Module 7.1 - Simple and Compound Interest

Your investments may be at risk if stock and bond markets slump, as a story in the *Globe and Mail* predicts. You wonder if you should shift your money into relatively secure short-term investments until the market booms again. You consider your high-interest savings account, but realize that only the first $60,000 of your savings account is insured. Perhaps you should put some of that money into treasury bills instead.

Looking ahead, what income will you live on once you are no longer working? As your career develops, you need to save money to fund your lifestyle in retirement. Some day you will have $100,000 or more that you must invest and reinvest to reach your financial retirement goals.

To make such decisions, you must first understand how to calculate simple interest. Second, you need to understand the characteristics of the various financial options that use simple interest. Armed with this knowledge, you can make smart financial decisions!

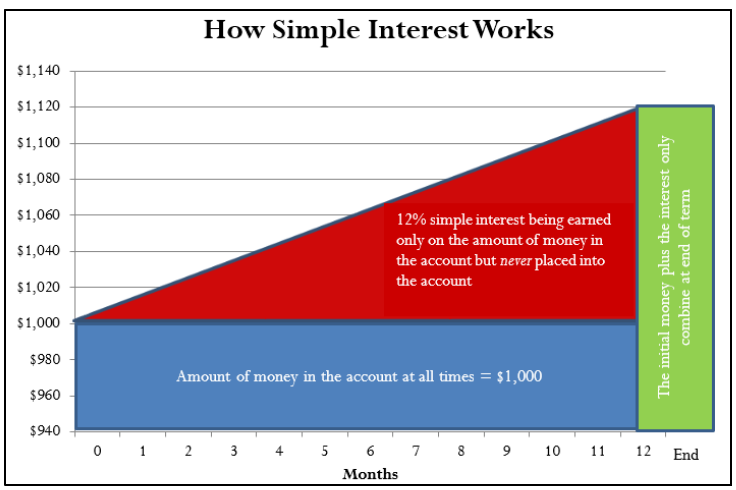
The world of finance calculates interest in two different ways:

1. **Simple Interest.** A **simple interest** system primarily applies to short-term financial transactions, with a time frame of less than one year. In this system, which is explored in this chapter, interest accrues but does not compound.
2. **Compound Interest.** A **compound interest** system primarily applies to long-term financial transactions, with a time frame of one year or more. In this system, which the next chapter explores, interest accrues and compounds upon previously earned interest.

Simple Interest

In a simple interest environment, you calculate interest solely on the amount of money at the beginning of the transaction. When the term of the transaction ends, you add the amount of the simple interest to the initial amount. Therefore, throughout the entire transaction the amount of money placed into the account remains unchanged until the term expires. It is only on this date that the amount of money increases. Thus, an investor has more money or a borrower owes more money at the end.

The figure illustrates the concept of simple interest. In this example, assume $1,000 is placed into an account with 12% simple interest for a period of 12 months. For the entire term of this transaction, the amount of money in the account always equals $1,000. During this period, interest accrues at a rate of 12%, but the interest is never placed into the account. When the transaction ends after 12 months, the $120 of interest and the initial $1,000 are then combined to total $1,120.



A loan or investment always involves two parties—one giving and one receiving. No matter which party you are in the transaction, the amount of interest remains unchanged. The only difference lies in whether you are earning or paying the interest.

- If you take out a personal loan from a bank, the bank gives you the money and you receive the money. In this situation, the bank earns the simple interest and you are being charged simple interest on your loan. In the figure, this means you must pay back not only the $1,000 you borrowed initially but an additional $120 in interest.
- If you place your money into an investment account at the bank, you have given the money and the bank has received the money. In this situation, you earn the simple interest on your money and the bank pays simple interest to your investment account. In the figure, this means the bank must give you back your initial $1,000 at the end plus an additional $120 of interest earned.

The Formula

The best way to understand how simple interest is calculated is to think of the following relationship:

Amount of simple interest = *How much* at *What simple interest rate* for *How long*

Notice that the key variables are the amount, the simple interest rate, and time. Formula 4.1.1 combines these elements into a formula for simple interest.

I is Interest Amount: The **interest amount** is the dollar amount of interest that is paid or earned. To interpret the interest amount properly, remember who you are in the transaction. If you are borrowing the money, this is the interest amount charged to you. If you are investing the money, this is the interest amount you earn.

P is Present Value or Principal: The amount of money at the beginning of the time period being analyzed is known as the **present value**, or **P**. If this is in fact the amount at the start of the financial transaction, it is also called the **principal**, which is the original amount of money that is borrowed or invested. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

Formula 8.1 – Simple Interest: $I = Prt$

r is Interest Rate: The **interest rate** is the rate of interest that is charged or earned during a specified time period. It is usually expressed in percent format. Unless noted otherwise, interest rates are expressed on an annual basis. An interest rate is the result of,

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}}$$

In this case, you calculate an annual interest rate in its decimal format as follows:

$$\text{Annual Interest Rate} = \frac{\text{Annual Interest Amount}}{\text{Principal}}$$

Thus, if you are earning \$3 of interest annually on a \$100 principal, then your annual interest rate is $3/100 = 0.03$ or 3%.

t is Time: The **time period** or term is the length of the financial transaction for which interest is charged or earned.

How It Works

Follow these steps when you calculate the amount of simple interest:

Step 1: Formula 4.1.1 has four variables, and you need to identify three for any calculation involving simple interest. If necessary, draw a timeline to illustrate how the money is being moved over time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If they are not already, you need to convert one of the two variables to the same units as the other.

Step 3: Apply Formula 4.1.1 and solve for the unknown variable. Use algebra to manipulate the formula if necessary.

Assume you have \$500 earning 3% simple interest for a period of nine months. How much interest do you earn?

Step 1: Note that your principal is \$500, or P = \$500. The interest rate is assumed to be annual, so $r = 3\%$ per year. The time period is nine months.

Step 2: Convert the time period from months to years: $t = \frac{9}{12}$.

Step 3: According to Formula 4.1.1, $I = \$500 \times 3\% \times \frac{9}{12} = \$11.25$. Therefore, the amount of interest you earn on the \$500 investment over the course of nine months is \$11.25.

Important Notes

Recall that algebraic equations require all terms to be expressed with a common unit. This principle remains true for Formula 4.1.1, particularly with regard to the interest rate and the time period. For example, if you have a 3% annual interest rate for nine months, then either

- The time needs to be expressed annually as 9/12 of a year to match the yearly interest rate, or
- The interest rate needs to be expressed monthly as 3%/12 = 0.25% per month to match the number of months.

It does not matter which you do so long as you express both interest rate and time in the same unit. If one of these two variables is your algebraic unknown, the unit of the known variable determines the unit of the unknown variable. For example, assume that you are solving Formula 4.1.1 for the time period. If the interest rate used in the formula is annual, then the time period is expressed in number of years.

Example 4.1A: How Much Interest Is Owed?

Julio borrowed $1,100 from Maria five months ago. When he first borrowed the money, they agreed that he would pay Maria 5% simple interest. If Julio pays her back today, how much interest does he owe her?

Plan

Calculate the amount of interest that Julio owes Maria (I).

Understand

What You Already Know

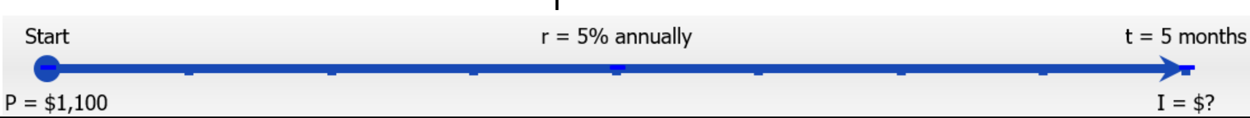
Step 1: The terms of their agreement are as follows:

$P = \$1,100$ $r = 5\%$ $t = 5$ months

How You Will Get There

Step 2: The rate is annual, and the time is in months. Convert the time to an annual number.

Step 3: Apply Formula 4.1.1.



Perform

Step 2: Five months out of 12 months in a year is $\frac{5}{12}$ of a year, or $t = \frac{5}{12}$.

Step 3: $I = \$1,100 \times 5\% \times \frac{5}{12} = \$1,100 \times 0.05 \times 0.41\overline{6} = \$22.92$

Present

For Julio to pay back Maria, he must reimburse her for the $1,100 principal borrowed plus an additional $22.92 of simple interest as per their agreement.

Example 4.1B: Do You Know Your Interest Rate?

A $3,500 investment earned $70 of interest over the course of six months. What annual rate of simple interest did the investment earn?

Plan

Calculate the annual interest rate (r).

Understand

What You Already Know

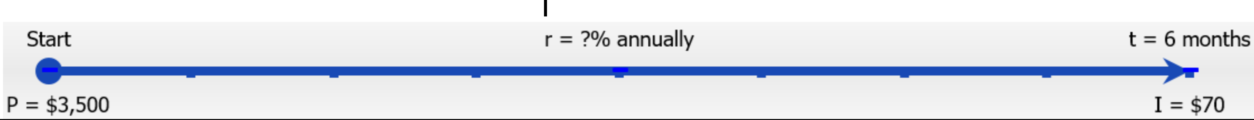
Step 1: The principal, interest amount, and time are known:

$P = \$3,500$ $I = \$70$ $t = 6$ months

How You Will Get There

Step 2: The computed interest rate needs to be annual, so you must express the time period annually as well.

Step 3: Apply Formula 4.1.1, rearranging for r.



Perform

Step 2: Six months out of 12 months in a year is $\frac{6}{12}$ of a year, or $t = \frac{6}{12}$.

Step 3: $\$70 = \$3{,}500 \times r \times \frac{6}{12}$ $\qquad r = \frac{\$70}{\$3{,}500 \times \frac{6}{12}} = \frac{\$70}{\$1{,}750} = 0.04$ or 4%

Present

For \$3,500 to earn \$70 simple interest over the course of six months, the annual simple interest rate must be 4%.

Example 4.1C: What Did You Start with?

What amount of money invested at 6% annual simple interest for 11 months earns \$2,035 of interest?

Plan

Calculate the amount of money originally invested, which is known as the present value or principal, symbolized by P.

Understand

What You Already Know

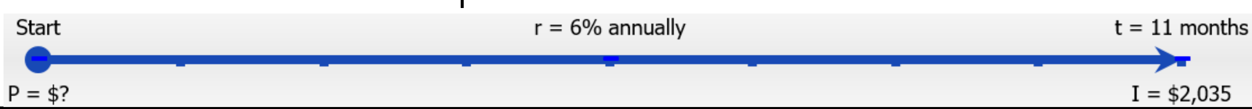
Step 1: The interest rate, time, and interest earned are known:

$r = 6\%$ $\quad t = 11$ months $\quad I = \$2{,}035$

How You Will Get There

Step 2: Convert the time from months to an annual basis to match the interest rate.

Step 3: Apply Formula 4.1.1, rearranging for P.



Perform

Step 2: Eleven months out of 12 months in a year is $\frac{11}{12}$ of a year, or $t = \frac{11}{12}$.

Step 3: $\$2{,}035 = \mathrm{P} \times 6\% \times \frac{11}{12}$ $\qquad \mathrm{P} = \frac{\$2{,}035}{6\% \times \frac{11}{12}} = \frac{\$2{,}035}{0.06 \times 0.91\overline{6}} = \$37{,}000$

Present

To generate \$2,035 of simple interest at 6% over a time frame of 11 months, \$37,000 must be invested.

Example 4.1D: How Long?

For how many months must \$95,000 be invested to earn \$1,187.50 of simple interest at an interest rate of 5%?

Plan

Calculate the length of time in months (t) that it takes the money to acquire the interest.

