# Module 7.2 Fundamentals of Annuities

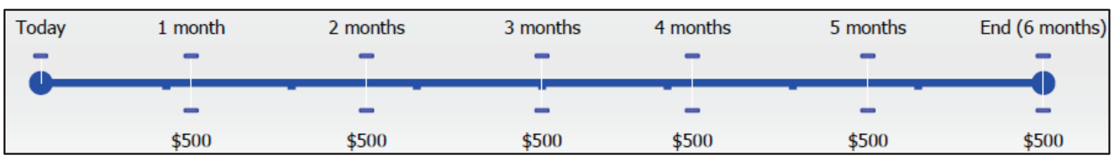
It is impossible to calculate payments accurately until you recognize a few of the key characteristics illustrated in the chapter introduction:

- All of the examples required a payment in the same amount on a regular basis, such as the $872.41 every month for your mortgage.
- The timing of the payments varied. The car lease required the first payment up front (at the beginning of the month), while the mortgage and mattress purchase had the first payment due a month after the purchase (at the end of the month).
- The frequency of the payments and the frequency of the interest rate varied. The mortgage had monthly payments with interest being charged semi-annually, while the car lease had monthly payments and monthly interest.

Unlike single payments for which there is only one formula, you solve series of payments by choosing the appropriate formula from four possibilities defined by the financial characteristics of the payments.

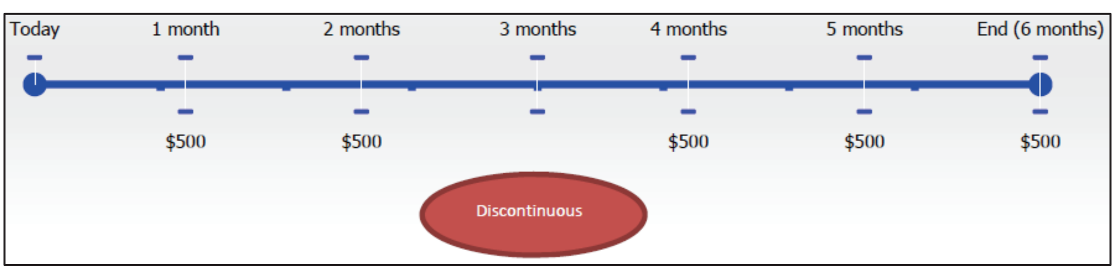
## What Are Annuities?

An **annuity** is a continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation. An **annuity payment** is the dollar amount of the equal periodic payment in an annuity environment. The figure below illustrates a six-month annuity with monthly payments. Notice that the payments are continuous, equal, periodic, and occur over a fixed time frame. If any one of these four characteristics is not satisfied, then the financial transaction fails to meet the definition of a singular annuity and requires other techniques and formulas to solve.

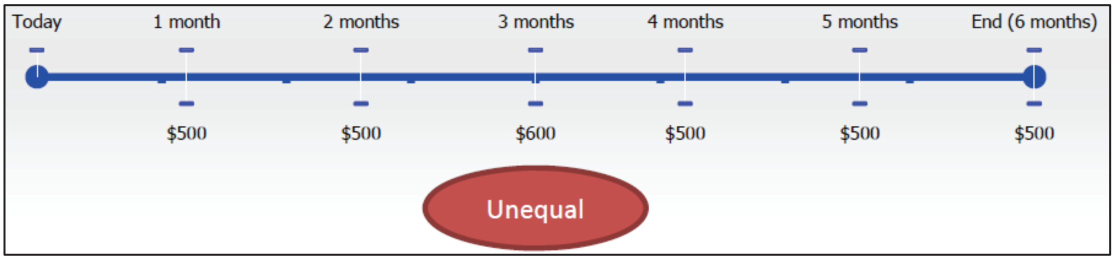



The examples below illustrate four timelines that look similar to the one above, but with one of the characteristics of an annuity violated. This means that none of the following in their entirety are considered an annuity:

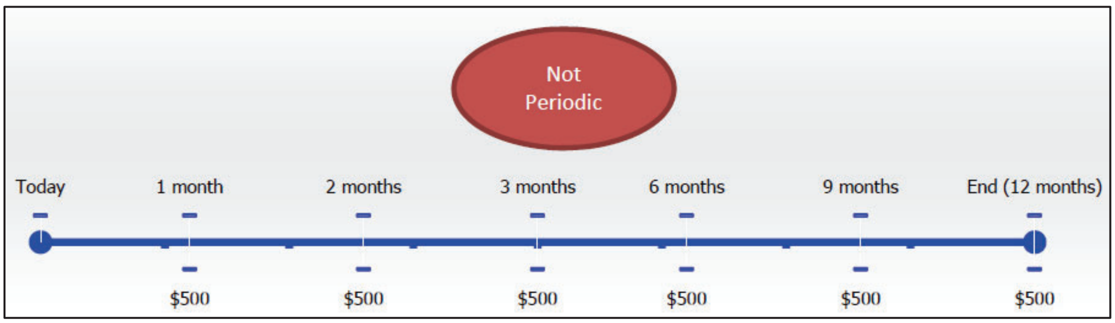
1. **Continuous.** Annuity payments are without interruption or breaks from the beginning through to the end of the annuity's term. In the figure above there are no breaks in the annuity since every month has an annuity payment. This next figure is not an annuity because the absence of a payment in the third month makes the series of payments discontinuous.



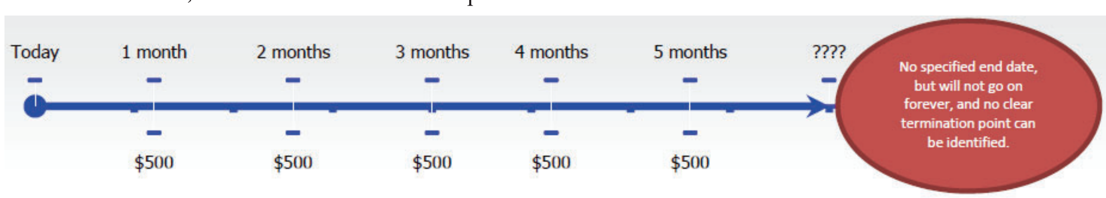

2. **Equal.** The annuity payments must be in the same amount every time from the beginning through to the end of the annuity's term. In the original figure, every monthly annuity is $500. Notice that in this next figure the payment amount varies and includes values of both $500 and $600.




3. ***Periodic.*** The amount of time between each continuous and equal annuity payment is known as the **payment interval**. Hence, a monthly payment interval means payments have one month between them, whereas a semi-annual payment interval means payments have six months between them. Annuity payments must always have the same payment interval from the beginning through to the end of the annuity's term. In the original figure there is exactly one month between each equal and repeated payment in the annuity. Notice that in this next figure the payments, although equal and continuous, do not occur with the same amount of time between each one. In fact, the first three payments are made monthly while the last three payments are made quarterly. (You may notice that if the first three payments are treated separately from the last three payments, then each separate grouping represents an annuity and therefore there are two separate annuities. However, as a whole this is not an annuity.)




4. **Specified Period of Time.** The annuity payments must occur within an identifiable time frame that has a specified beginning and a specified end. The annuity's time frame can be (1) known with a defined starting date and defined ending date, such as the annuity illustrated in the original figure, which endures for six periods; (2) known but nonterminating, such as beginning today and continuing forever into the future (hence an infinite period of time); or (3) unknown but having a clear termination point, for example, monthly pension payments that begin when you retire and end when you die—a date that is obviously not known beforehand. In the next figure, the annuity has no defined ending date, does not continue into the future, and no clear termination point is identifiable.




In summary, the first figure is an annuity that adheres to all four characteristics and can be addressed via an annuity formula. The next four figures are not annuities and need other financial techniques or formulas to perform any necessary calculations.

# Types of Annuities

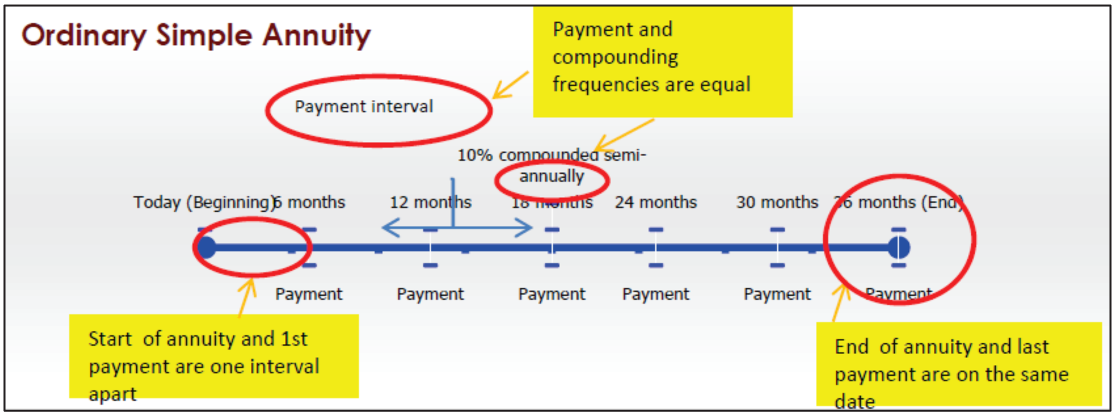
There are four types of annuities, which are based on the combination of two key characteristics: timing of payments and frequency. Let's explore these characteristics first, after which we will discuss the different annuity types.

