2.2 Trigonometric Functions: The Unit Circle Approach

There are two ways to define trig. functions:

- Unit Circle (this Section – general case)
- Right Triangle (Sec. 4.1 – special case)

- The Unit Circle

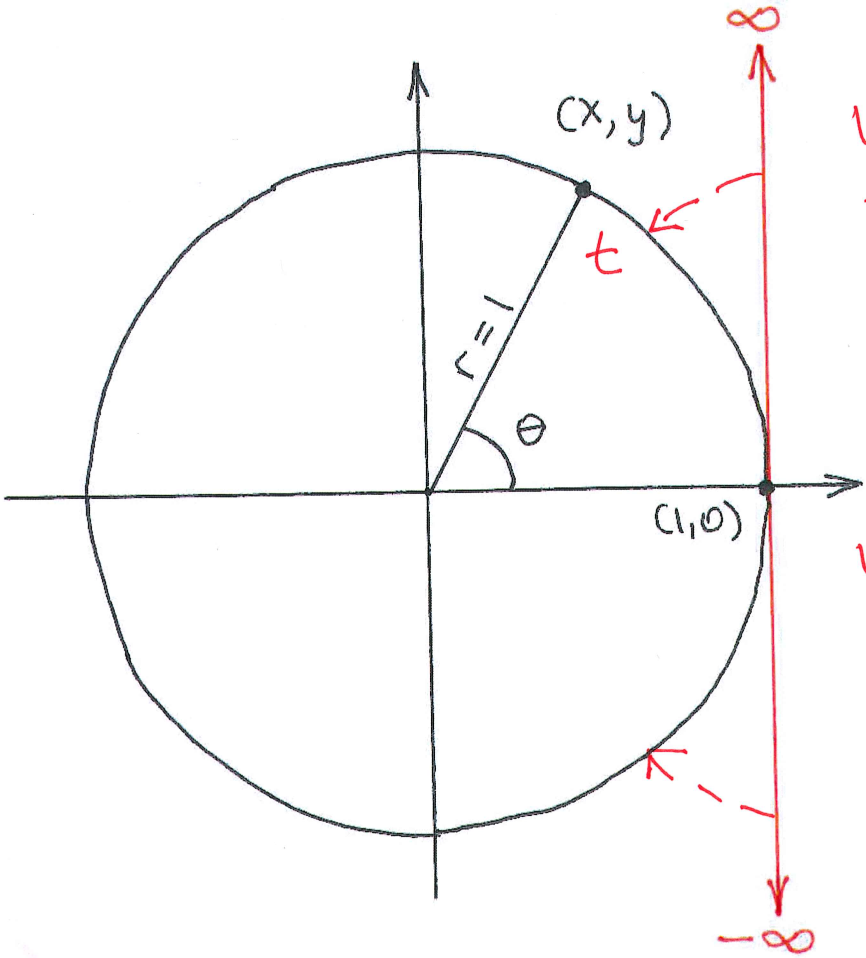
Recall from Algebra that $x^2+y^2=1$ is the equation of a circle with its center at the origin and with a radius of 1 unit

TO BE CONT'D

If we imagine a number line wrapped around a unit circle so that the positive numbers go around ccw and the negative numbers go around cw starting at the point $(1,0)$, then each number on the number line will correspond to central angle θ so that the arc length

$$t = r\theta.$$

And since $r=1$, then $t=\theta$.



Wrap positive number line around the circle ccw.

Wrap negative number line around the circle cw.

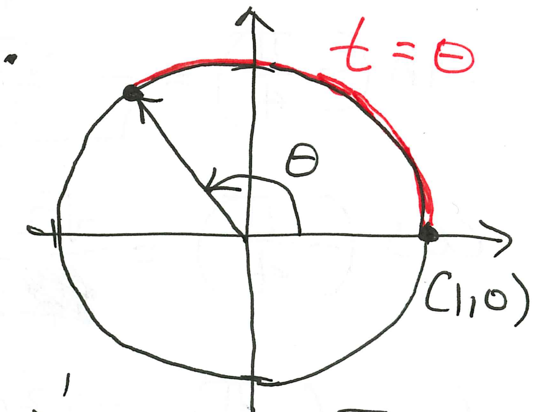
For example, if $\theta = \frac{\pi}{3}$, then $t = \frac{\pi}{3}$, and so on.

