MWF

3.6 B Half-Angle Formulas

Suppose we let $\theta = \frac{\alpha}{2}$ and re-write the power-reduction identities as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$$

$$\boxed{\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}}$$

Similarly, if we use $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$$\Rightarrow \boxed{\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}}$$

Finally, using $\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$

$$\Rightarrow \boxed{\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}$$

<u>NOTE</u>: (+) or (−) sign depends on the quadrant where $\frac{\alpha}{2}$ is located.

• Other Formulas for $\tan\left(\frac{\alpha}{2}\right)$

Two additional formular are:

$$\boxed{\tan\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{\sin\alpha}}$$

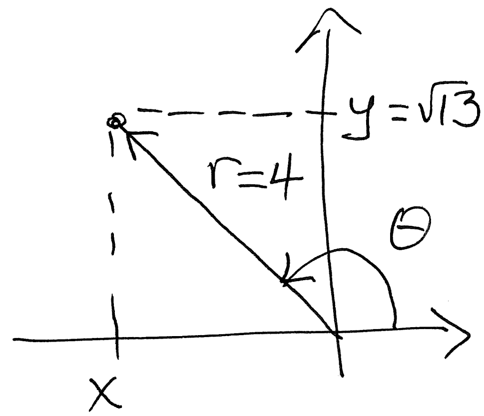
and

$$\boxed{\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1+\cos\alpha}}$$

<u>EX:</u> Given $\sin\theta = \frac{\sqrt{13}}{4}$, $\frac{\pi}{2} < \theta < \pi$. Find $\sin \theta/2$ exactly.

We are told that θ is in QII, therefore $\frac{\theta}{2}$ is in QI. Then, $\sin\frac{\theta}{2} = +\sqrt{\frac{1-\cos\theta}{2}}$.

We also know that $\sin\theta = \frac{\sqrt{13}}{4} \Rightarrow y = \sqrt{13},\ r = 4.$



$$\cos\theta = \frac{x}{r}$$

$$x = -\sqrt{r^2 - y^2} = -\sqrt{16-13}$$

$$x = -\sqrt{3}$$

Then, $\cos\theta = \frac{-\sqrt{3}}{4} = -\frac{\sqrt{3}}{4}$

Thus, $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{(1+\sqrt{3}/4)(4)}{2(4)}} = \sqrt{\frac{(4+\sqrt{3})}{8}\cdot\frac{2}{2}}$

$$= \sqrt{\frac{8+2\sqrt{3}}{16}} = \underbrace{\frac{\sqrt{8+2\sqrt{3}}}{4}} = \underbrace{\frac{1}{4}\sqrt{8+2\sqrt{3}}}$$

EX: Find exact value for cos 15° using a half-angle formula.

Let's write $15° = \frac{30°}{2} = \frac{\alpha}{2}$

Using $\cos\left(\frac{\alpha}{2}\right)$ with 30°, then

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

We know that $\frac{\alpha}{2} = 15°$, we must be in QI.

$$\cos\left(\frac{\alpha}{2}\right) = +\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\cos 15° = \sqrt{\frac{1+\cos 30°}{2}}$$

$$= \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$\cos 15° = \frac{\sqrt{2+\sqrt{3}}}{2} = \frac{1}{2}\sqrt{2+\sqrt{3}}$$

EX: Use half-angle formula to find exact value for $\tan\left(\frac{9\pi}{8}\right)$.

Let $\frac{\alpha}{2} = \frac{9\pi}{8}$, then $\alpha = \frac{9\pi}{4}$

Using $\tan\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{\sin\alpha}$

$$\tan\left(\frac{9\pi}{8}\right) = \frac{1-\cos\left(\frac{9\pi}{4}\right)}{\sin\left(\frac{9\pi}{4}\right)}$$

$$= \frac{1-\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{\left(1-\frac{\sqrt{2}}{2}\right)2}{\left(\frac{\sqrt{2}}{2}\right)2}$$

$$= \frac{2-\sqrt{2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-2}{2}$$

$$= \frac{\cancel{2}^{1}(\sqrt{2}-1)}{\cancel{2}_{1}}$$

$$\tan\left(\frac{9\pi}{8}\right) = \boxed{\sqrt{2}-1}$$

EX: Establish that $\cos\theta = \dfrac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$.

RS
$$\frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-\left(\frac{1-\cos\theta}{\sin\theta}\right)^2}{1+\left(\frac{1-\cos\theta}{\sin\theta}\right)^2}$$

$$= \frac{\left[1 - \frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}\right]\sin^2\theta}{\left[1+\frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}\right]\sin^2\theta}$$

$$= \frac{\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta}{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}$$

$$= \frac{1-\cos^2\theta - 1 + 2\cos\theta - \cos^2\theta}{2-2\cos\theta}$$

$$= \frac{-2\cos^2\theta + 2\cos\theta}{2-2\cos\theta} = \frac{\cos\theta(-2\cos\theta+2)}{2-2\cos\theta}$$

$$= \underline{\cos\theta} \quad \text{LS}$$