Understand

What You Already Know

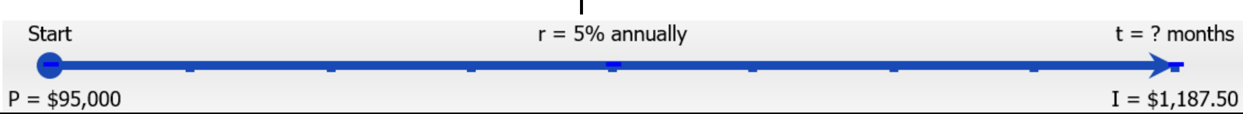
Step 1: The amount of money invested, interest earned, and interest rate are known:

$\mathrm{P} = \$95{,}000.00$ $\quad I = \$1{,}187.50$ $\quad r = 5\%$

How You Will Get There

Step 2: Express the time in months. Convert the interest rate to a "per month" format.

Step 3: Apply Formula 4.1.1, rearranging for t.



Perform

Step 2: 5% per year converted into a monthly rate is $r = \frac{0.05}{12}$

Step 3: $\$1{,}187.50 = \$95{,}000 \times \frac{0.05}{12} \times t$ $\qquad t = \frac{\$1{,}187.50}{\$95{,}000 \times \frac{0.05}{12}} = \frac{\$1{,}187.50}{\$95{,}000 \times 0.0041\overline{6}} = 3$ months

Present

For \$95,000 to earn \$1,187.50 at 5% simple interest, it must be invested for a three-month period.

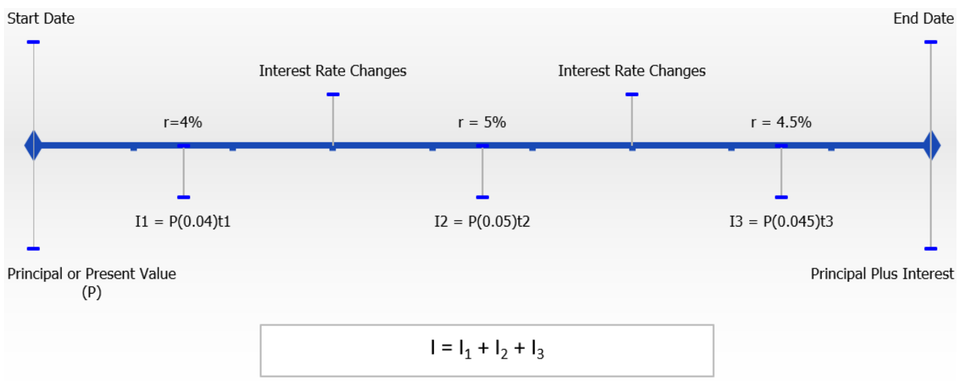
Variable Interest Rates

Not all interest rates remain constant. There are two types of interest rates:

1. **Fixed.** A **fixed interest rate** is an interest rate that is unchanged for the duration of the transaction.
2. **Variable.** A **variable interest rate** is an interest rate that is open to fluctuations over the duration of a transaction. This variable interest rate is usually tied to the **prime rate**, which is an interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans. Most people and companies are usually charged the prime rate plus a percentage (typically 0.5% to 5%) to arrive at the variable interest rate. The Bank of Canada periodically adjusts this rate because of economic and financial considerations in Canada.

In all of the previous examples in this section, the interest rate (r) was fixed for the duration of the transaction. When r fluctuates, you must break the question into time periods or time fragments for each value of r. In each of these time fragments, you calculate the amount of simple interest earned or charged and then sum the interest amounts to form the total interest earned or charged.

This figure illustrates this process with a variable rate that changes twice in the course of the transaction. To calculate the total interest amount, you need to execute Formula 8.1 for each of the time fragments. Then total the interest amounts from each of the three time fragments to calculate the interest amount (I) for the entire transaction.



Example 4.1E: A Floating Rate

Julio borrowed $10,000 on May 17 at a variable interest rate of prime + 3% when the prime rate was 2¼%. On June 22 the prime rate increased by ¼%, and on September 2 it increased again by ½%. How much interest will Julio pay when he repays his loan on October 4?

Plan

Calculate the amount of interest that Julio owes for the entire time frame of his loan, or I, from May 17 to October 4.

Understand

What You Already Know

Step 1: There is an initial interest rate and then it changes twice over the course of the transaction. That means there are three time periods and three different interest rates. The principal is also known: P = $10,000.

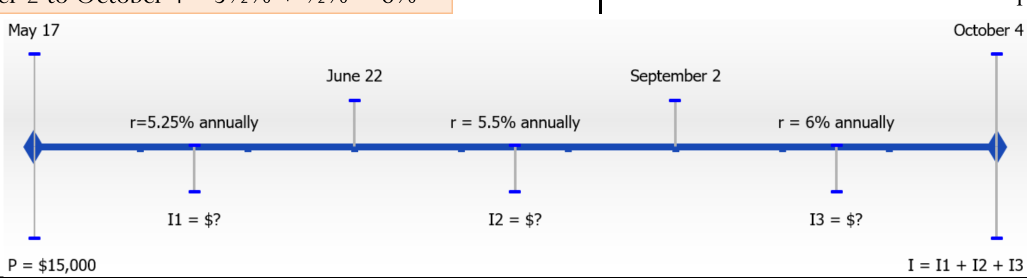
Date	Interest Rate
May 17 to June 22	2¼% + 3% = 5¼%
June 22 to September 2	5¼% + ¼% = 5½%
September 2 to October 4	5½% + ½% = 6%

How You Will Get There

Step 1 (continued): Calculate the number of days for each of the three time periods.

Step 2: The rates are annual while each of the times are in days. Convert each of the times to an annual number.

Step 3: For each time period, apply Formula 4.1.1. Step 3 (continued): Sum the three interest amounts to arrive at the total interest paid.



Perform

Dates	Step 1 (continued) Calculations	Total # of Days	Step 2 Convert t to annual	Step 3 Calculate interest
May 17 to June 22	May 17 to May 31 = 14 days May 31 to June 22 = 22 days	14 + 22 = 36 days	$\frac{36}{365}$	$I_1 = \$15{,}000(0.0525)\left(\frac{36}{365}\right) = \$77.671232$
June 22 to September 2	June 22 to June 30 = 8 days June 30 to July 31 = 31 days July 31 to August 31 = 31 days August 31 to Sept 2 = 2 days	8 + 31 + 31 + 2 = 72 days	$\frac{72}{365}$	$I_2 = \$15{,}000(0.055)\left(\frac{72}{365}\right) = \$162.739726$
September 2 to October 4	Sept 2 to Sept 30 = 28 days Sept 30 to October 4 = 4 days	28 + 4 = 32 days	$\frac{32}{365}$	$I_3 = \$15{,}000(0.06)\left(\frac{32}{365}\right) = \$78.904109$
			Step 3 (continued): TOTAL	$319.32

Present

Julio owes $319.32 of simple interest on his October 4 repayment date.

Maturity Value (or Future Value)

The **maturity value** of a transaction is the amount of money resulting at the end of a transaction, an amount that includes both the interest and the principal together. It is called a maturity value because in the financial world the termination of a financial transaction is known as the "maturing" of the transaction. The amount of principal with interest at some point in the future, but not necessarily the end of the transaction, is known as the **future value**.

The Formula

For any financial transaction involving simple interest, the following is true:

Amount of money at the end = Amount of money at the beginning + Interest

Applying algebra, you can summarize this expression by the following equation, where the future value or maturity value is commonly denoted by the symbol S:

$$S = P + I$$

Substituting in Formula 4.1.1, $I = Prt$, yields the equation

$$S = P + Prt$$

Simplify this equation by factoring P on the right-hand side to arrive at Formula 4.1.2.

S is Future Value or Maturity Value: This is the combined value of the principal and the simple interest together at a future point in time.

P is Present Value or Principal: The amount of money at the beginning of the time period being analyzed. If this is in fact the amount at the start of the financial transaction, it is the principal. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

Formula 4.1.2 – Simple Interest for Single Payments: $S = P(1 + rt)$

r is Interest Rate: The interest rate in Formula 4.1.2 operates in the same manner as in Formula 4.1.1. It is the simple rate of interest charged or earned during the specified time period. It is usually expressed annually and must share a common unit with the time period.

t is Time: This is the length of time for the financial transaction in which interest is charged or earned. It must be expressed in a common unit with the interest rate.

Depending on the financial scenario, what information you know, and what variable you need to calculate, you may need a second formula offering an alternative method of calculating the simple interest dollar amount.

I is Interest Amount: The interest amount is the dollar amount of interest earned or charged over the span of a financial transaction involving a single payment.

S is Future Value or Maturity Value: The amount of money in the future (or at maturity) including both the principal (or present value) and the interest together.

Formula 4.1.3 – Interest Amount for Single Payments: $I = S - P$

P is Present Value or Principal: The amount of money at the start of the transaction or at some other time before maturity that excludes the interest involved in the transaction.

How It Works

Follow these steps when working with single payments involving simple interest:

Step 1: Formula 4.1.2 has four variables, any three of which you must identify to work with a single payment involving simple interest.

- If either the P or S is missing but the *I* is known, apply Formula 4.1.3 to determine the unknown value.
- If only the interest amount *I* is known and neither P nor S is identified, recall that the interest amount is a portion and that the interest rate is a product of rate and time (e.g., 6% annually for nine months is an earned interest rate of 4.5%). Apply the calculation rate shown in Formula 4.1.1 to solve for the base which is the present value or P.
- If necessary, draw a timeline to illustrate how the money is being moved through time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If not, you need to convert one of the two variables to achieve the commonality.

Step 3: Apply Formula 4.1.2 and solve for the unknown variable, manipulating the formula as required.

Step 4: If you need to calculate the amount of interest, apply Formula 4.1.3.

Assume that today you have \$10,000 that you are going to invest at 7% simple interest for 11 months. How much money will you have in total at the end of the 11 months? How much interest do you earn?

Step 1: The principal is P = \$10,000, the simple interest rate is 7% per year, or $r = 0.07$, and the time is $t = 11$ months.

Step 2: The time is in months, but to match the rate it needs to be expressed annually as $t = \frac{11}{12}$.
Step 3: Applying Formula 4.1.2 to calculate the future value including interest,
$S = \$10,000 \times (1 + 0.07 \times \frac{11}{12}) = \$10,641.67$. This is the total amount after 11 months.
Step 4: Applying Formula 4.1.3 to calculate the interest earned, $I = \$10,641.67 - \$10,000.00 = \$641.67$. You could also apply Formula 4.1.1 to calculate the interest amount; however, Formula 8.3 is a lot easier to use. The \$10,000 earns \$641.67 in simple interest over the next 11 months, resulting in \$10,641.67 altogether.

Things To Watch Out For

As with Formula 4.1.1, the most common mistake in the application of Formula 4.1.2 is failing to ensure that the rate and time are expressed in the same units. Before you proceed, always check these two variables!

Paths To Success

When solving Formula 4.1.2 for either rate or time, it is generally easier to use Formulas 4.1.3 and 4.1.1 instead. Follow these steps to solve for rate or time:

1. If you have been provided with both the present value and future value, apply Formula 8.3 to calculate the amount of interest (I).
2. Apply and rearrange Formula 4.1.1 to solve for either rate or time as required.

Example 4.1F: Saving for a Down Payment on a Home

You just inherited \$35,000 from your uncle's estate and plan to purchase a house four months from today. If you use your inheritance as your down payment on the house, how much will you be able to put down if your money earns 4¼% simple interest? How much interest will you have earned?

Plan

Calculate the amount of money four months from now including both the principal and interest earned. This is the maturity value (S). Also calculate the interest earned (I).

