Module 9.2 - Linear Programming

Learning Objectives:

Objective function
Constraint equations
Feasible region
Corner points
Fundamental Theorem of Linear Programming
Solving a linear programming problem using a graph

Wouldn't it be nice if we could simply produce and sell infinitely many units of a product and thus make a never-ending amount of money? In business (and in day-to-day living) we know that some things are just unreasonable or impossible. Instead, our hope is to maximize or minimize some quantity, given a set of constraints.

In order to have a linear programming problem, we must have:

- Constraints, represented as inequalities
- An **objective function**, that is, a function whose value we either want to be as large as possible (want to maximize it) or as small as possible (want to minimize it).

Example 1

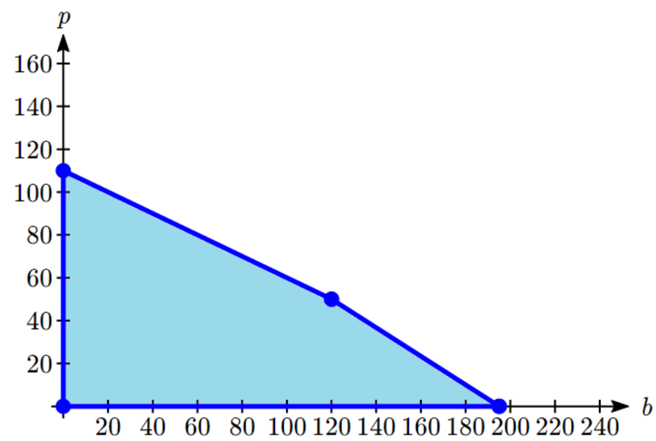
A company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. If the company has staffing for 3,900 minutes of assembly and 3,300 minutes of painting each week. They sell the basic products for a profit of \$30 and the premium products for a profit of \$40. How many of each version should be produced to maximize profit?

Let b = the number of basic products made, and p = the number of premium products made. Our objective function is what we're trying to maximize or minimize. In this case, we're trying to maximize profit. The total profit, P, is
$P = 30b + 40p$

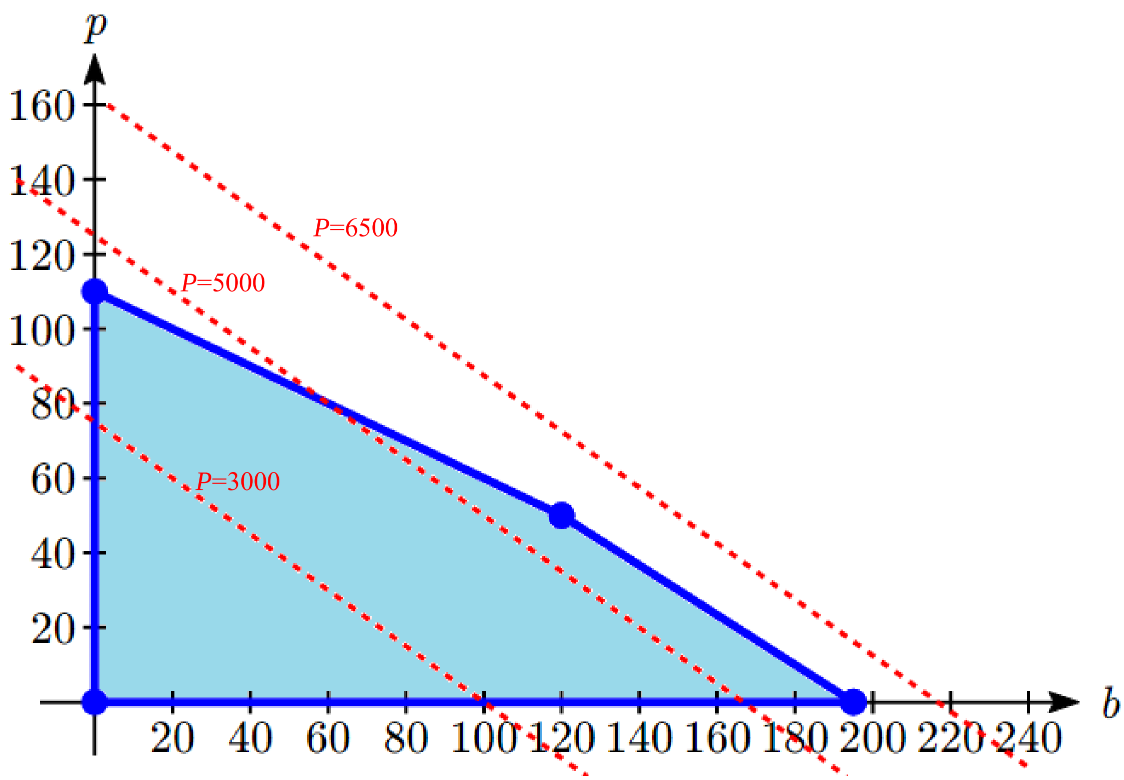
In the last section, the example developed our constraints. Together, these define our linear programming problem:

