A. Conventions and expository artifices

B. Composition and composition product

C. Matrix mappings

D. Pre- and post-quotient & zero spaces

E. Rank and nonsingularity

F. Polynomials and rational functions

A. Equality relationships

Type

Sets

Class Type?

$fn(\Theta)$ $\Theta \in fn(SM)$

Function designation in mapping to mapping transformation

Eg. trans: map → map′ $fn(trans(\Xi))$ written as trans ξ

$\xi := fn(\Xi)$

Use of function names in specification of dimensions

e.g. n in seq $\bar{N}[B, n]$ $(n: B \to \bar{N})$ expressions seq $\bar{N}[B, n+1]$

Sets and members of sets

seq : sequence {seq} set of members

S, T sets {S → T} mappings

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i) Allocation

ii) The use of globally declared variables in mapping specifications

iii) The use of function names alone in the specification of mappings (replace by $a \in fn\{K[B]^{n}_{m}\}$??)

iv) Algebraic operations over mappings

v) Relationships between functions over argument subdomains

B.

Composition

1] Set transformation

2] Composition products

3] Direct composition

4] The composition product expressed in terms of direct composition

5] Solubility conditions

1] Integer sequences

i) Special sequences

ii) $\bar{R}$-sequences

2] $\bar{R}$-mappings

3] Proper $\bar{R}$-sequence mappings

i)

ii) The base

iii) Special $\bar{R}$-sequence mappings

iv) Classes of $\bar{R}$-sequence mappings (nonsingular, unit, order, permutation and bounded mappings)

v) Unitary operations upon $\bar{R}$-sequence mappings

a) The order mapping

b) Order reversal

c) The ascension mapping

d) The transpose mapping

e) The spectrum

f) The inverse mapping

vi) Binary sequence operations over $\bar{R}$-sequence mappings

a) Sequence sums and differences

b) Direct composition

c) The composition product

vii) Binary algebraic operations over $\overline{N}$-sequence mappings

a) Addition, subtraction and multiplication of $\overline{N}$-sequence mappings

b) Algebraic operations involving $\overline{N}$-mappings and $\overline{N}$-sequence mappings

viii) Ordering and equality relationships over $\overline{N}$-sequence mappings

a) Setwise ordering

b) Subsequence ordering

c) Product ordering

d) Perpendicularity

e) Ordering according to magnitude

ix) Equivalence classes of $\overline{N}$-sequence mappings

(ud equiv) (comp equiv)

4] Improper $\overline{N}$-sequence mappings

i) Void $\overline{N}$-sequence mappings

ii) Improper $\overline{N}$-sequence mappings

iii) Special $\overline{N}$-sequence mappings

iv) Unitary operations upon improper $\overline{N}$-sequence mappings

a) 2 Seq v) Binary sequence operations over improper $\overline{N}$-sequence mappin!

a) Sequence sums and differences

b) Direct composition and the composition product

vi) Binary algebraic operations over improper $\overline{N}$-sequence mappings

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1] Properties of the base

i) Inclusion of the spectrum and order

ii) Invariance of the base under transformation

2] Relationships between $\bar{N}$-sequence mappings

i) Transitivity

ii) Inclusion

iii) Invariance under transformation

3] Characterisations and properties of $\bar{N}$-sequence mappings

4] Unitary operations

i) Permanence

ii) Reflection

iii) Invariance of order under transformation

iv) Inversion and transposition

5] Binary operations

i) Existence conditions for direct composition and the formation of the composition product

ii) Distributive property of direct composition

iii) Associative properties of $\bar{N}$-sequence operations

iv) Associative, commutative and distributive properties of algebraic operations

v) Partitioned transposition and multiplication

vi) Reversal of direct composition and composition products

vii) Transposition of composition products

viii) Preservation of nonsingularity

6] Decompositions

7] Solubility of composition and composition product relationships

8] Groups of $\bar{N}$-sequence mappings

C Matrix mappings

Sets of matrices $\mathbb{K}^{[n]}_{[m]}$ with $m,n \in \bar{\mathbb{N}}$ etc $0^{[n]}_{[m]}$..

Compound matrix mappings $[a. + b]$ etc (use in expressions to define spaces

Pre and post quotient spaces $\text{pre}\left[\frac{a}{0} | \mathbb{B}\right]$ etc

(Define $\mathbb{K}[\mathbb{B}]^{[n]}_{[m]}$ with $m,n \in \text{fn}\{\bar{\mathbb{N}}[\mathbb{B}]\}$ etc) $\mathbb{B}A$ $A\mathbb{B}$

$\mathbb{B}(AB) = (\mathbb{B}A)B$ etc

D Pre- and post-cofactor of zero spaces

Defs. $\text{preCFZ}\{n | \mathbb{B}\}_{[m]} := \text{pre}\left\{\frac{0^{[n]}_{[m]}}{n} | \mathbb{B}\right\}$ etc

Existence and description

Inclusion and equivalence domains for cofactor of zero spaces

(refers to $\text{pons}\,\mathbb{K}[\mathbb{B}]_{\langle n \rangle}$ etc.)

$\bar{C}$.

$\text{pre}\left\{\frac{a}{b} | \mathbb{B}\right\}$ b nonsing $\text{pre}\left\{\frac{a}{b} | \mathbb{B}\right\} = x + \text{pre}\left\{\frac{0^{[n]}_{[m]}}{b} | \mathbb{B}\right\}$

$\text{pre}\left\{\frac{a. + b}{c. + d}\right\} = \text{pre}\left\{\frac{a}{c} | \mathbb{B}\right\} \cap \text{pre}\left\{\frac{b}{d} | \mathbb{B}\right\}$ etc

structure e zerovor $\mathbb{B} \Rightarrow \text{pre}\left\{\frac{a. + b}{c. + d} | z\right\}$ or... void

$a \in \text{fn}\{nz\mathbb{K}[\mathbb{B}']^{n}_{m}\}$ $\text{pre}\left\{\frac{a}{b} | \mathbb{B}'\right\} \subseteq nz\mathbb{K}[\mathbb{B}]^{n}_{m}$

Algebraic sums and products of spaces

$\text{pre}\left\{\frac{a}{c} | \mathbb{B}\right\} \pm \text{pre}\left\{\frac{b}{c} | \mathbb{B}\right\} \equiv \text{pre}\left\{\frac{a \pm b}{c} | \mathbb{B}\right\} \cdots$

$\text{pre}\left\{\frac{a}{b}\right\}\text{pre}\left\{\frac{b}{c}\right\} \subseteq \text{pre}\left\{\frac{a}{c}\right\} \cdots$

Addition of terms to quotient spaces

$a + \text{pre}\left\{\frac{b}{c}\right\} \subseteq \text{pre}\left\{\frac{ac + b}{c}\right\}$..

Multiplication of spaces by factors

$a\,\text{pre}\left\{\frac{b}{c}\right\} \subseteq \text{pre}\left\{\frac{ab}{c}\right\} \cdots$ + nonsingular square (but prens sufficient?)

c.

Defs $\mathbb{K}[\mathbb{B}]^{[r]}_{[k]}$ $\mathbb{K}[\mathbb{B}]_{[k]}$ $\mathbb{K}[\mathbb{B}]^{[r]}$ $\mathbb{K}[\mathbb{B},k]$

$\mathbb{K}'[\mathbb{B}]^{[r]}_{[k]}$ $C:\mathbb{B}\to\mathbb{K}^{[r]}_{[k]}$ for which $C(z)\neq 0^{[r]}_{[k]}$ at least one $z\in\mathbb{B}$

$nz\mathbb{K}[\mathbb{B}]^{[r]}_{[k]}$ $C:\mathbb{B}\to\mathbb{K}^{[r]}_{[k]}\setminus 0^{[r]}_{[k]}$ $ns\mathbb{K}[\mathbb{B},k]$

Def

i) $\mathrm{PS}\{C,\mathbb{B}\}^{[k]} := \mathrm{pre}\left\{\dfrac{C\mathbb{K}[\mathbb{B}]_{[m]}}{\mathbb{K}'[\mathbb{B}]_{[k]}}\right\}$ ($\exists x$ $[b.+C]x=0_{[m]}$ with $x_{[k]}\neq 0_{[k]}$ some $\mathbb{B}\in\mathbb{I}$

ii) $\mathrm{PNS}\{C,\mathbb{B}\}^{[k]} := \mathbb{K}[\mathbb{B}]^{[k]}_{[m]}\setminus \mathrm{PS}\{C,\mathbb{B}\}^{[k]}$

$([b.+C]x=0_{[m]} \Rightarrow x_{[k]}=0_{[k]})$

k specified : $b\in\mathrm{prens}\{C,\mathbb{B}\}$ void C $b\in\mathrm{prens}\{\mathbb{B}\}$

$\mathrm{prerank}(p,C) := \mathrm{rank}[p.+C]-\mathrm{rank}[C]$

$\mathrm{def}_{[h]}\mathbb{K}[\mathbb{B}]^{[r]}_{[k]}$ $\mathrm{def}^{[n]}\mathbb{K}[\mathbb{B}]^{[r]}_{[k]}$: $\mathrm{rank}(C)<k-h$ $(r-n)$

$\mathbb{K}[\mathbb{B},h]^{[j]}_{[i]}$ $\mathrm{rank}[x]=h+1$ $\mathbb{K}[\mathbb{B},h]^{[k]}_{[h]}=ns\mathbb{K}[\mathbb{B},h]$

= Extension of quotient denominators

$\mathrm{pre}\left\{\frac{a}{b}\right\}$ nonvoid $\mathrm{post}\left\{\dfrac{0_{[m]}}{c.+d}\right\}_{(n,k+n]}\cap\{\mathbb{K}_{[k]}\setminus 0_{[k]}\}$ void

$\Rightarrow \mathrm{pre}\left\{\dfrac{[a.+b]}{[c.+d]}\right\}$ nonvoid

$k=0$ $z\in\mathbb{B}$ $\mathrm{post}\left\{\dfrac{0_{[m]}}{c.+d}\Big|z\right\}\cap\{\mathbb{K}\setminus 0\}$ void $\Leftrightarrow$ $\mathrm{post}\left\{\frac{d}{c}\Big|z\right\}$ void

=

Further defs. of prens: $\mathrm{post}\left\{\dfrac{0_{[m]}|z}{a.+b}\right\}_{(n,k+n]}\cap\{\mathbb{K}_{[k]}\setminus 0_{[k]}\}$ void

b prens wrt a. ~~post~~ (pons similarly) a void b prens

$b: B \to \mathbb{K}^{[k]}_{[m]}$ prens wrt $a \Rightarrow k \le m - \text{rank}[a]$

b prens wrt a iff

a) $\text{rank}[a.+b^{(h)}] \ne \text{rank}[a.+b^{[k]}]$ $h := [k]$ ($h := [k]$ for fns. h, k?)

b) $\text{post}\left\{\dfrac{b^{(h)}}{a.+b^{(k)}} \middle| z\right\}$ $h := [k]$ void $z \in B$ missing: $\text{rank}[a.+b] = \text{rank}[b] + k+1$

c) $\text{rank}[a.+b^{[k]}] = \text{rank}[a] + h+1 \langle h := [k] \rangle$

Further defs & prens

=

$\text{rank}[A.+B] = \text{rank}[A] \Rightarrow \exists X : AX = B$ || i.e. $\text{prerank}(B, A) = 0$?

$B = B.' + b \quad AX.' + (A + AX')x = A[X.' + (I + X')x] - AX$

=

d prens wrt a: $y_{(n, n+k+1]}$ same for all $y \in \text{post}\left[\dfrac{f}{c.+d}\right]$

$\text{post}\left\{\dfrac{0}{c}\right\} \cap O_{[:]}$ void $\text{post}\left[\dfrac{f}{c.+d}\right]$ consists of one member

$\text{post}\left\{\dfrac{f}{c.+d}\right\}_{(n, k+n+1]} \cap \text{post}\left\{\dfrac{0^{[r]}_{[n+1]}}{d}\right\}$ nonvoid then $\text{rank}[c.+f] = \text{rank}[c]$

=

$x \in \text{prens}[B]^r_k$ ($\Rightarrow r \le k$)

a) $b \in \text{prens}(C, B)^{[k]} \Rightarrow bx \in \text{prens}(C, B)^{[r]}$

b) $r = k$ $b \in \text{prens}(C, B)^{[k]}$ iff $bx \in \text{prens}(C, B)^{[k]}$

=

$p = bx + Cx'$ $\text{rank}[x] = \min\{\text{prerank}[p, C], k+1\}$

=

$b \in \text{prens}(C, B) \Rightarrow b\Xi$ prens (C, B) all $|\Xi| \le k$; $b^{(k)} \in \text{prens}(C, B)$

if $b \in \text{prens}(C, B)$ $k < m - \text{rank}[C] \Rightarrow \exists \bar{b}_m$ s.t. $[b.+\bar{b}] \in \text{prens}(C, B)$

Untrue that $\text{rank}[B] > \text{rank}[A] \Rightarrow B\mathbb{K}^{[r]}_{[n]} \supseteq A\mathbb{K}^{[r]}_{[j]}$

also $[b+\bar{b}.+C]\mathbb{K}^{[r]}_{[n+k+2]} \supset [b.+C]\mathbb{K}^{[r]}_{[n+k+1]}$ || $B = \begin{matrix}1 & 0\\ 0 & 1\\ 0 & 0\end{matrix}$ $A = \begin{matrix}0\\0\\1\end{matrix}$ $A\mathbb{K}_0$ contai[ns] $\begin{matrix}0\\0\\a\end{matrix}$ not in B'

F. Polynomials and rational functions

$P(x) = \sum c(\nu) x^{\nu} \langle \nu := [m] \rangle \quad cC = 0^{[n]} \quad C := [\lambda(\nu)^{\nu}]^{\nu := [m]}_{\nu := [n]}$ ch

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Boundaries

$u \in fn\{K[B]^{[n]}_{[k]}\}, C \in fn\{K[B]^{n}_{m}\}, p \in fn\{K[B]^{r}_{m}\}$

~~b∈fn~~ $b : B \to K^{[k]}_{[m]}$ st. a) $b \in prens\{C, B\}$

b) $post\{\frac{p}{b.+C}\} \cap poCFZ\{u, B\}^{[r]}$ nonvoid is boundary u/C

$b \leq p \{(u/C) B\}^{[k]} \quad b \equiv p \{..\}$

$u = 0^{[n]}_{[k]} \quad b) \Rightarrow post\{\frac{p}{b.+C}\}$ nonvoid $\quad b \leq p \{C, B\}^{[k]}$

Conditions defining boundaries presented in terms of rank

$(a, b) \Leftrightarrow$ a') $rank[b.+C] = rank[C]+k+1$

b') $rank[[0^{[k]}_{[k]}.+u].+[b.+C]] = rank[[0^{[k]}_{[k]}.+u.+0^{[r]}_{[k]}].+[b.+C.+p]]$

If above conditions satisfied $rank[p.+C] = rank[b.+C.+p]$

$rank[p.+C] = rank[C] \Leftrightarrow post\{\frac{p}{C}\}$ void

$rank[p.+C] = rank[C]+r+1 \Leftrightarrow p \in prens\{C|B\}$

Function and boundary spaces

$FS\{(u/C)B\}^{[k,r]} := prens\{C, B\}^{[k]} K[B]^{[r]}_{[k]} + C\,poCFZ\{u, B\}^{[r]}$

$FS'\{(u/C)B, s\}^{[k,r]} := prens\{C, B\}^{[k]} K[B, s]^{[r]}_{[k]} +$ "

$prensFS\{(u/C)B\}^{[k,r]} := prens\{C, B\}^{[k]} prens K[B]^{[r]}_{[k]} +$ "

$prenFS\{(u/C)B\}^{[r]} := C\,poCFZ\{u, B\}^{[r]}$

$BS\{p(u/C)B\}^{[k]} := pre\left\{\frac{p - C\,poCFZ\{u, B\}^{[r]}}{K[B]^{[r]}_{[k]}}\right\} \setminus FS\{C|B\}^{[k]}$

$EBS\{b(u/C)\mathbb{B}\} := bns\mathbb{K}[\mathbb{B},k] + Cpo\mathbb{CFZ}\{u,\mathbb{B}\}^{[k]}$

$\mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[r]} := b\mathbb{K}[\mathbb{B}]^{[r]}_{[k]} + \quad '' ^{[r]}$

$\mathbb{CFS}'\{b(u/C)\mathbb{B},s\}^{[r]} := b\mathbb{K}[\mathbb{B},s]^{[r]}_{[k]} + \quad ''$

$\text{prens}\mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[r]} := b\,\text{prens}\,\mathbb{K}[\mathbb{B}]^{[r]}_{[k]} + \cdots \quad ''$

$\text{pren}\mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[r]} := Cpo\mathbb{CFZ}\{u,\mathbb{B}\}^{[r]}$

$u = O^{[n]}_{[k]} \Rightarrow \mathbb{FS}\{C,\mathbb{B}\}^{[k,r]} := \text{prens}\{C,\mathbb{B}\}^{[k]}\mathbb{K}[\mathbb{B}]^{[r]}_{[k]} + C\mathbb{K}[\mathbb{B}]^{[r]}_{[n]}$ etc

Function and boundary spaces

i) Above spaces nonvoid

iia) $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ space of $p:\mathbb{B}\to\mathbb{K}^{[r]}_{[m]}$: $\exists$ $b:\mathbb{B}\to\mathbb{K}^{[k]}_{[m]}$

for which $b \leq p\ \{(u/C)\mathbb{B}\}$ exist

b) $\mathbb{FS}'\{\ldots s\}\ldots p$ with $\text{prerank}[p,C] = s+1\ \langle\mathbb{B}\rangle\ldots$

c) $\text{prens}\mathbb{FS}\ \cdots\ p\in\text{prens}(C,\mathbb{B})\ldots$

$\text{prens}\mathbb{FS}\ldots = \text{prens}\{C,\mathbb{B}\}\cap\mathbb{FS}\{\ldots\}^{[k,r]}$ (b, d similarly as intersection

d) $\text{pren}\mathbb{FS}$ prenugatory

iiia) $\mathbb{BS}\{p(u/C)\mathbb{B}\}^{[k]}$: $b:\mathbb{B}\to\mathbb{K}^{[k]}_{[m]}$ $b\leq p\ \{(u/C)\mathbb{B}\}$

b) $EBS\{b\ldots$ $\qquad b'$ $\qquad b'\leq p\ldots$ also $b'\leq b$

iva) $\mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[r]}$: p s.t. $b\leq p\ \{(u/C)\mathbb{B}\}$

b) $\mathbb{CFS}'\{\ldots s\}$ $\uparrow$ with $\text{prerank}[p,C] = s+1$

c) $\text{prens}\mathbb{CFS}\{\ldots\}$ $\uparrow$ $p\in\text{prens}\{C,\mathbb{B}\}$

d) $\text{pren}\mathbb{CFS}\{\ldots\}$ $\uparrow$ with p prenugatory

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Inclusion properties of function spaces

$u:\mathbb{B}\to\mathbb{K}^{[n]}_{[k]}$ $\quad C:\mathbb{B}\to\mathbb{K}^{[n]}_{[m]}$

ia) $k := [m-\text{rank}[C]-1)$ $\quad \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]} \subset \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k+1,r]}$

b) $s\in[m-\mathrm{rank}[C]-1)$ ↑ $k := [s, m-\mathrm{rank}[C]-1)$ for $\mathbb{FS}'\{..\ s]$

c) a) for prens

i) $k'\in((m-\mathrm{rank}[C]))$; $k\in[k')$ $b\in\mathrm{prens}\{C,B\}^{[k]}$ $b'\in\mathrm{prens}\{C,B'\}^{[k}$

a) $\mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[r]}\subset\mathbb{CFS}\{b'\ldots\}$

b) ↑ for prens $\mathbb{CFS}$ ... also: $s\in[k]$ for $\mathbb{CFS}'\{..\ \mathbb{B},s\}^{[r]}$

Constrained function spaces deriving from special boundaries

$C:\mathbb{B}\to\mathbb{K}^{[n]}_{[m]}$ $b:\mathbb{B}\to\mathbb{K}^{[k]}_{[m]}$

i) $h=k+n-m\geq 0$ $u:\mathbb{B}\to\mathbb{K}^{[n]}_{[h]}$ $b\in\mathrm{prens}\{C,B\}$

$[O^{[k]}_{[h]}.+u]+.[b.+C]$ nonsingular $\langle\mathbb{B}\rangle\Rightarrow\mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[r]}=\mathbb{K}[\mathbb{B}]^{[r}_{[}$

ii) $m=k+n$ $[b.+C]$ nonsing $\langle\mathbb{B}\rangle\Rightarrow\mathbb{CFS}\{b,C\|\mathbb{B}\}^{[r]}=\mathbb{K}[\mathbb{B}]^{[r]}_{[m]}$

Ex(i)^

$$\begin{array}{ccccc} 0 & 0 & 0 & 1 & u \\ 0 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ & & C & & \end{array}$$

$m=2$ $n=1$ $k=1$ $h=0$

Properties

i) Reflection $p\in p\{(u/C)\mathbb{B}\}\Leftrightarrow p\in\mathrm{prens}\{C,B\}$

ii) Commutativity

$p\in\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ $b\in\mathbb{BS}\{p(u/C)\mathbb{B}\}^{[k]}$ ← ? required

a) $p\in b\{(u/C)\mathbb{B}\}$ for one $b\in\mathbb{BS}\{\text{"}\}\Rightarrow p\in\mathrm{prens}\{C,B\}$

b) $p\in\mathrm{prens}\{C,B\}\Rightarrow p\in b\ldots$ all $b\in\mathbb{BS}\{..\}$

iii) Transitivity

$p\in\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ $q\in\mathbb{FS}\{(u/C)\mathbb{B}\}^{[r,s]}$ $p\in q\{(u/C)\mathbb{B}\}$

$b\in p\{(u/C)\mathbb{B}\}\Rightarrow b\in q\{..\}$ also $b=p\{\ldots\}$

iv) Nonzero properties of the boundary

$b\in p\{(u/C)\mathbb{B}\}\Rightarrow b\neq O^{[k]}_{[m]}\langle\mathbb{B}\rangle$ $\{$i.e. $\mathbb{BS}\{..\}\subseteq\mathbb{K}[\mathbb{B}]$? $\}$

Closure and invariance properties of function and boundary spaces

$u : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$ $C : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$

i) $b \in \text{prens}\{C, \mathbb{B}\}$

a) $\mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[r]} \mathbb{K}[\mathbb{B}]^{[s]}_{[r]} \subseteq \mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[s]}$

b) $b \in p, p'\{(u/C)\mathbb{B}\} \Rightarrow b \in (p \pm p')\{(u/C)\mathbb{B}\}$

ii) $k \in [m - \text{rank}[C] - 1)$ $p \in \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$

a) $\mathbb{BS}\{p(u/C)\mathbb{B}\}^{[k]} = \mathbb{BS}\{p(uD/CD)\mathbb{B}\}^{[k]}$ all $D \in \text{ns}\mathbb{K}[\mathbb{B}, n]$

b) $\mathbb{BS}\{Bp(u/BC)\mathbb{B}\}^{[k]} = B\ \mathbb{BS}\{p(u/C)\mathbb{B}\}^{[k]}$

The polynomial factor

$\mathbb{RD}_{[k]} := \text{def}_{[k]} \mathbb{K}^{[n]}_{[m]}$ $\mathbb{BD}^{[k]}\{C\} = \text{prens}\{C|\mathbb{B}\}^{[k]}$

$\mathbb{P}\{b, C\} = [b.+C]\ \mathbb{K}[\mathbb{B}]^{[r]}_{[k+n+1]}$

$\text{pfact} : \mathbb{P}\{\mathbb{BD}^{[k]}\{\mathbb{RD}_{[k]}\}, \mathbb{RD}_{[k]}\} \times \mathbb{BD}^{[k]}\{\mathbb{RD}_{[k]}\} \times \mathbb{RD}_{[k]} \to \mathbb{K}[\mathbb{B}]^{[r]}_{[k]}$

$\text{pfact}[(p/b)C] = x_{[k]}$ $x \in \text{post}\{\frac{p}{b.+C} | \mathbb{B}\}$

Existence and rank of the polynomial factor

$p \in [b.+C]\mathbb{K}[\mathbb{B}]^{[r]}_{[k+n+1]}$ (mixed fn. x mapping : clear up)

i) $x_{[k]}$ same for all $x \in \text{post}\{\frac{p}{b.+C} | \mathbb{B}\}$

ii) $\text{rank}(\text{pfact}[(p/b)C]) = \min\{\text{prerank}[p, C], k+1\}$

Invariance properties of the polynomial factor

$p \in [b.+C]\ \mathbb{K}[\mathbb{B}]^{[r]}_{[k+n+1]}$ (why not $p \in \mathbb{CFS} \ldots$?)

i) $\text{pfact}[(Bp/Bb)BC] = \text{pfact}[(p/b)C]$ all $B \in \text{prens}\{\mathbb{B}\}^{[m]}$

ii) $\text{pfact}[(p/b)CD] = \text{pfact}[(p/b)C]$ all $D \in \text{prns}\{\mathbb{B}\}_{[n]}$

Homogeneous constraint systems

$\mathrm{pre}\left\{\frac{u}{C} \mid \mathbb{B}\right\}$ nonvoid $\quad C \in \mathrm{H}\{u, \mathbb{B}\}_{[m]}$

Rank properties of a homogeneous constraint

$$C \in \mathrm{H}\{u, \mathbb{B}\}_{[m]} \Leftrightarrow \mathrm{rank}[u+.C] = \mathrm{rank}[C]$$

Structure and properties of quotient spaces

$C \in \mathrm{H}\{u, \mathbb{B}\}_{[m]}$ $\quad \swarrow$ $\mathrm{pre} \notin \mathbb{C}F\mathbb{Z}\{..\}$?

i) $y \in \mathrm{pre}\left\{\frac{u}{C} \mid \mathbb{B}\right\} : \mathrm{pre}\left\{\frac{u}{C} \mid \mathbb{B}\right\} = y + \mathrm{pre}\left\{\frac{O^{[n]}_{[h]}}{C} \mid \mathbb{B}\right\}$

iia) $\mathrm{pre}\left\{\frac{u}{BC} \mid \mathbb{B}\right\} B \subseteq \mathrm{pre}\left\{\frac{u}{C} \mid \mathbb{B}\right\}$ all $B : \mathbb{B} \to \mathbb{K}^{[m]}_{[r]}$

b) $B \in \mathrm{prens}\{\mathbb{B}\}^{[m]} \circlearrowright =$

iia) $\mathrm{pre}\left\{\frac{u}{C} \mid \mathbb{B}\right\} \subseteq \mathrm{pre}\left\{\frac{uD}{CD} \mid \mathbb{B}\right\}$ all $D : \mathbb{B} \to \mathbb{K}^{[r]}_{[m]}$

b) $D \in \mathrm{pons}\{\mathbb{B}\}_{[n]} \circlearrowright =$

The coefficient space

$$\mathbb{CS}\{(d/b)(u/C)\} := \mathrm{pre}\left\{\frac{d.+u}{b.+C} \mid \mathbb{B}\right\}$$

Existence and structure

$C \in \mathrm{det}_{[k]} \mathbb{K}[\mathbb{B}]^{[n]}_{[m]} \quad u \in \mathbb{K}[\mathbb{B}]^{[m]}_{[h]} C \quad b \in \mathrm{prens}\{C, \mathbb{B}\}^{[k]} \quad d : \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$

i) $\mathbb{CS}\{..\}$ nonvoid

ii) Select $c \in \mathbb{CS}\{..\} : \mathbb{CS}\{..\} = c + \mathrm{pre}\,\mathbb{C}F\mathbb{Z}\{b.+C, \mathbb{B}\}_{[h]}$

Invariance of coefficient spaces

i) $e\,\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\} \underset{\smile}{=} \mathbb{CS}\{(ed/b)(eu/C)\mathbb{B}\}$

all $e \in \mathrm{ns}\mathbb{K}[\mathbb{B}, h]$

ii) $\mathbb{CS}\{(d/Bb)(u/BC)\mathbb{B}\} = \mathbb{CS}\{(d/b)(u/C)\mathbb{B}\}$ all $B \in \mathrm{prens}\{\mathbb{B}\}^{k}$

iiia) $\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\} \subseteq \mathbb{CS}\{(dg/bg)(u/C)\mathbb{B}\}$ all $g \in \mathrm{prens}\{\mathbb{B}, k\}$

b) $g \in \mathrm{ns}\{\mathbb{B}, k\} \circlearrowright =$

iva) $\mathbb{CS}\{(d/b)(u/C)\}\mathbb{B}\} \subseteq \mathbb{CS}\{(d/b)(uD/CD)\mathbb{B}\}$ all $D \in K[\mathbb{B}]^{r}_{[n]}$

b) $D \in \text{pns}\{\mathbb{B}, n\}$? $\Rightarrow$

v) $\mathbb{CS}\{(d/b + C\,\text{post}\{\frac{0^{[k]}_{[h]}}{u}|\mathbb{B}\})(u/C)\mathbb{B}\} = \mathbb{CS}\{(d/b)(u/C)\mathbb{B}\}$

$\text{ps}\mathbb{CFZ}\{u \ldots$?

Inclusion domains for coefficient spaces and boundary spaces deriving from associated functions

$b \in \text{prens}\{C, \mathbb{B}\}^{[k]}$ $\quad p \in \mathbb{CFS}\{b(u/C)\mathbb{B}\}^{[r]}$ $\quad c \in$?

ia) $\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\} \subseteq c + \text{pre}\mathbb{CFZ}\{p.+C, \mathbb{B}\}_{[h]}$

$\subseteq c + \text{pre}\mathbb{CFZ}\{p, \mathbb{B}\}_{[h]}$

b) $r \geq k$, $p \in \text{prens}\{C|\mathbb{B}\} \Rightarrow \mathbb{CS}\{\ldots\} = c + \text{pre}\mathbb{CFZ}\{p.+C|\mathbb{B}\}$

ii) $k = r$ $\quad c \in \mathbb{CS}\{\ldots\}$

a) $b + c\,\text{post}\{\frac{0^{[k]}_{[h]}}{u}|\mathbb{B}\} \subseteq \mathbb{BS}\{p(u/C)\mathbb{B}\} \cap \text{post}\{\frac{d}{C}|\mathbb{B}\}$

$\text{ps}\mathbb{CFZ}\{\ldots$?

b) $d \in \text{prens}\{\mathbb{B}\}$, $p \in \text{prens}\{C|\mathbb{B}\}$? $\Rightarrow$

Polynomials

$u: \mathbb{B} \to K^{[m]}_{[h]}$ $\quad C \in \mathbb{H}\{u, \mathbb{B}\}_{[m]}$ $\quad p: \mathbb{B} \to \mathbb{K}^{[r]}_{[m]}$ $\quad b \leq p\{(u/C)\mathbb{B}\}^{[k]}$

$d: \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$ $\quad P = \text{pol}[d(p/b)C]$?

Polynomial $P: \mathbb{B} \to K^{[r]}_{[h]}$ defined by setting

$P := d\,\text{pfact}[(p/b)C]$ $\quad (\text{pfact}: \mathbb{B} \to \mathbb{K}^{[r]}_{[k]})$

$P = cp$ for all $c \in \mathbb{CS}\{(d/b)(u/C)\}$ $\quad (c: \mathbb{B} \to \mathbb{K}^{[m]}_{[h]})$

i.e. $P = \mathbb{CS}\{(d/b)(u/C)\}$ $\quad (P: \mathbb{B} \to \mathbb{K}^{[r]}_{[h]})$

$$[b. + C][x_{[k]} + . x_{(k, kmn]}] = p$$

$$c[b. + C] = d. + u$$

$$cp = c[b. + C][x_{[k]} + . x_{(k, kmn]}]$$

$$= [d. + u][x_{[k]} + x_{(k, kmn]}]$$

$$= dx_{[k]} \quad \text{since } ux_{(k, kmn]} = O^{[r]}_{[h]}$$

$[P. + c]$ satisfies the relationship

$$[P. + c][[(-I[r]) . + O^{[kmn]}_{[r]}] + . [p. + b. + C]] = [O^{[r]}_{[h]} . + d. + u]$$

i.e.

$$\left[P. + \operatorname{pre}\left\{\frac{d. + u}{b. + C} \mid \mathbb{B}\right\}\right] = \operatorname{pre}\left\{\frac{[O^{[r]}_{[h]} . + d. + u]}{[[(-I[r]) . + O^{[kmn]}_{[r]}] + . [p. + b. + C]]} \mid \mathbb{B}\right\}$$

$$\mathbb{CS}\{(d/b)(u/C)\}$$

$k = r = h = 0 \quad m = n - 1 \quad \Rightarrow u = O^{[n]} \quad [b. + C]$ nonsingular

$$P = \frac{|p. + C|}{|b. + C|} d \quad \langle B \rangle$$

Equiconstrained polynomials | Theory based on the belief that P depends on u?

$C' = BCD \qquad u' = uD \qquad \| \; c \in \mathbb{CS}\{(d/b)(u/C)\}$

i) $b \leq p'\{(u/C)\mathbb{B}\} \quad P' = \operatorname{pol}[d(p'/b)C] \quad P' = cp'$

ii) $P' = \operatorname{pol}[d'(p/b)C] \quad P' = cp' \quad p' = \left(\frac{d'}{d}\right)p \quad \langle NS(d) \rangle$ dimensions

iii) $b' \leq p\{(u/C)B\} \quad u' = \left(\frac{d}{cb'}\right)u$ (dimensions?)

$P' = cp' \quad p' = \frac{d}{cb'} p \; \langle B \rangle$

iv) $u' = \frac{d}{P} u$ $\quad P' = pol[d(\frac{p}{?}/p)C]$

$P' = cp'$ $\quad p' = (\frac{d}{P})b$ $\langle NS(P)\rangle$

v) $P' = pol[d(p/Bb)BC]$ $\quad P' = cp'$ $\quad p' = B^{-1}p$

vi) a) $P' = pol[d(p/b)CD]$ $\quad P' = cp$

b) $P' = pol[kd(p/b)C]$ $\quad P' = cp'$ $\quad p' = kp$

—

Transformations

$P = pol[d(p/b)C|\mathbb{B}]$

i) the identity transformation

$P = pol[P(p/p)C|\mathbb{B}]$ $\langle NS(P)\rangle$ ($p \in prem(C, \mathbb{B}) \rightarrow P \neq 0 \langle \mathbb{B}\rangle$??)

ii) Conjugation

$d = pol[P(b/p)C|NS(P)]$?

iii) Transformations of function mapping, boundary, constraint system and normalising factor C^n_m p_m $b \in p\{(u/C)\mathbb{B}\}$

a) $pol[d(kp/b)C|\mathbb{B}] = kP$ $\langle\mathbb{B}\rangle$?

b) $b' : \mathbb{B} \rightarrow K_{[m]}$: $b' = [b. + C][1 + . z]$?

$pol[d(p/b')C|\mathbb{B}] = P\langle\mathbb{B}\rangle$ all $z : \mathbb{B} \rightarrow \{post\{\frac{0}{u}\} \Leftarrow$??

c) $pol[d(p/b)CD|\mathbb{B}] = P\langle\mathbb{B}\rangle$ all $D : \mathbb{B} \rightarrow ns K[x]$? with nD ?

d) $pol[d(Bp/Bb)BC|\mathbb{B}] = P\langle\mathbb{B}\rangle$ all $B : \mathbb{B} \rightarrow ns K[m]$

e) $pol[kd(p/b)C] = kP$ all $k : \mathbb{B} \rightarrow K \backslash 0$ (ku?)

iv) Addition and subtraction

a) $b \leq p' \{(u/C)B\}$ $P' := \mathrm{pol}[d(p'/b)|C]$ . .

$\mathrm{pol}[d((p \pm p')/b)C] = P \pm P' \langle B \rangle$

b) $P' := \mathrm{pol}[d'(p/b)C]$

$\mathrm{pol}[(d \pm d')(p/b)C] = P \pm P' \langle B \rangle$

$\overline{C}^n_m$ $b\, p\, q : B \to K[m]$ $d, e, f : B \to K$

i) Multiplication

$b \leq p \leq q \{(u/C)B\}$

$\mathrm{pol}[e(q/p)C]\,\mathrm{pol}[d(p/b)C] = \mathrm{pol}[ed(q/b)C]$

ii) Factorisation

$b \leq q \{C, B\}$ choose p: $b \leq p \leq q$ $\mathrm{pol}[ed(q/b)C]$ has above factorisation

iii) Rational decomposition

$p \leq q \{C|B\}$. Select $b \leq p \{C|B\}$

$$\mathrm{pol}\left[\frac{e}{f}(q/p)C\right] = \frac{\mathrm{pol}[e(q/b)C]}{\mathrm{pol}[f(p/b)C]} \langle \text{ns}(P') \rangle \ (P' = \text{denom. pol})$$

iv) Reciprocation

$b \leq p$

$$\frac{1}{\mathrm{pol}[d(p/b)C]} = \mathrm{pol}\left[\frac{1}{d}(b/p)C\right] \langle \text{NS}(P) \rangle$$

Rational functions

$\mathrm{rat}[(q/b)C]$ treated as $\mathrm{pol}[1(q/b)C]$

Clarify treatment of functions of mappings over identical source domains in terms of mapping functions involved. (e.g. with $C \in \mathrm{fn}\{\mathbb{K}[\mathbb{B}]$ $\mathrm{pol}[\ldots C]$ etc $\}$ i.e. pol: ... $\rightarrow$ defined over mapping sets but functions feature in definition of pol)

clarify $h := [k]$ when $h, k : \mathbb{B} \rightarrow \overline{\mathbb{K}}$ etc

$\mathbb{K}[\mathbb{B}]^n_m$ as class $\{\mathbb{K}[\mathbb{B}]^n_m\}$ as set (i.e. type declaration assumed) and then $\mathrm{fn}\{\mathbb{K}[\mathbb{B}]^n_m\}$? (but fn. does not belong to set)

pol fact (p $\therefore$ pol) is $x_{[k]}$ where $[b.+C]x = p$ $\therefore$ independent of u. u constrains b. Examine theory of polynomials, eliminating u where necessary.

Extend present theory of polynomials (in which $r = k = h = 0$) to general case

Construct theory of mapping system (+, etc) including use of functions in c+ pre$\{\vdots \; |\mathbb{B}\}$ etc, sequence multiplication $\circledast A, \ldots$ $|\circledast|$ for base($\circledast$) A^{n}_{m} $m = |A|'$ $n = |A|''$? $\circledast : \mathbb{B} \rightarrow \mathrm{seq}\{\leq |A|' \,|\mathbb{N}\}$ etc.?

Construct theory of nonsingularity & rank $\mathrm{post}\{\frac{O}{A}\} = \mathrm{post}\{\frac{O}{B}\}$ $\Rightarrow \mathrm{rank}(A) = \mathrm{rank}(B)$ $\circledast A : \circledast A$ for all $\circledast \Rightarrow \Leftrightarrow$?

(A)

Equality relationships

A relationship "=" connecting objects, for which

i) $a=a$

ii) $a=b$ if and only if $b=a$ and

iii) $a=c$ if both $a=b$ and $b=c$

for all objects, pairs of objects and triads of objects concerned is called an equality relationship

Type

"=" being a prescribed equality relationship, an object which features in a relationship of the form $a=b$ is said to be of type $(=)$ or, with "type$(=)$" written as $\mathbb{T}$, a $\mathbb{T}$ number.

Sets

A $\mathbb{T}$-set M is an ordered pair $(\{M\},=)$, $\{M\}$ being a collection of $\mathbb{T}$-numbers and $=$ being the relationship defining $\mathbb{T}$

Conventions and expository artifices

In the following $\mathbb{B}$ is a prescribed set, $\bar{\mathbb{N}}$ the integer set ~~sequence~~ $\{0, 1, \ldots\}$ and $\mathbb{K}$ a prescribed field.

fn. m+n+1 etc set($\bar{\mathbb{N}}$) etc

Θ being a prescribed mapping, fn(Θ) is its function (thus if Θ is $\Theta: \mathbb{B} \to \mathbb{K}$, fn($\Theta$) is Θ.)

SM being a prescribed set of mappings, the notation $\Theta \in$ fn(SM) indicates that Θ is the function occurring in one of them.

The functions defined under mapping to mapping transformations are indicated ~~expressed~~ by the same symbol as that used to express the transformation, but without parentheses. Thus, map and map' being two sets of mappings occurring as source and target domains in the mapping trans: map $\to$ map', ~~fn(trans(Ξ))~~ and ξ being fn(Ξ), fn(trans(Ξ)) is written as trans ξ and used, for example, in the form trans $\xi(z)$, or trans $\xi(z|\omega)$, or... as is appropriate, in the implementation of trans(Ξ).

Functions are used in the specification of ~~dependent source and target domains. Thus ~~seq[n]~~ the function n occurring in the mapping n: $\mathbb{B} \to \bar{\mathbb{N}}$ is occurs in the mapping declaration $\xi: \mathbb{B} \to \bar{\mathbb{N}}$seq[n($\mathbb{B}$)]~~ sets of mappings. Thus the function n occurring in the mapping n: $\mathbb{B} \to \bar{\mathbb{N}}$ occurs in the specification of the set seq$\bar{\mathbb{N}}$[B, n]. Expressions involving functions are used in the same way. Thus with m: $\mathbb{B} \to \bar{\mathbb{N}}$ also prescribed, seq$\bar{\mathbb{N}}$[B, m+n+1] is

$\mathrm{seq}\,\bar{\mathbb{N}}\,[\mathbb{B}, r]$ where $r = \mathrm{MHH}\ \langle \mathbb{B} \rangle$

seq being a prescribed sequence, $\{\mathrm{seq}\}$ is the set ~~formed~~ of its members.

$\mathbb{S}$ and $\mathbb{T}$ being specified ~~source and target~~ sets, $\{\mathbb{S} \to \mathbb{T}\}$ is the set of mappings in which $\mathbb{S}$ and $\mathbb{T}$ feature as source and target sets respectively

iv) Algebraic operations over mappings

Algebraic operations defined over target domains of fixed type ~~directly~~ may be extended pointwise to mappings with such target domains. Thus, ~~with~~ the pointwise product of $a, b : \mathbb{B} \to \mathbb{K}$ ~~may be defined to be~~ is $c : \mathbb{B} \to \mathbb{K}$ where $c(z) = a(z)b(z)$ for each $z \in \mathbb{B}$. Pointwise operations over ~~doma~~ mappings with equivalent target domains of variable type may be defined without difficulty. It is possible to define, over mappings, operations which are not simple reflections of properties of target domains of fixed type. Thus, $\mathbb{B}$ and $\mathbb{K}$ in the above being suitably prescribed, a product may be defined in terms of a convolution integral over $\mathbb{B}$. Target domains of variable type permit the definition of operations of even greater variety. In subsequent exposition, operations are defined over mappings. Possible reduction of such operations to target domains may, in each relevant case, be accomplished by the use of ~~constant mappings the artifice~~ the interpretation of ~~of interpreting~~ members of target domains of fixed type as constant mappings. Such possible reduction is tacitly implied in the definition of operations over mappings; it is not the subject of extensive comment.

v) Allocation

Allocation is effected by use of the symbol ":=". Thus it may occur that $\mathbb{N}'$, which features both as an independent source ~~source domain $\mathbb{N}'$~~ domain and in the specification of a dependent source and ~~domain~~ target domains, ~~might be def~~ is ~~defined~~ declared by means of the allocation $\mathbb{N}' := \bar{\mathbb{N}}$. Again, a mapping $c : \mathbb{B} \to \mathbb{K}$ may be defined in terms of mappings $a, b : \mathbb{B} \to \mathbb{K}$ by setting $c := ab$, indicating that $c(z) = a(z)b(z)$ for each $z \in \mathbb{B}$.

iii) The use of function names alone in the specification of mappings

A fixed source–target domain pair defines a type of mapping, equality within the type being defined in terms of pointwise equality of function values over the source domain. Thus with $m,n \in \bar{\mathbb{N}}$ prescribed, a mapping ~~A~~ of the form $a: \mathbb{B} \to \mathbb{K}^n_m$, where $\mathbb{K}^n_m$ represents the set of ~~and~~ matrices over $\mathbb{K}$ with $m+1$ rows and $n+1$ columns, belongs to a mapping type which may be ~~belongs to a set of just described~~ denoted by $\mathbb{K}[\mathbb{B}]^n_m$. ~~That A of the type indicated may be indicated by use of the relationship $A \in \mathbb{K}[\mathbb{B}]^n_m$. However, since $\mathbb{B}$ and $\mathbb{K}^n_m$ are fixed, the notation $a \in \mathbb{K}[\mathbb{B}]^n_m$ suffices for this indication, without the declaration of A as a mapping.~~

~~Thus the notation~~

Since $\mathbb{B}$ and $\mathbb{K}^n_m$ are fixed, the notation $a \in \mathbb{K}[\mathbb{B}]^n_m$ suffices to indicate that A is a mapping of type $\mathbb{K}[\mathbb{B}]^n_m$, without the preliminary definition of A as a mapping. Use of function names alone in the specification of mappings of types determined by fixed source-target domain pairs is ~~ado~~ adopted consistently in the following exposition.

ii) The use of globally declared variables in mapping specifications definitions

It may occur that although the ~~genera~~ complete declaration of a mapping involves many variables, a number of them may be held fixed during a certain passage of exposition without inconvenience. In such cases mapping ~~declarations~~ specifications definitions may be economised by the use of globally declaration of the variables held fixed in the subsequent theory.

~~Thus, the mapping declaration $a: \bar{\mathbb{N}} \times \ldots \to \mathbb{K}$, although which leads to~~ with $\bar{\mathbb{N}}' := \ldots$ ~~$a: \mathbb{N}' \times \mathbb{B} \to \mathbb{T}(\mathbb{N}')$ where the target domain depend.~~

~~the use of function values $a(j,k)$ of two variables, may be~~

~~abbreviated by the preliminary declaration of $j \in \bar{N}$ and the subsequent use of the mapping $a' : \bar{N} \to K$ and it's induced function values $a'(k)$ if, indeed, the variation of k~~ ~~is~~ ~~is the predomin influence~~ ~~on~~ ~~upon the behaviour of $a(j,k)$ in the~~ ~~passage~~ ~~immediately subsequent passage of exposition.~~

~~$a : B \to T(\bar{N}) \quad a : N' \times B \to T(N') \quad a : B \to T(j)$~~

Thus, with $N' := \bar{N}$, the declaration $a : N' \times B \to T(N')$ for ~~which~~ ~~of $a(j,z)$~~ ~~the function value $a(j,z)$ lies is in $T(j)$~~ which involves ~~involving~~ a dependent target domain and for which the function value $a(j,z)$ lies in $T(j)$, may be abbreviated by the preliminary declaration of $j \in \bar{N}$, ~~and the subsequent use~~ ~~of~~ the specification $a' : B \to T(j)$ ~~if, indeed, the variation~~ and the ~~subsequent~~ reference to ~~use of the univariate function a'~~ function values $a'(z)$ if, indeed, ~~the variation of z is the predominant influence upon the behaviour of $a(j,z)$ in subsequent exposition~~.

the dependence of $a(j,z)$ upon z is the principal concern of later ~~subsequent~~ exposition.

v) — Relationships between functions ~~values~~ over argument subdomains

Relationships between ~~mappin~~ function values over a prescribed argument domain are indicated by the use of function names alone, ~~and~~ the domain in question being juxtaposed in angular brackets. Thus with $a, b, c : B \to K$, ~~being pe~~ ~~the~~ and $B' \subseteq B$, the notation

$$a = bc \qquad \langle B' \rangle$$

indicates that $a(z) = b(z)c(z)$ for each $z \in B'$.

