0^{n-1} \frac{i^{-\nu}}{(z-n)!} \frac{B_\nu^{(n)}(u)}{(n-\nu-1)!\,\nu!} (-1)^{\nu-z} \frac{e^{-2\pi h-2\pi i u}}{1-e^{-2\pi h-2\pi i u}} h^{z-\nu-1}$$

$$= \frac{i^{-z+1+n}(-1)^{z+n-1}}{(z-n)!\,(n-1)!} \int_0^\infty \frac{h^{z-n} B_{n-1}^{(n)}(u-ih)}{e^{2\pi(h+iu)}-1}\,dh \qquad \mathrm{Im}(u)\leq 0$$

$$b_z^{(n)-}(u) = \overline{b_z^{(n)+}(u)} \qquad h \to -h'$$

$$b_z^{(n)-}(u) = -\frac{i^{-z+n+1}}{(z-n)!\,(n-1)!} \int_{-\infty}^0 \frac{h^{z-n} B_{n-1}^{(n)}(u+ih)}{e^{2\pi(iu-h)}-1}\,dh \qquad \mathrm{Im}(u)\leq 0$$

i.e. $\bar{b}_z^{(n)+}(u) + \bar{b}_z^{(n)-}(u) =$

$$\frac{i^{n-z+1}}{(z-n)!\,(n-1)!} \int_{-\infty}^{\infty} h^z w(h)\,dh$$

where
$$w(h) = \frac{h^{-n} B_{n-1}^{(n)}(u+ih)}{e^{2\pi(h-iu)}-1} \qquad 0<h<\infty$$
$$w(h) = -\frac{h^{-n} B_{n-1}^{(n)}(u+ih)}{e^{2\pi(iu-h)}-1} \qquad -\infty<h<0$$
$$\left.\right\} \quad z>n, \quad \mathrm{Im}(u)=0$$

this can hardly be correct: function represented is analytic for $\mathrm{Re}(u) \gtrless 0$ & $\bar{b}_z^{(n)}(u)$ has discontinuous derivative at integer points $u>n$

can be correct: $\bar{b}_z^{(n)}(u) = \frac{1}{(z-n)!} *$ ~~Stieltjes m~~ Hamburger moment

$$\overline{E}_{\nu}^{(n+1)}(x) = \frac{2^{n-1}}{(n-1)!} \sum_{s=0}^{n-1} \frac{(-1)^{n-s+1}}{s!} \overline{E}_{\nu+s}(x) \Delta_x^s \left[(x-1)\dots(x-n+1)\right]$$

$$B_{n-1}^{(n)}(x) = (x-1)\dots(x-n+1) \qquad \Delta_x^s B_{n-1}^{(n)}(x) = (n-1)(n-2)\dots(n-s) B_{n-s-1}^{(n)}(x)$$

$$\overline{E}_{\tau}^{(n)}(x) = \frac{2^{n-1}}{(n-1)!} \sum_{j=0}^{n-1} \frac{(-1)^{n-j-1}}{j!} (n-1)\dots(n-j) B_{n-j-1}^{(n)}(x) \overline{E}_{\tau+j}(x)$$

$$\nu = n-1-j \qquad n-j = \nu+1 \qquad \tau+j = \tau+n-\nu-1$$

$$\overline{E}_{\tau}^{(n)}(x) = \frac{2^{n-1}}{\cancel{(n-1)!}} \sum_{\nu=0}^{n-1} \frac{(-1)^{\nu}}{(n-\nu-1)!} \frac{\cancel{(n-1)!}}{\nu!} B_{\nu}^{(n)}(x) \overline{E}_{\tau+n-\nu-1}(x)$$

$$\frac{\overline{E}_{\tau+n-\nu-1}(x)}{(\tau+n-\nu-1)!} = 2\Big[\sum_{\nu=1}^{\infty} \{(2\nu-1)\pi i\}^{-\tau-n+\nu} e^{(2\nu-1)\pi i u}$$
$$+ (-1)^{\tau+n-\nu} \sum_{\nu=1}^{\infty} \{(2\nu-1)\pi i\}^{-\tau-n+\nu} e^{-(2\nu-1)\pi i u}\Big]$$

$$\overline{E}_{\tau}^{(n)+}(u) = \frac{2^{n}}{(n-1)!} \sum_{\nu=0}^{n-1} \frac{(-1)^{\nu}}{(n-\nu-1)!} \frac{\cancel{(n-1)!}}{\nu!} B_{\nu}^{(n)}(u) . (\tau+n-\nu-1)! \, i^{-\tau-n+\nu}$$
$$\frac{1}{(\tau+n-\nu-1)!} \sum_{j=1}^{\infty} \int_0^{\infty} h^{\tau+n-\nu-1} e^{-(2j-1)\pi h + (2j-1)\pi i u}$$

$$= -\frac{2^{n} i^{-\tau-n}}{\cancel{(n-1)!}} \int_0^{\infty} \sum_{\nu=0}^{n-1} \frac{i^{-\nu}}{(n-\nu-1)!} \frac{B_{\nu}^{(n)}(u)}{\nu!} h^{\tau+n-\nu-1} \frac{h^{\tau+n-\nu-1}}{e^{\pi\{iu-h\}} - e^{-\pi\{iu-h\}}}$$

$$= -\frac{2^{n} i^{-\tau-n} i^{-n+1}}{(n-1)!} \int_0^{\infty} \frac{B_{n-1}^{(n)}(u+ih)\, h^{\tau}}{2i} \operatorname{cosec}\{\pi\{u+ih\}\} dh$$

$$\boxed{= \frac{(-2)^{n-1} i^{-\tau}}{(n-1)!} \int_0^{\infty} h^{\tau} B_{n-1}^{(n)}(u+ih) \operatorname{cosec}\{\pi\{u+ih\}\} dh}$$

$$\sum_{j=k}^{n} \binom{m+j-n-k}{m-n} b^{(n)}_{n-j}(u)\, \bar{b}_{m-n+j+z}(u) = \cancel{}\ b_{m+z}(k,n;u)$$

$$b^{+}_{m+z}(k,n;u) = -\sum_{j=k}^{n} \binom{m+j-n-k}{m-n} b^{(n)}_{n-j}(u) \sum_{1} (2i\nu\pi)^{-m+n-j-z} e^{2i\nu\pi u}$$

$$= -\sum_{j=k}^{n} \binom{m+j-n-k}{m-n} b^{(n)}_{n-j}(u)\, i^{-m+n-j-z} \frac{1}{\Gamma(z+j+m-n)} \int_0^\infty h^{m+z+j-n-1} e^{-2\nu\pi h} e^{2i\nu\pi u}$$

$$= -\int_0^\infty \sum_{j=k}^{n} \frac{(m+j-n-k)!}{(m-n)!\,(j-k)!} \frac{B^{(n)}_{n-j}(u)}{(n-j)!} \frac{h^{m+z+j-n-1}}{(m+j+z-n-1)!} \frac{i^{-m+n-j-z}}{e^{2\pi(h-iu)}-1} dh$$ integrating by parts

$$= i^{-m+n-z}\cdot(-)(-1)^{z-1}\int_0^\infty \sum_{j=k}^{n} \frac{(m+j-n-k)!}{(m-n)!\,(j-k)!} \frac{B^{(n)}_{n-j}(u)}{(n-j)!} \frac{h^{m+j-n-k}}{(m+j-n-k)!} \partial_h^{k+z-1} \frac{i^{-j}}{e^{2\pi(h-iu)}-1} dh\,(-1)^k$$

$$= i^{-m+n-z}\frac{(-1)^z}{(m-n)!}\int_0^\infty \sum_{j=k}^{n} \frac{B^{(n)}_{n-j}(u)}{(n-j)!}\frac{h^{j-k}}{(j-k)!}\, h^{m-n}\, \partial_h^{k+z-1}\frac{i^{-j}}{e^{2\pi(h-iu)}-1}(-1)^k dh$$

$$= \lim_{\lambda=1} \partial_\lambda^{k}\, \frac{i^{-m+n-z}(-1)^z}{(m-n)!}\int_0^\infty \sum_{j=0}^{n} \frac{B^{(n)}_{n-j}(u)}{(n-j)!} \frac{i^{-j}\lambda^j}{j!} h^j\, h^{m-n-k}\, \partial^{k+z+1} \frac{1}{e^{\cdots}}\, (-1)^k$$

$$= \frac{(-1)^{k+z}\, i^{-m+n-z}}{(m-n)!} \lim_{\lambda=1}\partial_\lambda^{k} \int_0^\infty \frac{B^{(n)}_{n}(u-i\lambda h)}{n!}\, h^{m-n-k}\, \partial_h^{k+z+1}\frac{1}{e^{2\pi(h-iu)}-1} dh$$

$$z = -i\lambda h \qquad dz = -i\lambda\, dh = -ih\, d\lambda \qquad \frac{d}{d\lambda} = \frac{h}{\lambda}\frac{d}{dh} = \frac{d}{dz}\cdot\frac{d}{-ih\,d\lambda}$$

$$\frac{d}{d\lambda} = -ih\frac{d}{dz}$$

$$= \frac{(-1)^z}{(m-n)!}\, i^{-m+n-z+k}\int_0^\infty \frac{B^{(n)}_{n-k}(u-ih)}{(n-k)!}\, h^{m-n}\, \partial_h^{k+z+1}\frac{1}{e^{2\pi(h-iu)}-1} dh$$

$$b^{+}_{m+z}(k,n;u) =$$

$$-\int_0^\infty \sum_{j=k}^{n} \frac{(m+j-n-k)!}{(m-n)!\,(j-k)!}\,\frac{B^{(n)}_{n-j}(u)}{(n-j)!}\,\frac{h^{m+z+j-n-1}}{(m+j+z-n-1)!}\,\frac{i^{-m+n-j-z}}{e^{2\pi(h-iu)}-1}\,dh$$

$$\frac{h^{m+z+j-n-1}}{(m+j+z-n-1)!} = \underbrace{\iiint_0^h \cdots \int}_{k+z-1 \text{ times}} \frac{h'^{\,m+j-n-k}}{(m+j-n-k)!}$$

$$b^{+}_{m+z}(k,n;u) =$$

$$-\frac{(-1)^{k+z-1}}{(m-n)!}\, i^{-m+n-z} \int_0^\infty \sum_{j=k}^{n} \frac{h^{m+j-n-k}}{(j-k)!}\,\frac{B^{(n)}_{n-j}(u)}{(n-j)!} \underbrace{\iiint_0^h}_{(k+z-1)\text{ times}} \frac{1}{e^{2\pi(h-iu)}-1}\, i^{-j}\, dh$$

$$\underbrace{\iint_0^h \cdots \int}_{k+z-1} \frac{1}{e^{2\pi(h-iu)}-1} = \int_0^h \frac{(h-h')^{k+z-2}}{(k+z-2)!}\,\frac{1}{e^{2\pi(h'-iu)}-1}\,dh'$$ $u=0$ · dangerously continue

even if this succeeded, cannot get rid of factor $\frac{1}{(k+z-2)!}$

$$\overline{E}^{(n)+}_{z} = 2^n \sum_{\nu=0}^{n-1} \frac{(-1)^{\nu}}{(n-\nu-1)!}\,\frac{B^{(n)}_{\nu}(u)}{\nu!}\,\frac{(z+n-\nu-1)!}{(z+n-\nu-2)!}\, i^{-z-n+\nu} \sum_{j=1}^{\infty} \int_0^\infty \frac{h^{z+n-\nu-2}\, e^{(2j-1)\pi\{iu-h\}}}{(2j-1)\pi}$$

$$z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots = \arctan(z)$$

$$z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots = i^{-1}\arctan(iz) \qquad z = e^{\pi\{iu-h\}}$$

$$\overline{E}^{(n)+}_{z} = \frac{2^n}{\pi}\, i^{-1-z-n} \int_0^\infty \sum_{\nu=0}^{n-1} \frac{B^{(n)}_{\nu}(u)}{\nu!}\,\frac{(z+n-\nu-1)}{(n-\nu-1)!}\, i^{-\nu}\, h^{z+n-\nu-2} \arctan\{i e^{\pi\{iu-h\}}\}\,dh$$

$$i^{-n+1}\frac{2^n}{\pi} i^{-n-\tau-1}\tau\int_0^\infty \sum_{\nu=0}^{n-1}\frac{B_\nu^{(n)}(u)}{\nu!} i^{n-\nu-1}h^{n-\nu-1}h^{\tau-1}\arctan\{ie^{\pi\{iu-h\}}\}dh$$

$$+\frac{2^n}{\pi} i^{-n+2} i^{-n-\tau-1}\int_0^\infty \sum_{\nu=0}^{n-2}\frac{B_\nu^{(n)}(u)}{\nu!} i^{n-\nu-2}h^{n-\nu-2}h^{\tau} \arctan\{i \;''\;\}dh$$

$$=\frac{(-1)^n 2^n}{\pi}\frac{i^{-\tau}\tau}{(n-1)!}\int_0^\infty B_{n-1}^{(n)}(u+ih)\,h^{\tau-1}\arctan\{ie^{\pi\{iu-h\}}\}dh$$

$$+\frac{(-1)^n 2^n}{\pi}\frac{i^{-\tau+1}}{(n-2)!}\int_0^\infty B_{n-2}^{(n)}(u+ih)\,h^{\tau}\arctan\{ie^{\pi\{iu-h\}}\}dh$$

$$\int_0^\infty \bar{B}_r^{(n)}(u-t)e^{-\eta t}dt = \sum \bar{b}_{1,\nu+1}^{(n)}(u)(-\eta)^\nu$$

$$\int_0^\infty \bar{b}_r^{(n)}(u-t)e^{-\eta t}dt = \sum_{k=1}^{n}\int_{-1/(2\pi)}^{1/(2\pi)}\frac{d\sigma(k,r,n;u|t)}{(1+i\eta u)^k}$$

$$(-1)^j\int_0^\infty b_r^{(n)}(u-t)e^{-\eta t}t^j dt = \sum_{k=1}^{n}(-1)^j k(k+1)\cdots(k+j-1)\int_{-1/2\pi}^{1/2\pi}\frac{i^j u^j d\sigma(k,r,n;u|t)}{(1+i\eta u)^{k+j}}$$

$$\lim_{\eta\to 0}\int_0^\infty b_m^{(n)}(u-t)\,t^{\tau}e^{-\eta t}dt = \sum_{k=1}^{n} i^j k(k+1)\cdots(k+j-1)\int_{-1/(2\pi)}^{1/(2\pi)} u^j d\sigma(k.\,.\,.)$$

A(i) rep of $\bar{b}_-^{(n)}$ as Hamburger moment

(ii) rep of $\bar{b}_-^{(n)}$ as compact moment over finite interval

(iii) $\int_0^\infty \bar{B}_m^{(n)}(y-t)e^{-\eta t}dt=\sum_{k=1}^{n}$

$f(z)$ analytic from a to z ; ϕ of degree n in t

$$\frac{d}{dt}\sum_{m=1}^{n}(-1)^m(z-a)^m\phi^{(n-m)}(t)\,f^{(m)}\{a+t(z-a)\}$$

$$=-(z-a)\phi^{(n)}(t)\,f'\{a+t(z-a)\}+(-1)^n(z-a)^{n+1}\phi(t)\,f^{(n+1)}\{a+t(z-a)\}$$

$$\phi^{(n)}(t)=\phi^{(n)}(0)\ :\ \int_0^1\cdots dt$$

$$\phi^{(n)}(0)\{f(z)-f(a)\}=\sum_{m=1}^{n}(-1)^{m-1}(z-a)^m\left\{\phi^{(n-m)}(1)\,f^{(m)}(z)-\phi^{(n-m)}(0)\,f^{(m)}(a)\right\}$$

$$+(-1)^n(z-a)^{n+1}\int_0^1\phi(t)\,f^{(n+1)}\{a+t(z-a)\}dt$$

(Taylor series : $\phi(t)=(t-1)^n$; $n\to\infty$)