Objective function: $P = 30b + 40p$
Constraints:
$20b + 30p \leq 3900$
$15b + 30p \leq 3300$
$b \geq 0,\ p \geq 0$

In this section, we will approach this type of problem graphically. We start by graphing the constraints to determine the **feasible region** – the set of possible solutions. Just showing the solution set where the four inequalities overlap, we see a clear region.

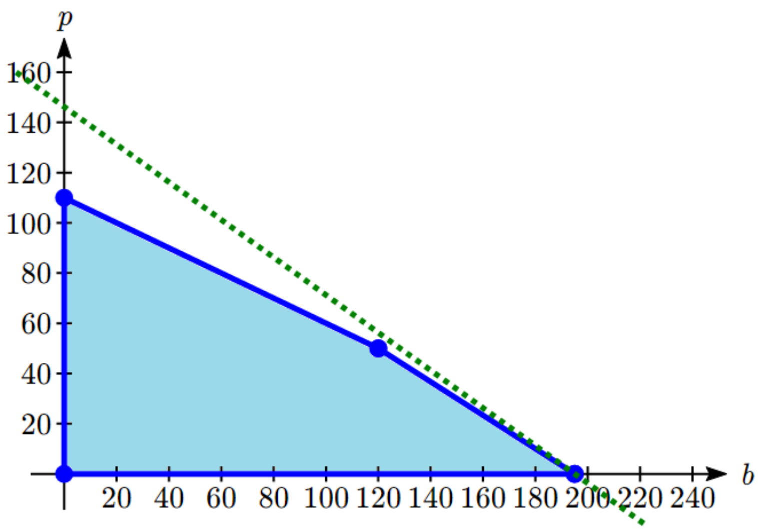


To consider how the objective function connects, suppose we considered all the possible production combinations that gave a profit of P = \$3000, so that $3000 = 30b + 40p$. That set of combinations would form a line in the graph. Doing the same for a profit of \$5000 and \$6500 would give additional lines. Graphing those on top of our feasible region, we see a pattern:



Notice that all the constant-profit lines are parallel, and that in general the profit increases as we move up to upper right. Notice also that for a profit of \$5000 there are some production levels inside the feasible region for that profit level, but some are outside. That means we could feasibly make \$5000 profit by producing, for example, 167 basic items and no premium items, but we can't make \$5000 by producing 125 premium items and no basic items because that falls outside our constraints.

The solution to our linear programming problem will be the largest possible profit that is still feasible. Graphically, that means the line furthest to the upper-right that still touches the feasible region on at least point. That solution is the one below:



This profit line touches the feasible region where b = 195 and p = 0, giving a profit of $P = 30(195) + 40(0) = \$5850$.

