

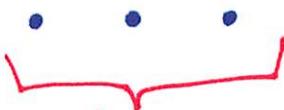
# 5.0. Sigma Notation: $\Sigma$ .

Motivation: As a first-grader, K.F. Gauss was given the assignment to add the integers from 1 to 100:

$$1 + 2 + 3 + 4 + 5 + \dots + 99 + 100 = ?$$

I hope you all understand that (...) above stands for "the sum of integers from 6 to 98".

Math abhors vaguenesses of interpretation such as that implicit in the interpretation of

  
Never has a precise meaning!

Question: How can one capture in precise mathematical notation long sums like the one above?

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"Long" sums are expressed using  
SIGMA NOTATION.

Denote any integer between 1 & 100  
by the so-called "dummy variable"  $k$ :

$k$  takes the values 1, 2, 3, ..., 100.

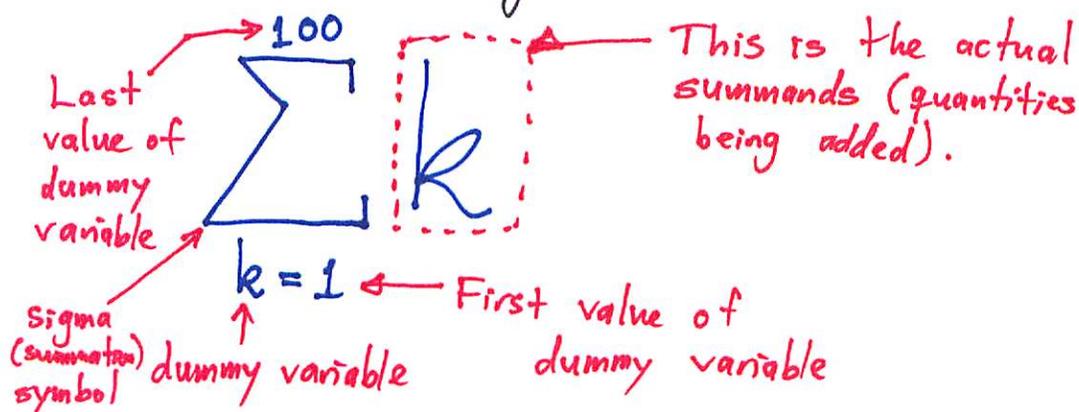
The capital (Greek) letter  $\Sigma$  corresponds to "S"  
in the Latin alphabet, reminding us of the word "sum".

We wish to indicate the sum of the  
numbers  $k$  where  $k$  ranges from 1 to 100.

Notation:

$$1 + 2 + \dots + 100$$

is written using  $\Sigma$  notation:  $\sum_{k=1}^{100} k$



## More Examples:

Your little brother or sister may ask:

How much is 5 plus 5 plus 5 plus 5 plus 5 plus...plus 5 one hundred times?

(In the minds of children 3-4 years old, one hundred is just about the largest number of all!)

The youngster is asking about the value of

$$\underbrace{5 + 5 + 5 + \dots + 5}_{100 \text{ times}}$$

You, as a teenager or so, know that his/her question can be stated mathematically precisely as: What is the value of

$$\sum_{i=1}^{100} 5 \quad ?$$

\* Note that I have chosen the dummy variable  $i$  this time. The dummy variable is just a placeholder for numbers (integers) in the range 1 through 100. Any variable (not yet used in the problem) may be used.

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Another important comment:

Sigma notation does not, by itself, answer any question about finding the value of a long sum.

Just because you rephrase your sibling's question from (vague)  $\underbrace{5 + 5 + \dots + 5}_{100 \text{ times}}$  to the mathematically

precise sigma expression  $\sum_{i=1}^{100} 5$ , it does not mean you have found the answer!