(B)

Composition $a: S \to T$

~~If~~ S and T being two sets with $T \subseteq S$, the composition ~~a~~ of the ordered pair of ~~two~~ mappings $b, c: S \to T$ is defined by setting

$$a(z) := \cancel{b}^{c}(\cancel{b}(z))$$

If ~~the condition $T \subseteq S$ is violated, either because the~~ S and T are of the same type but do not satisfy the semi inclusion relationship $T \subseteq S$, or S and T are of different types, simple composition as just defined is infeasible. It is nevertheless possible to implement composition in cases in which the condition $T \subseteq S$ is violated by introducing ~~means use of~~ a ~~type~~ set conversion ~~operator mapping~~ operator ω for which $\omega(T) \subseteq S$.

The following notes concern extended composition of mappings with dependent target domains. Some general definitions are given, and the case in which T is an ~~sequence~~ $\bar{N}$-sequence mapping is treated in detail

~~Set conversion transformation.~~

~~S and T are prescribed types. [T] is a set of T-sets. M, M′~~

~~M is an S-set and M′ a T-set~~

~~i) let the mapping $\omega: [T] \times \{[T]\} \to M$ be prescribed, the set to set mapping $\omega: [T] \subseteq M$ is defined by setting~~

~~$$\omega(T^*) := \{\omega(T^*|z) \langle z := T' \rangle\}$$~~

~~$T': B \subseteq M'$ being a prescribed~~

~~b) The domain to set transformation mapping $\omega: \{B \subseteq M'\} \to \{B \subseteq M\}$ is defined by setting, with $T': B \subseteq M'$,~~

~~$$\omega(T'|z) := \omega(T'(z))$$~~

1] Set transformation

A number of set conversion mappings may be presented as type extensions of an elementary mapping whose function ω has, as arguments, ~~mapping~~ ω whose ~~arguments~~ ~~are~~ ~~source~~ ~~arguments are~~ a set and an individual member of the set. The value of ω may, in particular, ~~be~~ ~~determined in terms of the~~ depend upon ~~the~~ a prescribed relationship of the member to the set of which it is a constituent.

$\mathbb{S}$ and $\mathbb{T}$ are prescribed types ~~sets~~. $[\mathbb{T}]$ is a set of $\mathbb{T}$ sets.

M is an $\mathbb{S}$-set and M' a $\mathbb{T}$-set.

Let the mapping $\omega : [\mathbb{T}] \times \{[\mathbb{T}]\} \to M$ be prescribed.

i) ~~A $\mathbb{T}$-set T for which~~ When $\mathbb{S} \equiv \mathbb{T}$, a $\mathbb{T}$-set T for which

$$\omega(T|\tau) = \tau \qquad \langle \tau := T \rangle$$

is said to be pointwise invariant under ω

ii) The set to set mapping $\omega : [\mathbb{T}] \subseteq M$ is defined by setting

$$\omega(T) := \{\omega(T|\tau) \ \langle \tau := T \rangle\}$$

b) When $\mathbb{S} \equiv \mathbb{T}$, a $\mathbb{T}$-set T for which

$$\omega(T) \equiv T$$

is said to be setwise invariant under ω

iii) The domain to set mapping transformation $\omega : \{B \subseteq M'\} \to \{B \subseteq M\}$ is defined by type extension of the preceding mapping. With $T' : B \subseteq M$, $\omega(T') : B \subseteq M$ is defined by setting

$$\omega(T'|z) := \omega(T'(z))$$

b) When $\mathbb{S} = \mathbb{T}$, a set mapping $T' : B \subseteq M$ such that for each $z \in B$

$$\omega(T'|z) \equiv T'(z)$$

... under ω

2] Composition products

~~S and T being $\mathfrak{S}$- and $\mathfrak{T}$-sets respectively, $M[S,T|\mathbb{B}]$ is~~

~~a set of mappings A of the form $a:\mathbb{B}\to\{S$~~

M and M' are $\mathfrak{S}$- and $\mathfrak{T}$-sets respectively.

S and T being set mappings of the form $S:\mathbb{B}\subseteq M$ and $T:\mathbb{B}\subseteq M'$, the notation $A\in M[S,T|\mathbb{B}]$ indicates that A is of the form ~~$M[S,T|\mathbb{B}]$ is a set of mappings A of the form~~ $a:\mathbb{B}\to\{S(\mathbb{B})\to T(\mathbb{B})\}$. Further ~~sets~~ mapping ~~description~~ classes $M[S^*,T'|\mathbb{B}]$, $M[S',T^*|\mathbb{B}]$ ..., similarly restricted, are defined in the same way

i ~~a~~) The ~~product~~ composition product $A\in M[S^*,T|\mathbb{B}]$ of $B\in M[S,T'|\mathbb{B}]$ and $C\in M[S',T^*|\mathbb{B}]$ where $\omega(T')\subseteq S'\langle\mathbb{B}\rangle$, is defined, with $a=fn(A)$, ~~b=fn~~..., $c=fn(C)$ by setting

$$a(z|\tau):=\overset{c}{\cancel{b}}(z|\omega(\{\overset{b}{\phi}(z)\}|\overset{b}{\phi}(z|\tau)))$$

and A is written as BCB

~~b) With $T\subseteq S$, S and T' of the same type and $T'\subseteq S$, the direct composition $A\in M$ A of B and C as specified above is defined by setting~~

~~$$a(z|\tau):=b(z|c(z|\tau))$$~~

~~The direct composition A is written as B[C].~~

ii ~~a~~) The formation of the composition product ~~as defined above~~ is associative. Let $B\in M[S,T'|\mathbb{B}]$, $C\in M[S',T^*|\mathbb{B}]$ and $D\in M[S'',T''|\mathbb{B}]$ with $\omega(T')\subseteq S'$, $\omega(T^*)\subseteq S''$: ~~Set E = DC and F = CB~~

$$~~(DC)~~\ ~~D(CB)=(DC)B~~\quad (BC)D=B(CD)$$

The product $\overset{BCD}{DCB}\in M[S^*,T''|\mathbb{B}]$ is unambiguously defined

~~An analogous result holds with regard to the direct composition B[C[D]].~~

iii a) Assuming in (a) that $\omega(T') \overset{\equiv S}{\subseteq} \langle B\rangle$. A mapping $I \in M[S^{*}, T' | B]$ for which, with $i := fn(I)$,

$$\omega(\{i(z)\} | i(z|\tau)) = \tau \quad \langle \tau := S^{*}(z)\rangle$$

for each $z \in B$ is said to be a unit element. For this, $\overset{C}{I B} = \overset{C}{B}$ for all $B \in M[S, T | B]$

iii b) Let $A \in M[S^{*}, T | B]$ and $\overset{C}{B} \in M[S^{*}, T^{*} | B]$ be such that $B \in M[S^{*}, T' | B]$, with $\omega(T') \leq S' \langle B\rangle$, exists for which $A = \overset{BC}{C B}$.

The equation $A = \overset{BC}{C B}$ is said to be soluble in $M[S^{*}, T' | B]$

b) Let $\overset{C}{B} \in M[S', T | B]$. If for all $A \in M[S^{*}, T | B]$ such that the equation $A = \overset{BC}{C B}$ is soluble, the corresponding solution $\overset{B}{B} \in M[S^{*}, T' | B]$ is unique, $\overset{C}{B}$ is said to be nonsingular with respect to $M[S^{*}, T' | B]$.

v) Let $\omega(T) \overset{e}{\leq} S$

a) ~~Let $\omega(T) \leq S$.~~ A set $\hat{M}$ of mappings of the class $M[S, T | B]$ for which $\overset{BC}{C B} \in \hat{M}$ for all $\overset{B,C}{C B} \in \hat{M}$ is said to be closed with respect to the composition product

b) ~~Let $\omega(T) \leq S$.~~ A set M of mappings of the class $M[S, T | B]$ for which the equation $A = \overset{BC}{C B}$ is soluble in $M[S, T | B]$ for all $A, \overset{C}{B} \in \tilde{M}$ is said to be soluble with respect to the composition product

3] Direct Composition

In the preliminary specifications of the previous section, let S and T be identical types and $T' \leq S' \langle B\rangle$

The direct composition $A \in M[S^{*}, T | B]$ of $B \in M[S, T | B]$ and $C \in M[S', T^{*} | B]$ is defined, with $a := fn(A), \ldots, c := fn(C)$ by setting

$$a(z|\tau) := \overset{c}{B}(z \,|\, \overset{b}{B}(z|\tau)) \quad \langle \tau := S^{*}(z)\rangle$$

and is written as $\overset{C[B]}{B[C]}$.

Direct composition is associative. With $B \in M[S, T'|\mathbb{B}]$, $C \in M[S', T''|\mathbb{B}]$, and $D \in M[S'', T''|\mathbb{B}]$ and ~~T'≤S'~~ $T'' \leq S''$, set $\overset{A}{B} := C[\overset{B}{B}]$ and $E := \overset{D}{B}[C]$; then $\overset{D}{B}[\overset{A}{B}] = E[\overset{B}{B}]$: the expression $\overset{D}{B}[C[\overset{B}{B}]]$ is $\in M[S, T''|\mathbb{B}]$ unambiguous

~~Assuming that T'≤S'~~ The mapping $I \in M[S^{\#}, S^{\#}|\mathbb{B}]$ for which, with $i := fn(I)$, $i(z|z) = z$ $\langle z := S^{\#}(z) \rangle$ for each $z \in \mathbb{B}$

? is a unit element. $\overset{C}{B}[I] = \overset{C}{B}$ for all $\overset{C}{B} \in M[S, T|\mathbb{B}]$

With ~~If~~ $A \in M[S^{\#}, T|\mathbb{B}]$ and $B \in M[S^{\#}, T'|\mathbb{B}]$ ~~be~~ such that $C \in M[S', T^{\#}|\mathbb{B}]$ ~~with T'≤S'~~ exists for which $A = \overset{C[B]}{B[C]}$, the equation $A = \overset{C[B]}{B[C]}$ is said to be soluble in $M[S^{\#}, T'|\mathbb{B}]$. Nonsingularity is ~~as described~~ mutatis mutandis described as in (iv b) ~~(f)~~ above.

With $T \leq S$, sets $\hat{M}$ and $\tilde{M}$ of mappings that are closed and soluble with respect to direct composition ~~and soluble with respect to composition~~ respectively are defined in analogy with (v) above.

4] The composition product expressed in terms of direct composition

Let $\hat{M}$ be a set of mappings $\overset{B}{B}$ in the class $M[S^{\#}, T'|\mathbb{B}]$. Define the set $\tilde{M}$ of mappings in the class $M[S^{\#}, \omega(T')|\mathbb{B}]$ by the condition that $\Theta \in \tilde{M}$ if and only if $\overset{B}{B} \in \hat{M}$ exists for which

$$\theta(z|\tau) = ~~c~~ \omega(\{\overset{b}{\phi}(z)\} | \overset{b}{\phi}(z|z)) \langle z := S^{\#}(z) \rangle$$

for each $z \in \mathbb{B}$, where $\theta = fn(\Theta)$ and $\overset{b}{\phi} = fn(\overset{B}{B})$.

Define the mapping $\Omega : \hat{M} \to \tilde{M}$ by use of the construction defining the members of $\tilde{M}$: $\Omega(\mathbf{B})$ is Θ.

Let $\omega(T') \leq S'\langle B\rangle$. The composition product ~~of~~ of $B \in M[S, T'|B]$ and $C \in M[S', T^{*}|B]$ may be expressed ~~as~~ in terms of direct composition

$$BC = C[\Omega(B)]$$
$$\cancel{CB = B[\Omega(C)]}.$$

5] ~~By introducing the set~~ Solubility conditions

By use of the set transformation operator ω it is, in certain cases, possible to form a composition product where direct composition is infeasible: BC ~~CB~~ may be defined when $C[B]$ ~~$B[C]$~~ is not, either because T' and S' are of different types or because $\omega(T') \leq S'$ but $T' \not\subseteq S'$. $\bar{M}$ being a set of mappings ~~C~~ $B \in M[S, T|B]$ and ~~When~~ T' and S' ~~are~~ being of the same class ~~and~~ with $T' \leq S'$, two sets of mappings of the class $M[S^{*}, T|B]$ may be defined. ~~$CB = B[\Omega(C)]$ for all $B \in \bar{M}$ and $C \in \hat{M}$ and~~ The first is the set whose members have the form $BC = C[\Omega(B)]$ ~~$CB = B[\Omega(C)]$~~ for $C \in \bar{M}$ and $B \in \hat{M}$. The second is the set whose members have the form $C[B]$ for $C \in \bar{M}$ and $B \in \hat{M}$. ~~The second~~ These sets may not be identical. In particular it may occur that $C[B]$ is defined while no $B' \in \hat{M}$ for which $B'C = C[B]$ exists. The conditions under which the equations $A = C[B]$ and $A = B'C$ are soluble may differ.

To illustrate the remarks of the preceding paragraph, consider the case in which $S, T, \ldots$ $b, c, \ldots$ are constant over B, S and S' ~~kk~~ being the constant integer set $\{0, 1, 2\}$. When T' is the integer set $\{3, 17\}$ and ~~C is~~, for B, $b(0) = b(2) = 3$, $b(1) = 17$, the formation of $C[B]$ is infeasible, since the values of $b(0), \ldots, b(2)$ lie outside the argument range of c. Define ω by taking $\omega(T|\tau)$ to be the number of members of T less than τ. Then, with $z \in B$,

$\omega(\cancel{c}\{b(z)\}\,|\,b(z,z))$ for $z := 0, 1, 2$ takes the values $0, 2, 0$. The latter values lie within the argument range of $\cancel{b}$ c: $A := C[\Omega(B)]$ ~~$B[\Omega(C)] = CB$~~ BC may be formed. Now select another $\cancel{C}$ B for which $b(10) = b(12) = 0$, $b(11) = 1$. $A := C[B]$ ~~$B[C]$~~ may be formed. ~~However no C' for which $C'B = B[C]$ exist~~ When $\cancel{B}$ C is nonsingular (i.e. $\cancel{b}$ c(0), $\cancel{b}$ c(1), $\cancel{b}$ c(2) are distinct) ~~no C' for which $C'B = B[C]$ exists~~, $C[B]$ ~~$B[C]$~~ $=$ $B'C$ ~~$C'B$~~ only when $\cancel{C}$ B $= \Omega(\cancel{C}$ B$'$

However, no $\cancel{C}''$ B$'$ with $\cancel{c}'$ b $= fn(B')$ for which $\omega(\{b'(z)\}\,|\,b'(z;z))$ for $z = 0, 1, 2$ takes the values $0, 1, 0$ exists. The values $z = 0, 2$ in this expression correspond to the minimum values in $\{b'(z)\}$. The ~~no~~ remaining value of ω is either 0, when all ~~members of $\{c'(z)\}$ are equal~~ $b'(z|z)$ are equal, or 2 when this is not so. Thus no $\cancel{B}'$ B$'$ for which $C[B]$ ~~$B[C]$~~ $=$ $B'C$ ~~$C'B$~~ exists. In the first of the above numerical examples the equation $A = BC$ ~~CB~~ ~~was~~ is soluble, the equation $A = C[B]$ ~~$B[C]$~~ ~~was~~ is not; in the second the converse is true.

1] Integer sequences

i) Special sequences

a) With $i, j \in \bar{\mathbb{N}}$ and $i \leq j$, $[i, j]$ is the sequence $\langle i, \ldots, j \rangle$; with $i < j$, $[i, j)$ and $(i, j]$ are the sequences $\langle i, \ldots, j-1 \rangle$ and $\langle i+1, \ldots, j \rangle$ respectively; with $i < j-1$, $((i, j))$ is the sequence $\langle i+1, \ldots, j-1 \rangle$.

b) When $i = 0$ in the above $[i, j]$ is written simply as $[j]$ ~~; the~~ ~~reduced forms~~ ~~notations $[j)$, $(j]$ and $((j))$ are similarly obtained.~~

and $[i, j)$, $(i, j]$, (i, j) as $[j)$, $(j]$, $((j))$

ii) $\bar{\mathbb{N}}$-sequences

A mapping Ξ' of the form $\xi' : \{[i]\} \to \bar{\mathbb{N}}$ for some $i \in \bar{\mathbb{N}}$ depending on Ξ', is called an $\bar{\mathbb{N}}$-sequence. $\bar{\mathbb{N}}$seq$[i]$ is ~~the~~ a set ~~class~~ of such mappings with source domain $\{[i]\}$

2] $\bar{\mathbb{N}}$-mappings

A mapping of the form $n : \mathbb{B} \to \bar{\mathbb{N}}$ is called an $\bar{\mathbb{N}}$-mapping Ξ. $\bar{\mathbb{N}}[\mathbb{B}]$ is the class of such mappings. In the following the declarations $m, n \in fn\{\bar{\mathbb{N}}[\mathbb{B}]\}$ are tacit.

3] ~~$\bar{\mathbb{N}}$-sequence mappings~~ Proper $\bar{\mathbb{N}}$-sequence mappings

~~i) Let $n \in fn\{\bar{\mathbb{N}}[\mathbb{B}]\}$.~~

i) A mapping Ξ of the form

$$\xi : \mathbb{B} \to \bar{\mathbb{N}}\text{seq}[n(\mathbb{B})]$$

is called an $\overline{\mathbb{N}}$-sequence mapping. With $z \in \mathbb{B}$, $\xi(z)$ is a sequence $\langle \xi(z|0), \xi(z|1), \ldots, \xi(z|n(z)) \rangle$ of members of $\overline{\mathbb{N}}$. (This method of representing the terms of $\xi(z)$ is retained in the sequel.)

$\mathrm{seq}\,\overline{\mathbb{N}}[\mathbb{B}]$ is the class of $\overline{\mathbb{N}}$-sequence mappings as just described. Where n is of critical interest, the abbreviation $\mathrm{seq}\,\overline{\mathbb{N}}[\mathbb{B},n]$ is used.

In the following, ~~where Ξ is~~ $\Xi, \Theta, \overline{\Phi}, \Psi \in \mathrm{seq}\,\overline{\mathbb{N}}[\mathbb{B}]$ are used as representative $\overline{\mathbb{N}}$-sequence mappings, the declarations ~~$\xi := fn(\Xi)$ is tacit~~ $\xi := fn(\Xi), \ldots, \psi := fn(\Psi)$, where relevant, being tacit.

iii) Special $\overline{\mathbb{N}}$-sequence mappings

a) With $i, j \in fn\{\overline{\mathbb{N}}[\mathbb{B}]\}$ such that $i \le j \langle \mathbb{B} \rangle$, $[i,j]$ is the $\overline{\mathbb{N}}$-sequence mapping whose function value, for each $z \in \mathbb{B}$, is the sequence $\langle i(z), \ldots, j(z) \rangle$; with $i < j \langle \mathbb{B} \rangle$, $[i,j)$ and $(i,j]$ are $[i,j-1]$ and $[i+1,j]$ respectively; with $i < j-1 \langle \mathbb{B} \rangle$, (i,j) is $[i+1, j-1]$.

b) When $i = 0 \langle \mathbb{B} \rangle$ in the above, $[i,j]$ is written simply as $[j]$ and $[i,j)$, $(i,j]$, (i,j) as $[j)$, $(j]$, $((j))$.

ii) The base $:= 1.1$?

The mapping

$$\mathrm{base} : \mathrm{seq}\,\overline{\mathbb{N}}[\mathbb{B}] \to \mathrm{seq}\,\overline{\mathbb{N}}[\mathbb{B}]$$

is defined by selecting $\Xi \in \mathrm{seq}\,\overline{\mathbb{N}}[\mathbb{B},n]$, ~~declaring~~ with $\mathrm{base}\,\xi := fn(\mathrm{base}(\Xi))$ when, for each $z \in \mathbb{B}$, $\mathrm{base}\,\xi(z)$ is the sequence $[n(z)]$.

is called an $\bar{\mathbb{N}}$-sequence mapping. With $z \in \mathbb{B}$, $\xi(z)$ is a sequence $\langle \xi(z|0), \xi(z|1), \ldots, \xi(z|n(z)) \rangle$ of members of $\bar{\mathbb{N}}$. (This method of representing the ~~components~~ terms of $\xi(z)$ is retained in the sequel.)

$\operatorname{seq}\bar{\mathbb{N}}[\mathbb{B}, n]$ is the class of $\bar{\mathbb{N}}$-sequence mappings as just described. Where n is not of critical interest, the abbreviation $\operatorname{seq}\bar{\mathbb{N}}[\mathbb{B}]$ is used.

In the following, where Ξ is used as as a representative ~~$\bar{\mathbb{N}}$-sequence mapping, the definition $\xi := fn(\Xi)$ is tacit.~~

iii) Classes of $\bar{\mathbb{N}}$-sequence mappings (nonsingular, unit, order, ~~permanent~~ permutation and bounded mappings).

A mapping $\Xi \in \operatorname{seq}\bar{\mathbb{N}}[\mathbb{B}]$ for which, for each $z \in \mathbb{B}$

a) $\xi(z)$ consists of distinct members of $\bar{\mathbb{N}}$ is said to be nonsingular

b) $\xi(z)$ is a strictly increasing sequence is said to be a unit mapping

~~γ) $\xi(z|\nu)$ is the number of members of $\{\phi(z)\}$ less than $\phi(z|\nu)$ for some $\Phi \in \operatorname{seq}\bar{\mathbb{N}}[\mathbb{B}, n]$, where $\phi := fn(\Phi)$,~~ for $\nu := [n(z)]$,

With $\Xi \in \operatorname{seq}\bar{\mathbb{N}}[\mathbb{B}, n]$,

c) $\Phi \in \operatorname{seq}\bar{\mathbb{N}}[\mathbb{B}, n]$ with $\phi := fn(\Phi)$ exists such that, for $\nu := \text{base}\,\xi(z)$, $\xi(z|\nu)$ is the number of members of $\{\phi(z)\}$ less than $\phi(z|\nu)$, is said to be an order sequence mapping

d) $\xi(z)$ is a rearrangement of the integer sequence base $\xi(z)$ is said to be a permutation sequence mapping

e) $i \in fn\{\bar{N}[B]\}$ exists for which $\xi(z|\nu) \leq i(z) \langle \nu := [\text{base } \xi(z)] \rangle$ is

said to be a bounded sequence mapping.

In the following ns $\bar{N}[B,n]$, unit $\bar{N}[B,n]$, ord $\bar{N}[B,n]$, perm $\bar{N}[B,n]$ and seq $\bar{N}[\leq i|B,n]$ are the classes of such mappings respectively. Where the context permits, the abbreviations ns $\bar{N}[B]$, unit $\bar{N}[B]$, ord $\bar{N}[B]$, perm $\bar{N}[B]$ and seq $\bar{N}[\leq i|B]$ are used.

~~iii) Equivalence classes of $\bar{N}$ sequence mappings~~

~~a) ord equiv (Ξ) is the class of $\bar{N}$-sequence mappings~~

~~$\bar{\Phi} \in$ seq $\bar{N}[B,n]$ for which~~

iii) Unitary operations upon $\bar{N}$-sequence mappings

a) The order mapping

The mapping

$$\text{ord}: \{\text{seq } \bar{N}[B]\} \rightarrow \{\text{seq } \bar{N}[B]\}$$

is defined in terms of ord $\xi : fn(\text{ord}(\Xi))$: for each $z \in B$, ord $\xi(z|\omega)$ is the number of members of $\xi(z)$ less than $\xi(z|\omega) \langle \omega := [\text{base } \xi(z)] \rangle$. ord (Ξ) in this case is said to be the order mapping of Ξ.

b) Order reversal

The order reversal mapping, indicated by means of a superposed tilde,

$$\tilde{}: \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B};\mathfrak{N}] \to \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B};\mathfrak{N}]$$

is defined by setting, with $\tilde{\xi} := \mathrm{fn}(\tilde{\Xi})$,

$$\tilde{\xi}(z|\omega) := \xi(z|n(z)-\omega) \quad \langle \omega := [n(z)]\ \mathrm{base}\ \xi(z) \rangle$$

for each $z \in \mathbb{B}$.

c) The ~~unit~~ ascension mapping

The mapping

$$\overset{\mathrm{asc}}{\cancel{\mathrm{unit}}} : \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B};\mathfrak{N}] \to \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B};\mathfrak{N}]$$

is defined ~~by setting~~ in terms of $\mathrm{unit}\,\xi := \mathrm{fn}(\mathrm{unit}(\Xi))$: for each $z \in \mathbb{B}$, ~~ord~~ $\mathrm{unit}\,\xi(z)$ is obtained by rearranging the terms of $\xi(z)$ in ascending order of magnitude.

d) The transpose mapping

The transpose mapping, indicated by means of an affixed "T",

$$^{\mathrm{T}}: \mathrm{ns}\,\bar{\mathbb{N}}[\mathbb{B};\mathfrak{N}] \to \mathrm{ns}\,\bar{\mathbb{N}}[\mathbb{B};\mathfrak{N}]$$

is defined in terms of $\xi^{\mathrm{T}} := \mathrm{fn}(\Xi^{\mathrm{T}})$: for each $z \in \mathbb{B}$

$$\xi^{\mathrm{T}}(z|\mathrm{ord}\,\xi(z|\omega)) = \mathrm{unit}\,\xi(z|\omega) \qquad \langle \omega := [n(z)]\ \mathrm{base}\ \xi(z) \rangle$$

To facilitate exposition (for example, in the use of expressions having the properties of similarity products) Ξ^{T} is occasionally written as Ξ^{-1} and $\mathrm{fn}(\Xi^{-1})$ as ξ^{-1}.

e) The spectrum

The mapping

$\text{spect} : \text{seq}\,\overline{\mathbb{N}}[\mathbb{B}] \to \text{seq}\,\overline{\mathbb{N}}[\mathbb{B}]$

is defined in terms of $\text{spect}\,\xi := \text{fn}(\text{spect}(\Xi))$: for each $z \in \mathbb{B}$, spect $\xi(z)$ is obtained from ~~unit~~ are $\xi(z)$ in the following ways. ~~Suppose~~ With $[n(z)] := \text{base}\,\xi(z)$, ~~that~~ let $s(z)$ be the increasing sequence of all $\nu \in [n(z)]$ for which ~~unit $\xi(z|\nu) >$~~ unit $\xi(z|\nu-1) <$ unit $\xi(z|\nu)$. spect $\xi(z)$ is

~~$\langle 0 + s(z) + n(z)$ (if $n(z) \notin s(z)$) $\rangle$~~

0 followed by the $s(z)$ followed by, if $n(z) \notin s(z)$, $n(z)$.

f) The inverse mapping

Let $\overline{\mathbb{N}}\mathbb{S} \subseteq \text{ns}\,\overline{\mathbb{N}}[\mathbb{B}]$ and define the auxiliary mapping $\text{fnset} : \overline{\mathbb{N}}\mathbb{S} \times \mathbb{B} \subseteq \overline{\mathbb{N}}$ by setting $\text{fnset}(\Xi, z) := \{\xi(z)\}$.

The inverse mapping

$\text{inv} : \overline{\mathbb{N}}\mathbb{S} \to \{\mathbb{B} \times \text{fnset}(\overline{\mathbb{N}}\mathbb{S}, \mathbb{B}) \to \{n(\mathbb{B})\}\}$

is defined by taking $\text{inv}(\Xi)$ to be

$\text{inv}\,\xi : \mathbb{B} \times \{\xi(z)\} \to \{n(\mathbb{B})\}$

where $\text{inv}\,\xi(z, \xi(z|\omega)) = \omega \quad \langle \omega := [n(z)] \ni \text{base}\,\xi(z) \rangle$

(Thus, if for some $z' \in \mathbb{B}$, $\xi(z')$ is $\langle a, b \rangle$, $\text{inv}\,\xi(z', a) = 0$ and $\text{inv}\,\xi(z', b) = 1$. $\text{inv}(\Xi)$ is not an $\overline{\mathbb{N}}$-sequence mapping: $\text{inv}\,\xi$ is a function of two variables: its value is an integer)

~~iv) Equivalence classes of $\overline{\mathbb{N}}$ sequence mappings~~

~~a) ord equiv(Ξ) is the class of $\overline{\mathbb{N}}$ sequence mappings $\Phi \in \text{seq}\,\overline{\mathbb{N}}[\mathbb{B}, n]$ for which, with~~

~~iv)~~ vi) Binary sequence operations ~~upon~~ over $\bar{\mathbb{N}}$-sequence mappings

~~a) Sequence~~ Let $\underline{\Phi} \in \operatorname{seq} \bar{\mathbb{N}}[B, m]$ with $\phi = fn(\underline{\Phi})$. In the following $\psi = fn($

a) Sequence sums and differences

$\underline{\Phi} \in \operatorname{seq}\bar{\mathbb{N}}[B,m]$ and $\Xi \in \operatorname{seq}\bar{\mathbb{N}}[B,n]$

The sequence sum of $\underline{\Phi}$ ~~and Ξ~~, written as $\langle \underline{\Phi} + \Xi \rangle$, is $\underline{\Psi} \in \operatorname{seq} \bar{\mathbb{N}}[B, m+n+1]$ where, ~~with $\psi = fn(\underline{\Psi})$,~~ $\psi(z)$ is constructed by placing $\xi(z)$ after $\phi(z)$ for each $z \in B$. ~~fn $\langle \underline{\Phi} + \Xi \rangle$ is written as~~ $\langle \phi + \xi \rangle$.

(defined as a proper $\bar{\mathbb{N}}$-sequence mapping ~~does defined~~ when $\{\phi\} \not\subseteq \{\xi\} \langle B \rangle$ and)

The ~~sequence~~ set difference of $\underline{\Phi}$ and Ξ, written as $\underline{\Phi} \setminus \Xi$, is $\underline{\Psi} \in \operatorname{seq} \bar{\mathbb{N}}[B]$ where, ~~with $\psi = fn(\underline{\Psi})$,~~ $\psi(z)$ is constructed for each $z \in B$ by removing from $\phi(z)$ ~~those terms present in $\xi(z)$~~ all terms having one or more counterparts in $\xi(z)$, and retaining those terms in $\psi(z)$ having no such counterparts in their order of appearance in $\phi(z)$. ~~$\underline{\Phi}\setminus\Xi$ may be void over certain subdomains of~~
~~defined as a proper $\bar{\mathbb{N}}$-sequence mapping when~~

The sequence difference ~~Φ~~ of $\underline{\Phi}$ and Ξ, ~~written as $\langle \underline{\Phi} - \Xi \rangle$,~~ is $\underline{\Psi} \in \operatorname{seq}\bar{\mathbb{N}}[B]$ where ~~with $\psi = fn(\underline{\Psi})$,~~ $\psi(z)$ is constructed for each $z \in B$ by removing from $\phi(z)$ those terms $\phi(z|\nu)$ having a counterpart $\xi(z|\nu)$ ~~in ξ with equal index~~ $= \phi(z|\nu)$ with equal index. ~~Again $\langle \underline{\Phi} - \Xi \rangle$ may be void over certain subdomains of B,~~

(Thus $\underline{\Phi}$ and Ξ being the constant $\bar{\mathbb{N}}$-sequence mappings $\langle 0,1,2,0,1,3 \rangle$ and $\langle 0,1 \rangle$, $\underline{\Phi} \setminus \Xi$ is $\langle 2,3 \rangle$ and $\langle \underline{\Phi} - \Xi \rangle$ is $\langle 2,0,1,3 \rangle$.)

(defined as a proper $\bar{\mathbb{N}}$-sequence mapping when, for each $z \in B$, $\phi(z)$ is neither $\xi(z)$ nor $\xi(z)$ followed by another sequence of numbers in $\bar{\mathbb{N}}$, is written as $\langle \underline{\Phi} - \Xi \rangle$. It

b) Direct composition

~~With $\Xi \in \mathrm{seq}\,\overline{\mathbb{N}}[\le m \mid \mathbb{B}, n]$~~, The direct composition $\phi[\Xi]$ is

~~$\Psi \in \mathrm{seq}\,\overline{\mathbb{N}}[\mathbb{B}, n]$~~ defined by setting, for each $z \in \mathbb{B}$,

$$\psi(z|\omega) := \phi(z \mid \xi(z|\omega)) \qquad \langle \omega := [\![n(z)]\!] \text{ base } \xi(z) \rangle$$

Compositions of the form $\phi[[i, j]]$ for suitably ~~declared~~ $i, j \in \mathrm{fn}\{\overline{\mathbb{N}}[\mathbb{B}]\}$ are written as $\phi[i, j]$. The notations $\phi[i, j), \ldots, \phi[j], \ldots, \phi((j))$ have similar meanings.

(To avoid possible confusion it is remarked that for suitably declared Θ with $\theta = \mathrm{fn}(\Theta)$, ord $\theta[\Xi]$ is $\phi[\Xi]$ where $\Phi = \mathrm{ord}(\Theta)$. ord $(\theta[\Xi])$ is ord(Φ) where $\Phi := \Theta[\Xi]$. The same remark relates ~~applies~~, mutatis mutandis, to other unitary operations.)

c) The composition product,

~~With $n \le m \langle \mathbb{B} \rangle$~~, The composition product $\Xi\Phi$ is $\phi[\mathrm{ord}(\Xi)]$.

b) Direct composition

With $\Xi \in \mathrm{seq}\,\bar{\mathbb{N}}[\leq m|\mathbb{B}], n]$, the direct composition $\Phi[\Xi]$ is $\Psi \in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}, n]$ defined by setting, for each $z \in \mathbb{B}$,

$$\psi(z|\omega) := \phi(z|\xi(z|\omega)) \qquad \langle \omega := [n(z)] \rangle$$

~~Compositions of the form $\phi[[i,j]]$ for suitably defined $i,j \in \mathrm{fn}\{\bar{\mathbb{N}}[\mathbb{B}]\}$ are written as $\phi[i,j]$. The notations $\phi[i,j)$, ... $\phi[j)$, ..., $\phi((j))$ have similar meanings.~~

c) The composition product

With $n \leq m$, the composition product $\Xi\Phi$ is Ψ ~~$\in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}, n]$~~

~~defined by setting, for each $z \in \mathbb{B}$,~~

$$\cancel{\psi(z|\omega) := \phi(z|\mathrm{ord}\,\xi(z|\omega))}$$

$$\cancel{\Phi[\mathrm{ord}(\Xi)]}$$

vii) ~~vi)~~ Binary algebraic operations over $\bar{\mathbb{N}}$-sequence mappings $\Xi \in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}, n]$ and $\Phi \in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}, m]$

~~Let $\Phi \in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}, m]$ with $\phi := \mathrm{fn}(\Phi)$. In the following $\psi := \mathrm{fn}(\Psi)$.~~

a) Addition, subtraction and multiplication of $\bar{\mathbb{N}}$-sequence mappings

Define $r \in \mathrm{fn}\{\bar{\mathbb{N}}[\mathbb{B}]\}$ by setting $r(z) := \min\{m(z), n(z)\}$. The algebraic sum of Φ and Ξ, written as $\Phi + \Xi$, is $\Psi \in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}, r]$

where, for each $z \in \mathbb{B}$,

$$\psi(z|\omega) := \phi(z|\omega) + \xi(z|\omega) \qquad \langle \omega := [r(z)] \rangle$$

The algebraic difference and product, $\Phi - \Xi$ and $\Phi \times \Xi$, are defined similarly.

In cases in which a ~~the~~ sequence sum $\langle \Phi + \Xi \rangle$ and ~~the~~ an algebraic sum $\Phi + \Xi$ occur in the same expression, the latter is enclosed in parentheses: $(\Phi + \Xi)$. ~~to avoid confusion~~ The difference $\Phi - \Xi$ is treated in the same way.

b) Algebraic operations involving $\bar{\mathbb{R}}$-mappings $r : B \to \bar{\mathbb{R}}$ and $\bar{\mathbb{R}}$-sequence mappings

The algebraic sum of Ξ and $\cancel{r \in fn\{\bar{\mathbb{R}}[B]\}}$, is written as $\Xi + r$, is $\bar{\Psi} \in seq\, \bar{\mathbb{R}}[B, n]$ where, for each $z \in B$,

$$\psi(z|\omega) = \xi(z|\omega) + r(z) \qquad \langle \omega := [n(z)]_\nu \text{ base } \xi(z) \rangle$$

The algebraic difference and product, $\Xi - r$ and Ξr, are defined similarly.

viii)/ix)/x) Ordering and equality relationships over $\bar{\mathbb{N}}$-sequence mappings.

a) ~~Component~~ ~~Let $\Phi \in seq\,\bar{\mathbb{N}}[B,m]$ with $\phi = fn(\Phi)$, $\Theta \in seq\,\bar{\mathbb{N}}[B,n]$ with $\phi = fn(\Phi)$, $\theta = fn(\Theta)$.~~

a) Setwise ~~inclusion~~ orderings

The notation $\Theta \leq \Phi$ (comp) indicates that for each $z \in B$

$$\{\theta(z)\} \leq \{\phi(z)\}$$

The notations $\Theta = \Phi$ and $\Theta \subset \Phi$ indicate that $\Theta \leq \Phi$ ~~which~~ while $\Phi \leq \Theta$ in the first case and $\Phi \nleq \Theta$ in the second.

b) Subsequence ~~inclusion~~ ordering

The notation $\Theta \leq \Phi$ (subseq.) indicates that ~~for each $z \in B$~~ ~~for each $z \in B$~~ a $T \in unit\,\bar{\mathbb{N}}[B]n$ ~~with~~ exists such that

$$\Theta = \Phi[T]$$

~~$\theta(z|\tau(z,\omega)) = \phi(z|\tau(z,\omega))$ $\langle \omega := [n(z)] \rangle$~~

The notations $\Xi = \Phi$ and $\Xi < \Phi$ indicate that $\Xi \leq \Phi$ while ~~the further relationship $\Phi \leq \Xi$ is and is not satisfied satisfied in the first case but not in the s~~ $\Phi \leq \Xi$ in the first case and $\Phi \nleq \Xi$ in the secon

~~c) The notation $\Xi \leq \Phi$~~

c) Product ordering

The notation $\Theta \leq \Phi$ (prod.) indicates that

~~$unit\,\Phi[spect(\Theta$~~

$$unit\,\Theta[spect(\Theta)] = unit\,\Phi[spect(\Theta)]$$

d) Perpendicularity

The notation $\Theta \perp \Phi$ indicates that for each $z \in B$ the sets $\{\theta(z)\}$ and $\{\phi(z)\}$ are disjoint.

e) Ordering according to magnitude

The notation $\Theta \lessdot \Phi$ (magnitude) indicates that for each $z \in B$

$$\theta(z|\omega) < \phi(z|\nu) \quad \langle \omega := [n(z)],\ \nu := [m(z)] \rangle$$

viii) Equivalence classes of $\overline{\aleph}$-sequence mappings

With $\Xi \in \mathrm{seq}\,\overline{\aleph}[B,n]$,

a) ord equiv(Ξ) is the class of $\overline{\aleph}$-sequence mappings $\underline{\Phi} \in \mathrm{seq}\,\overline{\aleph}[B,n]$ ~~$\underline{\Phi} \in \mathrm{seq}\,\overline{\aleph}[B,n]$~~ for which,

$$\mathrm{ord}(\underline{\Phi}) = \mathrm{ord}(\Xi)$$

b) comp equiv(Ξ) is the class of $\overline{\aleph}$-sequence mappings

$\underline{\Phi} \in \mathrm{seq}\,\overline{\aleph}[B]$ for which

~~ord equiv(Ξ) and comp equiv(Ξ) are the classes of $\overline{\aleph}$-sequence mappings~~

$$\underline{\Phi} \equiv \Xi .$$

~~4] Void sequences and sequence mappings~~

~~The existence of void $\overline{\aleph}$-sequence mappings is postulated~~

~~Let $B = B' \cup B''$, where B', B'' are nonvoid and disjoint~~

~~The $\overline{\aleph}$-sequence mapping $\Xi \in \mathrm{seq}\,\overline{\aleph}[B']$ may be extended as an improper $\overline{\aleph}$-sequence mapping over B by imposing the condition that Ξ should be void over B''.~~

~~Such extension is used to define special $\overline{\aleph}$-sequence mappings and differences of $\overline{\aleph}$-sequence mappings under conditions weaker than those given above.~~

~~a) Let $i, j \in fn\{\overline{\aleph}[B]\}$ with $i \leq j$ $\langle B' \rangle$ and $j < i$ $\langle B'' \rangle$. $[i, j]$ is defined as an $\overline{\aleph}$-sequence mapping over B', as~~

4] Improper $\bar{N}$-sequence mappings

~~In the preceding section~~

The preceding section deals exclusively with proper $\bar{N}$-sequence mappings. ~~Special conditions were~~ In certain cases it was necessary to impose special conditions to ensure that the $\bar{N}$-sequence mappings defined are proper (as, for example, in the case of the special mapping $[i, j]$) and that those resulting from binary operations are also proper (as in the definitions of the set and sequence differences $\bar{\Phi} \setminus \Xi$ and $\bar{\Phi} - \Xi$).

~~The exposition of later theory is greatly simplified by the introduction of improper $\bar{N}$-sequence mappings, together with appropriate rules for operating upon them. To consider give an example of such simplification, consider the special mapping [...] let $i, j, k \in fn\{\bar{N}[B]\}$ with $i \leq k \langle B \rangle$, so that $[i, k]$ is well defined in $seq\bar{N}[B]$. When $i \leq j \leq k$~~

In later theory, compound binary structures, having the form of a sequence sum of $\bar{\Phi}$ and ~~such as $\langle \bar{\Phi} + \Xi \rangle$ that~~ ~~and many others~~, are encountered. Over a prescribed ~~certain~~ subdomain of B, this structure is to be interpreted as $\langle \bar{\Phi} + \Xi \rangle$ defined as above as a proper $\bar{N}$-sequence sum, over another subdomain as $\bar{\Phi}$ and over yet another as Ξ, the three domains being disjoint, their union being B. ~~The structure defined in this way is~~ In this way a proper $\bar{N}$-sequence mapping is defined. The exposition of treating this case ~~and many others~~ is greatly simplified by ~~defining~~ $\bar{\Phi}$ and Ξ as improper $\bar{N}$-sequence mappings that are ~~null~~ void over ~~suitable~~ appropriate subdomains of B and defining sequence summation over improper $\bar{\mathbb{N}}$-sequence mappings, so that the structure just defined & may be expressed as $\langle \bar{\Phi} + \Xi \rangle$ in an extended sense

Other compound binary structures are encountered; and their treatment may be simplified in the same way.

i) Void $\bar{\mathbb{N}}$-sequence mappings

The existence of void $\bar{\mathbb{N}}$-sequence mappings is postulated.

ii) Improper $\bar{\mathbb{N}}$-sequence mappings

Let $\mathbb{B}' \subseteq \mathbb{B}$. The nonvoid $\bar{\mathbb{N}}$-sequence mapping $\Xi \in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}']$ is extended as an improper $\bar{\mathbb{N}}$-sequence mapping over $\mathbb{B}$ by imposing the condition that Ξ should be void over $\mathbb{B} \setminus \mathbb{B}'$.

iii) Special $\bar{\mathbb{N}}$-sequence mappings

Let ~~$\mathbb{B}', \mathbb{B}'' \subseteq \mathbb{B}$ be disjoint~~. $\mathbb{B} \equiv \mathbb{B}' \cup \mathbb{B}''$ where $\mathbb{B}', \mathbb{B}''$ are disjoint and

~~Let~~ $i, j \in \mathrm{fn}\{\bar{\mathbb{N}}[\mathbb{B}]\}$, with $i \leq j \, \langle \mathbb{B}' \rangle$ and $j < i \, \langle \mathbb{B}'' \rangle$.

$[i, j]$ is defined as an $\bar{\mathbb{N}}$-sequence mapping over $\mathbb{B}'$, as a void $\bar{\mathbb{N}}$-sequence mapping over $\mathbb{B}''$ and in this way as an improper $\bar{\mathbb{N}}$-sequence mapping over $\mathbb{B}$. $[i, j), \ldots, ((j))$ are treated in the same way.

iv) Unitary operations upon improper $\bar{\mathbb{N}}$-sequence mappings

Ξ being a proper $\bar{\mathbb{N}}$-sequence mapping over $\mathbb{B}'$ and void over $\mathbb{B}''$, ~~ord~~ base(Ξ) is defined over $\mathbb{B}'$ and $\mathbb{B}''$, and hence over $\mathbb{B}$, in the same way. $\overset{\mathrm{ord}(\Xi)}{\nu}\tilde{\Xi}$, $\mathrm{unit}(\Xi)$, Ξ^T, $\mathrm{spect}(\Xi)$ and $\mathrm{inv}(\Xi)$ are similarly extended as improper $\bar{\mathbb{N}}$-sequence mappings. ~~in the~~

v) Binary sequence operations over ~~$\bar{\mathbb{N}}$-sequence~~ improper $\bar{\mathbb{N}}$-sequence mappings

~~a) Sequence sums and differences~~

Let $\hat{\mathbb{B}} \subseteq \mathbb{B}$ and ~~Ξ be a Φ~~ $\Xi \in \mathrm{seq}\,\bar{\mathbb{N}}[\hat{\mathbb{B}}]$ be extended over $\mathbb{B}$ by taking Ξ ~~to~~ be void over $\mathbb{B} \setminus \hat{\mathbb{B}}$. Let $\tilde{\mathbb{B}}^{\subseteq \mathbb{B}}$ and Φ be similarly related.

a) Sequence sums and differences

Over $\hat{\mathbb{B}} \cap \tilde{\mathbb{B}}$, $\langle \bar{\Phi} + \Xi \rangle$ is defined directly as an $\bar{\aleph}$-sequence sum, over $\tilde{\mathbb{B}} \setminus \hat{\mathbb{B}}$ to be $\bar{\Phi}$, over $\hat{\mathbb{B}} \setminus \tilde{\mathbb{B}}$ to be Ξ and over $\mathbb{B} \setminus \{\hat{\mathbb{B}} \cup \tilde{\mathbb{B}}\}$ to be void.

(In the above it may occur that $\hat{\mathbb{B}}, \tilde{\mathbb{B}}$ are nonvoid and $\mathbb{B} \equiv \hat{\mathbb{B}} \cup \tilde{\mathbb{B}}$. In this case the sequence sum of two improper $\bar{\aleph}$-sequence mappings is a proper $\bar{\aleph}$-sequence mapping.)

related. ~~Direct~~

a) Sequence sums and differences

~~Over $\hat{\mathbb{B}}\cap\tilde{\mathbb{B}}$, $\langle\underline{\Phi}+\Xi\rangle$ is defined directly as an seq $\bar{\mathbb{N}}$-sequence sum, over $\tilde{\mathbb{B}}\setminus\hat{\mathbb{B}}$ as $\underline{\Phi}$, over to be $\underline{\Phi}$, over $\hat{\mathbb{B}}\setminus\tilde{\mathbb{B}}$ to be Ξ and over $\mathbb{B}\setminus\{\hat{\mathbb{B}}\cup\tilde{\mathbb{B}}\}$ to be void.~~

Let $\xi := fn(\Xi)$ and $\phi := fn(\underline{\Phi})$. ~~Define let $\mathbb{B}'\cup\mathbb{B}'' = \hat{\mathbb{B}}\cap\tilde{\mathbb{B}}$, $\mathbb{B}'$ and $\mathbb{B}''$ being disjoint~~, Let the disjoint domains $\mathbb{B}', \mathbb{B}''$ be such that $\mathbb{B}'\cup\mathbb{B}'' = \hat{\mathbb{B}}\cap\tilde{\mathbb{B}}$, with $\{\phi\}\not\subseteq\{\xi\}\langle\mathbb{B}'\rangle$ and $\{\phi\}\subseteq\{\xi\}\langle\mathbb{B}''\rangle$. The set difference $\underline{\Phi}\setminus\Xi$ is defined directly as an $\bar{\mathbb{N}}$-sequence mapping set difference over $\mathbb{B}'$, to be void over $\mathbb{B}''$, to be $\underline{\Phi}$ over $\tilde{\mathbb{B}}\setminus\hat{\mathbb{B}}$ and to be void over $\mathbb{B}\setminus\tilde{\mathbb{B}}$.