1. **Timing of Payments.** An example best illustrates this characteristic. Assume that you take out a loan today with monthly payments. If you were to make your first annuity payment on the day you take out the loan, the amount of principal owing would be immediately reduced and you would accumulate a smaller amount of interest during your first month. This is called making your annuity payment at the *beginning* of a payment interval, and this payment is known as a **due**. However, if a month passes before you make your first monthly payment on the loan, your original principal

accumulates more interest than if the principal had already been reduced. This is called making your payment at the *end* of the payment interval, and this payment is known as an **ordinary** because it is the most common form of annuity payment. Depending on when you make your payment, different principal and interest amounts occur.

2. **Frequency.** The frequency of an annuity refers to a comparison between the payment frequency and the compounding frequency. A **payment frequency** is the number of annuity payments that would occur in a complete year. Recall from Chapter 9 that the *compounding frequency* is the number of compounds per complete year. If the payment frequency is the same as the compounding frequency, this is called a **simple annuity**. When interest is charged to the account monthly and payments are also made monthly, you determine principal and interest using simplified formulas. However, if the payment frequency and the compounding frequency are different, this is called a **general annuity**. If, for example, you make payments monthly while interest is compounded semi-annually, you have to use more complex formulas to determine principal and interest since the paying of principal and charging of interest do not occur simultaneously.

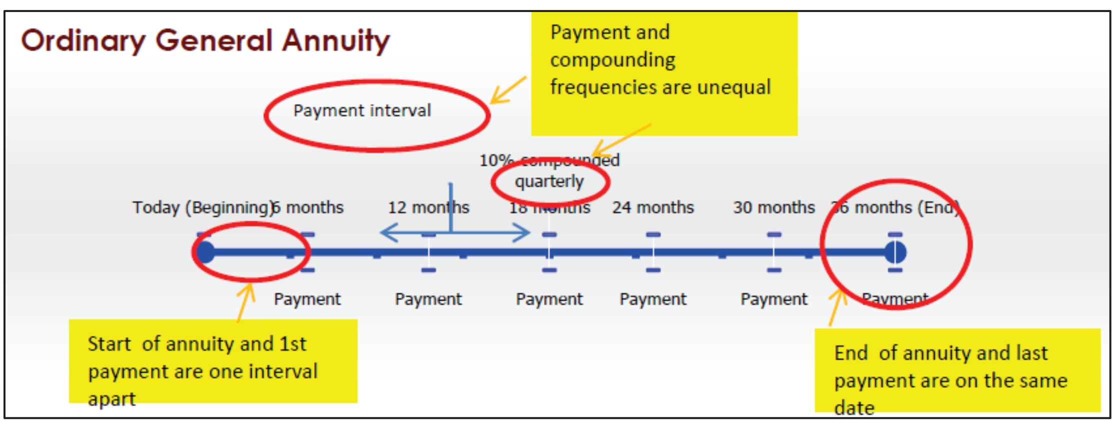
Putting these two characteristics together in their four combinations creates the four types of annuities. Each timeline in these figures assumes a transaction involving six semi-annual payments over a three-year time period.




**Ordinary Simple Annuity** An **ordinary simple annuity** has the following characteristics:

- Payments are made at the end of the payment intervals, and the payment and compounding frequencies are equal.
- The first payment occurs one interval after the beginning of the annuity.
- The last payment occurs on the same date as the end of the annuity.

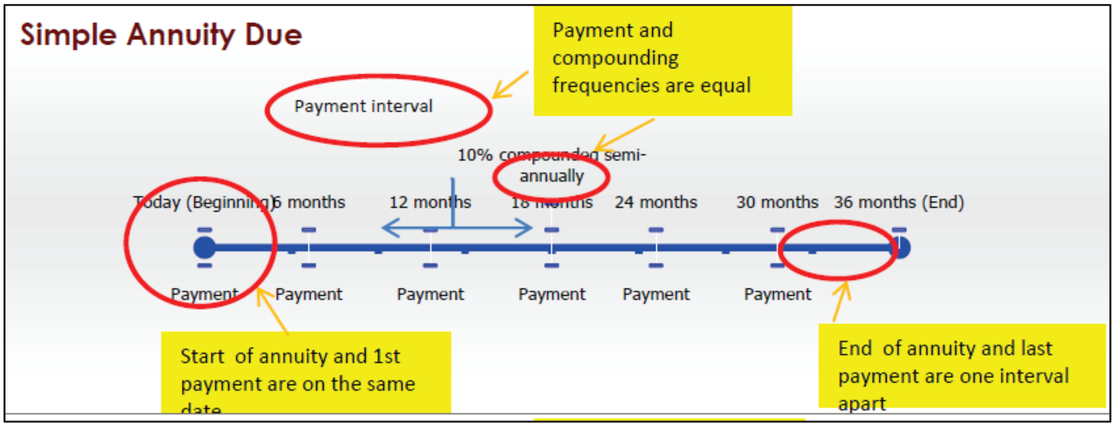
For example, most car loans are ordinary simple annuities where payments are made monthly and interest rates are compounded monthly. As well, car loans do not require the first monthly payment until the end of the first month.




**Ordinary General Annuity** An **ordinary general annuity** has the following characteristics:

- Payments are made at the end of the payment intervals, and the payment and compounding frequencies are unequal.
- The first payment occurs one interval after the beginning of the annuity.
- The last payment occurs on the same date as the end of the annuity.

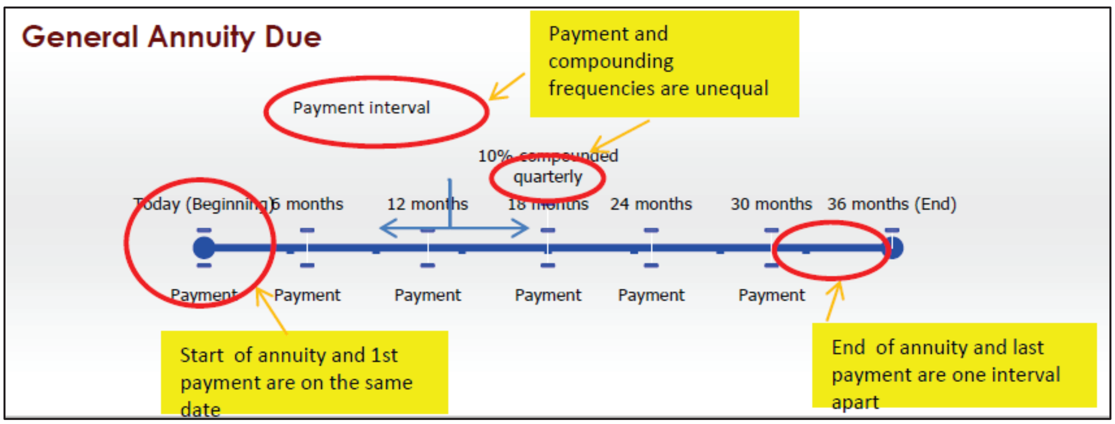
For example, most mortgages are ordinary general annuities, where payments are made monthly and interest rates are compounded semi-annually. As with car loans, your first monthly payment is not required until one month elapses.




**Simple Annuity Due** A **simple annuity due** has the following characteristics:

- Payments are made at the beginning of the payment intervals, and the payment and compounding frequencies are equal.
- The first payment occurs on the same date as the beginning of the annuity.
- The last payment occurs one payment interval before the end of the annuity.

For example, most car leases are simple annuities due, where payments are made monthly and interest rates are compounded monthly. However, the day you sign the lease is when you must make your first monthly payment.




***General Annuity Due***. A **general annuity due** has the following characteristics:

- Payments are made at the beginning of the payment intervals, and the payment and compounding frequencies are unequal.
- The first payment occurs on the same date as the beginning of the annuity.
- The last payment occurs one payment interval before the end of the annuity.

For example, many investments, like your RRSP, are general annuities due where payments (contributions) are typically made monthly but the interest compounds in another manner, such as annually. As well, when most people start an RRSP they pay into it on the day they set it up, meaning that their RRSP commences with the first deposited payment.

