

# 5.1/2 Area and Estimating with Finite Sums & Limits

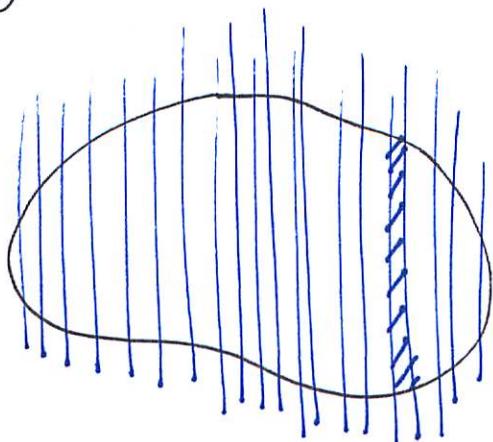
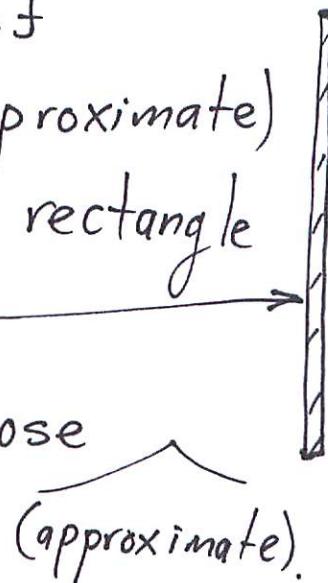
Fundamental Question:

How can one compute the area of a planar region bounded by one or more curves (or lines) ?

General Strategy:

Slice the region into bazillions of extremely thin stripes in the (approximate) shape of an extremely elongated rectangle like this one.

Then add up the areas of those rectangles.



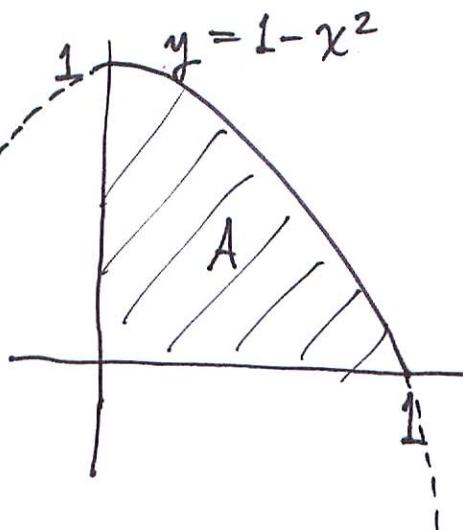
Area  $\approx$  sum of areas of region (quasi)rectangles.

5.182...

(2)

## Worked Example.

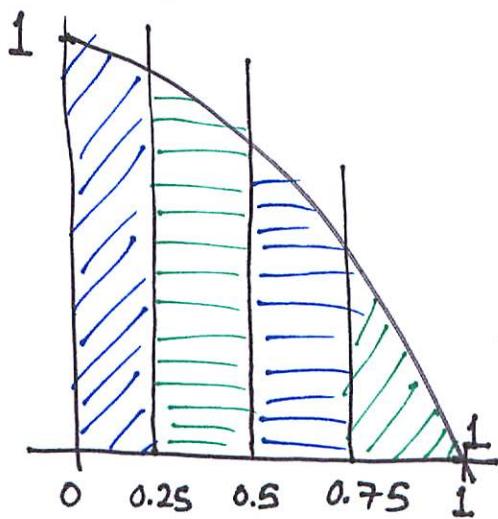
Estimate the area A of the "parabolic triangle" in the 1<sup>st</sup> quadrant, bounded by  $y = 1 - x^2$  (and the x- & y-axes).



\* Pick a number  $n$  of thin slices to chop into.

We take  $n=4$  (admittedly, a very small number).

\* Draw a picture of the chopping.



Very loosely speaking, each of the  $n=4$  pieces is roughly a rectangle. The larger  $n$  becomes, the more these pieces look like thin slices quite (elongated) rectangular in shape.

5.182...

(3)

\* Estimate the area of each of the  $n$  pieces.

Note: There is no "universal" method to approximate the area of an irregular shape.

Clearly, we don't (currently) have a way to exactly find each of the areas of the pieces.

There are various methods / techniques that one can apply.

We start by choosing currently:

Method I: Approximate Area ~~from~~<sup>by</sup> Overestimation

We will approximate the area of a thin slice ~~slice~~ by enclosing it in a rectangle.