(In the above it may occur that $\hat{\mathbb{B}}=\tilde{\mathbb{B}}=\mathbb{B}$, so that $\underline{\Phi}$ and Ξ are proper $\bar{\mathbb{N}}$-sequence mappings. In this case, $\underline{\Phi}\setminus\Xi$ is defined directly as an $\bar{\mathbb{N}}$-sequence mapping over $\mathbb{B}'$ and to be void over $\mathbb{B}''$, where now $\mathbb{B}'\cup\mathbb{B}''=\mathbb{B}$. If $\mathbb{B}''$ is nonvoid, the set difference of two proper $\bar{\mathbb{N}}$-sequence mappings is defined as an improper $\bar{\mathbb{N}}$-sequence mapping.)

The sequence difference $\underline{\Phi}-\Xi$ is treated in the same way.

b) Direct composition and the composition product

~~Over $\hat{\mathbb{B}}\cap\tilde{\mathbb{B}}$, $\phi[\Xi]$ is defined directly by direct composition~~

Let $\hat{\mathbb{B}}\subseteq\tilde{\mathbb{B}}$ ~~with $\underline{\Phi}\in seq\,\bar{\mathbb{N}}[\tilde{\mathbb{B}},m]$~~, $m\in fn\{\bar{\mathbb{N}}|\tilde{\mathbb{B}}\}$, $n\in fn\{\bar{\mathbb{N}}|\hat{\mathbb{B}}\}$ with $\underline{\Phi}\in seq\,\bar{\mathbb{N}}[\tilde{\mathbb{B}},m]$, $\Xi\in seq\,\bar{\mathbb{N}}[\le m|\hat{\mathbb{B}},n]$.

Over $\hat{\mathbb{B}}$, $\phi[\Xi]$ is defined ~~directly~~ by direct composition and, over $\mathbb{B}\setminus\hat{\mathbb{B}}$, to be void.

The composition product is treated in the same way

vi) Binary algebraic operations over improper $\overline{\mathbb{R}}$-sequence mappings

a) Let $\hat{\mathbb{B}}, \tilde{\mathbb{B}}, \Phi$ and Ξ be as described at the commencement of the preceding clause.

~~the algebraic sum~~ $\cancel{\Phi+\Xi}$ ~~of~~ $\cancel{\Phi}$ ~~and~~ $\cancel{\Xi}$ ~~is defined to be the~~ $\cancel{\Phi+\Xi}$ ~~the~~

$\Phi+\Xi$ is defined directly to be the algebraic sum of Φ and Ξ over $\hat{\mathbb{B}}\cap\tilde{\mathbb{B}}$ and to be void over $\mathbb{B}\setminus\{\hat{\mathbb{B}}\cap\tilde{\mathbb{B}}\}$.

~~The also~~ $\Phi-\Xi$ and $\Phi\times\Xi$ are defined similarly

b) Let $\hat{\mathbb{B}}, \tilde{\mathbb{B}}, \Xi$ be as above and $r\in fn\{\overline{\mathbb{R}}[\tilde{\mathbb{B}}]\}$.

$\Xi+r$ is defined to be the algebraic sum of Ξ and r over $\hat{\mathbb{B}}\cap\tilde{\mathbb{B}}$ and to be void over $\mathbb{B}\setminus\{\hat{\mathbb{B}}\cap\tilde{\mathbb{B}}\}$. $\Xi-r$ and Ξr are treated similarly.

In the following, where relevant, $\Xi, \Theta, \Phi \in \mathrm{seq}\,\overline{\mathbb{N}}[\mathbb{B}]$, $\xi := \mathrm{fn}(\Xi), \dots,$ $\phi := \mathrm{fn}(\Theta)$ and $m, n \in \mathrm{fn}\{\overline{\mathbb{N}}[\mathbb{B}]\}$

1] Properties of the base

i) Inclusion of the spectrum and order

$$\mathrm{spect}(\Xi), \mathrm{ord}(\Xi) \leq \mathrm{base}(\Xi)$$

ii) Invariance of the base under transformation

a) $\mathrm{base}(\Xi) = \mathrm{base}(\mathrm{ord}(\Xi)) = \mathrm{base}(\tilde{\Xi}) = \mathrm{base}(\mathrm{asc}(\Xi))$

b) If $\Xi \in \mathrm{ns}\,\overline{\mathbb{N}}[\mathbb{B}]$ then $\mathrm{base}(\Xi) = \mathrm{base}(\Xi^T)$

2] Relationships between $\overline{\mathbb{N}}$-sequence mappings

i) Transitivity

The relationship $\Theta \leq \Phi$ (comp) is transitive: if $\Xi \leq \Theta$, $\Theta \leq \Phi$ (comp) then $\Xi \leq \Phi$ (comp)

The same holds true with regard to the relationships

$$\Theta \leq \Phi \text{ (subseq)},\ \Theta \leq \Phi \text{ (prod)},\ \Theta \leq \Phi \text{ (magnitude)}$$

ii) Inclusion

If $\Theta \leq \Phi$ (subseq) then $\Theta \leq \Phi$ (comp)

iii) Invariance under transformation

a) The three conditions

$$\Xi, \mathrm{asc}(\Xi), \tilde{\Xi} \leq \mathrm{base}(\Phi)$$

are equivalent

If $\Xi \in \mathrm{ns}\,\overline{\mathbb{N}}[\mathbb{B}]$ then $\Xi^T \leq \mathrm{base}(\Phi)$ if and only if $\Xi \leq \mathrm{base}(\Phi)$

b) The ~~four~~ three conditions

$$\Theta \leq \Phi,\ \mathrm{asc}(\Theta) \leq \mathrm{asc}(\Phi),\ \tilde{\Theta} \leq \tilde{\Phi} \text{ (comp)}$$

are equivalent

If $\Theta, \Phi \in \mathrm{ns}\,\overline{\mathbb{N}}[\mathbb{B}]$ then $\Theta^T \leq \Phi^T$ (comp) if and only if $\Theta \leq \Phi$ (comp)

3] Characterisations and properties of $\overline{\mathbb{N}}$-sequence mappings.

ia) $\Xi \in \mathrm{ns}\,\overline{\mathbb{N}}[\mathbb{B}]$ if and only if $\mathrm{spect}(\Xi) = \mathrm{base}(\Xi)$.

[illegible] Ξ, $\mathrm{ord}(\Xi)$, $\tilde{\Xi}$, $\mathrm{asc}(\Xi)$, $\Xi^T \in \mathrm{ns}\,\overline{\mathbb{N}}[\mathbb{B}]$ are equivalent.

iia) If $\Xi \in \overset{\text{base}}{\cancel{\text{unit}}}\,\overline{\aleph}[\mathbb{B}]$ $\overset{\text{then}}{\text{if}}$ ~~and only if~~ $\overset{\text{asc}}{\cancel{\text{unit}}}(\Xi) = \Xi$

b) $\overset{\text{asc}}{\cancel{\text{unit}}}\{\overset{\text{ns}}{\cancel{\text{sq}}}\,\overline{\aleph}[\mathbb{B}, n]\} = \text{unit}\,\overline{\aleph}[\mathbb{B}, n]$

iii) $\Xi \in \text{ord}\,\overline{\aleph}[\mathbb{B}]$

a) if and only if $\text{ord}(\Xi) = \Xi$

b) if and only if

$$\overset{\text{asc}}{\cancel{\text{unit}}}\,\xi\,[\text{spect}(\Xi)] = \text{spect}(\Xi)$$

iv) If either $\Xi \in \text{unit}\,\overline{\aleph}[\mathbb{B}]$ or $\tilde{\Xi} \in \text{unit}\,\overline{\aleph}[\mathbb{B}]$ then $\Xi^T = \Xi$.

v) $\text{perm}\,\overline{\aleph}[\mathbb{B}, n] \equiv \text{ord}\,\overline{\aleph}[\mathbb{B}, n] \cap \text{ns}\,\overline{\aleph}[\mathbb{B}, n]$

A[3] Unitary operations

i) Permanence

a) $\text{base}(\text{base}(\Xi)) = \text{base}(\Xi)$

b) $\overset{}{\underset{\text{asc}}{\cancel{\text{unit}}}}(\underset{\text{asc}}{\cancel{\text{unit}}}(\Xi)) = \underset{\text{asc}}{\cancel{\text{unit}}}(\Xi)$

c) $\text{ord}(\text{ord}(\Xi)) = \text{ord}(\Xi)$

ii) Reflection

a) $\bar{\Phi} = \tilde{\Xi}$ if and only if $\Xi = \tilde{\bar{\Phi}}$

b) $\bar{\Phi} = \Xi^T$ if and only if $\Xi = \bar{\Phi}^T$

iii) Invariance of order under transformation

$$\widetilde{\text{ord}(\Xi)} = \text{ord}(\tilde{\Xi})$$

$$\text{ord}(\Xi)^T = \text{ord}(\Xi^T)$$

iv) Inversion and transposition

~~For each $z \in \mathbb{B}$~~ Let $\Xi \in \overset{\text{ns}}{\cancel{\text{sq}}}\,\overline{\aleph}[\mathbb{B}]$. For each $z \in \mathbb{B}$

a) $$\text{inv}\,\xi(z \mid \xi(z)) = \cancel{\mathbb{B}(z)}\ \text{base}\,\xi(z)$$

b) $$\text{inv}\,\xi(z \mid \xi^T(z)) = \text{ord}\,\xi(z)$$

~~c) $\text{inv}\,\xi(z \mid [n(z)]) = \xi^T(z)$~~

c) (if $\Xi \in \text{perm}\,\bar{\mathbb{N}}[B]$)

$$\text{inv}\,\xi\,(z \mid [n(\ldots)]) = \xi^{T}(z) \quad \text{base}\ \xi(B)$$

4] Binary operations

ii) Distributive properties of ~~the~~ direct composition

Let ~~$\Phi \in \text{seq}\,\bar{\mathbb{N}}[B, r]$ and $\Theta, \Xi \in \text{seq}\,\bar{\mathbb{N}}[\le r \mid B]$~~. $\Xi, \Theta \le \text{base}(\Phi)$

$$\Phi[\langle \Theta + \Xi \rangle] = \langle \Phi[\Theta] + \Phi[\Xi] \rangle$$

iii) Associative properties of $\bar{\mathbb{N}}$-sequence operations

b) Direct composition $\Xi \le \text{base}(\Theta)$ and $\Theta \le \text{base}(\Phi)$

Let ~~$\Phi \in \text{seq}\,\bar{\mathbb{N}}[B, r]$, $\Theta \in \text{seq}\,\bar{\mathbb{N}}[\le r \mid B^{\xi}_{\xi}, m]$ and $\Xi \in \text{seq}\,\bar{\mathbb{N}}[\le m]$~~

Set $\Psi := \Theta[\Xi]$ and $\Lambda := \Phi[\Theta]$.

mixed prods ?

$\Phi[\Xi\Theta] = \Xi\Phi[\Theta]$

$$\Phi[\Psi] = \Lambda[\Xi]$$

The compound composition $\Phi[\Theta[\Xi]]$ is uniquely defined.

a) ~~Addition~~ $\bar{\mathbb{N}}$-sequence mapping addition

$$\langle \Phi + \langle \Theta + \Xi \rangle \rangle = \langle \langle \Phi + \Theta \rangle + \Xi \rangle$$

The $\bar{\mathbb{N}}$-sequence mapping sum $\langle \Phi + \Theta + \Xi \rangle$ is uniquely defined

c) ~~Multiplication~~ The composition product $\text{ord}(\Xi) \le \text{base}(\Theta)$ and $\text{ord}(\Xi\Theta) \le \text{base}(\Phi)$

Let ~~$\Phi \in \text{seq}\,\bar{\mathbb{N}}[B, r]$, $\Theta \in \text{seq}\,\bar{\mathbb{N}}[B, m]$ and $\Xi \in \text{seq}\,\bar{\mathbb{N}}[B, n]$~~

~~Let $B', B'', B''' \le B$ be disjoint with $B' \cup B'' \cup B''' = B$. Let~~

~~$n \le m, r\langle B' \rangle$;~~ for each $z \in B''$ let all $\xi(z \mid \omega)$ ~~$\langle \omega := [n(z)] \rangle$ be~~

equal; ~~for~~ ~~let $n \le m\langle B''' \rangle$ and for each~~

i) Existence conditions for direct composition and the formation of the composite prod.

a) $\Phi[\Xi]$ is well defined if and only if $\Xi \le \text{base}(\Phi)$

b) ... well defined if and only if $\text{ord}(\Xi) \le \text{base}(\Phi)$

~~with $\text{ord}(\Xi) \in \text{seq}\,\bar{\aleph}[\leq m \mid \mathbb{B}]$ and $\text{ord}(\Theta[\text{ord}(\Xi)]) \in \text{seq}\,\bar{\aleph}[\leq r \mid \mathbb{B}]$~~

(these conditions are satisfied, in particular, ~~if $n \leq m, r \langle \mathbb{B} \rangle$~~ if $\Theta \in \text{seq}\,\bar{\aleph}[\mathbb{B}, n]$, $\Theta \in \text{seq}\,\bar{\aleph}[\mathbb{B}, m]$, $\Phi \in \text{seq}\,\bar{\aleph}[\mathbb{B}, r]$ with $n \leq m \leq r \langle \mathbb{B} \rangle$

$$\Xi(\Theta\Phi) = (\Xi\Theta)\Phi$$

The ~~$\bar{\aleph}$-sequence map~~ composition product $\Xi\Theta\Phi$ is uniquely defined.

~~d) Algebraic operations~~

~~Let $\Phi \in \text{seq}$~~

~~$$\Phi + (\Theta + \Xi) = (\Phi + \Theta) + \Xi$$~~

~~and~~

~~$$\Phi \times (\Theta \times \Xi) =$$~~

iv) Associative, commutative and distributive properties of algebraic operations

$$\Phi + \Theta = \Theta + \Phi \;,\; \Phi + (\Theta + \Xi) = (\Phi + \Theta) + \Xi$$

$$\Phi \times \Theta = \Theta \times \Phi \;,\; \Phi \times (\Theta \times \Xi) = (\Phi \times \Theta) \times \Xi$$

$$\Phi \times (\Theta + \Xi) = \Phi \times \Theta + \Phi \times \Xi$$

Similar relationships, such as $\Phi + r = r + \Phi$ where $r \in \text{fn}\{\bar{\aleph}[\mathbb{B}]\}$, hold with regard to mixed algebraic operations involving $\bar{\aleph}$-sequence mappings and $\bar{\aleph}$-mappings.

v) Partitioned transposition and multiplication

Let $\Xi \in \text{seq}\,\bar{\aleph}[\mathbb{B}, n]$ ~~and $m < n \langle \mathbb{B} \rangle$~~ $\text{ord}(\Xi) \leq \text{base}(\Phi)$ ~~$\Phi \in \text{seq}\,\bar{\aleph}[\mathbb{B}, r]$ with $\text{ord}(\Xi) \in \text{seq}\,\bar{\aleph}[\leq n \mid \mathbb{B}]$~~ and $m < n \langle \mathbb{B} \rangle$

a) If $\Xi[m] < \Xi(m, n]$ (magnitude) then

$$\text{ord}(\Xi) = \langle \text{ord}(\Xi[m]) + (m + 1 + \text{ord}(\Xi(m, n])) \rangle$$

~~while if $\Xi(m,n]$~~

$$\Xi^T = \langle \Xi[m]^T + \Xi(m,n]^T \rangle$$

$$\Xi\bar{\Phi} = \langle \Xi[m]\bar{\Phi}[m] + \Xi(m,n]\bar{\Phi}(m,n] \rangle$$

b) If $\Xi(m,n] < \Xi[m]$ (magnitude) then

$$\mathrm{ord}(\Xi) = \langle \mathrm{ord}(\Xi(m,n]) + (n-m+\mathrm{ord}(\Xi[m])) \rangle$$

$$\Xi^T = \langle \cancel{\Xi(m,n]^T + \Xi[m]^T} \rangle \langle \Xi[n-m)^T + \Xi[n-m,n]^T \rangle$$

$$\cancel{\Xi\bar{\Phi} = \langle \Xi(m,n]\bar{\Phi}(m,n]}$$

$$\Xi\bar{\Phi} = \langle \Xi[n-m)\bar{\Phi}[n-m) + \cancel{\Xi[m]\bar{\Phi}[m]} \; \Xi[n-m,n]\bar{\Phi}[n-m,n] \rangle$$

~~5) Multiplicative decompositions~~

vi) Reversal of ~~subsequences and products~~ direct composition and composition products

~~a) Let $\bar{\Phi} \in \mathrm{seq}\,\bar{\mathbb{R}}[\mathbb{B}, m]$, and $\mathrm{ord}(\Xi) \in \mathrm{seq}\,\bar{\mathbb{R}}[\leq m \mid \mathbb{B}]$.~~

$$\cancel{\widetilde{\bar{\Phi}[\Xi]} = \bar{\Phi}[\tilde{\Xi}]}$$

a) Let ~~$\Xi \in \mathrm{seq}\,\bar{\mathbb{R}}[\leq m \mid \mathbb{B}]$~~ $\Xi \subseteq \mathrm{base}(\bar{\Phi})$

$$\widetilde{\bar{\Phi}[\Xi]} = \bar{\Phi}[\tilde{\Xi}]$$

b) Let ~~$\mathrm{ord}(\Xi) \in \mathrm{seq}\,\bar{\mathbb{R}}[\leq m \mid \mathbb{B}]$~~ $\mathrm{ord}(\Xi) \subseteq \mathrm{base}(\bar{\Phi})$

$$\widetilde{\Xi\bar{\Phi}} = \tilde{\Xi}\bar{\Phi}$$

vii) Transposition of composition products

Let ~~$\bar{\Phi} \in \mathrm{seq}\,\bar{\mathbb{R}}[\mathbb{B}, m]$ and $\mathrm{ord}(\Xi) \in \mathrm{seq}\,\bar{\mathbb{R}}[\leq m \mid \mathbb{B}]$~~ $\mathrm{ord}(\Xi) \subseteq \mathrm{base}(\bar{\Phi})$

a) $$(\Xi\bar{\Phi})^T = \bar{\Phi}^T \Xi^T \mathrm{unit}(\bar{\Phi})$$

b) If $\overset{asc}{\text{unit}}(\Xi) = \overset{asc}{\text{unit}}(\Phi)$ (this condition is satisfied when $\Xi, \Phi \in ns\bar{N}[B]$ and $\Xi \equiv \Phi$)

$$(\Xi\Phi)^T = \Phi^T \Xi^T$$

viii) Preservation of nonsingularity.

Let $\Xi,\Phi \in ns\bar{N}[B], n \in \Delta$, ~~$\Xi \subseteq \text{base}(\Phi)$~~ ~~$\Xi \in ns\bar{N}[\ldots B]$~~

a) if $\Xi \subseteq \text{base}(\Phi)$,

$$\Phi[\Xi] \in ns\bar{N}[B]$$

b) If $\text{ord}(\Xi) \subseteq \text{base}(\Phi) \leftarrow \Xi\Phi \in ns\bar{N}[B]$

6] [5] Multiplicative Decompositions

a) $\Xi[\text{base}(\Xi)] = \Xi$.

b) $\Xi = \text{ord}(\Xi)\,\overset{asc}{\text{unit}}(\Xi)$

$$\{\text{ord equiv}(\Xi)\}\{\overset{asc}{\text{unit}}(\Xi)\}$$

b) Let $\Xi \in \text{seq}\bar{N}[B,n]$.

$$\Xi = \{\text{unit}\bar{N}[B,n]\}\Xi$$

7] [6] Solubility of composition and composition product relationships

a) $\Xi \in \text{seq}\bar{N}[B]$ exists such that $\Phi[\Xi] = \Theta$ if and only if

$$\Theta \leq \Phi \ (\text{comp})$$ false if entry independent

If this condition is satisfied and $\Phi \in ns\bar{N}[B]$ then Ξ is uniquely determined by this relationship

b) $\Xi \in \text{seq}\bar{N}[B]$ exists such that $\Xi\Phi = \Theta$ if and only if

$$\Theta \leq \Phi \ (\text{prod})$$

Let this condition be satisfied and $\overset{\Theta}{\Phi} \in \text{seq}\bar{N}[B, n]$.

$$\text{pre}\left\{\frac{\Theta}{\Phi}\middle| B\right\} = \{\text{unit}\bar{N}[B,n]\}\Xi\{\text{unit}\bar{N}[B,n]\}$$
$$= \text{unit}\bar{N}[B,n]\,\text{pre}\left\{\frac{\Theta}{\Phi}\middle| B\right\}\text{unit}\bar{N}[B,n]$$

§ ∇] Groups of $\bar{\mathbb{N}}$-sequence mappings

Let $\Theta \in ns\bar{\mathbb{R}}[\mathbb{B}]$. The members of comp equiv(Θ) form a group in which the unit member are ~~unit~~(Θ) and the inverse ~~of Θ^{m} is Θ~~ of Ξ is Ξ^{-1}.

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replace base by $|\ |$; thus: $\Xi \le |\bar{\Phi}|$ for existence of $\Xi\bar{\Phi}$

$|\Theta| \le |\Delta|$?

$\Xi \angle \Theta$ (magnitude) $|\Xi| = |\bar{\Phi}|\ |\Theta| = |\Delta| \Rightarrow \langle \Xi + \Theta \rangle \langle \bar{\Phi} + \Delta \rangle$

$= \Xi\bar{\Phi} + \Theta\Delta$

$= \langle \Theta + \Xi \rangle \langle \Delta + \bar{\Phi} \rangle$

		2	3		
0	1			2	3
1	0	3	2	2	3

$\langle \bar{\Phi}[m] + \bar{\Phi}(m,n] \rangle = \bar{\Phi}[n]$ $\quad \langle \bar{\Phi}[i,j] + \bar{\Phi}(j,k] \rangle = \bar{\Phi}[i,k]$

$\langle\langle \Theta + \Xi \rangle - \Theta \rangle = \Xi$ $\quad \Theta \perp \Xi \quad \langle\langle \Xi + \Theta \rangle - \Theta \rangle = \Xi$

$\Theta \perp \Xi \quad \bar{\Phi}\langle \Theta + \Xi \rangle \setminus \Theta = \bar{\Phi}\Xi$? $\quad \bar{\Phi}\langle\Theta$

$(\bar{\Phi}\langle \Theta + \Xi \rangle) \setminus \Delta\Xi = \bar{\Phi}\Theta$?

$\Xi = \text{ord}(\Phi)$ iff $\text{unit}(\Xi) = \text{ord}(\text{unit}\,\Phi)$

~~RTP~~ show ord idempotent. $\text{ord}(\text{ord}\,\Phi) = \text{ord}(\Phi)$ $\quad \Theta = \text{ord}(\Xi)$

$\xi(z|\nu)$: number of $\phi(z|\tau)$ less than $\phi(z|\nu)$

$\theta(z|\nu)$: " " $\xi(z|\tau)$ " " $\xi(z|\nu)$

0 0 2 2 2 5 5

$\phi\xi(z|\tau)$ take values $\sum r(z|\omega)$ $\langle \omega := [i] \rangle$ $\langle i := [k] \rangle$

~~$r(z|0) = 0$ $r(z|\omega) > 0 \langle \omega := (k] \rangle$~~ $r(z|0) \geq 0$ $r(z|\omega) > 0 \langle \omega := (k] \rangle$

$\phi\xi(z|\tau) = \sum r(z|\omega)$ $\langle \omega := [i] \rangle$ for $\tau \in M_i$ $\quad M_i$ disjoint $M_0 \cup M_1 \cup \ldots \cup M_k = \{[m($

$\xi(z|\tau)$ take values $\sum s(z|\omega)$ $\langle \omega := [i] \rangle$ $\langle i := [k] \rangle$

$s(z|0) = 0$ $\quad s(z|\omega) > 0$ $\langle \omega := (k] \rangle$

$\xi(z|\tau) = \sum s(z|\omega)$ $\langle \omega := [i] \rangle$ for $\tau \in M_i$

$s(z|\omega+1)$ = number of distinct members of τ in M_ω

$\omega := [k]$

$\theta(z|\tau)$ take values $\sum s'(z|\omega)$

$\theta(z|\nu)$ for $\nu \in M_0$ have value zero since $\xi(z|\nu) = 0$ for these and zero is minimum value of the $\xi(z|\tau)$ for $\tau \in [m(z)]$

i.e $\theta(z|\nu) = \xi(z|\omega)$ for all $\nu \in M_0$. For ν' outside this range $\xi(z|\nu') \neq 0$ and $\theta(z|\nu') \neq 0$

$\phi(z|\nu')$ does not have minimum value,

suppose have shown that $\theta(z|\nu) = \xi(z|\nu)$ for all $\nu \in M_0 \cup M_1 \cup \ldots \cup M_{i-1}$

for some $i \in (k]$. so that $s'(z|\omega) = s(z|\omega)$ $\omega := [i)$

and also that $\theta(z|\nu') > \theta(z|\nu)$ for all ν' outside this range

for $\nu \in$ ~~over~~ M_i, $\xi(z|\nu)$ has the value $\sum s(z|\omega)$ $\langle \omega := [i] \rangle$

$\theta(z|\nu)$ is the number of values of $\xi(z|\tau)$ less than $\xi(z|\nu)$

these values ~~must be~~ belong to two sets $\tau \in M_0 \cup M_1 \cup \ldots \cup M_{i-2}$

and $\tau \in M_{i-1}$. The number of values belonging to the first set is ~~ ~~ $\Theta(z|\nu)$ for $\nu \in M_{i-2}$. The number of values belonging to the second set is the number of values of $\xi(z|\nu)$ for $\nu \in M_{i-1}$ i.e. $s(z|i)$. Hence for $\nu \in M_i$, $\Theta(z|\nu) = \sum s(z|\omega)\ \langle \omega := [i] \rangle$ $= \xi(z|\nu)$. For all ν' outside the range $\nu' \in M_0 \cup M_1 \cup \ldots \cup M_i$, $\phi(z|\nu')$ takes values greater than those corresponding to index values

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$\mathrm{ord}(\Xi) = \cancel{\mathrm{ord}}\{\Xi\}$ if and only if $\Xi \in \mathrm{ord}\,\bar{N}[B]$.

If $\Xi = \mathrm{ord}(\Xi)$ then Ξ is the order of some sequence in $\mathrm{seq}\,\bar{N}[B]$, namely itself, and hence $\Xi \in \mathrm{ord}\,\bar{N}[B]$

Suppose $\Xi = \mathrm{ord}(\Phi)$ for some $\Phi \in \mathrm{seq}\,\bar{N}[B]^{,m}$. Set $\phi = \mathrm{fn}(\Phi)$. ~~Select $z \in B$~~ Select $z \in B$. $\{\phi(z)\}$ has the following structure: $k+1$ disjoint sets of integers $M_i \langle i := [k] \rangle$ exist such that $M_0 \cup M_1 \cup \ldots \cup M_k = \{[m(z)]\}$ and for which $\phi(z|\tau) = \sum r(z|\omega)\ \langle \omega := [i] \rangle$ for all $\tau \in M_i\ \langle i := [k] \rangle$ where $r(z|0) \geq 0$, $r(z|\omega) > 0\ \langle \omega := (k] \rangle$

Let $s(z|\omega+1)$ be the number of distinct ~~num~~ integers in M_ω $\langle \omega := [k) \rangle$ and set $s(z|0) = 0$. For each $i \in [k]$ and ~~For~~ all $\tau \in M_i$, $\xi(z|\tau)$ takes the value $\sum s(z|\omega)\ \langle \omega := [i] \rangle$. ~~Set $\Theta = \mathrm{ord}(\Xi)$~~ ~~and let~~ The value of this sum depends only upon the $s(z|\omega)$ (i.e. the number of distinct members of the sets $\{M_i\}$) The description of $\xi(z)$ serves equally for $\Theta(z)$: ~~$\Theta(z) = \xi(z)$~~ $\Theta(z) = \xi(z)$ for each $z \in B$

Set $\Theta := \mathrm{ord}(\Xi)$ and $\Theta = \mathrm{fn}(\Theta)$.

and $\Theta = \Xi$.

The spectrum of a sequence

The mapping

$\text{spect}: \text{seq}\bar{\mathbb{N}}[\mathbb{B}] \to \text{seq}\bar{\mathbb{N}}[\mathbb{B}]$

is ~~obtained~~ defined by setting $\text{fn}(\text{spect}(\Xi)) = \text{spect}\,\xi$ where, for each $z \in \mathbb{B}$, $\text{spect}\,\xi(z)$ is obtained from $\text{unit}\,\xi(z)$ in the following way. Supposing that $\Xi \in \text{seq}\bar{\mathbb{N}}[\mathbb{B}, n]$, let $s(z)$ be the increasing sequence of all $\nu \in [n(z)]$ for which $\text{unit}\,\xi(z|\nu) > \text{unit}\,\xi(z|\nu-1)$. ~~sig~~ $\text{spect}\,\xi(z)$ is

$\langle 0 + s(z) + n(z) \,(\text{if } n(z) \notin s(z))\rangle$.

=

$\Xi \in \text{ord}\bar{\mathbb{N}}[\mathbb{B}]$ if and only if

$\text{unit}(\Xi)[\text{spect}(\Xi)] = \text{spect}(\Xi)$.

Select $z \in \mathbb{B}$.
From the definition of the spectrum

$\text{ord unit}\,\xi(z|\tau) = \text{spect}\,\xi(z|\nu) \quad \langle \tau := [\text{spect}(z|\nu), \text{spect}(z|\nu+1))$

for ~~...~~ $\nu := [r(z))$ where $\text{spect}(\Xi) \in \text{seq}\bar{\mathbb{N}}[\mathbb{B}, r]$

If the stated condition holds

(*) $\text{unit}\,\xi(z|\text{spect}\,\xi(z|\nu)) = \text{spect}\,\xi(z|\nu)$

for $\nu := [r(z)]$ and in consequence

$\text{unit}\,\xi(z|\tau) = \text{spect}\,\xi(z|\nu) \quad \langle \tau := [\text{spect}(z|\nu), \text{spect}(z|\nu+1))$

for $\nu := [r(z))$. Hence

$\text{ord unit}\,\xi(z|\tau) = \text{unit}\,\xi(z|\tau) \quad \langle \tau := [n(z)]\rangle$

where $\Xi \in \text{ord}\bar{\mathbb{N}}[\mathbb{B}, n]$ & and in consequence $\text{ord}(\text{unit}(\Xi)) = \text{unit}(\Xi)$.

If the stated condition is not satisfied, there is at least one value of $\nu \in [r(z)]$ for which condition (*) is violated, and for this ν

~~ord unit $\xi(z|\nu) \neq$~~

$$\text{ord unit}\,\xi(z|\text{spect}(z|\nu)) \neq \text{unit}\,\xi(z|\text{spect}(z|\nu))$$

so that $\text{ord}(\text{unit}(\Xi)) \neq \text{unit}(\Xi)$

In conclusion, $\text{ord}(\text{unit}(\Xi)) = \text{unit}(\Xi)$ if and only if the stated condition holds. ~~But ord (unit Ξ~~ Hence $\text{unit}(\Xi) \in \text{ord}\bar{\mathbb{N}}[\mathbb{B}]$ if and only if this condition holds and the required result follows.

Let $\Phi \in \text{seq}\bar{\mathbb{N}}[\mathbb{B}, n]$, $\Psi \in \text{seq}\bar{\mathbb{N}}[\mathbb{B}, m]$ with $m \le n$ $\langle \mathbb{B} \rangle$.

$\text{pre}\left\{\frac{\Psi}{\Phi}\middle|\mathbb{B}\right\}$ is nonvoid if and only if

$$\text{unit}(\Phi)[\text{spect}(\Psi)] = \text{filt}(\Psi) \qquad \cancel{\text{and } \phi' = fn(\Phi'), \psi' = fn(\Psi')}$$

~~Set unit $\Phi' := \text{unit}(\Phi)$, and $\Psi' := \text{unit}(\Psi)$~~ ~~If the above~~ stated condition is satisfied, $\Xi' \in \text{unit}\bar{\mathbb{N}}[\mathbb{B}\xi, m]$ with $\xi' := fn(\Xi')$ (ξ, select $z \in \mathbb{B}$) exists for which

$$\psi'(z|\tau) = \phi'(z|\xi'(z, \tau)) \qquad \langle \tau := [m(z)] \rangle$$

~~Also $\text{spect}(\Xi') = \text{spect}(\Psi)$ and $\text{filt}(\Xi') = \text{filt}(\Psi')$~~

~~$\psi'(z|\tau)$ varies only~~

As τ increases, $\psi'(z|\tau)$ increases at values of τ belonging to $\text{spect}\psi^{\xi}(z)$. It is possible so to choose Ξ' that $\xi'(z|\tau)$ behaves in the same way and then $\text{spect}(\Xi') = \text{spect}(\Psi^{\xi})$. ~~for values of τ belonging to $\text{spect}(\Psi^{\xi})$~~ Also, from the stated condition

$$\psi'(z|\text{spect}\psi(z|\nu)) = \phi'(z|\text{spect}\,\psi(z|\nu)) \qquad \langle \nu := [r(z)] \rangle$$

Thus where $\text{spect}(\Psi) \in \text{seq}\bar{\mathbb{N}}[\mathbb{B}, r]$.

~~$\xi'(z,\tau) = \operatorname{spect}\xi'(z|\tau)$~~

Set $\Phi' := \operatorname{unit}(\Phi)$, $\Psi' := \operatorname{unit}(\Psi)$ and $\phi' := \operatorname{fn}(\Phi')$, $\psi' := \operatorname{fn}(\psi')$.

91 Select $z \in \mathbb{B}$.

The stated condition may be presented in the form

$$\psi'(z|\operatorname{spect}\psi'(z|\nu)) = \phi'(z|\operatorname{spect}\psi'(z|\nu)) \quad \langle \nu := [r(z)] \rangle$$

where $\operatorname{spect}(\Psi') \in \operatorname{seq}\bar{\mathbb{R}}[\mathbb{B}, r]$. Assuming this condition to hold it is possible to select $\Xi' \in \operatorname{unit}\bar{\mathbb{R}}[\mathbb{B}, m]$ with $\xi' := \operatorname{fn}(\Xi')$ for which

$$\psi'(z|\tau) = \phi'(z|\xi'(z,\tau)) \qquad \langle \tau := [m(z)] \rangle$$

As τ increases, $\psi'(z|\tau)$ increases at values of z belonging to $\operatorname{spect}\psi'(z)$. It is possible to choose Ξ' such that $\xi'(z|\tau)$ behaves in the same way and then $\operatorname{spect}\xi'(z) = \operatorname{spect}\psi'(z)$.

Also

$$\operatorname{spect}\psi'(z|\nu) = \xi'(z|\operatorname{spect}\psi'(z|\nu)) \qquad \langle \nu := [r(z)] \rangle$$

Hence

$$\operatorname{spect}\xi'(z|\nu) = \cancel{\operatorname{spect}}\,\xi'(z|\operatorname{spect}\xi'(z|\nu)) \quad \langle \nu := [r(z)] \rangle$$

or, since $\Xi' \in \operatorname{unit}\bar{\mathbb{R}}[\mathbb{B}, m]$

$$\operatorname{unit}(\Xi')[\operatorname{spect}(\Xi')] = \operatorname{spect}(\Xi')$$

and $\Xi' \in \operatorname{ord}\bar{\mathbb{R}}[\mathbb{B}]$. Furthermore ~~$\Phi \cdot \Phi$~~ ~~$\Psi = \Xi'$~~ $\Psi' = \Phi'[\Xi']$ so that $\Psi' = \Xi'\Phi'$. Hence ~~$\Phi = \operatorname{ord}\psi^{-1}\Xi' \operatorname{ord}\phi^{-1}\Phi$~~ $\Psi = \Xi\Phi$ where $\Xi = \operatorname{ord}\psi^{-1}\Xi'\operatorname{ord}\phi^{-1}$, ψ and ϕ being $\operatorname{fn}(\Psi)$ and $\operatorname{fn}(\Phi)$ respectively: if the stated condition holds $\operatorname{pre}\left\{\frac{\Psi}{\Phi}\middle|\mathbb{B}\right\}$ is nonvoid.

If $\mathrm{pre}\left\{\frac{\Psi}{\Phi}\middle|\mathbb{B}\right\}$ is nonvoid, $\Psi = \Phi[\Xi]$ for some $\Xi \in \mathrm{ord}\,\bar{\mathbb{N}}[\mathbb{B}]$ and in consequence $\Psi' = \Phi'[\Xi']$ for some $\Xi' \in \mathrm{unit}\,\bar{\mathbb{N}}[\mathbb{B}] \cap \mathrm{ord}\,\bar{\mathbb{N}}[\mathbb{B}]_{\leq}$ so that

$$\phi'(z|\xi'(z|\tau)) = \psi'(z|z) \qquad \langle \tau := [m(z)] \rangle$$

where $\xi' = m(\Xi')$.
As τ increases through $[m(z)]$, $\xi(z|\tau)$ must change when $\psi'(z|\tau)$ changes: all points of $\mathrm{spect}\,\psi'(z)$ belong to $\mathrm{spect}\,\xi'(z)$. Since $\Xi' \in \mathrm{ord}\,\bar{\mathbb{N}}[\mathbb{B}]$,

$$\xi'(~~\mathrm{spec}~~\, z|\mathrm{spect}\,\xi'(z|\nu)) = \mathrm{spect}\,\xi'(z|\nu)$$

for all ~~$\tau \in \mathrm{spect}\,\xi'(z)$~~. $\nu \in [r(z)]$, where $\mathrm{spect}(\Xi') \in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}, r]$.

Hence

$$\psi'(z|\mathrm{spect}\,\psi'(z|\nu)) = \phi'(z|\xi'(z|\mathrm{spect}\,\psi'(z|\nu)))$$
$$= \phi'(z|\mathrm{spect}\,\psi'(z|\nu))$$

for all $\nu \in [r'(z)]$ where $\mathrm{spect}(\Psi') \in \mathrm{seq}\,\bar{\mathbb{N}}[\mathbb{B}, r']$. The stated condition has been obtained.

$S(z) = [m(z)] \to \mathbb{N}[t(z)]$

$\mathrm{ident}\,\bar{\mathbb{N}}[\mathbb{B}, m]$ $\quad \lambda(z|\tau) = \tau \quad \langle \tau := [m(z)] \rangle$

$\mathrm{unit}(\Phi)[\mathrm{spect}(\Psi)] = \mathrm{filt}\,\Psi \quad (= \mathrm{unit}(\Psi)[\mathrm{spect}(\Psi)])$ $\qquad \mathbb{B}[\mathbb{C}[\mathbb{D}]]$

written as $\Psi \leq \Phi\ (\mathrm{spect})$ $\qquad E = C[D]$

$\mathrm{unit}(\Xi)[\mathrm{spect}(\Xi)] = \mathrm{spect}(\Xi)$ $\qquad F = \mathbb{B}[C]$

$\equiv$ ident ~~unit~~ $[\mathrm{spect}(\Xi)] = \mathrm{filt}(\Xi) \equiv \Xi \leq \mathrm{ident}$ ~~unit~~ (spect)

$\mathbb{B}[E] = F[D]$

show $\Psi \leq \Phi$ transitive

$b(z) = \tau \qquad S \notin T \qquad b(z|\tau) = \tau \qquad b(z|c(z|\tau)) = c(z|\tau)$

$c(z|b(z|\tau)) = c(z|\tau)$

Ⓒ Sets of matrices

$K^{[n]}_{[m]}$ $\mathbb{K}_{[m]}$ $\mathbb{K}^{[n]}$ $\mathbb{K}[m]$ $ns\mathbb{K}[m]$ $O^{[n]}_{[m]}$ $O_{[m]}$ $O^{[n]}$ $I[n]$?

Compound matrix mappings

i) $a.+b$ ii) $a+.b$ iii) conjunction iv) void constituents

v) Compound matrix mapping spaces

Pre and post quotients spaces

i) pre $\left\{\frac{e}{0}\middle| \mathbb{B}\right\}$ ii) post $\left\{\frac{e}{f}\middle| \mathbb{B}\right\}$

Pre and post nonsingular mappings

i) $a \in$ prens $\{b|\mathbb{B}\}^{[k]}$ (ii) $a \in$ pons $\{b|\mathbb{B}\}$

iii) $a \in$ prens $\{\mathbb{B}\}^{[k]}$ ..

$a \in$ prens $\{b|\mathbb{B}\}^{[k]} \to k \le m - \text{rank}[b]$

pons $\{c|\mathbb{B}\}_{[k]}$ $k \le n - \text{rank}[b]$

$a \in$ prens $\{b|\mathbb{B}\}^{[k]}$ iff a) $\text{rank}[a^{[k]}.+b] \neq \text{rank}[a^{[k]}+b]$

b) post $\left\{\frac{a^{(k)}}{a^{[k]}_{k+1}+b}\middle| z\right\}$ void $z \in \mathbb{B}$ c) $\text{rank}[a^{[k]}.+b] = \text{rank}[a] + k+1$

a) (missing) $\text{rank}[a.+b] = \text{rank}[b] + k+1$

~~also missing~~

ii) $a \in$ pons $\{b|\mathbb{B}\}$...

missing: $a \in$ prens $\{b|\mathbb{B}\}^{[k]} \Rightarrow a \equiv \in$ prens $\{b|\mathbb{B}\}$ in particular $a^{[k]} \in$ prens $\{b|\mathbb{B}\}$;

Existence and structure of quotient spaces

i) $b \in ns\mathbb{K}[\mathbb{B}] \Rightarrow$ pre $\left\{\frac{a}{b}\middle| \mathbb{B}\right\} = ab^{-1}$ post similarly

$a \in ns\mathbb{K}[\mathbb{B}]$ both ~~nons~~ inverses ... $NS(a)$

ii) pre $\left\{\frac{a.+b}{c.+d}\right\} = \frac{b.+a}{d.+c} = \frac{a}{c} \cap \frac{b}{d}$ post similarly

iii)a) e zero over $\mathbb{B} \Rightarrow$ at least one of pre $\left\{\frac{a.+b}{c.+d}\right\}$, post $\left\{\frac{c+.e}{d+.b}\right\}$ void

similar result for singular e

iv) $\text{pre}\left\{\frac{a}{b}\right\} = x + \text{pre}\left\{\frac{0}{b}\right\}$ post also (refer to CFR space)

v) a nonzero over $B' \subseteq B$ $\text{pre}\left\{\frac{a}{b} / B'\right\} \subseteq B' \to R.. \setminus 0..$ post also

Algebraic sums and products of spaces

i) $\text{pre}\left\{\frac{a}{c}\right\} \pm \text{pre}\left\{\frac{b}{c}\right\} = \text{pre}\left\{\frac{a \pm b}{c}\right\}$ post simly

ii) $\text{pre}\left\{\frac{a}{b}\right\} \text{pre}\left\{\frac{b}{c}\right\} \subseteq \text{pre}\left\{\frac{a}{c}\right\}$ post simly

Addition of terms to quotient spaces

i) $a + \text{pre}\left\{\frac{b}{c}\right\} \subseteq \text{pre}\left\{\frac{ac+b}{c}\right\}$ post simly

Multiplication of spaces by factors

$a\,\text{pre}\left\{\frac{b}{c}\right\} \subseteq \text{pre}\left\{\frac{ab}{c}\right\}$ post simly

$\text{pre}\left\{\frac{b}{c}\right\} \subseteq \text{pre}\left\{\frac{ba}{ca}\right\}$ post simly

---

$\text{rank}[A.+B] = \text{rank}[A] \Rightarrow \exists X$ st $AX = B$

$a \in \text{prens}\{c, B\} \Leftrightarrow x_{[k]}$ same for all $x \in \text{post}\left\{\frac{f}{a.+b}\right\}$

$x \in \text{prens}^{[r]}_{[k]}$

a) $b \in \text{prens}\{c, B\}^{[k]}$ $bx \in \text{prens}\{c, B\}^{[r]}$

b) $r = k$ $b \in \text{prens}\{C, B\}^{[k]}$ if $bx \in \text{prens}\{C, B\}^{[k]}$?

$p = bx + Cx'$ $\text{rank}[x] = \min\{\text{prerank}[p, C], k+1\}$

$\text{prerank}(b, C) = \text{rank}[b.+C] - \text{rank}[C]$

$b \in \text{prens}(C, B)$ $b\Xi \in \text{prens}(C, B)$ all $|\Xi| \leq k$ $b^{(k)} \in \text{prens}(C, B)$

$b \in \text{prens}(C, B)$ $k < m - \text{rank}[C]$ $\exists \bar{b}$ for which $[b + \bar{b}] \in \text{prens}(C, B)$

$[b + \bar{b} + C] K^{[r]}_{[n+k+2]} \supset [b.+C] K^{[r]}_{[n+k+1]}$ / $B \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$ $A = \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$

(not true that $\text{rank}[B] > \text{rank}[A] \Rightarrow BK^{[j]}_{[n]} \geq AK^{[r]}_{[j]}$ $\begin{matrix} 0 \\ 0 \\ x \end{matrix} \in AK^{0}_{0}$ not in BK_{1}^{0}

(c) $\bar{N}$ $[i]$ $\{[i]\}$

a) integer sequences $\bar{N}$ seq us$\bar{N}$ seq perm$\bar{N}$ seq $|\Xi_i|$

Addition of integer sequences

Rotation transposition

b) Sets of matrices

Compound ~~matrices~~ mappings

Matrix mappings

Sequence mappings

operations ~~determined~~ defined pointwise in terms of operations ~~in~~ ~~the~~ target set

Source and target sets

$\bar{N}$ $[i]$ $\{[i]\}$

integer sequences

Sets of matrices

Mappings

Sequence mappings

Matrix mappings

Compound matrix mappings

~~The Rank of a matrix mapping~~

Operations upon sequence mappings

Products of sequence and matrix mappings

Pre and post nonsingularity

The rank of a matrix mapping

deficiency

nonsingularity

nonzero

Pre and post quotient spaces

Pre and post cofactors of zero

(c) Arithmetic operations upon $\bar{N}$-integer mappings ~~$\bar{N}$-sequences~~ are defined pointwise: the product of $j: B \to \bar{N}$ and $k: B \to \bar{N}$ is $jk: B \to \bar{N}$ where $jk(z) := j(z)k(z)$

$\mathrm{inv}\,\Xi : \sqrt{\{\xi(B)\}} \xrightarrow{\times B} \{n(B)\}$

Sequences Φ Θ: Θ, Φ $\Theta : [\Theta] \to \{\Theta\}$ $\phi : \ldots$

$\langle \Theta + \Phi \rangle$ is Ψ for which $\Psi = |\Theta| + |\Phi| + 1$ $\psi : [\Psi] \to \{\Theta\} \cup \{\Phi\}$

$\mathrm{inv}\,\Xi(z | \xi(z|\omega)) = \omega$

ord, transpose, inv

$\langle \theta(0), \ldots, \theta(|\Theta|), \phi(0), \ldots, \phi(|\Phi|)$

$\mathrm{ord} : \mathrm{ns\,seq}\,\bar{N}[B, n] \to \mathrm{perm}\,\bar{N}[B, n]$

with $|\Theta| = |\Phi|$, $\Theta\Phi$ is ...

perm $\bar{N}$ seq: sequence Ξ' for which ~~$[\Xi'] = \{\Xi\}$~~ the integers in T:

$\{[\xi\Xi]\} \to \{\Xi\}$ Ξ' is a rearrangement of $[\Xi']$ $\xi(0) = 3$ $\xi(1) = 0$ $\xi(2) = 1$

; $[\Xi]$ as set of integers in $[i]$. $\mathrm{inv}(\cdot\ 0) = 1$ etc

ord $\Phi \in$ perm ord : $\bar{N}$ seq' $\to$ perm $\bar{N}$ seq ns N seq sequences of distinct ~~numbers~~

ord(Φ) $\psi : [\Phi] \to [\Phi]$ $\psi(k) =$ number of members of Φ less than $\phi(k)$

Ξ 2 0 1 inv(0) = 1 inv(1) = 2 inv(2) =
-1 0 = 1 1 = 2 2 =

$\Theta \Phi$ is Ξ: $\xi : [\Theta] \to \{\Theta\}$ $\xi(k) = \Theta(\psi(k))$ $\psi = \mathrm{fn}(\mathrm{ord}(\Phi))$?

other way round for $\Theta\Phi A = \Theta(\Phi A)$

Ξ	ord	T	inv
0	3	2	1	1
1	0	0	3	2
2	1	1	0	0

Ξ $\Theta^{-1} = \Theta^{T}$ $\phi\{\mathrm{ord}(\Theta)\}$ $\theta = \mathrm{fn}$ ~~$\xi(\mathrm{ord}\,\theta$~~ $\xi(\phi(k)) = \theta(k)$

$\phi = \mathrm{fn}\{\mathrm{ord}(\Theta)\}$ $A^{\phi(c)}_{\phi(\omega)}$ $\Phi A \Phi^{-1}$ $\phi = \mathrm{fn}(\Xi^{-1})$

ΘA $A\Phi$ Φ $\Theta(AB) = (\Theta A)B$ $(AB)\Phi = A(B\Phi)$

$\Theta(A\Phi) = (\Theta A)\Phi$ $\mathrm{inv}(z | \xi(z|\omega)) = \phi(z|\omega)$ iff $\Xi \in$ perm

$\Theta(\Phi A) = (\Theta\Phi)A$

$(A\Phi)B = A(\Phi B)$ requires $\Phi \in$ perm $\bar{N}$seq

~~$A\Theta$~~ $(A\Phi)\Theta = A(\Phi\Theta)$

$|\Phi| = n$ A^{n}_{m}

$(\Theta A)^{T} = A^{T}\Theta^{T}$ $(A\Phi)^{T} = \Phi^{T}A^{T}$

$\xi(z | \phi(z, \omega))$?

