LEARNING OBJECTIVES

In this section, you will:

- Identify end behavior of power functions.

MODULE 4.1 - MORE ON POWER FUNCTIONS AND POLYNOMIAL FUNCTIONS

Identifying End Behavior of Power Functions

Figure 1 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

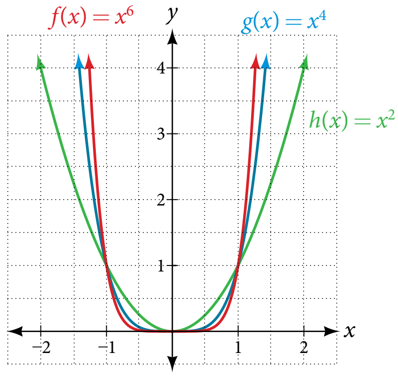


Figure 1 Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the positive even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, f(x) \to \infty$$

Figure 2 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

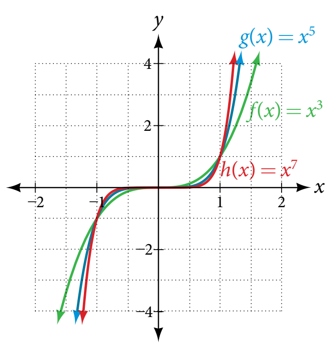


Figure 2 Odd-power functions

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in **Figure 1** we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In **Figure 2** we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty, \quad f(x) \to -\infty \qquad \text{as } x \to \infty, \quad f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

Figure 3 shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

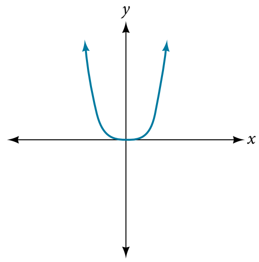
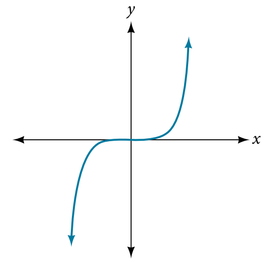
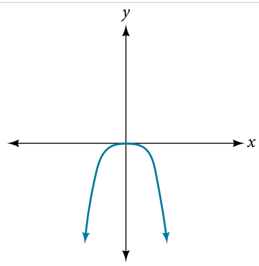
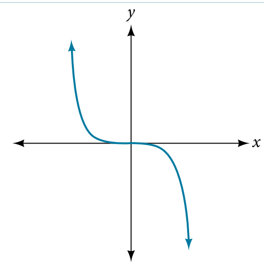
	Even power	**Odd power**
Positive constant $k > 0$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to \infty$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to -\infty$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to -\infty$

Figure 3

How To...

Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use **Figure 3** to identify the end behavior.

Example 1 Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty, f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty, f(x) \to \infty$. We can graphically represent the function as shown in **Figure 4**.

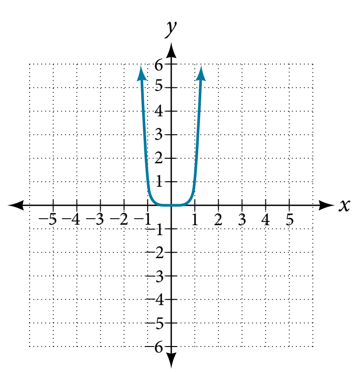


Figure 4

Example 2 Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution The exponent of the power function is 9 (an odd number). Because the coefficient is -1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. **Figure 5** shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty, \quad f(x) \to \infty$$
$$\text{as } x \to \infty, \quad f(x) \to -\infty$$

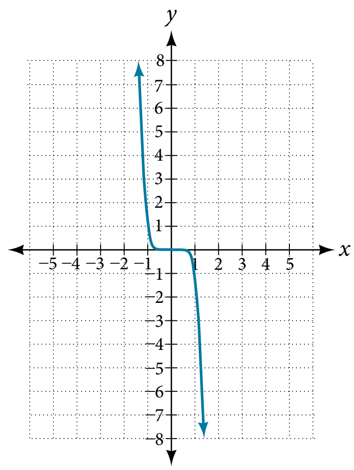


Figure 5

Analysis *We can check our work by using the table feature on a graphing utility.*

x	$f(x)$
-10	1,000,000,000
-5	1,953,125
0	0
5	$-1,953,125$
10	$-1,000,000,000$

Table 1

*We can see from **Table 1** that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).*

Try It #1

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the power function consisting of the leading term. See **Table 2**.

