MWF

3.6A Double-Angle Formulas

Recall that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If we let $\alpha = \beta = \theta$

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

Similarly, $\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$ (1)

This last formula can also be written as:

$$\cos(2\theta) = 1-\sin^2\theta - \sin^2\theta$$

$$\boxed{\cos(2\theta) = 1 - 2\sin^2\theta} \quad (2)$$

OR

$$\cos(2\theta) = \cos^2\theta - (1-\cos^2\theta)$$

$$= \cos^2\theta - 1 + \cos^2\theta$$

$$\boxed{\cos(2\theta) = 2\cos^2\theta - 1} \quad (3)$$

Finally, recall that $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

Since $\theta = \alpha = \beta$

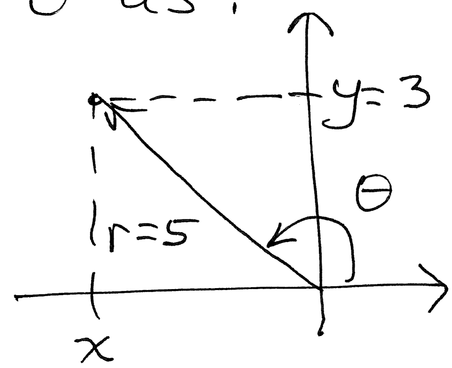
$$\boxed{\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}}$$

EX: Given $\sin\theta = \frac{3}{5}$, $\frac{\pi}{2} \le \theta < \pi$.
(QII)

FIND: The exact value for $\tan(2\theta)$.

We know that $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Because $\sin\theta = \frac{3}{5} = \frac{y}{r}$, and θ is in QII, then we can sketch the angle θ as:



Then, $\tan\theta = \dfrac{y}{x}$, $x = -\sqrt{r^2 - y^2}$

$$= -\sqrt{25-9}$$

$$x = -4$$

$$\tan\theta = \frac{3}{-4} = \underline{-\frac{3}{4}}$$

Finally, $\tan(2\theta) = \dfrac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}$

$$= \frac{-\frac{3}{2}}{1-\frac{9}{16}} = \frac{-\frac{3}{2}}{\frac{7}{16}}$$

$$= -\frac{3}{\cancel{2}_1} \cdot \frac{\cancel{16}^{8}}{7}$$

$$\underline{\tan(2\theta) = -\frac{24}{7}}$$

EX: Solve $\sin(2\theta) - \sin\theta = 0$, θ in $[0, 2\pi)$.

$$\underbrace{2\sin\theta\cos\theta} - \sin\theta = 0$$

$$\sin\theta\,(2\cos\theta - 1) = 0$$

$\sin\theta = 0$ or $2\cos\theta - 1 = 0$

$\Rightarrow \theta = 0, \pi$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Sol. Set $= \underline{\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}}$

EX: Solve $\sin\theta - \cos\theta = \sqrt{2}$, θ in $[0, 2\pi)$.

$$(\sin\theta - \cos\theta)^2 = (\sqrt{2})^2$$

$$\underline{\sin^2\theta} - \underbrace{2\sin\theta\cos\theta} + \underline{\cos^2\theta} = 2$$

$$1 - \sin(2\theta) = 2$$

$$-\sin(2\theta) = 1$$

$$\sin(2\theta) = -1$$

$$\Rightarrow 2\theta = \frac{3\pi}{2} + 2\pi k$$

$$\theta = \frac{3\pi}{4} + \pi k$$

For $k = 0$, $\theta = \frac{3\pi}{4}$ ✓

$k = 1$, $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$ ✓

Must check for extraneous solutions:

For $\theta = \frac{3\pi}{4}$

$\sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} \stackrel{?}{=} \sqrt{2}$

$\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)$

$\sqrt{2} = \sqrt{2}$

True

actual sol.

For $\theta = \frac{7\pi}{4}$

$\sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} \stackrel{?}{=} \sqrt{2}$

$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

$-\sqrt{2} = \sqrt{2}$

False

$\Rightarrow$ extraneous sol.

Sol. Set $= \left\{\frac{3\pi}{4}\right\}$

- Power Reduction Formulas

These are variations of double-angle formulas. For example, we wrote earlier that the $\cos(2\theta) = 1 - 2\sin^2\theta$.

Then, $\cos(2\theta) + 2\sin^2\theta = 1$

$$2\sin^2\theta = 1 - \cos(2\theta)$$

$$\boxed{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$$

Similarly, $\boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$

Finally, $\boxed{\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}}$

EX: Establish $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

LS $\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$

$= (1)(\cos(2\theta)) = \underline{\cos(2\theta)}$ RS

EX! Develop a formula for $\sin(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

$$\sin(3\theta) = \sin(2\theta + \theta)$$

Using the "sum of two angles" formula we get

$$= \sin(2\theta)\cos(\theta) + \cos(2\theta)\sin(\theta)$$

$$= (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin(\theta)$$

$$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$$

$$\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta$$