$\Theta[A. + B] = [\Theta A. + \Theta B]$ $[A. + B]\Phi = [A\Phi +. B\Phi]$

$X[A. + B] = XA. + XB$ $[A +. B]Y = AY +. BY$

ⓒ $m, n \in$

~~In the following $m, n \in \bar{\mathbb{N}}$ and $\mathbb{B} \subseteq \mathbb{K}$ $\mathbb{B}$ is a prescribed set~~

ⓒ Pre and post quotient spaces

Let $h, k \in \bar{\mathbb{N}}$ /m and $\mathbb{B}$ be a prescribed set $e : \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$. (i) Let $g : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$. $\operatorname{pre}\left[\frac{e}{g}\middle|\mathbb{B}\right]$ is the set of all $f : \mathbb{B} \to \mathbb{K}^{[m]}_{[h]}$ satisfying the relationship

$$e = f \overset{\circ}{\underset{\circ}{}} g \langle \mathbb{B} \rangle$$

ii) Let $f : \mathbb{B} \to \mathbb{K}^{[m]}_{[h]}$. $\operatorname{post}\left[\frac{e}{f}\middle|\mathbb{B}\right]$ is the set of all $g : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ satisfying the above relationship

The ~~ab~~ above notations are used in connection with point domains: ~~thus~~ the symbol $\operatorname{pre}\left[\frac{e}{0}\middle|z\right]$ where $z \in \mathbb{K}$, ~~for example~~ may be encountered, ~~for examples~~

ⓐ Sets of ~~matrices~~

With $m, n \in \bar{\mathbb{N}}$, $\mathbb{K}^{[n]}_{[m]}$ is the set of $(m+1)$-rowed, $(n+1)$-columned matrices whose elements are in $\mathbb{K}$. $\mathbb{K}^{[0]}_{[m]}$, ~~the set of~~ and $\mathbb{K}^{[n]}_{[0]}$, the sets of column vectors with $\{m+1\}$ elements and row vectors with $n+1$ elements respectively, are ~~with~~ written as $\mathbb{K}_{[m]}$ and $\mathbb{K}^{[n]}$ respectively. The set of square matrices $\mathbb{K}^{[m]}_{[m]}$ is written as $\mathbb{K}[m]$ ~~and~~; the subset of nonsingular matrices in this set is denoted by $\mathbb{K}'[m]$

~~Compound matrices~~ matrix mappings

The zero matrix in $\mathbb{K}^{[n]}_{[m]}$ is written as $O^{[n]}_{[m]}$. $O_{[m]}$ and $O^{[n]}$ are zero column and row vectors respectively.

(b) Compound matrix mappings

Let $h, k, m, n \in \bar{N}$ and $\mathbb{B}$ be a prescribed set

i) With $a : \mathbb{B} \to K^{[h]}_{[m]}$ and $b : \mathbb{B} \to K^{[k]}_{[m]}$, $[a \overset{+}{\times} b]$ is the compound matrix mapping $c : \mathbb{B} \to K^{[h+k]}_{[m]}$ whose successive columns are those of a in order followed by those of b in order

ii) With $d : \mathbb{B} \to K^{[n]}_{[h]}$ and $e : \mathbb{B} \to K^{[n]}_{[k]}$, $[d \bar{\times} e]$ is the corresponding compound matrix mapping $f : \mathbb{B} \to K^{[n]}_{[h+k]}$ formed by row adjunction.

iii) The above notations are used in conjunction. Thus with $m > k$ and $a : \mathbb{B} \to K^{[h]}_{[m]}$, $b : \mathbb{B} \to K^{[n]}_{[k]}$, $c : \mathbb{B} \to K^{[n]}_{[m-k-1]}$, $[a \overset{+}{\times} [b \bar{\times} c]]$ is formed by column adjunction of a and $[b \bar{\times} c]$, the latter mapping being formed by row adjunction.

iv) Compound matrix mappings whose constituents are void are permitted. Thus, in the preceding example, ~~it is~~ c becomes void when $m = k$, $[b \bar{\times} c]$ reduces to b and $a \overset{+}{\times} [b \bar{\times} c]$ to $[a \overset{+}{\times} b]$.

v) (treated before proofs of prl. results)

$\Rightarrow$ use of expressions to define spaces.

App.m ~~Certain~~ It may occur that ~~compound~~ matrix mappings may be compounded in different ways to yield the same result.

Thus, with $a : \mathbb{B} \to K^{[h]}_{[m]}$, $b : \mathbb{B} \to K^{[h]}_{[n]}$, $c : \mathbb{B} \to K^{[k]}_{[m]}$ and $d : \mathbb{B} \to K^{[k]}_{[n]}$

$$[[a+b]-[c+d]] = [[a-c]+[b-d]]$$

$$\cancel{[a \overset{+}{\times} b] \overset{+}{\times} [c \bar{\times} d] = [a \times c] \times [b \times d]} \quad \langle \mathbb{B} \rangle$$

Appropriate selection of a ~~form is~~ representation is often suggested

D Pre- and post- cofactor of zero spaces

Let $h, n \in \bar{\mathbb{N}}$. Set $M :=$ set, and $N' := \bar{\mathbb{N}}$.

i) The ~~precofactor of zero~~ mapping

$$\mathrm{pre}\mathbb{CFZ} : \mathbb{K}[M]^{[n]}_{[h]} \times M \times N' \subseteq \mathbb{K}[M]^{[h]}_{[N']}$$

is defined by setting

$$\mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[m]} := \mathrm{pre}\left\{ \frac{O^{[n]}_{[m]}}{u} \,\middle|\, \mathbb{B} \right\}$$

ii) The ~~postcofactor of zero~~ mapping

$$\mathrm{po}\mathbb{CFZ} : \mathbb{K}[M]^{[n]}_{[h]} \times M \times N' \subseteq \mathbb{K}[M]^{[N']}_{[n]}$$

is defined by setting

$$\mathrm{po}\mathbb{CFZ}\{u \mid \mathbb{B}\}^{[r]} := \mathrm{post}\left\{ \frac{O^{[r]}_{[h]}}{u} \,\middle|\, \mathbb{B} \right\}$$

Existence and description

Let $h, m, n, r \in \bar{\mathbb{N}}$ and $u : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$.

i) The spaces $\mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[m]}$ and $\mathrm{po}\mathbb{CFZ}\{u \mid \mathbb{B}\}^{[r]}$ are nonvoid, containing $O^{h}_{[m]}$ and $O^{[r]}_{[h]}$ respectively.

iia) $\mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[m]}$ is the space in $\mathbb{K}[\mathbb{B}]^{[h]}_{[m]}$ of precofactor x of $O^{[n]}_{[m]}$ corresponding to the postfactor u: $xu = O^{[n]}_{[m]}$ for all such x

b) $\mathrm{po}\mathbb{CFZ}\{u \mid \mathbb{B}\}^{[r]}$ is the space in $\mathbb{K}[\mathbb{B}]^{[r]}_{[n]}$ of postcofactors y of $O^{[r]}_{[m]}$ corresponding to the prefactor u: $uy = O^{[r]}_{[h]}$ for all such y

Inclusion and equivalence domains for cofactor ~~spaces~~ of zero spaces

Let $h, j, k, m, n, r \in \bar{\mathbb{N}}$ and $u : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$

ia) Let $v : \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$

$$\mathrm{pre}\mathbb{CFZ}\{u .+ v \mid \mathbb{B}\}_{[m]} = \mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[m]} \cap \mathrm{pre}\mathbb{CFZ}\{v \mid \mathbb{B}\}_{[m]}$$

b) Let $w : \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$.

$$\mathrm{po}\mathbb{CFZ}\{u +. w \mid \mathbb{B}\}^{[r]} = \mathrm{po}\mathbb{CFZ}\{u \mid \mathbb{B}\}^{[r]} \cap \mathrm{po}\mathbb{CFZ}\{w \mid \mathbb{B}\}^{[r]}$$

iia) $$\mathbb{K}[\mathbb{B}]^{[m]}_{[j]}\, \mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[m]} \subseteq \mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[j]}$$

With $j \le m$,

$$G\, \mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[j]} = \mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[j]}$$

for all $G \in \mathrm{pons}\,\mathbb{K}[\mathbb{B}]^{[m]}_{[j]}$,

$$\mathrm{pons}\,\mathbb{K}[\mathbb{B}]^{[m]}_{[j]}\, \mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[m]} = \mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[j]}$$

and the above semi-inclusion relationship becomes one of equivalence.

b) $$\mathrm{pre}\mathbb{CFZ}\{g u \mid \mathbb{B}\}_{[m]}\, g \subseteq \mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[m]}$$

for all $g \in \mathbb{K}[\mathbb{B}]^{\langle h \rangle}$. This relationship becomes one of equivalence for all $g \in \mathrm{pens}\,\mathbb{K}[\mathbb{B}]^{\langle h \rangle}$.

c) $$\mathrm{pre}\mathbb{CFZ}\{u \mid \mathbb{B}\}_{[m]} \subseteq \mathrm{pre}\mathbb{CFZ}\{u D, \mathbb{B}\}_{[m]}$$

for all $D \in \mathbb{K}[\mathbb{B}]_{\langle n \rangle}$. Also

$$\mathrm{pre}\mathbb{CFZ}\{u\, \mathrm{pons}\,\mathbb{K}[\mathbb{B}]_{\langle n \rangle}, \mathbb{B}\}_{[m]} = \mathrm{pre}\mathbb{CFZ}\{u, \mathbb{B}\}_{[m]}$$

iiia) $$\mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}^{[m]}\,\mathbb{K}[\mathbb{B}]^{[j]}_{[m]} \subseteq \mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}^{[j]}$$

With $j \leq m$,

$$\mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}^{[m]} F = \mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}^{[j]}$$

for all $F \in \mathrm{prens}\,\mathbb{K}[\mathbb{B}]^{[j]}_{[m]}$

$$\mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}^{[m]}\,\mathrm{prens}\,\mathbb{K}[\mathbb{B}]^{[j]}_{[m]} = \mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}^{[j]}$$

and the above semi-inclusion relationship becomes one of equivalence

~~b) DpoCF~~

b) $$\mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}^{[m]} \subseteq \mathrm{po}\mathbb{CFZ}\{gu|\mathbb{B}\}^{[m]}$$

for all $g \in \mathbb{K}[\mathbb{B}]^{\langle h\rangle}$. Also

$$\mathrm{po}\mathbb{CFZ}\{\mathrm{prens}\,\mathbb{K}[\mathbb{B}]^{\langle h\rangle} u|\mathbb{B}\} = \mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}$$

c) $$D\,\mathrm{po}\mathbb{CFZ}\{uD|\mathbb{B}\}^{[j]} \subseteq \mathrm{po}\mathbb{CFZ}\{u|\mathbb{B}\}^{[j]}$$

for all $D \in \mathbb{K}[\mathbb{B}]_{\langle n\rangle}$. This relationship becomes one of equivalence for all $D \in \mathrm{prens}\,\mathbb{K}[\mathbb{B}]_{\langle n\rangle}$.

(D)

Existence ~~and structure~~ and structure

i) Let $m, n \in \bar{\mathbb{N}}$, and $b : \mathbb{B} \to \mathbb{K}'[m]$

a) Let $a : \mathbb{B} \to \mathbb{K}^{[m]}_{[n]}$. $\mathrm{pre}\left\{\frac{a}{b} \middle| \mathbb{B}\right\}$ consists of the single member ab^{-1}

$b^{-1}a$.

b) Let $a : \mathbb{B} \to \mathbb{K}^{[m]}_{m}$. $\mathrm{post}\left\{\frac{a}{b} \middle| \mathbb{B}\right\}$ consists of the single member ~~ab^{-1}~~ $b^{-1}a$

ii) c) If $m = n$, both single members described above are nonsingular over $NS(a)$.

iv) ~~Structure~~ Let $m, n, h \in \bar{\mathbb{N}}$ and $a : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$.

a) Let ~~$a : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$~~ $b : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$ and $x \in \mathrm{pre}\left\{\frac{a}{b} \middle| \mathbb{B}\right\}$.

$$\mathrm{pre}\left\{\frac{a}{b} \middle| \mathbb{B}\right\} = x + \mathrm{pre}\left\{\frac{0^{[n]}_{[m]}}{b} \middle| \mathbb{B}\right\}$$

b) Let $b : \mathbb{B} \to \mathbb{K}^{[h]}_{[m]}$ and $y \in \mathrm{post}\left\{\frac{a}{b} \middle| \mathbb{B}\right\}$

$$\mathrm{post}\left\{\frac{a}{b} \middle| \mathbb{B}\right\} = y + \mathrm{post}\left\{\frac{0^{[n]}_{[m]}}{b} \middle| \mathbb{B}\right\}$$

iii) ~~?~~Lets.

~~c) Let $b : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$ and~~

v) Let $m, n, h \in \bar{\mathbb{N}}$ and $a : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$ be nonzero over $\mathbb{B}' \subseteq \mathbb{B}$

a) Let $b : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$.

$$\mathrm{pre}\left\{\frac{a}{b} \middle| \mathbb{B}'\right\} \subseteq \left\{\mathbb{B}' \to \mathbb{K}^{h\,[m]}_{[h]\,m} \setminus 0^{h\,[m]}_{[h]\,m}\right\}$$

b) Let $b : \mathbb{B} \to \mathbb{K}^{[h]}_{[m]}$

$$\mathrm{post}\left\{\frac{a}{b} \middle| \mathbb{B}'\right\} \subseteq \left\{\mathbb{B}' \to \mathbb{K}^{hn\,[n]}_{[h]\,h} \setminus 0^{hn\,[n]}_{[h]\,h}\right\}$$

Commutativity and intersection

iii) ~~Let~~ Let $h, k, m, n \in \bar{\mathbb{N}}$ and $a, e : \mathbb{B} \to \mathbb{K}[h]$, $b : \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$

$c : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$, $d : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$

a) ~~Let~~ Let a be nonzero ~~(nonsingular)~~ over $\mathbb{B}$. If ~~$e : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$~~

e is zero ~~(singular)~~ over $\mathbb{B}$ at least one of the spaces

$$\text{pre}\left\{\frac{a.+b}{c.+d}\Big|\,\overset{z}{\cancel{\mathbb{B}}}\right\},\ \text{post}\left\{\frac{c+.e}{d+.b}\Big|\,z\right\}$$

is void at each $z\in\mathbb{B}$.

b) Let $h=n$ and a be nonsingular over $\mathbb{B}$. If e is ~~zero~~ singular ~~with regard to the spaces~~

~~b) If~~ ~~The same holds true with regard to the spaces~~

$$\text{post}\left\{\frac{a+.c}{b+.d}\Big|\,z\right\},\ \cancel{\text{pre}\left\{\frac{b.+e}{d.+c}\Big|\,z\right\}}$$

over $\mathbb{B}$ at least one of the above spaces is void at each $z\in\mathbb{B}$.

Intersection

~~Commutativity and intersection~~

ii) Let $h,k,m,n\in\bar{N}$ and $a:\mathbb{B}\to\mathbb{K}^{[n]}_{[h]}$, $b:\mathbb{B}\to\mathbb{K}^{[k]}_{[h]}$, $c:\mathbb{B}\to\mathbb{K}^{[n]}_{[m]}$

and $d:\mathbb{B}\to\mathbb{K}^{[k]}_{[m]}$.

a) $$\text{pre}\left\{\frac{a.+b}{c.+d}\Big|\,\mathbb{B}\right\}=\text{pre}\left\{\frac{b.+a}{d.+c}\Big|\,\mathbb{B}\right\}$$

$$=\text{pre}\left\{\frac{a}{c}\Big|\,\mathbb{B}\right\}\cap\text{pre}\left\{\frac{b}{d}\Big|\,\mathbb{B}\right\}$$

b) $$\text{post}\left\{\frac{a+.c}{b+.d}\Big|\,\mathbb{B}\right\}=\text{post}\left\{\frac{c+.a}{d+.b}\Big|\,\mathbb{B}\right\}$$

$$=\text{post}\left\{\frac{a}{b}\Big|\,\mathbb{B}\right\}\cap\text{post}\left\{\frac{c}{d}\Big|\,\mathbb{B}\right\}$$

c) If the first space mentioned in subclause (a) is nonvoid, the last two spaces mentioned are nonvoid. A similar remark applies to subclause (b).

Algebraic (products and sums) of spaces

ii) Let ~~h, m,~~ $h, k, m, n \in \overline{N}$

a) Let $a : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$, $b : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$ and $c : \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$.

$$\operatorname{pre}\left\{\frac{a}{b}\middle|\mathbb{B}\right\}\operatorname{pre}\left\{\frac{b}{c}\middle|\mathbb{B}\right\} \subseteq \operatorname{pre}\left\{\frac{a}{c}\middle|\mathbb{B}\right\}$$

b) Let $a : \mathbb{B} \to \mathbb{K}^{[m]}_{[n]}$, $b : \mathbb{B} \to \mathbb{K}^{[h]}_{[n]}$ and $c : \mathbb{B} \to \mathbb{K}^{[k]}_{[n]}$.

$$\operatorname{post}\left\{\frac{\cancel{a}\,b}{\cancel{b}\,c}\middle|\mathbb{B}\right\}\operatorname{post}\left\{\frac{a\,\cancel{b}}{\cancel{c}\,b}\middle|\mathbb{B}\right\} \subseteq \operatorname{post}\left\{\frac{a}{c}\middle|\mathbb{B}\right\}$$

iii) Let $m, n, k \in \overline{N}$

a) Let $a, b : \mathbb{B} \to \mathbb{K}^{[m]}_{[k]}$ and $c : \mathbb{B}^{[\cancel{m}]}_{[\cancel{k}]} \to \mathbb{K}^{[m]}_{[n]}$. Assuming two of the three spaces involved to be nonvoid

$$\operatorname{pre}\left\{\frac{a}{\cancel{b}\,c}\middle|\mathbb{B}\right\} \pm \operatorname{pre}\left\{\frac{b}{c}\middle|\mathbb{B}\right\} = \operatorname{pre}\left\{\frac{a \pm b}{c}\middle|\mathbb{B}\right\}$$

b) Let $a, b : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ and $c : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$. In a similar way

$$\operatorname{post}\left\{\frac{a}{c}\middle|\mathbb{B}\right\} \pm \operatorname{post}\left\{\frac{b}{c}\middle|\mathbb{B}\right\} = \operatorname{post}\left\{\frac{a \pm b}{c}\middle|\mathbb{B}\right\}$$

Multiplication of spaces by factors

Let ~~k~~ $h, k, m, n \in \overline{N}$ and $a : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$, $b : \mathbb{B} \to \mathbb{K}^{[m]}_{[h]}$, $c : \mathbb{B} \to \mathbb{K}^{[m]}_{[k]}$

i) Let ~~$a : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$ $b, c : \mathbb{B} \to \mathbb{K}^{[k][m]}_{[k]}$~~.

$$\operatorname{pre}\left\{\frac{a\,b}{a\,c}\middle|\mathbb{B}\right\} \subseteq \operatorname{pre}\left\{\frac{b}{c}\middle|\mathbb{B}\right\}$$

~~b) If $m = n$ and a is nonsingular over $\mathbb{B}$, the above inclusion relationship becomes one of equivalence~~

Multiplication of spaces ~~by~~ and factors

Let $h, k, m, n \in \bar{\mathbb{N}}$ and $a: \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$

$\cancel{\text{i) Let } b: \mathbb{B} \to \mathbb{K}^{[m]}_{[h]},\ c: \mathbb{B} \to \mathbb{K}^{[m]}_{[k]}}$

$$\cancel{\mathrm{pre}\left\{\frac{b}{c} \,\middle|\, \mathbb{B}\right\} \subseteq \mathrm{pre}\left\{\frac{ba}{ca} \,\middle|\, \mathbb{B}\right.}$$

i) Let $b: \mathbb{B} \to \mathbb{K}^{[h]}_{[n]}$, $c: \mathbb{B} \to \mathbb{K}^{[h]}_{[k]}$

$$a\,\mathrm{pre}\left\{\frac{b}{c} \,\middle|\, \mathbb{B}\right\} \subseteq \mathrm{pre}\left\{\frac{ab}{c} \,\middle|\, \mathbb{B}\right\}$$

ii) Let $b: \mathbb{B} \to \mathbb{K}^{[m]}_{[h]}$, $c: \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$

$$\mathrm{post}\left\{\frac{b}{c} \,\middle|\, \mathbb{B}\right\} a \subseteq \mathrm{post}\left\{\frac{ba}{c} \,\middle|\, \mathbb{B}\right\}$$

iii) Let $b: \mathbb{B} \to \mathbb{K}^{[m]}_{[h]}$, $c: \mathbb{B} \to \mathbb{K}^{[m]}_{[k]}$

$$\mathrm{pre}\left\{\frac{b}{c} \,\middle|\, \mathbb{B}\right\} \subseteq \mathrm{pre}\left\{\frac{ba}{ca} \,\middle|\, \mathbb{B}\right\}$$

iv) Let $b: \mathbb{B} \to \mathbb{K}^{[h]}_{[n]}$, $c: \mathbb{B} \to \mathbb{K}^{[k]}_{[n]}$

$$\mathrm{post}\left\{\frac{b}{c} \,\middle|\, \mathbb{B}\right\} \subseteq \mathrm{post}\left\{\frac{ab}{ac} \,\middle|\, \mathbb{B}\right\}$$

v) If $m = n$ and a is nonsingular over $\mathbb{B}$, the above inclusion relationships become ones of equivalence.

Addition of terms to quotient spaces

Let $k, m, n \in \bar{\mathbb{N}}$ and $a: \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$

i) Let $b: \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ and $c: \mathbb{B} \to \mathbb{K}^{[k]}_{[n]}$

$$a + \mathrm{pre}\left\{\frac{b}{c} \,\middle|\, \mathbb{B}\right\} \subseteq \mathrm{pre}\left\{\frac{ac+b}{c} \,\middle|\, \mathbb{B}\right\}$$

ii) Let $b: \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$ and $c: \mathbb{B} \to \mathbb{K}^{[m]}_{[k]}$

$$a + \mathrm{post}\left\{\frac{b}{c} \,\middle|\, \mathbb{B}\right\} \subseteq \mathrm{post}\left\{\frac{b+ca}{c} \,\middle|\, \mathbb{B}\right\}$$

Proofs

i) $x := ab^{-1} : \mathbb{B} \to \mathbb{K}$ satisfies the relationship $xb = a \langle \mathbb{B} \rangle$ and is uniquely determined by it. $x = b^{-1}a : \mathbb{B} \to \mathbb{K}$ satisfies the relationship $bx = a$ in the same way. If a is nonsingular, so is x.

ii) Select $x : \mathbb{B} \to \mathbb{K}^{[m]}_{[h]}$ for which $x[c.+d] = [a.+b] \langle \mathbb{B} \rangle$ then $xc = a$, $xd = b \langle \mathbb{B} \rangle$: $x[\cancel{d}b.+\cancel{c}a] = [b.+a] \langle \mathbb{B} \rangle$ ~~and conversely~~. Also x is in the intersection of the two further spaces. The converse assertion follows in the same way. Select $x : \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$ for which $[b.+d]x = [a+.c]$ and proceed as above

iii) Select $x : \cancel{\mathbb{B} \to \mathbb{K}} \in \mathbb{K}^{[m]}_{[h]}$ in $\mathrm{pre}\left\{\frac{a.+b}{c.+d} \,\middle|\, z\right\}$ and $y \in \mathbb{K}^{[n]}_{[k]}$ in $\mathrm{post}\left\{\frac{c+.e}{d+.b} \,\middle|\, z\right\}$ so that $xc = a$, $xd = b$, $dy = c$, $by = e$ at z. Then $a = xc = xdy = by = e$ at z. If a is nonzero at z and e is zero at z, ~~see~~ the twin selection is impossible. Similarly for (b).

iv) Select $x' : \mathbb{B} \to \mathbb{K}^{[h]}_{[m]}$ in $\mathrm{pre}\left\{\frac{a}{b} \,\middle|\, \mathbb{B}\right\}$ so that $\cancel{t}\, x'b^{\#} = a \langle \mathbb{B} \rangle$. Then $(x' - x^{\#})b = \cancel{0^{[n]}}\, a - a = 0^{[n]}_{[m]}$: $x' = x^{\#} + x''$ for some $x'' \in \mathrm{pre}\left\{\frac{0^{[n]}_{[m]}}{b} \,\middle|\, \mathbb{B}\right\}$: $\mathrm{pre}\left\{\frac{a}{b} \,\middle|\, \mathbb{B}\right\} \subseteq x + \mathrm{pre}\left\{\frac{0^{[n]}_{[m]}}{b} \,\middle|\, \mathbb{B}\right\}$. This inclusion relationship may be reversed. Select $y' : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$ in $\mathrm{post}\left\{\frac{a}{b} \,\middle|\, \mathbb{B}\right\}$ so that $by' = a \langle \mathbb{B} \rangle$. Proceed as above

v) Select ~~as~~ $x : \mathbb{B} \to \mathbb{K}^{[h]}_{[k]}$ in $\mathrm{pre}\left\{\frac{a}{b} \,\middle|\, \mathbb{B}\right\}$ so that $xb = a \langle \mathbb{B} \rangle$ If $a \neq 0^{[m]}_{[k]} \langle \mathbb{B}' \rangle$, $x \neq 0^{[h]}_{[k]} \langle \mathbb{B}' \rangle$. Select $x : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$ in post$\{\frac{a}{b}|\mathbb{B}\}$ so that $bx=a\ \langle\mathbb{B}\rangle$. Proceed as before

—

(i) Select $x:\mathbb{B}\to\mathbb{K}^{[n]}_{[h]}$, $x':\mathbb{B}\to\mathbb{K}^{[n]}_{[h]}$ in pre$\{\frac{a}{c}|\mathbb{B}\}$, pre$\{\frac{b}{c}|\mathbb{B}\}$, so that $xc=a$, $x'c=b\ \langle\mathbb{B}\rangle$. Then $(x\pm x')c = a\pm b\ \langle\mathbb{B}\rangle$. The semi inclusion relationship derived from (a) holds and can be reversed. Select $x:\mathbb{B}\to\mathbb{K}^{[k]}_{[n]}$, $x':\mathbb{B}\to\mathbb{K}^{[k]}_{[n]}$ in post$\{\frac{a}{c}|\mathbb{B}\}$, post$\{\frac{b}{c}|\mathbb{B}\}$ so that $cx=a$, $cx'=b\ \langle\mathbb{B}\rangle$ and proceed as before.

ii) Select $x:\mathbb{B}\to\mathbb{K}^{[h]}_{[m]}$, $x':\mathbb{B}\to\mathbb{K}^{[k]}_{[h]}$ in pre$\{\frac{a}{b}|\mathbb{B}\}$, pre$\{\frac{b}{c}|\mathbb{B}\}$ so that $xb=a$ $x'c=b\ \langle\mathbb{B}\rangle$. Then $xx'c=a\ \langle\mathbb{B}\rangle$. Select $x:\mathbb{B}\to\mathbb{K}^{[m]}_{[h]}$, $x':\mathbb{B}\to\mathbb{K}^{[h]}_{[k]}$ in post$\{\frac{a}{b}|\mathbb{B}\}$, post$\{\frac{b}{c}|\mathbb{B}\}$ so that $bx=a$, $cx'=b\ \langle\mathbb{B}\rangle$. Then $cx'x=a\ \langle\mathbb{B}\rangle$

—

i) Select $x:\mathbb{B}\to\mathbb{K}^{[n]}_{[m]}$ in pre$\{\frac{b}{c}|\mathbb{B}\}$ so that $xc=b\ \langle\mathbb{B}\rangle$. Then

$\cancel{c(a+x)=a}$ $(a+x)c=ac+b\ \langle\mathbb{B}\rangle$

ii) Select $x:\mathbb{B}\to\mathbb{K}^{[n]}_{[m]}$ in post$\{\frac{b}{c}|\mathbb{B}\}$ so that $cx=b\ \langle\mathbb{B}\rangle$.

Then $c(a+x)=b+ca\ \langle\mathbb{B}\rangle$.

—

i) Select $x:\mathbb{B}\to\mathbb{K}^{[k]}_{[n]}$ in pre$\{\frac{b}{c}|\mathbb{B}\}$ so that $xc=b\ \langle\mathbb{B}\rangle$

Then $axc=ab\ \langle\mathbb{B}\rangle$

ii) Select $x:\mathbb{B}\to\mathbb{K}^{[m]}_{[k]}$ in post$\{\frac{b}{c}|\mathbb{B}\}$ so that $cx=b\ \langle\mathbb{B}\rangle$.

Then $cxa=ba\ \langle\mathbb{B}\rangle$

iii) Select $x:\mathbb{B}\to\mathbb{K}^{[k]}_{[h]}$ in pre$\{\frac{b}{c}|\mathbb{B}\}$ so that $xc=b\ \langle\mathbb{B}\rangle$.

Then $xca=ba\ \langle\mathbb{B}\rangle$

iv) Select $x : \mathbb{B} \to \mathbb{K}^{[k]}_{[k]}$ in post$\left\{\frac{b}{c} \mid \mathbb{B}\right\}$ so that $cx = b \langle \mathbb{B} \rangle$.

Then $acx = ab \langle \mathbb{B} \rangle$

With regard to clause (v)

v) Select $z : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ in pre$\left\{\frac{ab}{c} \mid \mathbb{B}\right\}$ so that $zc = ab \langle \mathbb{B} \rangle$

$z = ax \langle \mathbb{B} \rangle$. Also

~~Then~~ $\cancel{(a^{-1}z)c}$ Set $x := a^{-1}z$. Then $xc = b \langle \mathbb{B} \rangle$ ~~where~~ $\cancel{z = ax}$, so that $x \in$ pre$\left\{\frac{b}{c} \mid \mathbb{B}\right\}$. The remaining clauses are treated in the same way

(SVE) Preliminary definitions

$h, k, m, n, r \in \bar{\mathbb{N}}$. $\mathbb{B}$ is a prescribed set

$\mathbb{K}[\mathbb{B}]^{[r]}_{[k]}$ is the set of all mappings of the form $C: \mathbb{B} \to \mathbb{K}^{[r]}_{[k]}$.

$\mathbb{K}[\mathbb{B}]^{[0]}_{[k]}$, $\mathbb{K}^{[r]}_{[0]}$ and $\mathbb{K}^{\cancel{t}}[\mathbb{B}]^{[k]}_{[k]}$ are written as $\mathbb{K}[\mathbb{B}]_{[k]}$, $\mathbb{K}[\mathbb{B}]^{t[r]}$ and $\mathbb{K}[\mathbb{B},k]$ respectively. ~~$\mathbb{K}'[\mathbb{B},k]$ is the set of all mappings of the form $C:\mathbb{B}\to\mathbb{K}'[k]$.~~

$\mathbb{K}'[\mathbb{B}]^{[r]}_{[k]}$ is the set of all mappings of the form $C:\mathbb{B}\to\mathbb{K}^{[r]}_{[k]}$ for which $C(z) \neq O^{[r]}_{[k]}$ for at least one $z \in \mathbb{B}$. $nz\mathbb{K}[\mathbb{B}]^{[r]}_{[k]}$ is ~~$\mathbb{K}'[\mathbb{B}]^{t[r]}$ is similarly defined. $\mathbb{K}[\mathbb{B},k]$~~ the set of all mappings of the form $C:\mathbb{B}\to\mathbb{K}^{[r]}_{[k]}$. ~~and $\mathbb{K}'[\mathbb{B},k]$ are~~

$\mathbb{K}'[\mathbb{B}]_{[k]}$, ~~and~~ $\mathbb{K}'[\mathbb{B}]^{[r]}$ ~~are similarly~~ ,..... and $nz\mathbb{K}[\mathbb{B},k]$ are similarly defined.

$ns\mathbb{K}''[\mathbb{B},k]$ is the set of all mappings of the form $C:\mathbb{B}\xrightarrow{ns}\mathbb{K}''[k]$.

(i) Let $m,n\in\bar{\mathbb{N}}$. The mapping $\mathbb{PS}: \mathbb{K}[\overset{set}{\mathbb{B}}]^{[n]}_{[m]} \times set \times \bar{\mathbb{N}} \to \mathbb{K}[set]^{[\bar{\mathbb{N}}]}_{[m]}$ is defined by setting

$$\mathbb{PS}\{C,\mathbb{B}\}^{[k]} := pre\left\{\frac{C\,\mathbb{K}[\mathbb{B}]_{[n]}}{\mathbb{K}[\mathbb{B}]_{[k]}}\right\}$$

The above expression represents the space of all mappings $b\in\mathbb{K}[\mathbb{B}]^{[k]}_{[m]}$ that are pre singular with respect to $\overset{\sim\mathbb{B}}{C}$ in the sense that $x \in \mathbb{K}[\mathbb{B}]_{[k\times n]}$ exists for which

$$[b.+C]x = O_{[m]} \quad \langle\mathbb{B}\rangle$$

and $x^{[z]}_{[k]} \neq O^{[z]}_{[k]}$ for some $z\in\mathbb{B}$

(iia) The mapping $\mathbb{PNS}: \mathbb{K}[set]^{[n]}_{[m]}\times set\times\bar{\mathbb{N}}\to\mathbb{K}[set]^{[\bar{\mathbb{N}}]}_{[m]}$ is defined by setting

$$\mathbb{PNS}\{C,\mathbb{B}\}^{[k]} := \mathbb{K}[\mathbb{B}]^{[k]}_{[m]} \setminus \mathbb{PS}\{C,\mathbb{B}\}^{[k]}$$

The above expression represents the space of all mappings $b \in \mathbb{K}[\mathbb{B}]^{[k]}_{[m]}$ that are prenonsingular with respect to C over $\mathbb{B}$ in the sense that for all $x \in \mathbb{K}[\mathbb{B}]_{[k+n+1]}$ for which

$$[b. + C]x = O_{[m]} \qquad \langle \mathbb{B} \rangle$$

$\cancel{x_{[k]} \in nz\mathbb{K}[\mathbb{B}]}$. $x_{[k]} = O_{[k]} \ \langle \mathbb{B} \rangle$.

b) With ~~k~~ the mapping $b \in \mathbb{K}[\mathbb{B}]^{[k]}_{[m]}$ specified, the notation $b \in \text{prens}\{C, \mathbb{B}\}$ indicates that ~~the~~ the mapping ~~in~~ is in $\text{PNS}\{C, \mathbb{B}\}^{[k]}$.

This notation is also used in connection with void C. The notation $b \in \text{prens}\{\mathbb{B}\}$ indicates that

$$\text{post}\left\{ \frac{O_{[m]}}{b} \middle| \overset{z}{\mathbb{B}} \right\} \setminus O_{[k]}$$

is void for all $z \in \mathbb{B}$

Let $m, n, r \in \bar{\mathbb{N}}$ and $\mathbb{B}$ set. The mapping $\text{prerank}: \mathbb{K}[\mathbb{B}]^{[r]}_{[m]} \times \mathbb{K}[\mathbb{B}]^{[n]}_{[m]} \to \bar{\mathbb{N}}$ is defined by setting

$$\text{prerank}(p, C) = \text{rank}[p. + C] - \text{rank}[C]$$

$\text{def}\mathbb{K}_{[h]}[\mathbb{B}]^{[r]}_{[k]}$ and $\text{def}\mathbb{K}^{[n]}[\mathbb{B}]^{[r]}_{[k]}$ are ~~is~~ the sets of all mappings of the form $C: \mathbb{B} \to \mathbb{K}^{[r]}_{[k]}$ for which $\text{rank}(C) \leq k-h$ and $\text{rank}(C) < r-n$ respectively over $\mathbb{B}$.

$k > 0$ $h \in [k)$ $\qquad h \leq \min(i,j)$

$\mathbb{K}[\mathbb{B}, h]^{[j]}_{[i]}$ is the set of all mappings of the form $x: \mathbb{B} \to \mathbb{K}^{[j]}_{[i]}$ for which $\text{rank}[x] = h+1 \ \langle \mathbb{B} \rangle$. $\mathbb{K}[\mathbb{B}, h]^{[h]}_{[h]}$ written as $ns\mathbb{K}[\mathbb{B}, h]$

(E) Let $h, k, m, n \in \bar{N}$, B be a set, and $a : B \to \mathbb{K}^{[n]}_{[h]}$, $b : B \to \mathbb{K}^{[k]}_{[h]}$.

$c : B \to \mathbb{K}^{[n]}_{[m]}$ and $d : B \to \mathbb{K}^{[k]}_{[m]}$

i) Let $\mathrm{pre}\left\{\frac{a}{c} \middle| B\right\}$ be nonvoid.

a) If, for each $z \in B$,

$$\mathrm{post}\left\{\frac{O_{[m]}}{[c+d]} \middle| z\right\}_{(n, k+n+1]} \cap \left\{\mathbb{K}_{[k]} \setminus O_{[k]}\right\}$$

is void then

$$\mathrm{pre}\left\{\frac{[a+b]}{[c+d]} \middle| B\right\}$$

is nonvoid

b) Let $k = 0$ and $z \in B$

$$\mathrm{post}\left\{\frac{O_{[m]}}{[c+d]} \middle| z\right\}_{(n+1]} \cap \left\{\mathbb{K} \setminus O\right\}$$

is void if and only if

$$\mathrm{post}\left\{\frac{d}{c} \middle| z\right\}$$

is void

ii) Let $\mathrm{post}\left\{\frac{a}{b} \middle| B\right\}$ be nonvoid

a) If, for each $z \in B$,

$$\mathrm{pre}\left\{\frac{O^{[k]}}{b+|d} \middle| z\right\}^{(h, h+m+1]} \cap \left\{\mathbb{K}^{[m]} \setminus O^{[m]}\right\}$$

is void, then

$$\mathrm{post}\left\{\frac{a+|c}{b+|d} \middle| B\right\}$$

is nonvoid

b) Let $m=0$ and $z\in\mathbb{B}$.

$$\mathrm{pre}\left\{\frac{0^{[k]}}{b+|d}|z\right\}^{(h+1)}\cap\{\mathbb{K}\backslash 0\}$$

is void if and only if

$$\mathrm{pre}\left\{\frac{d}{b}|z\right\}$$

is void

Proof

ia) Select $z\in\mathbb{B}$ and $t\in[k]$. The condition imposed in subclause (ia) implies that there is no ~~ξ~~ $x\in\mathbb{K}_{[k+n]}$ for which

$$[c|+d][x_{[n+t]}+|1+|x_{(n+t,k+n]}]=0_{[m]}$$

at z. Hence

$$\mathrm{post}\left\{\frac{d^{(t)}}{c|+d^{(-t)}}|z\right\}$$

is void for $t:=[k]$ and in consequence $\mathrm{rank}\,[c|+d]\neq\mathrm{rank}\,[c|+d^{(-t)}]$ at z. But $\mathrm{rank}\,[c|+d]\geq\mathrm{rank}\,[c|+d^{(-t)}]$ so that $\mathrm{rank}\,[c|+d]\geq\mathrm{rank}\,[c|+d^{(-t)}]+1$ $\langle t:=[k]\rangle$ Select $y\in\mathrm{pre}\left\{\frac{a}{c}|\mathbb{B}\right\}$ and $r\in[h]$ so that ~~By~~ $y_{(r)}c=a_{(r)}\langle\mathbb{B}\rangle$.

~~$[1+|(-y_{(r)})$~~

$$[1|+(-y_{(r)})]\{+|[0_{[m]}|+I[m]][[a_{(r)}|+b_{(r)}]+|[c|+d]]=$$

$$[[0^{[m]}|+(b_{(r)}-y_{(r)}d)]+|[c|+d]]$$

Set ~~$R_{\neq}$~~ $R(r):=\mathrm{rank}\,[[a_{(r)}|+b_{(r)}]+|[c|+d]]$. Since the first left hand factor is nonsingular, ~~R~~ $R(r)$ is equal to the

(E) Relationships between quotient spaces

Let $h, k, m, n \in \bar{N}$, $\mathbb{B}$ be a set, and $a : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$, $b : \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$,

$c : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$ and $d : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$.

i) Commutativity and intersection

a) $$\mathrm{pre}\left\{\frac{[a.+b]}{[c.+d]}\middle|\mathbb{B}\right\} \equiv \mathrm{pre}\left\{\frac{[b.+a]}{[d.+c]}\middle|\mathbb{B}\right\}$$

$$\equiv \mathrm{pre}\left\{\frac{a}{c}\middle|\mathbb{B}\right\} \cap \mathrm{pre}\left\{\frac{b}{d}\middle|\mathbb{B}\right\}$$

b) $$\mathrm{post}\left\{\frac{a+.c}{b+.d}\middle|\mathbb{B}\right\} = \mathrm{post}\left\{\frac{c+.a}{d+.b}\middle|\mathbb{B}\right\}$$

$$= ~~\mathrm{pre}~~\,\mathrm{post}\left\{\frac{a}{b}\middle|\mathbb{B}\right\} \cap \mathrm{post}\left\{\frac{c}{d}\middle|\mathbb{B}\right\}$$

ii) Nonsingularity

a) Let a be nonzero over $\mathbb{B}' \subseteq \mathbb{B}$.

$$\mathrm{pre}\left\{\frac{a}{b}\middle|\mathbb{B}'\right\} \subseteq \left\{\mathbb{B}' \to \mathbb{K}^{[m]}_{[h]} \setminus O^{[m]}_{[h]}\right\}$$

$$\mathrm{post}\left\{\frac{a}{b}\middle|\mathbb{B}'\right\} \subseteq \left\{\mathbb{B} \to \mathbb{K}^{[n]}_{[k]} \setminus O^{[n]}_{[k]}\right\}$$

b) Let $a, b : \mathbb{B} \to \mathbb{K}^{[m]}_{[m]}$ and ~~a be n~~ $NS(a) = \mathbb{B}'$

$$NS\left\{\mathrm{pre}, \mathrm{post}\left\{\frac{a}{b}\middle|\mathbb{B}\right\}\right\} = \mathbb{B}'$$

The extension of quotient denominators.

If $(c.+d) : \mathbb{B} \to \mathbb{K}[m]$ then, for ~~suitably declared~~ mappings a, b of suitable dimension

$$\mathrm{pre}\left\{\frac{a.+b}{c.+d}\middle|\mathbb{B}\right\}$$

is nonvoid, without further specification of, in particular, b.

It is possible to obtain a result of this nature without imposing the condition that $c.+d$ should be nonsingular.

... may be treated in the same way.

Pre and post nonsingular mappings

Let $k, m, n \in \bar{N}$ and $a : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$.

i) A mapping $b : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ such that for each $z \in \mathbb{B}$

$$post\left\{ \frac{O_{[m]}}{a. + b} | z \right\}_{(n, k+n+1]} \cap \{ K_{[k]} \setminus O_{[k]} \}$$

is void is said to be prenonsingular with respect to a over $\mathbb{B}$.

ii) A mapping $b : \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$ such that for each $z \in \mathbb{B}$

$$pre\left\{ \frac{O^{[n]}}{a + . b} | z \right\}^{(m, k+m+1]} \cap \{ K^{[k]} \setminus O^{[k]} \}$$

is void is said to be postnonsingular with respect to a over $\mathbb{B}$

iii) The above definitions are extended to cases in which a features in a void mapping. A mapping $b : \mathbb{B} \to K^{[k]}_{[m]}$ such that for each $z \in \mathbb{B}$

$$post\left\{ \frac{O_{[m]}}{b} | z \right\} \cap \{ \mathbb{K}_{[k]} \setminus O_{[k]} \}$$

is void is said to be prenonsingular over $\mathbb{B}$. $b : \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$ for which, for each $z \in \mathbb{B}$,

$$pre\left\{ \frac{O^{[n]}}{b} | z \right\} \cap \{ K^{[k]} \setminus O^{[k]} \}$$

is void is said to be postnonsingular over $\mathbb{B}$.

Let $k, m, n \in \bar{N}$ and $a : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$.

ia) If $b : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ is prenonsingular with respect to a over $\mathbb{B}$,

$k \le m - rank[a] \langle \mathbb{B} \rangle$

b) If $b : \mathbb{B} \to K^{[n]}_{[k]}$ is post nonsingular with respect to a over $\mathbb{B}$,

ii) $b : \mathbb{B} \to K^{[k]}_{[m]}$ is prenonsingular with respect to a over $\mathbb{B}$

a) if and only if

$$\operatorname{rank}[a.+b^{(h)}] \neq \operatorname{rank}[a.+b^{[h]}] \quad \langle h := [k]; \mathbb{B} \rangle$$

b) if and only if

$$\operatorname{post}\left\{ \frac{b^{(h)}}{a.+b^{[h)}} \,\middle|\, z \right\} \quad \langle h := [k] \rangle$$

is void ~~at each $z \in \mathbb{B}$ and~~ at each $z \in \mathbb{B}$

c) if and only if

$$\operatorname{rank}[a.+b^{[h]}] = \operatorname{rank}[a] + h + 1 \quad \langle h := [k]; \mathbb{B} \rangle$$

~~at each $z \in \mathbb{B}$ in all cases~~

iii) $b : \mathbb{B} \to K^{[n]}_{[k]}$ is postnonsingular with respect to a over $\mathbb{B}$

a) if and only if

$$\operatorname{rank}[a+.b_{(h)}] \neq \operatorname{rank}[a+.b_{[h]}] \quad \langle h := [k]; \mathbb{B} \rangle$$

b) if and only if

$$\operatorname{pre}\left\{ \frac{b_{(h)}}{a+.b_{[h)}} \,\middle|\, z \right\} \quad \langle h := [k] \rangle$$

is void at each $z \in \mathbb{B}$

c) if and only if

$$\operatorname{rank}[a+.b_{[h]}] = \operatorname{rank}[a] + h + 1 \quad \langle h := [k]; \mathbb{B} \rangle.$$

Proof.

ii) Select $z \in \mathbb{B}$ and $h \in [k]$. If b is prenonsingular with respect to $\mathbb{B}$ there is, in particular, no $y \in K^{[n+h]}$ for which

$$[a.+b][(-y)+.1+.0_{[k-h)}] = 0_{[m]}$$

i.e.

$$[a. + b^{[h)}]y = b^{(h)}$$

at z: the condition stated in a) is satisfied. Assume that this condition holds and that for some $y \in K_{[n+k+1]}$ for which

$$- - \cancel{\ldots}\ y_{(n, k+n+1]} \neq 0_{[k]},$$

$$[a. + b]y = 0_{[m]}$$

at $z \in B$.