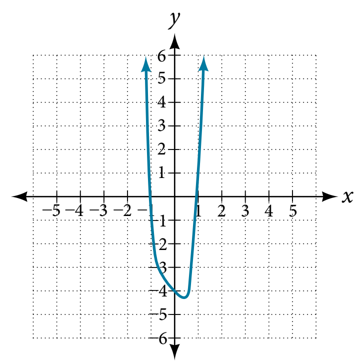
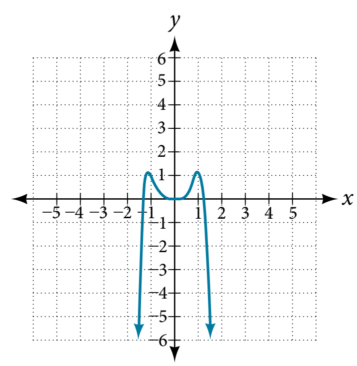
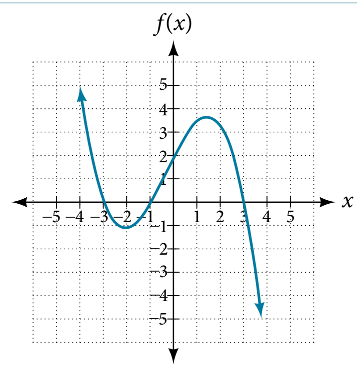
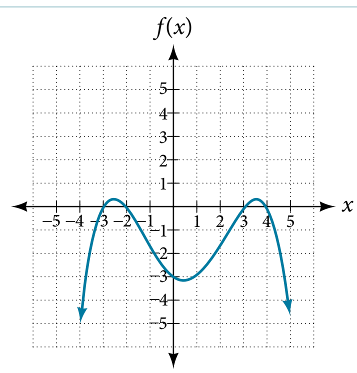
Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Table 2

Example 3 Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in **Figure 6**.

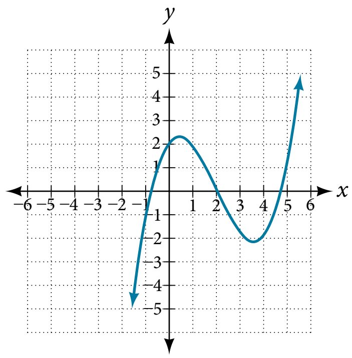


Figure 6

Solution As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty, \quad f(x) \to -\infty$$
$$\text{as } x \to \infty, \quad f(x) \to \infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Try It #2

Describe the end behavior, and determine a possible degree of the polynomial function in **Figure** 7.

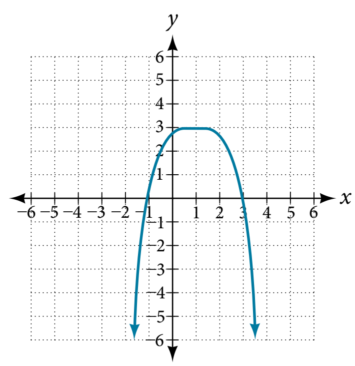


Figure 7

Example 4 Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x - 1)(x + 4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x - 1)(x + 4) \\ &= -3x^2(x^2 + 3x - 4) \\ &= -3x^4 - 9x^3 + 12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (−3), so the end behavior is

$$\text{as } x \to -\infty, \quad f(x) \to -\infty$$
$$\text{as } x \to \infty, \quad f(x) \to -\infty$$

Try It #3

Given the function $f(x) = 0.2(x - 2)(x + 1)(x - 5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See **Figure 8**.