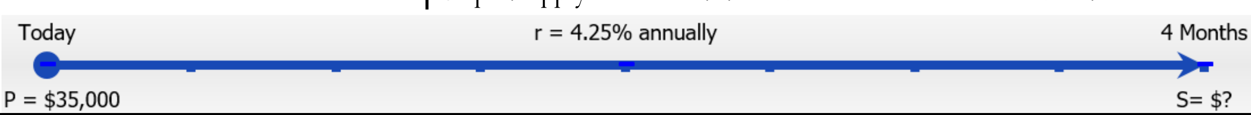
Understand

What You Already Know

Step 1: The principal, interest rate, and time frame are known:
P = \$35,000 r = 4¼% per year
t = 4 months

How You Will Get There

Step 2: While the rate is annual, the time is in months. Convert the time to an annual number.
Step 3: Apply Formula 4.1.2.
Step 4: Apply Formula 4.1.3 to calculate the interest amount.



Perform

Step 2: Four months out of 12 months in a year is $t = \frac{4}{12}$

Step 3: $S = \$35,000 \times (1 + 4\tfrac{1}{4}\% \times \frac{4}{12}) = \$35,000 \times (1 + 0.0425 \times 0.\overline{3}) = \$35,495.83$

Step 4: $I = \$35,495.83 - \$35,000.00 = \$495.83$

Present

Four months from now you will have \$35,495.83 as a down payment toward your house, which includes \$35,000 in principal and \$495.83 of interest.

Example 4.1G: Saving for Tuition

Recall the section opener, where you needed \$8,000 for tuition in the fall and the best simple interest rate you could find was 4.5%. Assume you have eight months before you need to pay your tuition. How much money do you need to invest today?

Plan

Calculate the principal amount of money today (P) that you must invest such that it will earn interest and end up at the \$8,000 required for the tuition.

Understand

What You Already Know

Step 1: The future value, interest rate, and time period are known:

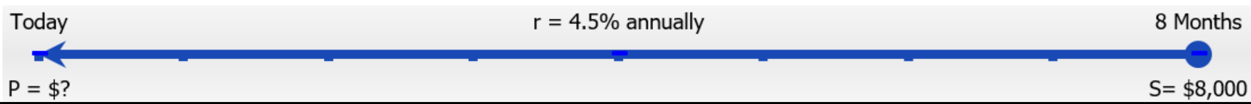
S = \$8,000 r = 4.5% per year

t = 8 months

How You Will Get There

Step 2: While the rate is annual, the time is in months. Convert the time to an annual number.

Step 3: Apply Formula 4.1.2, rearranging for P.



Perform

Step 2: Eight months out of 12 months in a year is $t = \frac{8}{12}$

Step 3: $\$8{,}000 = P(1 + 4.5\% \times \frac{8}{12})$

$$P = \frac{\$8{,}000}{(1+0.045 \times 0.\overline{6})} = \$7{,}766.99$$

Present

If you place \$7,766.99 into the investment, it will grow to \$8,000 in the eight months.

Example 4.1H: What Exactly Are You Being Offered?

You are sitting in an office at your local financial institution on August 4. The bank officer says to you, "We will make you a great deal. If we advance that line of credit and you borrow \$20,000 today, when you want to repay that balance on September 1 you will only have to pay us \$20,168.77, which is not much more!" Before answering, you decide to evaluate the statement. Calculate the simple interest rate that the bank officer used in her calculations.

Plan

Determine the rate of interest that you would be charged on your line of credit, or r.

Understand

What You Already Know

Step 1: The principal, maturity amount, and time frame are known:

P = \$20,000.00

S = \$20,168.77

t = August 4 to September 1

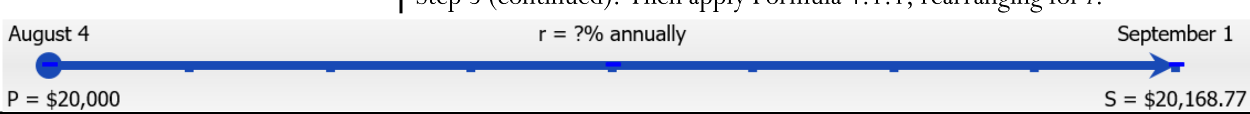
How You Will Get There

Step 1 (continued): Calculate the number of days in the transaction.

Step 2: Since interest rates are usually expressed annually, convert the time from days to an annual number.

Step 3: Use the combination of Formulas 4.1.3 and 4.1.1. First calculate the interest amount using Formula 4.1.3.

Step 3 (continued): Then apply Formula 4.1.1, rearranging for r.



Perform

Step 1 (continued):

Start Date	End Date	Days between Dates
August 4	August 31	$31 - 4 = 27$
August 31	September 1	$1 - 0 = 1$
TOTAL		$27 + 1 = 28$ days

Step 2: 28 days out of 365 days in a year is $t = \frac{28}{365}$

Step 3: $I = \$20{,}168.77 - \$20{,}000.00 = \$168.77$

Step 3 (continued): $\$168.77 = \$20{,}000 \times r \times \frac{28}{365}$

$$r = \frac{\$168.77}{\$20{,}000.00 \times \frac{28}{365}} = 0.110002 \text{ or } 11.0002\%$$

Calculator Instructions

Assume the year 2011 and use the DATE function:

DT1	DT2	DBD	Mode
08.0411	09.0111	Answer: 28	ACT

Present

The interest rate on the offered line of credit is 11.0002% (note that it is probably exactly 11%; the extra 0.0002% is most likely due to the rounded amount of interest used in the calculation).

Equivalent Payments

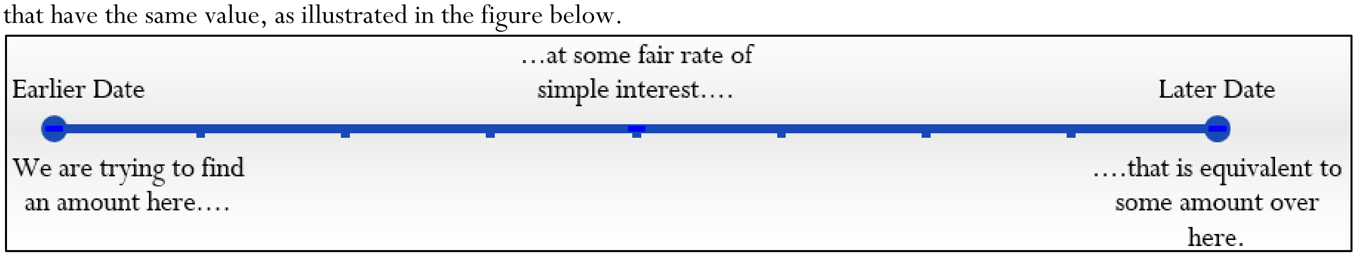
Life happens. Sometimes the best laid financial plans go unfulfilled. Perhaps you have lost your job. Maybe a reckless driver totalled your vehicle, which you now have to replace at an expense you must struggle to fit into your budget. No matter the reason, you find yourself unable to make your debt payment as promised.

On the positive side, maybe you just received a large inheritance unexpectedly. What if you bought a scratch ticket and just won $25,000? Now that you have the money, you might want to pay off that debt early. Can it be done?

Whether paying late or paying early, any amount paid must be equivalent to the original financial obligation. As you have learned, when you move money into the future it accumulates simple interest. When you move money into the past, simple interest must be removed from the money. This principle applies both to early and late payments:

- **Late Payments.** If a debt is paid late, then a financial penalty that is fair to both parties involved should be imposed. That penalty should reflect a current rate of interest and be added to the original payment. Assume you owe $100 to your friend and that a fair current rate of simple interest is 10%. If you pay this debt one year late, then a 10% late interest penalty of $10 should be added, making your debt payment $110. This is no different from your friend receiving the $100 today and investing it himself at 10% interest so that it accumulates to $110 in one year.
- **Early Payments.** If a debt is paid early, there should be some financial incentive (otherwise, why bother?). Therefore, an interest benefit, one reflecting a current rate of interest on the early payment, should be deducted from the original payment. Assume you owe your friend $110 one year from now and that a fair current rate of simple interest is 10%. If you pay this debt today, then a 10% early interest benefit of $10 should be deducted, making your debt payment today $100. If your friend then invests this money at 10% simple interest, one year from now he will have the $110, which is what you were supposed to pay.

Notice in these examples that a simple interest rate of 10% means $100 today is the same thing as having $110 one year from now. This illustrates the concept that two payments are **equivalent payments** if, once a fair rate of interest is factored in, they have the same value on the same day. Thus, in general you are finding two amounts at different points in time that have the same value, as illustrated in the figure below.



How It Works

The steps required to calculate an equivalent payment are no different from those for single payments. If an early payment is being made, then you know the future value, so you solve for the present value (which removes the interest). If a late payment is being made, then you know the present value, so you solve for the future value (which adds the interest penalty).

Paths To Success

TOP SECRET

Being financially smart means paying attention to when you make your debt payments. If you receive no financial benefit for making an early payment, then why make it? The prudent choice is to keep the money yourself, invest it at the best interest rate possible, and pay the debt off when it comes due. Whatever interest is earned is yours to keep and you still fulfill your debt obligations in a timely manner!

Example 4.1J: Making a Late Payment

Erin owes Charlotte \$1,500 today. Unfortunately, Erin had some unexpected expenses and is unable to make her debt payment. After discussing the issue, they agree that Erin can make the payment nine months late and that a fair simple interest rate on the late payment is 5%. How much does Erin need to pay nine months from now? What is the amount of her late penalty?

Plan

A late payment is a future value amount (S). The late penalty is equal to the interest (*I*).

Understand

What You Already Know

Step 1: The payment today, the agreed-upon interest rate, and the late payment timing are known:

P = \$1,500 r = 5% annually

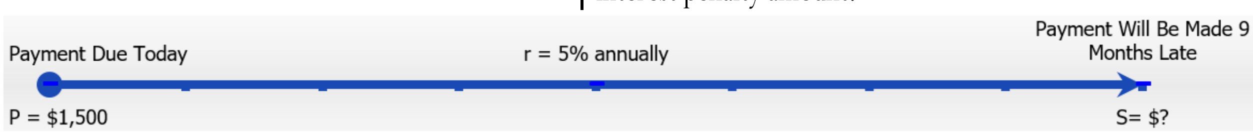
t = 9 months

How You Will Get There

Step 2: While the rate is annual, the time is in months. Convert the time to an annual number.

Step 3: Apply Formula 4.1.2.

Step 4: Apply Formula 4.1.3 to calculate the late interest penalty amount.



Perform

Step 2: Nine months out of 12 months in a year is $t = \frac{9}{12}$.

Step 3: $S = \$1,500 \times (1 + 5\% \times \frac{9}{12}) = \$1,500 \times (1 + 0.05 \times 0.75) = \$1,556.25$

Step 4: $I = \$1,556.25 - \$1,500.00 = \$56.25$

Erin's late payment is for \$1,556.25, which includes a \$56.25 interest penalty for making the payment nine months late.

Example 4.1K: Making an Early Payment

Rupert owes Aminata two debt payments: $600 four months from now and $475 eleven months from now. Rupert came into some money today and would like to pay off both of the debts immediately. Aminata has agreed that a fair interest rate is 7%. What amount should Rupert pay today? What is the total amount of his early payment benefit?

Plan

An early payment is a present value amount (P). Both payments will be moved to today and summed. The early payment benefit will be the total amount of interest removed (*I*).