Notice that this is slightly larger than the profit that would be made by completely utilizing all staffing at b = 120, p = 50, where the profit would be $5600.

The objective function along with the four corner points above forms a **bounded** linear programming problem. That is, imagine you are looking at three fence posts connected by fencing (black point and lines, respectively). If you were to put your dog in the middle, you could be sure it would not escape (assuming the fence is tall enough). If this is the case, then you have a bounded linear programming problem. If the dog could walk infinitely in any one direction, then the problem is unbounded.

In the past example, you can see that the line of maximum profit will always touch the boundary of the feasible region. That observation inspires the fundamental theorem of linear programming.

Fundamental Theorem of Linear Programming

- If a solution exists to a bounded linear programming problem, then it occurs at one of the corner points.
- If a feasible region is unbounded, then a maximum value for the objective function does not exist.
- If a feasible region is unbounded, and the objective function has <u>only</u> positive coefficients, then a minimum value exists.

In the last example we solve the problem somewhat intuitively by "sliding" the profit line up. Typically we use a more procedural approach.

Solving a Linear Programming Problem Graphically

1. Define the variables to be optimized. The question asked is a good indicator as to what these will be.
2. Write the objective function, first in words, then convert to a mathematical equation
3. Write the constraints, first in words, then convert to mathematical inequalities
4. Graph the constraints inequalities, and shade the feasible region
5. Identify the corner points by solving systems of linear equations whose intersection represents a corner point.
6. Test all corner points in the objective function. The "winning" point is the point that optimizes the objective function (biggest if maximizing, smallest if minimizing)

Try it Now

1. Maximize $P = 14x + 9y$ subject to the constraints:

$x + y \leq 9$

$3x + y \leq 15$

$x \geq 0, y \geq 0$

Example 2

A health-food business would like to create a high-potassium blend of dried fruit in the form of a box of 10 fruit bars. It decides to use dried apricots, which have 407 mg of potassium per serving, and dried dates, which have 271 mg of potassium per serving. The company can purchase its fruit through in bulk for a reasonable price. Dried apricots cost \$9.99/lb. (about 3 servings) and dried dates cost \$7.99/lb. (about 4 servings). The company would like the box of bars to have at least the recommended daily potassium intake of about 4700 mg, and contain at least 1 serving of each fruit. In order to minimize cost, how many servings of each dried fruit should go into the box of bars?

We begin by defining the variables. Let
x = number of servings of dried apricots
y = number of servings of dried dates

We next work on the objective function.

For apricots, there are 3 servings in one pound. This means that the cost per serving is \$9.99/3 = \$3.33. The cost for x servings would thus be $3.33x$.

For dates, there are 4 servings per pound. This means that the cost per serving is \$7.99/4 = \$2.00. The cost for y servings would thus be $2.00y$.

The total cost, C, for apricots and dates would be
$C = 3.33x + 2.00y$

Normally we would have constraints $x \geq 0$ and $y \geq 0$ since negative servings can't be used. But in this case, we're further restricted. In words:

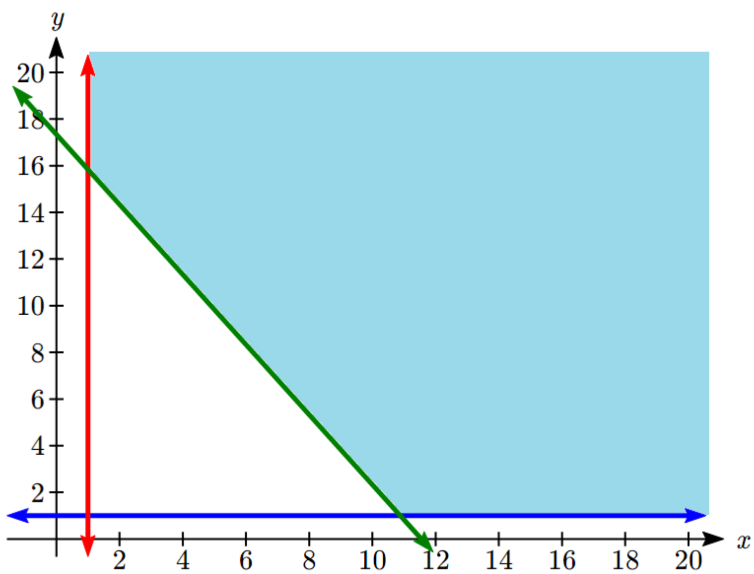
- There must be at least 1 serving of each fruit
- The product must contain at least 4700 mg of potassium

