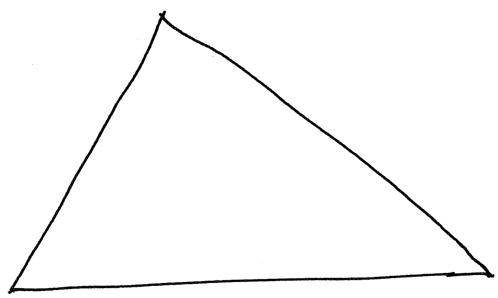
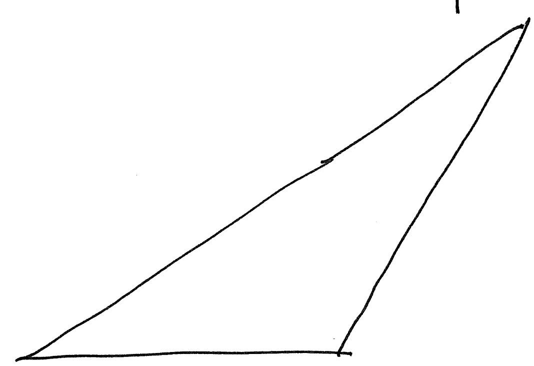
4.2 The Law of Sines

In this section and the next one we will extend the use of trig. functions to solve oblique triangles.

Oblique Triangle – a triangle that is not a right triangle. For example:



Acute Triangle



Obtuse Triangle

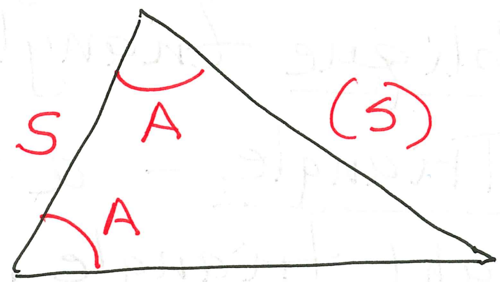
To solve any oblique triangle means to find the length of any side not given and the measure of any missing angles.

To do this we need:

- length of one side and
- two other facts, either
  - two angles
  - two other sides
  - one angle and one other side

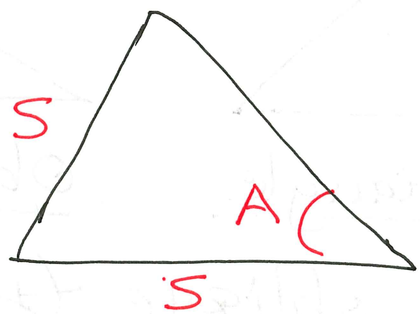
There are <u>four</u> possibilities in solving oblique triangles:

- <u>Case 1</u>



one side and two angles, either ASA or SAA.

- <u>Case 2</u>



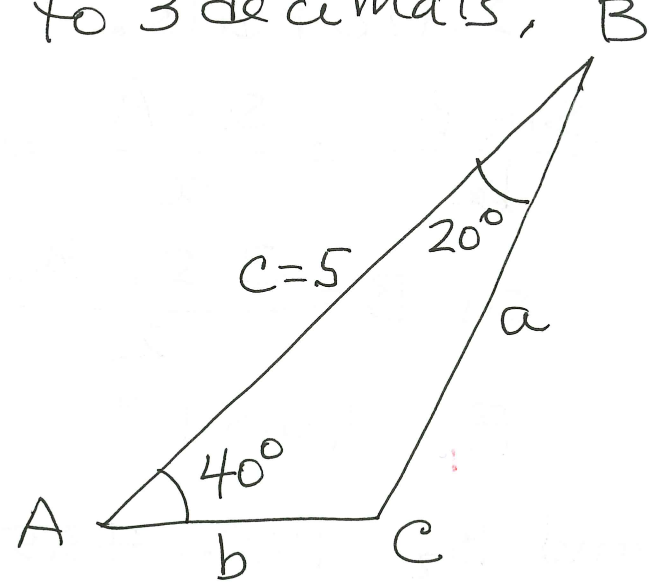
Two sides and an angle opposite <u>one</u> of the sides, SSA.

