7.6 Exp. Growth and Decay; Newton's Law of Cooling; Logistic Models

- Exp. Growth & Decay

Many phenomena follow exponential behavior described by: $A(t) = A_0 e^{kt}$,

where t = time

A_0 = initial or original amount at $t=0$.

A = future amount at any time t.

k = growth rate in decimal equivalent if $k>0$, decay rate if $k<0$.

EX: The population P of a city is increasing according to the model: $P(t) = 250{,}000\, e^{0.0293t}$, where $t=0$ represents 1990.

(a) What was the population in 1990? $P = 250{,}000$

(b) What was the % growth rate? 2.93% / year

(c) What was the population in 2007? $t = 2007-1990$, $t = 17$

$$P(17) = 250{,}000\, e^{0.0293(17)} = 411{,}398$$

(d) How long after 1990 will it take for the population to double?

$$500{,}000 = 250{,}000\, e^{0.0293t}$$

$$2 = e^{0.0293t}$$

$$\ln 2 = 0.0293t$$

$$\frac{\ln 2}{0.0293} = t = 23.7 \text{ years}$$

EX: The number of internet users in the U.S. was estimated to be 54 million in 1999, In 2002 the estimate was 85 million.

(a) Find the model of the form $f(t) = ce^{kt}$ to represent this uninhibited growth.

Let $t=0$ represent 1999 $\Rightarrow$

t	$f(t)$
0	54
3	85

Find c: $f(0) = 54 = ce^{0}$

$54 = c \Rightarrow f(t) = 54e^{kt}$

Find k: $f(3) = 85 = 54e^{3k}$

$$\frac{85}{54} = e^{3k}$$

$$\ln\left(\frac{85}{54}\right) = 3k$$

$$\frac{\ln\left(\frac{85}{54}\right)}{3} = k \simeq 0.1512$$

$$\Rightarrow \underbrace{f(t) = 54e^{0.1512t}}$$

(b) Use the model to estimate the number of internet users in 2019,

$t = 2019 - 1999 = 20$

$$f(20) = 54e^{0.1512(20)} = \underbrace{1,111 \text{ million}}$$

If we are not careful and use "old" data, exponential models will predict exaggerated or impossible values?

EX: The populated of mosquitoes obeys the law of uninhibited growth. If there are 1000 mosquitoes initially and there are 1700 after one day; 7.6 Continued:

(a) What is the size of the population after 3 days?

For exp. growth, we know $A(t) = A_0 e^{kt}$

$$\Rightarrow A(t) = 1000\, e^{kt}$$

t	$A(t)$
1	1700

Find k: $A(1) = 1700 = 1000\, e^{k}$

$$1.7 = e^{k}$$

$$\ln 1.7 = k \simeq 0.53063$$

$$\Rightarrow A(t) = 1000\, e^{0.53063t}$$

$$A(3) = 1000\, e^{0.53063(3)} = 4{,}913 \text{ mosquitoes.}$$

(b) How long will it take for this population to grow to 30,000 mosquitoes?

$$30{,}000 = 1000\, e^{0.53063t}$$

$$30 = e^{0.53063t}$$

$$\ln 30 = 0.53063t$$

$$\frac{\ln 30}{0.53063} = t \approx 6.4 \text{ days.}$$

EX: Iodine-131 is a radioactive isotope that decays according to the function:

$$A(t) = A_0 e^{-0.087t}, \quad t = \text{no. of days.}$$

(a) If $A_0 = 100$ grams, how much Iodine-131 is left after 9 days?

$$A(9) = 100\, e^{-0.087(9)} = \underbrace{45.7 \text{ grams}}$$

(b) What is the % decay rate? $\underbrace{-8.7\% / \text{day}}$

(c) What is the half-life of Iodine-131?

$$50 = 100\, e^{-0.087t}$$

$$\frac{1}{2} = e^{-0.087t}$$

$$\ln\left(\tfrac{1}{2}\right) = -0.087t$$

$$\frac{\ln\left(\frac{1}{2}\right)}{-0.087} = t \simeq \underbrace{7.97 \text{ days}}$$

(d) How long will it take for 99% of the Iodine-131 to decay?

$$1 = 100\, e^{-0.087t}$$

$$\frac{1}{100} = e^{-0.087t}$$

$$\ln\left(\tfrac{1}{100}\right) = -0.087t$$

$$\frac{\ln\left(\frac{1}{100}\right)}{-0.087} = t \simeq \underbrace{52.93 \text{ days}}$$

EX: A certain radioactive element has a half-life of 2,628 years.

(a) Find the decay rate to 6 decimals.

$$A(t) = A_0 e^{kt}$$

$$1 = 2e^{2628k}$$

$$\frac{1}{2} = e^{2628k}$$

$$\ln\left(\frac{1}{2}\right) = 2628k$$

$$\frac{\ln\left(\frac{1}{2}\right)}{2628} = \underbrace{k \approx -0.000264}$$

(b) Write the decay model.

$$\underbrace{A(t) = A_0 e^{-0.000264t}}$$

(c) Use this model to find out how long will it take for 25% of a sample to decay?

$$3 = 4e^{-0.000264t}$$

$$\frac{3}{4} = e^{-0.000264t}$$

$$\ln\left(\frac{3}{4}\right) = -0.000264t$$

$$\frac{\ln\left(\frac{3}{4}\right)}{-0.000264} = t$$

$$\underbrace{1{,}090 \text{ years} = t}$$

- Newton's Law of Cooling

The temperature Θ of an object at time t can be modeled by:

$$\Theta(t) = T + (\Theta_0 - T)e^{kt} \text{ for } k<0,$$

where T = constant temperature of the surroundings

Θ_0 = initial temperate of an object.

EX: A thermometer reading 72°F is placed in a refrigerator where the temperature is 38°F.

(a) If the thermometer reads 60°F after being in the refrigerator for 2 minutes, what is the temperature indicated by the thermometer after 7 minutes?

$$\Theta(2) = 38° + (72° - 38°)e^{2k} = 60°$$

$$38 + 34e^{2k} = 60$$

$$34e^{2k} = 22$$

$$e^{2k} = \frac{22}{34}$$

$$2k = \ln\left(\frac{22}{34}\right)$$

$$k = \frac{\ln(22/34)}{2} = -0.21766$$

Model: $\Theta(t) = 38 + 34e^{-0.21766t}$

$$\Theta(7) = 38 + 34e^{-0.21766(7)} = \underline{45.4°F}$$