Mathematically,

- Since there must be at least 1 serving of each fruit, $x \geq 1$ and $y \geq 1$
- There are $407x$ mg of potassium in x servings of apricots and $271y$ mg of potassium in y servings of dates. The sum should be greater than or equal to 4700 mg of potassium, or $407x + 271y \geq 4700$

Thus we have,
Objective function: $C = 3.33x + 2.00y$
Constraints:
$407x + 271y \geq 4700$
$x \geq 1, y \geq 1$

We graph the constraints and shade the feasible region:



The region is unbounded, but we will be able to find a minimum still. We can see there are two corner points.

The one in the upper left is the intersection of the lines $407x + 271y = 4700$ and $x = 1$.
Solving for the intersection using substitution:

$407(1) + 271y = 4700$

$y \approx 15.8$

Point: (1, 15.8)

The one in the lower right is the intersection of the lines $407x + 271y = 4700$ and $y = 1$.

$407x + 271(1) = 4700$

$x \approx 10.9$

Point: (10.9, 1)

Testing the objective function at each of these corner points:

Point	Cost, $C = 3.33x + 2.00y$
(10.9, 1)	$C = 3.33(10.9) + 2.00(1) = \$38.30$
(1, 15.8)	$C = 3.33(1) + 2.00(15.8) = \$34.96$

The company can minimize cost by using 1 serving of apricots and 15.8 servings of dates.

Try it Now

2. A company makes two products. Product A requires 3 hours of manufacturing and 1 hour of assembly. Product B requires 4 hours of manufacturing and 2 hours of assembly. There are a total of 84 hours of manufacturing and 32 hours of assembly available. Determine the production to maximize profit if the profit on product A is \$50 and the profit on product B is \$60.

Example 3

A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is \$20 per unit for a chair and \$30 for a table, how many units of each should be manufactured to maximize profit?

We begin by defining the variables. Let
c = number of chairs made
t = number of tables made

The profit, P, will be $P = 20c + 30t$.

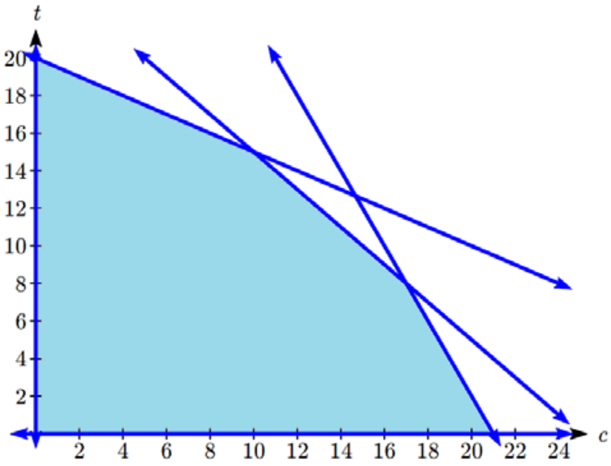
For cutting, c chairs will require $1c$ hours and t tables will require $2t$ hours. We can use at most 40 hours, so $c + 2t \leq 40$.

For assembly, c chairs will require $2c$ hours and t tables will require $1t$ hours. We can use at most 42 hours, so $2c + t \leq 42$.