The table below summarizes the four types of annuities and their characteristics for easy reference.

| Annuity Type | Timing of Payments in a Payment Interval | Payment Frequency and Compounding Frequency | Start of Annuity and First Payment Same Date? | End of Annuity and Last Payment Same Date? |
|---|---|---|---|---|
| Ordinary Simple Annuity | End | Equal | No, first payment one interval later | Yes |
| Ordinary General Annuity | End | Unequal | No, first payment one interval later | Yes |
| Simple Annuity Due | Beginning | Equal | Yes | No, last payment one interval earlier |
| General Annuity Due | Beginning | Unequal | Yes | No, last payment one interval earlier |

## Paths To Success

One of the most challenging aspects of annuities is recognizing whether the annuity you are working with is ordinary or due. This distinction plays a critical role in formula selection later in this chapter. To help you recognize the difference, the table below summarizes some key words along with common applications in which the annuity may appear.

| Type | Key Words or Phrases | Common Applications |
|---|---|---|
| Ordinary | -…payments are at the end…<br>-…payments do not start today…<br>-…payments are later…<br>-…first payment next interval… | - bank loans of any type<br>- mortgages<br>- bonds<br>- Canada Pension Plan (CPP) |
| Due | -…payments are at the beginning…<br>-…payments start today…<br>-…payments are in advance…<br>-…first payment today…<br>-…payments start now… | - any kind of lease<br>- any kind of rental<br>- RRSPs (usually)<br>- membership dues<br>- insurance |

## The Formula

**N is Total Number of Annuity Payments:** The total number of payments from the start of the annuity to the end of the annuity, inclusive.

**Years is Term of the Annuity:** The length of the annuity from start to end, expressed as the number of years.

**Formula 4.2.1 – Number of Annuity Payments:** $$N = PY \times Years$$

**PY is Payments per Year or Payment Frequency:** The number of annuity payments per complete year.

## How It Works

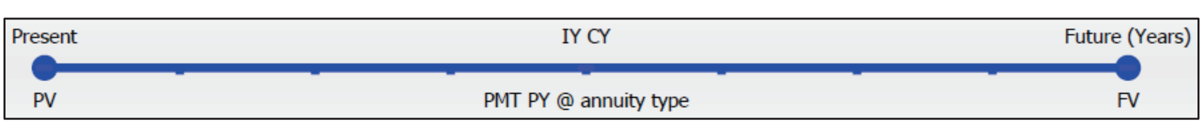
On a two-year loan with monthly payments and semi-annual compounding, the payment frequency is monthly, or 12 times per year. With a term of two years, that makes $N = 2 \times 12 = 24$ payments. Note that the calculation of N for an annuity does not involve the compounding frequency.

# Adapting Timelines to Incorporate Annuities

Annuity questions can involve many payments. For example, in a typical 25-year mortgage with monthly payments, that would be $25 \times 12 = 300$ payments in total. How would you draw a timeline for these? Clearly, it would be impractical to draw 300 payments.

A good annuity timeline should illustrate the present value (PV), future value (FV), number of annuity payments (*N*), nominal interest rate (IY), compounding frequency (CY), annuity payment (PMT), and the payment frequency (PY). One of these variables will be the unknown.

As well, a good timeline requires a clear distinction between ordinary annuities and annuities due. END is used to represent ordinary annuities, since payments occur at the *end* of the payment interval. Similarly, BGN is used to represent annuities due, since payments occur at the *beginning* of the payment interval. The figure below illustrates the adapted annuity timeline format.



## The Formula

First, you need to know how many times interest is converted to principal throughout the transaction. You can then calculate the future value. Use Formula 4.2.2 to determine the number of compound periods involved in the transaction

**N is Number of Compound Periods:** This is the measure of time. Calculating the maturity value requires knowing how many times interest is converted to principal throughout the transaction. Whereas simple interest expresses time in years, note that compound interest requires time to be expressed in periods.

**Formula 4.2.2 – Number of Compound Periods For Single Payments:** $$N = CY \times Years$$

**CY is Compounds per year (Compound Frequency):** Recall that the compounding frequency represents how many compound periods fall within a single year. The words that accompany the nominal interest rate determine this number.

**Years is Number of Years in Transaction (The Term):** Multiply the compounding frequency by the term of the transaction, expressed as a number of years. Express partial years as mixed fractions. For example, 1 year and 9 months is $1\frac{9}{12}$ years.

Once you know N, substitute it into Formula 4.2.3, which finds the amount of principal and interest together at the end of the transaction, or the maturity value.

**FV is Future Value or Maturity Value:** This is the principal and compound interest together at the future point in time.

**PV is Present Value or Principal:** This is the starting amount upon which compound interest is calculated. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some earlier point in time. In any case, the amount excludes the future interest.

**Formula 4.2.3 – Compound Interest For Single Payments:** $$FV = PV(1 + i)^N$$

**i is Periodic Interest Rate:** From Formula 4.3.1, the principal accrues interest at this rate every compounding period. For accuracy, you should never round this number.

**N is Number of Compound Periods:** From Formula 4.2.2, this is the total number of compound periods involved in the term of the transaction.

The $(1+i)^N$ performs the actual compounding of the money. The $(1 + i)$ determines the percent increase in the principal while the exponent (N) compounds the increase an appropriate number of times.

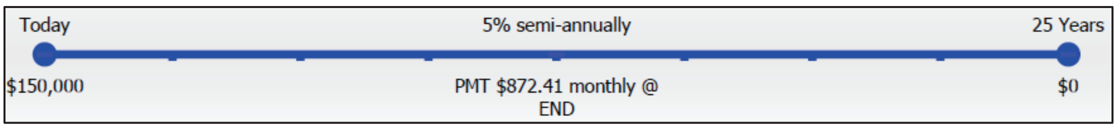
Sometimes a variable will change partway through the period of an annuity, in which case the timeline must be broken up into two or more segments. When you use this structure, in any time segment the annuity payment PMT is interpreted to have the same amount at the same payment interval continuously throughout the entire segment. The number of annuity payments N does not directly appear on the timeline since it is the result of a formula. However, its two components (Years and PY) are drawn on the timeline.

### How It Works

A mortgage is used to illustrate this new format. For now, focus strictly on the variables and how to illustrate them in a timeline. Do not focus on any mortgage calculations yet.

- As per the chapter introduction, you purchased a $150,000 home (PV) with a 25-year mortgage (Years). The mortgage rate is 5% (IY) compounded semi-annually (CY = 2), and the monthly payments (PY = 12) are $872.41 (PMT).
- After 25 years you will own your house and therefore have no balance remaining. This sets the future value to $0 (FV).
- Mortgages are ordinary in nature, meaning that payments occur at the end of the payment intervals (END).

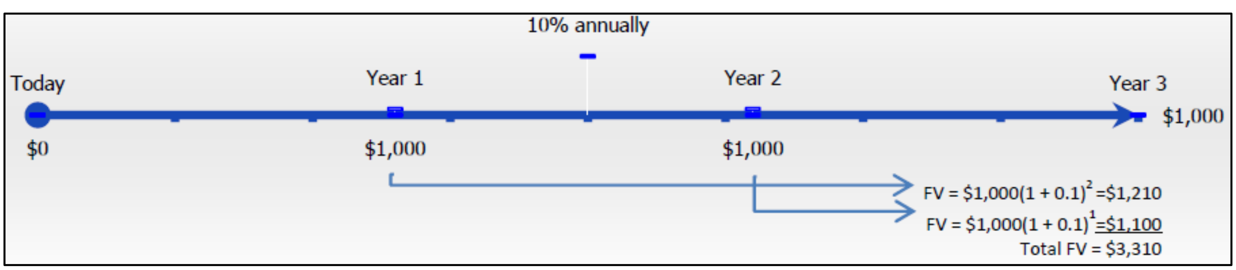
The figure below places all of these mortgage elements into the new timeline format. Once you have drawn the timeline, you can determine the following:




- The number of payments (*N*), which you calculate using Formula 4.2.1. Since both PY and Years are known, you have $N = 25 \times 12 = 300$.
- Whether the annuity is simple or general, which depends on the PY below the timeline and CY above the timeline. If they match, the annuity is simple; if they differ, as in this example, it is general.

## Ordinary Annuities

The **future value of any annuity** equals the sum of all the future values for all of the annuity payments when they are moved to the end of the last payment interval. For example, assume you will make $1,000 contributions at the end of every year for the next three years to an investment earning 10% compounded annually. This is an ordinary simple annuity since payments are at the end of the intervals, and the compounding and payment frequencies are the same. If you wanted to know how much money you have in your investment after the three years, the figure below illustrates how you would apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values to arrive at the future value.




While you could use this technique to solve all annuity situations, the computations become increasingly cumbersome as the number of payments increases. In the above example, what if the person instead made quarterly contributions of $250? That is 12 payments over three years, resulting in 11 separate future value calculations. Or if they made monthly payments, the 36 payments over three years would result in 35 separate future value calculations! Clearly, solving this would be tedious and time consuming—not to mention prone to error. There must be a better way!