—o—

$$t\,\frac{e^{zt}-1}{e^t-1}=\sum_1\frac{\phi_n(z)t^n}{n!}=\sum\frac{B_\nu(z)}{\nu!}t^\nu-\sum\frac{B_\nu(0)}{\nu!}t^\nu$$

$$\frac{te^{zt}}{e^t-1}=\sum\frac{B_\nu^{(1)}(z)t^\nu}{\nu!}\qquad\phi_\nu(z)=B_\nu(z)-B_\nu(0)$$

—o—

$$\phi(t)=\phi_n(t)\qquad\phi_n^{(n-k)}(1)=\phi_n^{(n-k)}(0)$$

$$\phi^{(n-2k-1)}(0)=0\qquad\phi_n^{(n-2k)}(0)=\frac{n!}{(2k)!}(-1)^{k-1}B_k$$

$$\phi_n^{(n-1)}(0)=-\frac{1}{2}n!\qquad\phi_n^{(n)}(0)=n!$$

$$(z-a)f'(a)=f(z)-f(a)-\frac{(z-a)}{2}\{f'(z)-f'(a)\}$$

$$+\sum_{m=1}^{n-1}\frac{(-1)^{m-1}B_m(z-a)^{2m}}{(2m)!}\left\{f^{(2m)}(z)-f^{(2m)}(a)\right\}-\frac{(z-a)^{2n+1}}{(2n)!}\int_0^1\phi_{2n}(t)\,f^{(2n+1)}\{a+(z-a)t\}dt$$

ω for $x - a$ $F(x) = f'(x)$

$$\int_a^{a+\omega} F(x)dx = \frac{1}{2}\omega\left\{F(a)+F(a+\omega)\right\} + \sum_{m=1}^{n-1}\frac{(-1)^m B_m \omega^{2m}}{(2m)!}\left\{F^{(2m-1)}(a+\omega) - F^{(2m-1)}(a)\right\}$$

$$+ \frac{\omega^{2n+1}}{(2n)!}\int_0^1 \phi_{2n}(t)F^{(2n)}(a+\omega t)dt$$

$a+\omega, a+2\omega, \ldots a+(r-1)\omega$ for a

$$\int_a^{a+r\omega} F(x)dx = \omega\left\{\frac{1}{2}F(a) + F(a+\omega) + \cdots + \frac{1}{2}F(a+r\omega)\right\}$$

$$+ \sum_{m=1}^{n-1}\frac{(-1)^m B_m \omega^{2m}}{(2m)!}\left\{F^{(2m-1)}(a+r\omega) - F^{(2m-1)}(a)\right\} + \frac{\omega^{2n+1}}{(2n)!}\int_0^1 \phi_{2n}(t)\sum_{m=0}^{r-1}F^{(2n)}(a+m\omega+\omega t)\,dt.$$

$\phi(t) = B_n^{(2)}(u+t)$ $\quad x := z\omega \quad a := z$

$$n!\left\{f(z+\omega) - f(z)\right\} = \sum_{m=1}^{n}(-1)^{m-1}\omega^m\left\{\frac{n!}{m!}B_{n-m}^{(2)}(u+1)f^{(m)}(z+\omega) - \frac{n!}{m!}B_{n-m}^{(2)}(u)f^{(m)}(z)\right\}$$

$\phi(t) = B_n^{(2)}(u-t)$

$$(-1)^n n!\left\{f(z+\omega) - f(z)\right\} = \sum_{m=1}^{n}(-1)^{m-1}\omega^m(-1)^{n-m}\left\{\frac{n!}{m!}B_{n-m}^{(2)}(u^{-1})f^{(m)}(z+\omega) - \frac{n!}{m!}B_{n-m}^{(2)}(u)f^{(m)}(z)\right\}$$

$\sum \frac{B_\nu^{(r)}(u-h)}{\nu!}\Delta^{\nu-r}\phi(x+h\omega)\omega^\nu$ independent of h

$$\nabla_u^m \xi(u)\left\{I(z+h\omega)-I(z)\right\}$$

$$=-\sum_{\nu=1}^{m}\left\{\nabla_u^{m-\nu}\xi(u-h)\,\nabla_z^{\nu} I(z+h\omega)-\nabla_u^{m-\nu}\xi(u)\,\nabla_z^{\nu} I(z)\right\}$$

$$+\,\omega^{m+1}h\int_0^1 \xi(u-hv)\,\nabla_z^{m+1} I(z+v\omega h)\,dv$$

$z := z-u\omega \quad h=1$

$$\nabla_u^m \xi(u)\left\{I(z-u\omega+\omega)-I(z-u\omega)\right\}$$

$$=-\sum_{\nu=1}^{m}\left\{\nabla_u^{m-\nu}\xi(u-1)\,\nabla_z^{\nu} I(z-u\omega+\omega)-\nabla_u^{m-\nu}\xi(u)\,\nabla_z^{\nu} I(z-u\omega)\right\}$$

$$+\,\omega^{m+1}\int_0^1 \xi(u-v)\,\nabla_z^{m+1} I(z-u\omega+v\omega)\,dv$$

simply $h=1$

$$\nabla_u^m \xi(u)\left\{I(z+\omega)-I(z)\right\}$$

$$=-\sum_{\nu=1}^{m}\left\{\nabla_u^{m-\nu}\xi(u-1)\,\nabla_z^{\nu} I(z+\omega)-\nabla_u^{m-\nu}\xi(u)\,\nabla_z^{\nu} I(z)\right\}$$

$$+\,\omega^{m+1}\int_0^1 \xi(u-v)\,\nabla_z^{m+1} I(z+v\omega)\,dv$$

$\nabla_u^m \xi(u) = \text{const}$

$$\sum_{\nu=0}^{m}\nabla_u^{m-\nu}\xi(u-h)\,\nabla_z^{\nu} I(z+h\omega)-\sum_{\nu=0}^{m}\nabla_u^{m-\nu}\xi(u)\,\nabla_z^{\nu} I(z)$$

$$=\omega^{m+1}h\int_0^1 \xi(u-hv)\,\nabla_z^{m+1} I(z+v\omega h)\,dv$$

$\phi(t) = B_n^{(\omega)}(u - ht) \quad f(z + h\omega)$ function of h ?

$$f(z+zh) \qquad \frac{d}{dz'} = \frac{d}{dz}\frac{dz}{dz'} = h\frac{d}{dz'}$$

$$\frac{d}{dt} = \frac{d}{du}\cdot\frac{du}{dt}$$

$$= -\frac{d}{du}h$$

$$z = \omega \qquad a = 0$$

$$f^{(n+1)}\{z + 0 + th\omega\}$$

$$(-1)^{n} h^{n} n!\{f(z+h\omega) - f(z)\} =$$

$$\sum_{m=1}^{n}(-1)^{n-1}\omega^{m}(-1)^{n-m}\Big\{\frac{n!}{m!}B_m^{(N)}(u-h).h^{\cancel{n-m}}\cancel{h}^{m} f^{(m)}(z+h\omega)$$

$$-\frac{n!}{m!}B_m^{(N)}(u)h^{\cancel{n-m}}\cancel{h}^{m} f^{(m)}(z)\Big]$$

$$+(-1)^{n}\omega^{n+1}\int_0^1 B_n^{(N)}(u-ht)f^{(n+1)}\{z+th\omega\}dt.h^{n+1} \qquad ht = t' \quad dt = \frac{dt'}{h}$$

$$t = 1 = t' = h$$

$$(-1)^{n}\frac{\omega^{n+1}}{n!}h^{n}\int_0^h B_n^{(N)}(u-t)f^{(n+1)}(z+\omega t)dt$$

$$f(z+h\omega) + \sum_{m=1}^{n}\frac{B_m^{(N)}(u-h)}{m!}f^{(m)}(z+h\omega)\omega^{m}$$

$$- f(z) - \sum_{m=1}^{n}\frac{B_m^{(N)}(u)}{m!}f^{(m)}(z) = \frac{\omega^{n+1}}{n!}\int_0^h B_n^{(N)}(u-t)f^{(n+1)}(z-\omega t)dt$$

$$f(z\sum_{m=0}^{n}\frac{B_m^{(N)}(u)}{m!}f^{(m)}(z+h\omega)\omega^{m} - \sum_{m=0}^{n}\frac{B_m^{(N)}(u+h)}{m!}f^{(m)}(z)$$

$$= \frac{\omega^{n+1}}{n!}\int_0^h B_n^{(N)}(u+h-t)f^{(n+1)}(z-\omega t)dt$$

$$= \frac{\omega^{n+1}}{n!}\int_0^1 B_n^{(N)}(u+h-ht)f^{(n+1)}(z\,\omega t)dt$$

$$\Delta_\omega^N \sum_{m=0}^{n}{}' \frac{B_m^{(N)}(u)}{m!} f^{(m)}(z)\,\omega^m = \sum^{n} \frac{\Delta_h^N B_m^{(N)}(u+h)}{m!} f^{(m)}(z)$$

$$+ \frac{\omega^{n+1}}{n!} \sum_{\nu=0}^{N} (-1)^{N-\nu}\binom{N}{\nu}\int_0^\nu B_n^{(N)}(u\nu - t)\, f^{(n+1)}(z-\omega t)$$

$$\frac{\omega^{n+1}}{n!}\left[\int_0^\infty B_n^{(N)}(u+h-ht)\, f^{(n+1)}(z+\omega t)\,dt - \int_1^\infty B_n^{(N)}(u+h-ht)\, f^{(n+1)}(z+\omega t)\,dt\right]$$

$$t^* = t'+1 \qquad u+h-ht = u+h-ht'-h$$

$$\frac{\omega^{n+1}}{n!}\left[\int_0^\infty B_n^{(N)}(u+h-ht)\, f^{(n+1)}(z+\omega t)\,dt - \int_0^\infty B_n^{(n)}(u-ht)\, f^{(n+1)}(z+\omega t+\omega h)\,dh\right.$$

$$\sum_{m=0}^{n}{}' \frac{B_m^{(N)}(u)}{m!} f^{(m)}(z+h\omega)\,\omega^m + \frac{\omega^{n+1}}{n!}\int_0^\infty B_n^{(n)}(u-t)\, f^{(n+1)}(z+\omega t+\omega h)\,dt$$

$$= \sum_{m=0}^{n}{}' \frac{B_m^{(N)}(u+h)}{m!} f^{(m)}(z) + \frac{\omega^{n+1}}{n!}\int_0^\infty B_n^{(N)}(u+h-t)\, f^{(n+1)}(z+\omega t)\,dt$$

$$\Delta_\omega^N \left\{\sum_{m=0}^{n} \frac{B_m^{(N)}(u)}{m!} f^{(m)}(z)\,\omega^m + \frac{\omega^{n+1}}{n!}\int_0^\infty B_n^{(N)}(u-t)\, f^{(n+1)}(z+\omega t)\,dt\right\}$$

$$= \omega^N \left\{\sum \frac{\Delta^N B_m^{(N)}(u)}{m!} f^{(m)}(z)\,\omega^m + \frac{\omega^{n+1}}{n!}\int_0^\infty \Delta^N B_n^{(N)}(u-t)\, f^{(n+1)}(z+\omega t)\,dt\right\}$$

$$f(a+h) = \sum_{m=0}^{n-1} \frac{h^m}{m!} f^{(m)}(a) + \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(a+th)\,dt$$

ok if set $u=0$ after differencing

$a := x \quad z := \cancel{z+h\omega} \quad t := u + hv \quad \phi := \xi \quad f := I \quad z + z'h\omega$

$$\lim_{v=0} \partial_v^m \xi(u-hv)\{I(\cancel{z+h\omega}) - I(z)\}$$

$I(z+z'h\omega) \quad A=0 \quad Z=1$

$A + v\cancel{(Z-A)} =$

$$= \sum_{\nu=1}^{m} (-1)^{\nu-1}\omega^{\nu}\Big\{\lim_{v=1}\partial_v^{m-\nu}\xi(u-hv)\,\partial_z^{\nu} I(z+h\omega) - \lim_{v=0}\partial_v^{m-\nu}\xi(u+hv)\,\partial_z^{\nu} I(z)\Big\}$$

$$+ (-1)^m \omega^{m+1}\int_0^1 \xi(u-hv)\,\partial_z^{m+1} I\{z+vh\omega\}\,dv$$

$$I\{z + (A+v(Z-A))h\omega\} = I\{z+vh\omega\}$$

$$\lim_{z'=1} \partial_{z'}^{\nu} I(z+z'h\omega) \qquad \lim$$

try $A=0 \quad Z=\omega_0 \quad I\{z+z'h\} \quad I(A) = I\{z\} \quad I(Z) = I(z+h\omega)$

$$\lim_{v=0}\partial_v^m \xi(u-hv)\{I(z+h\omega) - I(z)\}$$

$\cancel{A} \; z + \{A + v\omega\}h$

$$= \sum_{\nu=1}^{m}(-1)^{\nu-1}\omega^{\nu}\Big\{\lim_{v=1}\partial^{m-\nu}\xi(u-hv)\lim_{y=\omega}\partial_y^{\nu} I(z+yh)$$

$$- \lim_{v=0}\partial^{m-\nu}\xi(u-hv)\lim_{y=0}\partial_y^{\nu} I(z+yh)$$

$$+ (-1)^m\omega^{m+1}\int_0^1 \xi(u-hv)\,\partial_{v\omega}^{m+1} I(z+v\omega h)\,dv$$

$x = u - hv$

$dx = -h\,dv = du$

$\frac{d}{du} = -\frac{1}{h}\frac{d}{dv}$

$\frac{d}{dv} = -h\frac{d}{du}$

$$\cancel{(-1)^m}\,\cancel{h^m}\,\partial_u^m \xi(u)\,\{I(z+h\omega) - I(z)\}$$

$$= \sum_{\nu=1}^{m}(-1)^{\nu-1}\omega^{\nu}\Big\{\cancel{(-1)^{m-\nu}}\,\cancel{h^{m-\nu}}\,\partial_u^{m-\nu}\xi(u-h)\,h^{\nu}\partial_z^{\nu} I(z+h\omega)$$

$$- h^{m-\nu}\cancel{(-1)^{m-\nu}}\partial^{m-\nu}\xi(u)\,h^{\nu}\partial_z^{\nu} I(z)\Big\}$$

$$+ \cancel{(-1)^m}\,\omega^{m+1}\int_0^1 \xi(u-hv)\,\partial_z^{m+1} I(z+v\omega h)\,h^{m+1}\,dv$$

$$= \omega^{m+1} h \int_0^\infty \xi(u-hv)\, \Delta_z^{m+1} I(z+v\omega h)\,dv -$$

$$-\omega^{m+1} h \int_1^\infty \xi(u-hv)\, \Delta_z^{m+1} I(z+v\omega h)\,dv$$

$$v' = v-1$$

$$-\omega^{m+1} h \int_0^\infty \xi(u+h-hv)\, \Delta_z^{m+1} I(z+\omega h+vh\omega)\,dv$$

$$\sum_{\nu=0}^{m} \Delta_u^{m-\nu} \xi(u-h)\, \Delta_z^{\nu} I(z+h\omega)\omega^{\nu} + h\omega^{m+1} \int_0^\infty \bar{\xi}(u-h-hv)\, \Delta_z^{m+1} I(z+\omega h+vh\omega)\,dv$$

$$= \sum_{\nu=0}^{m} \Delta_u^{m-\nu} \xi(u)\, \Delta_z^{\nu} I(z)\omega^{\nu} + h\omega^{m+1} \int_0^\infty \bar{\xi}(u-hv)\, \Delta_z^{m+1} I(z+vh\omega)\,dv$$

$$\sum_{\nu=0}^{m} \Delta_u^{m-\nu} \xi(u)\, \Delta_z^{\nu} I(z+\tau\omega)\omega^{\nu} + h\omega^{m+1} \int_0^\infty \bar{\xi}(u-hv)\, \Delta_z^{m+1} I(z+\tau\omega+vh\omega)\,dv$$