For finishing, c chairs will require $1c$ hours and t tables will require $1t$ hours. We can use at most 25 hours, so $c + t \leq 25$.

Since we can't produce negative items, $c \geq 0, t \geq 0$.

Graphing the constraints, we can see the feasible region.



There are five corner points for this region.
Point 1: In the lower left, where $t = 0$ crosses $c = 0$. Point: (0, 0)

Point 2: In the upper left, where $c = 0$ crosses $c + 2t = 40$.
Using substitution, $0 + 2t = 40$, so $t = 20$.
Point: (0, 20)

Point 3: In the lower right, where $t = 0$ crosses $2c + t = 42$.
Using substitution, $2c + 0 = 42$, so $c = 21$.
Point: (21, 0)

Point 4: Where $c + 2t = 40$ crosses $c + t = 25$.
We can solve this as a system using any techniques we know. We could solve the second equation for c, giving $c = 25 - t$, then substitute into the first equation:

$$(25 - t) + 2t = 40$$

$$25 + t = 40$$

$$t = 15$$

Then $c = 25 - 15 = 10$.
Point: (10, 15)

Point 5: Where $2c + t = 42$ crosses $c + t = 25$.
We can solve this as a system using any techniques we know. Using a different technique this time, we could multiply the bottom equation by -1 then add it to the first:

$$2c + t = 42$$
$$\underline{-c - t = -25}$$
$$c = 17$$

Then using $c + t = 25$, we have $17 + t = 25$, so $t = 8$.
Point: (17, 8)

Testing the objective function at each of these corner points:

Point	Profit, $P = 20c + 30t$
(0, 0)	$P = 20(0) + 30(0) = \$0$
(0, 20)	$P = 20(0) + 30(20) = \$600$
(21, 0)	$P = 20(21) + 30(0) = \$420$
(10, 15)	$P = 20(10) + 30(15) = \$650$
(17, 8)	$P = 20(17) + 30(8) = \$580$

The profit will be maximized by producing 10 chairs and 15 tables.

For the next examples we will focus on setting up the objective function and constraints and interpreting the solution, and omit the details of solving.

Example 4

A catering company is to make lunch for a business meeting. It will serve ham sandwiches, light ham sandwiches, and vegetarian sandwiches. A ham sandwich has 1 serving of vegetables, 4 slices of ham, 1 slice of cheese, and 2 slices of bread. A light ham sandwich has 2 serving of vegetables, 2 slices of ham, 1 slice of cheese and 2 slices of bread. A vegetarian sandwich has 3 servings of vegetables, 2 slices of cheese, and 2 slices of bread. A total of 10 bags of ham are available, each of which has 40 slices; 18 loaves of bread are available, each with 14 slices; 200 servings of vegetables are available, and 15 bags of cheese, each with 60 slices, are available. Given the resources, how many of each sandwich can be produced if the goal is to maximize the number of sandwiches?

We wish to maximize the number of sandwiches, so let:
x = number of ham sandwiches
y = number of light ham sandwiches
z = number of vegetarian sandwiches

The total number of sandwiches is given by: $S = x + y + z$

The constraints will be given by considering the total amount of ingredients available. That is, the company has a limited amount of ham, vegetables, cheese, and bread.

In total, the company has $10(40)=400$ slices of ham, $18(14)=252$ slices of bread, 200 servings of vegetables, and $15(60)=900$ slices of cheese available. At most, the company can use the above amounts.

There are two sandwiches that use ham – the first requires 4 slices of ham and the second requires only 2, per sandwich, and the total number of slices of ham cannot exceed 400:
$4x+2y\le 400$

Each sandwich requires 2 slices of bread so:
$2x+2y+2z\le 252$

The ham sandwiches have 1 and 2 servings of vegetables, respectively, and the vegetarian sandwich has 3 servings of vegetables. So,
$1x+2y+3z\le 200$

Both ham sandwiches require one slice of cheese, and the vegetarian sandwich requires two slices of cheese, so,
$1x+1y+2z\le 900$

Our final setup is:
Maximize: $S=x+y+z$
Subject to:
$4x+2y\le 400$
$2x+2y+2z\le 252$
$2x+2y+2z\le 252$

Solving this, we get
Optimal Solution: S = 126; x = 100, y = 0, z = 26

We find that 100 ham sandwiches, 26 vegetarian sandwiches, and 0 light ham sandwiches should be made to maximize the total number of sandwiches made.