## The Formula

The formula for the future value of an ordinary annuity is indeed easier and faster than performing a series of future value calculations for each of the payments. At first glance, though, the formula is pretty complex, so the various parts of the formula are first explored in some detail before we put them all together.

Portion = Base × Rate
$\$1{,}210 = \$1{,}000 \times (1 + 0.1)^2$

1. **Numerator: The Percent Change Overall: $\left[(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}}\right]^{N} - 1$.** This part of the formula determines the percent change from the start of the annuity to the end of the annuity. It has three critical elements:
   a. **Interest Rate Conversion $(1+\mathrm{i})^{\frac{\mathrm{CY}}{\mathrm{PY}}}$.** The rate of interest that occurs with each payment must be known. All annuity calculations require the compounding period to equal the payment interval. If this is not already the case, then you must convert the expressed interest rate into an equivalent interest rate.
      - For simple annuities, no conversion is necessary since the frequencies are the same: CY = PY. The exponent of $\frac{\mathrm{CY}}{\mathrm{PY}}$ always equals 1 and has no effect.
      - For general annuities, recall Formula 9.4 for calculating equivalent interest rates. Here the old compounding frequency forms the numerator ($\mathrm{CY}_{\mathrm{Old}}$) and the new compounding frequency (which matches the payment frequency) forms the denominator ($\mathrm{CY}_{\mathrm{New}}$) = PY. Thus $\frac{\mathrm{CY}_{\mathrm{Old}}}{\mathrm{CY}_{\mathrm{New}}}$ becomes $\frac{\mathrm{CY}}{\mathrm{PY}}$.

   b. **The Compounding N.** The exponent *N* compounds the periodic interest rate (which matches the payment interval) in accordance with the number of annuity payments made. For example, assume there are two end-of-year payments at 10% compounded semi-annually. If the interest rate is left semi-annually, there are four compounds over the two years, which does not match the payments. The interest rate is converted within the brackets from 10% compounded semi-annually to its equivalent 10.25% compounded annually rate. The end result is that interest will now compound twice over the two years, matching the number of payments.
   c. **Removing the Starting Point (−1).** Since you added 1 to perform the compounding, mathematically you now need to remove the 1. The end result is that you now know (in decimal format) how much larger the future value is relative to its starting value.

2. **Denominator: The Percent Change with Each Payment: $(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}} - 1$.** The denominator in the formula shows the percent change in the account with each payment made. It too ensures that the denominator has a periodic rate matching the payment interval.

3. **The Quotient** By taking the numerator and dividing by the denominator, the percent change overall is divided by the percent change with each payment. This establishes a ratio between what is happening overall in the annuity versus what is happening with each transaction. Thus, this computation determines how much bigger the final value is relative to what happens with each payment. This ratio (the rate) is then multiplied against the annuity payment (the base) to get the final balance (the portion).

You are now in a position to see the whole formula by which you calculate the future value of any ordinary annuity. This single formula applies regardless of the number of payments that are made. Formula 4.2.2 summarizes the variables and calculations required.

**$FV_{ORD}$ is Future Value of an Ordinary Annuity:** This is the amount of money in the account including all of the payments and all interest at some future point in time. If the future point is at the end of the annuity, the future value is the maturity value.

**CY is Compounds Per Year or Compounding Frequency:** The number of compounding periods per complete year.

**N is Number of Annuity Payments:** The total number of payments made during the time segment of the annuity. It results from Formula 4.2.1.

**Formula 4.2.4 – Ordinary Annuity Future Value:**

$$FV_{ORD} = PMT\left[\frac{\left[(1+i)^{\frac{CY}{PY}}\right]^{N} - 1}{(1+i)^{\frac{CY}{PY}} - 1}\right]$$

**PMT is Annuity Payment Amount:** The amount of money that is invested or paid after each payment interval.

**i is Periodic Interest Rate:** This is the rate of interest that is used in converting the interest to principal.

**PY is Payments Per Year or Payment Frequency:** The number of payment intervals per complete year.

## How To

There is a five-step process for calculating the future value of any ordinary annuity:

**Step 1**: Identify the annuity type. Draw a timeline to visualize the question.

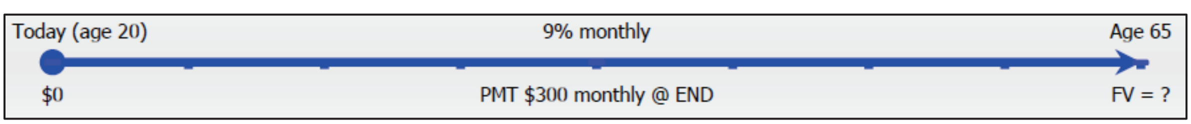
**Step 2:** Identify the known variables, including PV, IY, CY, PMT, PY, and Years.

**Step 3**: Use Formula 4.1.4 to calculate $i$.

**Step 4:** If PV = \$0, proceed to step 5. If there is a nonzero value for PV, treat it like a single payment.

**Step 5**: Use Formula 11.1 to calculate $N$ for the annuity. Apply Formula 4.2.2 to calculate the future value. If you calculated a future value in step 4, combine the future values from steps 4 and 5 to arrive at the total future value.

Revisiting the RRSP scenario from the beginning of this section, assume you are 20 years old and invest \$300 at the end of every month for the next 45 years. You have not started an RRSP previously and have no opening balance. A fixed interest rate of 9% compounded monthly on the RRSP is possible.
Step 1: This is a simple ordinary annuity since the frequencies match and payments are at the end of the payment interval.




Step 2: The known variables are PV = \$0, IY = 9%, CY = 12, PMT = \$300, PY = 12, and Years = 45.
Step 3: The periodic interest rate is $i$ = 9% ÷ 12 = 0.75%.
Step 4: Since PV = \$0, skip this step.
Step 5: The number of payments is $N$ = 12 × 45 = 540. Applying Formula 4.2.4 gives the following:

$$\mathrm{FV_{ORD}} = \$300\left[\frac{\left[(1+0.0075)^{\frac{12}{12}}\right]^{540} - 1}{(1+0.0075)^{\frac{12}{12}} - 1}\right] = \$2{,}221{,}463.54$$

Hence, 540 payments of \$300 at 9% compounded monthly results in a total saving of \$2,221,463.54 by the age of retirement.

## Important Notes

**Calculating the Interest Amount.** For investment annuities, if you are interested in knowing how much of the future value is principal and how much is interest, you can adapt Formula 4.1.3, where $I$ = S – P = FV – PV.

- The FV is the solution to Formula 4.2.4.
- The PV is the principal (total contributions) calculated by taking $N$ × PMT + PV. In the figure above, $N$ × PMT + PV = 540 payments × \$300 + \$0 = \$162,000 in principal. Therefore, $I$ =\$2,221,463.54 − \$162,000 = \$2,059,463.54, which is the interest earned.

1. You now have a value for PMT. Be sure to enter it with the correct cash flow sign convention. When you invest, the payment has the same sign as the PV. When you borrow, the sign of the payment is opposite that of PV.
2. The P/Y is no longer automatically set to the same value as C/Y. Enter your values for P/Y and C/Y separately. If the values are the same, as in the case of simple annuities, then taking advantage of the "Copy" feature on your calculator let's you avoid having to key the value in twice.
3. If a present value (PV) is involved, by formula you need to do two calculations. If you input values for both PV and PMT, the calculator does these calculations simultaneously, requiring only one sequence to solve.

## Things To Watch Out For

In many annuity situations there might appear to be more than one unknown variable. Usually the extra unknown variables are "unstated" variables that can reasonably be assumed. For example, in the RRSP illustration above, the statement "you have not started an RRSP previously and have no opening balance" could be omitted. If something were saved already, the number would need to be stated. As another example, it is normal to finish a loan with a zero balance. Therefore, in a loan situation you can safely assume that the future value is zero unless otherwise stated.

## Paths To Success

The ability to recognize a simple annuity can allow you to simplify Formula 4.2.2. Since CY = PY, these two variables form a quotient of 1 for the exponent. For a simple annuity, you can simplify any appearances of the following algebraic expressions in any annuity formula (not just Formula 11.2) as follows:

$$(1+\ i)^{\frac{\mathrm{CY}}{\mathrm{PY}}} \text{ in the numerator can be simplified to } (1+i)$$

and

$$(1+i)^{\frac{\mathrm{CY}}{\mathrm{PY}}} - 1 \text{ in the denominator can be simplified to just } i.$$

Thus, for simple annuities only, you simplify Formula 11.2 as

$$\mathrm{FV_{ORD}} = \mathrm{PMT}\left[\frac{(1+i)^N - 1}{i}\right]$$

## Example 4.2A: Future Value of an Investment Account

A financial adviser is reviewing one of her client's accounts. The client has been investing $1,000 at the end of every quarter for the past 11 years in a fund that has averaged 7.3% compounded quarterly. How much money does the client have today in his account?

