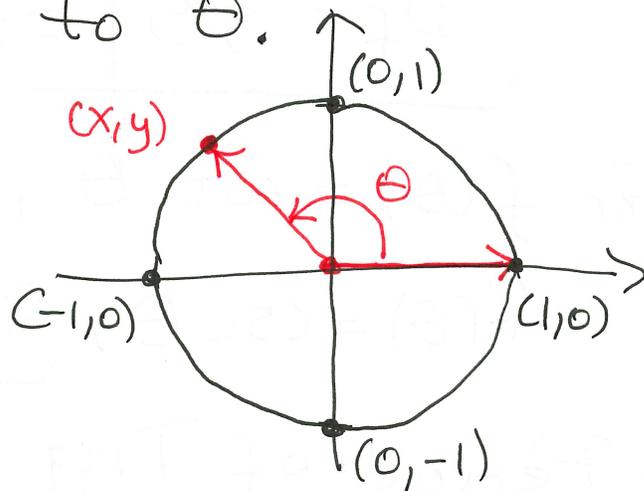


2.3 Properties of Trig. Functions

Let θ be an angle in standard position and $P = (x, y)$ be a point on the Unit circle to correspond to θ .



For $f(\theta) = \sin \theta$ or $f(\theta) = \cos \theta$

$$\boxed{D_f = (-\infty, \infty), \quad R_f = [-1, 1]}$$

For $f(\theta) = \tan \theta$ or $f(\theta) = \sec \theta$, $f(\theta)$ does not exist when $x=0$, which is at $\theta = \frac{\pi}{2}$ or any other ^{odd} multiple of this angle, therefore

$$\boxed{D_f = \left\{ \theta \mid \theta \neq \frac{\pi}{2} + n\pi, n = \text{odd integer} \right\}}$$

For $f(\theta) = \tan \theta$, $\boxed{R_f = (-\infty, \infty)}$

$f(\theta) = \sec \theta$, $\boxed{R_f = (-\infty, -1] \cup [1, \infty)}$

For $f(\theta) = \cot \theta$ or $f(\theta) = \csc \theta$, f does not exist for $\theta = 0$, $\theta = \pi$, or any other integral multiple of 2π .

$$D_f = \{ \theta \mid \theta \neq n\pi, n = \text{integer} \}$$

For $f(\theta) = \cot \theta$, $R_f = (-\infty, \infty)$

$f(\theta) = \csc \theta$, $R_f = (-\infty, -1] \cup [1, \infty)$

• Period of Trig. Functions

For any angle θ measured in radians, we have already said that $\sin \theta = \sin(\theta + 2\pi) = \sin(\theta + 4\pi) = \dots = \sin(\theta + 2\pi k)$, $k = \text{integer}$. Functions that exhibit this behavior are said to be periodic.

Def - a function f is periodic if there is a positive number p such that $f(\theta + p) = f(\theta)$.

The smallest such p -value is called the fundamental period of f .

$$\text{For } \sin(\theta + 2\pi) = \sin \theta$$

$$\cos(\theta + 2\pi) = \cos \theta$$

\Rightarrow fundamental period is 2π .

$$\text{Similarly, } \tan(\theta + \pi) = \tan \theta$$

$$\cot(\theta + \pi) = \cot \theta$$

\Rightarrow fundamental period is $\underline{\pi}$.