<u>Def.</u> – Let t be in $\mathbb{R}$ and (x,y) be <u>any</u> point on the Unit Circle corresponding to t, <u>then</u>:

(sine) $\sin t = y$	(cosecant) $\csc t = \frac{1}{y}$
(cosine) $\cos t = x$	(secant) $\sec t = \frac{1}{x}$
(tangent) $\tan t = \frac{y}{x}$	(cotangent) $\cot t = \frac{x}{y}$

<u>NOTE:</u> every trig. function has an argument t. A trig. function written <u>without</u> an argument is <u>meaningless</u> and thus incorrect math notation.

EX: Given a point $P = \left(-\frac{5}{11}, \frac{4\sqrt{6}}{11}\right)$ on the Unit Circle. Use the definitions to evaluate the six trig. functions.



$\sin t = y = \frac{4\sqrt{6}}{11}$

$\cos t = x = -\frac{5}{11}$

$\tan t = \frac{y}{x} = \frac{4\sqrt{6}/11}{-5/11} = -\frac{4\sqrt{6}}{\cancel{11}} \cdot \frac{\cancel{11}}{5} = -\frac{4\sqrt{6}}{5}$

$\cot t = \frac{x}{y} = \frac{-5/\cancel{11}}{4\sqrt{6}/\cancel{11}} = -\frac{5}{4\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = -\frac{5\sqrt{6}}{24}$

$\csc t = \frac{1}{y} = \frac{1}{4\sqrt{6}/11} = \frac{11}{4\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{11\sqrt{6}}{24}$

$\sec t = \frac{1}{x} = \frac{1}{-5/11} = -\frac{11}{5}$

EX: Use the Unit Circle to evaluate the 6 trig. functions for $t = \frac{2\pi}{3}$.

$\sin\left(\frac{2\pi}{3}\right) = y = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{2\pi}{3}\right) = x = -\frac{1}{2}$

$\tan\left(\frac{2\pi}{3}\right) = \frac{y}{x} = \frac{\sqrt{3}/\cancel{2}}{-1/\cancel{2}}$

$= -\sqrt{3}$

$\csc\left(\frac{2\pi}{3}\right) = \frac{1}{y} = \frac{1}{\sqrt{3}/2}$

$= \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\sec\left(\frac{2\pi}{3}\right) = \frac{1}{x} = -2$

$\cot\left(\frac{2\pi}{3}\right) = \frac{x}{y} = \frac{-1/\cancel{2}}{\sqrt{3}/\cancel{2}}$

$= -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

EX: Evaluate the 6 trig. functions for $t = -\frac{\pi}{4}$.

$\sin\left(-\frac{\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$

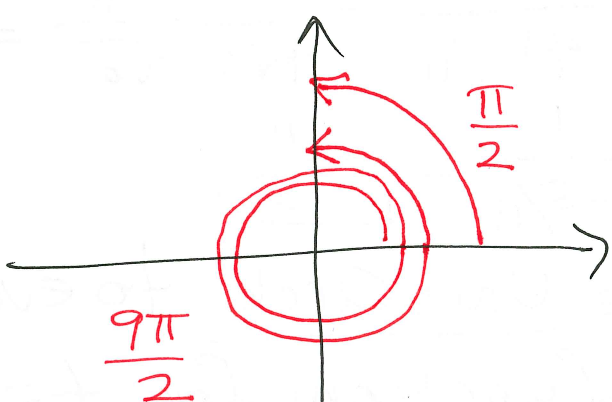
$\csc\left(-\frac{\pi}{4}\right) = -\frac{2}{\sqrt{2}} = -\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{-\sqrt{2}}$

$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sec\left(-\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$

$\tan\left(-\frac{\pi}{4}\right) = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = \boxed{-1}$, $\cot\left(-\frac{\pi}{4}\right) = \boxed{-1}$

EX: Find the exact value for $\sin\left(\frac{9\pi}{2}\right)$ without using a calculator.



Since $\frac{9\pi}{2}$ and $\frac{\pi}{2}$ are coterminal angles, then they they have the same values for all 6 trig. functions.

$$\Rightarrow \sin\left(\frac{9\pi}{2}\right) = \sin\left(\frac{8\pi}{2} + \frac{\pi}{2}\right) = \sin\left(4\pi + \frac{\pi}{2}\right)$$

$$= \sin\left(\frac{\pi}{2}\right) = \boxed{1}$$

- <u>Using a Calculator to Evaluate Trig. Functions</u>

Make sure the we have the correct $\boxed{\text{MODE}}$ setting, either <u>degree</u> or <u>radian</u>.