**Plan**

Step 1: The payments are at the end of the payment intervals, and both the compounding period and the payment intervals are the same. This is an ordinary simple annuity. Calculate its value at the end, which is its future value, or $FV_{ORD}$.

**Understand**

What You Already Know

Step 1 (continued): The timeline shows the client's account.

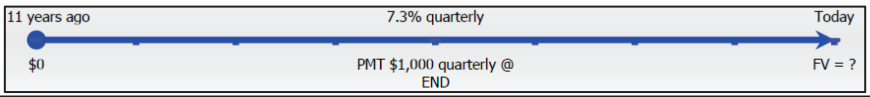
Step 2: PV = $0, IY = 7.3%, CY = 4, PMT = $1,000, PY = 4, Years = 11

How You Will Get There

Step 3: Apply Formula 4.1.4.

Step 4: Skip this step since PV = $0.

Step 5: Apply Formula 4.2.1 and Formula 4.2.4.




**Perform**

Step 3: $i = 7.3\% \div 4 = 1.825\%$

Step 5: $N = 4 \times 11 = 44$ payments

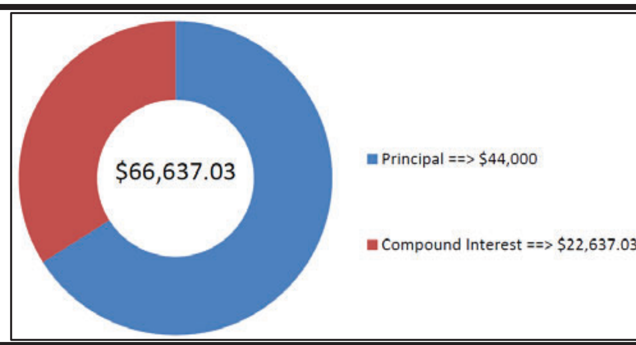
$$FV_{ORD} = \$1000\left[\frac{\left[(1+0.01825)^{\frac{4}{4}}\right]^{44} - 1}{(1+0.01825)^{\frac{4}{4}} - 1}\right] = \$66{,}637.03$$

Calculator Instructions

| N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|
| 44 | 7.3 | 0 | −1000 | Answer: 66,637.03449 | 4 | 4 |

**Present**

The figure shows how much principal and interest make up the final balance. After 11 years of $1,000 quarterly contributions, the client has $66,637.03 in the account.




## Example 4.2.2B: Future Value of a Savings Annuity

A savings annuity already contains $10,000. If an additional $250 is invested at the end of every month at 9% compounded semi-annually for a term of 20 years, what will be the maturity value of the investment?

**Plan**

Step 1: The payments are at the end of the payment intervals, and the compounding period and payment intervals are different. This is an ordinary general annuity. Calculate its value at the end, which is its future value, or $FV_{ORD}$.

**Understand**

What You Already Know

Step 1 (continued): The timeline appears below.

Step 2: PV = $10,000, IY = 9%, CY = 2, PMT = $250, PY = 12, Years = 20

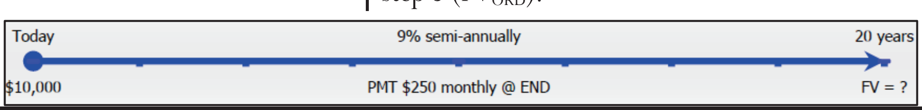
How You Will Get There

Step 3: Apply Formula 4.1.4.

Step 4: Apply Formula 4.2.2 and Formula 4.2.3.

Step 5: Apply Formula 4.2.1 and Formula 4.2.4.

The final future value is the sum of the answers to step 4 (FV) and step 5 ($FV_{ORD}$).



**Perform**

Step 3: $i = 9\% \div 2 = 4.5\%$

Step 4: $N = 2 \times 20 = 40$ compounds

$FV = \$10{,}000(1 + 0.045)^{40} = \$58{,}163.64538$

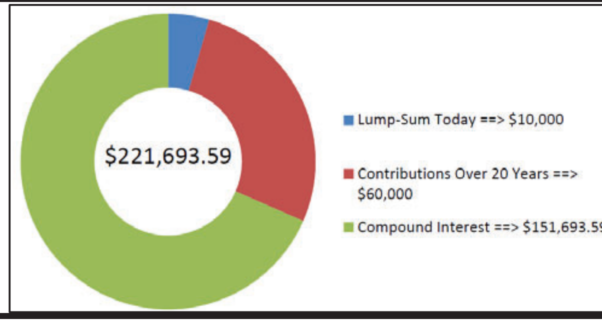
Step 5: $N = 12 \times 20 = 240$ payments

$$FV_{ORD} = \$250\left[\frac{\left[(1+0.045)^{\frac{2}{12}}\right]^{240} - 1}{(1+0.045)^{\frac{2}{12}} - 1}\right] = \$163{,}529.9492$$

Total FV = \$58,163.64538 + \$163,529.9492 = \$221,693.59

**Present**

The figure shows how much principal and interest make up the final balance. The savings annuity will have a balance of \$221,693.59 after the 20 years.




## Important Notes

If any of the variables, including IY, CY, PMT, or PY, change between the start and end point of the annuity, or if any additional single payment deposit or withdrawal is made, a new time segment is created and must be treated separately. There will then be multiple time segments that require you to work left to right by repeating steps 3 through 5 in the procedure. The future value at the end of one time segment becomes the present value in the next time segment. Example 4.2.2C illustrates this concept.

## Paths To Success

When working with multiple time segments, it is important that you always start your computations on the side opposite the unknown variable. For future value calculations, this means you start on the left-hand side of your timeline; for present value calculations, start on the right-hand side.

## Example 4.2.2C: Saving Up for a Vacation

Genevieve has decided to start saving up for a vacation in two years, when she graduates from university. She already has \$1,000 saved today. For the first year, she plans on making end-of-month contributions of \$300 and then switching to end-of-quarter contributions of \$1,000 in the second year. If the account can earn 5% compounded semi-annually in the first year and 6% compounded quarterly in the second year, how much money will she have saved when she graduates?

**Plan**

Step 1: There is a change of variables after one year. As a result, you need a Year 1 time segment and a Year 2 time segment. In both segments, payments are at the end of the period. In Year 1, the compounding period and payment intervals are different. In Year 2, the compounding period and payment intervals are the same. This is an ordinary general annuity followed by an ordinary simple annuity. You aim to calculate the future value, or $FV_{ORD}$.

**Understand**

What You Already Know

Step 1 (continued): The timeline for her vacation savings appears below.

Step 2: Time segment 1: PV = \$1,000, IY = 5%, CY = 2, PMT = \$300, PY = 12, Years = 1

Time segment 2: PV = $FV_1$ of time segment 1, IY = 6%, CY = 4, PMT = \$1,000, PY = 4, Years = 1

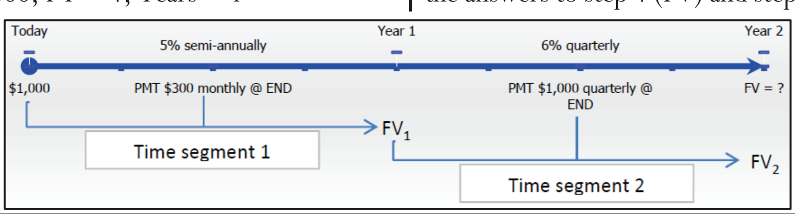
How You Will Get There

For *each* time segment repeat the following steps: Step 3: Apply Formula 4.1.4.

Step 4: Apply Formula 4.2.2 and Formula 4.2.3.

Step 5: Apply Formula 4.2.1 and Formula 4.2.4.

The total future value in any time segment is the sum of the answers to step 4 (FV) and step 5 ($FV_{ORD}$).