$$\text{Finally, } \sec(\theta + 2\pi) = \sec \theta$$

$$\csc(\theta + 2\pi) = \csc \theta$$

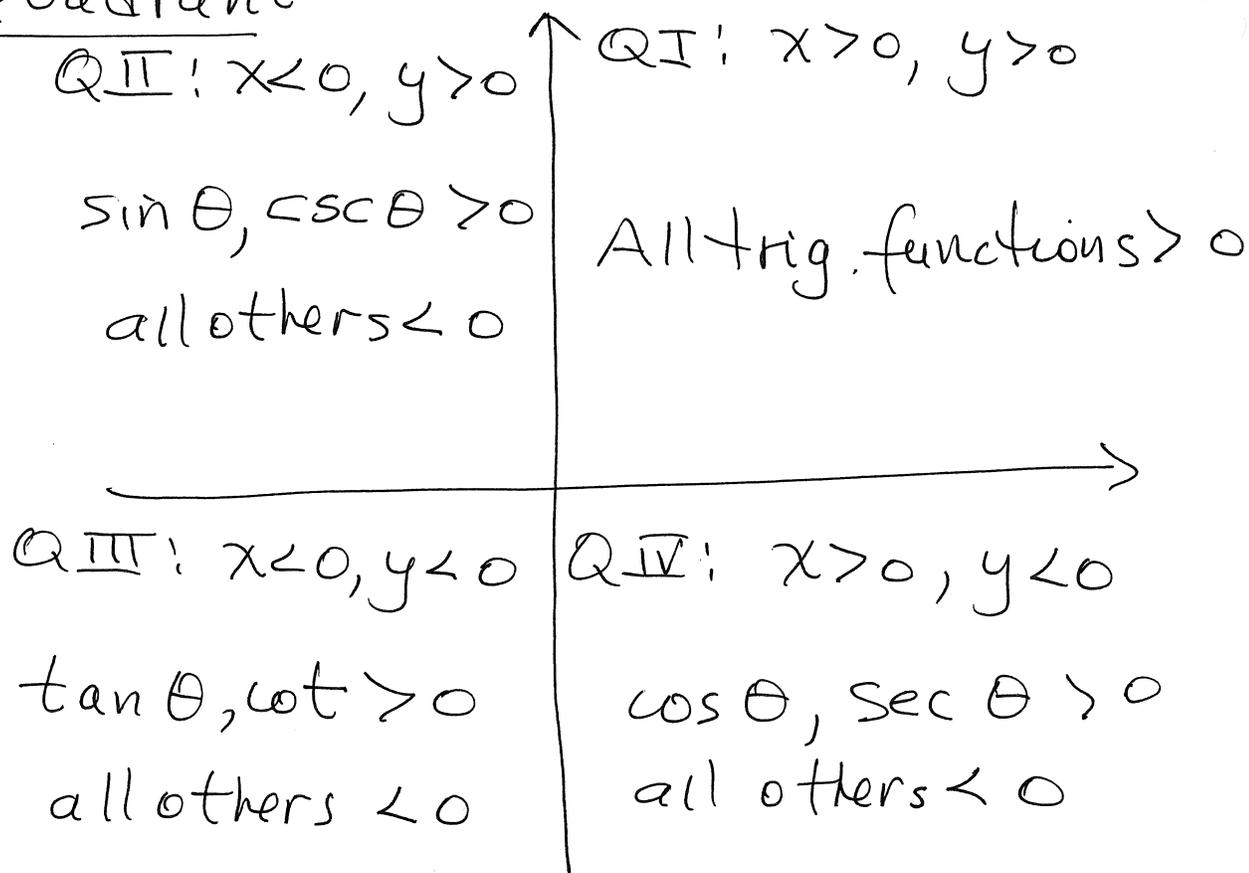
\Rightarrow fund. period is $\underline{2\pi}$

EX: Use fundamental period to find the exact values for

$$\begin{aligned} \text{(a) } \sin\left(\frac{17\pi}{4}\right) &= \sin\left(\frac{16\pi}{4} + \frac{\pi}{4}\right) = \sin\left(4\pi + \frac{\pi}{4}\right) \\ &= \sin\frac{\pi}{4} = \underline{\frac{\sqrt{2}}{2}} \end{aligned}$$

$$\begin{aligned} \text{(b) } \tan\left(\frac{15\pi}{4}\right) &= \tan\left(\frac{12\pi}{4} + \frac{3\pi}{4}\right) = \tan\left(3\pi + \frac{3\pi}{4}\right) \\ &= \tan\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = \underline{-1} \end{aligned}$$

• Signs of Trig. Functions in a Given Quadrant



EX: Find the quadrant in which θ is located if $\tan \theta < 0$ and $\csc \theta > 0$,

2.2 Continued Quadrant II TO BE CONT'D

• Fundamental Identities

– Reciprocal Identities

$$\csc \theta = \frac{1}{\sin \theta}, \sec \theta = \frac{1}{\cos \theta}, \cot \theta = \frac{1}{\tan \theta}$$

– Quotient Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \cot \theta = \frac{\cos \theta}{\sin \theta}$$

EX: Given $\sin \theta = -\frac{\sqrt{5}}{5}$, $\cos \theta = \frac{2\sqrt{5}}{5}$.

FIND: (a) in what quadrant is θ located? QIV

(b) the exact value for other 4 trig. using the previous identities.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-\cancel{\sqrt{5}}/5}{2\cancel{\sqrt{5}}/5} = \underline{-\frac{1}{2}}$$

$$\cot \theta = \frac{1}{\tan \theta} = \underline{-2}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{2\sqrt{5}/5} = \frac{\cancel{\sqrt{5}} \cdot \sqrt{5}}{2\sqrt{5}} = \underline{\frac{\sqrt{5}}{2}}$$

$$\csc \theta = \frac{1}{\sin \theta} = -\frac{5}{\sqrt{5}} = \underline{-\sqrt{5}}$$

Pythagorean Identities

If $P = (x, y)$ is a point on the Unit Circle corresponding to an angle θ , then we know that $y = \sin \theta$ and $x = \cos \theta$.

The equation of the unit circle is $x^2 + y^2 = 1$.

$$\text{Then, } (\cos \theta)^2 + (\sin \theta)^2 = 1.$$

The customary notation ^{for} $(\sin \theta)^2$ is $\sin^2 \theta$ and we use a similar notation for all the other trig. functions. Then,

$$\sin^2 \theta + \cos^2 \theta = 1$$

If we divide both sides by $\cos^2 \theta$,
we get:

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

We can also divide the first identity
above by $\sin^2 \theta$ to get:

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

These three Pythagorean identities
together with the Quotient and
Reciprocal identities are called the
Fundamental Identities

EX: Given $\sin \theta = \frac{1}{3}$ and $\cos \theta < 0$.