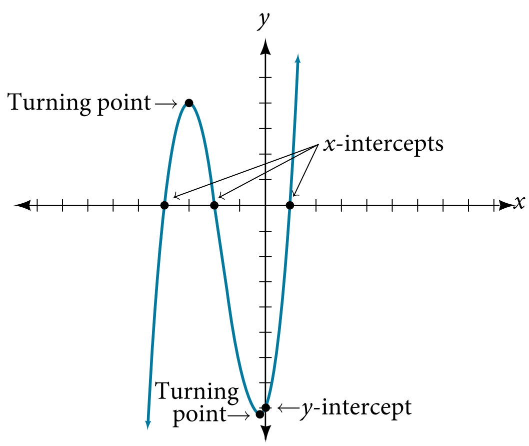


Figure 8

> ***intercepts and turning points of polynomial functions***
>
> A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The y-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

How To…

Given a polynomial function, determine the intercepts.

1. Determine the y-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the x-intercepts by solving for the input values that yield an output value of zero.

Example 5 **Determining the Intercepts of a Polynomial Function**

Given the polynomial function $f(x) = (x - 2)(x + 1)(x - 4)$, written in factored form for your convenience, determine the y- and x-intercepts.

Solution The y-intercept occurs when the input is zero so substitute 0 for x.

$$\begin{aligned} f(0) &= (0 - 2)(0 + 1)(0 - 4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The y-intercept is (0, 8).

The x-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$
$$x-2=0 \quad \text{or} \quad x+1=0 \quad \text{or} \quad x-4=0$$
$$x=2 \quad \text{or} \quad x=-1 \quad \text{or} \quad x=4$$

The x-intercepts are $(2, 0)$, $(-1, 0)$, and $(4, 0)$.

We can see these intercepts on the graph of the function shown in **Figure 9**.

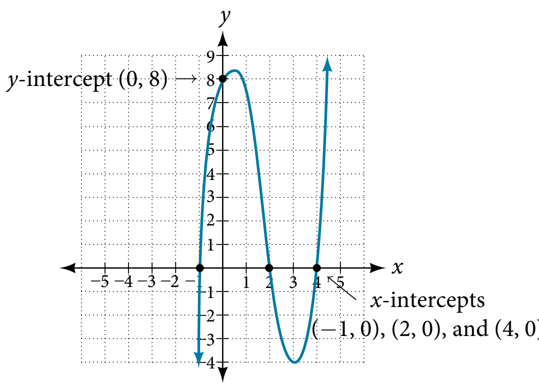


Figure 9

Example 6 Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the y- and x-intercepts.

Solution The y-intercept occurs when the input is zero.

$$f(0) = (0)^4 - 4(0)^2 - 45$$
$$= -45$$

The y-intercept is $(0, -45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$f(x) = x^4 - 4x^2 - 45$$
$$= (x^2-9)(x^2+5)$$
$$= (x-3)(x+3)(x^2+5)$$
$$0 = (x-3)(x+3)(x^2+5)$$
$$x-3=0 \quad \text{or} \quad x+3=0 \quad \text{or} \quad x^2+5=0$$
$$x=3 \quad \text{or} \quad x=-3 \quad \text{or} \quad \text{(no real solution)}$$

The x-intercepts are $(3, 0)$ and $(-3, 0)$.

We can see these intercepts on the graph of the function shown in **Figure 10**. We can see that the function is even because $f(x) = f(-x)$.

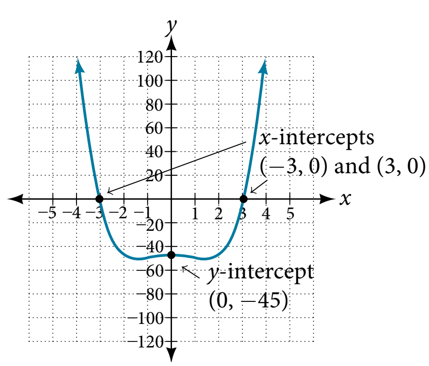


Figure 10

Try It #4

Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the y- and x-intercepts.