Understand

What You Already Know

Step 1: The two payments, interest rate, and payment due dates are known:

r = 7% annually

First Payment: S = $600

t = 4 months from now

Second Payment: S = $475

t = 11 months from now

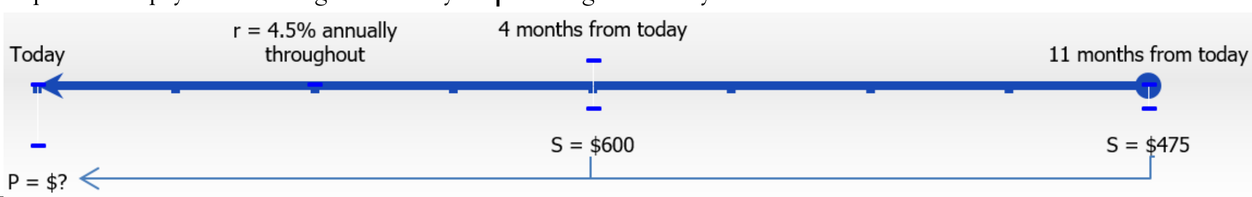
Replacement payment is being made today.

How You Will Get There

Do steps 2 and 3 for each payment:

Step 2: While the rate is annual, the time is in months. Convert the time to an annual number.

Step 3: Apply Formula 4.1.2, rearranging it to solve for P. Once you calculate both P, they can be summed to arrive at the total payment. Step 4: For each payment, apply Formula 4.1.3 to calculate the associated early interest benefit. Total the two values of *I* to get the early interest benefit overall.



Perform

Payment #1:

Step 2: Four months out of 12 months in a year is $t = \frac{4}{12}$

Step 3: $\$600 = P\left(1 + 7\% \times \frac{4}{12}\right)$

$$P = \frac{\$600}{(1+0.07\times 0.\overline{3})} = \$586.32$$

Payment #2:

Step 2: Eleven months out of 12 months in a year is $t = \frac{11}{12}$

Step 3: $\$475 = P\left(1 + 7\% \times \frac{11}{12}\right)$

$$P = \frac{\$475}{(1+0.07\times 0.91\overline{6})} = \$446.36$$

Totals:

P today = $586.32 + $446.36 = $1,032.68

Step 4:

Payment #1: *I*=S–P = $600 – $586.32 = $13.68

Payment #2: *I*=S–P = $475 – $446.36 = $28.64

I = $13.68 + $28.64 = $42.32

Present

To clear both debts today, Rupert pays $1,032.68, which reflects a $42.32 interest benefit reduction for the early payment.

The Concept of Compounding

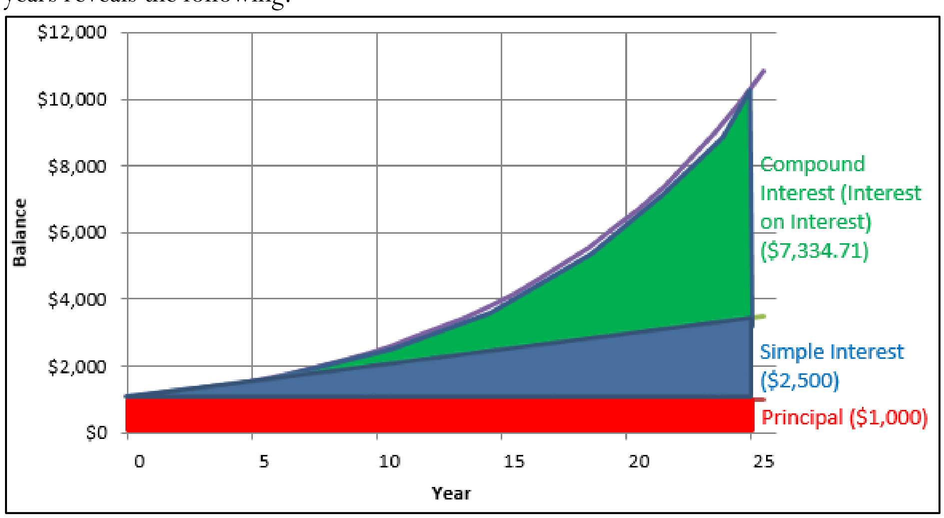
Simple interest, compound interest—what is the big difference? As you can see in the figure below, simple and compound interest calculations share the same fundamentals of time, interest rate, and placing interest into the account. But the accumulation is not the same, and over time the growth of compound interest will far outpace that of simple interest.

In both forms of interest, the principal is the starting amount that accumulates interest for a length of time at a specified interest rate. But you calculate simple interest in direct proportion to the starting amount, the interest rate, and the time period, whereas the calculation for compound interest is, well, not so simple!

Simple Interest	Characteristics	Compound Interest
✓	Principal	✓
Expressed in units of time (such as days, months, or years)	Time	Expressed as the number of times that interest is calculated
Per unit of time (such as days, months, or years)	Interest Rate	Per calculation of interest
At the end of the transaction's time frame	When Interest Is Placed into Account	With every calculation of interest

The critical difference is the placement of interest into the account. Under simple interest, you convert the interest to principal at the end of the transaction's time frame. For example, in a six-month simple interest GIC the balance in its account at any point before the maturity date is the original principal and nothing more. Only upon maturity does the interest appear. In contrast, a five-year compound interest GIC such as the one discussed in the section opener receives an interest deposit annually. After one year the principal increases from \$15,000 to \$15,892.50. This higher principal in the second year explains why the interest earned in the second year then increases as well.

Invest \$1,000 for 25 years at both 10% simple interest and 10% interest compounded annually. As demonstrated in the figure and table below, \$1,000 invested at 10% per year of simple interest has a \$3,500 balance after 25 years. This consists of the original \$1,000 principal plus \$2,500 in total interest (\$100 for each year the money was invested). However, \$1,000 invested at 10% compounded annually has a balance of \$10,834.71, which is \$7,334.71 more! This difference is the result of interest being converted to principal and thus earning even more interest for you. A close examination of the first two years reveals the following:

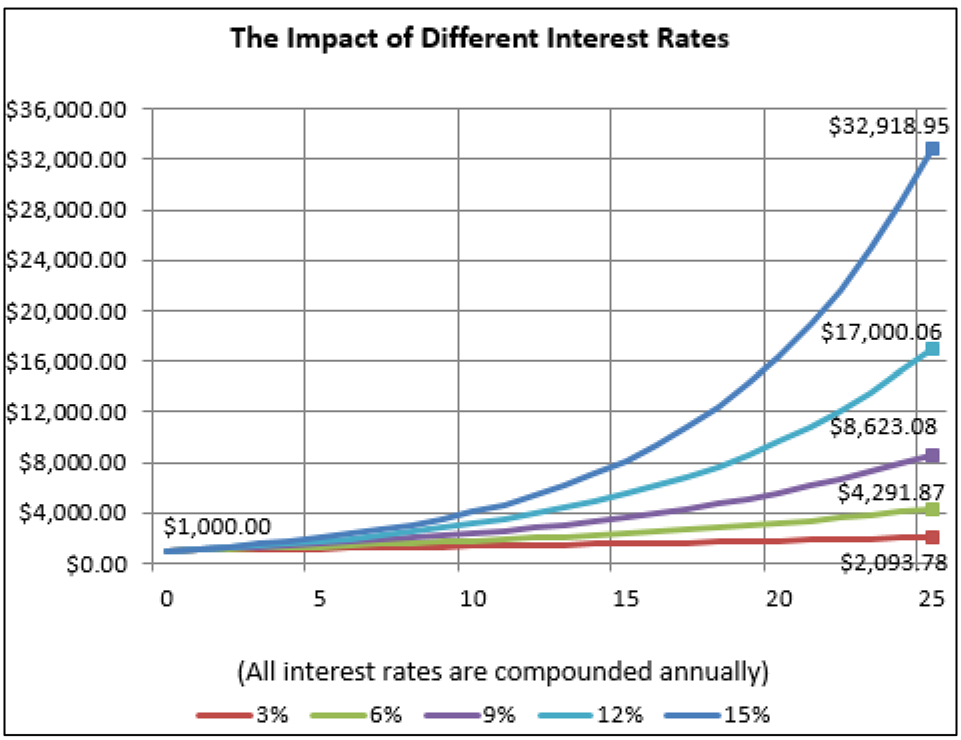


- In the first year, both investments have a \$1,100 balance. With simple interest, the \$100 of interest remains as accrued interest and is not placed into the account while the principal remains at \$1,000. With compound interest, the \$100 of interest is converted to principal, resulting in a \$1,100 principal for the second year.
- In the second year, the simple interest account still has a \$1,000 principal, which earns another \$100 of interest (\$1,000 × 10% = \$100). The account now has \$1,000 of principal plus \$200 in accrued interest. The compound interest account earns 10% on the new principal of \$1,100, or \$110 of interest (\$1,100 × 10% = \$110). This interest is placed into the account at the end of year two, making the principal \$1,210 for the third year . . . and so on.

End of Year	At 10% Simple Interest Principal	Accrued Interest	Balance	At 10% Compound Interest Principal	Accrued Interest	Balance
Start						$1,000.00
1	$1,000.00	$100.00	$1,100.00	$1,000.00	$100.00	$1,100.00
2	$1,000.00	$200.00	$1,200.00	$1,100.00	$110.00	$1,210.00
3	$1,000.00	$300.00	$1,300.00	$1,210.00	$121.00	$1,331.00
4	$1,000.00	$400.00	$1,400.00	$1,331.00	$133.10	$1,464.10
5	$1,000.00	$500.00	$1,500.00	$1,464.10	$146.41	$1,610.51
	…..	…..	…..	…..	…..	…..
10	$1,000.00	$1,000.00	$2,000.00	$2,357.95	$235.79	$2,593.74
	…..	…..	…..	…..	…..	…..
15	$1,000.00	$1,500.00	$2,500.00	$3,797.50	$379.75	$4,177.25
	…..	…..	…..	…..	…..	…..
20	$1,000.00	$2,000.00	$3,000.00	$6,115.91	$611.59	$6,727.50

Observe that the difference in the balance between simple interest and compound interest in the first five years is slight, but the gap widens over time.

Compound Interest Rates



If an equal amount of principal invested for 25 years earns an interest rate of 9% compounded annually versus 3% compounded annually, would the interest accumulated by the 9% investment be three times as large?

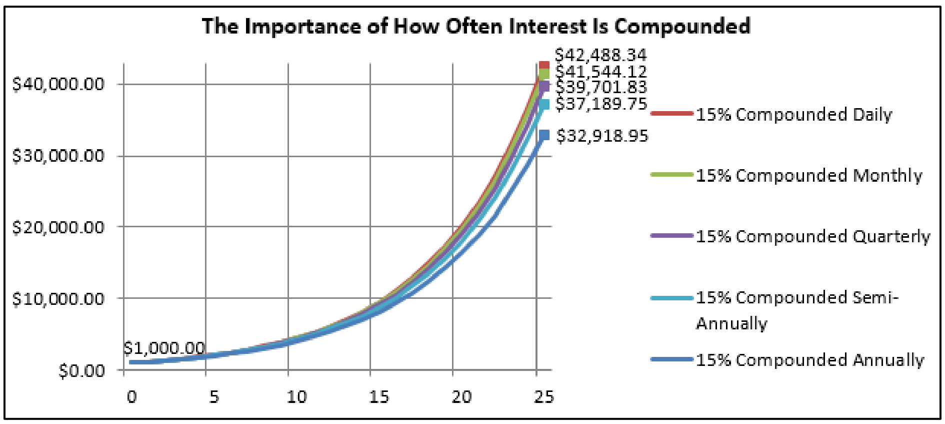
In a *simple interest* environment, this would be true since the principal never changes: in each of the 25 years the amounts of simple interest would be constant, and the amount for the 9% investment would be three times larger than for the 3% investment. However, in *compound interest* each subsequent year's interest is compounded on an increasingly larger principal. This chart shows $1,000 invested at five different annually compounded rates. Observe the following after 25 years:

- The investment at 9% compounded annually has a balance of $8,623.08 compared to $2,093.78 for the 3% compounded annually. Therefore, the interest earned is $7,623.08 versus $1,093.78, reflecting a ratio approximating 7 : 1.
- Comparing the 15% interest rate versus the 6% interest rate, many people imagine the final balances would be in the same ratio as the ratio of the rates, 15% : 6%, which simplifies to 2½ : 1. However, in the chart, the final balances are $32,918.95 versus $4,291.87, reflecting a ratio approximating 7⅔ : 1!