The <u>Law of Sines</u> is used to solve Cases 1 and 2 triangles. This Law says:

$$\boxed{\begin{gathered}\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \\ \text{OR} \\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\end{gathered}}$$

- Case 1 Example

EX: Solve the given triangle. Round final answers to 3 decimals,



$C = 180° - 40° - 20°$

$\underline{C = 120°}$

To find "a":

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{5 \sin 40°}{\sin 120°} = \underline{3.711}$$

To find "b":

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow b = \frac{5 \sin 20°}{\sin 120°} = \underline{1.975}$$

- Case 2 Examples

This case is known as the ambiguous case, there are three possibilities for thes SSA triangles. The given sides and one angle may for either:

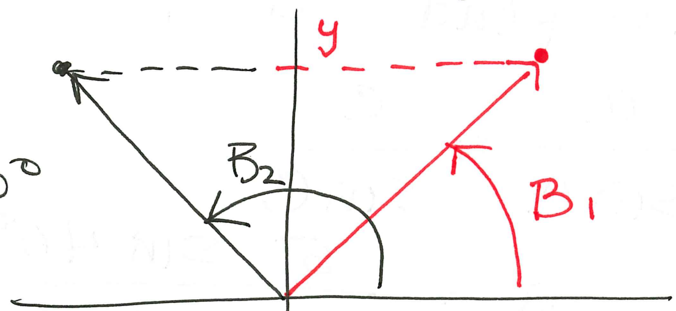
- one triangle only
- two triangles
- no triangles

EX: Solve the triangle with $a=2$, $b=3$ and $A=30°$. Round answers to 2 decimals.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{b \sin A}{a}$$

$$\sin B = \frac{3 \sin 30°}{2} \Rightarrow B = \sin^{-1}\left(\frac{3 \sin 30°}{2}\right)$$

$B_1 = 48.590°$



and $B_2 = 180° - 48.590°$

$= 131.410°$

$131.410° + 30° = 161.410° < 180°$

$\Rightarrow$ two triangles are possible

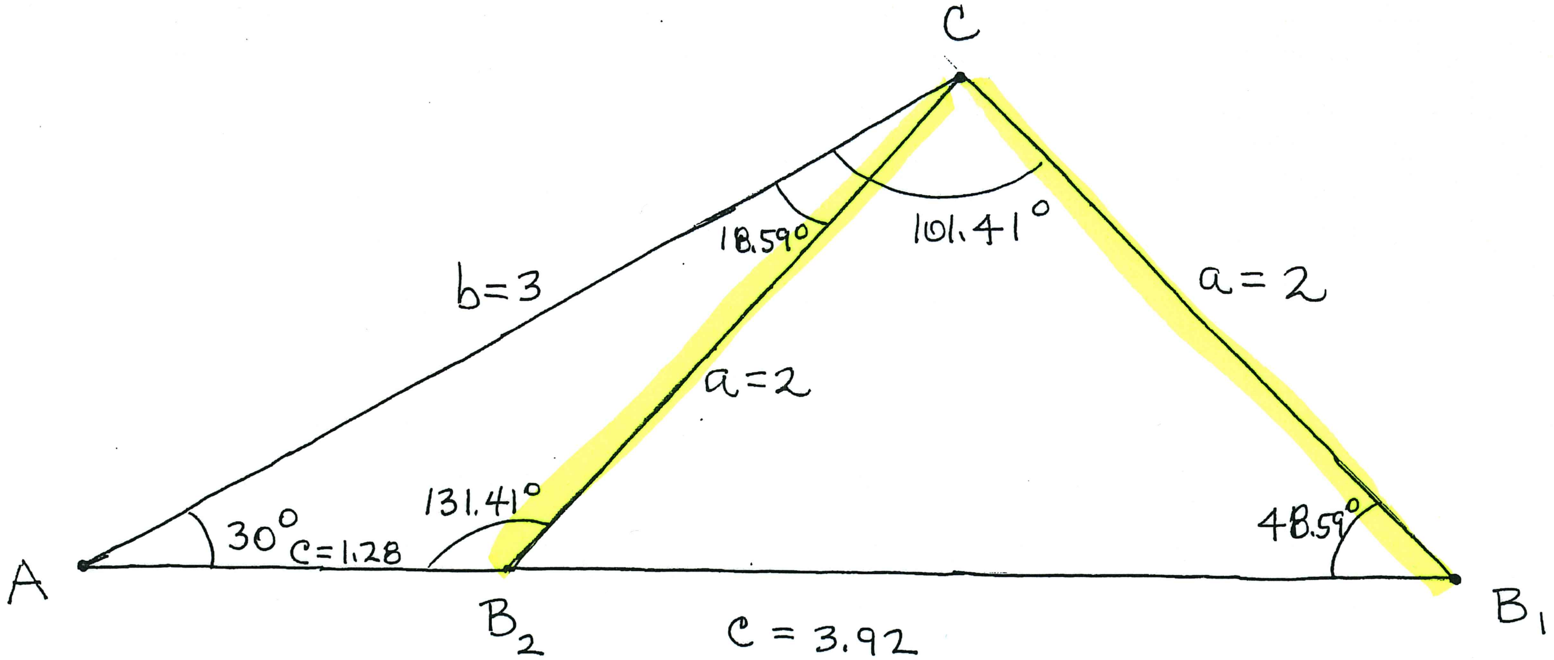
For $\underline{B_1 = 48.59°}$, $C_1 = 180° - 48.59° - 30° = \underline{101.41°}$