Of course the answer is:

$$\sum_{i=1}^{100} 5 = 100 \times 5 = 500.$$

BUT Sigma notation doesn't help you find the answer all by itself! You still need to know something quite beyond, namely multiplication is the answer to a repeated addition of equal quantities.

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A couple further examples:

Repeated addition a times of a fixed number b is simply  $a \cdot b$  (multiplication). (constant).

Example:  $7 + 7 + 7 = \sum_{j=1}^3 7 = 3 \times 7 = 21.$

and:  $\sum_{j=1}^a b = \underbrace{b + b + \dots + b}_{a \text{ times}} = a \cdot b.$

Gauss' question:

$$\sum_{k=1}^{100} k = 1 + 2 + \dots + 100 = 5,050.$$

[ (I hope you know how to figure this out! )

Gauss, as a first grader and without knowledge of any specific formulas, immediately walked to the teacher to tell him the answer to the question, back in the late 18<sup>th</sup> century. ]

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Example: Sums of odd numbers.

What is the sum of the first 5 odd numbers?

Easy:  $1+3+5+7+9=25$ .

How about the first 10 odd numbers?

A bit less easy:  $1+3+5+7+9+11+13+15+17+19=100$ .

How about the sum of the first  $n$  odd numbers?

(Here  $n$  could be any positive integer.

We just worked out the cases  $n=5$  &  $n=10$  above.)

We would like to use  $\Sigma$ -notation to phrase the question precisely.

ESSENTIAL STEP: We need to know a formula for successive odd numbers.

More precisely:

Find a formula for the  $k^{\text{th}}$  odd integer:

$k$	1	2	3	4	5	6	7	8	...
$k^{\text{th}}$ odd number	1	3	5	7	9	11	13	15	...

The  $k^{\text{th}}$  odd number is:  $2k-1$ .

The sum of the first  $n$  odd numbers, expressed in Sigma notation, is:

$$\sum_{k=1}^n (2k-1) = 1 + 3 + \dots + (2n-1)$$

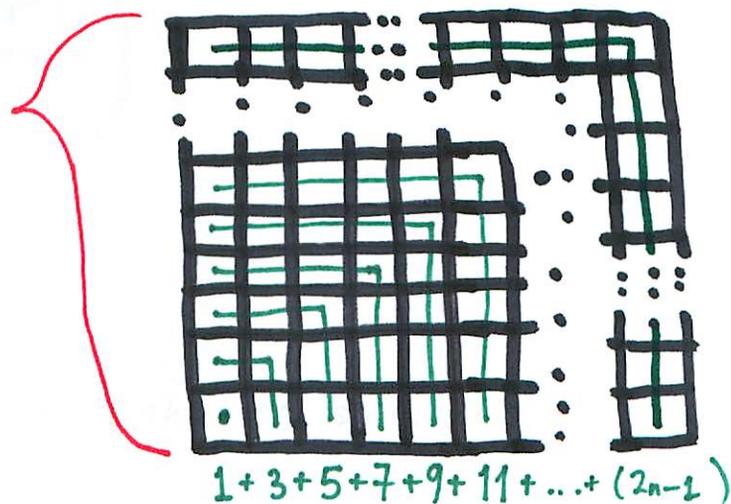
First odd #:  $1 = 2 \cdot 1 - 1$ 
2<sup>nd</sup> odd #:  $3 = 2 \cdot 2 - 1$ 
 $n^{\text{th}}$  odd #:  $2n - 1$

But, what is the value of the sum?!

Awesome cute answer:

$$\sum_{k=1}^n (2k-1) = n^2$$

The sum of the first  $n$  odd numbers equals the  $n^{\text{th}}$  square:  $n^2$ .



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More important formulas:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

[Gauss' problem is  $\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 50 \times 101 = 5,050$ .

Of course he did not know the formula beforehand!]

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k^3 = \left( \frac{n(n+1)}{2} \right)^2.$$

\*Crucial observation:

The dummy variable ("variable of summation") may never, under any circumstances whatsoever appear on the formula for a sum! (That is, on the "answer".)