These examples illustrate that in compound interest scenarios, the ratio of interest earned is not directly proportionate to the numerical ratio of the interest rates. Higher interest rates result in higher interest amounts, therefore yielding more principal upon which future interest is earned. Over a long time period, this growth in the balance is exponential.

How Often Interest Compounds

The examples so far have involved compound interest that has been compounded annually—the accrued interest is being converted to principal at the end of every year. But this is not the only option. Interest can be converted to principal at any frequency, including daily, weekly, monthly, quarterly (every three months), or semi-annually (every six months). Under any of these options the principal increases more frequently, which in turn results in more interest being earned.



The chart to the left illustrates the impact of different compounding frequencies. In the case of monthly versus annual compounding, the monthly compound increases the account balance every month whereas the annual compound increases the balance once every 12 months. To understand why the monthly compound earns more interest, note that after each month the principal increases and therefore earns more interest. Each of the 12 increases is small on its own, but the cumulative effect is large: by having the interest deposited every month instead of every year, over the course of 25 years an additional $\$41{,}544.12 - \$32{,}918.95 = \$8{,}625.17$ in interest is earned!

TOP SECRET

Paths To Success

The table below summarizes the desirable characteristics when either investing or borrowing.

Investing	Characteristic	Borrowing
Compound You want your interest to earn you more interest.	**Type of Interest**	Simple You don't want interest being converted to principal.
The higher the rate, the more you earn.	**Interest Rate**	The lower the rate, the less you pay.
The more often it compounds, the more principal that can earn interest. A daily compound would be best!	**Compounding**	The less often it compounds, the less principal that can earn interest. An annual compound would be best!
As the principal continually grows, you earn more interest. Longer time frames are desirable.	**Time (or Term)**	You don't want the principal to grow and earn more interest. Short time frames are desirable.

Calculating the Periodic Interest Rate

The first step in learning about investing or borrowing under compound interest is to understand the interest rate used in converting interest to principal. You commonly need to convert the posted interest rate to find the exact rate of interest earned or charged in any given time period.

The Formula

Formula 4.1.4 involves four key concepts, which are explained next, to do with conversion of the interest rate.

Formula 4.1.4 - Periodic Interest Rate: $$i = \frac{IY}{CY}$$

CY is Compounds Per Year (Compounding Frequency): The determination of CY involves two of the key concepts:
Key Concept #1: The Compounding Period. The amount of time that elapses between the dates of successive conversions of interest to principal is known as the **compounding period**. For example, a quarterly compounded interest rate converts interest to principal every three months. Therefore, the compounding period is three months.
Key Concept #2: The Compounding Frequency. The number of compounding periods in a complete year is known as the **compounding frequency**. For example, a quarterly compounded interest rate compounds every three months, or CY = 4 times in a single year.

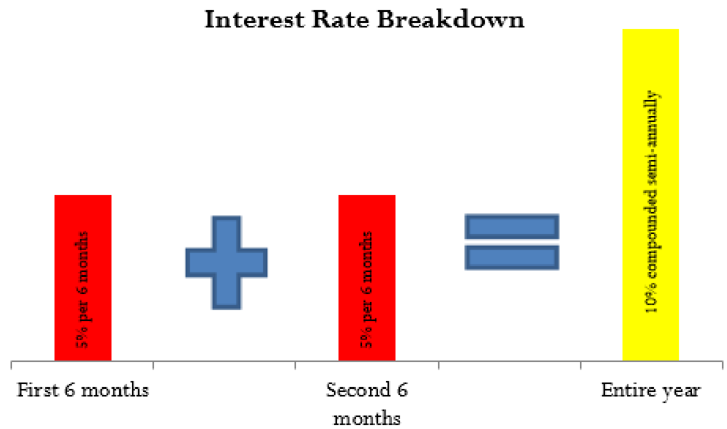
IY is Nominal Interest Rate Per Year: Key Concept #3: The Nominal Interest Rate. A compound interest rate consists of two elements: a nominal number for the annual interest rate, known as the **nominal interest rate**, and words that state the compounding frequency. For example, a 12% compounded quarterly interest rate is interpreted to mean that you accumulate 12% nominal interest per year but the interest is converted to principal every compounding period, or every three months. The word *nominal* is used because if compounding occurs more than once per year, the true amount of interest that you earn per year is greater than the nominal interest rate. Section 9.6 will help you calculate just how much more.

i is Periodic Interest Rate: Key Concept #4: The Periodic Interest Rate. The percentage of interest earned or charged at the end of each compounding period is called the **periodic interest rate**. You calculate it by taking the nominal interest rate and dividing by the compounding frequency. Continuing with the example of 12% compounded quarterly, it has a periodic interest rate of 12% ÷ 4 = 3%. This means that at the end of every three months, you calculate 3% interest and convert it to principal.

How It Works

To solve any question involving the periodic interest rate, follow these steps:

Step 1: Identify two of the three key variables—the nominal interest rate (IY), compounding frequency (CY), or periodic interest rate (i).



Step 2: Substitute into Formula 9.1, rearranging if needed, and solve for the unknown variable. If you calculate the compound frequency (CY), you must convert the number back into the compounding words associated with the frequency. For example, CY = 2 means twice per year and is stated as "semi-annually."

An example involving the calculation of the periodic interest rate for "10% compounded semi-annually" illustrates these steps.
Step 1: The wording "semi-annually" means the compounding period is every six months. One year contains two such compounding periods, making the compounding frequency twice per year, or CY = 2. The nominal annual interest rate is 10%, or IY = 10%.
Step 2: Applying Formula 9.1, calculate the periodic interest rate as $i = 10\% \div 2 = 5\%$. Figure 9.6 illustrates that every six months 5% interest is converted to principal. You could look at the nominal interest rate for the year, 10%, as the total of the periodic interest rates for each of the two periods. But remember, unlike with simple interest, the actual interest amount here will increase from one period to the next.

Important Notes

When you express a compound interest rate, you must always state the words for the compounding frequency along with the nominal annual number. For example, you must say "10% compounded semi-annually" and not just "10%." In the absence of an explicit compounding frequency, the number is interpreted by default to mean "compounded annually" except if an industry standard dictates otherwise. For example, in the case of mortgages in Canada, the default is semi-annual compounding, so when you hear of a 10% mortgage, you should assume it to be 10% compounded semi-annually. But in most industries "10%" with no words generally means "10% compounded annually."

Things To Watch Out For

It is common to confuse the compounding period and the compounding frequency. The table below shows the relationship between compounding periods and frequencies. Remember that to calculate the periodic interest rate you need the compounding frequency, not the compounding period.

Common Compounds	Compounding Period	Compounding Frequency (CY)
Annually	Every year	1
Semi-annually	Every 6 months	2
Quarterly	Every 3 months	4
Monthly	Every 1 month	12
Weekly	Every 1 week	52*
Daily	Every day	365*

*Note that although there are not exactly 52 weeks in a year and there are 366 days in a leap year, it is common practice to use these values for compounding frequency.

Example 4.1L: The Periodic Interest Rate

Calculate the periodic interest rate for the following nominal interest rates:
a. 9% compounded monthly b. 6% compounded quarterly

Plan

For each question, calculate the periodic interest rate (*i*).

What You Already Know

Step 1: For each question, use the following nominal interest rate and compounding frequency:
a. IY = 9% CY = monthly = 12 times per year
b. IY = 6% CY = quarterly = 4 times per year

How You Will Get There

Step 2: For each question apply Formula 4.1.4.

a. $i = \frac{9\%}{12} = 0.75\%$ per month

b. $i = \frac{6\%}{4} = 1.5\%$ per quarter

Present

a. Nine percent compounded monthly is equal to a periodic interest rate of 0.75% per month. This means that interest is converted to principal 12 times throughout the year at the rate of 0.75% each time.
b. Six percent compounded quarterly is equal to a periodic interest rate of 1.5% per quarter. This means that interest is converted to principal 4 times (every three months) throughout the year at the rate of 1.5% each time.

Example 4.1M: The Nominal Interest Rate

Calculate the nominal interest rate for the following periodic interest rates:
a. $0.58\overline{3}\%$ per month b. 0.05% per day

Plan

For each question, calculate the nominal interest rate (IY).

What You Already Know

Step 1: For each question, use the following periodic interest rate and compounding frequency:
a. $i = 0.58\overline{3}\%$ CY = monthly = 12 times per year
b. $i = 0.05\%$ CY = daily = 365 times per year

How You Will Get There

Step 2: For each question, apply Formula 4.1.4 and rearrange for IY.

a. $0.58\overline{3}\% = \frac{\text{IY}}{12}$ IY = $0.58\overline{3}\% \times 12 = 7\%$ compounded monthly

b. $0.05\% = \frac{\text{IY}}{365}$ IY = $0.05\% \times 365 = 18.25\%$ compounded daily

a. A periodic interest rate of $0.58\overline{3}\%$ per month is equal to a nominal interest rate of 7% compounded monthly.
b. A periodic interest rate of 0.05% per day is equal to a nominal interest rate of 18.25% compounded daily.

Example 4.1N: Compounds per Year	
Calculate the compounding frequency for the following nominal and periodic interest rates: a. nominal interest rate = 6%, periodic interest rate = 3% b. nominal interest rate = 9%, periodic interest rate = 2.25%	
Plan	For each question, calculate the compounding frequency (CY) and convert the calculated number to words.
	What You Already Know Step 1: For each question, use the following nominal and periodic interest rates: a. IY = 6% i = 3% b. IY = 9% i = 2.25% How You Will Get There Step 2: For each question, apply Formula 4.1.4 and rearrange for CY.
	a. $3\% = \frac{6\%}{\text{CY}}$ $\text{CY} = \frac{6\%}{3\%} = 2$ compounds per year = semi-annually b. $2.25\% = \frac{9\%}{CY}$ $\text{CY} = \frac{9\%}{2.25\%} = 4$ compounds per year = quarterly
	a. For the nominal interest rate of 6% to be equal to a periodic interest rate of 3%, the compounding frequency must be twice per year, which means a compounding period of every six months, or semi-annually. b. b. For the nominal interest rate of 9% to be equal to a periodic interest rate of 2.25%, the compounding frequency must be four times per year, which means a compounded period of every three months, or quarterly.