$u := u+\tau \quad h=\tau$

$$= \sum_{\nu=0}^{m} \Delta_u^{m-\nu} \xi(u+\tau)\, \Delta_z^{\nu} I(z)\omega^{\nu} + \omega^{m+1} \int_0^\infty \bar{\xi}(u+\tau-v)\, \Delta_z^{m+1} I(z+v\omega)\,dv$$

$F(m,u,z|\omega) =$

$$\sum_{\nu=0}^{m} \partial_u^{m-\nu} \xi(u)\, \Delta_z^{\nu} I(z)\omega^{\nu} + \omega^{m+1} \int_0^\infty \bar{\xi}(u-v)\, \Delta_z^{m+1} I(z+v\omega)\,dv$$

$$\Delta_\omega^{n} F(m,u,z|\omega) = \sum_{\nu=0}^{m} \left[\Delta_u^{n}\left\{\partial_u^{m-\nu} \xi(u)\right\}\right] \Delta_z^{\nu} I(z)\omega^{\nu}$$

$$+\omega^{m+1} \int_0^\infty \left[\Delta_u^{n} \bar{\xi}(u-v)\right] \Delta_z^{m+1} I(z+v\omega)\,dv$$

$$\sum_{\nu=0}^{m} \partial_u^{m-\nu}\xi(u)\,\partial_z^{\nu} I(z+h\omega)\omega^{\nu} + \omega^{m+1}\int_0^{\infty} \bar{\xi}(u-v)\,\partial_z^{m+1} I(z+\omega h+v\omega)\,dv$$

$$= \sum_{\nu=0}^{m} \partial_u^{m-\nu}\xi(u+h)\,\partial_z^{\nu} I(z)\omega^{\nu} + \omega^{m+1}\int_0^{\infty} \bar{\xi}(u+h-v)\,\partial_z^{m+1} I(z+v\omega)\,dv$$

coming from

$$\partial_u^m \xi(u+h)\left\{I(z+h\omega)-I(z)\right\}$$

$$= -\sum_{\nu=1}^{m}\left\{\partial_u^{m-\nu}\xi(u)\,\partial_z^{\nu} I(z+h\omega) - \partial^{m-\nu}\xi(u+h)\,\partial_z^{\nu} I(z)\right\}\omega^{\nu}$$

$$+\,\omega^{m+1} h\int_0^1 \xi(u+h-hv)\,\partial_z^{m+1} I(z+v\omega h)\,dv$$

obtained from Darboux with

$A=0 \quad Z=\omega \quad f(z')= I(z+z'h) \quad t=u-hv$

$I(z')$ analytic from $z'=z$ to $z'=z+h\omega$

$$\frac{d}{dv}\sum_{\nu=1}^{m} \partial_v^{m-\nu}\xi(u+h-v)\,\partial_v^{\nu} I(z+v\omega)$$

$$= \sum_{\nu=1}^{m} \partial_v^{m-\nu+1}\xi(u+h-v)\,\partial^{\nu} I(z+v\omega) + \cdots$$

$$\frac{d}{dv}\sum_{\nu=1}^{m}(-1)^{\nu}\,\partial_v^{m-\nu}\xi(u+h-v)\,\partial_v^{\nu} I(z+v\omega)$$

$$-\,\partial_v^{m}\xi(u+h-v)\,\partial_v^{1} I(z+v\omega) - \partial^{m-1}\xi(u+h+v)\,\partial_v^{2} I(z+v\omega)$$

$$+\,\partial_v^{m-1}\xi(u+h-v)\,\partial_v^{2} I(z+v\omega) + \partial^{m-2}\xi$$

$$+(-1)^m\,\xi(u+h-v)\,\partial_v^{m+1} I(z+v\omega)$$

$$(-1)^{m+1}\{I(z+h\omega)-I(z)\}+(-1)^m\int_0^h \xi(u+h-v)\,\partial_v^{m+1} I(z+v\omega)\,dv$$

$$=\sum_{\nu=1}^{m}{}'(-1)^\nu(-1)^{m-\nu}\left[\partial_u^{m-\nu}\xi(u)\,\partial_z^\nu I(z+h\omega)-\partial_u^{m-\nu}\xi(u+h)\,\partial_z^\nu I(z)\right]\omega^\nu$$

$$\boxed{\begin{aligned}&\sum_{\nu=0}^{m-1}\left\{\partial_u^{m-\nu-1}\xi(u)\,\partial_z^\nu I(z+h\omega)-\partial_u^{m-\nu-1}\xi(u+h)\,\partial_z^\nu I(z)\right\}\omega^\nu\\&=\xi\,\omega^{m+1}\int_0^h \xi(u+h-v)\,\partial_z^{m+1} I(z+v\omega)\,dv\end{aligned}}$$

$$=\omega^{m+1}\left[\int_0^\infty \bar{\xi}(u+h-v)\,\partial_z^{m+1}I(z+v\omega)\,dv-\int_h^\infty \bar{\xi}(u+h-v)\,\partial_z^{m+1}I(z+v\omega)\,dv\right.$$

$\bar{\xi}(u+h-v)=\xi(u+h-v)$ $\quad 0\le v\le h$ $\quad \xi$

$\bar{\xi}(v)=\xi(v)$ $\quad u\le v\le u+h$

$\bar{\xi}(v)$ extension of $\xi(v)$ over $-\infty<v\le u$

such that $\int_0^\infty \bar{\xi}(u+h-v)\,\partial_z^{m+1} I(z+v\omega)\,dv<\infty$

$$F^{m;}(u;z)=\sum_{\nu=0}^{m-1}{}'\partial_u^{m-\nu-1}\xi(u)\,\partial_z^\nu I(z)\,\omega^\nu+\omega^{m+1}\int_0^\infty \bar{\xi}(u-v)\,\partial_z^{m+1}I(z+v\omega)\,dv$$

$$F^{m;}(u;z+h\omega)=F(u+h;z)$$

$$F^{(n)}(z)=\sum_{\nu=0}^{m-1}{}'\,b_\nu^{(n)}(0)\,\partial^{\nu-n}\psi(z)\,\omega^\nu+\omega^{m+1}\int_0^\infty \bar{b}_{m-1}^{(n)}(-v)\,\partial_z^{m+1}I(z+v\omega)\,dv$$

$$I(z)=\partial^{-n}\psi(z)=\int_{a_n}^{z}\cdots\int_{a_1}^{z'}\psi(z')\,dz'$$

take $\xi(x)=b_m^{(n)}(x)$

$h=0,1,\dots,n$ $\quad \underset{\omega}{\Delta}^{n}F^{m;}(u,z)=\omega^{-n}\Delta_u^n F^{m;}(u,z)$

$$= \sum_{\nu=0}^{m-n-1} b_\nu^{(n)}(u)\, \mathcal{D}_z^\nu \psi(z)\, \omega^\nu + \omega^m \int_0 \Delta^{n} \bar{b}_{m-1}^{(n)}(u-v)\, \mathcal{D}_z \psi(z+v\omega)\, dv$$

~~$m \geq n$~~ $m > n$

ξ polynomial of degree $m-1$ such that $\mathcal{D}^{m-1}\xi(v) = 1$

$$\frac{d}{dv} \mathcal{D}_v \sum_{\nu=1}^{m-1} (-1)^\nu \mathcal{D}_v^{m-\nu-1} \xi(u+h-v)\, \mathcal{D}_v^\nu I(z+v\omega) =$$

$$(-1)^{m-1} \xi(u+h-v)\, \mathcal{D}_v^m I(z+v\omega) - \mathcal{D}^{m-1}\xi(u+h-v)\, \mathcal{D}_v I(z+v\omega)$$

$$R_{\rho\neq 0}^{(n,m)\prime}(\omega) = \int_1^\infty \int_\infty^\beta \frac{\bar{B}_{m-1}^{(n)}(\beta - t)}{(m-1)!} \mathcal{D}^m \psi(z+t\omega)\, d\rho(u)\, dt$$

definition also holds when $m = n+1$ even though $\bar{b}_n^{(n)}(v)$ discontinuous.

---

may be shown by use of explicit formulae for $\bar{b}_m^{(n)}(u)$ that

b) $\bar{b}_n^{(n)}$ discontinuous at $u = \dots -2, -1, 0, n, n+1, \dots$ with jumps

$$\bar{b}_n^{(n)}(u+0) - \bar{b}_n^{(n)}(u-0) = -\frac{u(u-1)(u-2)\dots(u-n+1)}{n!}$$

$$= -\binom{u-1}{n-1}$$

a) $\mathcal{D}_u \bar{b}_m^{(n)}(u) = \bar{b}_{m-1}^{(n)}(u)$ $m > n$ all real u ~~setting~~

b) " " when $m = n$ also for $u \neq \dots -2, -1, 0, n, n+1, \dots$

at the points u

integrating by parts $F(m+1; u; z+h\omega) = F(m; u; z+h\omega)$ for $m > n$

$$F^{(n)}(n+1;0;z+h\omega) =$$

$$\sum_{\nu=0}^{n} b_\nu^{(n)}(0)\,\mathfrak{d}^{\nu-n}\psi(z)\,\omega^\nu + \omega^{n+1}\left[\left[-b_{n+1}^{(n)}(-v)\,\mathfrak{d}_z^{1}\psi(z+v\omega)\right]_0^\infty + \omega\int_0^\infty b_n^{(n)}(-v)\,\mathfrak{d}_z^{2}\psi(z+v\omega)\,dv\right]$$

$$\omega^{n+1}\left[\int_0^\infty \bar{b}_n^{(n)}(-v)\,\mathfrak{d}_z^{n+1}I(z+v\omega)\,dv\right]$$

$$\omega^{n+1}\sum_{i=0}^{\infty}\int_{i}^{i+1}\bar{b}_n^{(n)}(-v)\,\mathfrak{d}_z\psi(z+v\omega)\,dv$$

$$\omega^{n}\sum_{i=0}^{\infty}\left[\left[\bar{b}_n^{(n)}(-v)\,\psi(z+v\omega)\,dv\right]_i^{i+1} + \int_{i}^{i+1} b_{n-1}^{(n)}(-v)\,\psi(z+v\omega)\,dv\right]$$

$$\bar{b}_n^{(n)}(-i-1+0)\,\psi(z+\overline{i+1}\,\omega) - \bar{b}_n^{(n)}(-i-0)\,\psi(z+i\omega)$$

$$\sum_{i=1}^{\infty}\bar{b}_n^{(n)}(-i+0)\,\psi(z+i\omega) - \sum_{i=0}\bar{b}_n^{(n)}(-i-0)\,\psi(z+i\omega)$$

$$-\bar{b}_n^{(n)}(-0)\,\psi(z) + \sum_{i=1}^{\infty}\left\{b_n^{(n)}(-i+0) - b_n^{(n)}(-i-0)\right\}\psi(z+i\omega)$$

$$-\bar{b}_n^{(n)}(0-)\,\psi(z) - \sum_{\nu=1}^{\infty}\binom{-\nu-1}{n-1}\psi(z+\nu\omega)$$

$$\cancel{b_n^{(n)}(0) = (-1)^{n}\,b_m^{(m)}}$$

$$\boxed{F^{(n)}(n+1;0;z+h\omega) = \sum_{\nu=0}^{n-1} b_\nu^{(n)}(0)\,\mathfrak{d}^{\nu-n}\psi(z)\,\omega^\nu - \omega^{n}\sum_{\nu=0}^{\infty}\binom{-\nu-1}{n-1}\psi(z+\nu\omega) + \omega^{n}\int_0^\infty b_{n-1}^{(n)}(-v)\,\psi(z+v\omega)\,dv}$$

$$\left\{\bar{b}_n^{(n)}(0+) - \bar{b}_n^{(n)}(0-)\right\}\psi(z)$$

$$F^{(n)}(m+1;0;z) = \sum_{\nu=0}^{m-1} b_\nu^{(n)}(0)\,\mathcal{D}^{\nu-n}\psi(z)\omega^\nu - \omega^n \sum_{\nu=0}^{\infty}\binom{-\nu-1}{n-1}\psi(z+\nu\omega)$$

$$+\,\omega^m \int_0^\infty b_{m-1}^{(n)}(-v)\,\mathcal{D}^{m-n}\psi(z+v\omega)\,dv$$

$m \neq n$? $m = n$

this does not go to zero as $v \to \infty$

$$\left[\omega^n - b_{m-1}^{(n)}(-v)\int_v^\infty \psi(z+v'\omega)\,dv'\right]_0^\infty - \omega^n \int_0^\infty b_{m-2}^{(n)}(-v)\int_v^\infty \psi(z+v'\omega)\,dv'\,dv$$

$z' = z + z''$ $\quad z' = a$ $\quad z'' = a - z$

$$\mathcal{D}^{-1}\psi(z) = \int_a^z \psi(z')\,dz' = \int_{a-z}^{0}\psi(z+z')\,dz'$$

~~$b_{m-1}^{(n)}$ etc term annihilated~~ left with $\omega^n \int_0^\infty b_{m-2}^{(n)}(-v)\, I^{(1)}(z+v\omega)\,dv$

$$I^{(1)}(z+v\omega) = \int_v^\infty \psi(z+v'\omega)\,dv' = \qquad v'\omega = z'$$

$$= \frac{1}{\omega}\int_{v\omega}^\infty \psi(z+z')\,dz' = \frac{1}{\omega}\int_{z+v\omega}^{\infty}\psi(z')\,dz'$$

$$b_{n-1}^{(n)}(0)\int_a^z \psi(z')\,dz' + b_{n-1}^{(n)}(0)\int_z^\infty \psi(z')\,dz'$$

$$= \left[\mathcal{D}^{-1}\psi(z+v\omega) - \mathcal{D}^{-1}\psi(z)\right]$$

$$\int_0^\infty t e^{-zt}\,dt = \left[-t\int_t^{\infty} e^{-zt'}\,dt'\right]_0^\infty + \int_0^\infty\left[\int_t^\infty e^{-zt'}\,dt'\right]dt = \frac{1}{z}\int_0^\infty e^{-zt}\,dt$$

$$\int_t^\infty e^{-zt'}\,dt' = \frac{e^{-zt}}{z}$$

$$\omega^n \int_0^\infty b_{m-2}^{(n)}(-v)\int_v^\infty \psi(z+v'\omega)\,dv'\,dv = \omega^n\left[\left[-b_{m-2}^{(n)}(-v)\int_v^\infty\int_{v'}^\infty \psi(z+v''\omega)\,dv''\right]_0^\infty\right.$$

$$\left. - \int_0^\infty b_{m-3}^{(n)}(-v)\int_v^\infty\int_{v'}^\infty \psi(z+v''\omega)\,dv''\,dv\right]$$