**Perform**

For the first time segment:

Step 3: $i = 5\% \div 2 = 2.5\%$

Step 4: $N = 2 \times 1 = 2$ compounds, $FV_1 = \$1{,}000(1 + 0.025)^2 = \$1{,}050.625$

Step 5: $N = 12 \times 1 = 12$ payments

$$FV_{ORD1} = \$300\left[\frac{\left[(1+0.025)^{\frac{2}{12}}\right]^{12} - 1}{(1+0.025)^{\frac{2}{12}} - 1}\right] = \$3{,}682.786451$$

Total $FV_1 = \$1{,}050.625 + \$3{,}682.786451 = \$4{,}733.411451$

For the second time segment:

Step 3: $i = 6\% \div 4 = 1.5\%$

Step 4: $N = 4 \times 1 = 4$ compounds, $FV_2 = \$4{,}733.411451(1 + 0.015)^4 = \$5{,}023.870384$

Step 5: $N = 4 \times 1 = 4$ payments

$$FV_{ORD2} = \$1{,}000\left[\frac{\left[(1+0.015)^{\frac{4}{4}}\right]^{4} - 1}{(1+0.015)^{\frac{4}{4}} - 1}\right] = \$4{,}090.903375$$

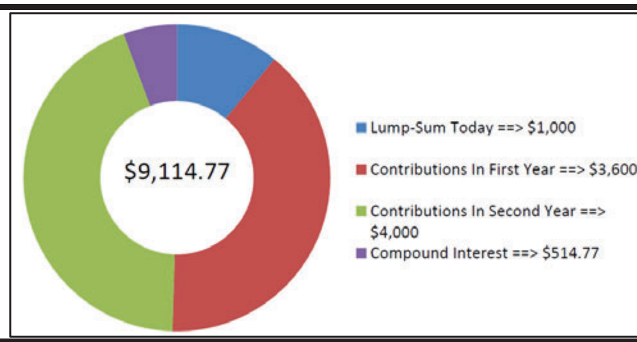
Total $FV_2 = \$5{,}023.870384 + \$4{,}090.903375 = \$9{,}114.77$

Calculator Instructions

| Time Segment | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|
| 1 | 12 | 5 | −1000 | −300 | Answer: 4,733.411451 | 12 | 2 |
| 2 | 4 | 6 | −4,733.411451 | −1000 | Answer: 9,114.773759 | 4 | 4 |

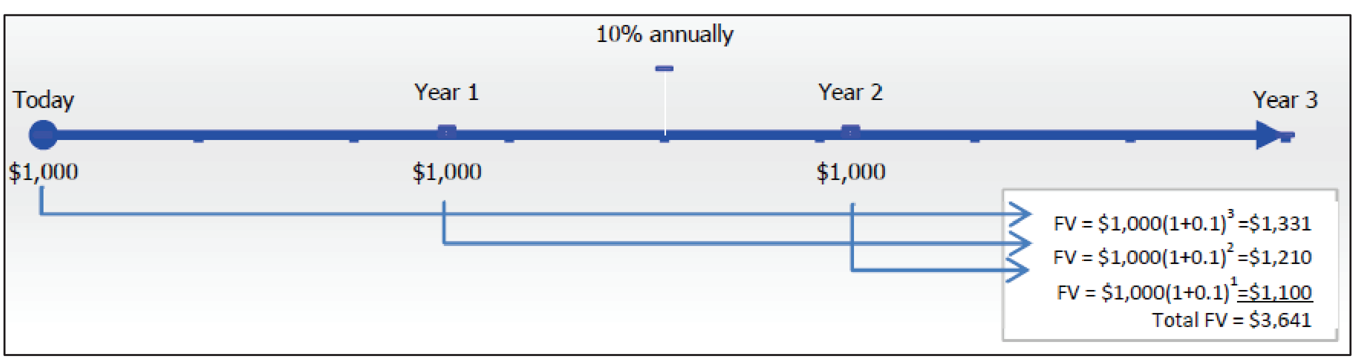
**Present**

The figure shows how much principal and interest make up the final balance. When Genevieve graduates she will have saved $9,114.77 toward her vacation.

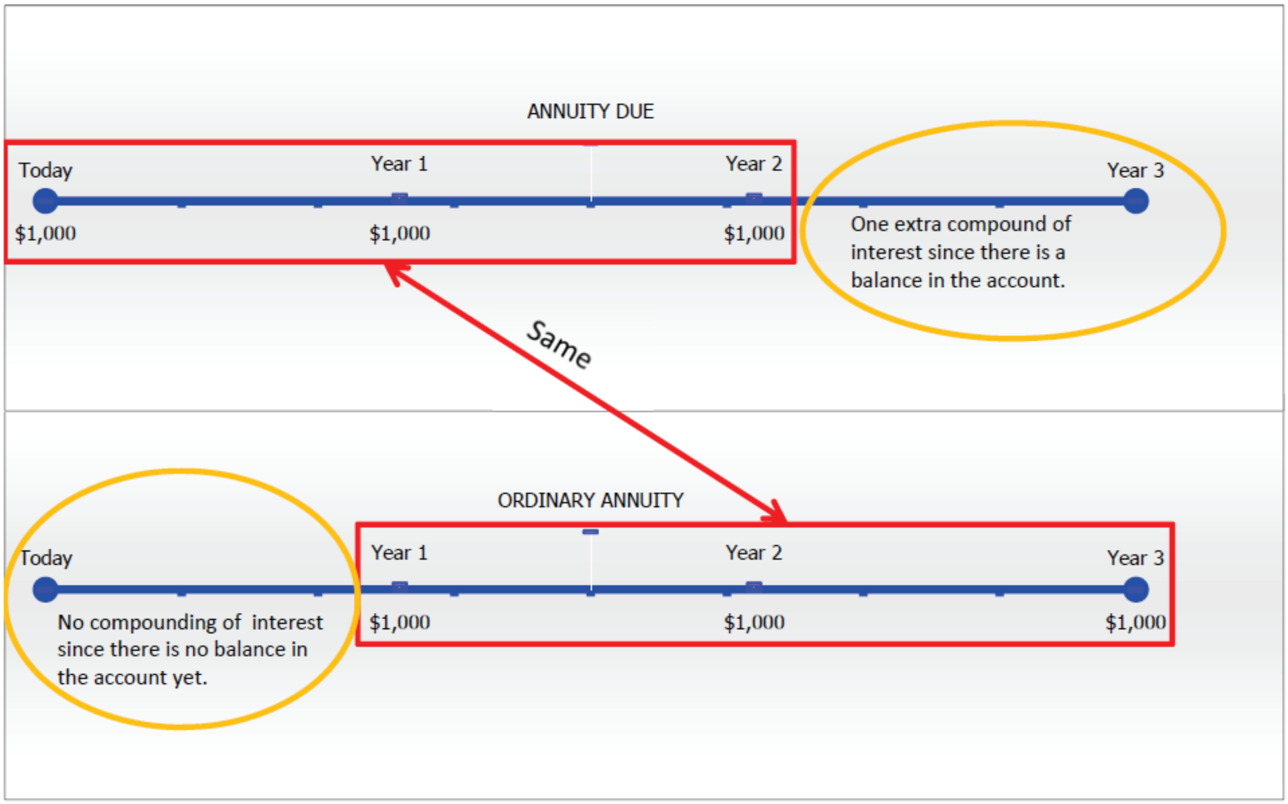



# Annuities Due

An annuity due occurs when payments are made at the beginning of the payment interval. To understand the difference this makes to the future value, let's recalculate the RRSP example from earlier in this section, but treat it as an annuity due. You want to know the future value of making $1,000 annual contributions at the beginning of every payment interval for the next three years to an investment earning 10% compounded annually. This forms a simple annuity due. The figure below illustrates how you apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values to arrive at the future value.



Note that you do not end up with the same balance of $3,310 achieved under the ordinary annuity. Instead, you have a larger balance of $3,641. Why is this? Placing the two types of annuities next to each other in the next figure demonstrates the key difference between the two examples.




Both annuities have an identical sequence of three $1,000 payments. However, in the ordinary annuity no interest is earned during the first payment interval since the principal is zero and the payment does not occur until the end of the interval. On the other hand, in the annuity due an extra compound of interest is earned in the last payment interval because of the existing principal at the end of the second year. If you take the ordinary annuity's final balance of $3,310 and give it one extra compound you have FV = $3,310(1 + 0.1) = $3,641. In summary, the key difference between the two types of annuities is that an annuity due always receives one extra compound of interest.

## The Formula

Adapting the ordinary annuity future value formula to suit the extra compound creates Formula 4.2.3. Note that all the variables in the formula remain the same; however, the subscript on the FV symbol is changed to recognize the difference in the calculation required.

**$FV_{DUE}$ is Future Value of an Annuity Due:** The payments and interest together at a future point in time.

**CY is Compounds Per Year or Compounding Frequency:** The number of compounding periods per complete year.

**N is Number of Annuity Payments:** The total number of payments made during the time segment of the annuity. It results from Formula 4.2.1.

**Formula 4.2.5**
**Annuity Due Future Value:**

$$\mathbf{FV_{DUE}} = \mathbf{PMT}\left[\frac{\left[(1+i)^{\frac{CY}{PY}}\right]^{N}-1}{(1+i)^{\frac{CY}{PY}}-1}\right]\times(1+i)^{\frac{CY}{PY}}$$

**PMT is Annuity Payment Amount:** The amount of money that is invested or paid at the beginning of each payment interval.