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of n^{th} degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

total number turning points of polynomials

A polynomial of degree n will have, at most, n x-intercepts and $n-1$ turning points.

Example 7 Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution The polynomial has a degree of 10, so there are at most 10 x-intercepts and at most 9 turning points.

Try It #5

Without graphing the function, determine the maximum number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$

Example 8 Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in **Figure 11** based on its intercepts and turning points?

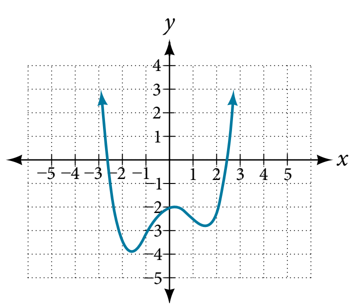


Figure 11

Solution The end behavior of the graph tells us this is the graph of an even-degree polynomial. See **Figure 12**.

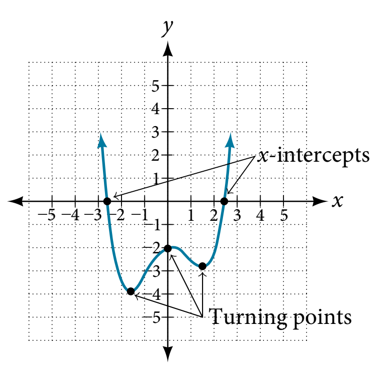


Figure 132

The graph has 2 x-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

Try It #6

What can we conclude about the polynomial represented by the graph shown in **Figure 13** based on its intercepts and turning points?

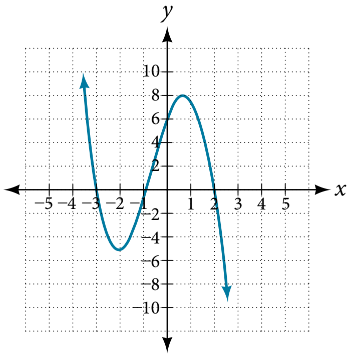


Figure 13

Example 9 Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x + 3)(x - 4)$, determine the local behavior.

Solution The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0 + 3)(0 - 4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0, 0)$.

The x-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x + 3)(x - 4) \\ x = 0 \quad &\text{or} \quad x + 3 = 0 \quad \text{or } x - 4 = 0 \\ x = 0 \quad &\text{or} \quad x = -3 \quad \text{or} \quad x = 4 \end{aligned}$$

The x-intercepts are $(0, 0)$, $(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

Try It #7

Given the function $f(x) = 0.2(x - 2)(x + 1)(x - 5)$, determine the local behavior.

Access these online resources for additional instruction and practice with power and polynomial functions.

- Find Key Information About a Given Polynomial Function (http://openstaxcollege.org/l/keyinfopoly)
- End Behavior of a Polynomial Function (http://openstaxcollege.org/l/endbehavior)
- Turning Points and x-intercepts of Polynomial Functions (http://openstaxcollege.org/l/turningpoints)
- Least Possible Degree of a Polynomial Function (http://openstaxcollege.org/l/leastposdegree)

POWER FUNCTIONS AND POLYNOMIAL FUNCTIONS EXERCISES

VERBAL

2. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

3. In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

4. What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

5. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.

ALGEBRAIC

For the following exercises, determine the end behavior of the functions.

6. $f(x) = x^4$

7. $f(x) = x^3$

8. $f(x) = -x^4$

9. $f(x) = -x^9$

10. $f(x) = -2x^4 - 3x^2 + x - 1$

11. $f(x) = 3x^2 + x - 2$

12. $f(x) = x^2(2x^3 - x + 1)$

13. $f(x) = (2 - x)^7$

For the following exercises, find the intercepts of the functions.