$z + z'' \cdot \psi(z')$

$v'\omega = z''$

$z' = z + z''$

$z' = z + z''$

$z'' = z''$

$$\int_v^\infty \int_{z+v'\omega}^\infty \psi(z')\,dz'\,dv' = \frac{1}{\omega}\int_{z+v\omega}^\infty \int_{z''}^\infty \psi(+z')\,dz'\,dz''$$

$z + v'\omega' = z''$ $\quad v' = v$ $\quad z'' = z + v\omega$

$$F^{(n)}(n-1,0;z)=\sum_{\nu=0}^{n-2} b_\nu^{(n)}(0)\,\partial^{\nu-n}\psi(z)\,\omega^{\nu}-\omega^{n}\sum_{\nu=0}^{\infty}\binom{-\nu-1}{n-1}\psi(z+\nu\omega)$$

$$+\,b_{n-1}^{(n)}(0)\left\{\int_a^z\psi(z')dz'+\int_z^\infty\psi(z')dz'\right\}\omega^{n-1}-\underbrace{\omega^n\int_0^\infty b_{n-2}^{(n)}(-v)\int_v^\infty\psi(z+v'\omega)dv'dv}_{*}$$

$$b_{n-1}^{(n)}(0)\,I^{(1)}(a,\infty)$$

$$* = -\omega^n\left[\frac{1}{\omega^2}b_{n-2}^{(n)}(0)\int_z^\infty\int_{z'}^\infty\psi(z'')dz''dz'-\int_0^\infty b_{n-3}^{(n)}(-v)\int_v^\infty\int_{v'}^\infty\psi(z+v''\omega)dv''dv'dv\right]$$

$$\boxed{\int_z^\infty\int_{z_{r-1}^*}^\infty\cdots\int_{z_1}^\infty\psi(z_1^*)dz_1\cdots dz_{r-1}dz=\int_z^\infty\frac{(z-z')^{r-1}}{(r-1)!}\psi(z')dz'\,(-1)^{r-1}\ ?}$$

$$\psi(z_1)=\frac{1}{z_1^n}\qquad\frac{1}{(n-1)(n-2)\cdots(n-r)}\frac{1}{z^{n-r}}=\frac{(n-r-1)!}{(n-1)!}\frac{1}{z^{n-r}}\qquad z^{-n+\nu}$$

$$\int_z^\infty\sum_{\nu=0}^{n-1}\binom{r-1}{\nu}\frac{1}{(r-1)!}\frac{1}{z'^{\,n-\nu}}z^{r-1-\nu}(-1)^\nu\qquad\qquad\int_z^\infty\frac{1}{z'^n}dz'=\frac{1}{n-1}\cdot\frac{1}{z^{n-1}}\ ?$$

$$\sum_{\nu=0}^{r-1}\frac{(r-1)!}{\nu!\,(r-\nu-1)!}(-1)^\nu\frac{1}{(r-1)!}\cdot\frac{1}{(n-\nu-1)}\cdot\frac{z^{r-\nu-1}}{z^{n-\nu-1}}$$

$$=\frac{1}{(r-1)!}z^{r-n}\sum_{\nu=0}^{r-1}\binom{r-1}{\nu}(-1)^\nu\frac{1}{n-\nu-1}$$

$$=\frac{1}{(r-1)!}z^{r-n}\frac{(n-r-1)!}{(n-1)!}\ \checkmark$$

$$\Delta\frac{1}{\nu+a}=-\frac{1}{(\nu+a)(\nu+a+1)}$$

$$\Delta^2\frac{1}{\nu+a}=(-1)^2\frac{1}{(\nu+a)(\nu+a+1)(\nu+a+2)}$$

$$\Delta^z\frac{1}{\nu+a}=(-1)^z\frac{(\nu+a-1)!}{(\nu+a+z)!}$$

$$a=-n-1$$

$$\partial^{-z}\psi(z)=I^{(z)}(a,z)=\int_a^z\frac{(z-z')^{z-1}}{(z-1)!}dz'$$

$$\Delta\frac{1}{a-\nu}=\frac{1}{(a-\nu)(a-\nu-1)}$$

$$I^{(z)}$$

$$\Delta^z\frac{1}{a-\nu}=\frac{(a-\nu-z-1)!}{(a-\nu)!}$$

$$a=n+1$$

$$F^{(n)}(n+1;0;z) = \sum_{\nu=0}^{n-1} b_\nu^{(n)}(0)\, \partial^{\nu-n}\psi(z)\omega^\nu - \omega^n \sum_{\nu=0}^{\infty} \binom{-\nu-1}{n-1}\psi(z+\nu\omega)$$

$$+\,\omega^n \int_0^\infty b_{n-1}^{(n)}(-v)\,\psi(z+v\omega)\,dv$$

$$F^{(n)}(n;0;z) =$$

$$= \sum_{\nu=0}^{n-2} b_\nu^{(n)}(0)\,\partial^{\nu-n}\psi(z)\omega^\nu + b_{n-1}^{(n)}(0)\int_a \psi(z')\,dz'\,\omega^{n-1} - \omega^n \sum_{\nu=0}^{\infty}\binom{-\nu-1}{n-1}\psi(z+\nu\omega)$$

$$-\omega^n \int_0^\infty b_{n-2}^{(n)}(-v)\int_v^\infty \psi(z+v'\omega)\,dv'\,dv \quad \overset{?}{=} -\omega^{n-1}\int_0^\infty b_{n-2}^{(n)}(-v)\,I^{(1)}(\psi, z+v\omega)\,dv$$

$$\sim \left\{-\omega^{n-2}\, I^{(2)}(\psi,z) + \omega^{n-2}\int_0^\infty b_{n-3}^{(n)}(-v)\,I^{(2)}(\psi, z+v\omega)\,dv\right\}$$

$$\boxed{\partial^{-\tau}\psi(z) = \int_a^z \frac{(z-z')^{\tau-1}}{(\tau-1)!}\psi(z')\,dz'}$$

$$\int_v^\infty \psi(z+v'\omega)\,dv' = \frac{1}{\omega}\int_{z+v\omega}^\infty \psi(z')\,dz'$$

$v' = v \equiv z'' = z+v\omega$

$z + v'\omega = z''$

$m = n = -$

$m = n-1 = +$

$$\int_v^\infty\int_{v'}^\infty \psi(z+v''\omega)\,dv''\,dv' = \frac{1}{\omega}\int_v^\infty\int_{z+v'\omega}^\infty \psi(z')\,dz'\,dv'$$

$$= \frac{1}{\omega^2}\int_{z+v\omega}^\infty\int_{z'}^\infty \psi(z')\,dz'\,dz''$$

$$-\omega^n\int_0^\infty b_{n-2}^{(n)}(-v)\int_v^\infty \psi(z+v'\omega)\,dv'\,dv =$$

$$-\omega^n\left[\frac{1}{\omega^2}\int_{z}^\infty\int_{z'}^\infty \psi(z')\,dz'' - \int_0^\infty b_{n-3}^{(n)}(-v)\int_v^\infty\int_{v'}^\infty \psi(z+v''\omega)\,dv''\,dv'\,dv\right]$$

$$\boxed{I^{(\tau)}(\psi,z) = \int_z^\infty\int_{z_{\tau-1}}^\infty \cdots \int_{z_1}^\infty \psi(z_\tau)\,dz_\tau\,dz_1\ldots dz_{\tau-1} = (-1)^{\tau-1}\int_z^\infty \frac{(z-z')^{\tau-1}}{(\tau-1)!}\psi(z')\,dz'}$$

$$\binom{-\tau}{n-1} = \frac{(-\tau)(-\tau-1)\ldots(-\tau-n+2)}{(n-1)!} = (-1)^{n-1}\binom{\tau+n-2}{n-1}$$

$u = 0$ yields

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \sum_{z=1}^{\prime} \frac{b_{n-1}^{(n)}(u+\nu)\,\psi(z+\nu\omega)}{\sin(\nu - z)} (-1)^{z}\, x^{-}$$

$$\sum_{z=1}^{\infty\,\prime} (-1)^{z}\, b_{n-1}^{(n)}(z)\,\psi(z+z\omega)\, x^{-z}$$

$$b_{n}^{(n)}(u)\,\psi(z) = (-1)^{n} \sum \binom{\nu+n-1}{n-1} \psi(z+\nu\omega) - \int_0^\infty \bar{b}_{n-1}^{(n)}(u-v)\,\psi(z+v\omega)\,dv$$

$$= \int_0^\infty \frac{\Phi(x)}{x\{\pi^2 + (\ln(x))^2\}}\,dx$$

where

$$\Phi(x) = \frac{1}{2i}\int_{-i\infty}^{i\infty} \frac{b_{n-1}^{(n)}(u+v)\,\{\psi(z+v\omega) - \psi(z)\}\, e^{i\pi u\, \mathrm{sgn}(-iv)}\, x^{-v}}{\sin\{\pi(u+v)\}}\,dv$$

$$u=0 \quad \Phi(x) = \frac{1}{2i}\int_{-i\infty}^{i\infty} \frac{b_{n-1}^{(n)}(v)\,\{\psi(z+v\omega) - \psi(z)\}\, x^{-v}}{\sin\{\pi v\}}\,dv$$

signs attached to ν?

$$= \sum_{1} (-1)^{\nu}\, b_{n-1}^{(n)}(\nu)\,\{\psi(z+\nu\omega) - \psi(z)\}\, x^{-\nu}\, dv$$

$$F^{(n)}(m;0;z) = \sum_{\nu=0}^{m-2} b_\nu^{(n)}(0)\Delta^{\nu-n}\psi(z)\omega^\nu + \sum_{\nu=m-1}^{n-1} b_\nu^{(n)}(0)\, I^{(n-\nu)}(\psi,a)\,\omega^\nu$$

$$+ (-\omega)^n \sum_{\nu=0}^{\infty} \binom{\nu+n-1}{n-1}\psi(z+\nu\omega)$$

$$+ (-1)^{n-m-1}\omega^{m-1}\int_0^\infty b_{m-2}^{(n)}(-v)\, I^{(n-m+1)}(\psi, z+v\omega)\,dv$$

$m := m-1$ in F

$$F^{(n)}(m;0;z) = \sum_{\nu=0}^{m-1} b_\nu^{(n)}(0)\Delta^{\nu-n}\psi(z)\omega^\nu + \sum_{\nu=m}^{n-1} b_\nu^{(n)}(0)\, I^{(n-\nu)}(\psi,a)\,\omega^\nu$$

$$+ (-\omega)^n \sum_{\nu=0}^{\infty} \binom{\nu+n-1}{n-1}\psi(z+\nu\omega) + (-1)^{n-m}\omega^{m}\int_0^\infty b_{m-1}^{(n)}(-v)\, I^{(n-m)}(\psi, z+v\omega)\,dv$$

$$F^{(n)}(m;0;z) = \sum_{\nu=0}^{m} b_\nu^{(n)}(0)\Delta^{\nu-n}\psi(z)\omega^\nu + \omega^{m+1}\int_0^\infty \bar{b}_m^{(n)}(-v)\,\Delta_z^{m-n+1}\psi(z+v\omega)\,dv$$

$$\sum_{\nu=0}^{m}\left\{\Delta_u^{m-\nu}\xi(u)\Delta_z^\nu I(z+h\omega) - \Delta_u^{m-\nu}\xi(u+h)\Delta_z^\nu I(z)\right\}\omega^\nu$$

$$= \omega^{m+1}\int_0^h \xi(u+h-v)\,\Delta_z^{m+1} I(z+v\omega)\,dv$$

$$\frac{d}{dv}\sum_{\nu=1}^{m}(-1)^\nu\Delta_v^{m-\nu}\xi(u+h-v)\Delta_v^\nu I(z+v\omega)$$

$$= (-1)^m\xi(u+h-v)\Delta_v^{m+1}I(z+v\omega) - \Delta_v^m\xi(u+h-v)\Delta_v I(z+v\omega)$$

$\xi(v)$ polynomial of m^{th} degree with $\Delta^m\xi(v) = 1$

$$F(m,n;u;z) = \sum_{\nu=0}^{m}\Delta_u^{m-\nu}\xi(u)\Delta_z^\nu I(z)\omega^\nu + \omega^{m+1}\int_0^\infty \bar{\xi}(u-v)\Delta_z^{m+1} I(z+v\omega)\,dv$$

$-z = z - z \qquad z' = z + z''$

Nörlund sets $\quad z' = a \quad z'' = a - z \qquad z - z' =$

$$F^{(n)}(z|\omega) = \int_a^\infty \hat{b}^{(n)}_{n-1}(z - z')\psi(z')dz' + (-\omega)^n \sum_{\nu=0}^\infty \binom{\nu+n-1}{n-1}\psi(z+\nu\omega)$$

$$= \int_{a-z}^\infty b^{(n)}_{n-1}(-z'')\psi(z+z'')dz'' + \quad ''$$

with $b^{(n)}_{n-1}$ differently defined

$$e^{ut}\frac{t\omega}{e^{\omega t}-1} = \sum \frac{\hat{B}_\nu(u)}{\nu!}t^\nu \qquad e^{ut\omega}\left\{\frac{t\omega}{e^{\omega t}-1}\right\}^n = \sum \frac{\hat{B}^{(n)}_\nu(\omega)}{\nu!}t^\nu$$

$$e^{ut\omega}\left\{\frac{t\omega}{e^{t\omega}-1}\right\} = \sum \frac{B^{(n)}_\nu(u)\omega^\nu t^\nu}{\nu!}$$

$$\hat{B}^{(n)}_\nu(u\omega) = B^{(n)}_\nu(u)\omega^\nu \qquad \hat{b}^{(n)}_{n-1}\{z - z'\} = b^{(n)}_{n-1}\left\{\frac{z-z'}{\omega}\right\}\omega^{n-1}$$

$$z' = z + v\omega$$

$$\boxed{F^{(n)}(z|\omega) = \omega^{n-1}\int_a^\infty b^{(n)}_{n-1}\left\{\frac{z-z'}{\omega}\right\}\psi(z')dz' + (-\omega)^n\sum \cdots}$$

$$\frac{z-z'}{\omega} = -v$$

$$dz' = \omega dv$$

$$= \omega^n \int_{\frac{z-a}{\omega}}^\infty b^{(n)}_{n-1}(-v)\psi(z+v\omega)dv$$

$$v = 0 \quad z' = z$$

$$= F(n;n;z|a)$$

$$= \omega^n \int_0^\infty b^{(n)}_{n-1}(-v)\psi(z+v\omega)dv + \omega^n \int_{\frac{z-a}{\omega}}^0 b^{(n)}_{n-1}(-v)\psi(z+v\omega)dv$$

$$+\omega^{n-1}\int_a^z b^{(n)}_{n-1}\left(\frac{z-z'}{\omega}\right)\psi(z')dz'$$

$$+\omega^{n-1}\sum_{\nu=0}^{n-1} b^{(n)}_\nu(0)\int_a^z \left\{\frac{z-z'}{\omega}\right\}^{n-\nu-1}\frac{\psi(z')}{(n-\nu-1)!}dz'$$

$$\omega^{n-1}\sum_{\nu=0}^{n-1} b^{(n)}_\nu(0)\,\mathcal{D}^{\nu-n}\psi(z)$$

from p156

$v = v' + h$

$$\int_0^h \bar{\xi}(u+h-v)\, {}^v\!\Delta_z^{m+1} I(z+v\omega)\,dv$$

$$= \int_{-\infty}^{h} \bar{\xi}(u+h-v)\, \Delta_z^{m+1} I(z+v\omega)\,dv - \int_{-\infty}^{0} \bar{\xi}(u+h-v)\, \Delta_z I(z+v\omega)\,dv$$

$$= \int_{-\infty}^{0} \bar{\xi}(u-v)\, \Delta_z^{m+1} I(z+h\omega+v\omega)\,dv - \int_{-\infty}^{0} \bar{\xi}(u+h-v)\, \Delta_z^{m+1} I(z+v\omega)\,dv$$

$$\boxed{F^{(n)}(m;u;z) = \sum_{\nu=0}^{m} \Delta_u^{m-\nu} \xi(u)\, {}^{\nu}\!\Delta_z I(z)\, \omega^{\nu} - \omega^{m+1} \int_0^{\infty} \bar{\xi}(u+v)\, \Delta_z I(z+v\omega)\,dv}$$

$$F^{(n)-}(m;u;z+h\omega) = F^{(n)-}(m;u+h;z)$$

$$\text{take } \boxed{F^{(n)}(m;0;z) = \sum_{\nu=0}^{m} b_\nu^{(n)}(0)\, \Delta^{\nu-n} \psi(z)\omega^{\nu} - \omega^{m+1} \int_0^{\infty} \bar{b}_m^{(n)}(v)\, \Delta_z^{m-n+1} \psi(z-v\omega)\,dv}$$