Future Value Calculations with No Variable Changes

The simplest future value scenario for compound interest is for all of the variables to remain unchanged throughout the entire transaction. To understand the derivation of the formula, continue with the opening scenario. If the money was borrowed two years ago, then the employee will owe two years of compound interest in addition to the original principal of \$4,000. That means PV = \$4,000. The compounding frequency is semi-annually, or twice per year, which makes the periodic interest rate $i = \frac{12\%}{2} = 6\%$. Therefore, after the first six months, your employee has 6% interest converted to principal. This a future value, or FV, calculated as follows:

Principal after one compounding period (six months) = Principal plus interest

$$
\begin{aligned}
\text{FV} &= \text{PV} + i(\text{PV}) \\
&= \$4{,}000 + 0.06(\$4{,}000) \\
&= \$4{,}000 + \$240 = \$4{,}240 \\
&= \$4{,}000 + 0.06(\$4{,}000) \\
&= \$4{,}000 + \$240 = \$4{,}240
\end{aligned}
$$

Now proceed to the next six months. The future value after two compounding periods (one year) is calculated in the same way. Note that the equation FV = PV + i(PV) can be factored and rewritten as FV = PV(1 + i).

$$\text{FV (after two compounding periods)} = \text{PV}(1 + i) = \$4{,}240(1 + 0.06) = \$4{,}240(1.06) = \$4{,}494.40$$

Since the PV = \$4,240 is a result of the previous calculation where PV(1 + i) = \$4,240, the following algebraic substitution is possible:

$$\text{FV} = \text{PV}(1 + i)(1 + i) = \$4{,}000(1.06)(1.06) = \$4{,}240(1.06) = \$4{,}494.40$$

Applying exponent rules from Section 2.4 and simplifying it algebraically, you get:

$$\text{FV} = \text{PV}(1 + i)(1 + i) = \text{PV}(1 + i)^2$$

Do you notice a pattern? With one compounding period, the formula has only one (1 + i). With two compounding periods involved, it has two factors of (1 + i). Each successive compounding period multiplies a further (1 + i) onto the equation. This makes the exponent on the (1 + i) exactly equal to the number of times that interest is converted to principal during the transaction.

The Formula

First, you need to know how many times interest is converted to principal throughout the transaction. You can then calculate the future value. Use Formula 4.1.5 to determine the number of compound periods involved in the transaction

N is Number of Compound Periods: This is the measure of time. Calculating the maturity value requires knowing how many times interest is converted to principal throughout the transaction. Whereas simple interest expresses time in years, note that compound interest requires time to be expressed in periods.

Formula 4.1.5 – Number of Compound Periods For Single Payments: $N = CY \times Years$

CY is Compounds per year (Compound Frequency): Recall that the compounding frequency represents how many compound periods fall within a single year. The words that accompany the nominal interest rate determine this number.

Years is Number of Years in Transaction (The Term): Multiply the compounding frequency by the term of the transaction, expressed as a number of years. Express partial years as mixed fractions. For example, 1 year and 9 months is $1\frac{9}{12}$ years.

Once you know N, substitute it into Formula 4.1.6, which finds the amount of principal and interest together at the end of the transaction, or the maturity value.

FV is Future Value or Maturity Value: This is the principal and compound interest together at the future point in time.

PV is Present Value or Principal: This is the starting amount upon which compound interest is calculated. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some earlier point in time. In any case, the amount excludes the future interest.

Formula 4.1.6– Compound Interest For Single Payments: $FV = PV(1 + i)^N$

i is Periodic Interest Rate: From Formula 4.1.4, the principal accrues interest at this rate every compounding period. For accuracy, you should never round this number.

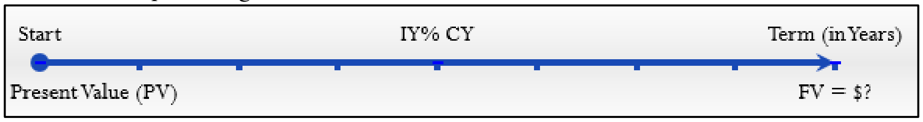
N is Number of Compound Periods: From Formula 9.2, this is the total number of compound periods involved in the term of the transaction.

The $(1+i)^N$ performs the actual compounding of the money. The $(1 + i)$ determines the percent increase in the principal while the exponent (N) compounds the increase an appropriate number of times.

How It Works

Follow these steps to calculate the future value of a single payment:

Step 1: Read and understand the problem. If necessary, draw a timeline similar to the one here identifying the present value, the nominal interest rate, the compounding, and the term.

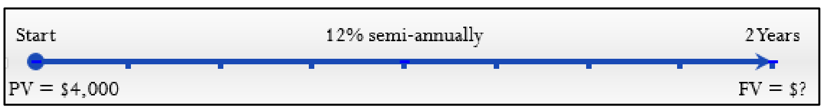


Step 2: Calculate the periodic interest rate (i) from Formula 4.1.4.

Step 3: Calculate the total number of compound periods (N) from Formula 4.1.5.

Step 4: Solve Formula 4.1.6.

Revisit the employee who had $4,000 outstanding for two years with interest at 12% compounded semi-annually. Step 1: Calculate the amount of the loan after two years (FV). Observe that PV = $4,000, IY = 12%, CY = 2 (every six months or twice per year), and Years = 2.



Step 2: According to Formula 4.1.4, $i = \frac{12\%}{2}$ 6%. Thus, interest at a rate of 6% is converted to principal at the end of each compounding period of six months.
Step 3: Applying Formula 4.1.5, N = CY × Years = 2 × 2 = 4. Interest is converted to principal four times over the course of the two-year term occurring at the 6, 12, 18, and 24 month marks.
Step 4: Calculate the maturity value using Formula 4.1.6:
$FV = \$4{,}000(1 + 0.06)^4 = \$5{,}049.91$

To pay off the loan the employee owes $5,049.91.

Things To Watch Out For

The most common error in the application of Formula 4.1.6 is to substitute the nominal interest rate for the periodic interest rate. Hence, for 12% compounded semi-annually you might inadvertently use a mistaken value of $i = 0.12$ instead of the appropriate $i = 0.06$. Formula 4.1.6 encompasses Formulas 4.1.4 and 4.1.5. If you write Formula 4.1.6 without requiring any substitution it appears as follows:

$$FV = PV\left(1 + \frac{IY}{CY}\right)^{(CY \times Years)}$$

Although you could use this equation instead of the one presented in Formula 4.1.6, most students find it best to use the sequence of three formulas. This requires the systematic approach involved in steps 2 to 4 of the process:

1. Calculate i.
2. Calculate N.
3. Calculate FV.

Use the phrase "**iN** the **F**uture" to remember this process.

Example 4.1O: Making an Investment

If you invested $5,000 for 10 years at 9% compounded quarterly, how much money would you have?

Plan

Calculate the principal and the interest together, which is called the maturity value (FV).

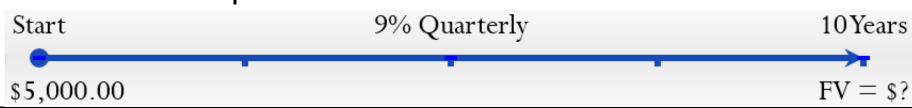
Understand

What You Already Know

Step 1: The principal, interest rate, and term, as illustrated in the timeline, are known.

How You Will Get There

Step 2: Calculate the periodic interest by applying Formula 4.1.4.
Step 3: Calculate the number of compound periods by applying Formula 4.1.5.
Step 4: Calculate the maturity value by applying Formula 4.1.6.



Perform

Step 2: $i = \frac{9\%}{4} = 2.25\% = 0.0225$
Step 3: N = 4 × 10 = 40
Step 4: FV = $5,000(1 + 0.0225)^{40} = $12,175.94

Calculator Instructions

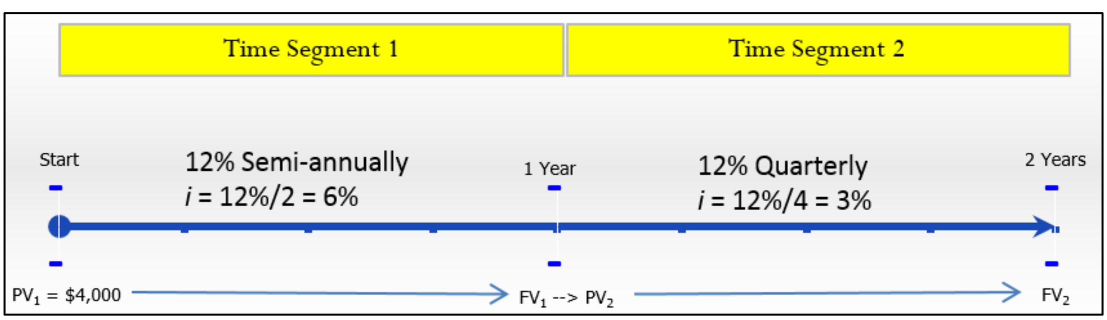
N	I/Y	PV	PMT	FV	P/Y	C/Y
40	9	−5,000	0	Answer: $12,175.94	4	4

After 10 years, the principal grows to $12,175.94, which includes your $5,000 principal and $7,175.94 of compound interest.

Future Value Calculations with Variable Changes

What happens if a variable such as the nominal interest rate, compounding frequency, or even the principal changes somewhere in the middle of the transaction? Formula 4.1.6 produces the correct final answer only when all variables remain unchanged. To illustrate this situation, assume your company modified its employee assistance plan one year after the money was borrowed, changing the interest rate in the second year from 12% compounded semi-annually to 12% compounded quarterly. Now how do you calculate the future value?

When any variable changes, you must break the timeline into separate time fragments at the point of the change. This timeline format is similar to those used involving variable simple interest rates. The timeline illustrates the employee's new scenario.



Solving for the unknown FV on the right of the timeline means that you must start at the left side of the timeline. To arrive at the solution, you need to work from left to right one time segment at a time using Formula 4.1.6. As noted on the timeline, at the one-year point the future value of the first time segment then becomes the present value for the second time segment since the interest is not just accrued but actually placed into the account.

How It Works

Follow these steps when variables change in calculations of future value based on lump-sum compound interest:

Step 1: Read and understand the problem. Identify the present value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.

Step 2: For each time segment, calculate the periodic interest rate (i) using Formula 4.1.4.

Step 3: For each time segment, calculate the total number of compound periods (N) using Formula 4.1.5.

Step 4: Starting with the present value in the first time segment (starting on the left), solve Formula 4.1.6.

Step 5: Let the future value calculated in the previous step become the present value for the next step. If the principal changes, adjust the new present value accordingly.

Step 6: Using Formula 4.1.6, calculate the future value of the next time segment.

Step 7: Repeat steps 5 and 6 until you obtain the final future value from the final time segment.

In the employee's new situation, he has borrowed \$4,000 for two years with 12% compounded semi-annually in the first year and 12% compounded quarterly in the second year.
Step 1: Figure above shows a timeline. In time segment one, $PV_1 = \$4{,}000$, IY = 12%, CY = 2, and the time segment is one year long. In time segment two, the only change is CY = 4.
Step 2: In the first time segment, the periodic interest rate is $i_1 = 12\%/2 = 6\%$. In the second time segment, the periodic interest rate is $i_2 = 12\%/4 = 3\%$.
Step 3: The first time segment is one year long; therefore, $N_1 = 2 \times 1 = 2$. The second time segment is also one year long; therefore, $N_2 = 4 \times 1 = 4$.
Step 4: Apply Formula 4.1.6 to the first time segment:

$$FV_1 = PV(1 + i_1)^{N1} = \$4{,}000(1 + 0.06)^2 = \$4{,}494.40$$

Step 5: Let $FV_1 = \$4{,}494.40 = PV_2$.
Step 6: Apply Formula 4.1.6 to the second time segment:

$$FV_2 = PV_2(1 + i_2)^{N2} = \$4{,}494.40(1 + 0.03)^4 = \$4{,}494.40 \times 1.03^4 = \$5{,}058.49$$

Step 7: You reach the end of the timeline. The employee needs to repay \$5,058.49 to clear the loan.

Things To Watch Out For

When you draw timelines, it is critical to recognize that any change in any variable requires a new time segment. This applies to changes in principal, the nominal interest rate, or the compounding frequency.