$$\frac{c_1}{\sin C_1} = \frac{a}{\sin A} \Rightarrow c_1 = \frac{2 \sin 101.41°}{\sin 30°} = \underline{3.92}$$

For $\underline{B_2 = 131.41°}$, $C_2 = 180° - 131.41° - 30° = \underline{18.59°}$

$$\frac{c_2}{\sin C_2} = \frac{a}{\sin A} \Rightarrow c_2 = \frac{2 \sin 18.59°}{\sin 30°} = \underline{1.28}$$

Here are the two possible triangles:



EX: Solve the triangle with $a=3$, $b=2$ and $A=40°$. Round answers to 2 decimals.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{2\sin 40^\circ}{3}$$

$$\Rightarrow B_1 = \sin^{-1}\left(\frac{2\sin 40^\circ}{3}\right) = 25.374^\circ$$

$$\text{and } B_2 = 180^\circ - 25.374^\circ = 154.626^\circ$$

$$A + B_2 = 40^\circ + 154.626^\circ > 180^\circ$$

$\Rightarrow$ only one possible triangle.

$$B = 25.37^\circ, \quad C = 180^\circ - 40^\circ - 25.374^\circ = 114.63^\circ$$

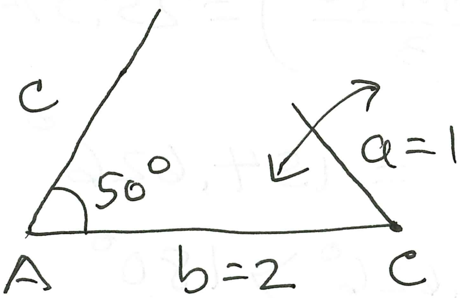
$$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow c = \frac{3\sin 114.626^\circ}{\sin 40^\circ} = 4.24$$

TO BE CONT'D

4.2 Continued

EX: Solve the triangle with $a=1$, $b=2$, and $A=50°$. Round answers to 2 decimals.

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \sin B = \frac{2\sin 50°}{1}$$



$$\Rightarrow B = \sin^{-1}(2\sin 50°) = \sin^{-1}(1.53)$$

$B =$ error ?

$\Rightarrow$ NO TRIANGLE POSSIBLE

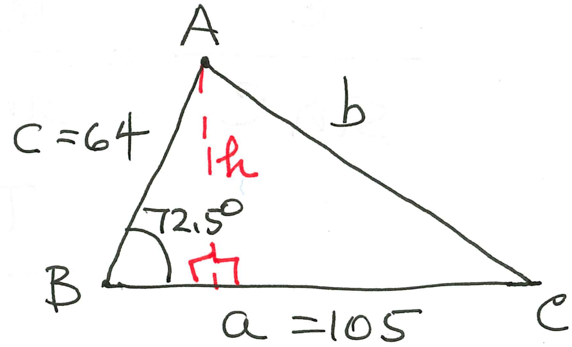
- Applications

EX: Find the area of a triangle with $B=72.5°$, $a=105$ and $c=64$. Find answer to nearest whole number.

From basic Geometry:

Area $= \frac{1}{2}$(base)(height)