$F^{(n)}(n;0;z)$

$$= \sum_{\nu=0}^{n} b_\nu^{(n)}(0)\, \Delta^{\nu-n} \psi(z)\omega^{\nu} - \omega^{n+1} \int_0^{n} \bar{b}_n^{(n)}(v)\, \Delta^{1} \psi(z-v\omega)\,dv \qquad \frac{\omega^{m+1}}{\omega}\left[-b_m^{(n)}(v)\Delta_z^{m-n}\psi(z)\right]$$

$$-\omega^{n+1} \sum_{i=n}^{\infty} \int_i^{i+1} b_n^{(n)}(v)\, {}^v\!\Delta_z^{1} \psi(z-v\omega)\,dv$$

$$-\omega^{n+1}\left[\sum_{i=n}^{\infty} \left[\frac{1}{\omega} \bar{b}_n^{(n)}(v)\psi(z-v\omega)\right]_i^{i+1} + \frac{1}{\omega}\int_i^{i+1} b_{n-1}^{(n)}(v)\psi(z-v\omega)\,dv\right]$$

$$-\frac{1}{\omega}\left[\bar{b}_n^{(n)}(i+1-0)\psi(z-\overline{i+1}\omega) - \bar{b}_n^{(n)}(i+0)\psi(z-i\omega)\right]$$

$$\frac{1}{\omega}\bar{b}_n^{(n)}(0+)\psi(z-i\omega) + \frac{1}{\omega}\sum_{i=1}^{\infty}\left\{\bar{b}_n^{(n)}(i+0) - \bar{b}_n^{(n)}(i-0)\right\}\psi(z-i\omega)$$

$$-\frac{1}{\omega}\left[\sum_{i=1} b_n^{(n)}(i-0)\psi(z-i\omega) - \sum_{i=0} b_n^{(n)}(i+0)\psi(z-i\omega)\right.$$

$$\frac{1}{\omega} b^{(n)}_{n}(0)\psi(z) - \frac{1}{\omega}\sum_{i=n} \binom{i-1}{n-1}\psi(z-i\omega)$$

$$\boxed{F^{(n)-}(n;0,z) = \sum_{\nu=0}^{n-1} b^{(n)}_{\nu}(0)\Delta^{\nu-n}\psi(z)\omega^{\nu} + \omega^{n}\sum_{i=n}^{\infty}\binom{i-1}{n-1}\psi(z-i\omega) - \omega^{n}\int_0^{\infty} b^{(n)}_{n-1}(v)\psi(z-v\omega)dv}$$

$$+\omega^{n}\sum_{\nu=0}^{\infty}\binom{\nu+n-1}{n-1}\psi(z-n\omega-\nu\omega)$$

$$\int_0^{\infty} b^{(n)}_{n-1}(v)\psi(z-v\omega)dv = \left[-b^{(n)}_{n-1}(v)\int_v^{\infty}\psi(z-v'\omega)dv'\right]_0 + \int_0^{\infty} b^{(n)}_{n-2}(v)\int_v^{\infty}\psi(z-v\omega)dv\,dv$$

$$= +b^{(n)}_{n-1}(0)\int_0^{\infty}\psi(z-v'\omega)dv' + b^{(n)}_{n-2}(0)\int_0^{\infty}\int_{v'}^{\infty}\psi(z-v''\omega)dv''dv'$$

$$-\int_0^{\infty} b^{(n)}_{n-3}(0)\int_v^{\infty}\int_{v'}^{\infty}\psi(z-v''\omega)dv''dv'dv$$

$$\int_0^{\infty}\psi(z-v'\omega)dv' = \frac{1}{\omega}\int_{-\infty}^{z}\psi(z')dz' \qquad \int_v^{\infty}\psi(z-v'\omega)dv' = \frac{1}{\omega}\int_{-\infty}^{z-v\omega}\psi(z')dz' \qquad z-v'\omega = z' \quad dv' = -\frac{dz'}{\omega}$$

$$b^{(n)}_{n-1}(0)\int_a^{z}\psi(z')dz'\omega^{n-1} - \omega^{n-1} b^{(n)}_{n-1}(0)\int_{-\infty}^{z}\psi(z')dz'\omega^{n-1}$$

$$z - v'\omega = z''$$

$$\int_v^{\infty}\int_{v'}^{\infty}\psi(z-v''\omega)dv''dv' = \frac{1}{\omega}\int_v^{\infty}\int_{-\infty}^{z-v'\omega}\psi(z')dz'$$

$$= \frac{1}{\omega^2}\int_{-\infty}^{z-v\omega}\int_{-\infty}^{z''}\psi(z')dz'dz''$$

$$\boxed{F^{(n)-}(m;0;z) = \sum_{\nu=0}^{m-1} b^{(n)}_{\nu}(0)\Delta^{\nu-n}\psi(z)\omega^{\nu} + \sum_{\nu=m}^{n-1} b^{(n)}_{\nu} I^{(n-\nu)-}(\psi,a)\omega^{\nu} + \omega^{n}\sum_{\nu=0}^{\infty}\binom{\nu+n-1}{n-1}\psi(z-n\omega-\nu\omega) - (-1)^{n-m}\omega^{m}\int_0^{\infty} b^{(n)}_{m-1}(v) I^{(n-m)}(\psi,z-v\omega)dv}$$

$$I^{(-z)-}(z) = \int_{-\infty}^{z}\int_{-\infty}^{z_{z-1}}\cdots\int_{-\infty}^{z_1}\psi(z')dz'dz_1\ldots dz_{z-1} = \int_{-\infty}^{z}\frac{(z-z')^{z-1}}{(z-1)!}\psi(z')dz'$$

$$\bar{B}^{(n)+}_{r}(u) = n\binom{r}{n} i^{n-r+1}\int_0^{\infty}\frac{(u+iv)_{n-1}\,v^{r-n}}{e^{2\pi(v-iu)}-1}\,dv$$

$$\bar{B}^{(n)-}_{r}(u) = n\binom{r}{n} i^{-n+r-1}\int_0^{\infty}\frac{(u-iv)_{n-1}\,v^{r-n}}{e^{2\pi(v+iu)}-1}\,dv$$

$$= n\binom{r}{n} i^{n-r-1}\int_{-\infty}^{0}\frac{(u+iv)_{n-1}\,v^{r-n}}{e^{2\pi(v+iu)}-1}\,dv$$

$\binom{r}{n}$; $r>n$

$r(r-1)\cdots(r-n+1)$

$n\binom{r}{n}$ above is $\dfrac{r!}{(r-n)!\,(n-1)!}$

$(-1)^n\dfrac{(1-r)(2-r)\cdots(n-1-r)}{n!}$

$\phi(x) = \psi(z+xw)$ $y=1$

$$\tfrac{1}{2}\phi(0)+\phi(1)y+\phi(2)y^2+\cdots-\int_0^{\infty} y^x\phi(x)\,dx = R(y)$$

$\dfrac{(n-1-r)(n-2-r)\cdots(1-r)}{n!}$

$(-1)^n\binom{n-1-r}{n}$

$$R(y) = i\int_0^{\infty}\frac{y^{iw}\phi(iw)-y^{-iw}\phi(-iw)}{e^{2\pi w}-1}\,dw$$

$$F^{(1)}(0;0,z) = \int_a^{\infty}\psi(z')\,dz' - \omega\sum{}'\psi(z+\nu\omega)$$

$$= \int_a^{z}\psi(z')\,dz' + \sum_{\nu=1}^{\infty} b^{(1)}_{\nu}(0)\,\partial^{\nu-1}\psi(z)\,\omega^{\nu}$$

$$= \int_a^{z}\psi(z')\,dz' + \omega\sum b^{(1)}_{\nu+1}(0)\,\partial^{\nu}\psi(z)\,\omega^{\nu}$$

$z' = z+x\omega$

$$\tfrac{1}{2}\psi(z)+\sum_{\nu=1}^{\infty}{}'\psi(z+\nu\omega)-\int_0^{\infty}\psi(z+x\omega)\,dx = i\int_0^{\infty}\frac{\psi(z+i\omega w)-\psi(z-i\omega w)}{e^{2\pi w}-1}\,dw$$

$$\int_z^\infty \psi(z')\,dz' - \omega \sum \psi(z+\nu\omega) = \omega \sum b^{(1)}_{\nu+1}(0)\, \Delta^\nu \psi(z)\,\omega^\nu =$$

$$-\frac{1}{2}\omega\psi(z) - i\omega \int_0^\infty \frac{\psi(z+i\omega w)-\psi(z-i\omega w)}{e^{2\pi w}-1}\,dw$$

$$\sum \frac{\Delta^\nu \psi(z)}{\nu!}\, i^\nu w^\nu - \sum \frac{\Delta^\nu \psi(z)}{\nu!}(-i)^\nu w^\nu$$

$$\int_0^\infty \frac{dx}{e^{2\pi x}-1} = \frac{1}{-2\pi}\cdot \ln 1 \qquad \int_0^\infty \frac{dx}{\gamma e^{\lambda x}+\delta} = \frac{1}{\lambda\delta}\ln\frac{\gamma+\delta}{\gamma} \quad \lambda>0 \;\; \delta\neq 0 \;\; \gamma(\gamma+\delta)>0$$

$$\int_a^\infty \frac{dx}{\gamma e^{\lambda x}+\delta} = -\frac{a}{\delta} + \frac{1}{\lambda\delta}\ln\left(e^{\lambda a} + \frac{\delta}{\gamma}\right) \quad \lambda>0 \;\; \gamma(\gamma e^{\lambda a}+\delta)>0$$

$\psi(z) = e^z$

$$\sum b^{(1)}_{\nu+1}(0)\,\omega^\nu = -\frac{1}{2} - i\int_0^\infty \frac{e^{iw\omega}-e^{-iw\omega}}{e^{2\pi w}-1}\,dw$$

$b_1^{(1)}(0) = -\frac{1}{2}$ thus [illegible] above

$$\omega \sum b^{(1)}_{\nu+2}(0)\,\Delta^{\nu+1}\psi(z)\,\omega^\nu = -i\int_0^\infty \frac{\psi(z+i\omega w)-\psi(z-i\omega w)}{e^{2\pi w}-1}\,dw$$

$$\sum b^{(1)}_{\nu+2}(u)\,\Delta^{\nu+1}\psi(z)\,\omega^\nu = -\frac{i}{\omega}\int_0^\infty \frac{\psi(z+u\omega+i\omega w)-\psi(z+u\omega-i\omega w)}{e^{2\pi w}-1}\,dw$$

$$+\, b_0^{(1)}(0)\,\Delta^{-1}\psi(z)\,\omega^{-2} + b_1^{(1)}(0)\,\psi(z)\,\omega^{-1}$$

$$-\, b_0^{(1)}(u)\,\Delta^{-1}\psi(z)\,\omega^{-2} - b_1^{(1)}(u)\,\psi(z)\,\omega^{-1}$$

$\psi(z) = e^{-tz}$

$\frac{x+y}{2}+1 = r-m+1 \qquad x = r+1;$

$\frac{x-y}{2}+1 = n \qquad y = r-2m+1$

$$\Delta^{r-n}\psi(z) = (-1)^{r-n}t^{r-n}e^{-tz}$$

$$\sum_{r=m}^{\infty}\frac{\bar{B}_r^{(n)+}(u)}{r!}\Delta^{r-n}\psi(z)\omega^r = \sum_{r=m}^{\infty}\frac{i^{n-r+1}(-1)^{r-n}t^{r-n}e^{-tz}}{(r-n)!\,(n-1)!}\omega^r\int_0^\infty\frac{(u+iv)_{n-1}v^{r-n}}{e^{2\pi(v-iu)}-1}dv$$

$$\frac{1}{r!} = \frac{r!}{\lambda^{r+1}} = \int_0^\infty r^{r}e^{-\lambda t}dr$$

$m = n \qquad r' = r-m = r-n$

$$\sum_{r=n}^{\infty}\frac{\bar{B}_r^{(n)+}(u)}{r!}\Delta^{r-n}\psi(z)\omega^r = \sum_{r=0}^{n-}\frac{i^{-r+1}(-1)^r t^r e^{-tz}}{r!\,(n-1)!}\omega^r\int_0^\infty\frac{(u+iv)_{n-1}v^{r}}{e^{2\pi(v-iu)}-1}dv$$

$$= \frac{i\omega^n}{(n-1)!}e^{-tz}\int_0^\infty\frac{e^{-itv\omega}(u+iv)_{n-1}}{e^{2\pi(v-iu)}-1}dv$$

allowing for divergence

$$\sum_{r=n}^{\infty}\frac{\bar{B}_r^{(n)-}(u)}{r!}\Delta^{r-n}\psi(z)\omega^r = \frac{-i\omega^n}{(n-1)!}e^{-tz}\int_0^\infty\frac{e^{itv\omega}(u-iv)_{n-1}}{e^{2\pi(v+iu)}-1}dv$$

$$\sum_{r=n}^{\infty}\frac{\bar{B}_r^{(n)}(u)}{r!}\Delta^{r-n}\psi(z)\omega^r = \frac{i\omega^n}{(n-1)!}\int_0^\infty\left\{\frac{(u+iv)_{n-1}\psi(z+iv\omega)}{e^{2\pi(v-iu)}-1} - \frac{(u-iv)_{n-1}\psi(z-iv\omega)}{e^{2\pi(v+iu)}-1}\right\}dv$$

appears only to work when ω is real also if ψ defined over a strip or over half-plane

$$F^{(n)}(\infty; u; z) = \sum_{\nu=0}^{\infty} \bar{b}_\nu^{(n)}(u)\, \Delta^{\nu-n} \psi(z)\, \omega^\nu \qquad \frac{1}{e^{-z}-1} + 1 = \frac{e^{-z}}{e^{-z}-1} = -\frac{1}{e^{z}-1}$$

$$F^{(n)+}(0, u; z) = \sum_{\nu=0}^{n-1} b_\nu^{(n)}(u)\, I^{(n-\nu)}(\psi, a)\, \omega^\nu \qquad \frac{1}{e^{-z}-1} = -1 - \frac{1}{e^{z}-1}$$

$$+ (-\omega)^n \sum_{\nu=0}^{\infty}{}' \binom{\nu+n-1}{n-1} \psi(z+\nu\omega)$$