Notice that this will effectively use up all of the bread, which is the first to go.

Example 5

A factory manufactures three products, A, B, and C. Each product requires the use of two machines, Machine I and Machine II. The total hours available, respectively, on Machine I and Machine II per month are 180 and 300. The time requirements and profit per unit for each product are listed below.

	A	B	C
Machine I	1	2	2
Machine II	2	2	4
Profit	20	30	40

How many units of each product should be manufactured to maximize profit, and what is the maximum profit?

As usual, we start by defining our variables:
A = number of units of product A manufactured
B = number of units of product B manufactured
C = number of units of product B manufactured

We are trying to maximize profit. Producing A units of item A will result in a profit of $20A$, producing B units of item B will profit $30B$, and C units of item C will profit $40C$, giving our objective function:
$P = 20A + 30B + 40C$

Next we consider the time available on each machine to establish constraints. Producing A units of item A will require $1A$ hours on Machine 1, producing B units of item B will require $2B$ hours, and producing C units of item C will require $2C$ hours. Together these need to not exceed the 180 hours available. This leads to the constraint:
$1A + 2B + 2C \leq 180$

We can construct a similar constraint using the hours on Machine 2:
$2A + 2B + 4C \leq 300$

Our final setup is:
Maximize $P = 20A + 30B + 40C$
Subject to:
$1A + 2B + 2C \leq 180$
$2A + 2B + 4C \leq 300$
$A \geq 0, B \geq 0, C \geq 0$

Solving this:
Optimal Solution: $P = 3300$; $A = 120$, $B = 30$, $C = 0$

We will maximize profit at $3300 by producing 120 units of item A, 30 units of item B, and no units of item C.

In addition to maximization problems, linear programming can also be used to solve minimization problems. When done by-hand, these would require a modification of the Simplex method shown in the last section, but these problems can be solved by most technologic methods.

Example 6

A company is creating a meal replacement bar. They plan to incorporate peanut butter, oats, and dried cranberries as the primary ingredients. The nutritional content of 10 grams of each is listed below, along with the cost, in cents, of each ingredient. Find the amount of each ingredient they should use to minimize the cost of producing a bar containing a minimum of 15g of each ingredient, at least 10g of protein and at most 14g of fat.

	Peanut Butter, 10g	Oats, 10g	Cranberries, 10g
Protein (grams)	2.5	1.7	0
Fat (grams)	5	0.7	0.1
Cost (cents)	6	1	2

We start by introducing variables:
p = number of 10g servings of peanut butter
a = number of 10g servings of oats
c = number of 10g servings of cranberries

The total cost, C, of producing the bar, in cents, will be $C = 6p + 1a + 2c$.

Our first constraints come from the requirement for 15g of each ingredient, which is 1.5 servings (1.5 servings at 10g per serving = 15g). Constructing those constraints:
$p \geq 1.5,\ a \geq 1.5,\ c \geq 1.5$

Next we look at the nutritional components. For protein, p servings of peanut butter will contain $2.5p$ grams of protein. Likewise, a servings of oats will have $1.7a$ grams of protein, and c servings of cranberries will have $0c$ grams of protein. Together, these need to be at least 10 grams, giving the constraint
$2.5p + 1.7a + 0c \geq 10$

We can construct a similar constraint for fat, in this case noting we want the fat to be at most 14g:
$5p + 0.7a + 0.1c \leq 14$