~~$y^{(n)} = 0$ and $y^{(r)} = 0 \langle r := (h, k]$~~

$y^{(r)} = 0 \langle r := (n+h, n+k+1] \rangle$ and $y^{(n+h)} \neq 0$ for some $h \in [k]$

Set $x = -\dfrac{y^{[n+h)}}{y^{(n+h)}}$. Then

$$[a. + b^{[h)}]x = b^{(h)}$$

so that rank $[a. + b^{[h)}] =$ rank $[a. + b^{[h]}]$ at z. The condition stated in a) is violated, the selection of y is impossible.

The condition stated in (a) implies and is implied by that stated in (b). ~~Since rank $[a. + b^{[h]}] \geq$ rank $[a. + b^{[h)}]$,~~ ~~rank $[a. + b$~~ rank $[a. + b^{[h)}] =$ rank $[a]$ when $h = 0$. Since rank $[a. + b^{[h]}] \geq$ rank $[a. + b^{[h)}]$, the condition of clause (c) follows from that stated in clause (a).

ia) From condition (ic) rank $[a. + b] =$ rank $[a] + k + 1$ at z when b is prenonsingular with respect to a over b. But rank $[a. + b] \leq m + 1$. Hence $k \leq m -$ rank $[a]$.

The proofs of clause (iii) and (ib) are analogous.

Let $h, k, m, n \in \bar{\mathbb{N}}$ and $a : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$, $b : \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$, $c : \mathbb{B} \to \mathbb{K}^{c}_{[m]}$

and $d : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$.

i) Let $\mathrm{pre}\left\{\frac{a}{c} \middle| \mathbb{B}\right\}$ be nonvoid and d be prenonsingular with respect to c over $\mathbb{B}$.

$$\mathrm{pre}\left\{\frac{\{a.+b\}}{c.+d} \middle| \mathbb{B}\right\}$$

is nonvoid

ii) Let $\mathrm{post}\left\{\frac{a}{b} \middle| \mathbb{B}\right\}$ be nonvoid and d be post nonsingular with respect to b over $\mathbb{B}$.

$$\mathrm{post}\left\{\frac{a_{\epsilon}+.c}{b+.d} \middle| \mathbb{B}\right\}$$

is nonvoid.

Proof $\overset{\text{i a)}}{\text{Se}}$ Select $z \in \mathbb{B}$ and $t \in [k]$. ...

Proof.

ia) Select $z \in \mathbb{B}$ and $t \in [k]$. The condition imposed in subclause (ia) implies that there is no $x' \in K_{[k-t)}$ for which

$$[c.+d][x'+.1+.0_{[k-t)}] = 0_{[m]}$$

at z, or, alternatively, for which

$$[c.+d^{[t]}][x'+.1] = 0_{[m]}$$

or, again, for which

$$[c.+d^{(t)}]x' = d^{[t]}$$

at z. In consequence $\operatorname{rank}[c.+d^{(t)}] \neq \operatorname{rank}[c.+d^{[t]}]$ at z. But $\operatorname{rank}[c.+d^{[t]}] \geq \operatorname{rank}[c.+d^{(t)}]$ so that $\operatorname{rank}[c.+d^{[t]}] \geq \operatorname{rank}[c.+d^{(t)}]+1$ at z.

~~From the stated conditions~~ Assume that $\operatorname{pre}\left\{ \dfrac{[a+b^{(t)}]}{[c+d^{(t)}]} \,\middle|\, \mathbb{B} \right\}$ is nonvoid ~~void~~ $: \mathbb{B} \to K^{[m]}_{[h]}$ ~~For this value of~~ Select $y(t) \in$ from this space and $r \in [h]$ so that $y(t)_{(r)}[c.+d^{(t)}] = [a.+b^{(t)}]_{(r)}$

$\langle \mathbb{B} \rangle$

$$[[1.+(-y(t)_{(r)})]+.[0_{[m]}.+I[m]]][[a.+b^{[t]}]_{(r)}+.[c.+d^{[t]}]]$$

$$[[0^{[m+t]}.+(b^{(t)}_{(r)} - y(t)_{(r)}d^{(t)})]+.[c.+d^{[t]}]]$$

at z.

Set $R(r,t) = \operatorname{rank}[[a.+b^{[t]}]_{(r)}+.[c.+d^{[t]}]]$. Since the first left hand factor above is nonsingular, $R(r,t)$ is equal to the rank of the right hand term. If $b^{(t)}_r = y(t)d^{(t)}$ at z, this rank is simply that of $[c.+d^{[t]}]$.

Let $k, m, n \in \bar{N}$ and $a: \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$, ~~$c: \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$~~

~~$b: \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$~~

~~i) pre$\left\{\frac{a}{c} \middle| \mathbb{B}\right\}$ being nonvoid, a mapping $d: \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$~~

$a^n_m \quad b^k_m$

b^n_k

~~such that for each $z \in \mathbb{B}$~~

~~$$\text{post}\left\{\frac{0_{[m]}}{[c+d]} \middle| z\right\}_{(n, k+n+1]} \cap \{\mathbb{K}_{[k]} \setminus 0_{[k]}\}$$~~

~~is void~~

~~is said to be prenonsingular with respect to a/c over $\mathbb{B}$.~~

~~ii) post$\left\{\frac{a}{b} \middle| \mathbb{B}\right\}$ being nonvoid, a mapping d~~

i) ~~Let $b: \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$ be such that pre$\left\{\frac{a}{b} \middle| \mathbb{B}\right\}$ is nonvoid.~~

A mapping $b: \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ such that for each $z \in \mathbb{B}$

$$\text{post}\left\{\frac{0_{[m]}}{[a+b]} \middle| z\right\}_{(n, k+n+1]} \cap \{\mathbb{K}_{[k]} \setminus 0_{[k]}\}$$

is void is said to be prenonsingular with respect to a over $\mathbb{B}$.

ii) ~~Let $b: \mathbb{B} \to \mathbb{K}^{[h]}_{[m]}$ be such that post$\left\{\frac{a}{b} \middle| \mathbb{B}\right\}$ is nonvoid.~~ A mapping $c: \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$ such that for each $z \in \mathbb{B}$

$$\text{pre}\left\{\frac{0^{[n]}}{[a+.b]} \middle| z\right\}_{(m, k+m+1]} \cap \{\mathbb{K}^{[k]} \setminus 0^{[k]}\}$$

is void is said to be post nonsingular with respect to a over $\mathbb{B}$.

iii) The above definitions are extended to cases in which a features in a void mapping. A mapping $b: \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ such that for each $z \in \mathbb{B}$,

$$\text{post}\left\{\frac{0_{[m]}}{b} \middle| z\right\} \cap \{\mathbb{K}_{[k]} \setminus 0_{[k]}\}$$

is void is said to be prenonsingular over B. ~~A mapping~~

~~while~~ If, for each $z \in B$,

$$\mathrm{pre}\left\{ \frac{O^{[k|n]}}{b} \Big| z \right\} \cap \{ \mathbb{K}^{[k]} \setminus O^{[k]} \}$$

is void, & the mapping is said to be post nonsingular over B.

$\mathrm{rank}[a.+b^{[h)}] = \mathrm{rank}[a]+h \quad \langle h := [k+1] \rangle$

$\mathrm{rank}[a]+h+1 \le m \qquad h \le m - \mathrm{rank}[a]-1 \qquad h < m-\mathrm{rank}[a]$

$m \; y \quad [a.+b^{[h)}]y = b^{(h)} \qquad h = k \qquad r := (n+h, n+k+1]$

$[a.+b]w = O_{[m]} \qquad w^{(k)} = 0 \quad k := (h, k]$

$b^{[h)}$ $\qquad w^{(h)} \ne 0 \quad \tilde{w} = -w^{[}$

$[a.+b]\tilde{w} = b^{(h)} \qquad w^{(n+h)} \ne 0 \quad \tilde{w} = -\frac{w^{[n+h)}}{w^{(n+h)}}$

$\mathrm{rank}[a.+b^{[h)}] = \mathrm{rank}[a.+b^{[h]}]$

$\mathrm{rank}[a.+b^{(h)}] \ne \mathrm{rank}[a.+b^{[h]}] \quad h := [k] \qquad n \;\; k=0$

$m=1 \quad 0 \; \cdot$

$\mathrm{post}\left\{ \frac{b^{(h)}}{a.+b^{[h)}} \Big| z \right\} \text{ void} \quad h := [k] \qquad 0 \; \cdot$

$\mathrm{rank}[a+.b_{[h)}] \ne \mathrm{rank}[a+.b_{[h]}] \quad h := [k]$

$\mathrm{pre}\left\{ \frac{b_{(h)}}{a+.b_{[h)}} \Big| z \right] \text{ void} \quad h := [k]$

$\mathrm{rank}[a_s+.b_{[h)}] = \mathrm{rank}[a]+h$

Otherwise

$$R(r,t) = \max\{\operatorname{rank}[c_{\cdot}+d^{[t]}], \operatorname{rank}[c_{\cdot}+d^{[t]}]+1\}$$

since d is pre nonsingular with respect to c

Hence, ~~from the result obtained in the above first paragraph.~~

$R(r,t) = \operatorname{rank}[c+d^{[t]}]$ at z. Thus, for each $r \in [h]$,

$x_{(r)} \in \mathbb{K}^{[m]}$ exists for which

$$x_{(r)}[c_{\cdot}+d^{[t]}] = [a_{\cdot}+b^{[t]}]_{(r)}$$

at z: $x \in \mathbb{K}^{[m]}_{[h]}$ exists for which

$$x[c_{\cdot}+d^{[t]}] = [a_{\cdot}+b^{[t]}]$$

at z: $x : \mathbb{B} \to \mathbb{K}^{[m]}_{[h]}$ exists for which this relationship holds

over $\mathbb{B}$: $\operatorname{pre}\left\{\dfrac{[a_{\cdot}+b^{[t]}]}{[c_{\cdot}+d^{[t]}]} \,|\, \mathbb{B}\right\}$ is nonvoid when $t=0$.

It has been shown that ~~when $t=0$~~ if, ~~when $t=0$~~

$\operatorname{pre}\left\{\dfrac{[a_{\cdot}+b^{[t]}]}{[c_{\cdot}+d^{[t]}]} \,|\, \mathbb{B}\right\}$ is nonvoid, $\operatorname{pre}\left\{\dfrac{[a_{\cdot}+b^{[t]}]}{[c_{\cdot}+d^{[t]}]} \,|\, \mathbb{B}\right\}$ is nonvoid.

From the condition stated in clause (i), the assumption is correct when $t=0$. The resulting conclusion shows that it is correct when $t=1$. By induction, the conclusion holds when $t=k$: the result of clause (ii) follows.

rank of the right hand term. Let Ξ be the set of $t \in [k]$ for which $b^{(t)}_{(r)} - y_{(r)} d^{(t)} \neq 0$.

$$R_{(r)} = \max\{\text{rank}[c+d], \text{rank}[d+d^{(-t)}]+1 \langle t := \Xi \rangle\}$$

$$= \text{rank}[c+d]$$

Thus, for each $r \in [k]$, $x_{(r)} \in K^{[n]}$ exists for which

$$x_{(r)}[c+d] = [a_{(r)} | + b_{(r)}]$$

$x \in K^{[m]}_{[h]}$ for which $x[c+d] = [a+b]$ at z exists and hence the result of the subclause follows.

ib) Select $z \in B$ and assume that post$\{\frac{d}{c}|z\}$ is void. If $x \in K_{[n+1]}$ exists such that $[c+d]x = 0_{[m]}$ and $x_{(n+1)} \neq 0$ at z, then

$$c\left(\frac{-x_{[n]}}{x_{(n+1)}}\right) = d$$

and post$\{\frac{d}{c}|z\}$ is nonvoid: no such x exists: the first space referred to is void. Assume that this ~~latter~~ first space is void. If $x \in K_{[n]}$ ~~exists~~ for which $cx = d$ at z exists then $[c|+d][x+1(-1)] = 0$ and the ~~latter~~ first space is nonvoid: no such x exists: post$\{\frac{d}{c}|z\}$ is nonvoid.

The proof of clause (ii) is parallel to that of its predecessor. ~~There is no~~ Assuming that there is no $x' \in K^{[h+t)}$ for which

$$[x'.+1+0^{[m-t)}][b+.d] = 0^{[k]}$$

it is deduced ~~to~~ that rank$[b+.d_{[t]}] \geq$ rank$[b+.d_{[t)}]+1$. Assumin that post$\left\{ \dfrac{[a+.c_{[t]}]}{[b+.d_{[t]}]} \middle| \mathbb{B} \right\}$ is nonvoid, $y(t) : \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$ is selected from this space so that, with $r \in [n]$, $[b+.d_{[t]}]y(t)^{(r)} = [a+.c_{[t]}]^{(r)}$. It then follows that

$$[[b+.d_{[t]}].+_{\xi}[a+.c_{[t]}]^{(r)}][[[I[k]+.0^{[k]}].+[\xi - y(t)^{(r)}+.1]]] =$$

$$[[b+.d_{[t]}].+[0_{[h+t]}+.(c_{(t)}^{(r)} - d_{(t)}y(t)^{(r)})]]$$

This relationship is used to show that ~~rank~~

$$\operatorname{rank}[[b+.d_{[t]}].+[a+.c_{[t]}]^{(r)}] = \operatorname{rank}[b+d_{[t]}]$$

and hence that post$\left\{ \dfrac{[a+.c_{[t]}]}{[b+.d_{[t]}]} \middle| \mathbb{B} \right\}$ is nonvoid. Finally the result of ~~clau~~ subclause (iia) is obtained.

For the proof of subclause (iib) it is first shown that if at $z \in \mathbb{B}$, $x \in \mathbb{K}^{[h+1]}$ exists such that $x[b+.d] = 0^{[k]}$ and $x^{(h+1)} \neq 0$ then pre$\left\{ \dfrac{d}{b} \middle| z \right\}$ is nonvoid. Again, if $x \in \mathbb{K}^{[h]}$ for which ~~$xb=d$~~ ~~at $z \in \mathbb{B}$ exists~~ ξ $[x.+(-1)][b+.d] = 0^{[k]}$ exists, the first space referred to is nonvoid.

(E) $\begin{matrix}1&1\\1&2\\1&3\end{matrix}$ $\begin{matrix}1\\2\\3\end{matrix}$ $\begin{matrix}1&2&3\\1&2&3\\1&2&3\end{matrix}$ $\begin{matrix}1&2&4\\2&4&8\\3&6&12\end{matrix}$ $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}\cancel{x\ y}\end{pmatrix}^{x\ y} = \begin{matrix}2&4\\4&8\\6&12\end{matrix}$

Suppose rank $[A\,b\,c\ \ d\,e] = \text{rank}[A]$ implies rank $[A\,b] =$ rank $[Ac] =$ rank $[Ae]$ $=$ rank $[A]$

then rank $[A\,b\,c \ldots d\,e\,f] = \text{rank}[A]$

$\Rightarrow$ rank $[A\,b\,c \ldots d\,e\,f] = \text{rank}[A\,b\,c \ldots d\,e]$

suppose

rank $[A, X^{[\mathcal{D},\omega]}] = \text{rank}[A]$

$\Rightarrow$ rank $[A, X^{[\mathcal{D},x]}] = \text{rank}[A]$ all $[\mathcal{D}, x] \leq [\omega]$

rank $[A, X^{[\omega+1]}] = \text{rank}[A]$

$\Rightarrow$ rank $[A, X^{[\omega+1]}] = \text{rank}[A X^{0}]$

$\Rightarrow$ rank $[A X^{(0)} + \cancel{X}^{[\ldots, x+1]}] = \text{rank}[A X^{0}]$

all $[\mathcal{D}, x] \leq [\omega]$

i.e rank $[A, X^{[x+1]}] = \text{rank}[A]$

must show rank $[A, X^{[\mathcal{D},\omega+1]}] = \text{rank}[A]$ $\mathcal{D}: (\omega+1]$

rank $[A, X^{[\mathcal{D},\omega+1]}] = \text{rank}[A .+ X^{[\mathcal{D},\omega]} .+ X^{(\omega+1)}]$

rank $[A\,b\,c] = \text{rank}[A\,b] = \text{rank}[A]$ does not yield

doch

$AX = [b\ c]$ $A = 1$ $b = 1$ $Ax_1 = b \Rightarrow x_1 = 1$

$1 \cdot (x_1, x_2) = 1\ \ 1$ $(1.\ 1.)\begin{pmatrix}y\\y'\end{pmatrix} = 1$

rank $[A.+B] = \text{rank}[A] \Rightarrow \exists X$ s.t. $AX = B$ ‖ suppose col dim $B = n+1$ and have proved for n

$B = B', b$ ~~rank $[A.+B'] \leq$~~ rank $[A] \leq \text{rank}[A+B'] \leq \text{rank}[A+B]$

$\Rightarrow$ rank $[A] = \text{rank}[A+B'] = X'$ $AX' = B'$

rank $[A+B'] = \text{rank}[A+B] \Rightarrow x: (A+B')x = b$

$B = \cancel{A}\,B' .+ b$ $AX'. + (A + AX')x = A[X'. + (I + X')x] = AX$

if conditions (a,b) hold,

$$\mathrm{rank}[c.+d] = \mathrm{rank}[[a.+b] + [c.+d]]$$

$a+b<1 \quad a+b+c<1$

$a-b>1 \quad a-(b+c)>1$

$a-c>1 \quad a>-b$

$a>-(b+c)$

$f := e+.f \qquad e: B \to K^{[r]}_{[k]} \quad f := K^{[r]}_{[m]}$

$a(b+1)>1 \quad a(b+c+1)>1$

$a(c+1)>1$

---

if conditions (a,b,c) hold

$$\mathrm{rank}[c.+d] = \mathrm{rank}[c.+d.+f]$$

$ay+by' = e$

$a+b$

$\mathrm{rank}[c.+f] = \mathrm{rank}[c]$ iff

$\mathrm{post}\left[\frac{f}{c.+d}\right] \cap \left[\cancel{\{B \to K\}}\ \mathrm{post}\left\{\frac{b}{a}\right\} +. \{B \to K^{[r]}_{[k]}\}\right.$

contains $y+.y'$ for which $y' \cancel{\ldots} \frac{0}{d}$ $\mathrm{post}\left\{\frac{0_{[m]}}{d}\right\}$

$[c.+d]\begin{bmatrix} y \\ y' \end{bmatrix} = f \Rightarrow cy = f \Rightarrow \mathrm{rank}[c.+f] = \mathrm{rank}[c]$

$cy + dy' = f \quad ay = b \qquad \Rightarrow \mathrm{rank}[c.+f] = \mathrm{rank}[c.+d]$

$c\frac{y}{y'} - \left(\frac{y}{y'}\right) + f\frac{1}{y'} = d \qquad \mathrm{rank}[c+f] = \mathrm{rank}[c+f+d]$

$= \mathrm{rank}[c+d]$

$[c.+f]\begin{bmatrix} -\frac{y}{y'} \\ \frac{1}{y'} \end{bmatrix} = d$

If $\mathrm{post}\left\{\frac{f}{c.+d}\right\}_{(n,k+n+1]} \cap \mathrm{post}\left\{\frac{0^{[r]}_{[m]}}{d}\right\}$ nonvoid then

$\mathrm{rank}[c.+f] \overset{\text{not}}{=} [c]$

$r=k$ If $\mathrm{post}\left\{\frac{f}{c.+d}\right\}_{(n,k+n+1]} \cap \left\{B \to K'^{[r]}_{[k]}\right\}$ nonvoid then

$\mathrm{rank}[c.+f] \neq \mathrm{rank}[c]$

$m=k \quad d: B \to K'[m] \qquad \mathrm{post}\left\{\frac{0^{[r]}_{[m]}}{d}\right\} \equiv 0^{[r]}_{[k]}$

$= \mathrm{rank}[c.+d]$

---

d prenonsingular wrt. a : $y_{(n,n+k+1]}$ same for all $y \in \mathrm{post}\left[\frac{f}{c.+d}\right]$

if $\mathrm{post}\left\{\frac{0}{c}\right\} \cap 0^{k}_{[\ ]}$ void $\mathrm{post}\left[\frac{f}{c.+d}\right]$ consists of one member

$p = \hat{p} + \tilde{p}$ $\quad \hat{p} \in \mathrm{prens}(C,B)^{[r]}$ $\quad \tilde{p} \in$ —

$\tilde{p} \in C\,\mathrm{post}\left\{\frac{O^{[r]}_{[h]}}{u}\right\}$ $\qquad \tilde{p} = Cy'$ $\qquad uy' = 0$

$p = bx + Cx'$ $\qquad \hat{p} = bx + C(x' - y')$

suppose ~~nonzero t~~ $x \notin \mathrm{prens}(B)$ ~~now~~ $\exists$ nonzero t for which

$xt = 0$ $\quad \hat{p}t = C(x'-y')t$ $\quad \hat{p} \notin \mathrm{prens}(C,B)^{[r]}$

$\therefore\ x \in \mathrm{prens}(B)$

suppose $x \in \mathrm{prens}(B) \Rightarrow bx \in \mathrm{prens}(C,B)^{[r]}$?

$bx \notin \mathrm{prens}(C,B)^{[r]} \Rightarrow \exists y \quad bxy + Cy = O_{[m]}$ with $y \neq 0$

$xy \neq 0$ since $x \in \mathrm{prens}(B) \Rightarrow b \notin \mathrm{prens}(C,B)$

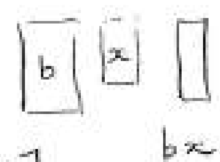
$\therefore \Rightarrow$ $b \in \mathrm{prens}(C,B)^{[k]}$ iff $bx \in \mathrm{prens}(C,B)^{[r]}$ $\quad k + \mathrm{rank}(C) + 1 \leq m$

$r + \mathrm{rank}(C) + 1 \leq m$

$x \in \mathrm{prens}(B)^{[r]}_{[k]}$ implying $r \leq k$

$bxy + Cy' = O_{[m]}$ for no nonzero $\overset{\circ}{y}_r \parallel \overset{\circ}{y'}_n$

if $bw + Cw' = O_{[m]}$ for nonzero $\overset{\circ}{w}_k \parallel \overset{\circ}{w'}_n$



$\exists y$ for which $xy = w$ yes if $\mathrm{rank}[x .+ w] = \mathrm{rank}[x]$

otherwise not.

$x \in \mathrm{prens}(B)^{[r]}_{[k]}$ (implying $r \leq k$) $\qquad C : B \to K^{[n]}_{[m]}$

a) if $b \in \mathrm{prens}(C,B)^{[k]}$ then $bx \in \mathrm{prens}(C,B)^{[r]}$

b) $r = k$: $b \in \mathrm{pren}(C,B)^{[k]}$ $\cancel{\Rightarrow}$ iff $bx \in \mathrm{prens}(C,B)^{[k]}$

a): no nonzero $\overset{\circ}{y}_k$ for which $by + Cy' = O_{[m]}$

suppose $\exists$ nonzero w_r for which $bxw + Cw' = O_{[m]}$

$x \in \mathrm{prens}(B)^{[r]}_{[k]} \to xw \neq O_{[k]} \Rightarrow b \notin \mathrm{prens}(C,B)^{[k]}$.

b) $bxy + Cy' = 0_{[m]}$ for no nonzero y°_{k} y'°_{n}

if $bw + Cw' = 0_{[m]}$ for nonzero w°_{k} ‖ w'°_{n}

determine y from $xy = w$ (possible since x square); y nonzero

then $bxy + Cy' = 0_{[m]}$ with nonzero y°_{k} w'°_{n} $\Rightarrow bx \notin \text{prens}(C \downarrow B)^{[k]}$

—

define $\overset{\text{post?}}{\text{pre}}$ rank(b, C) as rank$[b. + C]$ − rank$[C]$

$\exists\ X^{k}_{n}$ for which $CX = b$ iff $\overset{\text{post?}}{\text{pre}}$ rank$(b, C) = 0$

=

? rank(pfact $[(p/b)\ C]$) = pre rank(p, C) ⟨B⟩

$p = bx + Cx'$ ~~rank~~ ~~prank~~ prerank$[p, C]$ = prerank$[p - Cx', C]$

$p - Cx' = \hat{p}$ $\hat{p} = bx$ rank$[x]$ = ~~rank~~ pre rank$[\hat{p}]$

rank$[\hat{p}. + C]$ = rank$[x]$ + rank$[C]$?

$\hat{p}t = 0$ iff $bxt = 0$ iff $xt = 0$

? i.e. rank $\hat{p}$ = rank x

rank$[p. + b. + C]$ = rank$[b. + C]$ = rank$[C] + k + 1$

rank$[(p - Cx'). + b]$ = rank$[p - Cx']$ = rank$[x]$

rank$[p - Cx']$ = rank$[p. + C]$?

show rank$[x]$ = ~~prerank$[p, C]$~~ min{prerank$[p, C]$, $k+1$}

= min{ rank$[p. + C]$ − rank$[C]$, rank$[b. + C]$ − rank$[C]$ }

rank$[C]$ + rank$[x]$ = min{ rank$[p. + C]$, rank$[b + C]$ }

= rank$[p. + C]$

suppose have shown

$$p^{[r]} = bx^{[r]} + Cx'^{[r]}$$

for some r (may show for r=0 independently)

$$p^{(r+1)} = bx^{(r+1)} + Cx'^{(r+1)}$$

$r \leq k$, rank$[x]$ determines

pre rank$[p, C]$

pre rank $[p, C]$ = pre rank $[p - Cx', C$

= pre rank $[bx, C]$

$$\text{rank}[C] + \text{rank}(x^{(r+1)}) = \text{rank}[p^{(r+1)} \vdots + C]$$

$$\text{rank } x^{[r+1]} = \text{rank } x^{[r]} \quad \text{if } \exists y \quad x^{(r+1)} = x^{[r]}y$$

if such y exists $p^{(r+1)} - p^{[r]}y = C(x'^{(r+1)} - x'^{[r]}y)$

$$\begin{aligned} \text{rank}[p^{[r+1]} \vdots + C] &= \text{rank}[p^{[r]} \vdots + C] \\ &= \text{rank}[x^{[r]}] + \text{rank}[C] \\ &= \text{rank}[x^{[r+1]}] + \text{rank}[C] \quad ⊛ \end{aligned}$$

if no such y exists

suppose $\text{rank}[p^{[r]} \vdots + C] = \text{rank}[p^{[r+1]} \vdots + C]$

$\exists$ nonzero y for which $p^{(r+1)} = [p^{[r]} \vdots + C]y$

$$b\{x^{(r+1)} - x^{[r]}y\} = C\{x'^{(r+1)} - x'^{[r]}y\}$$

since $b \in$ prens$(C|B)$ $\quad x^{(r+1)} - x^{[r]}y = 0$ contrary to assumption

$$\therefore \text{rank}[p^{[r+1]} \vdots + C] = \text{rank}[p^{[r]} + C] + 1$$

⊛ again true

For all $b \leq p\{(u/C)B\}^{[k]}$, rank(pfact$[(p/b)C]$)

$$= \min\{\text{prerank}[p, C], k+1\}$$

implying both pfact$[(p/b)C] = 0$ when $\not\exists$ post$\{\frac{p}{C}|B\}$ nonvoid

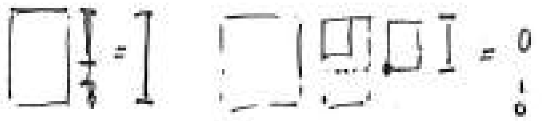
and pfact$[(p/b)C] \in$ prens$\{B\}$

when $p \in$ prens$\{C|B\}$

possible to find $\bar{b}$ such that post$\left\{\frac{\bar{b}}{b.+C} \middle| \mathbb{B}\right\}$ is void

e.g. $\bar{b} = A^{m}\left[\, O_{[h]} +.\, 1 +\, O_{[j]} \,\right]$ $\qquad A^{m}_{m} \left[\;\right]_{m}^{n+k+1} B^{n+k+1}_{n+k+1}$

where $h = \text{rank}[C]+k$ $\quad j = m - h - 1$



$h+1+1+j+1 = m+1$

$m - \text{rank}[C]$

i.e. $\text{rank}[b.+C+\bar{b}] \neq \text{rank}[b.+C] \Rightarrow \text{rank}[b.+C+\bar{b}] = \text{rank}[b.+C]+1$

$= \text{rank}[b.+C]+k+2$

if $b \in \text{prens}(C,\mathbb{B})$ $\quad b\,\Xi \in \text{prens}(C,\mathbb{B})$ all $|\Xi| \leq k$

in particular $b^{[k]} \in \text{prens}(C,\mathbb{B})$ $\qquad k = m \quad \text{rank}[C] = 0$

no extension possible

also if $b \in \text{prens}(C,\mathbb{B})$ with $k < m - \text{rank}[C]$ $\qquad m - k - \text{rank}[C] > 0$

then $\bar{b}_{m}$ exists such that $[b.+\bar{b}] \in \text{prens}(C,\mathbb{B})$

$\Rightarrow \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]} \subseteq \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k+1,r]}$

for $k := [m-1]$ // Select $p \in \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$, so that $p = bx + w'$

and form p

where $b \in \text{prens}\{C,\mathbb{B}\}^{[k]}$ $\quad x \in \mathbb{K}[\mathbb{B}]^{[r]}_{[k]}$ $\quad w' \in C\,\text{post}\left\{\frac{O^{[r]}_{[k]}}{u} \middle| \mathbb{B}\right\}$

$\exists\, \bar{b} : \mathbb{B} \to \mathbb{K}_{[m]}$ such that $[b.+\bar{b}] \in \text{prens}\{C,\mathbb{B}\}^{[k+1]}$

also $[x +.\, O^{[r]}] \in \mathbb{K}[\mathbb{B}]^{[r]}_{[k+1]}$

Since $[b+\bar{b}][x+.\,O^{[r]}] + w' = bx + w'$ $\;p$ features in $\mathbb{FS}\{..\}^{[k+1,r]}$

Hence $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]} \subseteq \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k+1,r]}$.

Select $p = [b.\,\cancel{+\bar{b}}\,]\{x + Cx' \in \mathbb{FS}\{\cdots\}^{[k+1,r]}$

with $x^{(h)}_{(k+1)} \neq O^{[r]}$ for some $h \in [r]$. Suppose it occurs in $\mathbb{FS}\{..\}^{[k,r]}$

It has the form $p = b'y + Cy'$ Then $bx + Cx' = b'y + Cy'$

and $bx^{(h)} + Cx'^{(h)} = b'\,y^{(h)} + Cy'^{(h)}$

Untrue that if $\operatorname{rank}[B] \geq \operatorname{rank}[A]$ $\quad B^{n}_{m}$ A^{j}_{m} then

$B\,\mathbb{K}^{[r]}_{[n]} \supseteq A\,\mathbb{K}^{[r]}_{[j]}$ e.g. $B \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}$ $A = \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$ $\quad \begin{matrix} 0 & n \\ b & C \end{matrix} \quad x \quad \begin{matrix} \cdot \\ p \end{matrix}$

$A\mathbb{K}^{\circ}_{0}$ contains $\begin{matrix} 0 \\ 0 \\ z \end{matrix}$ not contained by $B\mathbb{K}^{\circ}_{1}$

Show e.g. $\begin{pmatrix} 1 & \\ 0 & \bar{b} \\ 0 & \end{pmatrix} \mathbb{K}^{\circ}_{1}$ contains member of $\mathbb{K}_2$ not in $\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\, \mathbb{K}^{\circ}_{0}$ $\quad$ $\boxed{}$

$B \quad A\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} C \quad B.+\bar{b} = A\left[\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} C.+\bar{c}\right] \quad \bar{c} = A^{-1}\bar{b}$

$\operatorname{rank}[C] \leq \min(m,n)+$

$\leq \min(m$

$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} C$

$a.+b = a\left[I[m].+O^{[j]}_{[n]}\right] + b\left[O^{[n]}_{[j]}.+I[j]\right]$

$a^{n}_{m} \quad b^{j}_{m}$

$\operatorname{rank}[C] = n+1 \Rightarrow$

$\operatorname{rank}[b.+C] =$

$n+1+k+1 \leq m+1$

$n+k+1 \leq m$

furthermore $[b.+\bar{b}]\,\mathbb{K}^{[r]}_{[k+1]} \supset b\,\mathbb{K}^{[r]}_{[k]}$

take $[t+.t'] \in \mathbb{K}^{[r]}_{[k+1]}$ if $s \in \mathbb{K}^{[r]}_{[k]}$ exists for which $\quad t' \neq O^{[r]}$

$\operatorname{rank}[n+C] = \operatorname{rank}[C]$

$\cancel{t+\bar{t}'}$ $\quad bt + \bar{b}t' = bs$ then $\bar{b}t' = b(s-t)$ $\qquad \operatorname{rank}[\,\vdots\ \vdots\,] = k+n+2$

$t' \neq O^{[r]} \Rightarrow \exists\, t'^{(k)} \neq 0 \quad \bar{b}t'^{(k)} = b(s^{(k)} - t^{(k)})$ $\qquad \geq \operatorname{rank}[C]$

$[b.+\bar{b}]\left[(s^{(k)} - t^{(k)})+.\,t'^{(k)}\right] = O_{[m]}$ with $t'^{(k)} \neq 0$

means $[b.+\bar{b}] \notin \operatorname{prens}\{B\}$ $\qquad k+n+2 = \operatorname{rank}[\,\vdots\ \vdots\,] \geq \operatorname{rank}[n.+C]$

$A[\ ]C_0$

$[b.+\bar{b}.+C]\,\mathbb{K}^{[r]}_{[n+k+2]} \supset [b.+C]\,\mathbb{K}^{[r]}_{[n+k+1]}$ if $\operatorname{pre}\left\{\frac{n}{C}\middle|B\right\}$ nonvoid $\operatorname{rank}[n+.C] = \operatorname{rank}[C]$

any $\cancel{bs + Cs' \in \mathbb{K}^{[r]}_{[n+k+1]}}$ $[b.+C][s+.s'] \in$ this space occurs as

$[b.+\bar{b}+C][s+.O^{[r]}_{1}+.\,s'] $ in this space $\Rightarrow \supseteq$

F

yb nonzero $\quad r = \max\{\mathrm{rank}[b,C], \mathrm{rank}(C)+1\}$

$= \mathrm{rank}[b,C]$

$\mathrm{rank}\begin{bmatrix} 0 & u \\ b & C \end{bmatrix} = \mathrm{rank}[b,C]$ means $y[b+C] = [0\, u]$ for some $y \in K^{[m]}$

i.e. $yb = 0$ for some $y \in K^{[m]}$ then $yp = 0$

yb nonzero $\Rightarrow \mathrm{rank}[b\ C] \neq \mathrm{rank}[C] \qquad yb = 0 \Rightarrow yp = 0$

suppose $b = Cz$ then $uz = 0$ $\quad \mathrm{rank}\begin{bmatrix} 0 & u \\ b & C \end{bmatrix} = \mathrm{rank}\begin{bmatrix} u \\ C \end{bmatrix} = \mathrm{rank}(C)$

$$\begin{matrix} 0 & u \\ b & C \end{matrix} \quad \begin{matrix} 1 & 0 \\ -z & I \end{matrix} = \begin{matrix} 0 & u \\ 0 & C \end{matrix}$$

$$\begin{bmatrix} 0 & u & 0 \\ b & C & p \end{bmatrix} \qquad \begin{bmatrix} 0 & u & 0 \\ p & C & b \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & u \\ p & b & C \end{bmatrix}$$

a $\Rightarrow$ ~~rank $[0+u] =$~~

~~rank $[b+C]$~~ $= \mathrm{rank}[[0+u]-[b+C]] = \mathrm{rank}[[0+u+0]-[b+C+p]]$

b $\Rightarrow$ ~~rank $[b+C]$~~ $\neq \mathrm{rank}[C]$

For all $z \in B$ such that $p = [b+C][0 - x']$ ~~at z~~ for some

$x' \in \mathrm{post}\left\{\frac{0}{u} \middle| z\right\}$ at z,

$\overset{u+0}{\mathrm{rank}[[0+u]-[p+C]]} = \mathrm{rank}[C]$

|| $\mathrm{rank}[C] = \mathrm{rank}[p+C]$

$\mathrm{rank}[b+C] = \mathrm{rank}[b+C+p]$

always

and then $\quad \mathrm{rank}[[0+u]-[p+C]] \neq \mathrm{rank}[[0+u]-[b+C]]$

$[[0-p]+[u-C]] \neq \mathrm{rank}[[0-b]+[u-C]]$

$\square$

Let $C \in H\{u[m]B\}$ and $b, p : B \to K_{[m]}$.

i) Subject to condition (a) alone

$\mathrm{rank}[b+C] = \mathrm{rank}[[0+u]+.[b+C]] \quad \langle B \rangle$

ii) Conditions (a, b) may be reformulated in terms of rank as follows:

$b \leq p\ \{(u/C)B\}$ if and only if for each $z \in B$

$\mathrm{rank}[[0+u]+.[b+C]] = \mathrm{rank}[[0+u+0]+.[b+C+p]]$

$\neq \mathrm{rank}[C]$

iii) If the above conditions are satisfied

a) $$\operatorname{rank}[b_. + C] = \operatorname{rank}[b_. + C + p] \quad \langle \mathbb{B} \rangle$$

$$\neq \operatorname{rank}[C] \quad \langle \mathbb{B} \rangle$$

b) At each $z \in \mathbb{B}$,

$$\operatorname{rank}[p_. + C] = \operatorname{rank}[C]$$

if and only if $\operatorname{post}\left\{\frac{p}{\{b_. + C\}} \middle| \mathbb{B}\right\} \cap \left[K - \operatorname{post}\left\{\frac{O}{u}\right\}\right]$ contains $[x(0) - x']$ for which $x(0) = 0$ at z.

Proof

i) Select $y \in \operatorname{pre}\left\{\frac{u}{C} \middle| \mathbb{B}\right\}$. Since

$$[[1 + (-y)] - [O_{[m]} + I[m]]]\,[[O + u] - [b + C]]$$

$$= [[(-yb) + O^{[m]}] - [b + C]] \quad \langle \mathbb{B} \rangle$$

and the first factor is nonsingular over $\mathbb{B}$,

$$r = \operatorname{rank}[[O + u] - [b + C]] = \operatorname{rank}[[(-yb) + O^{[m]}] - [b + C]] \ \langle \mathbb{B}$$

If at $z \in \mathbb{B}$, $yb = 0$, $r = \operatorname{rank}[b + C]$. $\operatorname{rank}[b + C] \geq \operatorname{rank}[C]$ and, from condition (b), $\operatorname{rank}[b + C] \neq \operatorname{rank}[C]$ $\langle \mathbb{B} \rangle$. Thus $\operatorname{rank}[b + C] \geq \operatorname{rank}[C] + 1$.

If at $z \in \mathbb{B}$, $yb \neq 0$

$$r = \max\{\operatorname{rank}[b + C], \operatorname{rank}(C) + 1\}$$

~~if $\operatorname{rank}[b + C] \geq \operatorname{rank}(C) + 1$, $r = \operatorname{rank}[b + C]$.~~

and in this case $r = \operatorname{rank}[b + C]$ also. The result of clause (i) follows.

ii) Condition (a) is equivalent to the assertion that $[x(0) - x'] \in \left[K - \operatorname{post}\left\{\frac{O}{u} \middle| \mathbb{B}\right\}\right]$ exists such that

(+) $$[[O + u] - [b + C]]\,[x(0) - x'] = [O - p] \quad \langle \mathbb{B} \rangle$$

Hence, for each $z \in B$, the first relationship holds. ~~Condition (b)~~

~~~~ ~~~~ Condition (b) is equivalent to the assertion that for each $z \in B$, $\mathrm{rank}[b+C] \neq \mathrm{rank}[C]$. The second relationship follows from clause (i).

iii a) ~~The two~~ The first two relationships follow directly from ~~clause~~ conditions (a, b).

b) If relationship (+) is satisfied with $x(0) = 0$ at $z \in B$, then $Cx' = p$ at z : $\mathrm{rank}[p+C] = \mathrm{rank}[C]$ at z. Otherwise, since $bx(0) + Cx' = p$ with $x(0) \neq 0$, $\mathrm{rank}[p+C] = \mathrm{rank}[b+C] \neq \mathrm{rank}[C]$.

=

Homogeneous constraint systems

A mapping $C : B \to K^{[n]}_{[m]}$ for which $\mathrm{pre}\left\{\frac{u}{C} \,|\, B\right\}$ is nonvoid is said to be a homogeneous constraint system with respect to u. The notation $C \in H\{u[m]\,B\}$ indicates that C is such a system.

Let $C \in H\{u[m]\,B\}$

i) $\mathrm{rank}[u - C] = \mathrm{rank}[C]$ $\langle B \rangle$.

ii) Let $y \in \mathrm{pre}\left\{\frac{u}{C} \,|\, B\right\}$.

$$\mathrm{pre}\left\{\frac{u}{C} \,|\, B\right\} = y + \mathrm{pre}\left\{\frac{0^{[n]}}{C} \,|\, B\right\}$$

=

$$\mathrm{pre}\left\{\frac{[a+b]}{[c+d]}\right\} \subseteq \mathrm{pre}\left\{\frac{a}{c}\right\} \quad \mathrm{post}\left\{\frac{[a-b]}{[c-d]}\right\} \subseteq \mathrm{post}\left\{\frac{a}{c}\right\}$$

$$= \mathrm{pre}\left\{\frac{[b+a]}{[d+c]}\right\} = \qquad \mathrm{post}\left\{\frac{[b-a]}{[d-c]}\right\} =$$

$\mathrm{pre}\left\{\frac{a}{c}\right\}$ nonvoid $\mathrm{post}\left\{\frac{d}{c}\right\}$ void $\Rightarrow \mathrm{pre}\left\{\frac{a+b}{c+d}\right\}$ nonvoid

$\mathrm{post}\ \frac{e}{c+d}$ ~~non~~void $\to \mathrm{pre}\left\{\frac{a+b+e}{c+d+f}\right\}$ nonvoid etc
~~~~

Polynomials and rational functions

In the following $m, n \in \bar{N}$ ~~fixed~~, B is a prescribed set and $u: B \to K^{[n]}$.

Homogeneous constraint systems Def + Th.

Boundaries

Let $C: B \to K^{[n]}_{[m]}$, $C \in H\{u[m] | B\}$ and $p: B \to K_{[m]}$. A mapping $b: B \to K_{[m]}$

for which

$\mathrm{post}\left\{\frac{p}{b.+C} | B\right\} \cap \{K+.\ \mathrm{post}\{\frac{0}{u}\}$ is nonvoid

b) ~~$\mathrm{post}\left\{\frac{p}{[b \overset{+}{*} C]} | B\right\}$~~

is void

~~is nonvoid, and~~

a) b is prenonsingular with respect to C over B, and

~~b)~~ $\mathrm{post}\left\{\frac{b}{C} | z\right\}$

~~for all $z \in B$~~ is void ~~for all $z \in B$~~

is said to be a boundary of the homogeneous constraint system C with respect to the function ~~mapping~~ p over B.

$\mathrm{bound}\{p(u/C)B\}$ ~~$\{p, C | B\}$~~ is the set of all such boundaries. The relationship $b \in \mathrm{bound}\{p, C | B\}$ is written as ~~$b \leq p\{C|B\}$~~, $b \leq p\{(u/C)B\}$

The above conditions may be presented in terms of rank.

a') $\mathrm{rank}[p \overset{+}{*} b \overset{+}{*} C] = \mathrm{rank}[b \overset{+}{*} C]$ Th. $\square$

b') $\mathrm{rank}[[1 \overset{+}{*} 0^{[n]}]/[b \overset{+}{*} C]] = \mathrm{rank}[b \overset{+}{*} C]$

Existence and related boundaries // Properties // Transformations

of function mapping and constraint systems

equivalence of b) and $\mathrm{pre}\left\{\frac{[d \times 0^{[n]}]}{[b \times C]}\right\}$ nonvoid

let $C: B \to K^{[n]}_{[m]}$, $C \in H\{u[m] | B\}$ ~~and p~~ $b: B \to K_{[m]}$ ~~b~~ and $d: B \to K \setminus \{0\}$.

Δ $\mathrm{pre}\left\{\frac{[d \overset{+}{*} 0^{[n]}]}{[b \overset{+}{*} C]} | B\right\}$ is nonvoid if and only if

$\mathrm{post}\left\{\frac{b}{C} | z\right\}$ is void for all $z \in B$.