14. $f(t) = 2(t - 1)(t + 2)(t - 3)$

15. $g(n) = -2(3n - 1)(2n + 1)$

16. $f(x) = x^4 - 16$

17. $f(x) = x^3 + 27$

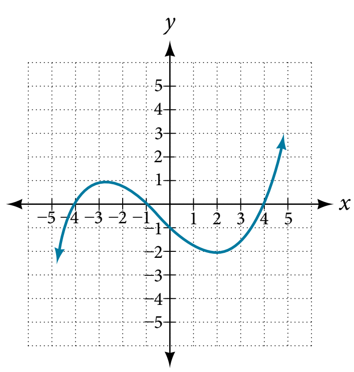
18. $f(x) = x(x^2 - 2x - 8)$

19. $f(x) = (x + 3)(4x^2 - 1)$

GRAPHICAL

For the following exercises, determine the least possible degree of the polynomial function shown.

20.



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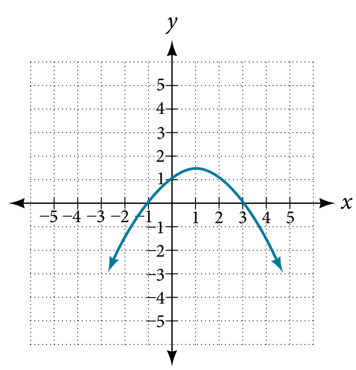

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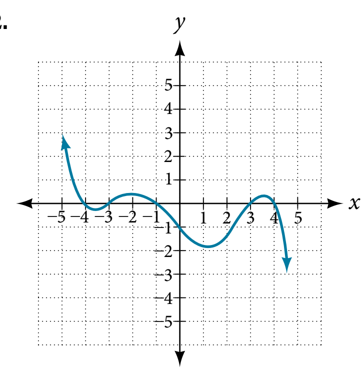

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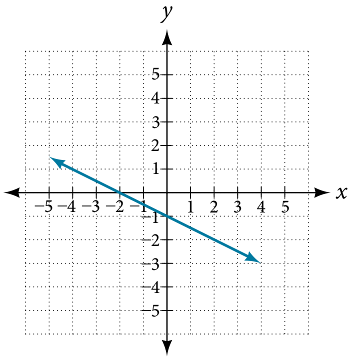

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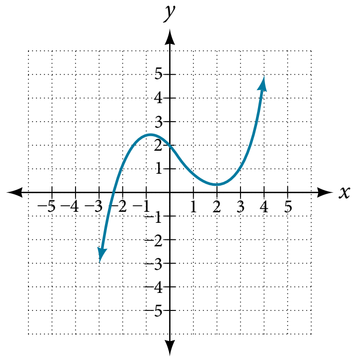

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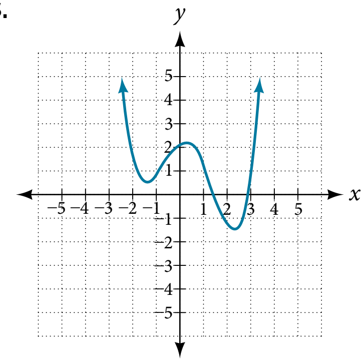

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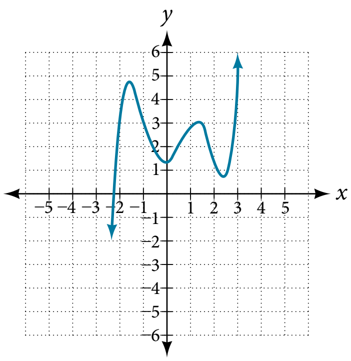

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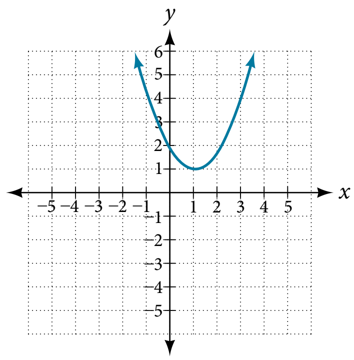

For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

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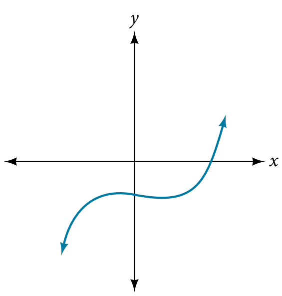