+ n in both cases ✓

$$F^{(n)}(n, u; z) = \sum_{\nu=0}^{n-1} b_\nu^{(n)}(u)\, \Delta^{\nu-n} \psi(z)\, \omega^\nu + (-\omega)^n \sum_{\nu=0}^{\infty}{}' \binom{\nu+n-1}{n-1} \psi(z+\nu\omega)$$

$$+ \omega^n \int_0^\infty \bar{b}_{n-1}^{(n)}(u-v)\, \psi(z+v\omega)\, dv$$

$$\sum_{\nu=n}^{\infty}{}' \bar{b}_\nu^{(n)}(u)\, \Delta^{\nu-n} \psi(z)\, \omega^\nu = \omega^n \int_0^\infty \bar{b}^{(n)}_{n-1}(u-v)\, \psi(z+v\omega)\, dv + (-\omega)^n \sum_{\nu=0}^{\infty}{}' \binom{\nu+n-1}{n-1} \psi(z+\nu\omega)$$

$$\int_0^\infty \bar{b}^{(n)}_{n-1}(u-v)\, \psi(z+v\omega)\, dv + (-1)^n \sum_{\nu=0}^{\infty}{}' \binom{\nu+n-1}{n-1} \psi(z+\nu\omega)$$

$$= i \int_0^\infty \left\{ \frac{\bar{b}^{(n)}_{n-1}(u+iv)\, \psi(z+iv\omega)}{e^{2\pi(v-iu)}-1} - \frac{\bar{b}^{(n)}_{n-1}(u-iv)\, \psi(z-iv\omega)}{e^{2\pi(v+iu)}-1} \right\} dv$$

$$\bar{B}_z^{(n)-}(u) = \frac{z!}{(z-n)(n-1)!}\, i^{n-z+1} \int_{-\infty}^{0} \left\{ + \frac{(u+iv)_{n-1}\, v^{z-n}}{e^{2\pi(v-iu)}-1} + (u+iv)_{n-1} v^{z-n} \right\}$$

as den. goes to -1 as $v \to -\infty$

$\therefore$ $\bar{B}_z^{(n)}(u)$ cannot be expressed as simple interval over doubly infinite range attending to divergence in p162 $\quad \int_z e^{-tz} dz' = \frac{1}{t} e^{-tz}$

$$\Delta^M \psi(z) = e^{-tz} \qquad \Delta^{z-n} \psi(z) = (-1)^{z-n-M} t^{z-n-M} e^{-tz}$$

$$\sum_{z=m}^{\infty} \frac{\bar{B}_z^{(n)+}(u)}{z!} \Delta^{z-n} \psi(z) \omega^z = \sum_{z=m}^{\infty} \frac{i^{n-z+1} (-1)^{z-n-M} t^{z-n-M} e^{-tz} \omega^z}{(z-n)!(n-1)!} \int_0^\infty \frac{(u+iv)_{n-1} v^{z-n}}{e^{2\pi(v-iu)}-1}$$

$$i^{n-z} (-1)^{z-n} = \left(\frac{-1}{i}\right)^{z-n} = i^{z-n}$$

$m = n+1 \qquad z' = z-n$

$$\frac{(-1)^{-M} i}{(n-1)!} e^{-tz} \omega^n t^{-M} \sum_{z=1}^{\infty} \frac{(+i)^z t^z \omega^z}{z!} \int_0^\infty \frac{(u+iv)_{n-1} v^z}{e^{2\pi(v-iu)}-1} dv$$

$$\frac{(-1)^{-M} i}{(n-1)!} e^{-tz} \omega^n t^{-M} \left[\int_0^\infty \frac{(u+iv)_{n-1}}{e^{2\pi(v-iu)}-1} \left\{ e^{+it\omega v} - 1 \right\} dv \right]$$

$$\sum_{z=n+1}^{\infty} \frac{\bar{B}_z^{-(n)}(u)}{z!} \Delta^{z-n} \psi(z) \omega^z = \frac{\omega^n i (-1)^M}{(n-1)!} \left[\int_0^\infty \left[\frac{(u+iv)_{n-1}}{e^{2\pi(v-iu)}-1} \left\{ \bar{I}^{(M)}(\psi, z - it\omega v) - I^{(M)}(\psi, z) \right\} \right.\right.$$

$$\left.\left. - \frac{(u-iv)_{n-1}}{e^{2\pi(v+iu)}-1} \left\{ I^{(M)}(\psi, z + i\omega v) - I^{(M)}(\psi, z) \right\} \right] dv \right]$$

$M = 0$

$$\frac{i \omega^n}{(n-1)!} \left[\int_0^\infty \left[\frac{(u+iv)_{n-1}}{e^{2\pi(v-iu)}-1} \left\{ \psi(z - iv\omega) - \psi(z) \right\} - \frac{(u-iv)_{n-1}}{e^{2\pi(v+iu)}-1} \left\{ \psi(z + iv\omega) - \psi(z) \right\} \right] dv \right]$$

$u = 0$

$$\frac{i\omega^n}{(n-1)!} \left[\int_0^\infty \frac{(iv)_{n-1} \psi(z - iv\omega) - (-iv)_{n-1} \psi(z + iv\omega) - \psi(z) \left\{ (iv)_{n-1} - (-iv)_{n-1} \right\}}{e^{2\pi v}-1} dv \right]$$

$$\sum_{\nu=n+1}^{\infty} \bar{b}_{\nu}^{(n)}(u)\,\Delta^{\nu-n}\psi(z)\omega^{\nu} = \omega^{n}\int_0^{\infty} \bar{b}_{n-1}^{(n)}(u-v)\psi(z+v\omega)\,dv + (-\omega)^{n}\sum_{\nu=0}^{\infty}\binom{\nu+n-1}{n-1}\psi(z+\nu\omega)$$

$$-\ \omega^{n}\,\bar{b}_{n}^{(n)}(u)\psi(z)$$

consider $\int_0^{\infty}\left[\frac{b_{n-1}^{(n)}(u+iv)}{e^{2\pi(v-iu)}-1} - \frac{b_{n-1}^{(n)}(u-iv)}{e^{2\pi(v+iu)}-1}\right]dv,\ \Delta_w$ yields

$$\frac{1}{(n-2)!}\int_0^{\infty}\left[\frac{(u+iv)^{n-2}}{e^{2\pi(v-iu)}-1} - \frac{(u-iv)^{n-2}}{e^{2\pi(v+iu)}-1}\right]dv$$ $u=0 \quad n=3$ this is non zero

$\therefore$ considered term not identically zero

$$\boxed{\begin{aligned}&-i\left[\int_0^{\infty}\left[\frac{b_{n-1}^{(n)}(u+iv)}{e^{2\pi(v-iu)}-1}\{\psi(z+iv\omega)-\psi(z)\} - \frac{b_{n-1}^{(n)}(u-iv)}{e^{2\pi(v+iu)}-1}\{\psi(z-iv\omega)-\psi(z)\}\right]dv\right]\\ &= \int_0^{\infty}\bar{b}_{n-1}^{(n)}(u-v)\psi(z+v\omega)\,dv + (-1)^{n}\sum\binom{\nu+n-1}{n-1}\psi(z+\nu\omega) - b_{n}^{(n)}(u)\psi(z)\end{aligned}}$$

$$B_1^{(1)}(u) = u - \frac{1}{2}$$

Handy: $\frac{1}{2}\psi(z) + \psi(z+1) + \psi(z+2) + \dots - \int_0^{\infty}\psi(z+x)\,dx = i\int_0^{\infty}\frac{\psi(z+iv)-\psi(z-iv)}{e^{2\pi v}-1}\,dv$

i.e. sign wrong somewhere (corrected)

$$b_{n}^{(n)}(u)\psi(z) + (-1)^{n-1}\sum\binom{\nu+n-1}{n-1}\psi(z+\nu\omega) - \int_0^{\infty}\bar{b}_{n-1}^{(n)}(u-v)\psi(z+v\omega)\,dv$$

$$= i\left\{\int_0^{\infty}\left[\frac{b_{n-1}^{(n)}(u+iv)}{e^{2\pi(v-iu)}-1}\{\psi(z+iv\omega)-\psi(z)\} - \frac{b_{n-1}^{(n)}(u-iv)}{e^{2\pi(v+iu)}-1}\{\psi(z-iv\omega)-\psi(z)\}\right]dv\right.$$

$$R = \tfrac{1}{2}\phi(0) + \Sigma_1 \phi(\nu) - \int \phi(x)\,dx$$

Poisson: $R = 2\sum_{n=1}^{\infty} \int_0^{\infty} \phi(x)\cos 2n\pi x\,dx$

Ramanujan: $R = \int_0^{\infty} \dfrac{\tfrac{1}{2}\phi(0) + \sum_{n=1}^{\infty}\phi(n)(-x)^n}{x\{\pi^2 + (\ln(x))^2\}}\,dx$

$$\boxed{\int_0^{\infty} \frac{dx}{x\{\pi^2 + \ln(x)^2\}} = \int_{-\infty}^{\infty} \frac{dy}{\pi^2 + y^2} = 1}$$

$0 < u < 1$ $\quad -u \quad 1-u$

$$\bar{b}^{(1)}_{\tau+1}(-u) = b^{(1)}_{\tau+1}(1-u)$$

$$i\int_0^{\infty}\left[\frac{b^{(n)}_{n-1}(u-iv)}{e^{2\pi(v+iu)}-1} - \frac{b^{(n)}_{n-1}(u+iv)}{e^{2\pi(v-iu)}-1}\right]dv \qquad n=1\ u=0 = 0$$

$$\int_0^{\infty} b^{(n)}_{n-1}(u-iv)\,\Sigma_1\, e^{-2\pi(\nu+1)(v+iu)}\,dv$$

$$\sum_{\tau=0}^{n-1} b^{(n)}_{n-\tau-1}(u)(-i)^{\tau} \sum_{\nu=0} \int_0^{\infty} e^{-2\pi(\nu+1)(v+iu)} \frac{v^{\tau}}{\tau!}\,dv \qquad -i = i^{-1}$$

$$\sum_{\tau=0}^{n-1} b^{(n)}_{n-\tau-1}(u)(i)^{\tau} \sum_{\nu=1} \frac{e^{-2\pi\nu u i}}{\{2\pi(\nu)\}^{\tau+1}}\,\frac{\tau!}{\tau!}$$

$$\bar{b}_{\tau}(u) = -\Sigma'\,(2i\nu\pi)^{-\tau} e^{2i\nu\pi u}$$

$$i\left[\tfrac{1}{2}\left\{\sum_{\tau=0}^{n-1} i\,b^{(n)}_{n-\tau-1}(u)\sum_{\nu=1}\frac{e^{-2\pi\nu u i}}{(2\pi\nu i)^{\tau+1}} + i\sum_{\tau=0}^{n-1} b^{(n)}_{n-\tau-1}(u)(-1)^{\tau+1}\sum_{\nu=1}\frac{e^{+2\pi\nu u i}}{(2\pi\nu i)^{\tau+1}}\right\}\right]$$

~~[illegible] than $0<u<1$~~

$$= +\sum_{\tau=0}^{n-1} b^{(n)}_{n-\tau-1}(u)\,\bar{b}^{(1)}_{\tau+1}(-u) = \sum_{\tau=0}^{n} b^{(n)}_{n-\tau}(u)\,b^{(1)}_{\tau}(1-u) - b^{(n)}_n(u)\,b^{(1)}_0(1-u)$$

~~this [illegible] = 0 when [illegible]~~ an integer

$$= b^{(n+1)}_n(1) - b^{(n)}_n(u) = \cancel{[illegible]} = -b^{(n)}_n(u)$$

$\neq 0$ when $n=1\ u=0$

appears that — examine presence of this term later (but should we move since $b_1(0)=0$ $b_1^{(1)} = -\frac{1}{2}$, in agreement with the dy)

$$\boxed{\begin{aligned}&\left\{b_n^{(n)}(u)+\bar{b}_n^{(n)}(u)\right\}\psi(z)+(-1)^{n-1}\sum\binom{2n-1}{n-1}\psi(z+\nu\omega)-\int_0^{\infty}\bar{b}_{n-1}^{(n)}(u-v)\,\psi(z+v\omega)\,dv\\&=i\int_0^{\infty}\left[\frac{b_{n-1}^{(n)}(u+iv)\,\psi(z+iv\omega)}{e^{2\pi(v-iu)}-1}-\frac{b_{n-1}^{(n)}(u-iv)\,\psi(z-iv\omega)}{e^{2\pi(v+iu)}-1}\right]dv\end{aligned}}$$

have not allowed for divergence

$$i\int_0^{\infty}\left[\sum_{r=0}^{n-1}\left[b_{n-r-1}^{(n)}(u)(-i)^r\sum\nolimits_1'\frac{e^{-2\pi\nu(v+iu)}}{r!}v^r\right.\right.$$

$$\left.\left.-\,b_{n-r-1}^{(n)}(u)(i)^r\sum\nolimits_1'\frac{e^{-2\pi\nu(v-iu)}}{r!}v^r\right]\right]dv$$

$$b_{n-1}^{(n)}(u)\sum\nolimits_1'\left\{e^{-2\pi\nu(v+iu)}-e^{-2\pi\nu(v-iu)}\right\}$$

$$\frac{1}{e^{2\pi(v+iu)}-1}-\frac{1}{e^{2\pi(v-iu)}-1}=\frac{e^{2\pi(v-iu)}-e^{2\pi(v+iu)}}{e^{4\pi v}-2e^{2\pi v}\cos 2\pi u+1}$$

$$-2i\sin(2\pi u)\int_0^{\infty}\frac{dv}{e^{2\pi v}+e^{-2\pi v}-2\cos 2\pi u}$$

$2\pi v = dv'$ $\quad \dfrac{dv'}{2\pi}=dv$

$-2\cos 2\pi u = 2\cos\left\{2\pi\left(u\mp\frac{1}{2}\right)\right\}$

$$-\frac{2i\sin 2\pi u}{2\pi}\int_0^{\infty}\frac{dv'}{e^{v'}+e^{-v'}+2\cos\left\{2\pi\left(u-\frac{1}{2}\right)\right\}}$$