$\Rightarrow$

$b, b' \in \mathrm{bound}\{p, C | B\}$.

iv ~~(4)~~) Addition and subtraction

a) Let $p' : B \to \{K\}_{[m]}$ ~~and~~ be such that $b \le p' \{\cancel{C|B}\}(u/C)B\}$ ~~$b \in \mathrm{bound}[p', C|B]$~~. Set $P' :=$

~~$\mathrm{pol}[p', b, C, d \,|\, B]$~~

$\mathrm{pol}[d(p'/b)|(u/C)B]$

~~$\mathrm{pol}[p \pm p', b, C, d \,|\, B]$~~ $= P \pm P' \langle B \rangle$

$\mathrm{pol}[d((p \pm p')/b)|C|B](u/C)B]$

~~iv) Multiplication~~

~~iii) Let $q : B \to \{K\}_{[m]}$ be such that $p \in \mathrm{bound}[q, C|B]$.~~

~~Set let $d', d'' : B \to K\backslash 0$ and set~~

~~$R := \mathrm{pol}[q, p, C, d' \,|\, B]$, $S := \mathrm{pol}[q, b, C, d'' \,|\, B]$~~

b) Let $d' : B \to K \backslash 0$ and set $P' =$ ~~$\mathrm{pol}[p, b, C, d' \,|\, B]$~~ $\mathrm{pol}[d'(p/b):C|$

~~then $\mathrm{pol}[p, b, C, d \pm d' \,|\, B]$~~ $= P \pm P' \langle B \rangle$

~~$S = \left\{\frac{d''}{dd'}\right\} PR \langle B \rangle$~~ $\mathrm{pol}[(d \pm d')(p/b):C|B]$

Existence and related boundaries

Let $C :$ ~~$B \to \{K\}$~~ $\{K\}^{[n]}_{[m]}$, $C \in H\{u[m]B\}$

i) Let $B' \subseteq B$. For all $p : B' \to \{K\}_{[m]}$ such that corresponding to each $z \in B'$ at least one $c \in \mathrm{pre}\left\{\frac{u^{[m]}}{C} \middle| B'\right\}$ can be found for which $cp \ne 0$ at z, p ~~itself is a boundary of C~~ ~~$p \in b\{C|B'\}$~~ $p \le p'\{(u/C)B\}$ ~~with $p \in \mathrm{bound}\{p, C|B'\}$~~ and $u = O^{[m]}_{[n]}$ For

ii) ~~All now~~ Let $n = m-1$. All mappings $b : B \to \{K\}_m$ for which $[b \overset{+}{\vdots} C]$ is nonsingular, ~~over B are bound~~ be bound$\{p, C|B$ ~~$b \le p\{C|B\}$~~ $b \le p\{(u/C)B\}$ for all $p : B \to \{K\}_{[m]}$.

iii) If one boundary is known, all others may be constructed in a simple way. Let $p : B \to \{K\}_{[m]}$ and ~~$b \in \mathrm{bound}\{p, C|B\}$~~ $b \le p\{(u/C)B\}$ ~~$b = p\{C|B\}$~~.

a) $b' \le p$ ~~$\{C|B\}$~~ $\{(u/C)B\}$ ~~$b' \in \mathrm{bound}\{p, C|B\}$~~ if and only if $W := [w(0) \vdots w'] : B \to K_{[n+1]}$ ~~$W : B \to [K]_{[n+1]}$~~ with and $w' \in \mathrm{post}\left\{\frac{C}{u} \middle| B\right\}$ $W(0) \ne 0$ over B exists such that $b' = [b \overset{+}{\vdots} C]W$.

formulate in terms of spaces: Foll. page (iii b)

Properties

Let $C : \cancel{\mathbb{B} \to \{\mathbb{K}\}_{[n]}} \in H\{u[m]\mathbb{B}\}_{[n]}^{\cdot}$, $p : \mathbb{B} \to \{\mathbb{K}\}_{[m]}$ and $\cancel{b \in \text{bound}[p, C|\mathbb{B}]}$ $b \leq p \{\cancel{C|\mathbb{B}}\}(u/C)\mathbb{B}$

iii) Nonsingularity

$b \neq O_{[m]} \langle \mathbb{B} \rangle$

ii) Reflexivity

If $\text{post}\left\{\frac{p}{[b \dot{+} C]} \middle| \mathbb{B}\right\} \cap \{\mathbb{K} +. \text{post}\{\tfrac{O}{u}\}\}_{=[x(0)+.x']}$ contains x' for which $x(0) \neq O \langle \mathbb{B} \rangle$

then $\cancel{p \in \text{bound}\{b, C|\mathbb{B}\}}$, $p \leq b \{\cancel{C|\mathbb{B}}\}\{(u/C)\mathbb{B}\}$

i) Transitivity

If $q : \mathbb{B} \to [\mathbb{K}]_{[m]}$ is such that $\cancel{p \in \text{bound}[q, C|\mathbb{B}]}$ $p \leq q \{\cancel{C|\mathbb{B}}\}\{(u/C)\mathbb{B}\}$ then

$\cancel{b \in \text{bound}[q, C|\mathbb{B}]}$ $\cancel{b \leq q \{C|\mathbb{B}\}}$ $b \leq q \{(u/C)\mathbb{B}\}$

Transformations of function mappings and constraint systems.

Let $C : \cancel{\mathbb{B} \to \{\mathbb{K}\}_{[m]}^{[n]}} \in H\{u[m]\mathbb{B}\}$, $p : \mathbb{B} \to \{\mathbb{K}\}_{[m]}$ and $\cancel{b \in \text{bound}[p, C|\mathbb{B}]}$ $b \leq p \{\cancel{C|\mathbb{B}}\}(u/C)\mathbb{B}$

i) $\cancel{b \in \text{bound}\{kp, C|\mathbb{B}\}}$ $b \leq kp \{\cancel{C|\mathbb{B}}\}(u/C)\mathbb{B}\}$ for all $k : \mathbb{B} \to \mathbb{K}$.

ii) If $p' : \mathbb{B} \to [\mathbb{K}]_{[m]}$ is such that $\cancel{b \in \text{bound}[p', C|\mathbb{B}]}$ $b \leq p' \{\cancel{C|\mathbb{B}}\}(u/C)\mathbb{B}\}$

then $b \in \text{bound}[p \pm p', C|\mathbb{B}]$ $b \leq p \pm p' \{\cancel{C|\mathbb{B}}\}(u/C)\mathbb{B}$

iii) $\text{bound}\{\cancel{p, CD|\mathbb{B}}\}$ $\{p\,(uD/CD)\mathbb{B}\}$ $= \text{bound}\{\cancel{p, C|\mathbb{B}}\}$ $\{p\,(u/C)\mathbb{B}\}$

for all $D : \mathbb{B} \to [\mathbb{K}|n]'$

iv) $B\,\text{bound}\{\cancel{p, C|\mathbb{B}}\}$ $\{p\,(u/C)\mathbb{B}\}$ $= \text{bound}\{\cancel{Bp, BC|\mathbb{B}}\}$ $\{Bp\,(u/C)\mathbb{B}\}$

for all $B : \mathbb{B} \to [\mathbb{K}|m]'$.

Prev. p. (iii b) $\text{bound}\{\cancel{p, C|\mathbb{B}}\}$ $\{p\,(u/C)\mathbb{B}\}$ $= [b \dot{+} C]\,[\cancel{\mathbb{K}\setminus O \dot{+} \mathbb{K}_{[n]}}]$ $[\{\mathbb{B} \to \mathbb{K}\setminus O\} +. \{\cancel{\mathbb{B} \to \mathbb{K}_{[n]}}\}]$ $\text{post}\{\tfrac{O}{u}|\mathbb{B}\}$

~~(b) ii) If one boundary is known, all others may be constructed in a simple way. Let $b \in \mathrm{bound}\{p, C|B\}$. $b' \in \mathrm{bound}\{p, C|B\}$ if and only if $z: B \to \mathrm{col}[K|n]$ with $z(\cdot) \neq 0$ over B exists such that $b' = [b \times C]z$.~~

In the following section use is made of the fact that if $b \leq p\{C|B\}\ \{(u/C)B\}$ ~~$b \in \mathrm{bound}\{p, C|B\}$~~ and $d: B \to K^{(n)}$, then

$$(+) \qquad \mathrm{pre}\left\{ \frac{[d \,\dot{\times}\, O^{[n]}]}{[b \,\dot{\times}\, C]} \right\}$$

is nonvoid. That this is so is a consequence of the second condition imposed upon boundaries. ~~established in the following theorem.~~

When $b, b' \leq p\{C|B\}$ ~~$b, b' \in \mathrm{bound}\{p, C|B\}$~~ and $d: B \to K^{(n)}$ the two spaces of the form (+) are nonvoid. Conditions necessary and sufficient to ensure that these two spaces intersect, and the nature of the intersection are established in the following theorem.

Let $C: B \to K^{[m]}_{[n]}$ $H\{u[n]B\}$, $p: B \to \mathrm{col}[K|m]$ and let $d: B \to K^{(n)}$ ~~and let $c: B \to \mathrm{row}[K|m]$ be such that $c[b\times$~~

i) Let $b \leq p\{C|B\}\ (u/C)B$ ~~$b \in \mathrm{bound}\{p, C|B\}$~~ and $c: B \to \mathrm{row}[K|m]$ be such that

$$c[b \,\dot{\times}\, C] = [d \,\dot{\times}\, O^{[n]}] \quad \langle B \rangle$$

Let $C \in H\{u[m]\mathbb{B}\}$, $p : \mathbb{B} \to \mathbb{K}_{[m]}$ and $d : \mathbb{B} \to \mathbb{K}$

i) Let $b \leq p\{(u/C)\mathbb{B}\}$ and $c : \mathbb{B} \to \mathbb{K}^{[m]}_{\cancel{[m]}}$ be such that

$$c[b.+C] = [d.+u] \quad \langle\mathbb{B}\rangle$$

~~a) Define $b' : \mathbb{B} \to \mathbb{K}_{[m]}$ by setting~~

~~$b' :=$~~

a) If $b' : \mathbb{B} \to \mathbb{K}_{[m]}$ satisfies the relationship

$$b' = [b.+c][1+.w'] \quad \langle\mathbb{B}\rangle$$

for some $w' \in \operatorname{post}\left\{\frac{0}{u}\,|\mathbb{B}\right\}$ then $b' \leq p\{(u/C)\mathbb{B}\}$ and

$cb' = d \ \langle\mathbb{B}\rangle$.

b) If $d \neq 0 \ \langle\mathbb{B}\rangle$ and $b' \leq p\{(u/C)\mathbb{B}\}$ with $cb' = d \ \langle\mathbb{B}\rangle$ then b' is given by the above expression with w' as described.

ii) Let $b, b' \leq p\{(u/C)\mathbb{B}\}$ satisfy the above relationship with w' as described:

$$\operatorname{pre}\left\{\frac{d.+u}{b.+C}\,|\mathbb{B}\right\} = \operatorname{pre}\left\{\frac{d.+u}{b'.+C}\,|\mathbb{B}\right\}$$

$m, n \in \bar{N}$ $\quad B \in K$ $\quad C_{\xi}: B \to [K|m,n]$ ~~such that~~ $\quad e\ g\ h, m\ e$

$e = fg$

$h, k \quad h, m \quad m\ k$

i) Let $e: B \to \overset{\text{row}[K|n]}{V},\ \text{pre}\left\{\dfrac{O^{[n]}}{e}|B\right\}$ ~~nonvoid.~~

The notation $\overset{e}{}$
$c \in \text{pre}\left\{\dfrac{O^{[n]}}{c}|B\right\}$ ~~is the space of~~ indicates that $c: B \to \text{row}[K|n$

satisfies the relationship $cC = O^{[n]} e \langle B\rangle$

ii) Let $f: B \to \text{col}[K|m]$.
The notation $x \in \text{post}\left\{\dfrac{O_{[m]}^{f}}{C}|B\right\}$ indicates that $x: B \to \text{col}[K|n]$ satisfies

$a\ \overset{h}{m}\ k/c \quad \overset{n}{b}\ k \quad c\ \overset{n}{m-k-1}$

the relationship

$$Cx = O_{[m]}^{f} \langle B\rangle$$

$h \quad k \quad k$
$m \quad m$
$n \quad k$
$n \quad n$

Homogeneous constraint systems.

A mapping $C: B \to [K|m,n]$ for which $\text{pre}\left\{\dfrac{O^{[n]}}{C}|B\right\}$?

is nonvoid is said to be a homogeneous constraint system

Let $\partial: B \to \text{col}[K|m]$ and $d \in \overset{B \to K}{}$ be nonzero over B.

$$\text{pre}\left\{\frac{[d \times O^{[n]}]}{[\partial \times C]}|B\right\} \text{ is nonvoid? if and only if}$$

? $\text{post}\left\{\dfrac{\partial}{C}\right\}$ is void

If no $y: B \to \text{col}[K_{[n]}]$ such that $Cy = \partial \langle B\rangle$ at $z \in B$ exists,

$\text{rank}[C] + \text{rank}[\overset{+}{\partial} \times C] \overset{\text{at } B}{\langle B\rangle}$. Since $\text{rank}[\overset{+}{\partial} \times C] \geq$

? $\text{rank}[C] \overset{\text{at } z}{\langle B\rangle}$, $\text{rank}[\overset{+}{\partial} \times C] \geq \text{rank}[C] + 1 \langle B\rangle$ at z.

$\text{rank}[[d \overset{+}{\times} O^{[n]}] \overleftrightarrow{\times} [\overset{+}{\partial} \times C]] = \max\{\text{rank}[\overset{+}{\partial} \times C], \text{rank}[C] +$

at $z \in B$.

Subject to the above condition $\underset{}{u \notin \text{post}\{\frac{b}{c}|z\}}$ ~~upon[?]~~, the right hand side expression reduces to $\text{rank}[\overset{+}{\partial} \times C]$ over B. Thus

$c : B \to \mathrm{row}[K|m]^{K[m]}$ for which

~~$c[[d$~~

$$c[\overset{+}{b} \ast C] = [\overset{+}{d} \ast O^{[n]}] \quad \langle B \rangle$$

exists and $\mathrm{pre}\left\{ \dfrac{[\overset{+}{d} \ast O^{[n]}]}{[\overset{+}{b} \ast C]} \middle| B \right\}$ is nonvoid

If the above relationship is satisfied,

$$c[\overset{+}{b} \ast C][1 \ast x] = d$$

at $z \in B$ for all $x : B \to \mathrm{col}[K|m]^{K[m]}$. Since $d \neq 0$ at z, no ~~such~~ x for which $b + Cx = 0_{[m]}$ at z ~~$\langle B \rangle$~~ exists. Hence $\mathrm{post}\left\{ \dfrac{b}{C} \right\}$ is void for all $z \in B$.

Let $C : B \to [K|m,n]$ be a homogeneous constraint system and $p : B \to \mathrm{col}[K|m]$. A mapping $\partial : B \to \mathrm{col}[K|m]$ for which

b) $\mathrm{post}\left\{ \dfrac{\partial}{C} \middle| B' \right\}$ for all $B' \subseteq B$ is void and

a) $\mathrm{post}\left\{ \dfrac{p}{\partial \times C} \middle| B \right\}$ is nonvoid

is said to be a boundary of C with respect to p over B.

The above conditions may be presented in terms of rank

b') $\mathrm{rank}\{[1 \times O^{[n]}] / [\partial \times C]\} = \mathrm{rank}[\partial \times C] \quad \langle B \rangle$.

Polynomials

Let $\cancel{C: B \to K^{[n]}_{[m]}}$ $C \in H\{u[m]B\}$, $p: B \to K_{[m]}$, $\cancel{b \in band\{p, C|B\}}$ $b \in p\{C|B\}\{(u/C)B\}$ and

$d: B \to K \setminus \{0\}$. The mapping $P: B \to K$ defined by setting

$$P := cp$$

where $c: B \to K_{[m]}$ is any mapping for which

$$(*) \qquad c[b \overset{+}{*} C] = [d \overset{+}{*} \overset{u}{e^{[n]}}] \qquad \langle B\rangle$$

is said to be the polynomial determined by d, p, $b_{u/C}$ and d $(u/C)B$

over B; it is denoted by $\cancel{pol[p, b, C, d|B]}$ $pol[d(p/b)|C|B]$

c is a coefficient of the polynomial, p its function, d its normalising factor and $b.+C$ its total constraint system.

Existence and properties

Let $\cancel{C: B \to K^{[n]}_{[m]}}$ $C \in H\{u[m]B\}$, $p: B \to K_{[m]}$, $\cancel{b \in band\{p, C|B\}}$ $b \in p\{\cancel{C|B}\}\{(u/C)B\}$ and

$d: B \to K \setminus \{0\}$. Set $P := \cancel{pol[p, b, C, d|B]}$ $P := pol[d(p/b)|C|B]$ $(u/C)B$

(i) Existence and alternative definitions

(a) P is independent of the selection of c from the mapping systems satisfying condition $(*)$. For all $x := [x(0) + .x'] \in post\{\frac{p}{b.+C}|B\}$

b) ~~The first elements of all~~ $\cancel{x \in post\{\frac{p}{[b*C]}|B\}}$ ~~have the same value~~ $x(0): B \to K$ ~~and~~ $P = \overset{+}{d}x(0)$ $\langle B\rangle$

$\cancel{P = dx(0) \quad \langle B\rangle}$ If $d \neq 0$ $\langle B\rangle$, the first elements of all such x have the same value.

c) $$P = pre\left\{\frac{[d \overset{+}{*} \overset{u}{e^{[n]}}]}{[b \overset{+}{*} C]}\Big|B\right\}p$$

$$= d\,post\left\{\frac{p}{[b \overset{+}{*} C]}\Big|B\right\}_{(0)}$$

(ii) Solution vectors of linear equation systems

(a) If c satisfies condition $(*)$, $[P \overset{+}{*} c]$ satisfies the equation

$$[P \overset{+}{*} c][[(-1) \overset{+}{*} 0^{[n+1]}] \overset{+}{*} [p \overset{+}{*} b \overset{+}{*} C]] = [0 \overset{+}{*} d \overset{+}{*} \overset{u}{e^{[n]}}] \qquad \langle B\rangle$$

b) $\left[P \overset{+}{\underset{\cdot}{*}} \mathrm{pre}\left\{ \frac{[d \overset{+}{*} \overset{w}{0}^{[n]}]}{[b \overset{+}{\underset{\cdot}{*}} C]} \middle| B \right\}\right] = \mathrm{pre}\left\{ \frac{[0 \overset{+}{*} d \overset{+}{*} \overset{w}{0}^{[n]}]}{[(-1) \overset{+}{*} 0^{[n+1]}] / [p \overset{+}{*} b \overset{+}{*} C]} \middle| B \right\}$

~~iii)~~ $\{ P = \cancel{\mathrm{pol}[p, p, C, P \mid NS(P)]}$ ~~$\mathrm{pol}[P(p/p) : C \mid B] NS(P)]$~~

iii) Relationships between coefficients constraint systems

~~v)~~ iv) Nonsingular ~~the~~ If $m = n-1$ and $[b \overset{+}{*} C]$ is nonsingular over B

$$P = \frac{|p \overset{+}{*} C|}{|b \overset{+}{*} C|} d \quad \langle B \rangle$$

Proof of clause (iii) now (i) of Th.

Any c satisfying condition $(*)$ is, in particular, in $\mathrm{pre}\left\{ \frac{0^{[n]}}{C} \middle| NS(P) \right\}$ and $cp \neq 0 \; \langle NS(P) \rangle$. From clause (i) of Th. $p \leq p\{C \mid NS(P)\}$ ~~$\mathrm{pe\,bound}\{p, C \mid NS(P)\}$~~. Since $P \neq 0 \; \langle NS(P) \rangle$, $P' := \mathrm{pol}[P(p/p) : C \mid NS(P)]$ ~~$\mathrm{pol}[p, p, C, P \mid NS(P)]$~~ is well defined, and $P' = c'p \; \langle NS($

where $c' : NS(P) \to K_{[m]}$ is any mapping satisfying $cp + 0^{[n]}$ the condition $c'[p \overset{+}{*} C] = \cancel{\{p \times 0^{[n]}\}}$ ~~taking~~ where c is any of the mappings used to define P. This relationship is, in particular, satisfied by $c' = c$. Thus $P' = cp = P \; \langle NS(P) \rangle$

Transformations

Rational functions

Muller (iv) Conjugacy

~~The~~ A polynomial and it's normalising factor are in conjugate relationship to each other.

$$\cancel{d = pol[b, p, C, P \mid B]} \overset{NS(P)}{} \langle NS(P)\rangle \; pol[P(p/b) : C \mid B] \quad NSG$$

(iii) ~~(v)~~ Relationships between coefficients

~~let $p' : B \to K_{[m]}$, $b' \leq p'$~~ $C' \in H\{u[n] B\}$

Let $n \in \mathbb{N}$, ~~$C' : B \to K^{[n]}_{[m]}$~~, $p' : B \to K_{[m]}$, $b' \leq p' \{(u/C')B\}$ ~~$\{C' \mid B\}$~~, Assume that either $d = d' \langle B\rangle$ or that $u = 0_{C}$ (n)

$d' : B \to K$ and set ~~$P' = pol[p, b', C, d' \mid B]$~~. The

a) The coefficients $c, c' : B \to K_{[m]}$ and $c' : B \to K_{[n]}$ in the two representations $P' = pol[d'(p'/b') : C' \mid B]$

$P := cp$ and $P' = c'p'$ are related:

$$d'c[b.+C] = dc'[b'.+C'] \; ~~[dd' + b'^{[n]}]~~ \langle B\rangle$$

b) If $n = m-1$ and $[b.+C]$ is nonsingular over B, c may be expressed directly in terms of c':

$$c = \left(\frac{d}{d'}\right) c' [b'.+C'][b.+C]^{-1} \quad \langle NS(d')\rangle$$

= ~~Two polynomials derived from the same constraint system Θ are said to be similar~~

= A change in normalisation is equivalent to a function change.

let $C : B \to K^{[n]}_{[m]}$, $p : B \to K_{[m]}$, $b' \leq p\{C' \mid B\}$ and $d' : B \to K \setminus 0$ and set $P' := pol[p, b', C, d' \mid B]$. P' may be represented in the form $P' = cp' \langle B\rangle$, having the same coefficient as P:

$$p' = \frac{d'}{(cp')} p \quad \langle B\rangle$$

Equiconstrained polynomials

Two polynomials ~~$\mathrm{pol}[p, b, d(p/b), C|\mathbb{B}]$, $\mathrm{pol}[d'(p'/b'), C|\mathbb{B}]$, where $p, p' : \mathbb{B} \to K_{[m]}$, $C, C' : \mathbb{B} \to K^{[n]}_{[m]}$, $b \leq p\ \{C|\mathbb{B}\}$ $b' \leq p'\ \{C'|\mathbb{B}\}$, and $d, d' : \mathbb{B} \to K \setminus 0$, for which~~ mappings with homogeneous constraint systems $C, C' : \mathbb{B} \to K^{[n]}_{[m]}$ such that $B : \mathbb{B} \to K[m]$, $D : \mathbb{B} \to K[n]$, both nonsingular over $\mathbb{B}$, exist ~~such that~~ for which

$C \in H\{n[m]|\mathbb{B}\}$
$C' \in H\{n'[m]|\mathbb{B}\}$

$$C' = BCD \quad \langle \mathbb{B} \rangle, \quad n' = nD \ \langle \mathbb{B} \rangle$$

are said to be equiconstrained over $\mathbb{B}$.

In the following, ~~joint~~ conjoint properties of equiconstrained polynomials are described and relationships involving such polynomials are stated.

Related polynomials with identical coefficients

Given two polynomials expressible in the forms

$$P := \sum_i c(\omega) p(\omega) \ \langle \omega := [m] \rangle \qquad P' := \sum_i c'(\omega) p'(\omega) \ \langle \omega := [m] \rangle \ \langle \mathbb{B} \rangle$$

it is, subject to evident conditions upon the $c(\omega)$, possible to express the latter, as a polynomial with coefficient c:

$P' = cp'' \ \langle \mathbb{B} \rangle$ ~~$P' = \sum_i c(\omega)$~~ where

$$p''(\omega) = \frac{c'(\omega)\, p'(\omega)}{c(\omega)} \quad \langle \omega := [m] \rangle ; B \rangle$$

If the two polynomials concerned are equiconstrained, the above representation can be attained without ~~use of~~ reference to the coefficient c'.

~~Related polynomials with ... coefficients~~

Let $C \in H\{u[m]\mathbb{B}\}$, $p : \mathbb{B} \to \mathbb{K}_{[m]}$, $b \leq p\{(u/C)\mathbb{B}\}$,

$d : \mathbb{B} \to \mathbb{K}$, and set $P := \mathrm{pol}[d(p/b)|(u/C)\mathbb{B}]$, and

select $c \in \mathrm{pre}\left\{\frac{d.+u}{b.+C}|\mathbb{B}\right\}$ so that P has the representation

$P = cp \langle \mathbb{B} \rangle$

i) Polynomials differing only in function

Let $p' : \mathbb{B} \to \mathbb{K}_{[m]}$ be such that $b \leq p'\{(u/C)\mathbb{B}\}$. Set

$P' := \mathrm{pol}[d(p'/b)|(u/C)\mathbb{B}]$. $P' = cp' \langle \mathbb{B} \rangle$.

ii) Polynomials differing only in normalising factor.

Let $d' : \mathbb{B} \to \mathbb{K}$ and set $P' :=$

$\mathrm{pol}[d'(p/b)|(u/C)\mathbb{B}]$. $P' = cp' \langle NS(d) \rangle$ where

$$p' := \left(\frac{d'}{d}\right) p \quad \langle NS(d) \rangle$$

iii) Polynomials differing in boundary

Let $b' \leq p\{(u/C)\mathbb{B}\}$ and set $P' := \mathrm{pol}[d(p/b')|((\frac{d}{cb'})u/C)$

$\mathbb{B}]$. $P' = cp' \langle \mathbb{B} \rangle$, where

$$p' = \frac{d}{cb'} p \quad \langle \mathbb{B} \rangle$$

iv) Interchange of function and boundary

Set $P' := \mathrm{pol}[d(b/p)|((\frac{d}{P})u/C)NS(P)]$. $P' = cp'$ where

$$p' = \left(\frac{d}{P}\right) b \quad \langle NS(P) \rangle$$

v) Polynomials differing in homogeneous constraint system and boundary.

Let $\mathcal{B}: \mathbb{B} \to \mathbb{K}'[n]$ and set $P' := \mathrm{pol}[d(p/\mathcal{B}b) \,|\, (u/\mathcal{B}C)\mathbb{B}]$.

$P' = cp'$ where

$$p' = \mathcal{B}^{-1}p \quad \langle \mathbb{B} \rangle$$

vi) Polynomials differing in ~~cons~~ ~~total~~ constraint system

a) Let $D: \mathbb{B} \to \mathbb{K}'[n]$ and set $P' := \mathrm{pol}[d(p/b) \,|\, (uD/CD)\mathbb{B}]$.

$P' = cp \ \langle \mathbb{B} \rangle$.

b) Let $k: \mathbb{B} \to \cancel{\mathbb{B}}\mathbb{K}$ and set $P' = \mathrm{pol}[kd(p/b) \,|\, (ku/C)\mathbb{B}]$.

$P' = cp' \ \langle \mathbb{B} \rangle$ where

$$p' = kp \quad \langle \mathbb{B} \rangle$$

transformations

Let $C : \mathbb{B} \to \cancel{\mathbb{K}[n]}$ $\in H[u[n]\mathbb{B}][n]$, $p : \mathbb{B} \to \{\mathbb{K}\}_{[m]}$, ~~$b \in$ bound$\{p, C \mid \mathbb{B}\}$~~ $b \leq p\,\{\cancel{C \mid \mathbb{B}}\}(u/C)\mathbb{B}\}$

$d : \mathbb{B} \to \mathbb{K}\setminus 0$ and $P := \cancel{\text{pol}[p, b, C, d \mid \mathbb{B}]}$ pol$[d(p/b) \mid C \mid \mathbb{B}](u/C)\mathbb{B}$

i) The identity transformation $P = \text{pol}[P(p/p) : C / NS(P)]$ $\langle \mathbb{B}NS(P)\rangle$

iii) Transformations of function mapping, boundary, constraint system and normalising factor

ii) Conjugation

A polynomial and its normalising factor are in conjugate relationship to each other

$d = \text{pol}[P(p/b) : C / NS(P)]$ b/p

a) $\text{pol}\,\cancel{[kp, b, C, d \mid \mathbb{B}]}\,[d(kp/b) \mid C \mid \mathbb{B}](u/C)\mathbb{B}] = kP \quad \langle \mathbb{B}\rangle$

for all $k : \mathbb{B} \to \mathbb{K}$

b) With $b' : \mathbb{B} \to \{\mathbb{K}\}_{[m]}$ defined by setting

$$b' = [b \overset{+}{*} C][1 \overset{+}{*} z]$$

$$\text{pol}\,\cancel{[p, b', C, d \mid \mathbb{B}]}\,[d(p/b') \mid C \mid \mathbb{B}](u/C)\mathbb{B}] = P \quad \langle \mathbb{B}\rangle$$

for all $z : \mathbb{B} \to \{\mathbb{K}\}_{[n]}$, $\text{post}\left\{\frac{0}{u}\right\}$

c) $\text{pol}\,\cancel{[p, b, CD, d \mid \mathbb{B}]}\,[d(p/b) \mid CD \mid \mathbb{B}](uD/CD)\mathbb{B}] = P \quad \langle \mathbb{B}\rangle$

for all $D : \mathbb{B} \to \cancel{[\mathbb{K} \mid n]}'$, $\mathbb{K}'[n]$.

d) $\text{pol}\,\cancel{[Bp, Bb, BC, d \mid \mathbb{B}]}\,[d(Bp/Bb) \mid BC \mid \mathbb{B}](u/BC)\mathbb{B}] = P \quad \langle \mathbb{B}\rangle$

for all $B : \mathbb{B} \to \cancel{[\mathbb{K} \mid m]}$, $\mathbb{K}'[m]$

e) $\text{pol}\,\cancel{[p, b, C, kd \mid \mathbb{B}]}\,[kd(p/b) \mid C \mid \mathbb{B}](ku/C)\mathbb{B}] = kP \quad \langle \mathbb{B}\rangle$

for all $k : \mathbb{B} \to \mathbb{K}\setminus 0$.

~~iii) from prev. page~~

~~ii) Reciprocation~~

~~If $P \neq 0 \ \langle \mathbb{B}\rangle$ then~~

~~$\text{pol}\left[\frac{b}{p}, p, C, \frac{1}{d} \mid \mathbb{B}\right] = \frac{1}{P} \quad \langle \mathbb{B}\rangle$~~

iv) Addition and subtraction

a) Let $p' : \mathbb{B} \to \mathbb{K}_{[m]}$ be such that $b \in p'\{(u/C)\mathbb{B}\}$. Set

$P' := \mathrm{pol}[d(p'/b) \mid (u/C)\mathbb{B}]$.

$$\mathrm{pol}[d((p \pm p')/b) \mid (u/C)\mathbb{B}] = P \pm P' \langle \mathbb{B} \rangle$$

b) Let $d' : \mathbb{B} \to \mathbb{K}$ and set $P' := \mathrm{pol}[d'(p/b) \mid (u/C)\mathbb{B}]$.

$$\mathrm{pol}[(d \pm d')(p/b) \mid (u/C)\mathbb{B}] = P \pm P' \langle \mathbb{B} \rangle$$

Let $C \Rightarrow K_{[m]}$, $b, p, q : B \to K_{[m]}$ and $d, e : B \to K$.

i) Multiplication

If of two polynomials formed from the same local constraint system the boundary of one is the function of the other, the product of the polynomials may be expressed as a polynomial in closed form. Let $b \leq p \leq q \; \{(u/C)B\}$

$$\mathrm{pol}\left[\textstyle\sum e(q \ddagger / p) \mid (u/C)B\right] \mathrm{pol}\left[\textstyle\sum d(p\ddagger / b) \mid (u/C)B\right]$$

$$= \mathrm{pol}\left[\textstyle\sum ed(q\ddagger / b) \mid (u/C)B\right] \qquad \langle B \rangle$$

ii) Factorisation

A polynomial may be expressed as the product of two factors polynomial factors, its function being the function of one factor, its boundary being the boundary of the other. Let $b \leq q \; \{(u/C)B\}$. With p so chosen that $b \leq p \leq q \; \{(u/C)B\}$, $\mathrm{pol}\left[ed(q/b) \mid (u/C)B\right]$ has the factor decomposition expressed above

~~$\mathrm{pol}[\sum e(q\ddagger/b) : C \mid B] =$~~

~~$\mathrm{pol}[\sum e(q\ddagger/p) : C \mid B]\, \mathrm{pol}[\sum 1(p\ddagger/b) : C \mid B] \quad \langle B \rangle$~~

iii) Rational decomposition

A polynomial may be expressed as the quotient of two polynomials. Its function is that of the numerator, its boundary is the function of the denominator. The numerator and denominator have a common boundary. Let $p \leq q \; \{B \mid C\}$. With $b \leq p \; \{B \mid C\}$

$$\mathrm{pol}\left[\textstyle\sum \tfrac{e}{f}(q\ddagger/p) \mid (u/C)\,NS(P')\right] = \frac{\mathrm{pol}\left[\sum e(q\ddagger/b) \mid (u/C)B\right]}{\mathrm{pol}\left[\sum f(p\ddagger/b) \mid (u/C)B\right]} \quad \langle NS(P') \rangle$$

P' being the denominator polynomial.

iv) Reciprocation

The reciprocal of a polynomial is another polynomial, formed by interchanging function and boundary and replacing the normalising factor by it's reciprocal. Let $b \leq p \{\cancel{C|B}\}.(u/C)B\}$

$$\frac{1}{\mathrm{pol}[\xi d(p\xi/b) \cancel{|C|B]}\,(u/C)B]} = \mathrm{pol}[\cancel{b;p}\,\xi\tfrac{1}{d}(b\xi/p)|\cancel{C|B]}\,(u/C)NS(d)] \langle NS(P)\rangle$$

Proofs.

~~i) Set $P := \mathrm{pol}[(d;p)/b:C|B]$, $Q := \mathrm{pol}[$~~

i) Denote the polynomials in order of appearance in the stated relationship by Q, P and R. $P = dx(0) \langle B\rangle$ for any $[x(0) \mp x']$ ~~such that~~ for which $p = [b+C][x(0) \mp x'] \langle B\rangle$; $Q = ey(0)$ for any $[y(0)+y']$ for which $q = [p+C][y(0) \mp y'] \langle B\rangle$; $R = edz(0) \langle B\rangle$ for any $[z(0)+z']$ for which $q = [b+C][z(0)+z'] \langle B\rangle$. $[z(0) \mp z'] = [x(0)y(0) \mp (x'y(0)+y')] \langle B\rangle$ satisfies the latter relationship.

ii) p can be selected (e.g. $p = b$ or $p = q$). That the stated relationship is satisfied follows by ~~setting~~ taking $d := 1 \langle B\rangle$ in clause (i)

iii) Since $b \leq p$, $p \leq q \{C|B\}$, $b \leq q \{C|B\}$ also. That the product of the right hand denominator and left hand term is equal to the right hand numerator follows from clause (i).

iv) Since $P = dx(0) \langle B\rangle$ for all $[x(0) \mp x'] \in \mathrm{post}\left\{\frac{p}{[b+C]} | B\right\}$, $x(0) \neq 0 \langle NS(P)\rangle$ and, from Th. clause (ii), $p \leq b \{C|NS(P)\}$: the right hand polynomial is well defined. From Th clause ( ), $\mathrm{pol}[\xi\tfrac{1}{d}(b\xi/b) \vdots C|B] = \frac{1}{d} \langle B\rangle$ and, from clause ( ) of Th. , $d\,\mathrm{pol}[\cancel{p;b}\,\xi 1(p\xi/b) \vdots C|B] = \mathrm{pol}[\xi d(p\xi/b) \vdots C|B]$. The required result follows by setting $q := b$, $e := \frac{1}{d}$ in clause a similar relationship expresses c' in terms of c, and the pairs c, c'' and c', c'' may be treated in the same way.

Rational functions

A rational function is a polynomial of special form

Let $C : \mathbb{B} \to K^{[n]}_{[m]}$, $q : \mathbb{B} \to K_{[m]}$ and $p \leq q \{C|\mathbb{B}\}$.

rat $[(q/b) : C|\mathbb{B}]$, the rational function determined by C, p and q over $\mathbb{B}$, is the polynomial pol$[1(q/b) : C|\mathbb{B}]$.

The theory of rational functions may be developed as a ~~special case of that~~ restricted version of that of polynomials, simply by discarding, where appropriate, reference to a variable normalising factor. In particular a rational function can be expressed as the product of two polynomial factors, or even of two rational function factors; it can also be expressed as the quotient of two polynomials or of two rational functions

v) Matrix mapping spaces are ~~compounded~~ Compound also denoted by use of the above notations. Thus, if $a: \mathbb{B} \to K^{[h]}_{[m]}$ and $\mathcal{S}$ is a prescribed space of mappings ~~to~~ of the form $b: \mathbb{B} \to K^{[k]}_{[m]}$, $[a+\mathcal{S}]$ ~~$a\times\mathcal{S}$~~ is a space of mappings of the form $[a+b]$ ~~$a\times b$~~, where $b \in \mathcal{S}$. The conventions concerning void mappings, described in the previous clause, are also observed in the treatment of matrix mapping spaces.

Proofs.

i) Since $p = [p+C][1-O^{*}_{[n]}]$ ~~$[p\times C][1/O_{[n]}]$~~ $\langle \mathbb{B} \rangle$ the first requirement is met by p. Suppose the second requirement is not met: $y: \mathbb{B} \to K_{[n]}$ exists for which $p = Cy$ at $z \in \mathbb{B}'$. Then $cp = cCy = O^{[n]}y = 0$ at z. But $cp \neq 0$ at z: no such y exists. The second requirement is met by p over $\mathbb{B}'$.

ii) If $[b+C]$ ~~$[b\times C]$~~ is nonsingular over $\mathbb{B}$, $p = [b \overset{+}{\times} C]x \langle \mathbb{B} \rangle$ where $x := [b\times C]^{-1}p$. Furthermore post$\{\frac{p}{C}\}\mathbb{B}\}$ is void.

iii) Since $b \in p\{C|\mathbb{B}\}$ ~~bound$\{p, C|\mathbb{B}\}$~~, $x: \mathbb{B} \to K_{[n+1]}$ exists ~~exists~~ for which $p = [b \overset{+}{\times} C]x \langle \mathbb{B} \rangle$. Assuming W as described to exist, $p = [b' \overset{+}{\times} C]x''$ where, with $x = [x(0) \overset{\sim}{\times} x']$, $W = [W(0) \overset{\sim}{\times} W'] \langle \mathbb{B} \rangle$, x'' has the decomposition ~~$x''(0) = \frac{x(0)}{z(0)}$ and~~ $x'' = [x''(0) \overset{\sim}{\times} \hat{x}] \langle \mathbb{B} \rangle$ where with $x''(0) = \frac{x(0)}{W(0)}$, $\hat{x} = x' - \frac{x(0)W'}{W(0)} \langle \mathbb{B} \rangle$. If $y: \mathbb{B} \to K_{[n]}$ for which $b' = Cy$ at $z \in \mathbb{B}$ exists, $b = Cy'$ at z, where ~~y' has the decom~~ $y' = \frac{y - W'}{W(0)}$ at z: b is not in bound$\{p, C|\mathbb{B}\}$.

With W as described, $b = [b' \overset{+}{\times} C]W'' \langle \mathbb{B} \rangle$ where $W'' = \frac{1}{W(0)}[1 \overset{\sim}{\times} (-W'$

The above argument may be used to show that if ~~$b \in \text{bound}\{p, C|\mathbb{B}\}$~~ $b' \leq p\{C|\mathbb{B}\}$. $b \leq p\{C|\mathbb{B}\}$ ~~$b \in \text{bound}\{p, C|\mathbb{B}\}$~~ also.

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iii) If $b = O_{[n]}$ at $z \in \mathbb{B}$, $\text{post}\left\{\frac{b}{C}\Big|z\right\}$ contains $O_{[n]}$.

$x = [x(0) \overset{+}{\not\leq} x']$ and

ii) If $p = [b \overset{+}{\not\leq} C]x \langle\mathbb{B}\rangle$ with $x(0) \neq 0 \langle\mathbb{B}\rangle$ then $b = [p \overset{+}{\not\leq} C]x'' \langle\mathbb{B}\rangle$

where $x'' = \frac{1}{x(0)}[1 \overset{+}{\not\leq} (-x')]$: $\text{post}\left\{\frac{b}{[p \overset{+}{\not\leq} C]}\Big|\mathbb{B}\right\}$ is nonvoid.

If $y : \mathbb{B} \to K^{\not\leq}_{[n]}$ for which $p = Cy$ at z exists, then $b = Cy''$ at z, where $y'' = \frac{y - x'}{x(0)}$: $b \notin \text{bound}\{p, C|\mathbb{B}\}$.

iii) If $p = [b \overset{+}{\not\leq} C]x$, $q = [p \overset{+}{\not\leq} C]v$ where $x, v : \mathbb{B} \to K_{[n+1]}$ have the decompositions $[x(0) \overset{+}{\not\leq} x']$ and $[v(0) \overset{+}{\not\leq} v']$ respectively, then $q = [b \overset{+}{\not\leq} C]u$ where $u = [x(0)v(0) \overset{+}{\not\leq} (v(0)x' + v')]$. Since $b \nleq p\{C|$ ~~$b \notin \text{bound}\{p, C|\mathbb{B}\}$~~, $\text{post}\left\{\frac{b}{C}\Big|\mathbb{B}\right\}$ is void

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i) If $p = [b \overset{+}{\not\leq} C]x \langle\mathbb{B}\rangle$, $kp = [b \overset{+}{\not\leq} C](kx) \langle\mathbb{B}\rangle$

ii) If $p = [b \overset{+}{\not\leq} C]x$, $p' = [b \overset{+}{\not\leq} C]x' \langle\mathbb{B}\rangle$ then $p \pm p' = [b \overset{+}{\not\leq} C](x \pm x') \langle\mathbb{B}\rangle$

where $ux' = 0$

iii) If $p = [b \overset{+}{\not\leq} C][x(0) \overset{+}{\not\leq} x'] \langle\mathbb{B}\rangle$ then $p = [b \overset{+}{\not\leq} CD][x(0) \overset{+}{\not\leq} D^{-1}x'] \langle\mathbb{B}\rangle$. If $b = CDy$ at $z \in \mathbb{B}$, then $b = Cy''$ at z where $y'' = Dy$: $b \notin \text{bound}\{p, C|\mathbb{B}\}$ $p(u/C|\mathbb{B}\}$. Also $(uD)(D^{-1}x') = 0 \langle\mathbb{B}\rangle$, this argument may be reversed.

Hence $\text{bound}\{p, C|\mathbb{B}\} \subseteq \text{bound}\{p, CD|\mathbb{B}\}$

iv) If $p = [b \overset{+}{\not\leq} C]x \langle\mathbb{B}\rangle$ then $Bp = [b' \overset{+}{\not\leq} BC]x \langle\mathbb{B}\rangle$ where $b' = Bb$. If $Bb = BCy$ at $z \in \mathbb{B}$, $b = Cy$ at z and $b \notin \text{bound}\{p$ etc $p(u/C)$

Hence $B\,\text{bound}\{p, C|\mathbb{B}\} \subseteq \text{bound}$ ~~$\{Bp, BC|\mathbb{B}\}$~~. This argument may be reversed. $\{p(u/C)\mathbb{B}\}$ $\{Bp\,(u/BC)\mathbb{B}\}$

i) If $b' = Cy^{ar,\, z\in B}\langle B\rangle$ then $b = C(y - gz')\langle B\rangle$ and $b \notin \text{bound}\{p, C|1$.

Thus no such y exists. This result together with the condition $b' = [b \times C][1/z']\langle B\rangle$ imply that $b' \in \text{bound}\{b, C|B\}$. Since $b \in \text{bound}\{p, C|B\}$, $b' \in \text{bound}\{p, C|B\}$. Since $cb = d$, $cC = 0^{[n]}\langle B\rangle$

$cb' = d\langle B\rangle$.

i) Since $b \le p\{(u/C)B\}$ $\cancel{b \in \text{bound}\{p, C|B\}}$ and $b' = [b \dotplus C][1 \dotplus z']\langle B\rangle$, $b' \le p\{(u/C)B\}$ ~~b′∈bound{p, C|B} also~~ (Th. clause (iii)). Since $cb = d$, $cC = 0^{[u]}\langle B\rangle$, it follows that $cb' = d\langle B\rangle$.

If $b, b' \le p\{(u/C)B\}$ ~~b′∈bound{p, C|B}, z exists such that b′=[b×C]z~~ $w: B \to K_{[n+1]}$ with $w(0) \ne 0\langle B\rangle$ for which $b' = [b \dotplus C]z\langle B\rangle$ exists (the same clause). Since $cb = cb' = d\langle B\rangle$, and $cC = 0^{[n]}\langle B\rangle$, $z(0) = 1\langle B\rangle$.

ii) Select $c \in \text{pre}\left\{\frac{[d \dotplus 0^{[n]}]}{[b \dotplus C]}\middle|B\right\}$ so that $cb = d$, $cC = 0^{[n]}\langle B\rangle$.

The relationship stated in (i) implies that $cb' = cb = d\langle B\rangle$. Hence $c \in \text{pre}\left\{\frac{[d \dotplus 0^{[n]}]}{[b' \dotplus C]}\middle|B\right\}$. This argument may be reversed.

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iii) Since, from clause (i) Th. , $b \le kp\{C|B\}$ ~~b∈bound{kp, C|B}~~ and $[d(kp/b):C|B]$ ~~pol[kp, b, C, d|B]~~ is well defined. ~~If c∈pre{[d×0^{[n]}]/[b×C]}~~

If $x \in \text{post}\left\{\frac{p}{[b \dotplus C]}\right\}$, $kx \in \text{post}\left\{\frac{kp}{[b \dotplus C]}\right\}$. Also $d\{k\{x(0)\} = k\{dx(0$

b) From Th clause (iii), $b' \le p[C|B]$ ~~b′∈bound{p, C|B} and pol[p, b′, C, d|B]~~ $\text{pol}[d(p/b'):C|B]$ is well defined. ~~From claus~~ From clause (ii) of Th , any $c \in \{$

$c \in pre\left\{\frac{[d \overset{+}{\ast} O^{[n]}]}{[b\overset{+}{\ast}C]}|B\right\}$ is in $pre\left\{\frac{[d\overset{+}{\ast}O^{[n]}]}{[b'\overset{+}{\ast}C]}|B\right\}$. P and $pol[d(p/b'):c|B]$ ~~$pol[p, b', C, d|B]$~~ are both expressible as cp, over B.

c) From the clause (iii), $b \le p\{CD|B\}$: ~~$b \in bound\{p, CD|B\}$~~: $pol[d(p/b):CD|B]$ ~~$pol[p, b, CD, d|B]$~~ is well defined. If $c[b\overset{+}{\ast}C]=[d\overset{+}{\ast}O^{[n]}]$, $c[b\overset{+}{\ast}CD]=[d\overset{+}{\ast}O^{[n]}]$ also.

d) From the clause (iv), $Bb \le Bp\{BC|B\}$ ~~$Bb \in bound\{Bp, BC|B\}$~~. If $c[b\overset{+}{\ast}C]=[d\overset{+}{\ast}O^{[n]}]\langle B\rangle$, $cB^{-1}[Bb\overset{+}{\ast}BC]=[d\times O^{[n]}]\langle B\rangle$. $P=cp$ and $pol[d(Bp/Bb):BC|B]$ ~~$pol[Bp, Bb, BC, d|B]$~~ $=(cB^{-1})Bp\ \langle B\rangle$.

e) If $c[b\overset{+}{\ast}C]=[d\overset{+}{\ast}O^{[n]}]\langle B\rangle$, $kc[b\overset{+}{\ast}C]=[kd\overset{+}{\ast}O^{[n]}]\langle B\rangle$.

ii) If $P\neq 0\ \langle B\rangle$, $post\left\{\frac{p}{[b\overset{+}{\ast}C]}|B\right\}$ contains x for which $x(0)\neq 0$ and, from the clause (ii), $p\le b\{C|B\}$: ~~$p\in bound\{b, C|B\}$~~: $P'=pol[\frac{1}{d}(b/p):C|B]$ ~~$pol[b, p, C, \frac{1}{d}|B]$~~ is well defined. $P=dx(0)\ \langle B\rangle$, where $p=[b\overset{+}{\ast}C]x$.

But $b=[p\overset{+}{\ast}C]z$ where $z(0)=\frac{1}{x(0)}\ \langle B\rangle$ and $P'=\frac{z(0)}{d}\ \langle B\rangle$. Thus $P'=\frac{1}{P}\ \langle B\rangle$.

iii) If $b\le p'\{C|B\}$ ~~$b\in bound\{p', C|B\}$~~ then $b\le p\pm p'\{C|B\}$ ~~$b\in bound\{p\pm p', C|B\}$~~, from the clause (ii). For any c for which $c[b\overset{+}{\ast}C]=[d\overset{+}{\ast}O^{[n]}]\langle B\rangle$, $c(p\pm p')=cp\pm cp'\ \langle B\rangle$.

iv) $P=dx(0)$, $R=d'w(0)$ and $S=d''u(0)\ \langle B\rangle$, where $p=[b\overset{+}{\ast}C]x$, $q=[p\overset{+}{\ast}C]w$ and $q=[b\overset{+}{\ast}C]u\ \langle B\rangle$.