The area of the rectangle overestimates the area of the slice and is called an upper estimate.



Area of enclosing rectangle is upper estimate for area of slice.

5.1 & 2...

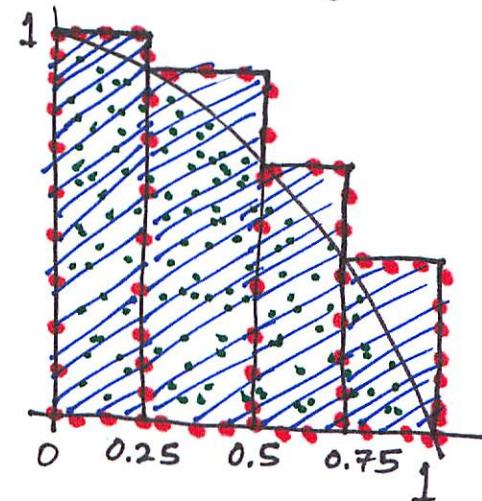
(4)

## Upper Approximation to area of region under

→ "upper"

$$\begin{aligned}U_1 &= \text{area of } 1^{\text{st}} \text{ rectangle} \\&= \text{base} \cdot \text{height} \\&= 0.25 \cdot f(0) \\&= 0.25 \cdot 1 = 0.25.\end{aligned}$$

---



$$\begin{aligned}U_2 &= \text{area of } 2^{\text{nd}} \text{ rectangle} \\&= 0.25 \cdot f(0.25) \\&= 0.25 \cdot 0.9375 \\&\approx 0.2344\end{aligned}$$

---

$$\begin{aligned}U_3 &= \text{area of } 3^{\text{rd}} \text{ rectangle} \\&= 0.25 \cdot f(0.5) \approx 0.1875\end{aligned}$$

---

$$\begin{aligned}U_4 &= \text{area of } 4^{\text{th}} \text{ rectangle} \\&= 0.25 \cdot f(0.75) \approx 0.1094\end{aligned}$$

---

$x$	$f(x) = 1 - x^2$
0	1
$\frac{1}{4} = 0.25$	$\frac{15}{16} = 0.9375$
$\frac{1}{2} = 0.5$	$\frac{3}{4} = 0.75$
$\frac{3}{4} = 0.75$	$\frac{7}{16} = 0.4375$
1	0

Upper Approximation with  $n=4$  slices:

$$U_4 = U_1 + U_2 + U_3 + U_4$$

$$U_4 = 0.25 + 0.2344 + 0.1875 + 0.1094$$

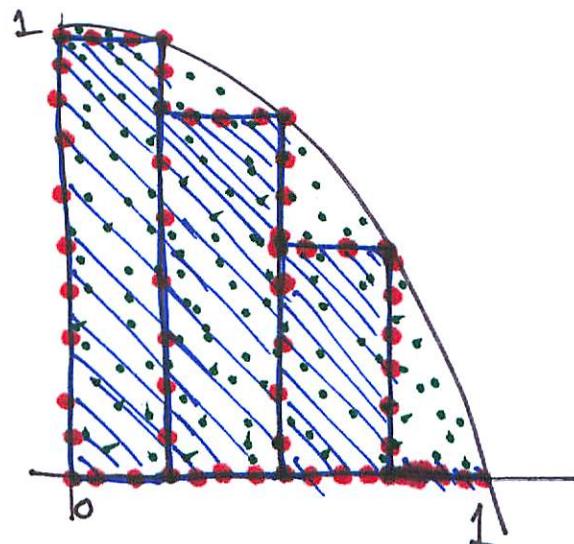
$$U_4 \doteq 0.7813.$$

5.1(b)2. Method II: Underestimate area of slices  
 Lower Approximation to area by inscribed rectangles (5)

$$l_1 \xrightarrow{\text{"lower"}} \begin{aligned} &= \text{area of } 1^{\text{st}} \text{ rectangle} \\ &= 0.25 \times f(0.25) \\ &= 0.25 \times 0.9375 \\ &\approx 0.2344 \end{aligned}$$

$$l_2 = \text{area of } 2^{\text{nd}} \text{ rectangle} \begin{aligned} &= 0.25 \times f(0.5) \\ &= 0.25 \times 0.75 \\ &\approx 0.1875 \end{aligned}$$

$$l_3 = \text{area of } 3^{\text{rd}} \text{ rectangle} \begin{aligned} &= 0.25 \times f(0.75) \\ &= 0.25 \times 0.4375 \\ &= 0.1094 \end{aligned}$$



$$l_4 = \text{area of } 4^{\text{th}} \text{ rectangle} \begin{aligned} &= 0.25 \times f(1) \\ &= 0.25 \times 0 \\ &= 0. \end{aligned}$$