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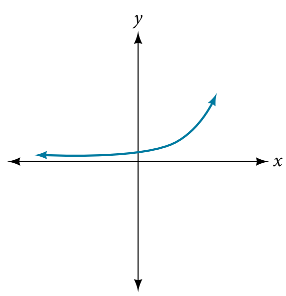

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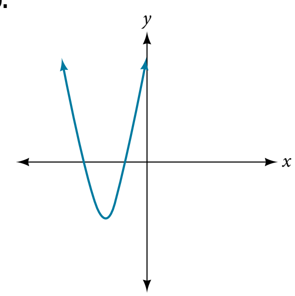
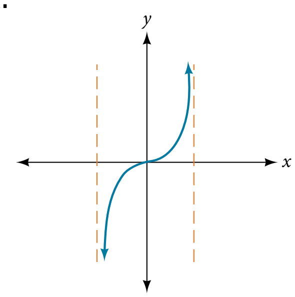


31.



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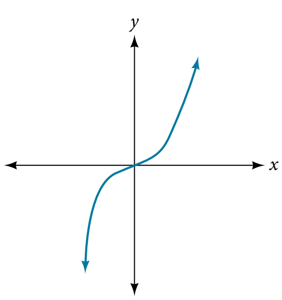

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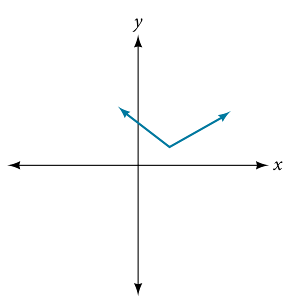

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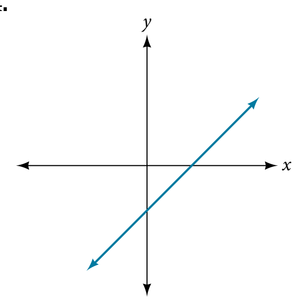

NUMERIC

For the following exercises, make a table to confirm the end behavior of the function.

35. $f(x) = -x^3$

36. $f(x) = x^4 - 5x^2$

37. $f(x) = x^2(1 - x)^2$

38. $f(x) = (x - 1)(x - 2)(3 - x)$

39. $f(x) = \dfrac{x^5}{10} - x^4$

TECHNOLOGY

For the following exercises, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.

40. $f(x) = x^3(x - 2)$

41. $f(x) = x(x - 3)(x + 3)$

42. $f(x) = x(14 - 2x)(10 - 2x)$

43. $f(x) = x(14 - 2x)(10 - 2x)^2$

44. $f(x) = x^3 - 16x$

45. $f(x) = x^3 - 27$

46. $f(x) = x^4 - 81$

47. $f(x) = -x^3 + x^2 + 2x$

48. $f(x) = x^3 - 2x^2 - 15x$

49. $f(x) = x^3 - 0.01x$

EXTENSIONS

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or -1. There may be more than one correct answer.

50. The y- intercept is $(0, -4)$. The x- intercepts are $(-2, 0)$, $(2, 0)$. Degree is 2. End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty, f(x) \to \infty$.

51. The y- intercept is $(0, 9)$. The x- intercepts are $(-3, 0)$, $(3, 0)$. Degree is 2. End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty, f(x) \to -\infty$.

52. The y-intercept is $(0, 0)$. The x-intercepts are $(0, 0)$, $(2, 0)$. Degree is 3. End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$.

53. The y-intercept is $(0, 1)$. The x-intercept is $(1, 0)$. Degree is 3. End behavior: as $x \to -\infty, f(x) \to \infty$, as $x \to \infty, f(x) \to -\infty$.

54. The y-intercept is $(0, 1)$. There is no x-intercept. Degree is 4. End behavior: as $x \to -\infty, f(x) \to \infty$, as $x \to \infty, f(x) \to \infty$.

REAL-WORLD APPLICATIONS

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

55. An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of d, the number of days elapsed.

56. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.

57. A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.

58. An open box is to be constructed by cutting out square corners of x-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.

59. A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).