We can now have our complete problem:
Minimize $C = 6p + 1a + 2c$
Subject to:
$2.5p + 1.7a + 0c \geq 10$
$5p + 0.7a + 0.1c \leq 14$
$p \geq 1.5,\ a \geq 1.5,\ c \geq 1.5$

Turning to technology, we get a solution:
Optimal Solution: $C = 15.6765$; $p = 1.5$, $a = 3.67647$, $c = 1.5$

Interpreting that result, the minimum cost of to produce the bar will be about 15.7 cents, by using 15 grams of peanut butter, 36.8 grams of oats, and 15 grams of dried cranberries.

Verifying our conditions, we can see that our recipe contains at least 1.5 servings of each ingredient. The protein content will be $2.5(1.5) + 1.7(3.68) + 0(1.5) \approx 10$ grams. The fat content will be $5(1.5) + 0.7(3.68) + 0.1(1.5) \approx 10.2$ grams.

In some cases we have to be clever with how we create our constraints to maintain the correct form of a linear programming problem while still meeting the requirements of the actual application.

Example 4

A distribution company needs to ship products from its two warehouses to three retailers. Warehouse A has 1000 products in stock, and Warehouse B has 1200 products. Retailer 1 needs 700 products, Retailer 2 needs 500 products, and Retailer 3 needs 600 products The cost to ship a product from each warehouse to each retailer is shown below. Find the number of products the company should ship from each warehouse to each retailer to minimize shipping costs.

	Retailer 1	Retailer 2	Retailer 3
Warehouse A	3	5	6
Warehouse B	4	7	5

To start this problem, we first need to define our variables. Since there are six different routes, we will need to define six variables:
A_1 = the number of products shipped from Warehouse A to Retailer 1
B_1 = the number of products shipped from Warehouse B to Retailer 1
A_2 = the number of products shipped from Warehouse A to Retailer 2
We can similarly define variables B_2, A_3, and B_3.

Our objective function is the total shipping cost. Shipping A_1 items from Warehouse A to Retailer 1 will cost \$3 per item, so a total cost of $3A_1$. Doing the same for the other variables gives our total cost equation:
$C = 3A_1 + 4B_1 + 5A_2 + 7B_2 + 8A_3 + 5B_3$

We know that Warehouse A has 1000 products in stock, so the total number of items shipped out of Warehouse A needs to be no more than 1000. Likewise Warehouse B can't ship more than 1200 items. These give the constraints:
$A_1 + A_2 + A_3 \le 1000$
$B_1 + B_2 + B_3 \le 1200$

Retailer 1 needs 700 products. While is technically a strict equality, we can set it up as an inequality, indicating the total number of product arriving at retailer 1 needs to be at least 700 products. Since we're minimizing cost, there's no way we'd end up shipping more than 700 items to the retailer. Setting up this constraint, and similar ones for the other three retailers:
$A_1 + B_1 \ge 700$
$A_2 + B_2 \ge 500$
$A_3 + B_3 \ge 600$

Our final problem setup is:

Minimize $C = 3A_1 + 4B_1 + 5A_2 + 7B_2 + 8A_3 + 5B_3$

Subject to:

$A_1 + A_2 + A_3 \le 1000$

$B_1 + B_2 + B_3 \le 1200$

$A_1 + B_1 \ge 700$

$A_2 + B_2 \ge 500$

$A_3 + B_3 \ge 600$

$A_1 \ge 0, B_1 \ge 0, A_2 \ge 0, B_2 \ge 0, A_3 \ge 0, B_3 \ge 0$

Solving this, we get the solution:

Optimal Solution: C = 7800; A_1 = 500, B_1 = 200, A_2 = 500, B_2 = 0, A_3 = 0, B_3 = 600

Try it Now

3. A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. If Food A costs \$2 per unit and Food B cost \$1 per unit, how many units of each food should be purchased to keep costs at a minimum?