Lower Approximation with  $n=4$  slices:

$$L_4 = l_1 + l_2 + l_3 + l_4$$

$$L_4 = 0.2344 + 0.1875 + 0.1094 + 0$$

~~$L_4 \approx 0.6213$~~

$$\underline{L_4 \doteq 0.5313.}$$

(6)

## 5.1b.2 General Method to Approximate Areas:

Rectangles whose height is a value of  $f(x)$ .

\* Split interval  
into  $n$  subintervals

(say  $n=4$ ,  
say subintervals equal)

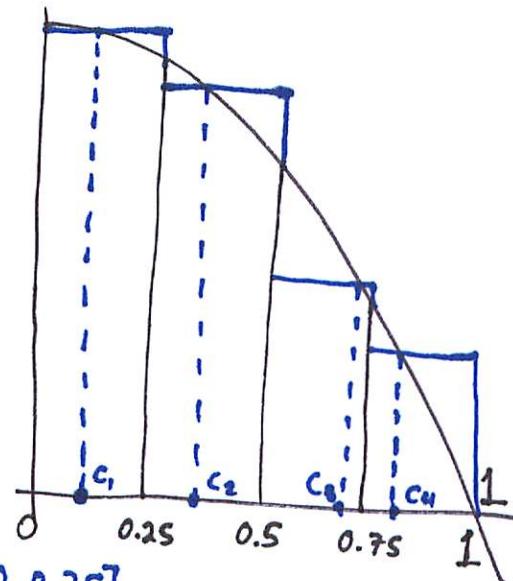
\* Pick ~~any~~ points:

$c_1$  in 1<sup>st</sup> interval  $[0, 0.25]$ ,

$c_2$  in 2<sup>nd</sup> interval  $[0.25, 0.5]$ ,

$c_3$  in 3<sup>rd</sup> interval  $[0.5, 0.75]$ ,

$c_4$  in 4<sup>th</sup> interval  $[0.75, 1]$ .



\* Approximate area of region using  $n$  rectangles:

Area of 1<sup>st</sup> rectangle  $A_1 = \text{base} \cdot \text{height} = 0.25 \times f(c_1)$

$$\text{--- } 2^{\text{nd}} \text{ --- } A_2 = 0.25 \times f(c_2)$$

$$\text{--- } 3^{\text{rd}} \text{ --- } A_3 = 0.25 \times f(c_3)$$

$$\text{--- } 4^{\text{th}} \text{ --- } A_4 = 0.25 \times f(c_4)$$

Approximation to area of region:

$$A \approx 0.25f(c_1) + 0.25f(c_2) + 0.25f(c_3) + 0.25f(c_4)$$

5.1&2...

(7)

## The Mid Point Rule

$$f(x) = 1 - x^2 \quad \text{for } 0 \leq x \leq 1, \underline{n=4}.$$

Take  $c_1 = \text{midpoint of } 1^{\text{st}} \text{ interval} = \frac{0+0.25}{2} = 0.125$

$$c_2 = \text{--- } 2^{\text{nd}} \text{ ---} = \frac{0.25+0.5}{2} = 0.375$$

$$c_3 = \text{--- } 3^{\text{rd}} \text{ ---} = \frac{0.5+0.75}{2} = 0.625$$

$$c_4 = \text{--- } 4^{\text{th}} \text{ ---} = \frac{0.75+1}{2} = 0.875$$

Midpoint approximation to area with  $n=4$  divisions:

$$A \approx 0.25 f(0.125) + 0.25 f(0.375) + 0.25 f(0.625) + 0.25 f(0.875)$$

$$= 0.25 \times 0.984 + 0.25 \times 0.859 + 0.25 \times 0.609 + 0.25 \times 0.234$$