Let K = area, h = height



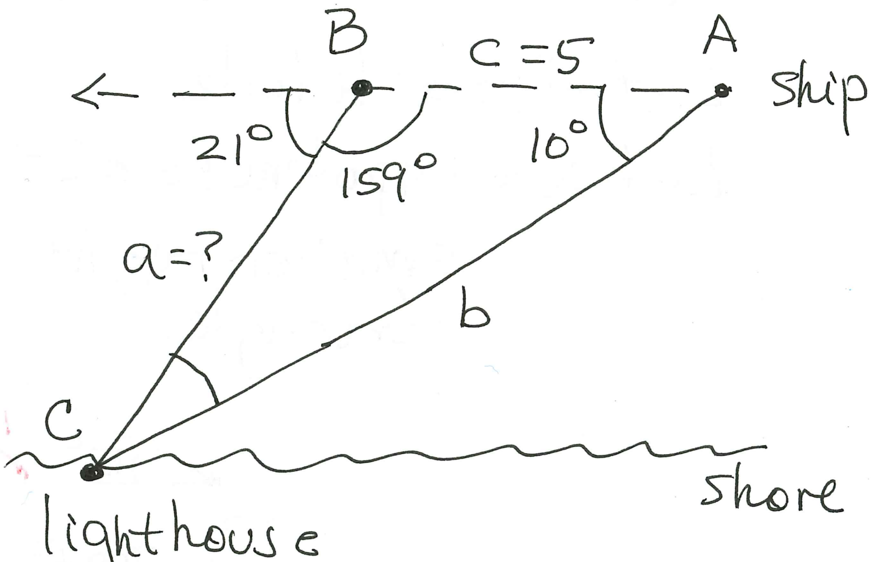
$$\sin B = \frac{h}{c} \Rightarrow h = c\sin B$$
$$= 64\sin 72.5°$$

$$K = \frac{1}{2}(\overset{a}{105})(\overset{c}{64}\sin \overset{B}{72.5°}) = \underbrace{3{,}204}$$

In general, the area of any SAS triangle is:

$$\boxed{K = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A}$$

EX: A ship, sailing parallel to the shore, sight a lighthouse at an angle of 10° from its directeón of travel. After travelling 5 miles further, this angle is 21°. At that time, how far is the ship from the lighthouse? Round answer to 2 decimals.



$C = 180° - 159° - 10°$

$C = 11°$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

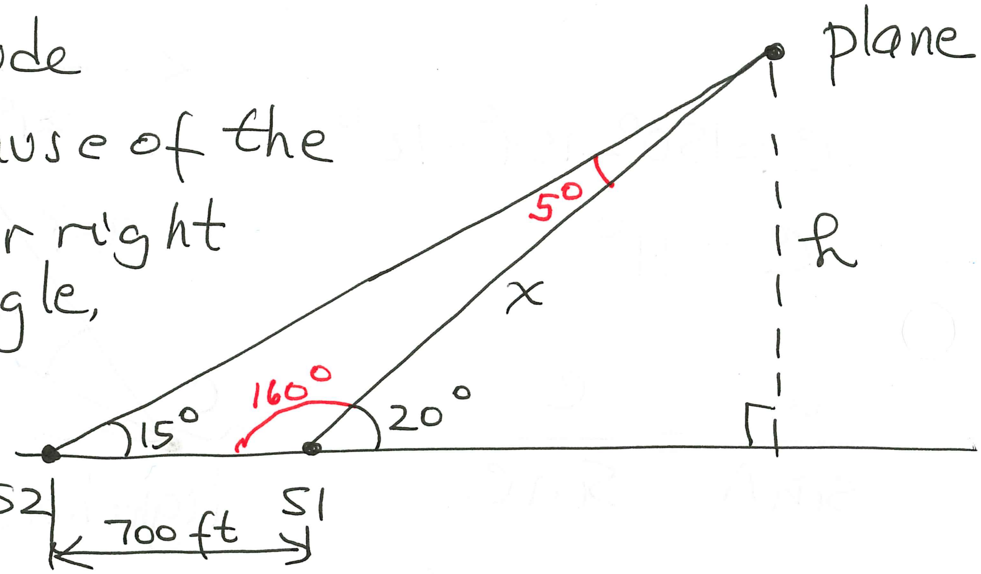
$$a = \frac{c \sin A}{\sin C} = \frac{5 \sin 10°}{\sin 11°}$$

$$\underbrace{a = 4.55 \text{ miles}}$$

EX: Two sensors are spaced 700 ft apart along the approach of a small landing strip. When a plane is landing, the angle of elevation from the first sensor to the plane is 20° and from the second sensor is 15°. Find the altitude of the plane at that instant of time to 2 decimals.

Let h = altitude

Let x = hypotenuse of the smaller right triangle.



$$\frac{x}{\sin 15^\circ} = \frac{700}{\sin 5^\circ} \Rightarrow x = \frac{700 \sin 15^\circ}{\sin 5^\circ}$$

$$\sin 20^\circ = \frac{h}{x} \Rightarrow h = x \sin 20^\circ$$

$$h = \frac{700 \sin 15^\circ \sin 20^\circ}{\sin 5^\circ} = \underline{710.97 \text{ ft}}$$