FIND: The exact value for the other 5 trig. functions.

Method I - use a circle of radius r .

We are given that the $\sin \theta = \frac{1}{3}$, $\cos \theta$ is (-).
Therefore, the terminal side of θ is in Q_{II} .

By definition, $\sin \theta = \frac{y}{r} = \frac{1}{3}$

$$\Rightarrow y = 1, r = 3$$

Equation of a circle of radius r centered at origin is:

$$x^2 + y^2 = r^2 \Rightarrow x^2 = r^2 - y^2$$

$$x = \pm \sqrt{r^2 - y^2} = -\sqrt{9 - 1} = -\sqrt{8}$$

$$x = -2\sqrt{2}$$

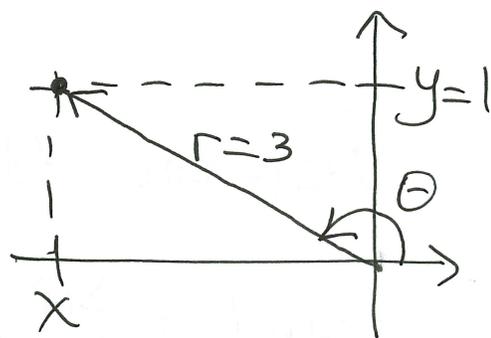
$$\cos \theta = \frac{x}{r} = \frac{-2\sqrt{2}}{3} = -\frac{2\sqrt{2}}{3}$$

$$\tan \theta = \frac{y}{x} = \frac{1}{-2\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

$$\cot \theta = \frac{x}{y} = -2\sqrt{2}$$

$$\sec \theta = \frac{r}{x} = \frac{3}{-2\sqrt{2}} = -\frac{3\sqrt{2}}{4}$$

$$\csc \theta = \frac{r}{y} = \frac{3}{1} = 3$$



$\sqrt{2} \cdot \sqrt{4}$
 $\sqrt{2} \cdot 2$

Method II - using Fundamental Identities

$$\text{Given } \sin \theta = \frac{1}{3}, \cos \theta < 0$$

$$\text{We know that } \sin^2 \theta + \cos^2 \theta = 1$$

$$\left(\frac{1}{3}\right)^2 + \cos^2 \theta = 1$$

$$\frac{1}{9} + \cos^2 \theta = 1$$

$$\cos^2 \theta = \frac{8}{9}$$

$$\cos \theta = \pm \sqrt{\frac{8}{9}} = \pm \frac{2\sqrt{2}}{3}$$

$$\text{We know that } \cos \theta < 0 \Rightarrow \cos \theta = -\frac{2\sqrt{2}}{3}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{1}{3}}{-\frac{2\sqrt{2}}{3}} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

$$\cot \theta = \frac{1}{\tan \theta} = -2\sqrt{2}, \quad \sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta} = 3 = -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4}$$

• Even or Odd Functions: From Algebra

- a function f is even iff $f(t) = f(-t)$
 \Rightarrow symmetry about the y-axis,

- a function f is odd iff $f(-t) = -f(t)$
 \Rightarrow symmetry about the origin.

The trig. functions sine, tangent, cotangent and cosecant are odd functions. The cosine and the secant are even functions.

Therefore, the six Even-Odd Identities ^{MWF12, F19} (or negative angle identities) are:

$\sin(-\theta) = -\sin \theta$	$\cot(-\theta) = -\cot \theta$
$\tan(-\theta) = -\tan \theta$	$\cos(-\theta) = \cos \theta$
$\csc(-\theta) = -\csc \theta$	$\sec(-\theta) = \sec \theta$

EX: Find the exact values using even-odd identities and positive angles on the unit circle.

(a) $\cos(-240^\circ) = \cos 240^\circ = \underline{-\frac{1}{2}}$

(b) $\csc\left(-\frac{5\pi}{3}\right) = -\csc\left(\frac{5\pi}{3}\right) = -\frac{1}{-\sqrt{3}/2}$
 $= \frac{2}{\sqrt{3}} = \underline{\frac{2\sqrt{3}}{3}}$ TO BE CONT'D

2.3 Continued

EX: Given $f(x) = \sin x \cos x$. Is the function f even, odd, or neither.

f is even iff $f(x) = f(-x)$

$$f(-x) = \sin(-x) \cos(-x) = \underline{-\sin x \cos x} \neq f(x)$$

$\Rightarrow f$ is not even

f is odd iff $f(-x) = -f(x)$

$$-f(x) = -(\sin x \cos x) = \underline{-\sin x \cos x}$$

$\Rightarrow f$ is odd