~~A ≈ 0.234~~

$$\boxed{A \approx 0.672.}$$

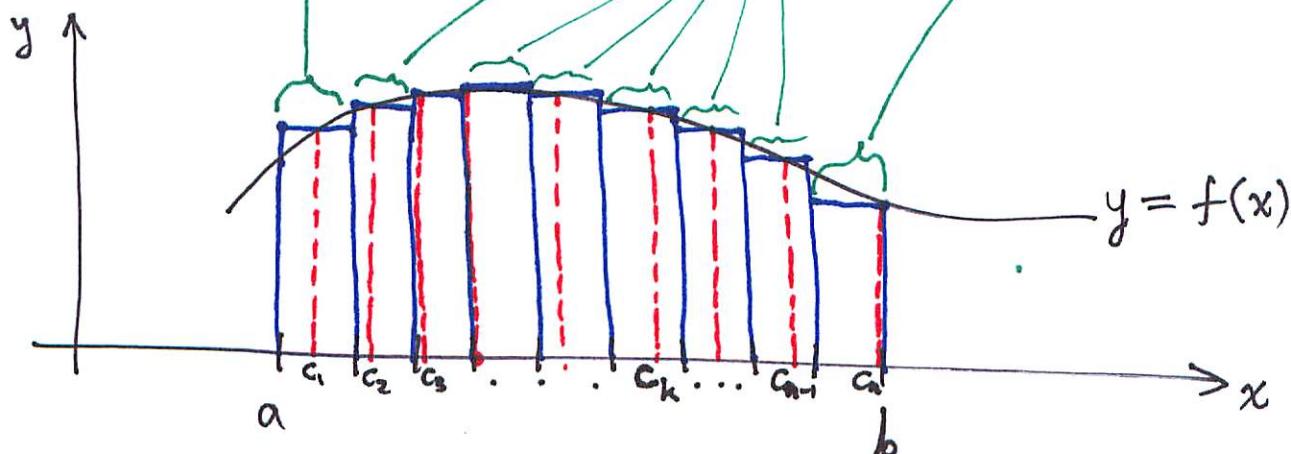
5.1Q2...

(Approximate!)

Summary: Area under curve  $y=f(x)$   
for  $a \leq x \leq b$ .

- \* Pick number  $n$  (ideally large) of parts.
- \* ~~Diff~~ Split the interval  $[a, b]$  into  $n$  subintervals.  
(say, if intervals are equal, then all of them have the same length  $= \frac{b-a}{n}$ .)
- \* Pick  $c_1$  in 1<sup>st</sup> interval,  $c_2$  in 2<sup>nd</sup>,  $c_3$  in 3<sup>rd</sup>, ...  
 $c_k$  in  $k^{\text{th}}$  interval, ...,  $c_{n-1}$  in  $(n-1)^{\text{st}}$ ,  $c_n$  in  $n^{\text{th}}$  interval.  
Here  $k$  stands for any number  $1, 2, 3, \dots, n$ .
- \* Compute areas of rectangles approximating slices of the region:  
 $f(c_1) \cdot \frac{b-a}{n}, f(c_2) \cdot \frac{b-a}{n}, \dots, f(c_k) \cdot \frac{b-a}{n}, \dots, f(c_n) \cdot \frac{b-a}{n}$ .
- \* Sum rectangle areas to approximate area under curve.

$$A \approx f(c_1) \cdot \frac{b-a}{n} + f(c_2) \cdot \frac{b-a}{n} + \dots + f(c_n) \cdot \frac{b-a}{n}.$$



5.182...

(9)

## Riemann Sums

+ Standard notation:  $\Delta x = \frac{b-a}{n}$

is the length of each of the  $n$  small (equal) intervals into which  $[a, b]$  is split.

(This is also the thin width of each of the slices.)

The area is approximated by

$$f(c_1)\Delta x + f(c_2)\Delta x + \dots + f(c_n)\Delta x$$

This is known as the **Riemann<sup>(\*)</sup> Sum**

for  $f(x)$  on  $[a, b]$  (with sample points  $c_1, c_2, \dots, c_n$ ).

### Sigma Notation for a Riemann sum:

$$A \approx \sum_{k=1}^n f(c_k)\Delta x = f(c_1)\frac{b-a}{n} + \dots + f(c_n)\frac{b-a}{n}$$

$\underbrace{\phantom{f(c_k)}}$  Height of rectangle approximating slice

$\Delta x = \frac{b-a}{n}$ : width of rectangle/slice

(\*) Bernhard Riemann, a XIX century mathematician.