**PY is Payments Per Year or Payment Frequency:** The number of payment intervals per complete year.

**i is Periodic Interest Rate:** The rate of interest used in converting the interest to principal.

## How It Works

The steps required to solve for the future value of an annuity due are almost identical to those you use for the ordinary annuity. The only difference lies in step 5, where you use Formula 4.2.5 instead of Formula 4.2.4. Examples 4.2D and 4.2E illustrate the adaptation.

### Example 4.2D: Lottery Winnings

The Set for Life instant scratch n' win ticket offers players a chance to win \$1,000 per week for the next 25 years starting immediately upon validation. If a winner was to invest all of his money into an account earning 5% compounded annually, how much money would he have at the end of his 25-year term? Assume each year has exactly 52 weeks.

**Plan**

Step 1: The payments start immediately, and the compounding period and payment intervals are different. Therefore, this a general annuity due. Calculate its value at the end, which is its future value, or $FV_{DUE}$.

**Understand**

What You Already Know

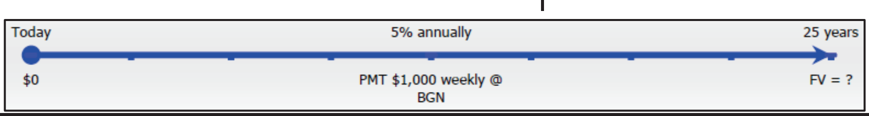
Step 1 (continued): The timeline for the lottery savings is below.

Step 2: PV = \$0, IY = 5%, CY = 1, PMT = \$1,000, PY = 52, Years = 25

How You Will Get There

Step 3: Apply Formula 4.1.4.

Step 4: Skip this step since there is no PV. Step 5: Apply Formula 4.2.1 and Formula 4.2.5.



**Perform**

Step 3: $i = 5\% \div 1 = 5\%$

Step 5: $N = 52 \times 25 = 1{,}300$ payments

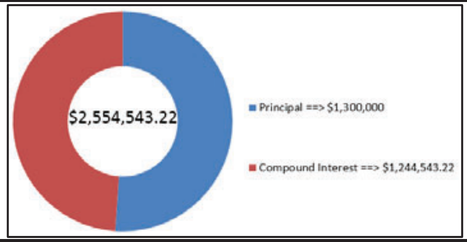
$$FV_{DUE} = \$1{,}000\left[\frac{\left[(1+0.05)^{\frac{1}{52}}\right]^{1300}-1}{(1+0.05)^{\frac{1}{52}}-1}\times(1+0.05)^{\frac{1}{52}}\right]$$
$$= \$2{,}544{,}543.22$$

Calculator Instructions

| Mode | N | I/Y | PV | PMT | FV | P/Y | C/Y |
|---|---|---|---|---|---|---|---|
| BGN | 1300 | 5 | 0 | −1000 | Answer: 2,544,543.218 | 52 | 1 |

**Present**

The figure shows how much principal and interest make up the final balance. If the winner was to invest all of his lottery prize money, he would have \$2,544,543.22 after 25 years.




## Example 4.2E: Saving into a Trust Fund with a Variable Change

When Roberto's son was born, Roberto started making payments of \$1,000 at the beginning of every six months to a trust fund earning 5.75% compounded monthly. After five years, he changed his contributions and started depositing \$500 at the beginning of every quarter. How much money will be in his son's trust fund when his son turns 18?

**Plan**

Step 1: There is a change of variables after five years. As a result, you need two time segments. In both segments, payments are at the beginning of the period and the compounding periods and payment intervals are different. Therefore, Roberto has two consecutive general annuities due. Combined, calculate the future value, or $FV_{DUE}$.

**Understand**

What You Already Know

Step 1 (continued): The timeline for the trust fund is below.

Step 2: Time segment 1: PV = \$0, IY = 5.75%, CY = 12, PMT = \$1,000, PY = 2, Years = 5

Time segment 2: PV = $FV_1$ of time segment 1, IY = 5.75%, CY = 12, PMT = \$500, PY = 4, Years = 13

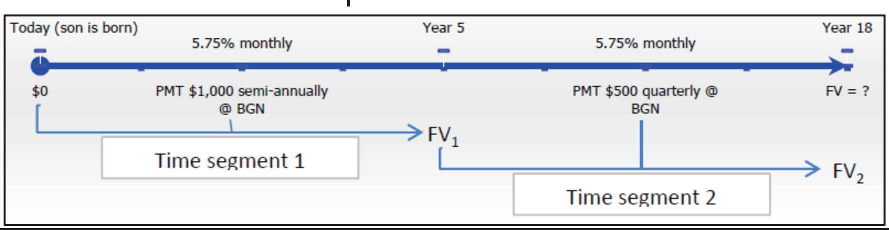
How You Will Get There

For *each* time segment, repeat the following steps:

Step 3: Apply Formula 4.1.4.

Step 4: If there is a PV, apply Formula 4.2.2 and Formula 4.2.3. Step 5: Apply Formula 4.2.1 and Formula 4.2.5.

The total future value in any time segment is the sum of the answers to step 4 (FV) and step 5 ($FV_{ORD}$).




**Perform**

Step 3: $i = 5.75\% \div 12 = 0.4719\overline{6}\%$

Step 4: No PV, so skip this step.

Step 5: $N = 2 \times 5 = 10$ payments

$$FV_{DUE1} = \$1{,}000\left[\frac{\left[(1+0.0047196\overline{6})^{\frac{12}{2}}\right]^{10}-1}{(1+0.0047196\overline{6})^{\frac{12}{2}}-1}\times(1+0.0047196\overline{6})^{\frac{12}{2}}\right] = \$11{,}748.47466 = FV_1$$

For the second time segment:

Step 3: $i$ remains unchanged $= 0.4719\overline{6}\%$

Step 4: $N = 12 \times 13 = 156$ compounds, $FV_2 = \$11{,}748.47466(1 + 0.0047196\overline{6})^{156} = \$24{,}765.17$

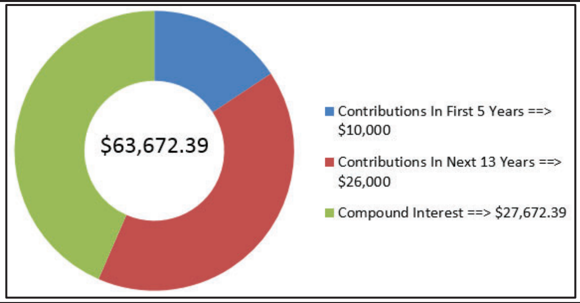
Step 5: $N = 4 \times 13 = 52$ payments

$$FV_{DUE2} = \$500\left[\frac{\left[(1+0.0047196\overline{6})^{\frac{12}{4}}\right]^{52}-1}{(1+0.0047196\overline{6})^{\frac{12}{4}}-1}\times(1+0.0047196\overline{6})^{\frac{12}{4}}\right] = \$38{,}907.21529$$

Total $FV_2 = \$24{,}765.17 + \$38{,}907.21529 = \$63{,}672.39$

**Present**

The figure shows how much principal and interest make up the final balance. When Roberto's son turns 18, the trust fund will have a balance of $63,672.39.




# Module 4.2 Exercises

## Mechanics

For questions 1–4, use the information provided to determine whether an annuity exists.

1. A debt of four payments of $500 due in 6 months, 12 months, 18 months, and 24 months.
2. A debt of four quarterly payments in the amounts of $100, $200, $300, and $400.
3. Contributions to an RRSP of $200 every month for the first year followed by $200 every quarter for the second year.
4. Regular monthly deposits of $250 to an RRSP for five years, skipping one payment in the third year.

For questions 5–8, determine the annuity type.

| | Compounding Frequency | Payment Frequency | Payment Timing |
|---|---|---|---|
| 5. | Quarterly | Semi-annually | Beginning |
| 6. | Annually | Annually | End |
| 7. | Semi-annually | Semi-annually | Beginning |
| 8. | Monthly | Quarterly | End |

For questions 9–10, draw an annuity timeline and determine the annuity type.

9. A $2,000 loan at 7% compounded quarterly is taken out today. Four quarterly payments of $522.07 are required. The first payment will be three months after the start of the loan.
10. A new RRSP is set up with monthly contributions of $300 for five years earning 9% compounded semi-annually. The RRSP will have $22,695.85 when complete. The first payment is today.

