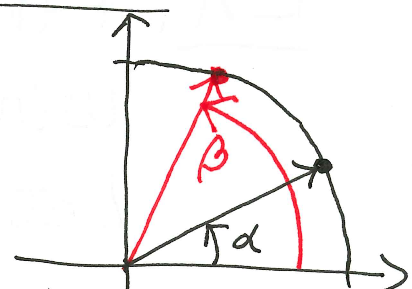
3.5 Sum and Difference Formulas

MWF

Suppose we draw two angles α and β on the unit circle. Then, the identities related to the sum and difference of these two angles are:



(1) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(2) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(3) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(4) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(5) $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

(6) $\tan(\alpha-\beta) = \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

EX: Using identities (1) and (3), derive (5).

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \underbrace{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

EX: Find the exact value for $\sin 15^\circ$ using a difference formula.

Let $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

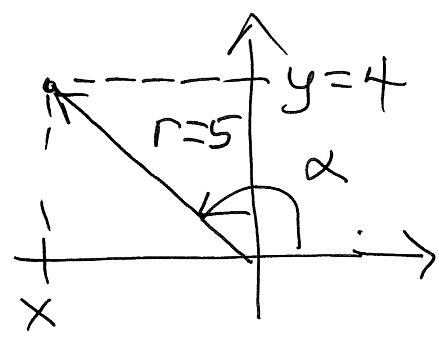
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \underbrace{\frac{\sqrt{6}-\sqrt{2}}{4}} \text{ OR } \underbrace{\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)}$$

EX: Find the exact value given that

$\sin\alpha = \frac{4}{5}$, $\frac{\pi}{2} < \alpha < \pi$ (QII) and $\sin\beta = -\frac{2\sqrt{5}}{5}$, $\pi < \beta < \frac{3\pi}{2}$.

(a) For $\cos\alpha$. (ch. 2 stuff)

$$\sin\alpha = \frac{4}{5} = \frac{y}{r}$$



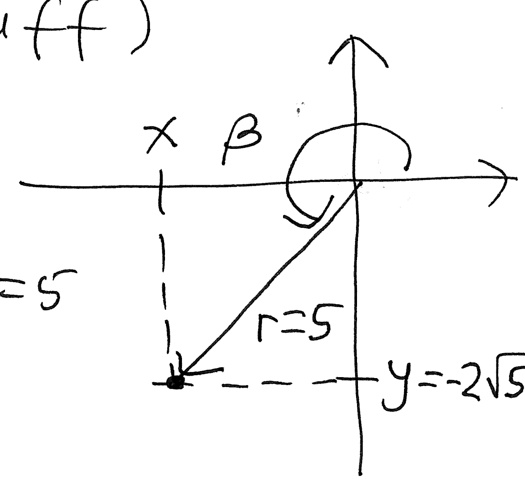
$$\cos\alpha = \frac{x}{r}, \quad x = -\sqrt{r^2 - y^2}$$

$$= -\sqrt{25-16} = -\sqrt{9}$$

$$\underline{x = -3}$$

$$\cos\alpha = \frac{-3}{5} = \underbrace{-\frac{3}{5}}$$

(b) Find $\cos\beta$. (Ch. 2 stuff)



$$\sin\beta = -\frac{2\sqrt{5}}{5}, \quad Q\,\text{III}$$

$$= \frac{y}{r} \Rightarrow y = -2\sqrt{5},\ r = 5$$

$$\cos\beta = \frac{x}{r} \Rightarrow x = -\sqrt{25-20}$$

$$x = -\sqrt{5}$$

$$\cos\beta = \frac{-\sqrt{5}}{5} = -\frac{\sqrt{5}}{5}$$

(c) Find $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$. (Ch. 3 stuff)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \left(-\frac{3}{5}\right)\left(-\frac{\sqrt{5}}{5}\right) - \left(\frac{4}{5}\right)\left(-\frac{2\sqrt{5}}{5}\right)$$

$$= \frac{3\sqrt{5}}{25} + \frac{8\sqrt{5}}{25} = \frac{11\sqrt{5}}{25}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \left(\frac{4}{5}\right)\left(-\frac{\sqrt{5}}{5}\right) + \left(-\frac{3}{5}\right)\left(-\frac{2\sqrt{5}}{5}\right)$$

$$= -\frac{4\sqrt{5}}{25} + \frac{6\sqrt{5}}{25} = \frac{2\sqrt{5}}{25}$$

<u>EX</u>: Establish that $\tan\left(\theta+\frac{\pi}{2}\right) = -\cot\theta$.

LS $\quad \tan\left(\theta+\frac{\pi}{2}\right) = \dfrac{\sin\left(\theta+\frac{\pi}{2}\right)}{\cos\left(\theta+\frac{\pi}{2}\right)}$

$$= \frac{\sin\theta \overset{0}{\cancel{\cos\frac{\pi}{2}}} + \cos\theta \overset{1}{\cancel{\sin\frac{\pi}{2}}}}{\cos\theta \underset{0}{\cancel{\cos\frac{\pi}{2}}} - \sin\theta \underset{1}{\cancel{\sin\frac{\pi}{2}}}}$$

$$= \frac{\cos\theta}{-\sin\theta} = -\frac{\cos\theta}{\sin\theta} = \underline{-\cot\theta} \quad \text{RS}$$

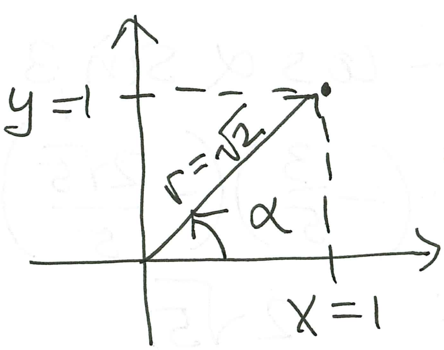
- <u>Using Sum / Diff. Formulas With Inverse Trig. Functeow</u>

<u>EX</u>: Evaluate <u>exactly</u> $\cos\left(\tan^{-1}1 + \cos^{-1}x\right)$

Let $\alpha = \tan^{-1}1$, $\beta = \cos^{-1}x$

Then, what we want is $\cos(\alpha+\beta)$.

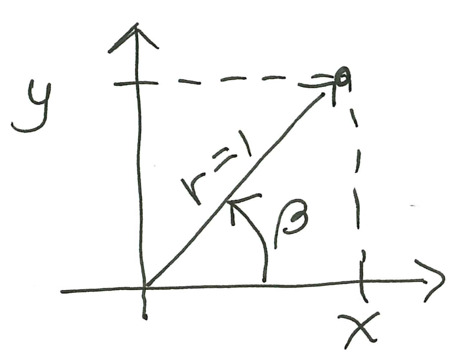
Since $\tan^{-1}1 = \alpha \Rightarrow \tan\alpha = 1 = \frac{1}{1} = \frac{y}{x}$



Also, $\beta = \cos^{-1}x$

$\Rightarrow \cos\beta = x = \frac{x}{1} = \frac{x}{r}$

$y = \sqrt{1-x^2}$



Then, using the two sketches for α & β:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{x}{1}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{1-x^2}}{1}\right)$$

$$= \frac{x}{\sqrt{2}} - \frac{\sqrt{1-x^2}}{\sqrt{2}} = \frac{x-\sqrt{1-x^2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\underbrace{\cos(\alpha+\beta) = \frac{x\sqrt{2}-\sqrt{2-2x^2}}{2}}$$