## Area = Limit of Riemann Sums.

It stands to reason that the area under  $y = f(x)$  between  $a$  &  $b$  should be very closely approximated by the Riemann sum, that is by the areas of rectangles approximating the shape of each thin slice when  $n$  is very large.

$$A = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

(if the limit exists).

5.1 & 2 ...

(1)

## Exact Area under $y = 1 - x^2$ , $0 \leq x \leq 1$ .

Let  $n =$  large # of pieces to chop  $[0, 1]$  into.

$$\Delta x = \frac{\text{interval length}}{n} = \frac{1-0}{n} = \frac{1}{n}.$$

For concreteness, take  $c_k =$  right endpoint  
(of the  $k^{\text{th}}$  interval).

Intervals: 1<sup>st</sup>:  $[0, \frac{1}{n}] \rightarrow c_1 = \frac{1}{n}$

2<sup>nd</sup>:  $[\frac{1}{n}, \frac{2}{n}] \rightarrow c_2 = \frac{2}{n}$

...

$k^{\text{th}}$ :  $[\frac{k-1}{n}, \frac{k}{n}] \rightarrow c_k = \frac{k}{n}$  } This is  
the key!

(last)  $n^{\text{th}}$ :  $[\frac{n-1}{n}, 1] \rightarrow c_n = \frac{n}{n} = 1$ .

Riemann sum:

$$f(x) = 1 - x^2$$

Constant!  
(As  $k$  varies)

$$\sum_{k=1}^n f(c_k) \Delta x = \sum_{k=1}^n (1 - c_k^2) \cdot \frac{1}{n} = \sum_{k=1}^n \left[1 - \left(\frac{k}{n}\right)^2\right] \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{k=1}^n \left(1 - \frac{k^2}{n^2}\right) = \frac{1}{n} \left[ \sum_{k=1}^n 1 - \sum_{k=1}^n \frac{k^2}{n^2} \right]$$

Constant factor  $\frac{1}{n}$  pulled out

Addition splits sum  
Subtraction

5.1 & 5.2 ...  $y = 1 - x^2$  on  $[0, 1]$  cont'd...

$$\begin{aligned} \text{Riemann sum} &= \frac{1}{n} \sum_{k=1}^n 1 - \frac{1}{n} \sum_{k=1}^n \frac{1}{n^2} k^2 \\ &= \frac{1}{n} \sum_{k=1}^n 1 - \frac{1}{n} \cdot \frac{1}{n^2} \sum_{k=1}^n k^2 \end{aligned}$$

Constant multiple  $\frac{1}{n^2}$  pulled out.

So far:

$$\text{Riemann Sum} = \frac{1}{n} \sum_{k=1}^n 1 - \frac{1}{n^3} \sum_{k=1}^n k^2$$

Exact area = limit as  $n \rightarrow \infty$  of Riemann sum:

$$A = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{k=1}^n 1 - \frac{1}{n^3} \sum_{k=1}^n k^2 \right).$$

We want to find this area!

As usual, there's no free lunch:

We must evaluate the sums in closed form.

It is not (ever) possible to pass directly from sigma notation to an answer by taking a direct limit as  $n \rightarrow \infty$ .

5.1 & 5.2

$y = 1 - x^2$  on  $[0, 1]$  cont'd..

(13)

Evaluate the Riemann sum in closed form:

$$\begin{aligned}
 & \frac{1}{n} \sum_{k=1}^n 1 - \frac{1}{n^3} \sum_{k=1}^n k^2 \\
 &= \frac{1}{n} \cdot n - \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
 &= 1 - \frac{(n+1)(2n+1)}{6n^2}.
 \end{aligned}$$

Sum of constant rule  
 &  
 Sum of squares formula

Final Stretch: Riemann sum, now evaluated in closed form!

$$\begin{aligned}
 A &= \lim_{n \rightarrow \infty} \left( 1 - \frac{(n+1)(2n+1)}{6n^2} \right) \\
 &= \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} \\
 &= 1 - \frac{1}{6} \lim_{n \rightarrow \infty} (1 + \gamma_n)(2 + \gamma_n)
 \end{aligned}$$

$\gamma = \frac{1}{6} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} = \frac{1}{6}(1 + \frac{1}{n})(2 + \frac{1}{n})$   
 $\lim_{n \rightarrow \infty} \gamma_n = 0$

$$\begin{aligned}
 &= 1 - \frac{1}{6} (1 + 0)(2 + 0) \\
 &= 1 - \frac{2}{6} = 1 - \frac{1}{3} = \frac{2}{3}.
 \end{aligned}$$