$2\pi\left(u\mp\frac{1}{2}\right)<\pi$

$u<1$

$$-\frac{2i\sin 2\pi u}{2\pi}\cdot\frac{\left|2\pi\left(u-\frac{1}{2}\right)\right|}{\sin\left(2\pi\left(u-\frac{1}{2}\right)\right)}=+2i\left|u-\frac{1}{2}\right|\quad 0<u<1$$

and term $=0$ when $u=0$ $\quad -\pi<2\pi\left(u-\frac{1}{2}\right)$

i.e. $2i\,b_1^{(1)}(u)$ $\quad 0<u$

$$i\int_0^\infty \sum_{z=0}^{n-1}\left[b_{n-z-1}^{(n)}(u)(-i)^z \sum_1 \frac{e^{-2\pi\nu(v+iu)-t(z+iv\omega)}v^z}{z!} \right.$$

$$\left. - b_{n-z-1}^{(n)}(u)\, i^z \sum_1 \frac{e^{-2\pi\nu(v-iu)-t(z-iv\omega)}v^z}{z!}\right] dv$$

$$i\int_0^\infty \sum_{z=0}^{n-1}\sum_1 \left[b_{n-z-1}^{(n)}(u)(-i)^z \frac{e^{-tz-2\pi\nu iu}}{(2\pi\nu+it\omega)^{z+1}} - b_{n-z-1}^{(n)}(u)\, i^z \frac{e^{-tz-2\pi\nu iu}}{(2\pi\nu-it\omega)^{z+1}}\right]$$

$$\oint_0^\infty \frac{x\cot(\mu x)}{x^2+a^2}dx = \frac{\pi}{e^{2\mu a}-1} \qquad a>0 \quad \mu>0$$

$$\frac{1+z}{1-z} - \frac{2}{1-z} = \frac{-1+z}{1-z} = -1$$

$$\frac{1+z}{1-z} = \frac{2}{1-z} - 1$$

$$\cot \mu x = i\,\frac{e^{i\mu x}+e^{-i\mu x}}{e^{i\mu x}-e^{-i\mu x}} = i\left\{\frac{1+e^{-2i\mu x}}{1-e^{-2i\mu x}}\right\} = \frac{i}{2}\left\{\frac{1}{1-e^{-2i\mu x}} - \frac{1}{2}\right\}$$

$$i\int_0^\infty\left[\frac{b_{n-1}^{(n)}(u+iv)\psi(z+iv\omega)}{e^{2\pi(v-iu)}-1}-\frac{b_{n-1}^{(n)}(u-iv)\psi(z-iv\omega)}{e^{2\pi(v+iu)}-1}\right]dv$$

$iv=v'$ $\quad dv=\frac{dv'}{i}$

$v=+i\infty \quad v'=\infty$

$$=i\sum_{\nu=1}^{\infty}\left\{\int_0^{\infty} b_{n-1}^{(n)}(u+iv)\psi(z+iv\omega)e^{-2\nu\pi(v-iu)}dv-\int_0^\infty b_{n-1}^{(n)}(u-iv)\psi(z-iv\omega)e^{-2\nu\pi(v+iu)}dv\right\}$$

$$\int_0^{-\infty i} \qquad -\int_0^{+i\infty}$$

$-iv=v' \quad v=i\infty \quad v'=\infty$

$dv=i\,dv'=-\frac{dv'}{i}$

rotate

$$\sum_{\nu=1}\int_0^\infty b_{n-1}^{(n)}(u+v')\psi(z+v'\omega)\left\{e^{2\pi i\nu\{v'+u\}}+e^{-2\pi i\nu\{v'+u\}}\right\}dv'$$

$$=2\sum_{\nu=1}\int_0^\infty b_{n-1}^{(n)}(u+v')\psi(z+v\omega)\cos\{2\nu\pi(v+u)\}dv \qquad \frac{1}{z+2n\pi i}+\frac{1}{z-2n\pi i}$$

$$\boxed{\begin{aligned}&\left\{b_n^{(n)}(u)+\bar b_n^{(n)}(u)\right\}\psi(z)+(-1)^{n-1}\sum\binom{\nu+n-1}{n-1}\psi(z+\nu\omega)-\int_0^\infty \bar b_{n-1}^{(n)}(u-v)\psi(z+v\omega)dv\\&=2\sum_1\int_0^\infty b_{n-1}^{(n)}(u+v)\psi(z+v\omega)\cos\{2\nu\pi(v+u)\}dv\end{aligned}}$$

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+2z\sum_1\frac{1}{z^2+4n^2\pi^2} \quad \Bigg| \quad \frac{1}{e^{2\pi(v+iu)}-1}=\frac{1}{2\pi(v+iu)}-\frac{1}{2}+\frac{4\pi(v+iu)}{4\pi^2}\sum_1\frac{1}{(v+iu)^2+n^2}$$

$$=\frac{1}{z}-\frac{1}{2}+\sum_1\left\{\frac{1}{z+2n\pi i}+\frac{1}{z-2n\pi i}\right\}$$

$$i\int_0^\infty\Bigg[b_{n-1}^{(n)}(u+iv)\psi(z+iv\omega)\left[\frac{1}{2\pi(v-iu)}-\frac{1}{2}+\sum_1\left\{\frac{1}{2\pi(v-iu)+2\pi n i}+\frac{1}{2\pi(v-iu)-2\pi n i}\right\}\right]$$

$$-b_{n-1}^{(n)}(u-iv)\psi(z-iv\omega)\left[\frac{1}{2\pi(v+iu)}-\frac{1}{2}+\sum_1\left\{\frac{1}{2\pi(v+iu)+2\pi n i}+\frac{1}{2\pi(v+iu)+2\pi n i}\right\}\right]\Bigg]$$

the becomes infinite

$$\int_0^\infty b_{n-1}^{(n)}(u+v)\psi(z+v)\left[\frac{1}{-2\pi i(v+u)} - \frac{1}{2} + \frac{1}{2\pi i}\sum{}'\left\{\frac{1}{-v-u+n} + \frac{1}{-v-u-n}\right\}\right.$$

$$\left. + \frac{1}{2\pi i(v+u)} - \frac{1}{2} + \frac{1}{2\pi i}\sum{}'\left\{\frac{1}{v+u+n} + \frac{1}{v+u-n}\right\}\right]dv$$

Ramanujan $\quad \ln(x) = v \quad x = e^{v} \quad \frac{dx}{x} = \frac{e^v dv}{e^v} = dv \quad x = 0 \;\&\; \ln(x) = -\infty$

$x = \infty$

$\ln(x) = v + \pi i \quad x = -e^v \quad \frac{dx}{x} = \frac{-e^v dv}{-e^v} = dv \quad (-x)^n = e^{nv}$

$\ln(x) = -v + \pi i \quad x = -e^{-v} \quad (-x)^n = e^{-nv} \quad \frac{dx}{x} = \frac{e^{-v}dv}{-e^{-v}} = -dv$

$x = 0 \quad v = \infty \quad x = \infty \quad v = -\infty + \pi i$

$x = 1 \quad v = \pi i$

$x = \infty \quad v = -\infty + \pi i$

$$R = \int_{-\infty + \pi i}^{\infty + \pi i} \frac{\frac{1}{2}\phi(0) + \sum_{n=1}^\infty \phi(n) e^{-nv}}{\pi^2 + (-v+\pi i)^2}$$

$$\int_0^1 \frac{\frac{1}{2}\phi(0) + \sum_{n=1}^\infty{}' \phi(n)(-x)^n}{x\{\pi^2 + (\ln(x))^2\}}dx + \int_1^\infty$$

$$\int_{-\infty-\pi i}^{0-\pi i} \frac{\frac{1}{2}\phi(0) + \sum_{n=1}^\infty{}' \phi(n) e^{nv}}{\{\pi^2 + (v+\pi i)^2\}}dv + \int_{-\infty+\pi i}^{\pi i} \frac{\frac{1}{2}\phi(0) + \sum_{n=1}^\infty{}' \phi(n) e^{-nv}}{}dv$$

$$\phi(n) = \int_0^\infty e^{-vt}d\sigma(t) \qquad \frac{1}{2}\phi(0) + \sum_{n=1}^\infty \int e^{-nt}\sigma(t)(-x)^n =$$

$$\frac{1}{2}\int_0^\infty \left[\frac{1}{2} - \frac{e^{-t}}{1+xe^{-t}}\right]d\sigma(t)$$

$$R = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left[1+\frac{1}{e^t+e^v}\right]\frac{1}{\pi^2+v^2}\,d(t)\,dv$$

$$\int_0^{\infty} b_{n-1}^{(n)}(u+v')\,\psi(z+v'\omega)\left\{\frac{e^{2\pi i(v'+u)}}{1-e^{2\pi i(v'+u)}}+\frac{e^{-2\pi i(v'+u)}}{1-e^{-2\pi i(v'+u)}}\right\}$$

$$= \frac{1}{e^{-2\pi i(v'+u)}-1}+\frac{1}{e^{2\pi i(v'+u)}-1} \quad \frac{e^{2\pi i(v'+u)}+e^{-2\pi i(v'+u)}-2}{2-2\cos\{2\pi(v'+u)\}} = -1$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-\pi|w|}}{\sin\pi i w}\phi(-iw)\,dw = i\int_0^{\infty}\frac{\phi(ih)-\phi(-ih)}{e^{2\pi h}-1}\,dh$$

$$i\left[\int_0^{\infty}\frac{b_{n-1}^{(n)}(u+iv)\,e^{-\pi(v-iu)}\hat\psi(z+iv\omega)}{e^{\pi(v-iu)}-e^{-\pi(v-iu)}}-\int_{-\infty}^{0}\frac{b_{n-1}^{(n)}(u+iv)\,\hat\psi(z+iv\omega)\,e^{\pi(v-iu)}}{e^{\pi(-v+iu)}-e^{-\pi(v+iu)}}\right]dv$$

$$i\int_{-\infty}^{\infty}\frac{b_{n-1}^{(n)}(u+iv)\,e^{-\pi|v|+iu\,\mathrm{sign}(v)\pi}}{e^{\pi(v-iu)}-e^{-\pi(v-iu)}}\hat\psi(z+iv\omega)\,dv$$

$$\pi(v-iu) = i\{-u-iv\}$$

$$\hat\psi(z+iv\omega) = \psi(z+iv\omega)-\psi(z)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{b_{n-1}^{(n)}(u+iv)\,e^{-\pi|v|\,iu\,\mathrm{sign}(v)\pi}\,\hat\psi(z+iv\omega)}{\sin\{\pi(u+iv)\}}\,dv$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{b_{n-1}^{(n)}(u+iv)}{\sin\{\pi(u+iv)\}}\hat\psi(z+iv\omega)\,e^{iu\,\mathrm{sign}(v)\pi}\int_0^{\infty}\frac{x^{-iv}}{x\{\pi^2+\log(x)^2\}}\,dx\,dv$$

$iu\,\mathrm{sign}(v'+u)$

$$= \int_0^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}\,\frac{b_{n-1}^{(n)}(u+iv)\,\psi(z+iv\omega)\,e^{iu\,\mathrm{sign}(v)\pi}}{\sin\{\pi(u+iv)\}} \quad \| \quad \Big]\,dv\,dx$$

$u+iv = iv'$ $\quad v=\infty$

$z - i\omega + i\omega(u+iv)$

$$\cancel{A_n} = \int_0^\infty \frac{\Phi_1(-x)dx}{x\{\pi^2+(\ln(x))^2\}}\, dx$$

$v' = iv$

where

$$\Phi_1(-x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{b_{n-1}^{(n)}(u+iv)\psi(z+ivw)e^{iu\,\mathrm{sgn}(v)\pi}}{\sin\{\pi(u+iv)\}} x^{-iv}\, dv \quad \Big\| \quad \phi_1(z+\cancel{\ell}uw, -x) = \frac{1}{2} \cdot (u+iv \quad \psi(z+(u+iv)w$$

$$\sum_1^\infty (-1)^n \phi(n) x^n = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi}{\sin \pi i w} x^{-iw}\phi(-iw)dw \text{ when } \phi(0) = 0$$

$$\Phi_1(-x) = \cancel{+}\frac{1}{2i}\int_{-i\infty}^{i\infty} \frac{b_{n-1}^{(n)}(u+\cancel{i}v)\hat{\psi}(z+vw)e^{iu\,\mathrm{sign}(iv)\pi} x^{\cancel{-a}-v'}}{\sin\{\pi(u+v)\}} dv$$

$e^{u\,\mathrm{sign}(-iv)}$ not analytic fn. of w

$$\Phi_1(-x) = \frac{1}{2i}\left\{\int_{\cancel{+}0}^{\infty} \frac{b_{n-1}^{(n)}(u+iv)\psi(z+ivw)e^{iu\pi}x^{-iv}dv}{\sin\{\pi(u+iv)\}} + \int_{-\infty}^{0} \cdots e^{-iu\pi}x^{-iv}dv\right.$$

$$e^{iu\pi}\{a+ib\} \qquad \xleftarrow{u\,;}\ \cancel{\phi_0}\ e^{iu\pi}(a+ib) + e^{-iu\pi}(a-ib)$$

$$(e^{iu\pi}+e^{-iu\pi})a + i(e^{iu\pi}-e^{-iu\pi})b$$

$$\int_{-\infty}^{0} \cdot \{a-ib\}$$

$$\int_{-\infty}^{\infty} \cdots = 2a$$

$$\Phi(u) = \cancel{\ldots}\{e^{iu\pi}+e^{-iu\pi}\}a(u) + i\{e^{iu\pi}-e^{-iu\pi}\}b(u)$$

$$a(u+1) = -a(u)$$

$$\cancel{\phi(u)\ \phi(u+1) = -e\,\phi(u)} \qquad \phi_1 = e^{iu\pi}\phi_n^+(u) + e^{-iu\pi}\phi_n^-(u)$$

ϕ

$$-\phi_n^+(u+1) = \int \frac{b_{n-1}^{(n)}(u+1\ iv)}{\sin\{\cdots\}}$$

$$-\phi_n^\pm(u+1) - \phi_n^\pm(u) = \phi_{n-1}^\pm(u)$$

from p156 with $\Xi = \psi$ $\qquad \xi = e_m^{(n)} \quad \nabla^z e_m^{(n)} = e_m^{(n-z)} \quad \mathcal{D}^r e_m^{(n)}(u) = e_{m-r}^{(n)}(u)^{43}$

$$\mathfrak{F}(m,n;u;z) = \sum_{\nu=0}^{m} \mathcal{D}^{m-\nu} \xi(u) \mathcal{D}_z^\nu \psi(z) \omega^\nu + \omega^{m+1} \int_0^\infty \xi(u-v) \mathcal{D}_z \psi(z+v\omega)$$

$$\cancel{\nabla_\omega^m F(m,n;u;z) = \sum_{\nu=0}^{m} e_{m-\nu}^{(n)}(u) \mathcal{D}_z^\nu}$$

$\xi = b_m^{(n)}(u) \quad \mathcal{D}^{m-\nu} \xi(u) = b_{m-m+\nu}^{(n)}(u) = b_\nu^{(n)}(u) \quad \nu = 0, \ldots, 1$ all proceed since $m-\nu \leq m$

$$\boxed{\mathfrak{F}(m,n;u;z) = \sum_{\nu=0}^{m} e_\nu^{(n)}(u) \mathcal{D}_z^\nu \psi(z) \omega^\nu + \omega^{m+1} \int_0^\infty \bar{e}_m^{(n)}(u-v) \mathcal{D}_z^{m+1} \psi(z+v\omega)\, dv}$$

$$\nabla_\omega^n \mathfrak{F}(m,n;u;z) = \sum_{\nu=0}^{m} e_\nu^{(0)}(u) \mathcal{D}_z^\nu \psi(z) \omega^\nu + \omega^{m+1} \int_0^\infty \nabla^n \bar{e}_m^{(n)}(u-v) \mathcal{D}_z^{m+1} \psi(z+v\omega)\, dv$$

$$\nabla_{\omega}^n F(m,n;0;z) = \psi(z) \quad \text{if } \nabla^n \bar{e}_m^{(n)}(u) = 0$$

$$\bar{E}_0^{(n)}(p+0) - \bar{E}^{(n)}(p-0) = (-1)^{p+n-1} 2^{?} \binom{p-1}{n-1} \qquad p = \ldots -2, -1, 0, n, n+1, \ldots$$