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### Shameless Plug:

The most important formula  
in all of mathematics.\*

(\*) Although, probably, you won't squeeze much out of it in Calculus I — but wait until Calculus II!

$$\sum_{k=1}^n r^k = \frac{r - r^{n+1}}{1 - r} \quad \text{if } r \neq 1.$$

This is the formula to add the <sup>first n</sup> terms of a geometric progression:  $r, r^2, r^3, \dots, r^n$  where  $r$  is the fixed ratio ( $r \neq 1$ ) between any two consecutive terms.

Example: What rational number is  $0.\overbrace{111\dots 1}^{100 \text{ ones}}$ ?

$$0.\underbrace{11\dots 1}_{100} = \sum_{k=1}^{100} 0.1^k = \frac{0.1 - 0.1^{101}}{1 - 0.1} = \frac{10^{-1} - 10^{-101}}{1 - 10^{-1}} = \frac{10^{-101}}{10^{-1}} \cdot \frac{10^{100} - 1}{10 - 1}$$

$$\text{So, } 0.\underbrace{11\dots 1}_{100} = \frac{10^{100} - 1}{9 \cdot 10^{100}} = \frac{\overbrace{99\dots 9}^{100 \text{ 9's}}}{\underbrace{900\dots 0}_{100 \text{ zeros}}}$$

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Example continued...

What is  $0.111\dots$ ?

Clearly  $0.111\dots = \lim_{n \rightarrow \infty} 0.\overbrace{11\dots 1}^{n \text{ ones}}$

$$= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n 0.1^k \right) = \lim_{n \rightarrow \infty} \frac{0.1 - 0.1^n}{1 - 0.1}$$

~~State~~ It is very easy to see

that ~~that~~  $\lim_{n \rightarrow \infty} 0.1^n = 0$ .

[For instance,  $0.1^n = e^{-n \cdot \ln(0.1)} = e^{-n \ln(10)}$ .

Since  $-n \cdot \ln(10) \rightarrow -\infty$  as  $n \rightarrow \infty$

and  $\lim_{x \rightarrow -\infty} e^x = 0$ , we get  $\lim_{n \rightarrow \infty} 0.1^n = 0$ .]

Thus,  $0.111\dots = \lim_{n \rightarrow \infty} \frac{0.1 - 0.1^n}{1 - 0.1} = \frac{0.1 - 0}{1 - 0.1} = \frac{10^{-1}}{1 - 10^{-1}}$

$$0.111\dots = \frac{10^{-1}}{1 - 10^{-1}} = \frac{1}{10 - 1} = \frac{1}{9}.$$

# Rules for Sums in $\Sigma$ -Notation

Sum / Difference:

$$\sum_{k=1}^n (a_k \pm b_k) = \sum_{k=1}^n a_k \pm \sum_{k=1}^n b_k.$$

("addition is commutative")

Constant Factor:  $c = \text{any fixed constant.}$

$$\sum_{k=1}^n c a_k = c \cdot \sum_{k=1}^n a_k.$$

("multiplication distributes over addition")