Example 4.1P: Delaying a Facility Upgrade

Five years ago Coast Appliances was supposed to upgrade one of its facilities at a quoted cost of \$48,000. The upgrade was not completed, so Coast Appliances delayed the purchase until now. The construction company that provided the quote indicates that prices rose 6% compounded quarterly for the first 1½ years, 7% compounded semi-annually for the following 2½ years, and 7.5% compounded monthly for the final year. If Coast Appliances wants to perform the upgrade today, what amount of money does it need?

Plan	Take the original quote and move it into the future with the price increases. You can view this as a single lump sum with multiple successive interest rates. The amount of money needed today is the maturity amount (FV).

Understand

What You Already Know

Step 1: The timeline below shows the original quote from five years ago until today. The principal, terms, and interest rates are known:

PV_1 = \$48,000

First time segment: IY = 6%

CY = quarterly = 4 Term = 1½ years

Second time segment: IY = 7%

CY = semi-annually = 2 Term = 2½ years

Third time segment: IY = 7.5%

CY = monthly = 12 Term = 1 year

How You Will Get There

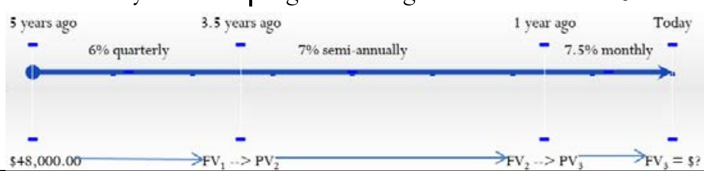
Step 2: For each time segment, calculate the periodic interest rate by applying Formula 4.1.4.

Step 3: For each time segment, calculate the number of compound periods by applying Formula 4.1.5.

Step 4/5: Calculate the future value of the first time segment using Formula 4.1.6. Let $FV_1 = PV_2$.

Step 6/5: Calculate the future value of the second time segment using Formula 4.1.6. Let $FV_2 = PV_3$.

Repeat Step 6/7: Calculate the future value of the third time segment using Formula 4.1.6. FV_3 is the final future value amount.



Perform

Step	1st Segment	2nd Segment	3rd Segment
2.	$i = \frac{6\%}{4} = 1.5\%$	$i = \frac{7\%}{2} = 3.5\%$	$i = \frac{7.5\%}{12} = 0.625\%$
3.	N = 4 × 1½ = 6	N = 2 × 2½ = 5	N = 12 × 1 = 12
4/5.	$FV_1 = \$48{,}000 \times (1 + 0.015)^6 = \$52{,}485.27667$		
6/5.		$FV_2 = \$52{,}485.27667 \times (1 + 0.035)^5 = \$62{,}336.04435$	
6/7.			$FV_3 = \$62{,}336.04435 \times (1 + 0.00625)^{12} = \$67{,}175.35$

Calculator Instructions

Segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	6	6	48000	0	Answer: −52,485.27677	4	4
2	5	7	52,485.27677	✓	Answer: −62,336.04435	2	2
3	12	7.5	62,336.04435	✓	Answer: −67,175.35	12	12

Present

Coast Appliances requires \$67,175.35 to perform the upgrade today. This consists of \$48,000 from the original quote plus \$19,175.35 in price increases.

Integer Compounding Periods

Some applications of solving for the number of compounding periods include the following:

- Determining the time frame to meet a financial goal
- Calculating the time period elapsing between a present and future value
- Evaluating the performance of financial investments

The Formula

To solve for the number of compounds you need Formula 4.1.6 one more time. The only difference from your previous uses of this formula is that the unknown variable changes from FV to N, which requires you to substitute and rearrange the formula algebraically.

How It Works

Follow these steps to compute the number of compounding periods (and ultimately the time frame):

Step 1: Draw a timeline to visualize the question. Most important at this step is to identify PV, FV, and the nominal interest rate (both IY and CY).

Step 2: Solve for the periodic interest rate (i) using Formula 4.1.1.

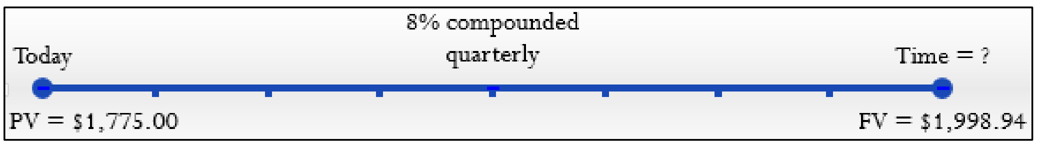
Step 3: Substitute into Formula 4.1.6, rearrange, and solve for N. Note that the value of N represents the number of compounding periods. For example, if the compounding is quarterly, a value of N = 9 is nine quarters.

Step 4: Take the value of N and convert it back to a more commonly expressed format such as years and months. When the number of compounding periods calculated in step 3 works out to an integer, handling step 4 involves applying the rearranged Formula 4.1.5 and solving for Years $= \frac{\text{N}}{\text{CY}}$

1. If the Years is an integer, you are done.
2. If the Years is a noninteger, the whole number portion (the part in front of the decimal) represents the number of years. As needed, take the decimal number portion (the part after the decimal point) and multiply it by 12 to convert it to months. For example, if you have Years = 8.25 then you have 8 years plus 0.25 × 12 = 3 months, or 8 years and 3 months.

Revisiting the opening scenario, your friend has saved $1,775 and needs it to become $1,998.94 at 8% compounded quarterly. How long will it take?

Step 1: The timeline illustrates this scenario. Note that IY = 8% and CY = quarterly = 4.



Step 2: The periodic interest rate is i = 8%/4 = 2%.

Step 3: Applying Formula 4.1.6, you have $\$1{,}998.94 = \$1{,}775(1 + 0.02)^{\text{N}}$ or N = 6 (details of the algebra can be found in subsequent examples).

Step 4: Applying the rearranged Formula 4.1.5, Years $= \frac{6}{4} = 1.5$. Your friend will be headed to the Mayan Riviera in 1½ years.

If you prefer to express this in months, it is 1 year plus 0.5 × 12 = 6 months, or 1 year and 6 months.

Example 4.1Q: Integer Compounding Period Investment

Jenning Holdings invested $43,000 at 6.65% compounded quarterly. A report from the finance department shows the investment is currently valued at $67,113.46. How long has the money been invested?

Plan

Determine the amount of time that the principal has been invested. This requires calculating the number of compounding periods (N).

Understand

What You Already Know

Step 1: The principal, future value, and interest rate are known, as illustrated in the timeline.

IY = 6.65% CY = 4

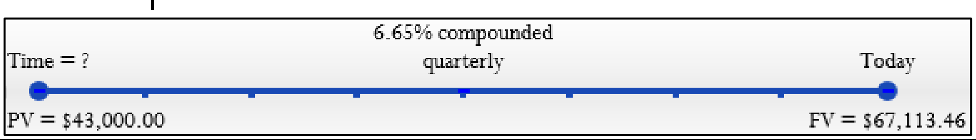
How You Will Get There

Step 2: Apply Formula .

Step 3: Substitute into Formula 4.1.6, rearrange, and solve for N.

Recall from Module 3 that to solve an equation for an unknown exponent you take the logarithm of both sides.

Step 4: Apply Formula 4.1.5, rearrange, and solve for Years.



Perform

Step 2: $i = \frac{6.65\%}{4} = 1.6625\%$

Step 3: $\$67{,}113.46 = \$43{,}000(1 + 0.016625)^{\text{N}}$

$1.560778 = 1.016625^{\text{N}}$

$\ln(1.560778) = \ln(1.016625^{\text{N}}) = \text{N} \times \ln(1.016625)$

$0.445184 = \text{N} \times 0.016488$

N = 26.999996 or 27 quarterly compounds

Step 4: 27 = 4 × Years

Years = 6.75, which is 6 years plus 0.75 × 12 = 9 months

Present

Jenning Holdings has had the money invested for six years and nine months.

Noninteger Compounding Periods

When the number of compounding periods does not work out to an integer, the method of calculating N does not change. However, what changes is the method by which you convert N and express it in more common terms (step 4 of the process). Typically, the noninteger involves a number of years, months, and days.

As summarized in the table below, to convert the compounding period into the correct number of days you can make the following assumptions:

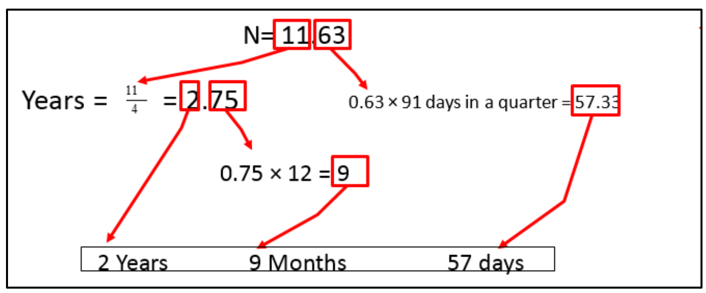
Compounding Period	# of Days in the Period
Annual	365
Semi-annual	182*
Quarter	91*
Month	30*
Week	7
Daily	1

How It Works

You still use the same four steps to solve for the number of compounding periods when N works out to a noninteger as you did for integers. However, in step 4 you have to alter how you convert the N to a common expression. Here is how to convert the value:

1. Separate the integer from the decimal for your value of N.
2. With the integer portion, apply the same technique used with an integer N to calculate the number of years and months.
3. With the decimal portion, multiply by the number of days in the period to determine the number of days and round off the answer to the nearest day (treating any decimals as a result of a rounded interest amount included in the future value).

The figure to the right illustrates this process for an N = 11.63 with quarterly compounding, or CY = 4. Thus, an N = 11.63 quarterly compounds converts to a time frame of 2 years, 9 months, and 57 days. Note that without further information, it is impossible to simplify the 57 days into months and days since it is unclear whether the months involved have 28, 29, 30, or 31 days.



Example 4.1R: Saving for Postsecondary Education

Tabitha estimates that she will need $20,000 for her daughter's postsecondary education when she turns 18. If Tabitha is able to save up $8,500, how far in advance of her daughter's 18th birthday would she need to invest the money at 7.75% compounded semi-annually?

Plan

Determine the amount of time in advance of the daughter's 18th birthday that the money needs to be invested. The number of compounding periods (N) must be calculated.

Understand

What You Already Know

Step 1: The principal, future value, and interest rate are known, as illustrated in the timeline.

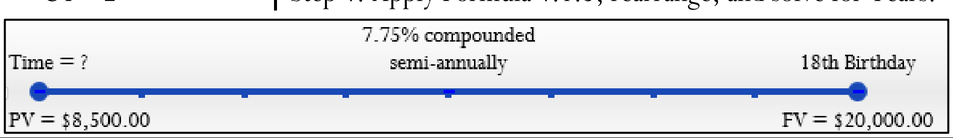
IY = 7.75% CY =2

How You Will Get There

Step 2: Apply Formula 4.1.4.

Step 3: Substitute into Formula 4.1.6, rearrange, and use the rule for logarithm of a power to solve for N.

Step 4: Apply Formula 4.1.5, rearrange, and solve for Years.



Perform

Step 2: $i = \frac{7.75\%}{2} = 3.875\%$

Step 3: $\$20{,}000 = \$8{,}500(1 + 0.03875)^N$

$2.352941 = 1.03875^N$

$\ln(2.352941) = N \times \ln(1.03875)$

$0.855666 = N \times 0.038018$

$N = 22.506828$ semi-annual compounds

Step 4: Take the integer: $22 = 2 \times$ Years

Years = 11

Take the decimal: semi-annual compounding according is 182 days, so 0.506828×182 days = 92 days

Present

If Tabitha invests the $8,500 11 years and 92 days before her daughter's 18th birthday, it will grow to $20,000.