$0.666\dots = \underline{\underline{A = 2/3.}}$

Recall approximations:  $A \approx 0.672$  (midpoint upper)  
 $A \approx 0.5313$  (lower)  
 $A \approx 0.7813$  (upper)

# Further applications of Areas under Curves.

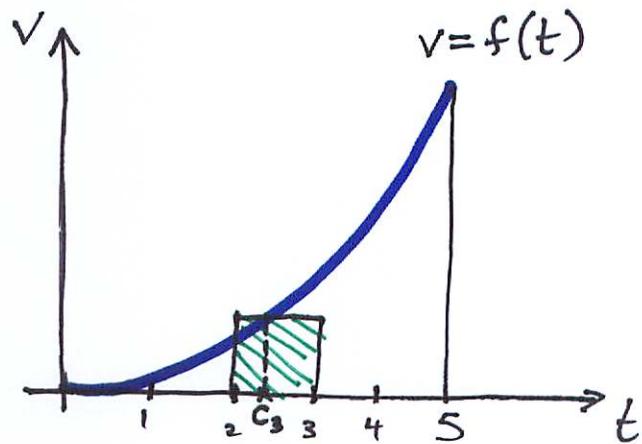
## 1) Distance Traveled.

Object (say, a car)  
moving at variable speed  
(say, accelerating - step on it!)

If velocity (say, speedometer readings)  
~~are~~ known at each instant,  
but needn't be constant, how is distance traveled found?

**DISTANCE = AREA UNDER VELOCITY CURVE**  
(as a function of time)

Why so? During a small time interval, say  
 $t$  between 2s & 3s, the distance traveled,  $\Delta x$ ,  
may be approximated by  $\Delta x \approx v(c_3) \Delta t$ .  
This is exactly the area of a rectangle  
approximating the area under the curve  $v = f(t)$ .  
As time intervals shrink ( $n \rightarrow \infty$ ), distance = sum of all small  $\Delta x$   
= Area under curve.



Velocity  
 $v = \frac{\text{displacement}}{\text{time}}$

## Further Application : Average of Function.

2) The average of a nonconstant function.

If  $f(x) = k$

is a constant, say on  $[a, b]$ ,

then the region under  
the graph is a rectangle.

If  $f(x)$  is nonconstant,

"flatten down" the graph

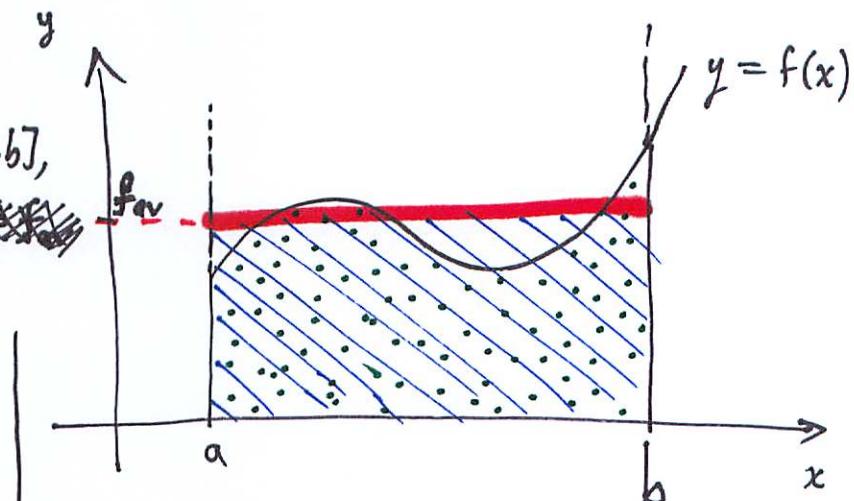
to a constant line

in such a manner as

to make the curved

top of the region flat

while keeping the total  
area the same.



$$\begin{aligned} \text{Area of rectangle } (b-a)f_{\text{avg}} \\ = \text{Area under curve } y=f(x) \end{aligned}$$

$$f_{\text{avg}} = \frac{\text{Area}}{b-a}$$

Original area under  $y=f(x)$  on  $[a, b] = A$ .

Area under rectangle with height  $h = (b-a)h$ .

Hence  $h(b-a) = A$  to make area of rectangle  
equal to  $A$ . This height  $h = \frac{A}{b-a}$  is  $f_{\text{avg}}$ ; Average of f.