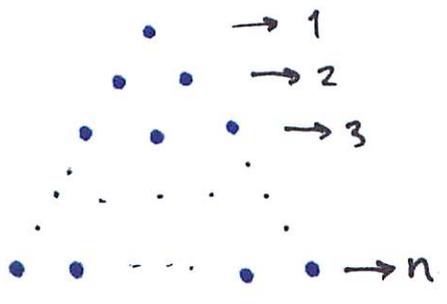
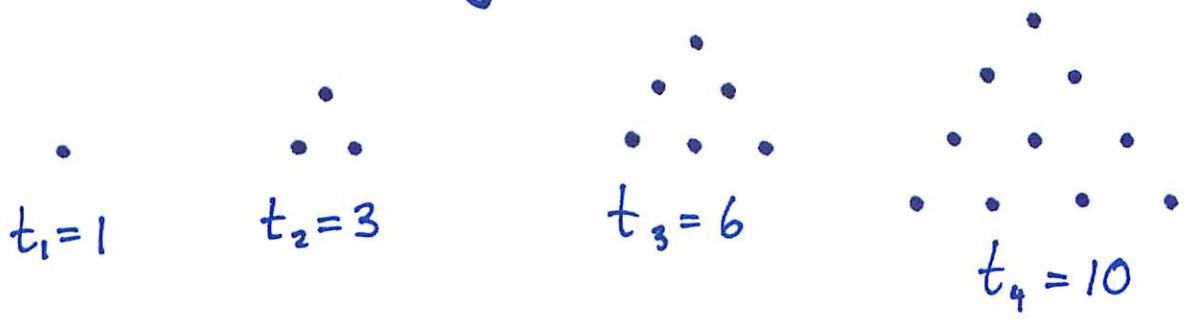
Constant Summand:  $c = \text{any fixed constant.}$

$$\sum_{k=1}^n c = n \cdot c.$$

("multiplication is repeated addition of a fixed number to itself")

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## Example: Triangular Numbers



$$t_n = \frac{n(n+1)}{2}$$

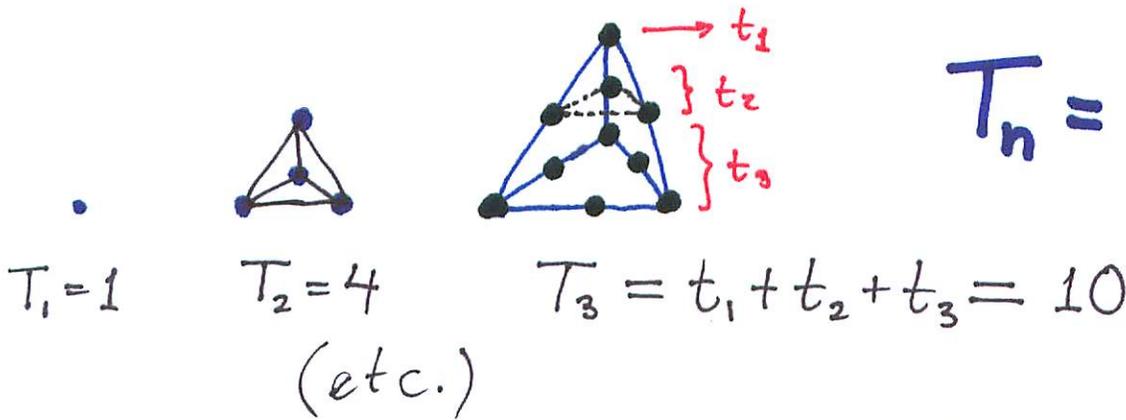
$$t_n = 1+2+\dots+n$$

$$t_n = 1+2+\dots+n = \sum_{k=1}^n k = \frac{n(n+1)}{2} .$$

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## Example: Tetrahedral Numbers.



$$T_n = \frac{n(n+1)(n+2)}{6}$$

$$T_n = t_1 + t_2 + \dots + t_n = \sum_{k=1}^n t_k$$

$$T_n = \sum_{k=1}^n \frac{k(k+1)}{2} = \sum_{k=1}^n \frac{1}{2}(k^2 + k)$$

$$\stackrel{\text{(constant multiple)}}{=} \frac{1}{2} \sum_{k=1}^n (k^2 + k) = \frac{1}{2} \left( \sum_{k=1}^n k^2 + \sum_{k=1}^n k \right) \quad \text{(Addition rule)}$$

$$= \frac{1}{2} \left[ \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right] \quad \text{(Formulas for sum of } k \text{ \& } k^2)$$

$$= \frac{n(n+1)(n+2)}{6} \quad \swarrow \text{Algebra.}$$