<u>EX</u>: Evaluate to 4 decimals.

(a) $\sin(32.6^\circ) = \underline{0.5388}$

(b) $\tan(1.5) = \underline{14.1014}$

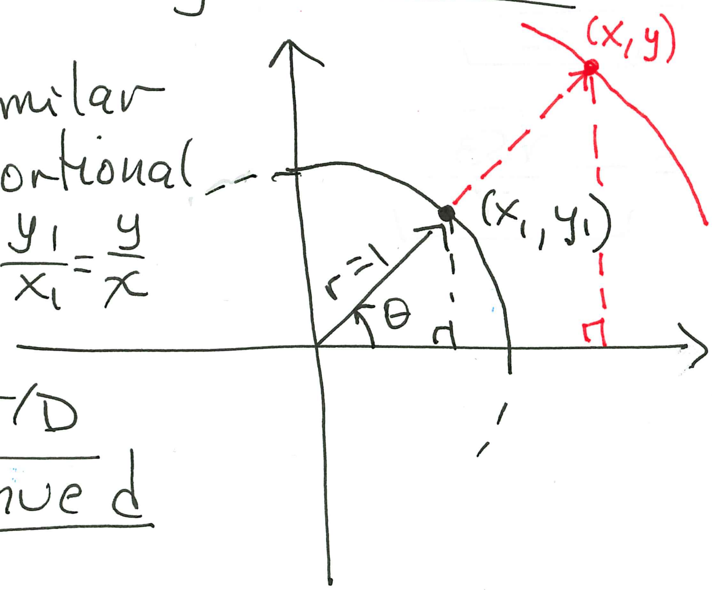
(c) $\csc(32.6^\circ) = \dfrac{1}{\sin(32.6^\circ)} = \underline{1.8561}$

- <u>Evaluating Trig. Functions for a Circle of Any Radius r</u>

Triangles are similar

$\Rightarrow$ sides are proportional

$\dfrac{y_1}{1} = \dfrac{y}{r}$, $\dfrac{x_1}{1} = \dfrac{x}{r}$, $\dfrac{y_1}{x_1} = \dfrac{y}{x}$



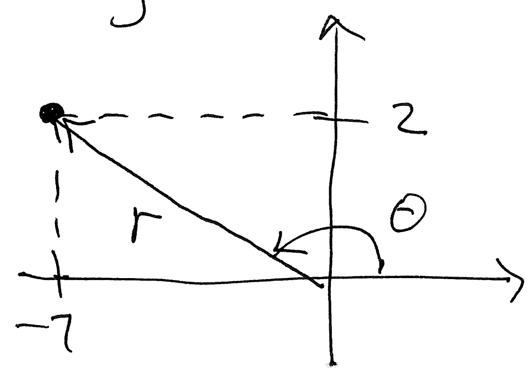
<u>TO BE CONT'D</u>

<u>2.2 Continued</u>

<u>Therefore,</u>

$$\boxed{\begin{array}{lll} \sin\theta = \dfrac{y}{r}, & \cos\theta = \dfrac{x}{r}, & \tan\theta = \dfrac{y}{x} \\[2ex] \csc\theta = \dfrac{r}{y}, & \sec\theta = \dfrac{r}{x}, & \cot\theta = \dfrac{x}{y} \end{array}}$$

EX: Find the exact value for the 6 trig. functions if $(-7,2)$ is a point on the terminal side of an angle θ.



$$x^2+y^2=r^2$$
$$\sqrt{x^2+y^2}=r$$
$$\sqrt{49+4}=r$$
$$\sqrt{53}=r$$

$$\sin\theta=\frac{y}{r}=\frac{2}{\sqrt{53}}\cdot\frac{\sqrt{53}}{\sqrt{53}}=\boxed{\frac{2\sqrt{53}}{53}}$$

$$\cos\theta=\frac{x}{r}=\frac{-7}{\sqrt{53}}=\boxed{-\frac{7\sqrt{53}}{53}}$$

$$\tan\theta=\frac{y}{x}=\frac{2}{-7}=\boxed{-\frac{2}{7}}$$

$$\cot\theta=\frac{x}{y}=\frac{-7}{2}=\boxed{-\frac{7}{2}}$$

$$\csc\theta=\frac{r}{y}=\boxed{\frac{\sqrt{53}}{2}}$$

$$\sec\theta=\frac{r}{x}=\boxed{-\frac{\sqrt{53}}{7}}$$