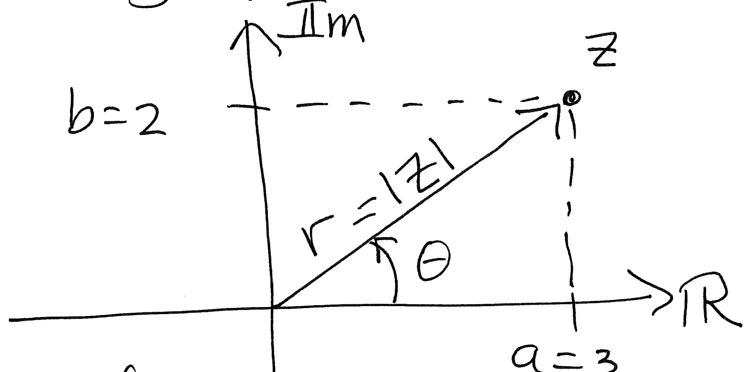
MWF12, F19

5.3 The Complex Plane; De Moivre's Theorem

Recall that a complex number has the form $a+bi$. A complex can be represented as a point, (a,b), on the complex plane. For example, $z=3+2i$ can be graphed as $(3,2)$ as follows:



Def: The absolute value of z, written $|z|$, given by $|a+bi| = \sqrt{a^2+b^2}$.

- Polar and Trig. Forms of Complex Numbers

Given $z=a+bi$, then $r=|z|=\sqrt{a^2+b^2}$ and $a=r\cos\theta$ and $b=r\sin\theta$. Therefore,

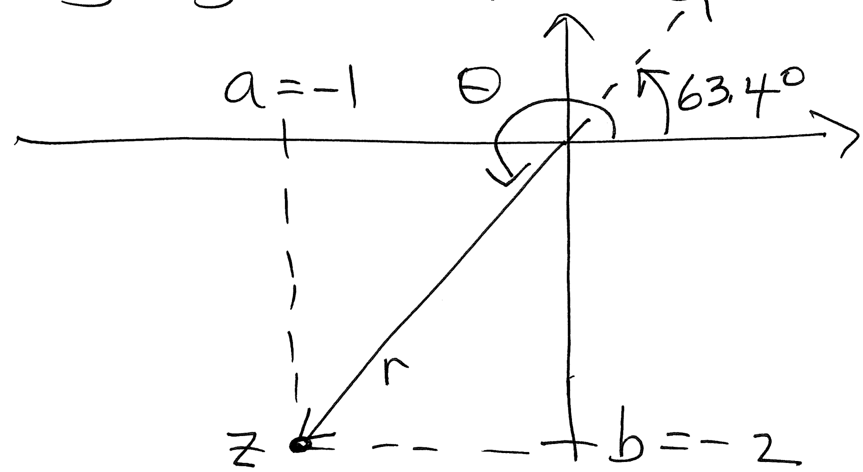
$$z=a+bi = r\cos\theta + r\sin\theta\, i$$

$$\boxed{z=r(\cos\theta+i\sin\theta)} \quad \text{(Trig. Form)}$$

$$\boxed{z=(r,\theta)} \quad \text{(Polar Form)}$$

In these two forms, r is called the absolute value of z or the modulus of z. The angle θ is referred to as the argument of z.

EX: Write $z = -1 - 2i$ in both trig. form and polar form using degrees approximated to one decimal.



$|z| = r = \sqrt{a^2 + b^2}$

$= \sqrt{1+4}$

$r = \sqrt{5}$

$\tan\theta = \frac{-2}{-1} = 2 \Rightarrow \theta = \tan^{-1} 2 = 63.4°$ which is in Q1. But, θ has to be in Q3.

$\Rightarrow \theta = 180° + 63.4° = 243.4°$

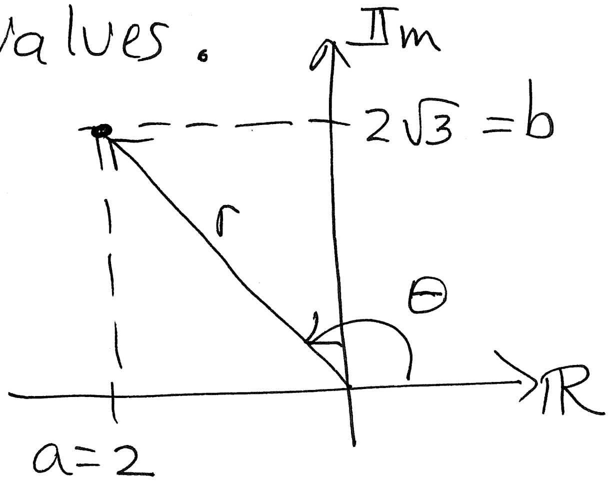
$z = r(\cos\theta + i \sin\theta)$

$$\underbrace{z = \sqrt{5}\,(\cos 243.4° + i \sin 243.4°)}_{\text{(Trig. Form)}}$$

$$\underbrace{z = (r, \theta) = (\sqrt{5}, 243.4°)}_{\text{(Polar Form)}}$$

NOTE: sometimes the terms "polar" and "trigonometric" are used interchangeably.

EX: Write $z = -2 + 2\sqrt{3}i$ in trig. form using radians and exact values.



$$r = \sqrt{a^2+b^2} = \sqrt{4+12}$$

$r = 4$ with θ in Q II.

$$\sin\theta = \frac{b}{r} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$\Rightarrow \underbrace{z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$$

EX: Express $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ in standard form, $a+bi$.

$$z = 2\left(\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$\underbrace{z = 1 + \sqrt{3}i}$$

- Multiplication / Division of Two Complex Numbers in Trig. Form

To multiply two complex numbers given in trig. form, we multiply the moduli (r's) and add the arguments (angles).

EX: Given $z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and $z_2 = 3\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$. Find $z_1 \cdot z_2$.

$$z_1 \cdot z_2 = (2)(3)\left[\cos\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right) + i\sin\left(\frac{2\pi}{3} + \frac{11\pi}{6}\right)\right]$$

$$= 6\left(\cos\frac{15\pi}{6} + i\sin\frac{15\pi}{6}\right)$$

$$= 6\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= 6(0 + i(1))$$

$$\underbrace{z_1 \cdot z_2 = 6i}$$

To divide two complex numbers in trig. form, we divide the moduli (r's) and subtract the arguments (angles).

EX: Given the same two complex numbers as in the previous example, to find z_1/z_2.

$$\frac{z_1}{z_2} = \frac{2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}{3\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)} = \frac{2}{3}\left[\cos\left(\frac{2\pi}{3} - \frac{11\pi}{6}\right) + i\sin\left(\frac{2\pi}{3} - \frac{11\pi}{6}\right)\right]$$

$$= \frac{2}{3}\left[\cos\left(-\frac{7\pi}{6}\right) + i\sin\left(-\frac{7\pi}{6}\right)\right]$$

$$= \frac{2}{3}\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right] = \frac{2}{3}\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]$$

$$\underbrace{\frac{z_1}{z_2} = -\frac{\sqrt{3}}{3} + \frac{1}{3}i}$$

• De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$, then

$$\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$$

EX: Given $z = \sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$.

FIND: z^4 in standard form with exact values.

$$z^4 = (\sqrt{2})^4\left(\cos\frac{4(5\pi)}{6} + i\sin\frac{4(5\pi)}{6}\right)$$

$$= 4\left(\cos\frac{20\pi}{6} + i\sin\frac{20\pi}{6}\right)$$

$$= 4\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$$

$$= 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 4\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$z^4 = -2 - 2\sqrt{3}\,i$$