Module 4.1 Exercises

For each of the following questions, round all money to two decimals and percentages to four decimals.

Mechanics

For questions 1–6, solve for the unknown variables (identified with a ?) based on the information provided.

	Interest Amount	Principal or Present Value	Interest Rate	Time
1.	?	$130,000.00	8%	9 months
2.	$4,000.00	?	2% per month	8 months
3.	$1,437.50	$57,500.00	? per year	4 months
4.	$103.13	$1,250.00	9%	? months
5.	?	$42,250.00	1½% per month	½ year
6.	$1,350.00	?	6¾%	3 months

For questions 7–10, solve for the unknown variables (identified with a ?) based on the information provided.

	Interest Amount	Principal / Present Value	Interest Rate	Time	Maturity / Future Value
7.	?	$16,775.00	0.5% per month	6 months	?
8.	?	?	9%	2 months	$61,915.00
9.	$1,171.44	?	5¾%	? months	$23,394.44
10.	$2,073.67	$41,679.00	?	227 days	?

For questions 11–13, solve for the future value at the end of the term based on the information provided.

	Principal	Interest Rate	Term
11.	$7,500	8% compounded quarterly	3 years
12.	$53,000	6% compounded monthly	4 years, 3 months
13.	$19,000	5.75% compounded semi-annually	6 years, 6 months

For questions 14-16 solve for the future value at the end of the sequence of interest rate terms based on the information provided.

	Principal	Interest Rates and Term
14.	$3,750	4.75% compounded annually for 3 years; then 5.5% compounded semi-annually for 2 years
15.	$11,375	7.5% compounded monthly for 2 years, 9 months; then 8.25% compounded quarterly for 3 years, 3 months
16.	$24,500	4.1% compounded annually for 4 years; then 5.15% compounded quarterly for 1 year, 9 months; then 5.35% compounded monthly for 1 year, 3 months

For questions 17 - 24 solve for the number of compounds involved in each transaction based on the information provided. Express your answer in a more common format.

	Present Value	Future Value	Nominal Interest Rate
17.	$68,000.00	$89,032.97	4.91% compounded monthly
18.	$41,786.68	$120,000.00	8.36% compounded quarterly
19.	$10,000.00	$314,094.20	9% compounded annually
20.	$111,243.48	$1,000,000.00	8.8% compounded semi-annually
21.	$25,000.00	$125,000.00	5.85% compounded monthly
22.	$8,000.00	$10,000.00	18% compounded daily
23.	$110,000.00	$250,000.00	6.39% compounded weekly
24.	$500,000.00	$2,225,500.00	7.1% compounded quarterly

Applications

25. Brynn borrowed $25,000 at 1% per month from a family friend to start her entrepreneurial venture on December 2, 2011. If she paid back the loan on June 16, 2012, how much simple interest did she pay.

26. Alex took out a variable rate $20,000 loan on July 3 at prime + 4% when prime was set at 3%. The prime rate increased ½% on August 15. How much simple interest does Alex owe if the loan is paid back on September 29?

27. How much simple interest is earned on $50,000 over 320 days if the interest rate is:
   a. 3 %
   b. 6%
   c. 9%
   d. What relationship is evident between simple interest amounts and rates in your three answers above?

28. If you placed $2,000 into an investment account earning 3% simple interest, how many months does it take for you to have $2,025 in your account?

29. Cuthbert put $15,000 into a nine-month term deposit earning simple interest. After six months he decided to cash the investment in early, taking a penalty of 1% on his interest rate. If he received $393.75 of interest, what was the original interest rate before the penalty on his term deposit?

30. If you want to earn $1,000 of simple interest at a rate of 7% in a span of five months, how much money must you invest?

31. Jessica decided to invest her $11,000 in two back-to-back three-month term deposits. On the first three-month term, she earned $110 of interest. If she placed both the principal and the interest into the second three-month term deposit and earned $145.82 of interest, how much higher or lower was the interest rate on the second term deposit?

32. On January 23 of a non–leap year, a loan is taken out for $15,230 at 8.8% simple interest. What is the maturity value of the loan on October 23?

33. An accountant needs to allocate the principal and simple interest on a loan payment into the appropriate ledgers. If the amount received was $10,267.21 for a loan that spanned April 14 to July 31 at 9.1%, how much was the principal and how much was the interest?

34. Suppose Robin borrowed $3,600 on October 21 and repaid the loan on February 21 of the following year. What simple interest rate was charged if Robin repaid $3,694.63?

35. How many weeks will it take $5,250 to grow to $5,586 at a simple interest rate of 10.4%? Assume 52 weeks in a year.

36. Jayne needs to make three payments to Jade requiring $2,000 each 5 months, 10 months, and 15 months from today. She proposes instead making a single payment eight months from today. If Jade agrees to a simple interest rate of 9.5%, what amount should Jayne pay?

37. Markus failed to make three payments of $2,500 scheduled one year ago, nine months ago, and six months ago. As his creditor has successfully sued Markus in small claims court, the judge orders him to pay his debts. If the court uses a simple interest rate of 1.5% per month, what amount should the judge order Markus to pay today?

38. Calculate the periodic interest rate if the nominal interest rate is 7.75% compounded monthly.

39. Calculate the compounding frequency for a nominal interest rate of 9.6% if the periodic interest rate is 0.8%. Calculate the nominal interest rate if the periodic interest rate is 2.0875% per quarter.

40. Lori hears her banker state, "We will nominally charge you 10.68% on your loan, which works out to 0.89% of your principal every time we charge you interest." What is her compounding frequency?

41. You just received your monthly MasterCard statement and note at the bottom of the form that interest is charged at 19.5% compounded daily. What interest rate is charged to your credit card each day?

42. Larry just purchased a new Ford Mustang from his local Ford dealer. His contract states that he will be charged interest at 0.658 3% per month. What is his nominal interest rate?

43. You are planning a 16-day African safari to Rwanda to catch a rare glimpse of the 700 remaining mountain gorillas in the world. The estimated cost of this once-in-a-lifetime safari is $15,000 including the tour, permits, lodging, and airfare. Upon your graduation from college, your parents have promised you a $10,000 graduation gift. You intend to save this money for five years in a long-term investment earning 8.3% compounded semi-annually. If the cost of the trip will be about the same, will you have enough money five years from now to pay for your trip?

44. Your investment of $9,000 that you started six years ago earned 7.3% compounded quarterly for the first 3¼ years, followed by 8.2% compounded monthly after that. How much interest has your investment earned so far?

45. What is the maturity value of your $7,800 investment after three years if the interest rate was 5% compounded semi-annually for the first two years, 6% compounded quarterly for the last year, and 2½ years after the initial investment you made a deposit of $1,200? How much interest is earned?

46. You just took over another financial adviser's account. The client invested $15,500 at 6.92% compounded monthly and now has $24,980.58. How long has this client had the money invested?

47. A debt of $7,500 is owed. Suppose prevailing interest rates are 4.9% compounded quarterly. How far in advance was the debt paid if the creditor accepted a payment of $6,721.58?

48. How long will it take $5,750 to become $10,000 at 6.25% compounded weekly?

49. A friend of yours just won the 6/7 category on the Lotto Max (matching six out of seven numbers), and her share of the prize was $275,000. She wants to pay cash for a new home that sells for $360,000. If she can invest the money at 7.45% compounded semi-annually, how long will she have to wait to purchase the home assuming its sale price remains the same?

50. Lakewood Properties anticipates that the City of Edmonton in the future will release some land for a development that costs $30 million. If Lakewood can invest $17.5 million today at 9.5% compounded monthly, how long will it take before it will have enough money to purchase the land?

Challenge, Critical Thinking, & Other Applications

51. The simple interest formula applies to any instance of constant unit growth. Assume that 200 units of product are sold today and you forecast a constant 20-unit sales increase every four weeks. What will be the sales in one year's time (assume a year has exactly 52 weeks)?

52. Marrina is searching for the best way to invest her $10,000. One financial institution offers 4.25% on three-month term deposits and 4.5% on six-month term deposits. Marrina is considering either doing two back-to-back three-month term deposits or just taking the six-month deposit. It is almost certain that interest rates will rise by 0.5% before her first three-month term is up. She will place the simple interest and principal from the first three-month term deposit into the second three-month deposit. Which option should Marrina pursue? How much better is your recommended option?

53. Evaluate each of the following $10,000 investment alternatives and recommend the best alternative for investing any principal regardless of the actual amount. Assume in all cases that the principal and simple interest earned in prior terms are placed into subsequent investments.

Alternative 1	Alternative 2	Alternative 3
6% for 1 year	5% for 6 months, then 7% for 6 months	5.25% for 3 months, then 5.75% for 3 months, then 6.25% for 3 months, then 6.75% for 3 months

What percentage more interest is earned in the best alternative versus the worst alternative?

54. Over a period of nine months, $40.85 of simple interest is earned at 1.75% per quarter. Calculate both the principal and maturity value for the investment.

55. Merina is scheduled to make two loan payments to Bradford in the amount of $1,000 each, two months and nine months from now. Merina doesn't think she can make those payments and offers Bradford an alternative plan where she will pay$775 seven months from now and another payment seven months later. Bradford determines that 8.5% is a fair interest rate. What is the amount of the second payment?

56. Betty just signed a contract for her business that will allow her to make interest-only payments for the next 12 months. If her interest rate is 12% compounded monthly and her outstanding principal is $5,000, how much will she pay in interest every month?

57. Geoff is shopping around for a car loan. At three different websites he saw posted rates of 6.14% compounded semi-annually, 3.06% compounded per six months, and 1.49% compounded every quarter. Which is the lowest nominal rate?

58. After a period of three months, Alese saw one interest deposit of $176.40 for a principal of $9,800. What nominal rate of interest is she earning?

59. You invested $5,000 10 years ago and made two further deposits of $5,000 each after four years and eight years. Your investment earned 9.2% compounded quarterly for the first two years, 8.75% compounded monthly for the next six years, and 9.8% compounded semi-annually for the remaining years. As of today, how much interest has your investment earned?

60. Suppose you placed $10,000 into each of the following investments. Rank the maturity values after five years from highest to lowest.

8% compounded annually for two years followed by 6% compounded semi-annually
8% compounded semi-annually for two years followed by 6% compounded annually
8% compounded monthly for two years followed by 6% compounded quarterly
8% compounded semi-annually for two years followed by 6% compounded monthly

61. How long will it take money to double if it earns 8.2% compounded quarterly? First use the Rule of 72 to estimate the time, then compute the actual time. What is the difference between the two numbers (assume 182 days in a half year)?

62. Your organization has a debt of $30,000 due in 13 months and $40,000 due in 27 months. If a single payment of $67,993.20 was made instead using an interest rate of 5.95% compounded monthly, when was the payment made?

63. A $9,500 loan requires a payment of $5,000 after 1½ years and a final payment of $6,000. If the interest rate on the loan is 6.25% compounded monthly, when should the final payment be made?

64. Your finance department has a policy of maturing its investments from longest held first to shortest when it needs the money for other purposes. Rank the following investments in the order that they should be matured as needed.

Investment	Amount Invested	Current Valuation	Nominal Interest Rate
A	$50,000	$75,549.62	7.94% compounded quarterly
B	$75,000	$104,830.46	6.83% compounded monthly
C	$35,000	$51,231.69	7.25% compounded semi-annually
D	$110,000	$161,643.96	8.1% compounded quarterly

65. A loan requires five payments of $1,000 today, $1,500 due in 9 months, $3,000 due in 15 months, $2,500 due in 21 months, and $4,000 due in 33 months. Using an interest rate of 4.4% compounded monthly, a single payment of $11,950 was made to clear all debts. When was the single payment made?