~~But $u=u(0)\times u'$ where $u(0)=x(0)w(0)$, $u'=w(0)x'+w'$~~ term

Setting $x=x(0)\overset{+}{\ast}x'$ and decomposing u and w in a similar way, $u(0)=x(0)w(0)$, $u'=w(0)x'+w'$. The required result follows from the first of these relations.

and $c'C = O^{[n]}$ $\langle B \rangle$. Then $P' = c'b = cb = d$ $\langle NS(P) \rangle$.

v) The coefficients c and c' satisfy the relationships

$$c[b+C] = [d + O^{[n]}] \quad \langle B \rangle$$

$$c'[b'+C] = [d' + O^{[n]}] \quad \langle B \rangle$$

Multiplying the first of the above relationships throughout by d' and the second by d the first ~~stated~~ result of (v) follows immediately, as does the second.

vi a) ~~The result~~

$$P' = cp' \text{ where } c \in \mathrm{pre}\left\{\frac{[d+O^{[n]}]}{[b+C]}\right\}$$

b) $P' = c'p$ ~~where c~~ for any $c' : B \to K_{[m]}$ satisfying

new proof

$$c'[b+C] = [d' + O^{[n]}] \quad \langle B \rangle$$

$c' := \left(\frac{d'}{d}\right)c$ may be chosen and then $c'p = cp'$ $\langle B \rangle$ ~~where~~ where p' is as described.

c) $P' = c'p$ for any $c' : B \to K_{[m]}$ satisfying

$$c'[b'+C] = [d + \cancel{O}^{[n]}] \quad \langle B \rangle \qquad \frac{d}{(cb')}u$$

$c' := \left(\frac{d}{cb'}\right)c$ may be chosen and then $c'p = cp'$ $\langle B \rangle$ where p' is as described.

Since $b' \leq p \{C|B\}$, $b' = [b+C]z$ $\langle B \rangle$ for some $z : B \to K_{[m]}$ for which $z(0) \neq 0$ over B, from the clause (iii). Then $cb' = [d + O^{[n]}]z = dz(0) \neq 0$ $\langle B \rangle$.

$c' = kc$

$c'[b+C] = [kd + ku]$

d) ~~From clause (iv)~~

~~$d = \mathrm{pol}[b, p, C, d\, P \mid NS(P)]$~~ $\quad c'p = c(kp)$

~~so that, from (vi c) above~~

~~$P' =$~~

d) As in the proof of clause (iv) it is shown that $p \leq b\{C|NS(P)\}$.

so that P' is well defined. $P' = c'b$ where for any $c' : B \to K_{[M]}$ satisfying

$$c'[p+C] = [d+0^{[M]}] \quad \langle NS(P) \rangle$$

$c' := \left(\frac{d}{\{cp\}}\right)c = \frac{d}{P}c$ may be chosen and then $c\{\} = cp' \overset{\langle B \rangle}{}$ where

p' is as described.

$q := b \quad f = \frac{1}{d}$

$p = [b+C]x \quad P = dx(0) \quad q = [p+C]y \quad R = fy(0)$

$q = [b+C]z \quad PR = \quad p = bx(0) + Cx' \quad q = py(0) + Cy'$

$q = bx(0)y(0) + C(x'y(0) + y') \quad z(0) = x(0)y(0) \quad z' = x'y(0) + y'$

p can be selected (e.g. $p = b$ or $p = q$). That the ...

Since $b \leq p, p \leq q\{C|B\}$, $b \leq q\{C|B\}$ also. That the product of the right hand denominator and left hand term is equal to the right hand numerator follows from th. i)

Since ~~B~~ $P = dx(0)$ $\langle B \rangle$ for all $x \in post\{\frac{p}{[b+C]}|B\}$, $x(0) \neq 0$ $\langle NS(P) \rangle$ and, from th. (ii) $p \leq b\{C|NS(P)\}$: the right hand polynomial is well defined

$q := b \quad f \cdot \frac{1}{d} \quad prl[\cancel{b}(\frac{1}{d}; b)/b; C|B] = \frac{1}{d}$

$$d\,prl[p, b, C, 1|B] = prl[(d;p)/b, C|B]$$
$$(1;p)/b, C|B$$

$P = cp \quad c[b + C] = [d + 0^{[n]}] \qquad c = c'B$

$c'[Bb + BC] \qquad c'p = cB^{-1}p = cp' \quad p' = B^{-1}p$

$c'[b + CD] = [d + 0^{[n]}] \quad c' = c$

e) Polynomials differing ~~only~~ in homogeneous constraint system and boundary

Let $B : \mathbb{B} \to \mathbb{K}'[m]$ ~~be nonsingular over $\mathbb{B}$~~ and set

$P' = \mathrm{pol}[d(p / Bb) : BC | \mathbb{B}]$. $P' = cp'$ where

$p' = B^{-1}p \quad \langle \mathbb{B} \rangle$

~~Let $D : \mathbb{B} \to$~~

f) Polynomials differing only in homogeneous constraint system

Let $D : \mathbb{B} \to \mathbb{K}'[n]$ and set $P' = \mathrm{pol}[d(p/b) : CD | \mathbb{B}]$. $P' = cp \langle \mathbb{B}$
$| (uD / CD) \mathbb{B}]$

Proofs. e) Since $b \leq p \{C | \mathbb{B}\}$, $Bb \leq p \{BC | \mathbb{B}\}$ from clause ( ) Th. :
P' is well defined. $P' = c'p \langle \mathbb{B} \rangle$ for any c' such that
$c'[Bb + BC] = [d + 0^{[n]}] \langle \mathbb{B} \rangle$. $c' := cB^{-1}$ may be chosen and then
$c'p = cp'$ where $p' := B^{-1}p$.

f) Since $b \leq p \{C | \mathbb{B}\}$, $b \leq p \{CD | \mathbb{B}\}$ from Th clause ( ) :
P' is well defined $P' = c'p \langle \mathbb{B} \rangle$ for any c' such that
$c'[b + CD] = [d + 0^{[n]}] \langle \mathbb{B} \rangle$. $c' = c$ may be chosen.

(F)

nonsingularity: require $[a.+b][c.+d]^{-1}f = e$

in particular this holds when $a.+b = 0$ $e=0$ i.e. $a=0$ $b=0$

Transitivity

$[a.+0][x'+.x''] = e$ $\quad [a.+0][w'+.w''] = e$

$[c.+d][x'+.x''] = f$ $\quad [c.+f][w'+w''] = fh$

$bx + Cx' = p$

$ux' = 0$

$\begin{array}{ccc} 0 & \boxed{u} & 0 \\ \boxed{b} & \boxed{C} & p \end{array}$

Less $cw' + \{cx' + dx''\}w'' = h$

$c\{w' + x'w''\} + dx''w'' = h$

$a\{w' + x'w''\} = e + ew''$

$u^n_h \quad b^k_m \quad C^n_m \quad k+n = h+m$

a) if $[0^{[k]}_{[h]}.+u]+.[b.+C]$ is nonsingular wrt B & BS ... nonvoid for all p^r_m, $ux' = 0^{[r]}_{[h]}$

transitivity requires $e = 0$

b) if $k+n \overset{+1}{=} m$

$[b.+C]$ nonsingular x'^r_n

BS ... nonvoid for all p^r_m

$0^{[r]}_{[h]}$ all $h \in \bar{N}$

reflection

$c.+d]$ $cx' + dx'' = f$ $\quad c(-x'(x'')^{-1}) + f(x'')^{-1} = d$

x'' nonsingular $\quad a(-x'(x'')^{-1}) = e$ again if $e = 0$

$[c.+d][y'+y''] = 0 \Rightarrow y'' = 0$

$[c.+f][w'+w''] = 0$ $\quad cw' + fw'' = 0 \Rightarrow \cancel{\{cw' + \{c(-x'(x'')^{-1}) + f(x'')^{-1}\}}w''$

$cw' + \{cx' + dx''\}w'' = 0$

$[c.+d][(w'+x') + x''w''] = 0$

if $w'' \neq 0$ $y'' = x''w'' \neq 0 \Rightarrow$ ~~d~~ d not prenonsing wrt c

Nonsingularity follows from

if d pre nonsing wrt c then $b \neq 0$

otherwise post $\left\{\dfrac{0}{c.+d}\right\}_{[n,k+n+1]}$ contains nonzero member $[0_{[k+n]}+.1]$

i) $d \le f \Rightarrow d \le fk$ all $k : B \to K^{1:1}_{1:1}$ $\quad [c.+d][x'k+.x''k] = fk$

again using $e = 0$ $\quad ax'k = 0$

ii) $[c_{\cdot}+d][(x'+w')+_{\cdot}(x''+w'')] = f+f'$ $a(x'+w')=0$ again $e=0$

iii) $\operatorname{post}\left\{\frac{0}{c_{\cdot}+d}\right\}_{(n,k+m+n]} = \operatorname{post}\left\{\frac{0}{cD+d}\right\}_{(n,k+m+n]}$ $Cx' = $

$\{cy'+dy''=0$ $cD(D^{-1}y')+dy''=0$ $ux'=0$

$aD.D^{-1}x' = e$

$cD.D^{-1}x'+dx''=f$

$Bp = Bbx + BCx'$

$Bb \in \operatorname{prens}\{BC|B\}$?

iv) $[Bc_{\cdot}+Bd][x'+x''] = Bf$

means $Bbx + BCx' = 0 \Rightarrow x=0$

$ax' = e$

$=$ $\operatorname{prens}\{BC|B\} \subseteq B\operatorname{prens}\{C|B\}$

b' $b'x + BCx' = 0$

$\Rightarrow x=0$

$=$

$y[c_{\cdot}+d] = \{\ \} [\cancel{n_{\cdot}+a}\ a_{\cdot}+n]$

$\boxed{yd=n}$

suppose $Bbw + BCw' = 0$ $w \neq 0$

can be so when $B=0$

$d' = [c_{\cdot}+d][w'+_{\cdot}I]$ $aw'=0$

a) $\underline{yd'} = [a_{\cdot}+n][w'+_{\cdot}I] = n$

b) $d' \leq f \ldots \Rightarrow d' = [c_{\cdot}+d][w'+w'']$ w'' nonsing $aw'=0$

$yd' = n$ $yd' = [a_{\cdot}+n][w'+w''] = nw''$ n nonsing $\Rightarrow w''=I$

$=$ ii) $y[c_{\cdot}+d] = [a_{\cdot}+n]$

$y[c_{\cdot}+d'] = yc_{\cdot}+yd' = \cancel{y[cw'+d}\ yc_{\cdot}+y(cw'+d)$

$= yc_{\cdot}+(ycw'+yd) = a_{\cdot}+(aw'+n) = a_{\cdot}+n$ $aw'=0$

and reverse

$\Rightarrow$

Pd Th.

i) $y[c_{\cdot}+d] = [a_{\cdot}+\overset{n}{\varphi}]$ $P=yf$ $1 = $ unit matrix $\dim = \dim \ni f =$ " " " p

$$[P_{\cdot}+y]\begin{bmatrix} 0 & -1 \\ [c_{\cdot}+d] & f \end{bmatrix} = [a_{\cdot}+\overset{n}{\varphi}_{\cdot}+0]$$

$aa' = a'a$ $an' = a'n$

$=$ relationships between coefficients $y'_{\cdot}[c_{\cdot}+d'] = [a'_{\cdot}+\overset{n'}{\varphi}]$ $a[a'+n'] = a'[a+n]$

~~ab~~ $\cancel{\Rightarrow a'y[c_{\cdot}+d] = ay'[c+d']}$ $\Rightarrow a'y[c_{\cdot}+d] = ay'[c+d']$

$\cancel{[c_{\cdot}+d], a' \text{ nonsingular} \Rightarrow y = (a')^{-1} a\, y'[c_{\cdot}+d'][c_{\cdot}+d]^{-1}}$

Nonsingular tot. contr. repr.

$$P = [a. + \overset{n}{\cancel{b}}][c. + d]^{-1} f$$

P scalar $P = \dfrac{\begin{vmatrix} a. \cancel{b}. 0 \\ [c. + d] f \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ [c. + d] f \end{vmatrix}}$ $a = 0$ $P = \dfrac{|\cancel{a}c. + f \cancel{b}|}{|c. + d|} n$

$$= \frac{\begin{vmatrix} a. n. 0 \\ [c. + d] f \end{vmatrix}}{|c. + d|} (-1)^?$$

Related polynomials

i) Determination of y : $y[c. + d] = [a. + n]$ does not depend on f

ii) $y \cdot [c. + d] [a. + n]$ $P = yf = [a. + n][x' + x''] = nx''$

$f = [c. + d][x' + x'']$

$P = nx''$ $P' = n'x''$ $P' = n'n^{-1}P = n'n^{-1}yf$

require $n'n^{-1}y = y\,n'n^{-1}$ $P' = \cancel{k} \, kP$

$n'n^{-1}y \cdot [c. + d] = [n'n^{-1}a + n']$ $= kyf = y(kf)$

$y\,\underset{\uparrow \; nn'^{-1}}{n'n^{-1}}[c. + d] = [a + n]$

require $n' = kn$ k scalar

$d' = [c + d][z' + z'']$ $z'' \neq 0$

$yd' = [a. + n][z' + z''] = nz''$

investigate z'': always nonsingular?

iii) $y'[c + d'] = [a. + n]$ $y'f$ $\cancel{kg}$ require yd' nonsingular

$n(yd')^{-1}$ scalar

$\dfrac{\cancel{x}a + n}{yd'}$

$y' = \cancel{y} \; n(yd')^{-1}y \Rightarrow y'd' = n(yd')^{-1}yd' = n$

$n(yd')^{-1}yc = n(yd')^{-1}a$ $y'f = n(yd')^{-1}yf$

iv) $y'[c. + f] = [\frac{n}{(yf)} a. + n]$ $y' = \frac{n}{yf} y$ $y'a$

$yc = a$ $\frac{n}{yf} \cdot yc = \frac{n}{yf} a$ $\frac{n}{yf} \cdot yf = n$ $n(yf)^{-1} yc = n(yf)^{-1} a$

$y' = \cancel{n}$ $P = yf$ $y' = nP^{-1} y$ $y'b = nP^{-1} yb$

$= y(nP^{-1} b)$

assume nP^{-1} scalar

then $P' = yp'$ where $p' = (nP^{-1})$

v) $y'[Bc. + Bd] = [a. + n]$ $y'B = y$ $y'f = yB^{-1} f$

$= yf'$ where $f' = B^{-1} f$

vi) a) $y'[cD. + d] = [aD. + n]$ $y = y'$ $yf = y'f$

b) $y'[c. + d] = [ka + kn]$ $ky = y'$ & $y'f = ykf$

if k scalar $P' = y'f = ka\, kn\, x''$

Transformations

i) $p \leq p$ NS(P) $p = [c. + p][x' + x'']$ can take $x'' = I$

$nI = nx''$ $= P$ $= px''$

ii) $f \leq b$ NS(P) $b = [c. + f][w' + w'']$

$y[c. + f] = [a. + P]$

$P' = $ pol $[P(b/f) | (u/C) BNS(P)]$

$P = yf$ $y[c. + b] = [a. + n]$ $y[c. + f] = [a. + P]$

$P' = yb = n$

$u^n_h \quad C^{[n]}_{[m]} \quad yC = u \quad y^m_h \quad C \in H\{u[m]B\} \quad C \in H\{u, B\}_{[m]}$

$p = [b. + C]x \quad b^k_m \quad x^r_{k+n+1} \quad \text{pfact}\,[(p/b)C] = x_{[k]}$

$p^r_m \quad ux_{-[k]} = 0^{[r]}_{[h]}$

$(yp)^r_h$

$b \leq p\ \{(u/C)B\} \quad b \leq p\ \{(u/C)[k]\}$

(i) $\left\{p \in \left\{\frac{u}{c}\,|B'\right\}p\right\} \quad 0^{[r]}_{[h]}$ nonvoid

$b \leq p\ \{(u/C)B\}^{[k]}$

(ii) $n+k=m$ $\quad$ ~~$u=0^{[m]}$ not required~~ $\quad u = 0^{[n]}_{[h]}$

$b. + C$ nonsingular

$HCS\{u/\}$ $\quad CS$ $\quad \text{bound}\{p(u/C)B\}^{[k]}$

$\{B \to K'[k]$

iii) $\text{bound}\{p(u/C)B\}^{[k]} = [b. + C]\left[\{B \to K^{[k]}_{[k]}\ 0^{[k]}_{[k]}\}\right.$

$+.\ \text{post}\left\{\frac{0^{[k]}_{[h]}}{u}\,|B\right\}$

$b' = [b. + C][w + w'] = bw + Cw'$

$p = bx + Cx' = \{b'w^{-1} - Cw'w^{-1}\} + Cx' \qquad w^k_k \quad w'^k_n$

$= [b'. + C][wx + (x' - w'w^{-1}x)] \Rightarrow \supseteq$

$p = [b'. + C]y = b'y + Cy'$ find w, w' such that $b' =$

$= bx + Cx'$

$(bw + Cw')y + Cy' = bwy + C(y' + w'y)$ $\quad \Big|\ vbx = vb'y$

$w = xy^{-1} \quad w' = (x' - y')y^{-1}$

$k=0$ if $x \neq 0$ $y=0$, w cannot be found

have shown $[b.+C][\{B \to K'[k]\} +.\ \text{post}\left\{\frac{0^{[k]}_{[h]}}{u}\,|B\right\} \leq$

$\text{bound}\{p(u/C)B\}^{[k]}$

define $\text{bound}'\{p(u/C)B\}^{[k]}_{[k]}$

all $b \in \text{bound}\{p(u/C)B\}^{[k]}$ for which $\text{pfact}\,[(p/b)C]$ nonsingular

$\text{bound}'\{p(u/C)B\}^{[k]} \subseteq [b.+C]\left[\{B \to K^{*}[k]\} +.\ \text{post}\qquad\right\}$

iii) $b \leq p\,\{(u/C)\,\mathbb{B}\}^{[k]}$

a) $[b.+C][\{\mathbb{B}\to\mathbb{K}'[k]\}+.\,\mathrm{post}\{\underset{u}{O^{[k]}_{[k]}}|\mathbb{B}\}] \leq \mathrm{bound}\{p(u/C)\,\mathbb{B}\}^{[k]}$

b) $\mathrm{bound}'\{p(u/C)\,\mathbb{B}\}^{[k]} \leq [b.+C][\{\mathbb{B}\to\mathbb{K}'[k]\}+\mathrm{post}.\{\underset{u}{O^{[k]}_{[k]}}|\mathbb{B}\}$

c) $b \in \mathrm{bound}'\{p(u/C)\,\mathbb{B}\}^{[k]}$

$\mathrm{bound}'\{p(u/C)\,\mathbb{B}\}^{[k]} = [b.+C][\{\mathbb{B}\to\mathbb{K}'[k]\}+.\,\mathrm{post}\{\underset{u}{O^{[k]}_{[k]}}|\mathbb{B}\}$.

a): select $w+.\,w'$ in $[..]$ $\qquad b^k_m \quad b'^k_m$

& $bx+Cx'=p \quad b'=bw+Cw'$ $\qquad x^r_k \quad x'^r_n \quad w^k_k \quad w'^k_n$

$= \{b'w^{-1}-Cw'w^{-1}\}x+Cx'$ $\qquad b\leq b'$ for all selected w,w'

$=[b'.+C][w^{-1}x+.(x'-w'w^{-1}x)] \Rightarrow b' \in$ rhs space

b) Select $b' \in$ post space

$p=b'y+Cy'$ y invertible $\quad p=bx+Cx'$ $\quad$ since rank x = rank y $\quad x$ invert.

$b'y+Cy'=bx+Cx' \quad b'=bxy^{-1}+C(x'-y')y^{-1}$

$=[b.+C][xy^{-1}+.(x'-y')y^{-1}] \in$ rhs space

c) as in b) but now $xy' \in \{\mathbb{B}\to\mathbb{K}'[k]\}$

as in a) " " $w^{-1}x \in$ "

b) $b'y=bx+C(x'-y')$ do w,w' exist s.t. $b'=bw+Cw'$

yes when y invertible then $b\leq b'$

a) $b\leq b'$ all selected w,w' $\quad b=b'w^{-1}-Cw'w^{-1}$

if $b'u+Cu'=O_{[m]}$ with nonzero u, then $bwu+Cw'u+Cu'$

$=bwu+C(w'u+u')=O_{[m]}$

with $wu\neq 0$ i.e. b not prenonsing wrt C

hence b' pre nonsing wrt. C and $b'\leq b$

i.e. $b\equiv b'$ all selected w,w'

$p=bx+Cx'=b'w+Cw'$

$AxB=w$

$bx+Cx'=b'AxB+Cw'$

a) all b^{u} for which $b \equiv p^{u}$ given by rnk?

$$[p.+C]\left[\{B \to K'[k] +. \operatorname{post}\left\{\frac{O^{[k]}_{[h]}}{u} \middle| B\right\}\right.$$ independent ~~of~~ mention ~~of p~~

b) $\operatorname{band}'\{p(u/C)B\}^{[k]}$: all $b' \equiv p$

? given by $[p.+C][B \to K'[k] +. \operatorname{post}\left\{\frac{O^{[k]}_{[h]}}{u} \middle| B\right\}$

—

p prenonsingular wrt $C \Rightarrow p \leq p\ \{(u/C)B\}$

$p = p\,I[r] + C\,O^{[r]}_{[n]}$

=

$\operatorname{post}\left\{\frac{p}{C} \middle| B\right\}$ nonvoid + $b \leq p\ \{(u/C)B\} \Rightarrow p\operatorname{fact}[(p/b)C]$ iff $= O^{[r]}_{[k]}$

containing $w'^{\,r}_{\in u}$

$b\,O^{[r]}_{[k]} + C w' = \cancel{O^{[r]}_{[m]}}\ p$ $uw' = 0$? then any $b \in \operatorname{pre}\{C, B\}$ $\Rightarrow p \leq p\ \{(u/C)B\}$.

If $p\operatorname{fact}[(p/b)C] = O^{[r]}_{[k]}$ then iff $\operatorname{post}\left\{\frac{p}{C}\right\}B\} \cap \operatorname{post}\left\{\frac{O}{u}\right\}B\}$ is nonvoid

=

$p \leq p$ p prenonsingular with respect to C

$$\begin{array}{ccc} k & w & r \\ \boxed{b} & \boxed{C} & \begin{bmatrix} x \\ x' \end{bmatrix} \end{array} = \begin{array}{c} r \\ \boxed{p} \end{array}$$

$p = bx + Cx'$ if $[p.+C][w+.w'] = O_{[m]}$ then $w = 0$

$pw + Cw' = O_{[m]}$

$\operatorname{rank}[C] + r \leq m$

$r \leq k$

$\operatorname{rank}[C] + k \leq m$

suppose $t^{\neq 0}$ for which $xt = 0$ exists

$pt = Cx't \Rightarrow t = 0$ no such t exists $\therefore x$ is nonsingular

$b \leq p\ \{(u/C)B\}$. $t^{\neq 0}_{r}\ \ x t\ \overset{0}{k}$

p prenonsingular with respect to C over B

iff $p\operatorname{fact}[(p/b)C]$ nonsingular over B

~~p~~ $b \equiv p$ suppose $pw + Cw' = 0$ for some nonzero w

$pw = bxw + Cx'w = 0$ x nonsing $w \neq 0 \Rightarrow xw \neq 0$

$\Rightarrow b \not\equiv p$

~~iff~~ p prenonsingular wrt C over $\mathbb{B}$ notation: $p \in \mathrm{pre}\{C, \mathbb{B}\}$

iff ~~b~~ $\mathrm{bound}\{p(u/C)\mathbb{B}\} = p\ \{(u/C)\mathbb{B}\}$

select $b \leq p\ \{(u/C)\mathbb{B}\}$ then $p \in \mathrm{pre}\{C, \mathbb{B}\} \Rightarrow p$ fact $[(p/b)C]$ nonsing

$\Rightarrow p \leq b \rightarrow \mathrm{bound}\{p(u/C)\mathbb{B}\} = p$

~~from as above~~ $\mathrm{bound}\{\ldots = p$ ~~now~~ if

$[p.+C]\{\mathbb{B}' \to \mathbb{K}'[k] +. p\ \mathrm{st} \ldots\} \cancel{= \mathrm{bound}}\{p$

$\mathrm{bound}\{p(u/C)\mathbb{B}\} = p \Rightarrow p \leq p\ \{(u/C)\mathbb{B}\} \Rightarrow$

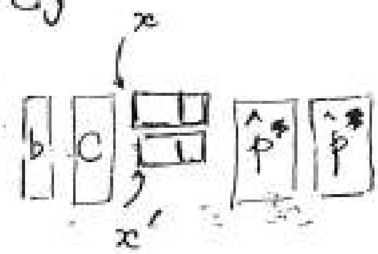
$p \in \mathrm{pre}\{C, \mathbb{B}\}$

$\hat{p}^{*}$ formed from column components of $p \in \mathrm{pre}\{C, \mathbb{B}\}$

$\tilde{p}^{*}$ " " " " " for which $p''(t) = Cy$

$[b. + C][x +. x'] = \hat{p}^{*}.+\tilde{p}^{*} = bx + Cx'$

~~$p \leq \mathbb{B}x$~~ $= b[\hat{x}.+\tilde{x}] + C[\hat{x}'.+\tilde{x}']$



$\hat{p} = b\hat{x} + C\hat{x}'$ $\tilde{p} = b\tilde{x} + C\tilde{x}'$ $\tilde{x} = 0$ x^{r}_{k} x'^{r}_{n}

Suppose $b\hat{x}$ not pre nonsingular wrt ~~b~~ C

$\exists\ y, y'$ st. $[b\hat{x}. + C]\underset{[y+.y']}{y} = 0_{[m]}$ for which y ~~[r]~~ $\neq 0$ $b\hat{x}^{r}_{m}$ C^{n}_{m}

y^{r}_{r} $\hat{x}y^{r}_{k}$

$[b.+C][\hat{x}y+.y'] = 0^{[r]}_{[m]}$ provide $\hat{x}y = 0$

p	b	y	x	b	C	x	p
0 0	1 0	0	0 0	1 0	0	$\begin{bmatrix}0&0&0\end{bmatrix}$	0 0 0
0 1	0 1	0	0 1	0 0	1	$\begin{bmatrix}0&1&0\end{bmatrix}$	a b c
a b	0 0	1	a b	0 1	0	$\begin{bmatrix}a&b&c\end{bmatrix}$	0 1 0

0 0 1
1 0 0
0 1 0

$p - Cx'$

0 0 0
0 0 0
0 1 0

$\mathrm{rank}[x] = \mathrm{rank}[b\overset{x'}{x}]$?

$$\begin{array}{c} b\ \ C \\ \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\end{array}\begin{bmatrix}0&1&0\\0&0&0\\a&b&c\end{bmatrix} = \begin{array}{c} p \\ \begin{matrix}0&0&0\\a&b&c\\0&1&0\end{matrix}\end{array}$$

Transformations

iii) must have k scalar $\quad P = yf \quad y[c.+d] = [a.+n]$

$$P' = ykf = kyf$$

b) $d' = [c.+d][w'+.1] = cw+d$

$$f = [c.+d][x'+x''] = \{cx'+dx'' = cx'+(d'-cw)x''$$

$$= [c.+d'][(x'-wx'')+x''] \qquad = c(x'-wx'')+d'x''$$

$$P' = nx'' = P \qquad ax' = aw = 0$$

c) $y.[c.+d] = [a.+n] \quad y[cD.+d] = [aD+n] \quad P = P' = yf$

must show if d prenonsingular with respect to c then d also prenonsingular with respect to cD

$$f = [c.D+d][w'+w''] = [c.+d][Dw'+w'']$$

d) $Bf = [Bc+Bd][x'+x''] \quad P = P' = n$

also ... then Bd prenonsingular with respect to Bc

e) $y'.[c.+d] = [ka+kn] \quad y' = ky \quad y'f = kyf \quad P' = kP$

iv) Addition and subtraction

a) $f' = [c.+d][w'+w''] \quad y(f \pm f'') = P' \pm P$

b) $P = nx'' \quad P' = n'x'' \quad f = [c.+d][x'+x'']$

$$P \pm P' = pd[(n \pm n')..]$$

$$mnx''y'' = nx''my$$

Multiplication

m scalar remove with

i) $b \leq p \leq q \qquad p = [c.+d][x'+.x''] \quad P = nx''$

$R = \dfrac{Q}{P}$

$q = [c.+d][y'+.y''] \quad Q = my'', \quad R = PQ \qquad Q = PR$

$q = [c.+d][w'+w''] \quad R = mnw'' \qquad R \Leftrightarrow Q$

$$q = cy' + py'' = cy' + \{cx'+dx''\}y'' = [c.+d][(y'+x'y'')+x''y'']$$

$$Q = PR \Rightarrow R = P^{-1}Q$$

iii) $R = P^{-1}Q$ with m scalar and P square

iv) $P = x x''$ $f = [c. + d][x'+ . x'']$

P^{-1} requires x, x'' invertible ?

use $b \leq d$ $d \leq d$ pd$[d^{-1}(b/b)$ $] = d^{-1}$

d pd $[I(p/b)\cdot\cdot] =$ pd$[d(p/b)\cdots]$

(F)

Nonsingularity.

If $b = O^{[k]}_{[m]}$ at some $z \in \mathbb{B}$, $b \notin \text{prens}\{C|\mathbb{B}\}$.

Transformations

i) If $b \leq p\{(u/C)\mathbb{B}\}$ then post$\{\cdots\}$ $bx + Cx' = p \langle \mathbb{B}\rangle$ for some $x \in \mathbb{K}(\mathbb{B})^{[r]}_{[k]}$, $x' \in \text{post}\left\{\frac{O^{[r]}_{[k]}}{u}|\mathbb{B}\right\}$. For all $g \in \mathbb{K}(\mathbb{B})^{[s]}_{[r]}$, $bxg + Cx'g = pg$ and $x'g \in \text{post}\left\{\frac{O^{[s]}_{[k]}}{u}|\mathbb{B}\right\}$

Since $b \in \text{prens}\{C|\mathbb{B}\}$, $b \leq pg\{(u/C)\mathbb{B}\}$

$=$

$b \leq pg\{(u/C)\mathbb{B}\}$ for all $g : \mathbb{B} \to \mathbb{K}^{[s]}_{[r]}$.

---

ii) If $b \leq p'\{(u/C)\mathbb{B}\}$ then $bw + Cw' = p' \langle \mathbb{B}\rangle$ for some $w \in \mathbb{K}(\mathbb{B})^{[r]}_{[k]}$

$w' \in \text{post}\left\{\frac{O^{[r]}_{[k]}}{u}|\mathbb{B}\right\}$. $[b, + C][(x \pm w), +, (x' \pm w')] = p \pm p' \langle \mathbb{B}\rangle$

— If $p' : \mathbb{B} \to \mathbb{K}^{[r]}_{[m]}$ is such that $b \leq p'\{(u/C)\mathbb{B}\}$ then

$b \leq \{p \pm p'\{(u/C)\mathbb{B}\}$.

$\sim$

iii) $\mathbb{BS}\{p(u/C)\mathbb{B}\} = \mathbb{BS}\{p(uD/CD)\mathbb{B}\}$

for all ~~$D : \mathbb{B} \to$~~ $D \in \mathbb{K}'[\mathbb{B}, n]$

If $p = bx + Cx' \langle \mathbb{B}\rangle$ for some $x \in \mathbb{K}(\mathbb{B})^{[r]}_{[k]}$ $x' \in \text{post}\left\{\frac{O^{[r]}_{[k]}}{u}|\mathbb{B}\right\}$ then $p = bx + CDy'$ where $y' = D^{-1}x' \langle \mathbb{B}\rangle$. Also $uDD^{-1}x' = ux' = O^{[r]}_{[k]} \langle \mathbb{B}\rangle$. Lastly $b \in \text{prens}\{CD|\mathbb{B}\}$ if $b \in \text{prens}\{C|\mathbb{B}\}$: $b \leq p\{(uD)/(CD)\mathbb{B}\}$. This argument can be reversed.

If $bw + CDw' = O_{[m]}$ for some ~~nonzero~~ $w \in \mathbb{K}(\mathbb{B})$ which is nonzero at some $z \in \mathbb{B}$, $b \notin \text{prens}\{C|\mathbb{B}\}$. Hence $\text{prens}\{C|\mathbb{B}\} \subseteq \text{prens}\{CD|\mathbb{B}\}$. If $D \in \mathbb{K}'[\mathbb{B}, n]$ this argument can be reversed.

Let $m, n, r \in \bar{N}$ and $C : \mathbb{B} \to K^{[n]}_{[m]}$ ~~and $b \in$ prens~~ $D : \mathbb{B} \to K^{[r]}_{[n]}$.

a) prens$\{C|\mathbb{B}\} \subseteq$ prens$\{\cancel{C+D}\, CD|\mathbb{B}\}$ for all $D \in K(\mathbb{B})^{[r]}_{[n]}$.

b) prens$\{C|\mathbb{B}\} =$ prens$\{CD|\mathbb{B}\}$ for all $D \in K'[\mathbb{B}, n]$

proof

Let $m, n \in \bar{N}$ and $C : \mathbb{B} \to K^{[n]}_{[m]}$.

$$\text{prens}\{BC|\mathbb{B}\} = B\,\text{prens}\{C|\mathbb{B}\}$$

for all $B \in K'[\mathbb{B}, m]$.

Select $b' \in$ prens$\{BC|\mathbb{B}\}^{[k]}$ so that $b'w + BCw' = 0_{[m]}$ only when $w = 0_{[k]}$. ~~Then b'~~ Set $b := B^{-1}b' \langle \mathbb{B} \rangle$. Then $bw + Cw' = 0_{[m]}$ only when $w = 0_{[k]}$: $b \in$ prens$\{C|\mathbb{B}\}^{[k]}$. Since $b' = Bb \langle \mathbb{B} \rangle$. prens$\{BC|\mathbb{B}\} \subseteq B$ prens$\{C|\mathbb{B}\}$

---

w) $\quad B\,BS\{p(u/C)\mathbb{B}\} = BS\{Bp(u/BC)\}$

for all $B \in K'[\mathbb{B}, n]$

Select $b \in BS\{p(u/C)\mathbb{B}\}$. Since $b \in$ prens$\{C|\mathbb{B}\}$, $Bb \in$ prens$\{BC|\mathbb{B}$

Since ~~$Bx +$ $bx + Cx'$~~ post$\left\{\dfrac{p}{b.+C}|\mathbb{B}\right\}$ is nonvoid, post$\left\{\dfrac{Bp}{Bb.+BC}|\mathbb{B}\right\}$ is nonvoid : $Bb \in Bp\,\{(u/BC)\mathbb{B}\}$. This argument can be reversed.

---

coefficients space $\quad CS\{d(p/b)|(u/C)\mathbb{B}] = $ pie$\left\{\dfrac{d.+u}{b.+C}|\mathbb{B}\right\}$ varies with b as case in which $[b.+C]$ nonsing indicates

$bx + Cx' = p \quad b'y + Cy' = p$

$dx = cp \quad dy = cp \quad dx = dy \Rightarrow x = y$ when ~~drawing~~

$d \in$ prens$(\mathbb{B})$

assuming $CS\{d(p/b)\ldots$; $CS\{d(p/b')$. intersect as in b)

$b' = bw + Cw'$ $\quad d = dw \Rightarrow w = 1$ $\quad b \le b'$ $\quad b \in \text{prens}\{C|B\}$ ⓑ

$b'y = b$ $\quad$ ~~All $b' \in \text{prens}\{C|B\}$ for which $b \le b'$~~ $\{(u/C)B\}$

~~given by~~ $b' = bw + Cw'$ ⓐ with $w \in \text{prens}\{B\}$.

$b'y + Cy' = by + Cw'y$ $\qquad$ ⓐ $\subseteq$ ⓑ $\quad \text{post}\left\{\begin{matrix} b' \\ b.+C \end{matrix} \middle| B\right\}$ nonvoid

$b'y + Cy' = p$ $\qquad$ if $b'y + Cy' = 0_{[m]}$ for some nonzero y then

$bwy + Cw'y + Cy' = p$ $\qquad$ $bwy + Cw'y + Cy' = 0_{[m]}$ $\quad wy$ may be zero

$w \in$ prens means $\qquad \therefore$ ⓐ $\nsubseteq$ ⓑ

$\exists b$ such that $bw = Cw' - b'$ $\qquad$ must have $w \in \text{prens}\{B\}$ as healed

$xw = y$ $\qquad$ $b \le b' \{(u/C)B\}$ means $b' = bw + Cw'$

$b' \in \text{prens}\{C|B\}$ means no $y \ne 0$ for which $b'y + Cy' = 0_{[m]}$

no $y \ne 0$ for which $bwy + Cw'y + Cy' = 0_{[m]}$

since $b \in \text{prens}\{C|B\}$ this relationship only satisfied when $wy = 0$ i.e. for no $y \ne 0$

is $wy = 0 \Rightarrow w \in \text{prens}\{B\}$. $\quad \text{rank}\begin{bmatrix} x \\ x' - y' \end{bmatrix} = \text{rank}$

$b' = b + Cw'$ $\quad b \le p \Rightarrow b' \le p$

a) $[b.+C][IL]+.\text{post}\left\{\begin{matrix} 0 \\ u \end{matrix} \middle| B\right\} \subseteq$

$B\$\{p(u/C)|B\} \cap \text{post}\left\{\begin{matrix} c \\ d \end{matrix}\right\}$

$\cancel{b}$. $p = b'y + Cy'$ & $cb' = d$

show $b' = \cancel{bw}\, b + C\cancel{w}'$

given that ~~bec~~

$p = bx + Cx'$ $\quad cb = d$

$bx + Cx' = b'y + Cy' \Rightarrow x = y$ if $d \in \text{prens}(B)$

$b'x + Cx' = bx + Cy' \Rightarrow b' = b + Cw'$ only if x ~~is~~ nonsing

b) $=$ if $d \in \text{prens}(B)$ and $p \in \text{prens}(C|B)$ $\qquad$ i.e. $p \in \text{prens}(C|B)$

ii) $b' = [b.+C][IL]+.w']$ $\quad w' \in \text{post}\left\{\begin{matrix} 0 \\ u \end{matrix} \middle| B\right\}$ $\qquad wb^{k}_{h}$

$\text{pre}\left\{\begin{matrix} d.+u \\ b.+C \end{matrix} \middle| B\right\} = \text{pre}\{b'.+$ $\quad \Big\|\quad$ $cC = u$ $\quad cb = d \Rightarrow cb' = d$ $\qquad x^{r}_{k}$

$b' = b + Cw'$

$b'x = bx + C(x' - y')$ $\qquad$ $x' - y' = w'x$ $\qquad$ $\boxed{w}\,\boxed{x} = \boxed{x' - y'}$

The coefficient space

Let $u: B \to K^{[n]}_{[h]}$, $C: B \to K^{[n]}_{[m]}$, $b: B \to K^{[k]}_{[m]}$ and ~~d~~ in ~~K[B]~~ $d: B \to K^{[k]}_{[h]}$. The coefficient space ~~$\mathbb{CS}\{(d/b)(u/C)B\}$~~ $K[B]^{[n]}_{[h]}$ generated by d, b, u and C over B is defined by setting

$$\mathbb{CS}\{(d/b)(u/C)B\} = \text{pre}\left\{\frac{d.+u}{b.+C}\,\Big|\,B\right\}$$

Let $C \in HC\{u|B\}_{[m]}$, $b \in \text{prens}\{C|B\}^{[k]}$ and $d: B \to K^{[k]}_{[h]}$.

i) $\mathbb{CS}\{(d/b)(u/C)B\}$ is nonvoid.

ii) Select $c \in \mathbb{CS}\{(d/b)(u/C)B\}$.

$\text{pre}\,\mathbb{CFZ}\{b.+C, B\}_{[h]}$

a) $$\mathbb{CS}\{(d/b)(u/C)B\} \equiv c + \cancel{\text{pre}\left\{\frac{O^{[k+n+1]}_{[h]}}{b.+C}\,\Big|\,B\right\}} \quad \langle\langle B\rangle\rangle$$

b) If ~~k~~ $p: B \to K^{[r]}_{[m]}$ is such that $b \in p\{(u/C)B\}$

$$\mathbb{CS}\{(d/b)(u/C)B\} \subseteq c + \cancel{\text{pre}}\left\{\frac{O^{[r+n+1]}_{[h]}}{p.+C}\,\Big|\,B\right\} \subseteq c + \cancel{\text{pre}}\left\{\frac{O^{[r]}_{[h]}}{p}\,\Big|\,B\right\}$$

$\text{pre}\,\mathbb{CFZ}\{p, B\}_{[}$ ~~$k \geq r$~~

c) If, furthermore, $r \geq k$ and $p \in \text{prens}\{C|B\}$,

$$\mathbb{CS}\{(d/b)(u/C)B\} \equiv c + \cancel{\text{pre}}\left\{\frac{\cancel{O}^{[k+n+1]}_{[h]}}{p.+C}\right\} \leftarrow \text{pre}\,\mathbb{CFZ}\{p.+C, B\}_{[h]}$$

Since $b \in \text{prens}\{C|B\}$ and $\text{pre}\left\{\frac{u}{C}|B\right\}$ is nonvoid, the result of clause (i) follows from th. clause . The result of subclause (iia) follows directly from Th. clause.

Under the conditions of subclause (ii b), $p = bx + Cx'$ for some $x \in K[B]^{[r]}_{[k]}$, $x' \in \text{post}\left\{\frac{O^{[r]}_{[n]}}{u}|B\right\}^{\langle B\rangle}$. For all $w \in K[B]^{[m]}_{[h]}$ for which $wb = O^{[k]}_{[h]}$ and $wC = O^{[n]}_{[h]}$, $wp = O^{[r]}_{[h]}$ also. Hence

$$\mathrm{pre}\left\{ \frac{O^{[k+m]}_{[h]}}{b.+C} \middle| \mathbb{B} \right\} \subseteq \mathrm{pre}\left\{ \frac{O^{[r+m]}_{[h]}}{p.+C} \middle| \mathbb{B} \right\}$$

The last ~~result~~ relationship follows from Th clause .

~~Under the conditions~~

The two conditions $wp = O^{[r]}_{[h]}$, $wC = O^{[m]}_{[h]}$ induce the condition $wbx = O^{[r]}_{[h]}$. Under the conditions of subclause (iic), x is ~~square~~ of rank $k+1$ ~~and nonsingular~~ over $\mathbb{B}$. Hence $wb = O^{[k]}_{[h]}$. The above semi-inclusion result becomes one of equivalence.

~~Let $p \in FS\{(u/C)\mathbb{B}\}^{[k,k]}$~~

Let $k \in \bar{N}$, $C \in HC\{u|\mathbb{B}\}_{[m]}$, ~~$p \in FS\{(u/C)\mathbb{B}\}^{[k,k]}$~~, ~~$b \in p\{(u/C)\mathbb{B}\}$~~, and $d \in K[\mathbb{B}]^{[k]}_{[h]}$

i) Let $p \in FS\{(u/C)\mathbb{B}\}^{[k,k]}$ and be $b \in p\,\{(u/C)\,\mathbb{B}\}$

(ia) Select $c \in CS\{(d/b)(u/C)\mathbb{B}\}$.

a) $b + C\,\mathrm{post}\left\{ \frac{O^{[k]}_{[h]}}{u} \middle| \mathbb{B} \right\} \subseteq \mathbb{BS}\{p(u/C)\mathbb{B}\} \cap \mathrm{post}\left\{ \frac{d}{c} \middle| \mathbb{B} \right\}$

b) If $d \in \mathrm{prens}\{\mathbb{B}\}$ and $p \in \mathrm{prens}\{C|\mathbb{B}\}$, the above semi-inclusion result becomes one of equivalence

ii) Let $b \in \mathrm{prens}\{C|\mathbb{B}\}^{[k]}$

(iia) Select $w' \in \mathrm{post}\left\{ \frac{O^{[k]}_{[h]}}{u} \middle| \mathbb{B} \right\}$ and set $b' = b + Cw'$.

Then $\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\} = \mathbb{CS}\{(d/b')(u/C)\mathbb{B}\}$.

That $b + C\,\mathrm{post}\left\{ \frac{O^{[k]}_{[h]}}{u} \middle| \mathbb{B} \right\} \subseteq \mathbb{BS}\{p(u/C)\mathbb{B}\}$ follows from Th clause . Also $c\left\{ b + C\,\mathrm{post}\left\{ \frac{O^{[k]}_{[h]}}{u} \middle| \mathbb{B} \right\} \right\} = d + u\,\mathrm{post}\left\{ \frac{O^{[k]}_{[h]}}{u} \middle| \mathbb{B} \right\}$

$$= d + O^{[k]}_{[h]} = d$$

Select b' ~~$\in \mathbb{BS}\{p(u$~~ b' in the left hand space of subclause (ia) so that $b'y + Cy' = p$ where $uy' = 0^{[k]}_{[h]}$ and $cb' = d \langle B \rangle$.

Under the conditions of subclause (ib), ~~y is~~ x and y are nonsingular over B. Also $bx + bx' = p$ where ~~x is set~~ $x \in K[B,k]$. Hence $b'y + Cy' = bx + Cx'$. Since $c\{bx + Cx'\} = c\{b'y + Cy'\} = cbx = cby = dx = dy$ and $d \in$ prens$\{B\}$, $x = y \langle B \rangle$. ~~Since~~ Hence $b' = b + C(x' - y'x^{-1}) \langle B \rangle$ and b' is in the left hand space denoted in subclause (ia).

~~Let $C \in HC\{u|B\}$ $k \in \bar{K}$, $b \in$ prens$\{C|B\}$ and $d \in K[B]^{[k]}_{[h]}$. Select $w' \in$ post$\left\{\frac{0^{[k]}_{[h]}}{u}\middle|B\right\}$~~

~~and set $b' := b + Cw'$.~~

~~$$CS\{(d/b)(u/C)B\} \equiv CS\{(d/b')(u/C)B\}$$~~

~~Since $b \in$ prens$\{C|B\}$ and $C \in HC\{u|B\}$, $CS\{(d/b)(u/C)B\}$ is nonvoid. Select c in this space~~

Let $C \in K[B]^{[n]}_{[m]}$, $b \in K[B]^{[k]}_{[m]}$ and $d \in K[B]^{[k]}_{[h]}$. Select $w' \in$ post$\left\{\frac{0^{[k]}_{[h]}}{u}\middle|B\right\}$ and set $b' := b + Cw' \langle B \rangle$.

i a) $CS\{(d/b)(u/C)B\} \equiv CS\{(d/b')(u/C)\}$

ii b) If $C \in HC\{u|B\}$ and $b \in$ prens$\{C|B\}$, both of the above spaces are nonvoid.

If pre$\left\{\frac{u}{C}\middle|B\right\}$ is ~~non~~ void, both spaces are void. If no $c \in$ pre$\left\{\frac{u}{C}\middle|B\right\}$ for which $cb = d \langle B \rangle$ exists, no such c for which $cb' = d \langle B \rangle$ exists, since $cb' = cb + uw' = cb \langle B \rangle$. In this case both spaces are void. Otherwise $cb' = cb = d$: all c in the left hand space belong ~~to~~ the right hand counterpart. This argument may be reversed. Clause (ii) is a consequence of th. clause .

let $C \in HC\{u|B\}_{[m]}$, $p \in FS\{(u/C)B\}^{[k,k]}$, $b \leq p\{(u/C)B\}^{[k}$ and $d \in K[B]^{[k]}_{[h]}$. Select ~~or~~ $c \in CS\{(d/b)(u/C)B\}$.

i) $b + C\,\mathrm{post}\left\{\frac{O^{[k]}_{[h]}}{u}|B\right\} \leq BS\{p(u/C)B\} \cap \mathrm{post}\left\{\frac{d}{c}|B\right\}$

ii) If $d \in \mathrm{prenv}\{B\}$ and $p \in \mathrm{prenv}\{C|B\}$ the above semi inclusion result becomes one of equivalence.