$u > 0$ $\qquad -1 \;\; 0 \;\; u$ $\qquad v = u+i \quad v' = i$

$u - v = -v' \quad v = u + v'$

$$\int_0^\infty \bar{e}_0^{(n)}(u-v) \mathcal{D}_v \psi(z+v\omega)\, dv$$

$$= \int_0^u \bar{e}_0^{(n)}(u-v) \mathcal{D}_v \psi(z+v\omega)\, dv + \sum_{i=0}^{\infty} \int_{u+i}^{u+i+1} \bar{e}_0^{(n)}(u-v) \mathcal{D}_v \psi(z+v\omega)\, dv$$

$\bar{E}_\nu^{(n)}(x) = O(x^{n-1})$ for $\nu = 0, 1, \ldots$ $\quad \Big| \quad \sum_{i=0} \int_i^{i+1} \bar{e}_0^{(n)}(-v') \mathcal{D}_v \psi(z + u\omega + v'\omega)$

$$\left[\bar{e}_0^{(n)}(u-v) \psi(z+v\omega)\right]_0^u + \int_0^u -\frac{d}{dv} \bar{e}_0^{(n)}(u-v) \psi(z+v\omega)\, dv$$

$$\bar{E}_i^{(n)}(u) = 2^{n-1} \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{(n-\nu-1)!} b_\nu^{(n)}(u) E_{2\, n-\nu-1}(u)$$

$$2^{1-n} \mathcal{D}_u \bar{E}_0^{(n)}(u) = \sum_{\nu=0}^{n-2} \frac{(-1)^\nu}{(n-\nu-2)!} b_\nu^{(n)}(u) E_{n-\nu-2}(u) + \sum_{\nu=1}^{n-1} \frac{(-1)^\nu}{(n-\nu-1)!} b_{\nu-1}^{(n)}(u) E_{n-\nu-1}(u)$$

$= 0$ for all $u \neq -2, -1, 0, n, n+1, \ldots$

$$\sum_{i=0}' \left[\psi(z+u\omega+\overline{i+1}\omega)\,\bar{e}_0^{(n)}(-i-1+0) - \psi(z+u\omega+i\omega)\,\bar{e}_0^{(n)}(-i-0) \right]$$

$$-\psi(z+u\omega)\,\bar{e}_0^{(n)}(0-) + \sum_{i=1}^{\infty}{}' \psi(z+u\omega+i\omega)\left\{ \bar{e}_0^{(n)}(-i+0) - \bar{e}_0^{(n)}(-i-0) \right\}$$

$$\bar{e}_0^{(n)}(0+)\psi(z+u\omega) - \underbrace{\bar{e}_0^{(n)}(u)}_{=1}\psi(z) \qquad (-1)^{-i+n-1}\binom{-i-1}{n-1}$$

$-\sum_0 e_0^{(n)}(u)\psi(z)$ cancels

$$\boxed{(-1)^{n-1}\sum_{\nu=0}^{\infty}{}'(-1)^{\nu}2^{n}\binom{-\nu-1}{n-1}\psi(z+u\omega+\nu\omega) = F(0,n;u;z)}$$

$$u=0 \quad F(0,n;0;z) = (-1)^{n-1}\sum_{\nu=0}^{\infty}{}'(-1)^{\nu}\binom{-\nu-1}{n-1}\psi(z+\nu\omega)$$

$r \le u < r+1$: same result if $r<n$

$$\frac{1}{2}\left\{F(0,1;u;z) + F(0,1;u;z\omega)\right\} = \psi(z+u\omega)$$

$n<u<n+1$

$$\int_0^{\infty}\bar{e}_0^{(n)}(u-v)\,\Delta_v\psi(z+v\omega)\,dv = \int_0^{u-n}\bar{e}_0^{(n)}(u-v)\,\Delta_v\psi(z+v\omega)\,dv + \int_{u-n}^{u} + \int_u^{\infty}$$

$$\bar{e}_0^{(n)}(u)=1 \quad 0<u<n \qquad E_0^{(n)}(n+0) - E_0^{(n)}(n-0) = -1.2^n$$

$$(-1)2^n\left[\psi(z+\overline{u-n}\omega) - \psi(z)\right] + \left[\psi(z+u\omega) - \psi(z+\overline{u-n}\omega)\right]$$

$$-\psi(z+u\omega)\,\bar{e}_0^{(n)}(0-) + \sum_{i=1}^{\infty}{}'\psi(z+u\omega+i\omega)(-1)^{i+n-1}2^n\binom{i-1}{n-1}$$

$$= (-1)^{n-1}\sum_{\nu=0}^{\infty}{}'(-1)^{\nu}2^{n}\binom{-\nu-1}{n-1}\psi(z+u\omega+\nu\omega) + \psi(z)\left\{1+2^n\right\} - 2^n\psi(z+\overline{u-n}\omega)$$

$$G^{(n)-}(m, \cancel{m}; u; z) = \sum_{\nu=0}^{m} e_{\nu}^{(n)}(u) \Delta_z^{\nu} \psi(z) \omega^{\nu} - \omega^{m+1} \int_0 \bar{e}_m^{(n)}(u+v) \Delta_z^{m+1} \psi(z - v\omega) d\omega$$

B

$B =$

$$+\int_0^{\infty} \bar{e}_0^{(n)}(u+v) \Delta_v \Psi(z - v\omega) dv$$

$n - u + i = n i \qquad v' = u + v$

$$\int^{n+i+1}_{n+i} \bar{e}_0^{(n)}(v') \Delta_v (z - (v' - u)\omega) dv'$$

$$\int_0^{n-u} \bar{e}_0^{(n)}(u+v) \Delta_v \Psi(z - v\omega) dv + \sum_{i=0}^{\infty} \int_{n-u+i}^{n-u+i+1} \bar{e}_0^{(n)}(u+v) \Delta_v \Psi(z - v\omega) dv$$

$$\Psi(z - (n-u)\omega) - \psi(z) + \sum_{i=0} \Big[\psi(z - (n+i+1-u)\omega) \bar{e}_0^{(n)}(n+i+1-0) - \psi(z - (n+i-u)\omega) \bar{e}(n+i+0) \Big]$$

$$\sum_{i=0} \psi(z - (n+i-u)\omega) \Big[\bar{e}(n+i-0) - \bar{e}(n+i+0) \Big] - \psi(z)$$

$$-2^n \sum_{i=0} (-1)^{i-1} \binom{n+i-1}{n-1} \psi(z - n\omega - i\omega + u\omega)$$

$$\boxed{G_1^{(n)}(0; u; z) = 2^n \sum_{\nu=0} \binom{n+\nu-1}{\nu} (-1)^{\nu} \psi(z - n\omega - \nu\omega + u\omega)}$$

$$\frac{1}{\sin(\cdot)} = \frac{2i}{e^{i}}$$

$$\bar{E}_z^{(n)}(u) = (-2)^{n-1} \Big[i^{-z} \int_0^{\infty} b_{n-1}^{(n)}(u+iv) v^z \operatorname{cosec}\{\pi(u+iv)\} dv + i^z \int_0^{\infty} b_{n-1}^{(n)}(u-iv) v^z \operatorname{cosec}\{\pi(u-iv)\} dv$$

$$= (-2)^{n-1} i^{-z} \int_{-\infty}^{\infty} b_{n-1}^{(n)}(u+iv) v^z \operatorname{cosec}\{\pi(u+iv)\} dv$$

$z = n+1, n+2, \ldots \quad (z = 1, 2, \ldots \;?)$

$$\bar{b}^{(n)}_z(u) = (-1)^{n-1} \sum_0^{n-1} \frac{\Gamma(z-\nu)}{\Gamma(z-n+1)\Gamma(n-\nu)} (-1)^\nu b^{(n)}_\nu \bar{b}_{z-\nu}(u)$$

$z-\nu < 0$ $\bar{b}_{z-\nu}(u) = 0$ $z > \nu$ $\Gamma(z-\nu)$ finite $\Gamma(z-n+1)$ ∞ ite for $z < n$

above formula only holds for $z = n, n+1, \ldots$

$\bar{E}^{(n)}_z(u)$ formula holds for $z = 1, 2, \ldots$ $\quad (iv t\omega)^\nu \quad \sum_{\nu=1} \frac{x^\nu}{\nu!} = e^x - 1$

$\psi(z) = e^{-tz}$

$$\sum_{\nu=1} e^{(n)}_\nu(u) \Delta^\nu \psi(z) \omega^\nu = \cancel{\Delta}$$

$$(-2)^{n-1} \int_{-\infty}^{\infty} b^{(n)}_{n-1}(u+iv) \frac{i^{-\nu}}{\nu!} v^\nu (-1)^\nu t^\nu e^{-tz} \omega^\nu \operatorname{cosec}\{\pi(u+iv)\} dv$$

$$(-2)^{n-1} \int_{-\infty}^{\infty} b^{(n)}_{n-1}(u+iv) e^{-tz} \operatorname{cosec}\{\pi(u+iv)\} \{e^{ivt\omega} - 1\} dv$$

$$(-2)^{n-1} \int_{-\infty}^{\infty} b^{(n)}_{n-1}(u+iv) \operatorname{cosec}\{\pi(u+iv)\} \{\psi(z - iv\omega) - \psi(z)\} dv$$

$$(-2)^{n-1} \left\{ \int_0^\infty \left[b^{(n)}_{n-1}(u+iv) \operatorname{cosec}\{\pi(u+iv)\} \psi(z - iv\omega) + b^{(n)}_{n-1}(u-iv) \operatorname{cosec}\{\pi(u-iv)\} \psi(z+iv\omega) \right] dv \right.$$

$$\left. + \int_0^\infty \cdots \psi(z) + \cdots \psi(z) \quad dv \right] \qquad \frac{1}{e^{i\pi u - \pi v} - e^{-i\pi u + \pi v}}$$

$$(-2)^{n-1} \cdot 2i \left\{ \sum_{z=0}^{n-1} b^{(n)}_{n-1-z}(u) \int_0^\infty \frac{(iv)^z}{z!} \frac{1}{e^{i\pi(u+iv)} - e^{-i\pi(u+iv)}} + \cdots (-iv) \right. \qquad = -\frac{e^{i\pi u - \pi v}}{1 - e^{i2\pi u - 2\pi v}}$$

$$\int_0^\infty \frac{(iv)^z}{z!} \sum_0 -e^{\pi(\nu+1)(iu - v)} + \cdots = i^z \sum_0 \frac{-e^{(\nu+1)i\pi u}}{\{\pi(\nu+1)\}^{z+1}} + i^{-z} \sum \cdots$$

$$i \sum_1 \frac{-e^{(\nu+1)i\pi u}}{\{\pi(\nu-1)i\}^{z+1}} (-1)^z + i \sum_{\nu=-\infty}^{0} \frac{-e^{(\nu-1)i\pi u}}{\{\pi(\nu-1)i\}^{z+1}} (-1)^z = \frac{i}{2} (-1)^z \bar{E}_z(u)$$

$$-(-2)^{n-1}\cdot 2i\cdot\frac{i}{2}\sum_{\tau=0}^{n-1} b_{n-1-\tau}^{(n)}(u)(-1)^{\tau}\bar{e}_{\tau}(u) = (-1)^{n}2^{n-1}(-1)^{n-1}\sum_{\nu=1}^{n-1} b_{\nu}^{(n)}(u)\bar{e}_{n-1-\nu}(u)(-1)^{\nu}$$

$$= +\bar{E}_0^{(n)}(u)$$

$$G(0,n;u;z) = \left\{E_0^{(n)}(u) \pm \bar{E}_0^{(n)}(u)\right\}\psi(z) +$$

$$(-2)^{n-1}\int_0^{\infty}\left[b_{n-1}^{(n)}(u+iv)\operatorname{cosec}\{\pi(u+iv)\}\psi(z-iv\omega) + b_{n-1}^{(n)}(u-iv)\operatorname{cosec}\{\pi(u-iv)\}\psi(z+iv\omega)\right]dv$$

$$\nabla_u G(0,n;u;z) = G(0,n-1;u;z) \quad \| \; n=1.$$

$$(-2)^{n-1}(-1)^{2i}\sum_{\nu=0}\left[\int_0^{\infty} b_{n-1}^{(n)}(u+iv)\psi(z-iv\omega)e^{(2\nu+1)\pi i(u-v)} - \int_0^{\infty} b_{n-1}^{(n)}(u-iv)\psi(z+iv\omega)e^{-(2\nu+1)\pi(iu+v)}\right]$$

$$\frac{1}{e^{i\pi u+\pi v} - e^{-i\pi u-\pi v}} = \frac{e^{-i\pi u-\pi v}}{1-e^{-i2\pi u-2\pi v}} \qquad \int_0^{+i\infty} - \int_0^{-i\infty}$$

$iv'=v \quad v=i\infty \quad v'=\infty$

$dv = i\,dv'$

$iv = -v'$

$$2i(-1)^{n}2^{n-1}i\sum_{\nu=0}\int_0^{\infty} b_{n-1}^{(n)}(u-v)\psi(z+v\omega)\left\{e^{(2\nu+1)\pi(iu-iv)} + e^{-(2\nu+1)\pi(iu-iv)}\right\} -$$

$$= (-2)^{n+1}\sum_{\nu=0}^{\infty}\int_0^{\infty} b_{n-1}^{(n)}(u-v)\psi(z+v\omega)\cos\{(2\nu+1)\pi(u-v)\}dv$$

$$\boxed{G(0,n;u;z) = \left\{E_0^{(n)}(u) - \bar{E}_0^{(n)}(u)\right\}\psi(z) + (-2)^{n+1}\sum_{\nu=0}^{\infty}\int_0^{\infty} b_{n-1}^{(n)}(u-v)\psi(z+v\omega)\cos\{(2\nu+1)\pi(u-v)\}dv}$$

$$\hat{\psi}(z+z') = \psi(z+z') - \psi(z) \qquad \cancel{iv = \xi v'} \quad \cancel{v = iv'} \quad v' = iv$$

$$(-2)^{n-1} \int_{-\infty}^{\infty} \frac{b_{n-1}^{(n)}(u+iv)\, \hat{\psi}(z-iv\omega)}{\sin\{\pi(u+iv)\}}\, dv$$

$$(-1)^{n-1} 2^{n-1} i^{-1} \int_{-i\infty}^{i\infty} \frac{b_{n-1}^{(n)}(u+v)\, \hat{\psi}(z-v\omega)}{\sin \pi(u+v)}\, dv = (-1)^{n} 2^{n-1} i \int_{-i\infty}^{i\infty} \frac{b_{n-1}^{(n)}(u-v)\, \hat{\psi}(z+v\omega)}{\sin \pi(u+v)}\, dv$$

$\underline{(-2)^{n-1}}$ | Ramanujan form for Boole's series not given.