For questions 11-14, calculate the future value.

| | Present Value | Interest Rate | Payments | Timing of Payment | Years |
|---|---|---|---|---|---|
| 11. | $0 | 7% quarterly | $2,000 quarterly | Beginning | 10 |
| 12. | $0 | 9% monthly | $375 monthly | End | 20 |
| 13. | $15,000 | 5.6% quarterly | $3,000 annually | End | 30 |
| 14. | $38,000 | 8% semi-annually | $1,500 monthly | Beginning | 8 |

For questions 15–18, calculate the future value.

| | Present Value | Interest Rate | Payments | Timing of Payment |
|---|---|---|---|---|
| 15. | $0 | 6% annually for four years; then 7% semi-annually for six years | $1,000 quarterly | End |
| 16. | $0 | 12% quarterly for six-and-a-half years; then 11% semi-annually for three-and-a-half years | $100 monthly | Beginning |
| 17. | $27,150 | 11% quarterly | $750 quarterly for five years; then $1,000 quarterly for 10 years | End |
| 18. | $50,025 | 8% annually for three years; then 10% annually for seven years | $5,000 annually for three years; then $4,000 annually for seven years | Beginning |

## Applications

For questions 19-23, draw an annuity timeline and determine the annuity type. Calculate the value of *N*.

19. Marie has decided to start saving for a down payment on her home. If she puts $1,000 every quarter for five years into a GIC earning 6% compounded monthly she will have $20,979.12. She will make her first deposit three months from now.

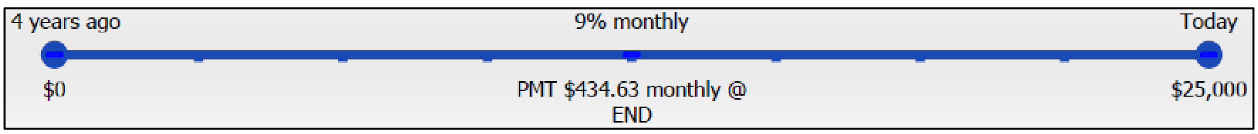
20. Laroquette Holdings needs a new $10 million warehouse. Starting today the company will put aside $139,239.72 every month for five years into an annuity earning 7% compounded semi-annually.

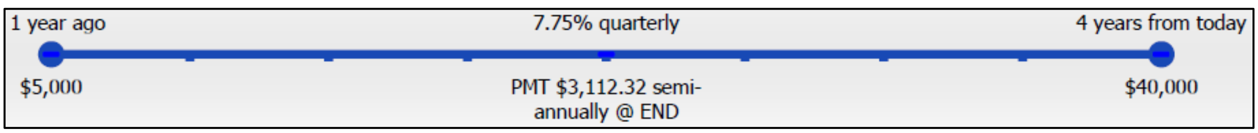
21. Brenda will lease a $25,000 car at 3.9% compounded monthly with monthly payments of $473.15 starting immediately. After three years she will still owe $10,000 on the vehicle.

22. Steve takes out a two-year gym membership worth $500. The first of his monthly $22.41 payments is due at signing and includes interest at 8% compounded annually.

23. Each year, Buhler Industries saves up $1 million to distribute in Christmas bonuses to its employees. To do so, at the end of every month the company invests $81,253.45 into an account earning 5.5% compounded monthly.

For questions 24–28, assign the information in the timeline to the correct variables and determine the annuity type. Calculate the value of N.

24.



25.
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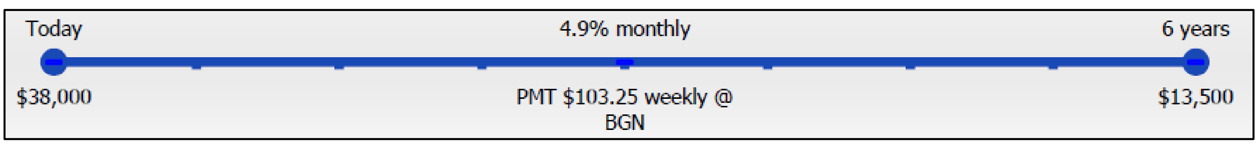
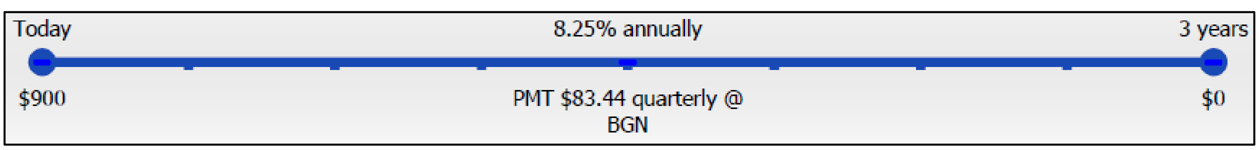
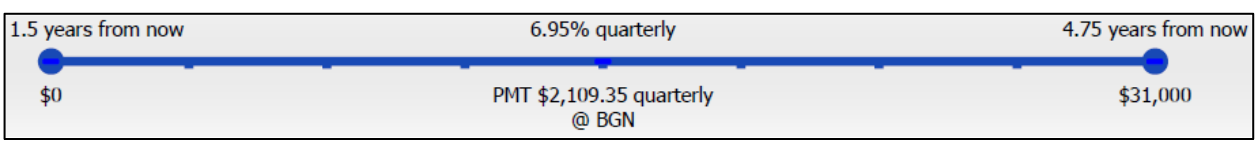



27.



28.



Use the appropriate annuity formulas in this module to answer question 29-34.

29. Nikola is currently 47 years old and planning to retire at age 60. She has already saved $220,000 in her RRSP. If she continues to contribute $200 at the beginning of every month, how much money will be in her RRSP at retirement if it can earn 8.1% compounded monthly? No deposit is made the day she turns 60.

30. You are a financial adviser. Your client is thinking of investing $600 at the end of every six months for the next six years with the invested funds earning 6.4% compounded semi-annually. Your client wants to know how much money she will have after six years. What do you tell your client?

31. The Saskatchewan Roughriders started a rainy day savings fund three-and-a-half years ago to help p y for stadium improvements. At the beginning of every quarter the team has deposited $20,000 into the fund, which has been earning 4.85% compounded semi-annually. How much money is in the fund today?

32. McDonald's major distribution partner, The Martin-Brower Company, needs at least $1 million to build a new warehouse in Medicine Hat two years from today. To date, it has invested $500,000. If it continues to invest $50,000 at the end of every quarter into a fund earning 6% quarterly, will it have enough money to build the warehouse two years from now?Show calculations to support your answer.

33. The human resource department helps employees save by taking preauthorized RRSP deductions from employee paycheques and putting them into an investment. For the first five years, Margaret has had $50 deducted at the beginning of every biweekly pay period. Then for the next five years, she increased the deduction to $75. The company has been able to average 8.85% compounded monthly for the first seven years, and then 7.35% compounded monthly for the last three years. What amount has Margaret accumulated in her RRSP after 10 years? Assume there are 26 biweekly periods in a year.

34. Joshua is opening up a Builder GIC that allows him to make regular contributions to his GIC throughout the term. He will initially deposit $10,000, then at the end of every month for the next five years he will make $100 contributions to his GIC. The annually compounded interest rates on the GIC in each year are 0.75%, 1.5%, 2.5%, 4.5%, and 7.25%. What is the maturity value of his GIC?

## Challenge, Critical Thinking, & Other Applications

35. When Shayla turned five years old, her mother opened up a Registered Education Savings Plan (RESP) and started making $600 end-of-quarter contributions. The RESP earned 7.46% semi-annually. At the end of each year, Human Resources and Skills Development Canada (HRSDC) made an additional deposit under the Canada Education Savings Grant (CESG) of 20% of her annual contributions into her RESP. Calculate the total maturity value available for Shayla's education when she turns 18.

36. Assume a 10-year ordinary annuity earning 10% compounded annually.
If $5,000 is deposited annually, what is the maturity value?
What is the maturity value if the deposits are doubled to $10,000? Compared to (a), what is the relationship between the size of the deposit and the maturity value, all other conditions being held equal?
What is the maturity value if the $5,000 deposits are made semi-annually? Compared to (b), what is the relationship between the frequency of payments and the maturity value, all other conditions being held equal?

37. Carlyle plans to make month-end contributions of $400 to his RRSP from age 20 to age 40. From age 40 to age 65, he plans to make no further contributions to his RRSP. The RRSP can earn 9% compounded annually from age 20 to age 60, and then 5% compounded annually from age 60 to age 65. Under this plan, what is the maturity value of his RRSP when he turns 65?

38. To demonstrate the power of compound interest on an annuity, examine the principal and interest components of the maturity value in your RRSP after a certain time period. Suppose $200 is invested at the end of every month into an RRSP earning 8% compounded quarterly.
Determine the maturity value, principal portion, and interest portion at 10, 20, 30, and 40 years. What do you observe?
Change the interest rate to 9% quarterly and repeat (a). Comparing your answers to (a), what do you observe?

39. Compare the following maturity values on these annuities due earning 9% compounded semi-annually:
Payments of $1,000 quarterly for 40 years.
Payments of $1,600 quarterly for 25 years.
Payments of $4,000 quarterly for 10 years.
Note in all three of these annuities that the same amount of principal is contributed. What can you learn about compound interest from these calculations?