$x' = N'x$

$$\begin{array}{ccccccc|cc} b & C & & & & p & & b & C \\ 1 & 0 & 0 & a & a' & a\,a' & & 1\;0 & 0\;a \\ 0 & 1 & 0 & b & b' & b\,b' & & 0\;1 & 0\;b \\ 0 & 0 & 1 & c & c' & c\,c' & & 0\;0 & 1\;c \end{array}$$

$b\ C$

show $\xi x' t = 0$ if $\xi x t = 0$

rank $a\ a'$ = rank $\begin{matrix} a & a' \\ b & b' \\ c & c' \end{matrix}$

$p \in \mathrm{prenv}(C$

~~$x \in \mathrm{prenv} \Rightarrow$~~

show ~~$tx' = 0$ only when $t = 0$~~

$$\begin{array}{ccccc} b & C & & p & \\ 1 & 0 & 0 & a & a' \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b & b' \end{array}$$

~~rank argument false~~

$\boxed{0\ a}$ A

$\boxed{\begin{matrix} 0 & b \\ 0 & c \\ 0 & d \end{matrix}}$ B

rank A = rank B

= rank $[A + B]$

$p = bx + Cx'$ an ~~and~~ but $yA = B$ has a solution

$xt = 0 \Rightarrow pt + Cx't = 0$

$\begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix}$ $\boxed{0\ a}$ $\begin{matrix} b & b \\ 0 & c \\ 0 & d \end{matrix}$

$y_1 a = b$
$y_2 a = c$
$y_3 a = d$

$\Rightarrow pt = 0$ this impossible for nonzero t

~~$\Rightarrow x't = 0$ if $C \in \mathrm{prenv}(B)$~~

$x't = 0 \Rightarrow pt = bxt$ possible $xt \neq 0$

$cpt = cbxt$

$Pt = dxt$

$b'x = bx + C(x' - y')$ $(x' - y')t = 0 \Rightarrow b'xt = bxt$

(1) Polynomials and rainmax functions

In many investigations of properties of polynomials of the form

$$P(x) = \sum c(\nu) x^{\nu} \langle \nu := [m] \rangle$$

the coefficient vector $c \in K_{[m]}$ is subjected to a homogeneous linear constraint ~~of the~~ such as

$$c\,C = 0^{[n]}$$

where $C \in K^{[n]}_{[m]}$.

Many uses of polynomials are based upon similar properties of their coefficient vector. ¶ For example, supposing the roots of P in a suitably extended field to be $\lambda(\nu) \langle \nu := [m] \rangle$ the $\lambda(\nu)$ being distinct, C may be taken to be

$$C := [\lambda(\nu)^{z}]^{\nu := [m]}_{z := [m]}$$

The factorisation properties of P may then be discussed in terms of the possibility of reducing C to block diagonal form by pre- and post-multiplication of C by suitable matrices with elements in K. Again, orthogonal polynomials derive their significance from orthogonal constraints ~~of~~ expressible in the above form imposed upon their coefficients. The coefficients of polynomials occurring in the theory of interpolation satisfy interpolatory constraints.

It is customary to select from the class of polynomials under investigation one representative member by imposing upon the coefficients a further inhomogeneous constraint, involving a boundary $b \in K_{[m]}$ and a normalising factor $d \in K \setminus 0$, of the form

$$cb = d$$

In investigations into factorisation properties, the selected polynom has monomial form: the coefficient of the highest power of the variable is unity. In the above constraint $b = [0^{[m]} + 1]$, $d = 1$. This is also the preferred normalisation in the algebraic theory of orthogonal polynomials. In some uses of interpolatory polynomials, the constant term in P is taken to be unity*. Now $b = [1 + 0^{[m]}]$, $d = 1$.

In the following a theory of generalised polynomials defined by the formula $P := cp$, where $c : \mathbb{B} \to K_{[m]}$, $p : \mathbb{B} \to K^{[m]}$, is presented in terms of constraint mappings of the form $C : \mathbb{B} \to K^{[n]}_{[m]}$ and similarly extended boundary and normalising factor mappings.

Polynomials and rational functions

Boundaries

~~Let h, m, n ∈ $\overline{\mathbb{N}}$ and M := set. Where relevant, $\mathbb{H}[M,k]$:=~~

~~def$_{[k]}$ $\mathbb{K}[M]^{[n]}_{[m]}$ with k ∈ [m],~~ ~~and~~ ~~$\mathbb{R}[M] := \mathbb{K}[M]^{[n]}_{[h]}$~~ and $\mathbb{B} \subseteq M$.

Let h, k, m, n, r ∈ $\overline{\mathbb{N}}$,

~~Let k, r ∈ $\overline{\mathbb{N}}$,~~ $u := \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$, $C := \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$ and $p: \mathbb{B} \to \mathbb{K}^{[r]}_{[m]}$

i a) A mapping $b: \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ for which

α) $b \in$ prens$\{C, \mathbb{B}\}$ and $p \circ \mathbb{C}FZ\{u, \mathbb{B}\}^{[r]}$

β) post$\left\{\frac{p}{b.+C} \mid \mathbb{B}\right\} \cap [\mathbb{K}[\mathbb{B}]^{[r]}_{[k]} +.$ ~~post$\left\{\frac{\mathbb{O}[h]}{u} \mid \mathbb{B}\right\}$~~]

is nonvoid

is said to be a boundary of the constraint system u/C

with respect to the function p over $\mathbb{B}$.

b) The notation $b \leq p \ \{(u/C)\mathbb{B}\}^{[k]}$ indicates that $b: \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$

~~is a boundary in the sense described. When the mapping~~

$b: \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ has been prescribed, the notation $b \leq p\{(u/C)\mathbb{B}\}$

is used.

c) The notation $b \equiv p \ \{(u/C)\mathbb{B}\}$ is used to indicate that

both $b \leq p$, $p \leq b \ \{(u/C)\mathbb{B}\}$

ii) Set $\mathbb{N}' := [m)$ and $\mathbb{N}'' := \overline{\mathbb{N}}$.

a) ~~Set $\mathbb{N}'' := \overline{\mathbb{N}}$.~~ The mapping

$\mathbb{FS}: \mathbb{R}[M] \times \mathbb{H}[M, \mathbb{N}'] \times M \times \mathbb{N}' \times \mathbb{N}'' \to \mathbb{K}[M]^{[\mathbb{N}'']}_{[m]}$

is a boundary in the sense described. The notation $b \leq p \{(u/C)\mathbb{B}\}$ indicates that the function b features in a mapping which is a boundary of u/C with respect to p over $\mathbb{B}$, the column dimension indicator k either having been already specified or not being of immediate interest.

c) The notation $b \equiv p \{(u/C)\mathbb{B}\}$ is used to indicate that both $b \leq p$, $p \leq b$ $\{(u/C)\mathbb{B}\}$.

ii) In the special case in which $u = O^{[n]}_{[k]}\langle\mathbb{B}\rangle$, $p \circ CF\mathbb{Z}\{u,\mathbb{B}\}^{[r]}$ reduces to $K[\mathbb{B}]^{[r]}_{[n]}$ and the requirement (β) reduces to the condition that

$$\operatorname{post}\left\{\frac{p}{b_{\cdot}+C}\middle|\mathbb{B}\right\}$$

should be nonvoid. In this case b is said to be a boundary of the constraint ~~system~~ C with respect to the function p over $\mathbb{B}$, the notation $b \leq p \{C,\mathbb{B}\}^{[k]}$ being used to denote this fact. The notations $b \leq p \{C,\mathbb{B}\}$ and $b \equiv p \{C,\mathbb{B}\}$ have similarly reduced meanings.

~~Exist~~

Conditions defining boundaries presented in terms of rank.

Let $k, h, m, n, r \in \overline{\mathbb{N}}$, $u : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$, $C : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$ and $p : \mathbb{B} \to \mathbb{K}^{[r]}_{[m]}$.

i) Conditions (ia,b) may be formulated in terms of rank as follows:

a') $$\operatorname{rank}[b.+C] = \operatorname{rank}[C] + k + 1 \qquad \langle \mathbb{B} \rangle$$

b') $$\operatorname{rank}\left[\left[O^{[k]}_{[h]}.+u\right]+.\left[b.+C\right]\right] =$$

$$\operatorname{rank}\left[\left[O^{[k]}_{[h]}.+u.+O^{[r]}_{[h]}\right]+.\left[b.+C.+p\right]\right] \qquad \langle \mathbb{B} \rangle$$

ii) If the above conditions are satisfied

$$\operatorname{rank}[b.+C] = \operatorname{rank}[b.+C.+p]$$

iiia) $$\operatorname{rank}[p.+C] = \operatorname{rank}[C] \qquad \langle \mathbb{B} \rangle$$

if and only if $\operatorname{post}\left\{\frac{p}{C} \middle| \mathbb{B}\right\}$ is ~~not~~ nonvoid

b) $$\operatorname{rank}[p.+C] = \operatorname{rank}[C] + r + 1 \qquad \langle \mathbb{B} \rangle$$

if and only if $p \in \operatorname{prens}\{C \mid \mathbb{B}\}$ (in which case necessarily $r \leq m - \operatorname{rank}[C]$).

Function and boundary spaces

Let $h, m, n \in \overline{\mathbb{N}}$ and $M :=$ set. Where relevant, $\mathbb{H}[M, k] := \mathrm{def}_k \mathbb{K}[M]^{[n]}_{[m]}$ with $k \in [n]$, $\mathbb{R}[M] := \mathbb{K}[M]^{[n]}_{[h]}$ and $\mathbb{B} \subseteq M$.

i) Set $\overline{\mathbb{N}} := [m]$ and $\mathbb{N}'' := \overline{\mathbb{N}}$.

a) The mapping

$$\mathbb{FS} : \mathbb{R}[M] \times \mathbb{H}[M, \mathbb{N}'] \times M \times \mathbb{N}' \times \mathbb{N}'' \subseteq \mathbb{K}[M]^{[\mathbb{N}'']}_{[m]}$$

is defined by setting

$$\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]} := \mathrm{prens}\{C,\mathbb{B}\}^{[k]}\,\mathbb{K}[\mathbb{B}]^{[r]}_{[k]} + C\,\mathrm{post}\Big\{\frac{O^{[r]}_{[h]}}{u}\Big|\mathbb{B}\Big\}^{\mathrm{CpoOF}\mathbb{Z}\{u,\mathbb{B}\}^{[r]}}$$

b) The mapping

$$\mathbb{FS}': \mathbb{R}[M]\times H[M,\mathbb{N}']\times M\times[\min(\mathbb{N}',\mathbb{N}'')]\times\mathbb{N}'\times\mathbb{N}''\to\mathbb{K}[M]^{[\mathbb{N}'']}_{[m]}$$

is defined by setting

$$\mathbb{FS}'\{(u/C)\mathbb{B},s\}^{[k,r]} := \mathrm{prens}\{C,\mathbb{B}\}^{[k]}\,\mathbb{K}[\mathbb{B},s]^{[r]}_{[k]} + C\,\mathrm{post}\Big\{\frac{O^{[r]}_{[h]}}{u}\Big|\mathbb{B}\Big\}^{\mathrm{CpoOF}\mathbb{Z}\{u,\mathbb{B}\}^{[r]}}$$

c) ~~The mapping~~ Set $\mathbb{N}''' := [\mathbb{N}']$. The mapping

$$\mathrm{prens}\mathbb{FS}: \mathbb{R}[M]\times H[M,\mathbb{N}']\times M\times\mathbb{N}'\times\mathbb{N}'''\to\mathbb{K}[M]^{[\mathbb{N}''']}_{[m]}$$

is defined by setting

$$\mathrm{prens}\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]} := \mathrm{prens}\{C,\mathbb{B}\}^{[k]}\,\mathrm{prens}\mathbb{K}[\mathbb{B}]^{[r]}_{[k]} + C\,\mathrm{post}\Big\{\frac{O^{[r]}_{[h]}}{u}\Big|\mathbb{B}\Big\}^{\mathrm{CpoOF}\mathbb{Z}\{u,\mathbb{B}\}^{[r]}}$$

d) → (ii) Let $r\in\mathbb{N}$ and set $\mathbb{N}' := [m)$

a) The mapping

$$\mathbb{BS}: \mathbb{FS}\{(\mathbb{R}[M]/H[M,\mathbb{N}'])M\}^{[\mathbb{N}',r]}\times\mathbb{R}[M]\times H[M,\mathbb{N}']\times M\times\mathbb{N}'\to \mathbb{K}[M]^{[\mathbb{N}']}_{[m]}$$

is defined by setting

$$\mathbb{BS}\{p(u/C)\mathbb{B}\}^{[k]} := \mathrm{pre}\left\{\frac{p - C\,\mathrm{post}\Big\{\frac{O^{[r]}_{[h]}}{u}\Big\}^{\mathrm{CpoOF}\mathbb{Z}\{u,\mathbb{B}\}^{[r]}}}{\mathbb{K}[\mathbb{B}]^{[r]}_{[h]}}\right\}\setminus\mathbb{PS}\{C|\mathbb{B}\}^{[k]}$$

b) Let $k\in\mathbb{N}$. The mapping

$$\mathbb{EBS}: \mathrm{prens}\{H[M,k],M\}^{[k]}\times\mathbb{R}[M]\times H[M,k]\times M\to\mathbb{K}[M]^{[k]}_{[m]}$$

is defined by setting

$$\mathrm{EBS}\{b(u/C)\mathbb{B}\} := b\, \mathrm{us}\mathbb{K}[\mathbb{B},k] + \overset{\mathrm{CpoCFZ}\{u,\mathbb{B}\}^{[k]}_{[k]}}{\cancel{C\,\mathrm{post}\left\{\frac{O_{[h]}}{u}\,|\mathbb{B}\right\}}}$$

iii) Let $k \in$ [illegible]$[M]$ and set $\bar{\mathbb{N}}' := \bar{\mathbb{N}}$.

a) The mapping

$$\mathrm{CFS} : \mathrm{prens}\{\mathbb{H}[M,k],M\} \times \mathbb{R}[M] \times \mathbb{H}[M,k] \times \mathbb{M} \times \mathbb{N}' \to \mathbb{K}[M]^{[\mathbb{N}']}_{[m]}$$

is defined by setting

$$\mathrm{CFS}\{b(u/C)\mathbb{B}\}^{[r]} := b\,\mathbb{K}[\mathbb{B}]^{[r]}_{[k]} + \overset{\mathrm{CpoCFZ}\{u,\mathbb{B}\}^{[r]}_{[r]}}{\cancel{C\,\mathrm{post}\left\{\frac{O_{[h]}}{u}\,|\mathbb{B}\right\}}}$$

b) The mapping

$$\mathrm{CFS}' : \mathrm{prens}\{\mathbb{H}[M,k],M\} \times \mathbb{R}[M] \times \mathbb{H}[M,k] \times \mathbb{M} \times [\min(k,\mathbb{N}')] \times \bar{\mathbb{N}}' \to \mathbb{K}[M]^{[\mathbb{N}']}_{[m]}$$

is defined by setting

$$\mathrm{CFS}'\{b(u/C)\mathbb{B},s\}^{[r]} := b\,\mathbb{K}[\mathbb{B},s]^{[r]}_{[k]} + \overset{\mathrm{CpoCFZ}\{u,\mathbb{B}\}^{[r]}}{\cancel{C\,\mathrm{post}\left\{\frac{O_{[h]}}{u}\,|\mathbb{B}\right\}}}$$

c) ~~The mapping~~ Set $\mathbb{N}'' := [k]$. The mapping

$$\mathrm{prensCFS} : \mathrm{prens}\{\mathbb{H}[M,k],M\} \times \mathbb{R}[M] \times \mathbb{H}[M,k] \times \mathbb{M} \times \mathbb{N}'' \to \mathbb{K}[M]^{[\mathbb{N}'']}_{[m]}$$

is defined by setting

$$\mathrm{prensCFS}\{b(u/C)\mathbb{B}\}^{[r]} := b\,\mathrm{prens}\mathbb{K}[\mathbb{B}]^{[r]}_{[k]} + \overset{\mathrm{CpoCFZ}\{u,\mathbb{B}\}^{[r]}}{\cancel{C\,\mathrm{post}\left\{\frac{O_{[h]}}{u}\,|\mathbb{B}\right\}}}$$

d) The mapping

$$\mathrm{prenCFS} : \mathrm{prens}\{\mathbb{H}[M,k],M\} \times \mathbb{R}[M] \times \mathbb{H}[M,k] \times \mathbb{M} \times \mathbb{N}' \to \mathbb{K}[M]^{[\mathbb{N}']}_{[m]}$$

is defined by setting

$$\mathrm{prenCFS}\{b(u/C)\mathbb{B}\}^{[r]} := \cancel{C\,\mathrm{post}\left\{\frac{O_{[h]}^{[r]}}{u}\,|\mathbb{B}\right\}}\ C\,\mathrm{poCFZ}\{u,\mathbb{B}\}^{[r]}$$

id) The mapping

$$\mathrm{preFS} : \mathbb{R}[M] \times H[M, n-1] \times M \times \mathbb{N}'' \to \mathbb{K}[M]^{[\mathbb{N}'']}_{[m]}$$

is defined by setting

$$\mathrm{preFS}\{(u/C)B\}^{[\cancel{k},r]} := \cancel{C\,\mathrm{post}\{\tfrac{O^{[r]}_{[h]}}{C} | B\}}\; C\,\mathrm{poCFZ}\{u, B\}^{[r]}$$

Write ~~FS{..}~~ as pres{..}?

~~$\mathrm{post}\{\frac{O^{[r]}_{[h]}}{u} | B\}$ as $\mathrm{POQ}\{u | B, r\}$~~ ?

pred for predegenerate? ~~prerogatory~~

---

iv) In the special case in which $u = O^{[n]}_{[h]}\langle B\rangle$, $\mathrm{poCFZ}\{u, B\}^{[r]}$ reduces to $\mathbb{K}[B]^{[r]}_{[n]}$ in the above.

With $\mathbb{N}' := [m]$ and $\mathbb{N}'' := \bar{\mathbb{N}}$, the special mapping

$$\mathrm{FS} : H[M, \mathbb{N}'] \times M \times \mathbb{N}' \times \mathbb{N}'' \subseteq \mathbb{K}[M]^{[\mathbb{N}'']}_{[m]}$$

is defined by setting

$$\mathrm{FS}\{C, B\}^{[k,r]} := \mathrm{prens}\{C, B\}^{[k]}\, \mathbb{K}[B]^{[r]}_{[k]} + C\,\mathbb{K}[B]^{[r]}_{[n]}$$

The number of arguments associated with the symbol FS clearly indicates whether reference is being made to the space defined under the general conditions of subclause (ia) or that defined in the presence of the restriction $u = O^{[n]}_{[h]}\langle B\rangle$

Reduced forms of the further mappings described above are defined in a similar way. The spaces concerned may be denoted in succession as $\mathrm{FS}'\{C, B : s\}^{[k,r]}$, $\mathrm{prensFS}\{C, B\}^{[k,r]}$, $\mathrm{prenFS}\{C, B\}^{[r]}$, $\mathrm{BS}\{p, C | B\}^{[k]}$, $\mathrm{EBS}\{b, C | B\}$, $\mathrm{CFS}\{b, C | B\}^{[r]}$, $\mathrm{CFS}'\{b, C | B, s\}^{[r]}$, $\mathrm{prensCFS}\{b, C | B\}^{[r]}$ and $\mathrm{prenCFS}\{b, C | B\}^{[r]}$.

Function and boundary spaces

i) With the source domains as specified in each case, the spaces defined in clauses (iia–d), (iiia,b), (iva–d) are nonvoid.

iia) $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ is the function space of mappings $p:\mathbb{B}\to K^{[r]}_{[m]}$ for which mappings $b:\mathbb{B}\to K^{[k]}_{[m]}$ such that $b\subseteq p\{(u/C)\mathbb{B}\}$ exist

b) $\mathbb{FS}'\{(u/C)\mathbb{B},s\}^{[k,r]}$ is the subspace of $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ composed of mappings $p:\mathbb{B}\to K^{[r]}_{[m]}$ for which prerank$[p,C]=s+1\ \langle B\rangle$

~~c) prens $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ is $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}\cap$ prens$\{C,\mathbb{B}\}$~~

~~d) pren $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[r]}$~~

c) prens $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ is the subspace composed of ~~$\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$~~

~~composed of mappings $p:\mathbb{B}\to K^{[r]}_{[m]}$ that are prenonsingular with~~

all mappings $p:\mathbb{B}\to K^{[r]}_{[m]}$ in $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ that are prenonsingular with respect to C over $\mathbb{B}$:

$$\text{prens}\,\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}\equiv\text{prens}\{C,\mathbb{B}\}\cap\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$$

d) pren $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[r]}$ is, ~~the~~ for each $k\in[m]$, the subspace composed of all mappings $p:\mathbb{B}\to K^{[r]}_{[m]}$ in $\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ that are prenugatory with respect to C ~~are prenugatory with respect to~~ over $\mathbb{B}$:

$$\text{pren}\,\mathbb{FS}\{(u/C)\mathbb{B}\}^{[r]}\equiv\text{pren}\{C,\mathbb{B}\}\cap\mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$$

iiia) $\mathbb{BS}\{p(u/C)\mathbb{B}\}^{[k]}$ is the boundary space of mappings $b:\mathbb{B}\to K^{[k]}_{[m]}$ for which $b\subseteq p\{(u/C)\mathbb{B}\}$

b) $\mathbb{E}\mathbb{B}\mathbb{S}\{b(u/C)\mathbb{B}\}$ is the equivalent boundary space of mappings $b':\mathbb{B}\to\mathbb{K}^{[k]}_{[m]}$ for which $b'\leq b\,\{(u/C)\mathbb{B}\}^{[k]}$ and is also the space of such mappings for which $b'\equiv b\,\{(u/C)\mathbb{B}\}^{[k]}$.

iva) $\mathbb{C}\mathbb{F}\mathbb{S}\{b(u/C)\mathbb{B}\}^{[r]}$ is the constrained function space of mappings $p:\mathbb{B}\to\mathbb{K}^{[r]}_{[m]}$ for which $b\leq p\,\{(u/C)\mathbb{B}\}$.

b) $\mathbb{C}\mathbb{F}\mathbb{S}'\{b(u/C)\mathbb{B},s\}$ is the subspace of $\mathbb{C}\mathbb{F}\mathbb{S}\{b(u/C)\mathbb{B}\}^{[r]}$ composed of mappings $p:\mathbb{B}\to\mathbb{K}^{[r]}_{[m]}$ for which $\operatorname{prerank}[p,C]=s+1\,\langle\mathbb{B}\rangle$

c) $\operatorname{prens}\mathbb{C}\mathbb{F}\mathbb{S}\{b(u/C)\mathbb{B}\}^{[r]}$ is the subspace composed of all mappings $p:\mathbb{B}\to\mathbb{K}^{[r]}_{[m]}$ in $\mathbb{C}\mathbb{F}\mathbb{S}\{b(u/C)\mathbb{B}\}^{[r]}$ that are prenonsingular with respect to C over $\mathbb{B}$:

$$\operatorname{prens}\mathbb{C}\mathbb{F}\mathbb{S}\{b(u/C)\mathbb{B}\}^{[r]}=\operatorname{prens}\{C,\mathbb{B}\}\cap\mathbb{C}\mathbb{F}\mathbb{S}\{b(u/C)\mathbb{B}\}^{[r]}$$

d) $\operatorname{pren}\mathbb{C}\mathbb{F}\mathbb{S}\{b(u/C)\mathbb{B}\}^{[r]}$ is the subspace composed of all mappings $p:\mathbb{B}\to\mathbb{K}^{[r]}_{[m]}$ in $\mathbb{C}\mathbb{F}\mathbb{S}\{b(u/C)\mathbb{B}\}^{[r]}$ that are prenugatory with respect to C over $\mathbb{B}$.

Inclusion properties of function spaces:

Let $h,m,n,r\in\bar{\mathbb{N}}$, $\mathbb{B}\in$ set, $u:\mathbb{B}\to\mathbb{K}^{[n]}_{[h]}$ and $C:\mathbb{B}\to\mathbb{K}^{[n]}_{[m]}$

ia) For $k:=[m-\operatorname{rank}[C]-1)$

$$\mathbb{F}\mathbb{S}\{(u/C)\mathbb{B}\}^{[k,r]}\subset\mathbb{F}\mathbb{S}\{(u/C)\mathbb{B}\}^{[k+1,r]}.$$

~~b) Similar inclusion relationships hold for the function spaces $\mathbb{F}\mathbb{S}'$ and $\operatorname{prens}\mathbb{F}\mathbb{S}^{2}$.~~

~~ii) Let $k\in[m-\operatorname{rank}[C]),\ k\in[k')$ and $b\in\operatorname{prens}\{C,\mathbb{B}\}^{[k]}$, $b'\in\operatorname{prens}\{C,\mathbb{B}'\}^{[k']}$, with $k\in[k')$~~

~~a) $CFS\{b(u/C)B\}^{[r]} \subset CFS\{b'(u/C)B\}^{[r]}$~~

b) Let $s \in [m-\mathrm{rank}[C]-1)$. An ~~simil~~ inclusion relationship similar to the above holds for $k := [s, m-\mathrm{rank}[C]-1)$ ~~with regard to the functions~~ for the function space $FS'\{(u/C)B, s\}^{[k,r]}$ as stated

c) Mutatis mutandis, the result of subclause (ia) holds as stated with regard to the function space prensFS.

ii) Let $k' \in ([m-\mathrm{rank}[C]])$, $k \in [k')$ and $b \in \mathrm{prens}\{C, B\}^{[k]}$, $b' \in \mathrm{prens}\{C, B'\}^{[k']}$

a) $CFS\{b(u/C)B\}^{[r]} \subset CFS\{b'(u/C)B\}^{[r]}$

b) ~~With $s \in [k]$~~ The above result holds with regard to the function space ~~CFS~~ prens CFS and, with $s \in [k]$, to the function space $CFS'\{b(u/C)B, s\}^{[r]}$.

— Constrained function spaces deriving from special boundaries

Let $k, m, n \in \overline{N}$, ~~$u: B \to K^{[n]}_{[h]}$~~, $C: B \to K^{[n]}_{[k]}$ and $b: B \to K^{[k]}_{[m]}$

i) Let $h = k+n-m \geq 0$, $u: B \to K^{[n]}_{[h]}$, $b \in \mathrm{prens}\{C|B\}$, and $[O^{[k]}_{[h]} + u] +. [b.+C]$ be nonsingular over B.

$$CFS\{b(u/C)B\}^{[r]} = K[B]^{[r]}_{[m]}$$

for all $r \in \overline{N}$

ii) Let $m = k+n$ and $[b.+C]$ be nonsingular over B.

$$CFS\{b, C|B\}^{[r]} \quad \cancel{CFS\{b(O^{[n]}_{[h]}/C)B\}^{[r]}} = K[B]^{[r]}_{[m]}$$

for all $h, r \in \overline{N}$.

Example of clause (i) of above theorem

$$\begin{array}{cccc} 0 & 0 & (0 & 1)\ u \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}$$

$\underbrace{0\ \ 1}_{b}\ \underbrace{0\ \ 0}_{C}$

$m=2\ \ n=1\ \ k=1\ \ h=0$

pre$\left\{\frac{u}{C}|B\right\}$ is void. In clause (i), $k+n \geq m$ and ~~$k+n \geq m+1$~~ and

Under the conditions of clause (i) ~~{rank[C] ≤ n since b ∈ prens{C|B}}~~

$b \in$ prens$\{C|B\}$. If rank$[C]=n+1$, ~~$[b.+C]$ is nonsingular and~~

rank$[b.+C] = n+1+k+1$. But rank$[b.+C] \leq m+1$, i.e $k+n \leq m-1$. Hence

rank$[C] \leq n$. $r=$ rank$\left[\left[0^{[k]}_{[h]}.+u\right]+.[b.+C]\right] = k+n+2$ since the matrix in question is nonsingular. Also $r \leq$ rank$[u+.C]+k+1$

If pre$\left\{\frac{u}{C}|B\right\}$ is nonvoid, rank$[u+.C]=$ rank$[C]$

Hence $k+n+2 \leq$ rank$[C]+k+1$, i.e rank$[C] \geq n+1$. Since rank$[C] \leq n$,

pre$\left\{\frac{u}{C}|B\right\}$ is void.

Suggest redefining a boundary by use of the conditions

~~post$\left\{\frac{0^{[k]}_{[h]}.+u}{b.+C}|B\right\}$ ...~~

post$\left\{\frac{0^{[r]}_{[h]}+.p}{\left[0^{[k]}_{[h]}.+u\right]+.[b.+C]}|B\right\}$ nonvoid

and

$\left[0^{[k]}_{[h]}+.b\right] \in$ prens$\{u+.C|B\}$

(the condition $b \in$ prens$\{C|B\}$ implies the above and is stronger.)

Unfortunately pre$\left\{\frac{d.+u}{b.+C}|B\right\}$ may be void for some d, eg. $d=0, u=b=C=1$

Properties

Let $k, \cancel{l}, m, n, r \in \bar{\mathbb{N}}$, $u : \mathbb{B} \to \mathbb{K}^{[n]}_{[k]}$, $C : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$ and

$\cancel{p \in FS\{(u/C)B\}^{[k,r]}}$

i) Reflection

$p \leq p \{(u/C)\mathbb{B}\}$ if and only if $p \in \text{prens}\{C, \mathbb{B}\}$

ii) Commutativity

Let $k, r \in \bar{\mathbb{N}}$, $p \in FS\{(u/C)\mathbb{B}\}^{[k,r]}$ and $b \in BS\{p(u/C)\mathbb{B}\}^{[k]}$

a) If $p \leq b \{(u/C)\mathbb{B}\}$ for one $b \in BS\{p(u/C)\}^{[k]}$ then

$p \in \text{prens}\{C, \mathbb{B}\}$

b) If $p \in \text{prens}\{C, \mathbb{B}\}$, then $\cancel{p \leq BS\{p(u/C)\}^{[k]}}$ $p \leq b \{(u/C)\mathbb{B}\}$ for all $b \in BS\{p(u/C)\}^{[k]}$

iii) Transitivity

Let $k, r, s \in \bar{\mathbb{N}}$ and $p \in FS\{(u/C)\mathbb{B}\}^{[k,r]}$, $q \in FS\{(u/C)\mathbb{B}\}^{[r,s]}$ with ~~be such that~~ $p \leq q \{(u/C)\mathbb{B}\}$. Let $b : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ be such that $b \leq p \{(u/C)\mathbb{B}\}$. Then $b \leq q \{(u/C)\mathbb{B}\}$. Also $b \equiv p \{(u/C)\mathbb{B}\}$.

iv) Nonzero properties of ~~the~~ the boundary.

For any $b : \mathbb{B} \to \mathbb{K}^{[k]}_{[m]}$ featuring in a relationship of the form $b \leq p \{(u/C)\mathbb{B}\}$, $b \neq 0^{[k]}_{[m]}$ at each point in $\mathbb{B}$.

Closure and invariance properties of function and boundary spaces

~~Transformations of functions mapp and constraints~~

Let $h, k, m, r \in \overline{\mathbb{N}}$, $u : \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$ and $C : \mathbb{B} \to \mathbb{K}^{[n]}_{[m]}$ and

~~$p \in \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$.~~

~~i) Let $s \in \overline{\mathbb{N}}$ and $b \le p\{(u/C)\mathbb{B}\}^{[k]}$. $b \le pg\{(u/C)\mathbb{B}\}^{[k]}$ for all $g : \mathbb{B} \to \mathbb{K}^{[s]}_{[r]}$.~~

~~ii) Let $b \le p\{(u/C)\mathbb{B}\}^{[k]}$ and let $p' \in \mathbb{CFS}\{b(u/C)\}^{[r]}$. Then $b \le p \pm p'\{(u/C)\mathbb{B}\}^{[k]}$~~

i) Let ~~$b \le p\{(u/C)\mathbb{B}\}^{[k]}$~~ $b \in \text{preno}\{C, \mathbb{B}\}$

a) The ~~memb~~ constrained function space $\mathbb{CFS}\{b(u/C)^{\mathbb{B}}\}$ is closed with respect to multiplication by mappings of suitable dimension: ~~CFS~~ with $s \in \overline{\mathbb{N}}$,

$$\mathbb{CFS}\{b(u/C)^{\mathbb{B}}\}^{[r]}\,\mathbb{K}[\mathbb{B}]^{[s]}_{[r]} \subseteq \mathbb{CFS}\{b(u/C)^{\mathbb{B}}\}^{[s]}$$

b) The space $\mathbb{CFS}\{b(u/C)^{\mathbb{B}}\}^{[r]}$ is closed with respect to addition and subtraction: if $p, p' : \mathbb{B} \to \mathbb{K}^{[r]}_{[m]}$ are such that $b \le p, p'\{(u/C)\mathbb{B}\}^{[k]}$, then $b \le (p \pm p')\{(u/C)\mathbb{B}\}$.

ii) Let $p \in \mathbb{FS}\{(u/C)\mathbb{B}\}^{[k,r]}$ ~~and~~ $k \in [m - \text{rank}[C] - 1)$ and

a) The boundary space $\mathbb{BS}\{p(u/C)\mathbb{B}\}$ is invariant with respect to simultaneous postmultiplication of ~~the~~ u and C by a nonsingular matrix mapping: ~~with $k \in [m - \text{rank}[C] - 1)$~~

$$\mathbb{BS}\{p(u/C)\mathbb{B}\}^{[k]} = \mathbb{BS}\{p(uD/CD)\mathbb{B}\}^{[k]}$$

for all $D \in \text{ns}\mathbb{K}[\mathbb{B}, h]$.

b) Simultaneous multiplication of p and C by a nonsingular matrix mapping is equivalent to premultiplication of the boundar space by the multiplicative factor,:

$$\mathbb{BS}\{\mathbb{B}p(u/\cancel{c}\mathbb{B}C)\mathbb{B}\}^{[k]} = \mathbb{B}\,\mathbb{BS}\{p(u/C)\mathbb{B}\}^{[k]}$$

for all $\mathbb{B} \in \mathrm{ns}\mathbb{K}[\mathbb{B}, m]$.

—

The polynomial factor

Let ~~k~~, $m, n, r \in \overline{\mathbb{N}}$. ~~Dk~~ and $k \in [m)$. Set $\mathbb{RD}_{[k]} := \mathrm{def}_{[k]}\mathbb{K}^{[n]}_{[m]}$.

Define the auxiliary mappings $\mathbb{BD}^{[k]} : \mathbb{RD}_{[k]} \subseteq \mathbb{K}[\mathbb{B}]^{[k]}_{[m]}$ and $\mathbb{P} : \mathbb{K}[\mathbb{B}]^{[k]}_{[m]} \times \mathbb{K}[\mathbb{B}]^{[n]}_{[m]} \subseteq \mathbb{K}[\mathbb{B}]^{[r]}_{[m]}$ by setting

$$\mathbb{BD}^{[k]}\{C\} = \mathrm{prens}\{C\,|\mathbb{B}\}^{[k]}$$

$$\mathbb{P}\{b, C\} = [b.+C]\,\mathbb{K}^{\$}[\mathbb{B}]^{(r)}_{[k:m+1]}$$

The mapping ~~p~~

$$\mathrm{pfact} : \mathbb{P}\{\mathbb{BD}^{[k]}\{\mathbb{RD}_{[k]}\}, \mathbb{RD}_{[k]}\} \times \mathbb{BD}^{[k]}\{\mathbb{RD}_{[k]}\} \times \mathbb{RD}_{[k]} \to \mathbb{K}[\mathbb{B}]^{[r]}_{[k:}$$

is defined by setting

$$\mathrm{pfact}[(p/b)C] := x_{[k]}$$

where ~~x is~~ $x : \mathbb{B} \to \mathbb{K}^{[r]}_{[k:m+1]}$ is any member of $\mathrm{post}\left\{\frac{p}{b.+C}\,|\mathbb{B}\right\}$.

— $\mathrm{pfact}[(p/b)C]$ is called the polynomial factor determined by the function p, the constraint ~~x~~ C and the boundary b.

— Let $m, n, r \in \overline{\mathbb{N}}$, ~~and~~ $k \in [m)$, $C \in \mathrm{def}_{[k]}\mathbb{K}^{[n]}_{[m]}$, $b \in \mathrm{prens}\{C, \mathbb{B}\}^{[k:}$ and $p \in [b.+C]\,\mathbb{K}[\mathbb{B}]^{[r]}_{[k:m+1]}$.

i) The polynomial factor ~~pf~~ $\mathrm{pfact}[(p/b)C]$ is uniquely determined in the sense that $x_{[k]}$ is the same for all ~~x~~ $x \in \mathrm{post}\left\{\frac{p}{b.+C}\,|\mathbb{B}\right\}$

ii) The rank of $\mathrm{pfact}[(p/b)C]$ depends upon the dimensions of the ~~§~~ relative rank of p and C and upon the dimension of ~~[illegible]~~ the boundary b being, in particular, invariant with respect to all boundaries of the same row dimension:

$$\operatorname{rank}(\operatorname{pfact}[(p/b)C]) = \min\{\operatorname{prerank}[p.C], k+1\} \quad \langle B \rangle$$

b) $\operatorname{pfact}[(p/b)C] \in \operatorname{prens}\{B\}$ if and only if $p \in \operatorname{prens}\{C, B\}$.

(Existence and rank of the polynomial factor

Invariance properties of the polynomial factor.

Let $m, n, r \in \bar{N}$, $k \in (m)$, $C \in \operatorname{def}_{[k]} K^{[n]}_{[m]}$, $b \in \operatorname{prens}\{C, B\}^{[k]}$ and $p \in [b.+C]K[B]^{[r]}_{[k+1]}$

i) $$\operatorname{pfact}[(Bp/Bb)BC] = \operatorname{pfact}[(p/b)C] \quad \langle B \rangle$$

for all $B \in \operatorname{prens}\{B\}^{[m]}$

~~ii) pfact[(p/b)CD] = pfact~~

~~[illegible]~~

ii) $$\operatorname{pfact}[(p/b)CD] = \operatorname{pfact}[(p/b)C] \quad \langle B \rangle$$

for all $D \in \operatorname{pons}\{B\}_{[n]}$

Homogeneous constraint systems

~~$u: \mathbb{B} \to K$~~ With $k, m, n \in \bar{\mathbb{N}}$ and $u: \mathbb{B} \to K^{[n]}_{[k]}$ prescribed, a mapping $C: \mathbb{B} \to K^{[n]}_{[m]}$ for which $\mathrm{pre}\left\{\frac{u}{C} \middle| \mathbb{B}\right\}$ is nonvoid is said to be a homogeneous constraint ~~system~~ with respect to u. The notation $C \in \mathbb{H}\{u, \mathbb{B}\}_{[m]}$ indicates that C is such a ~~system~~ constraint

Rank properties ~~and structure~~ of a homogeneous constraint. ~~and~~

Let $k, m, n \in \bar{\mathbb{N}}$, and $u: \mathbb{B} \to K^{[n]}_{[k]}$. ~~and $C \in H\{u, \mathbb{B}\}_{[m]}$~~.

$C \in H\{u, \mathbb{B}\}_{[m]}$ if and only if

$$\mathrm{rank}[u +. C] = \mathrm{rank}[C] \qquad \langle \mathbb{B} \rangle$$

Structure and properties of quotient spaces

Let $k, m, n, r \in \bar{\mathbb{N}}$, $u \in \mathbb{B} \to K^{[n]}_{[k]}$ and $C \in H\{u, \mathbb{B}\}_{[m]}$

i) Let $y \in \mathrm{pre}\left\{\frac{u}{C} \middle| \mathbb{B}\right\}$.

$$\mathrm{pre}\left\{\frac{u}{C} \middle| \mathbb{B}\right\} = y + \mathrm{pre}\left\{\frac{O^{[n]}_{[k]}}{C} \middle| \mathbb{B}\right\}$$

~~iia) to~~

iia) $$\mathrm{pre}\left\{\frac{u}{BC} \middle| \mathbb{B}\right\} B \subseteq \mathrm{pre}\left\{\frac{u}{C} \middle| \mathbb{B}\right\}$$

for all ~~$B \in: \mathbb{B} \to K^{[m]}_{[r]}$~~ ~~$B \in K[\mathbb{B}, m]$~~ $B: \mathbb{B} \to K^{[m]}_{[r]}$

b) For all $B \in \mathrm{prens}\{\mathbb{B}\}^{[m]}$ the above relationship becomes one of equivalence

iiia) $$\mathrm{pre}\left\{\frac{u}{C} \middle| \mathbb{B}\right\} \subseteq \mathrm{pre}\left\{\frac{uD}{CD} \middle| \mathbb{B}\right\}$$

for all $D: \mathbb{B} \to K^{[r]}_{[n]}$.

b) For all $D \in \mathrm{pons}\{\mathbb{B}\}_{[n]}$ the above relationship becomes one of equivalence

The coefficient space

Let $h, \cancel{k}, m, n \in \overline{N}$, $k \in [n]$ and set $H[M,k] :=\mathrm{def}_{[k]} \mathbb{K}[\cancel{B}]^{[n]}_{[m]} \cancel{B}$. The coefficient space mapping

$$CS: \mathbb{K}[B]^{[k]}_{[h]} \times \mathrm{prens}\{H[M,k], M\}^{[k]} \times \mathbb{K}[M]^{\overset{m}{\cancel{[M]}}}_{[h]} H[M,k] \times H[M,k] \times M \subseteq \mathbb{K}[M]^{[m]}_{[h]}$$

is defined by setting

$$CS\{(d/b)(u/C)B\} := \mathrm{pre}\left\{\frac{d.+u}{b.+C} \middle| B\right\}$$

Existence and structure

Let $h, \cancel{k}, m, n \in \overline{N}$, $\cancel{k \in \mathrm{def}_{[k]}}$ $k \in [n]$, $C \in \mathrm{def}_{[k]} \mathbb{K}[B]^{[n]}_{[m]}$, $u \in \mathbb{K}[B]^{[m]}_{[h]} C$, $b \in \mathrm{prens}\{C, B\}^{[k]}$ and $d: B \to \mathbb{K}^{[k]}_{[h]}$.

The coefficient space

Let $h,k,m,n \in \bar{\mathbb{N}}$, $u: \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$, $C \in \mathbb{HC}\{u|\mathbb{B}\}_{[m]}$, $b \in \mathrm{prens}\{C,\mathbb{B}\}^{[k]}$ and $d: \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$. The coefficient space in $\mathbb{K}[\mathbb{B}]^{[m]}_{[h]}$ generated by d, b, u and C over $\mathbb{B}$ is defined by setting

$$\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\} := \mathrm{pre}\left\{\frac{d.+u}{b.+C}\,\Big|\,\mathbb{B}\right\}$$

Existence and structure

Let $h,k,m,n \in \bar{\mathbb{N}}$, $u: \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$, $C \in \mathbb{HC}\{u|\mathbb{B}\}_{[m]}$, ~~$b \in \mathrm{prens}\{C,\mathbb{B}\}^{[k]}$ and $d: \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$~~.

i) $\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\}$ is nonvoid

ii) Select $c \in \mathbb{CS}\{(d/b)(u/C)\mathbb{B}\}$.

$$\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\} = c + \mathrm{pre}\left\{\frac{0^{[k+n+1]}_{[h]}}{b.+C}\,\Big|\,\mathbb{B}\right\} \quad \mathrm{pre}\mathbb{CFZ}\{b.+C,\mathbb{B}\}_{[h]}$$

Invariance of coefficient spaces

Let $h, m, n \in \bar{\mathbb{N}}$, ~~$u: \mathbb{B} \to \mathbb{K}^{[n]}_{[h]}$, $C \in \mathbb{HC}\{u|\mathbb{B}\}_{[m]}$~~, $k \in [m]$, $C \in \mathrm{def}_{[k]}\mathbb{K}[\mathbb{B}]^{[n]}_{[m]}$, $u \in \mathbb{K}[\mathbb{B}]^{[m]}_{[h]}C$, $b \in \mathrm{prens}\{C|\mathbb{B}\}^{[}$ and $d \in \mathbb{B} \to \mathbb{K}^{[k]}_{[h]}$.

i) $$e\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\} = \mathbb{CS}\{(ed/b)(eu/C)\mathbb{B}\}$$

for all $e \in \mathbb{K}[\mathbb{B}, n \times \mathbb{K}[\mathbb{B}, h]$

ii) $$\mathbb{CS}\{(d/Bb)(u/BC)\mathbb{B}\}B = \mathbb{CS}\{(d/b)(u/C)\mathbb{B}\}$$

for all $B \in \mathrm{prens}\{\mathbb{B}\}^{[m]}$

iiia) $$\mathbb{CS}\{(d/b)(u/C)\mathbb{B}\} \subseteq \mathbb{CS}\{(dg/bg)(u/C)\mathbb{B}\}$$

for all $g \in \mathrm{prens}\{\mathbb{B},k\}$

b) The above relationship is one of equivalence for all $g \in \mathrm{ns}\{\mathbb{B},k\}$.

iia) $\mathbb{CS}\{(d/b)(u/C)\} \subseteq \mathbb{CS}\{(d/b)(uD/CD)B\}$

for all $D \in \mathbb{K}[B]^{[r]}_{[n]}, r \in \bar{\mathbb{N}}$.

b) The above relationship is one of equivalence for all $D \in \mathrm{prns}\{B, n$

~~Function spaces derived from related boundaries~~

v) $\mathbb{CS}\{(d/b + C\mathrm{post}\{\frac{0^{[k]}_{[h]}}{u}|B\})(u/C)B\} = \mathbb{CS}\{(d/b)(u/C)B\}$

Inclusion domains for boundary spaces and coefficient spaces deriving from associated functions

Let $h, k, m, n, r \in \bar{\mathbb{N}}$, $k \in [m], C \in \mathrm{def}_{[k]}\mathbb{K}[B]^{[n]}_{[m]}, u \in \mathbb{K}[B]^{[m]}_{[h]}C$ ~~$u: B \to \mathbb{K}^{[n]}_{[h]}, C \in \mathbb{HC}\{u|B\}_{[m]}$~~, c?
$b \in \mathrm{prens}\{C, B\}^{[k]}$, ~~and~~ $d: B \to \mathbb{K}^{[k]}_{[h]}$ and $p \in \mathbb{CFS}\{b(u/C)B\}^{[r]}$

ia) $\mathbb{CS}\{(d/b)(u/C)B\} \subseteq c + \mathrm{pre}\{\frac{0^{[r+h]}_{[h]}}{p.+C}|B\} \subseteq c + \mathrm{pre}\{\frac{0^{[r]}_{[h]}}{p}|B\}$

$\mathrm{pre}\mathbb{CFZ}\{p.+C, B\}_{[h]}$ $\quad$ $\mathrm{pre}\mathbb{CFZ}\{p, B\}_{[h]}$

b) If $r \geq k$ and $p \in \mathrm{prens}\{C|B\}$

$\mathbb{CS}\{(d/b)(u/C)B\} = c + \mathrm{pre}\{\frac{0^{[r+h]}_{[h]}}{p.+C}|B\}$

ii) Let $k = r$ and select $c \in \mathbb{CS}\{(d/b)(u/C)B\}$.

a) $b + c\,\mathrm{post}\{\frac{0^{[k]}_{[h]}}{u}|B\} \subseteq \mathbb{BS}\{p(u/C)B\} \cap \mathrm{post}\{\frac{d}{c}|B\}$

b) If $d \in \mathrm{prens}\{B\}$ and $p \in \mathrm{prens}\{C|B\}$ the above semi-inclusion relationship becomes one of equivalence.